

Neural Networks: Basic Structure, Representation Power

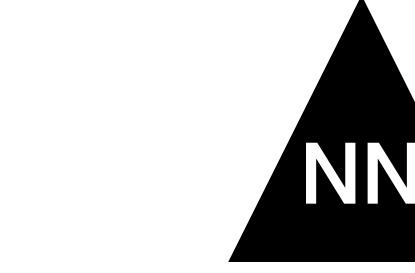
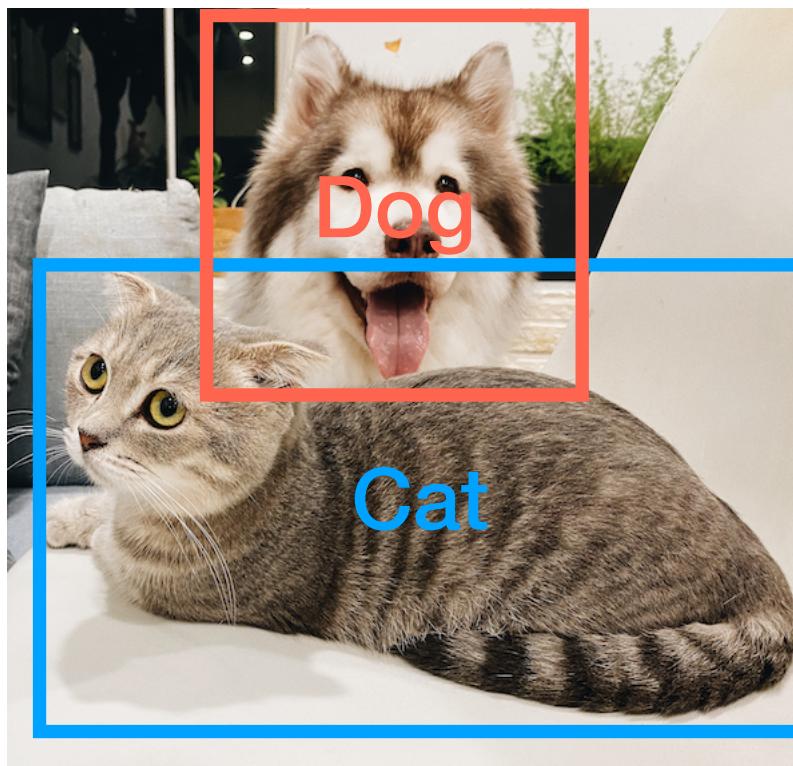
Machine Learning Course - CS-433
Oct 30, 2024
Nicolas Flammarion



Deep Learning: ML Breakthrough

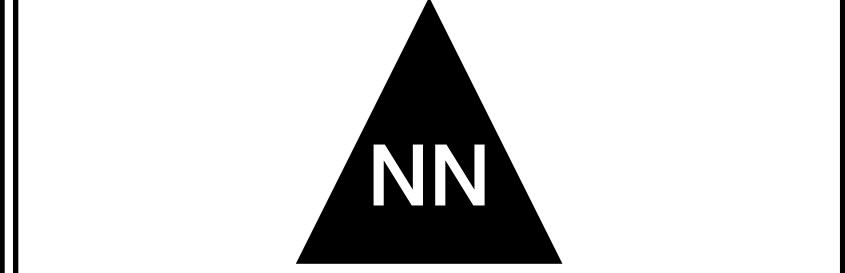
State of the art performance across a broad range of useful tasks

Object Detection



Translation

elle est très intelligente



she is very smart

Speech to Text

hey Siri next song

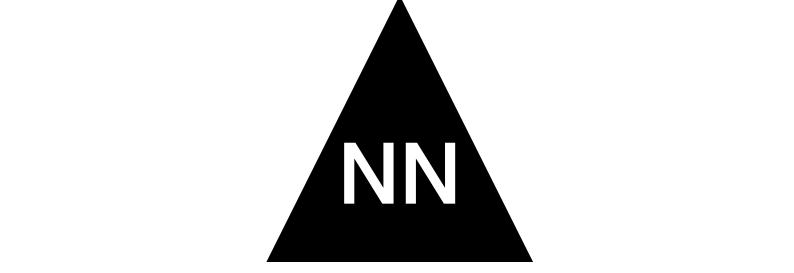
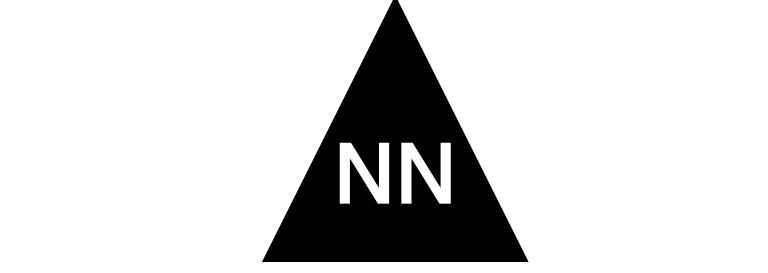
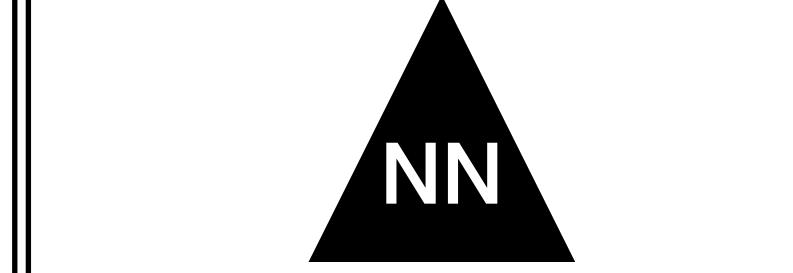


Image Generation

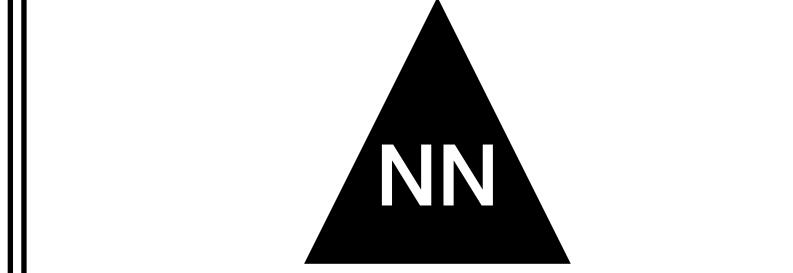
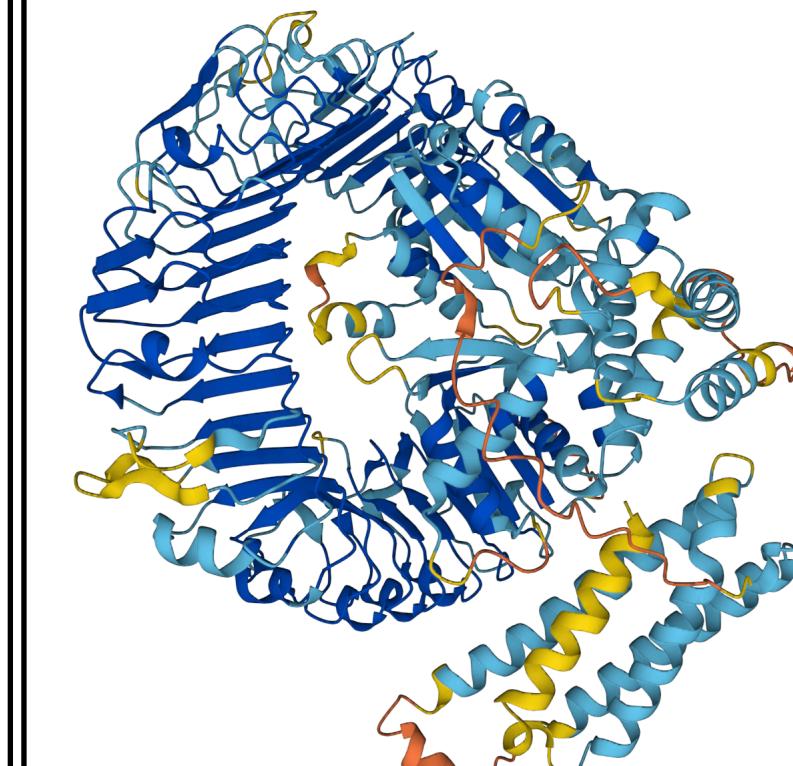


student studying machine learning

Playing Chess



Protein Folding



MAGELVSFAVNKLWDLLSHEYTLFQGVEDQ
VAELKSDLNLLKSFLKDADAKKHTSALVRY
CVEEIKDIVYDAEDVLETFVQKEKLGTTSGIR
KHIKRLTCIVPDRREIALYIGHVSKRITRVIRD
MQSFGVQQMIVDDYMHPLRNRREREIRRTF
PKDNESGFVALEENVKKLVGYFVEEDNYQV
VSITGMGLGLKTTLARQVFNHDMDVTKKFD
KLAWSVSQDFTLKNWQNILGDLKPKEEE
TKEEEKKILEMTEYTLQRELYQLLEMSKSLI
VLDDIWKKEDWEVIKPIFPPTKGWKLLLSR
NESIVAPNTKYFNFKPECLKTDDSWKLQF
RIAFPINDASEFEIDEEMEKLGEKMIEHCGG
LPLAIKVGGMLAEKYTSMDWRRRLSENIGS
HLVGGRTNFNDDNNNSCNYVLSLSFEELPS
YLKHCFLYLAHFPEDYEIKVENLSYYWAAEE
IFQPRHYDGEIIRDVGDVYIEELVRRNM.....

From Linear Models to NNs

Supervised learning: we observe some data $S_{\text{train}} = \{x_n, y_n\}_{n=1}^N \in \mathcal{X} \times \mathcal{Y}$

→ given a new x , we want to predict its label y

Linear prediction (with augmented features): $y = f_{\text{Lin}}(x) = \phi(x)^T w$

Features are given

Prediction with a neural network (NN):

$$y = f_{\text{NN}}(x) = f(x)^T w$$

Function implemented by the NN
parameters

First layers transform the input
into a good representation

Last layer is performing
a linear prediction

Why Neural Networks?

Linear models vs. neural networks:

$$y = f_{\text{Lin}}(x) = \phi(x)^T w \text{ (fixed } \phi\text{)} \quad \text{vs.} \quad y = f_{\text{NN}}(x) = f(x)^T w \text{ (learned } f\text{)}$$

Classical machine learning:

- Relatively simple models on top of **features handcrafted by domain experts**
- Only works well when used with **good handcrafted features**

Deep learning:

- Large neural networks that **learn features directly from the data**
- Can be viewed as a complicated feature extractor + linear prediction
- **Requires large amounts of data and compute to train**
- Quality often continues to **improve substantially** with more data and larger models

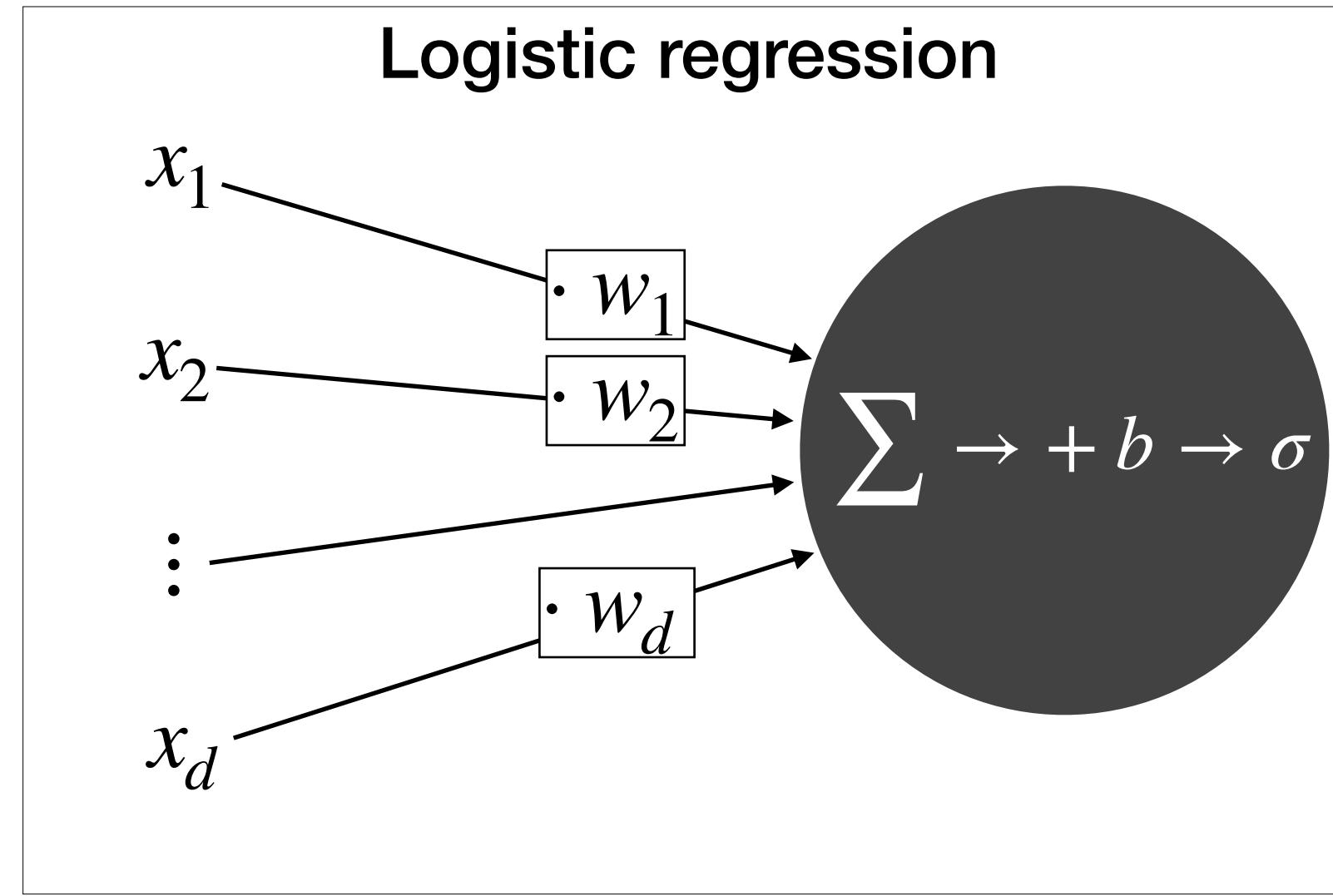
The Basic Structure of Neural Networks

Recap: Logistic Regression

Logistic Regression:

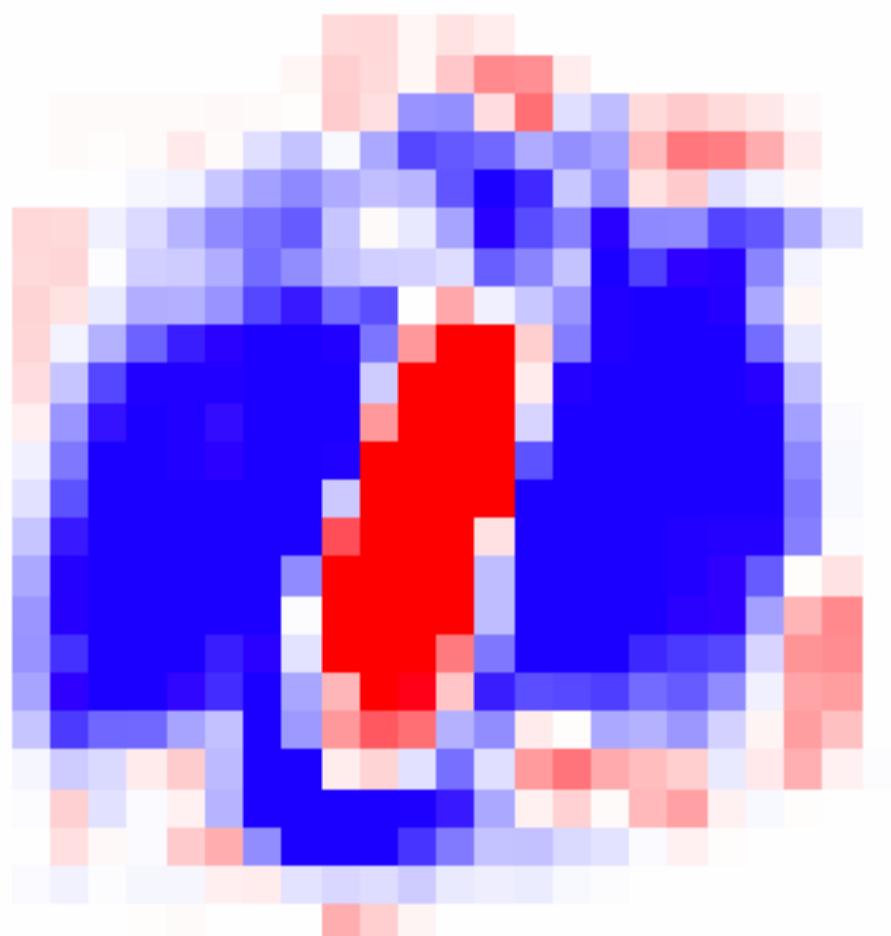
$$f_{LR}(x) = p(1 | x) = \sigma(x^T w + b) = \sigma\left(\sum_{i=1}^d x_i w_i + b\right)$$

where $\sigma(z) = (1 + \exp(-z))^{-1}$ is the *sigmoid*



Pattern-matching perspective:

- **Task:** classify between digit 1 and digit 0 using logistic regression
- We learn a **single pattern** w which we apply elementwise to each input x (very restrictive!)
- We classify the digit as 1 if $p(1 | x) \geq 0.5$ or, equivalently, if $\sum_{i=1}^d x_i w_i \geq -b$

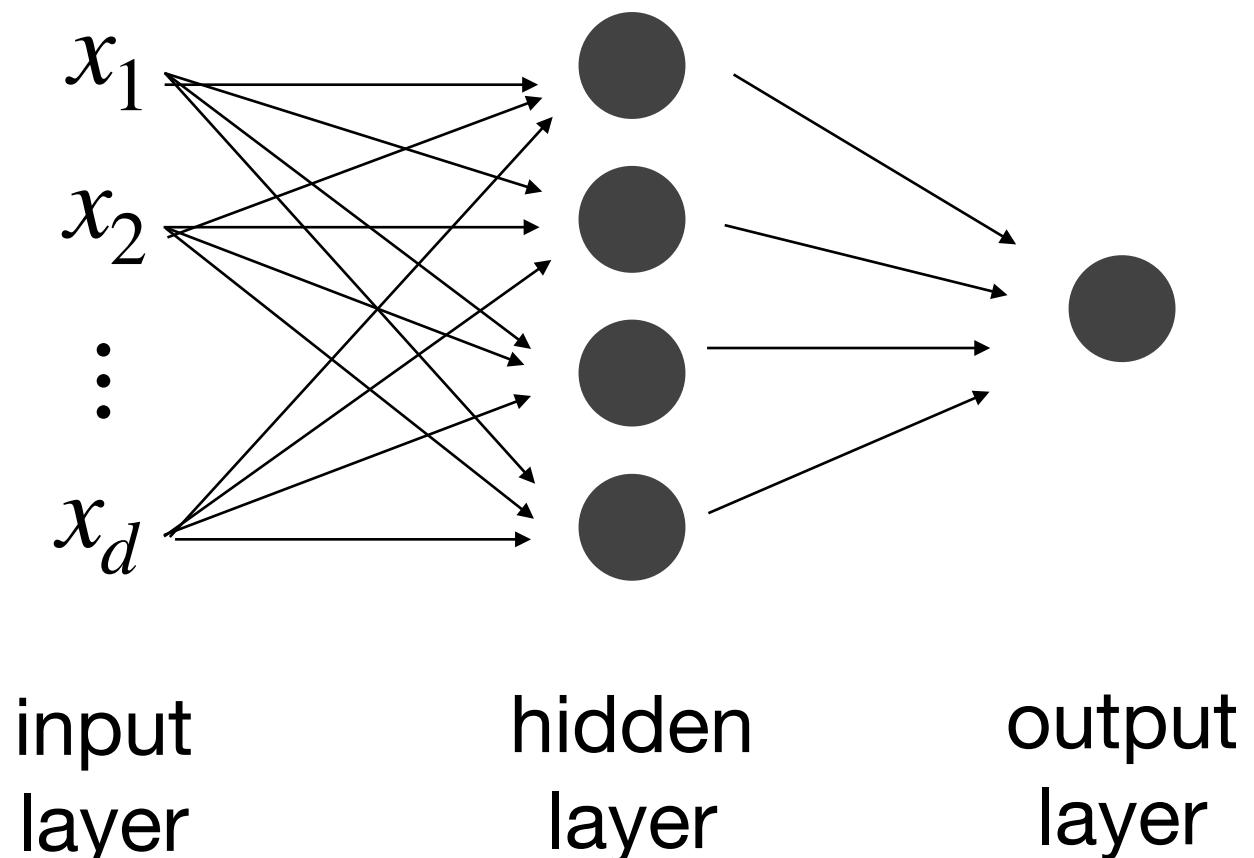


Two-Layer Neural Networks

Two-layer neural network (aka two-layer perceptron):

- Stack logistic regression units $\phi(x)_j = \sigma\left(\sum_{i=1}^d x_i w_{i,j}^{(1)} + b_j\right)$ (aka *hidden units* or *neurons*) in a *hidden layer*
- Aggregate the hidden units using a linear function $\phi(x)^\top w^{(2)}$

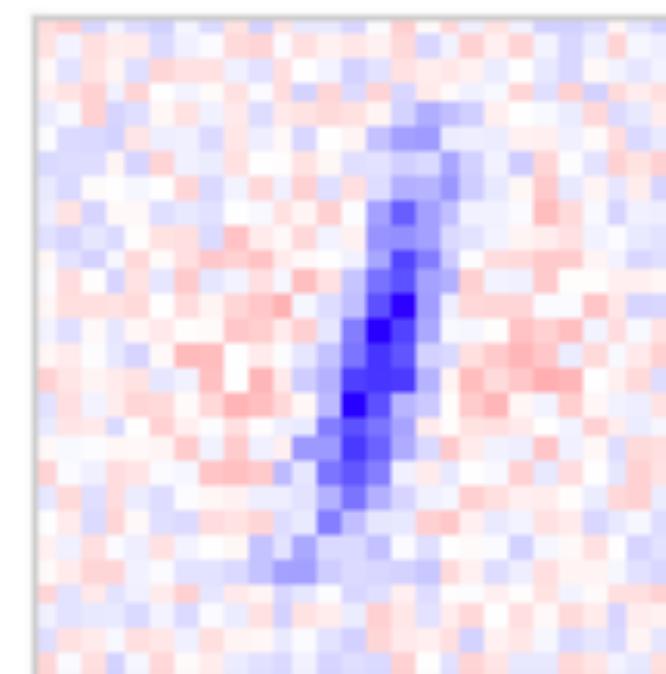
Two-layer neural network



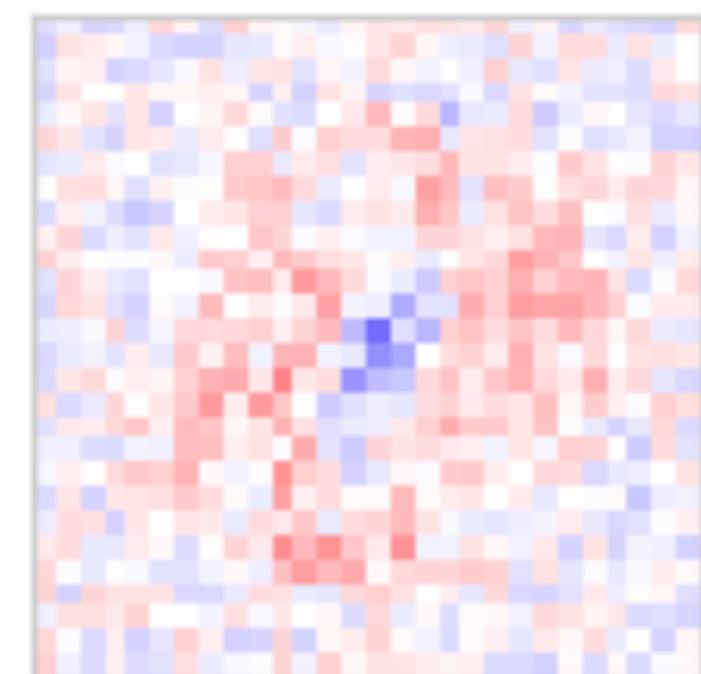
Pattern-matching perspective:

- **Task:** classify between digit 1 and digit 0 with a two-layer neural network
- **Each** hidden unit learns a **different pattern** (not necessarily interpretable!)
- We classify based on a linear combination of these patterns \Rightarrow much more flexible

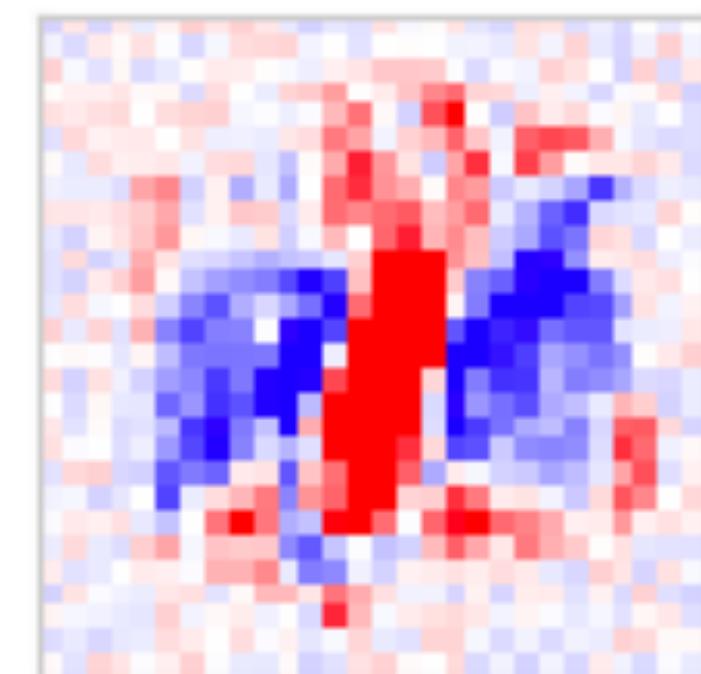
learned $w_{:,1}^{(1)}$



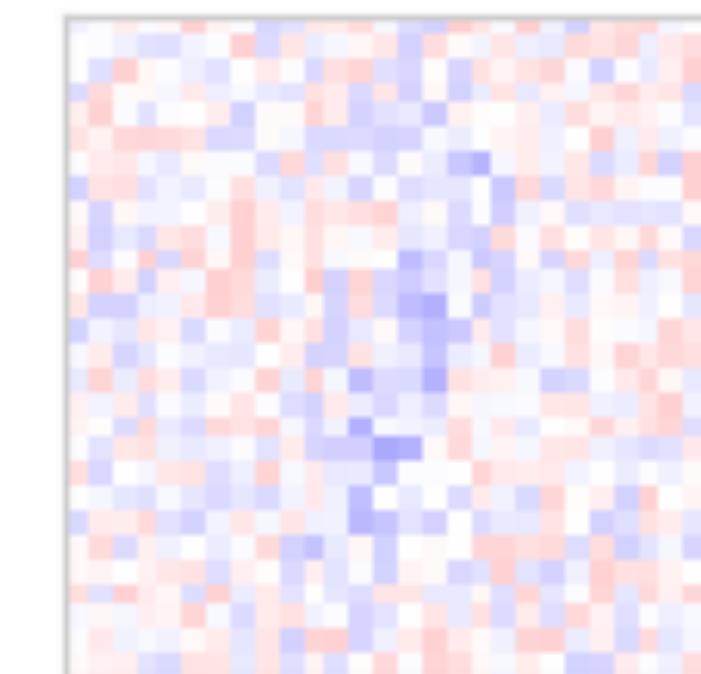
learned $w_{:,2}^{(1)}$



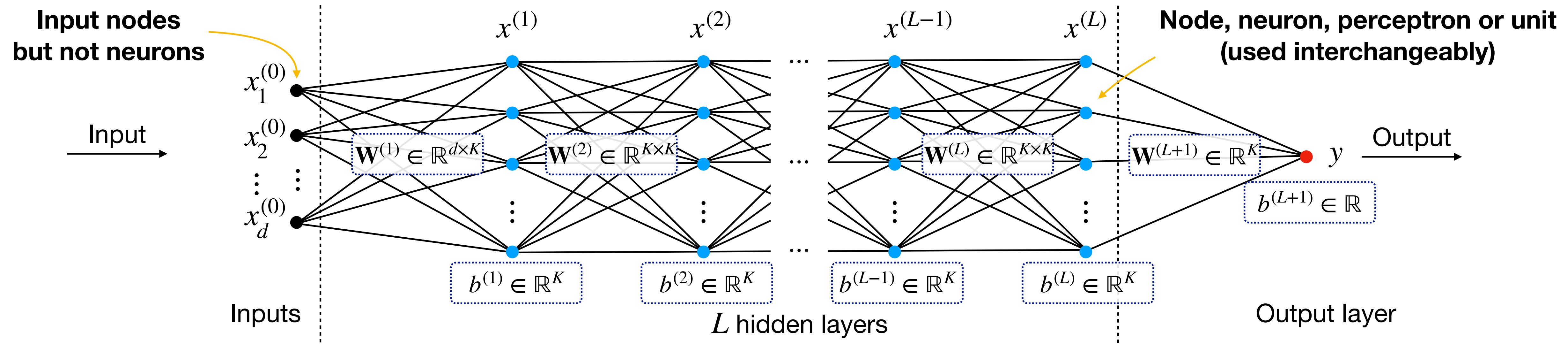
learned $w_{:,3}^{(1)}$



learned $w_{:,4}^{(1)}$



Multi-layer Neural Networks



Assume L hidden layers with K neurons each + output layer with single node

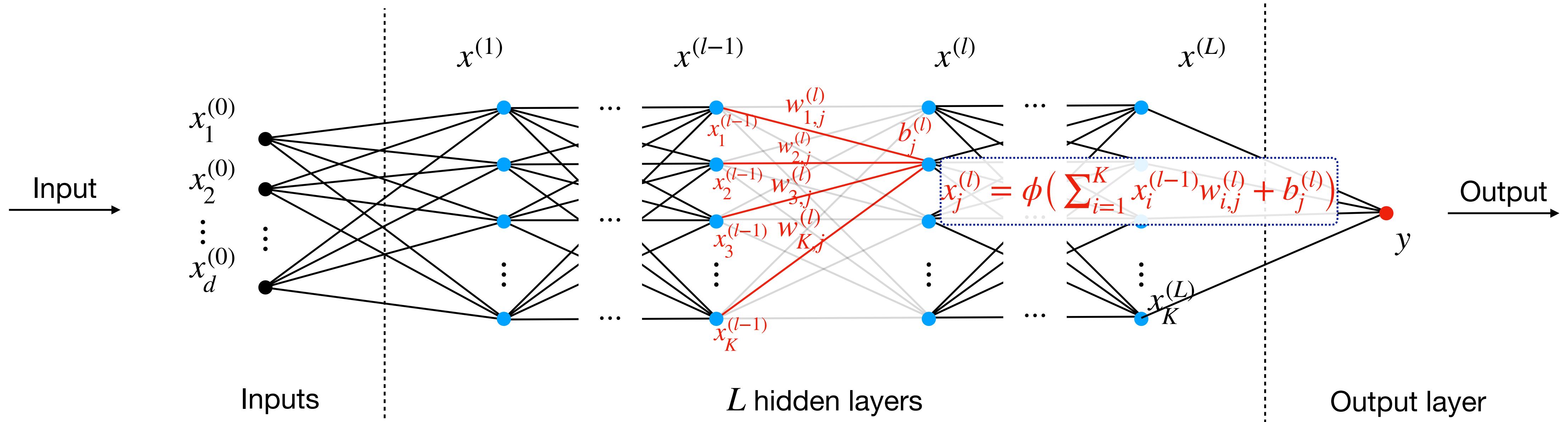
Learnable Parameters: Weight matrices $\mathbf{W}^{(l)}$ and bias vectors $b^{(l)}$ for $1 \leq l \leq L + 1$

- Each column of $\mathbf{W}^{(l)}$ corresponds to the weights of one perceptron

Outputs of hidden layer l given by vector: $x^{(l)} = f^{(l)}(x^{(l-1)}) := \phi((\mathbf{W}^{(l)})^\top x^{(l-1)} + b^{(l)})$

Elementwise

Single Neuron View



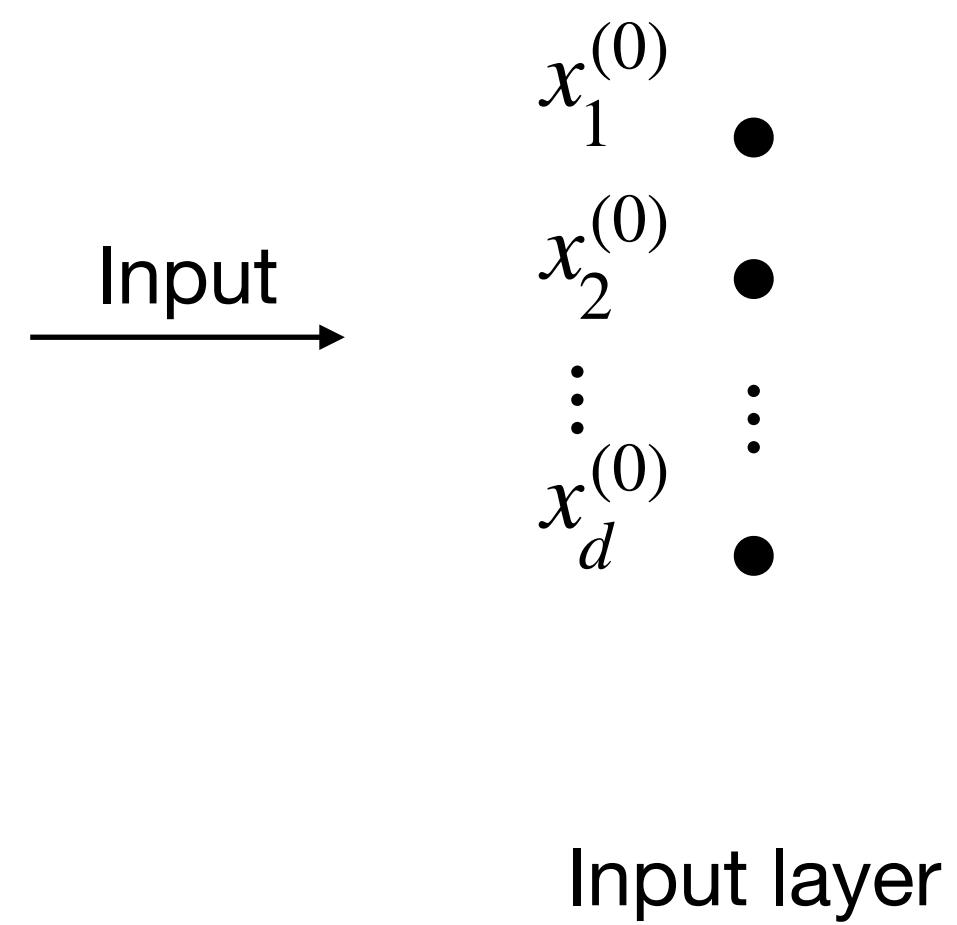
$$x_j^{(l)} = \phi\left(\sum_{i=1}^K x_i^{(l-1)} w_{i,j}^{(l)} + b_j^{(l)}\right)$$

Important: ϕ is non-linear
otherwise we can only
represent linear functions

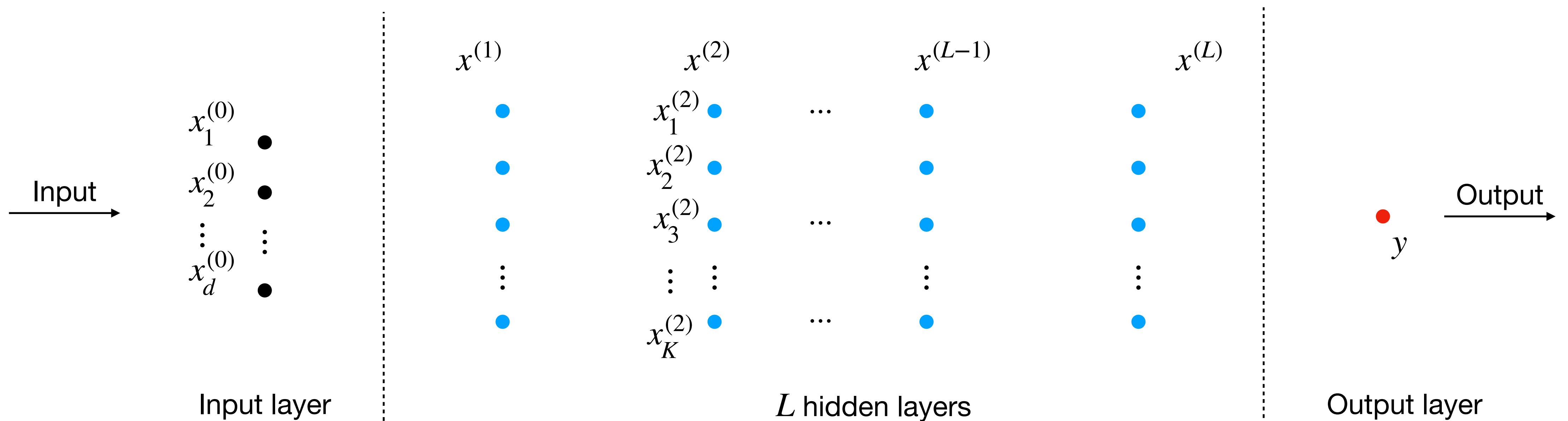
weight of the edge going from node i
in layer $l - 1$ to node j in layer l

bias term associated with
node j in layer l

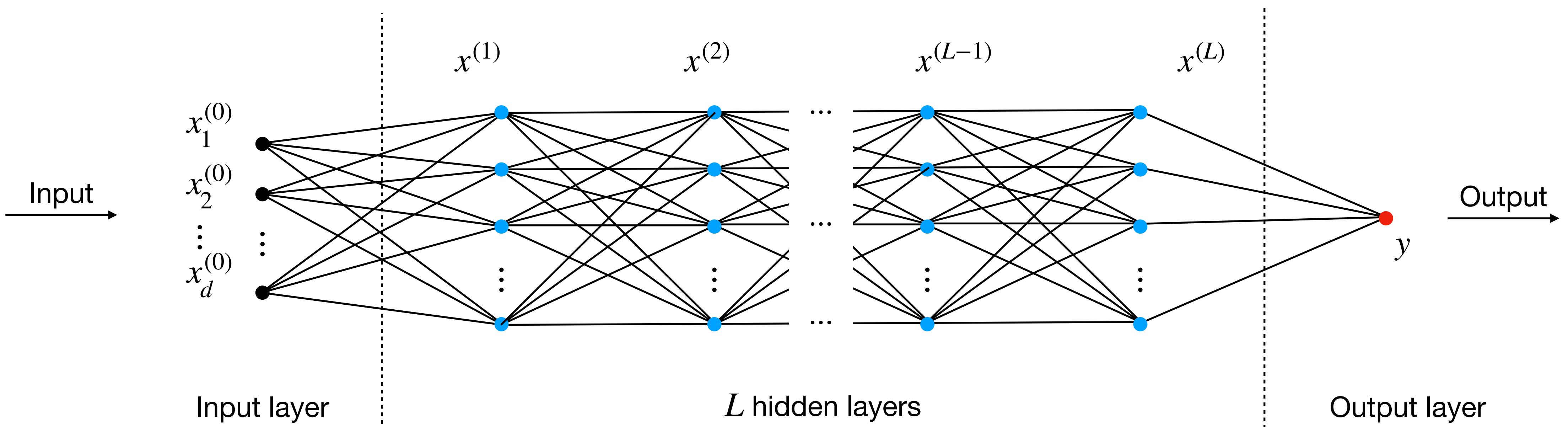
The input x corresponds to the input layer



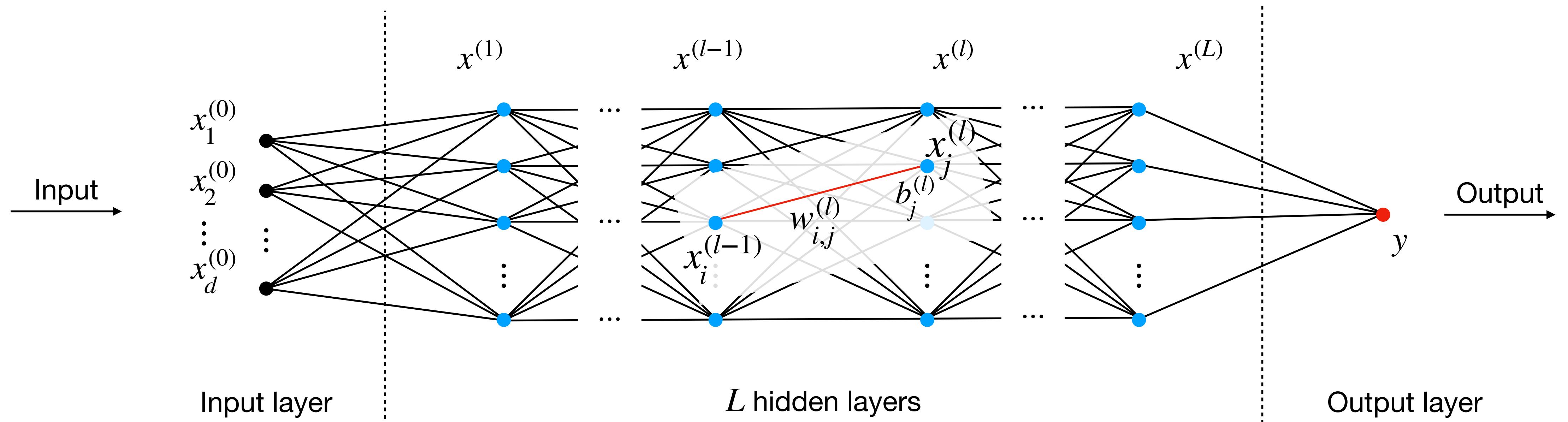
The signal propagates in many other layers



Fully connected NN: each node in one layer is connected to every nodes in the previous layer

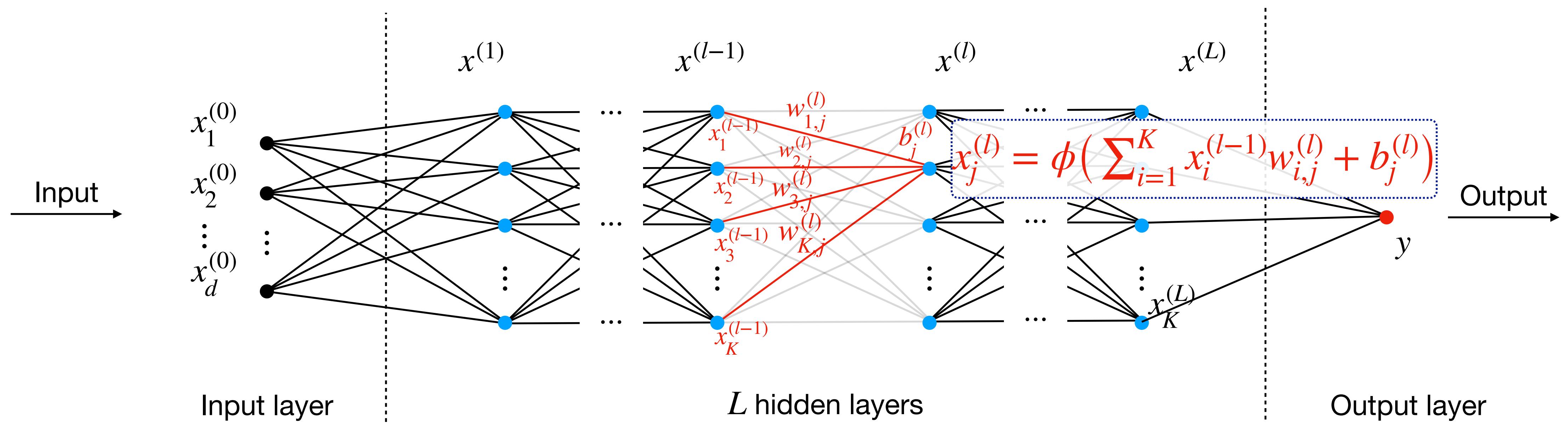


The weights and the biases are the parameters of the NN to be learnt



- $w_{i,j}^{(l)}$: weight of the edge going from node i in layer $l - 1$ to node j in layer l
- $b_j^{(l)}$: bias term associated with node j in layer l

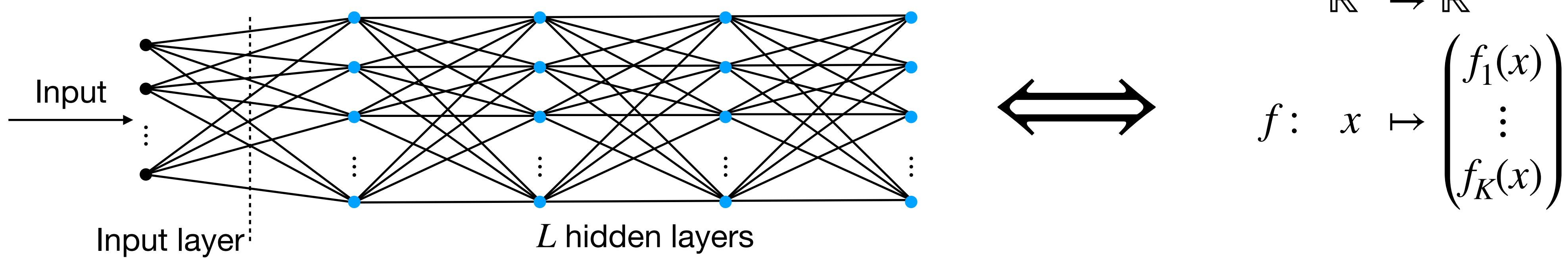
The function value at the l^{th} layer is defined recursively



NNs extract suitable features from the input

A NN can be decomposed into a feature extractor and the output layer.

Feature extractor from \mathbb{R}^d to \mathbb{R}^K :

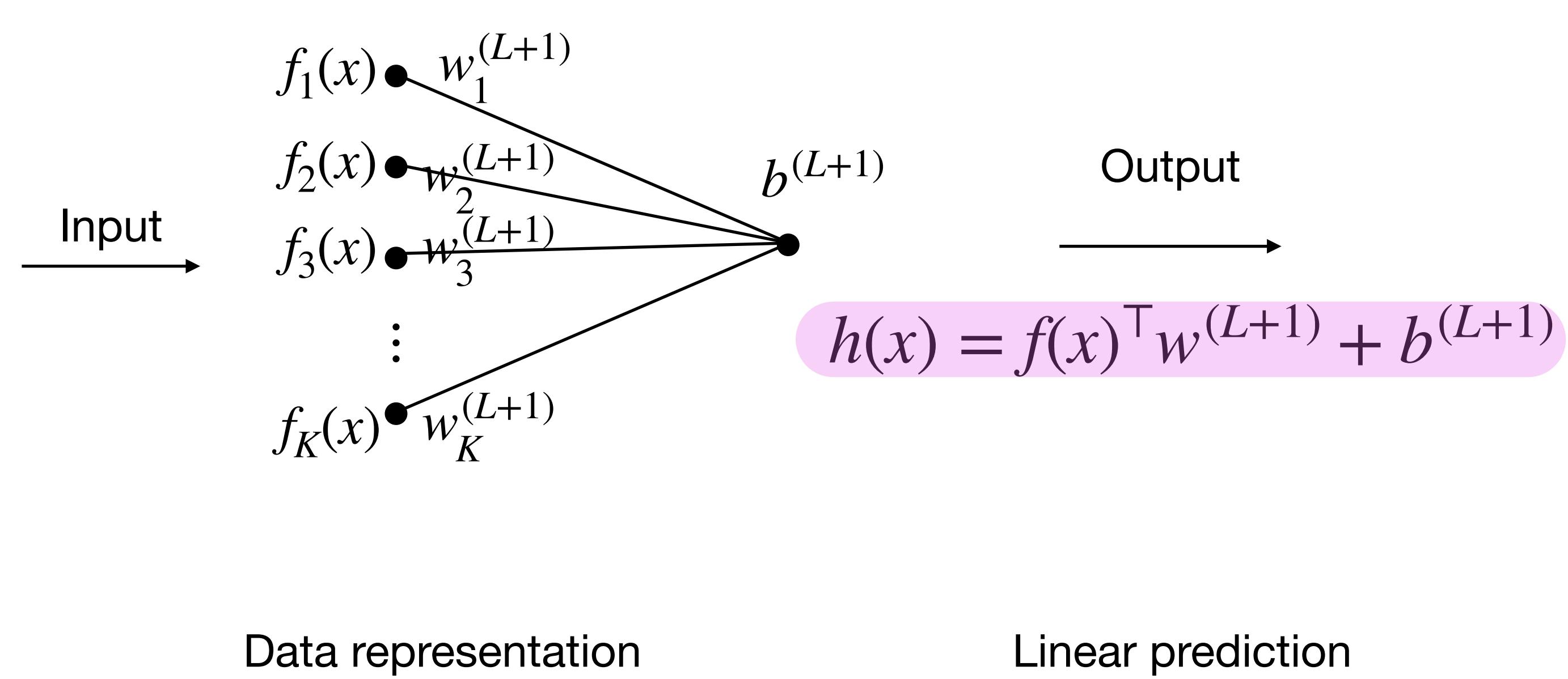


This function is defined by

- The biases $\{b^{(l)}\}_{l \in [L]}$ and weights $\{\mathbf{W}^{(l)}\}_{l \in [L]}$ we learn
→ $O(dK + K^2L) \approx O(K^2L)$ parameters
- The activation function ϕ we pick

In practice: both L and K are large - overparametrized NNs

Neural Network: Inference Time



Regression

$$h(x)$$

Binary Classification with $y \in \{-1, 1\}$

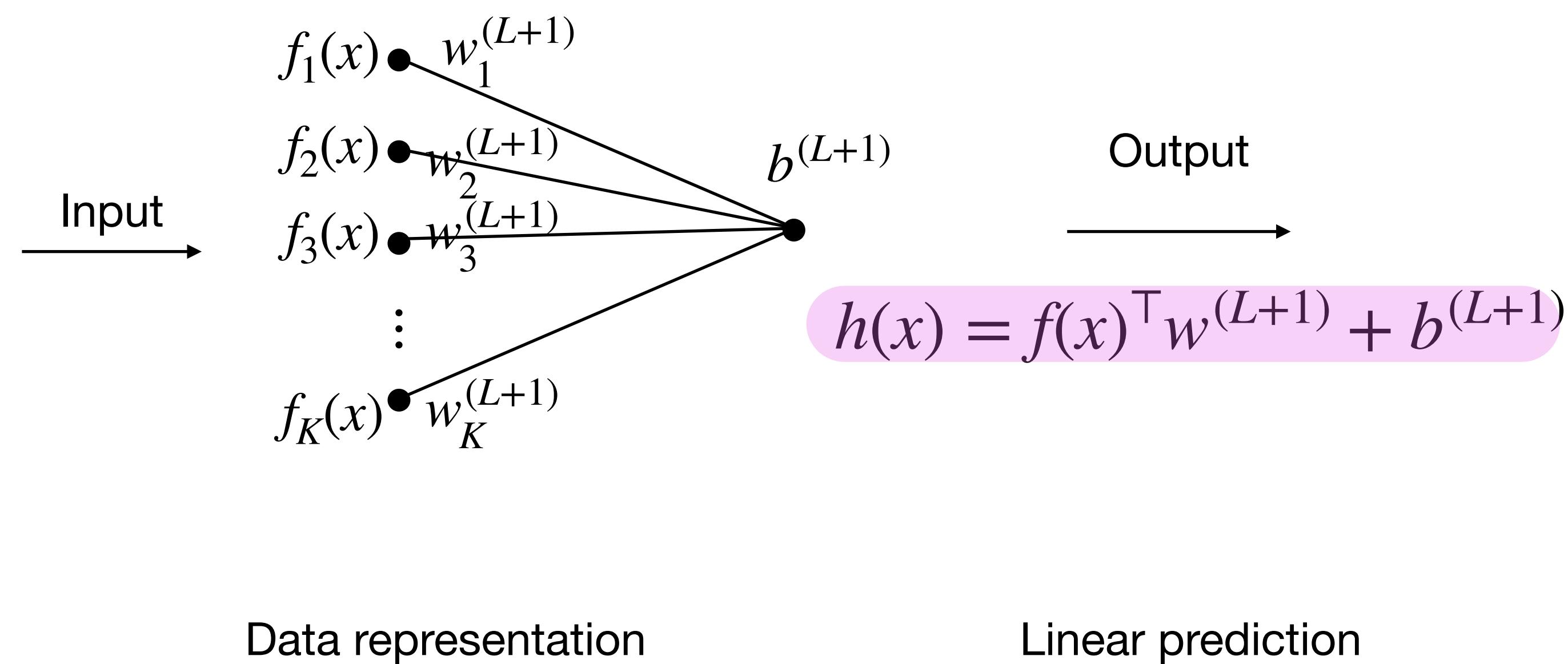
$$sign(h(x))$$

Multi-Class Classification with $y \in \{1, \dots, K\}$
(see the exercise!)

$$argmax_{c \in \{1, \dots, K\}} h(x)_c$$

With a suitable representation of the data $f(x)$, the last layer only performs a linear regression or classification step

Neural Network: Training Time



Regression

$$\ell(y, h(x)) = (h(x) - y)^2$$

Binary Classification with $y \in \{-1, 1\}$

$$\ell(y, h(x)) = \log(1 + \exp(-yh(x)))$$

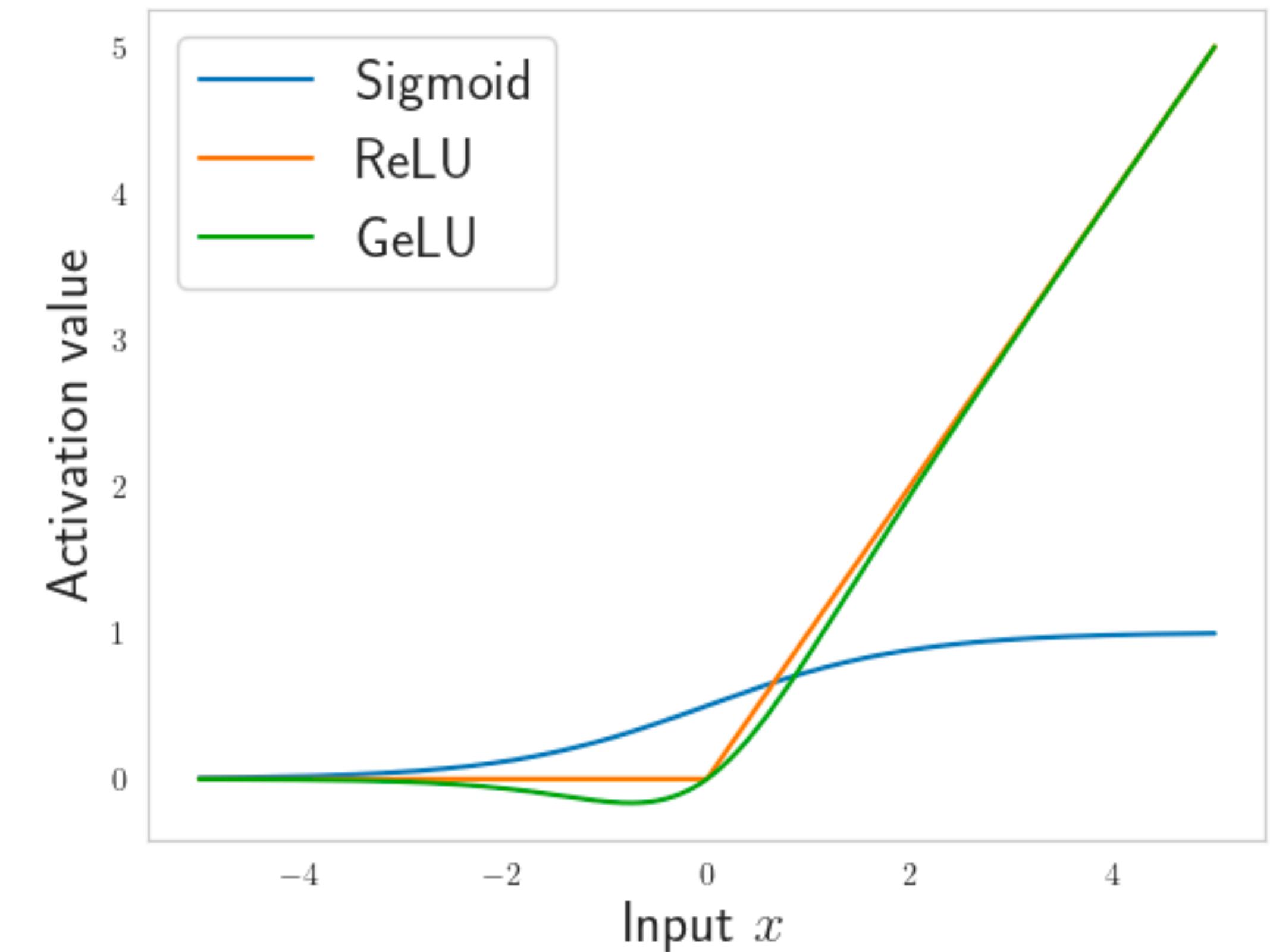
Multi-Class Classification with $y \in \{1, \dots, K\}$
(see the exercise!)

$$\ell(y, h(x)) = - \log \frac{e^{h(x)_y}}{\sum_{i=1}^K e^{h(x)_i}}$$

With a suitable representation of the data $f(x)$, the last layer only performs a linear regression or classification step

Popular activation functions

- Sigmoid: $\phi(x) = \sigma(x) = \frac{1}{1 + e^{-x}}$
- ReLU: $\phi(x) = (x)_+ = \max\{0, x\}$
- GELU: $\phi(x) = x \cdot \Phi(x) \approx x \cdot \sigma(1.702x)$
- In practice: ReLU and GeLU are the most widely used activation functions



Three theoretical questions in Deep Learning

- **Expressive power** of NNs: Why can neural networks approximate the functions we are interested in so effectively?
- **Success of naive optimization**: Why does gradient descent consistently lead to a good local minimum?
- **Generalization miracle**: Why don't neural networks overfit, despite having so many parameters?

Representational Power of NNs

Barron's universal approximation theorem

Let $f: \mathbb{R}^d \rightarrow \mathbb{R}$ and define $\hat{f}(\omega) = \int_{\mathbb{R}^d} f(x) e^{-i\omega^\top x} dx$, its Fourier transform

Assumption: $\int_{\mathbb{R}^d} |\omega| |\hat{f}(\omega)| d\omega \leq C$ (smoothness assumption)

Claim: For all $n \geq 1$ and $r > 0$, there exists a function f_n of the form

$$f_n(x) = \sum_{j=1}^n c_j \phi(x^\top w_j + b_j) + c_0$$

such that

$$\int_{|x| \leq r} (f(x) - f_n(x))^2 dx \leq \frac{(2Cr)^2}{n}$$

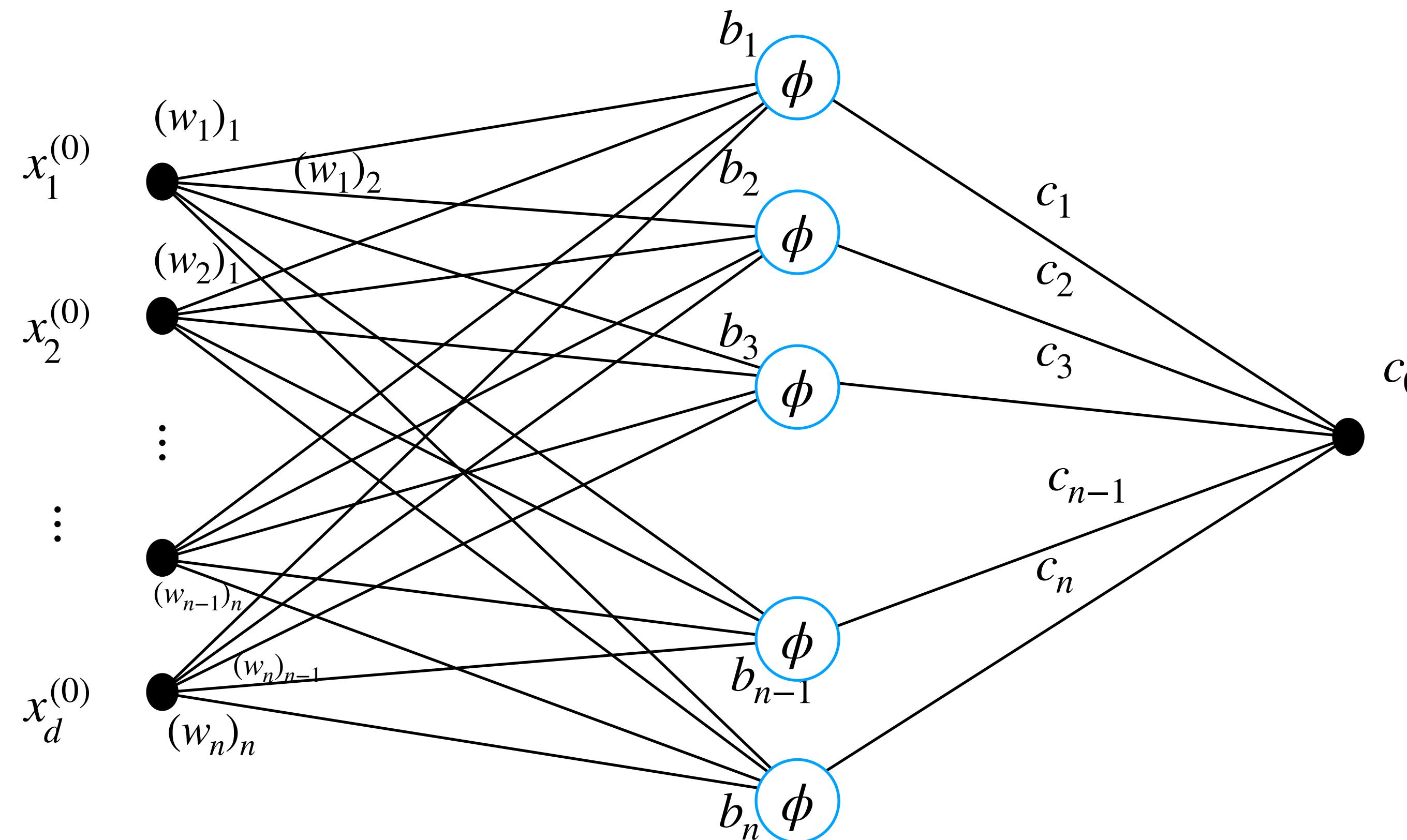
All sufficiently smooth function can be approximated by a one-hidden-layer NN

$$\int_{|x| \leq r} (f(x) - f_n(x))^2 dx \leq \frac{(2Cr)^2}{n}$$

- The more neurons allowed, the smaller the error.
- The smoother the function (the smaller C), the smaller the error
- The larger the domain (the larger r), the greater the error
- Approximation is in average (in ℓ_2 -norm)
- Applicable for any “sigmoid-like” activation function

The function f_n is a one-hidden-layer NN with n nodes

$$f_n(x) = \sum_{j=1}^n c_j \phi(x^\top w_j + b_j) + c_0 = c^\top \phi(W^\top x + b) + c_0$$



Intuitive proof: approximation in ℓ_1 -norm

Simple and intuitive explanation of a slightly different result:

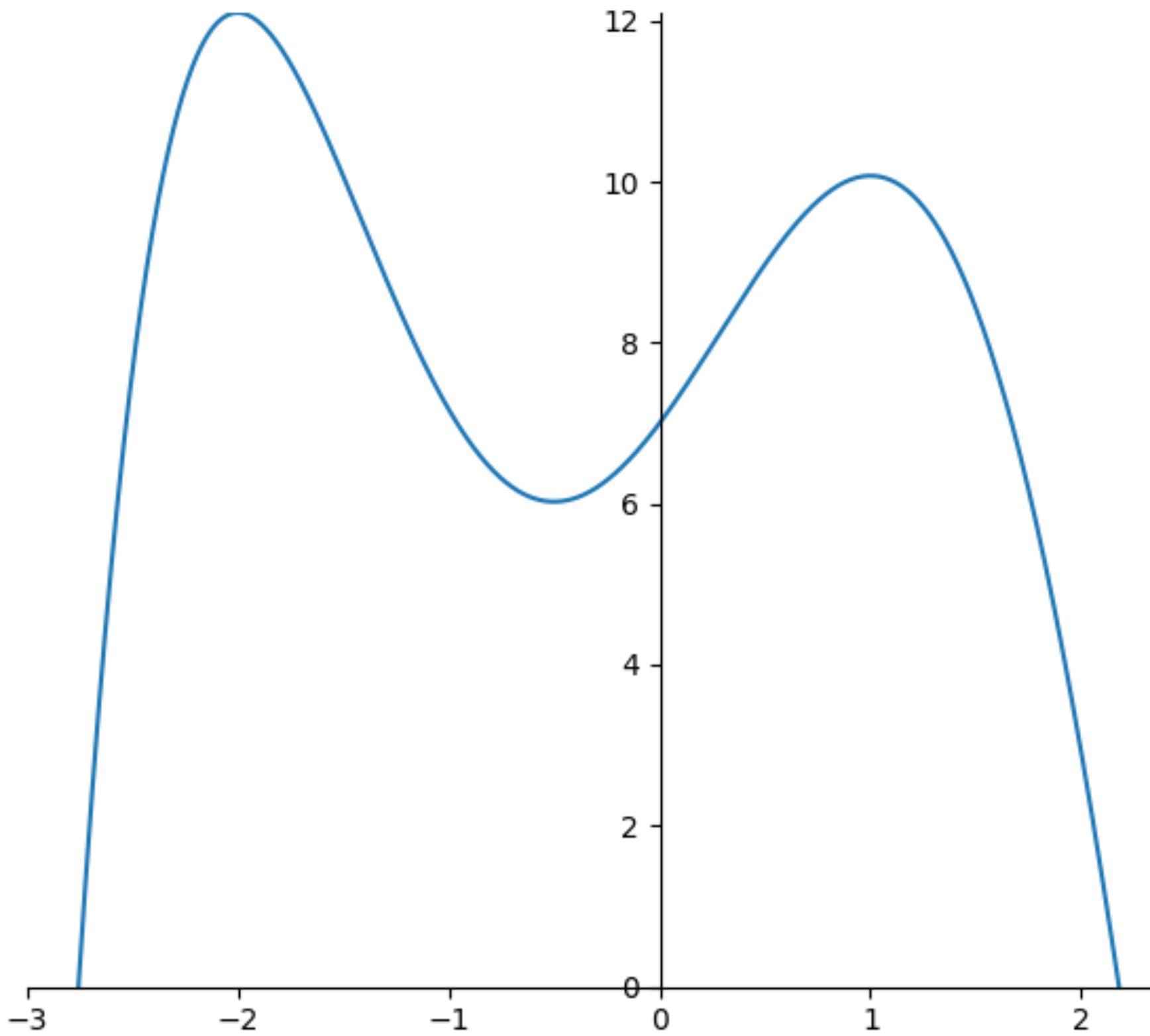
“A NN with sigmoid activation and at most two hidden layers can approximate well a smooth function in ℓ_1 -norm”

Approximation in ℓ_1 -norm:

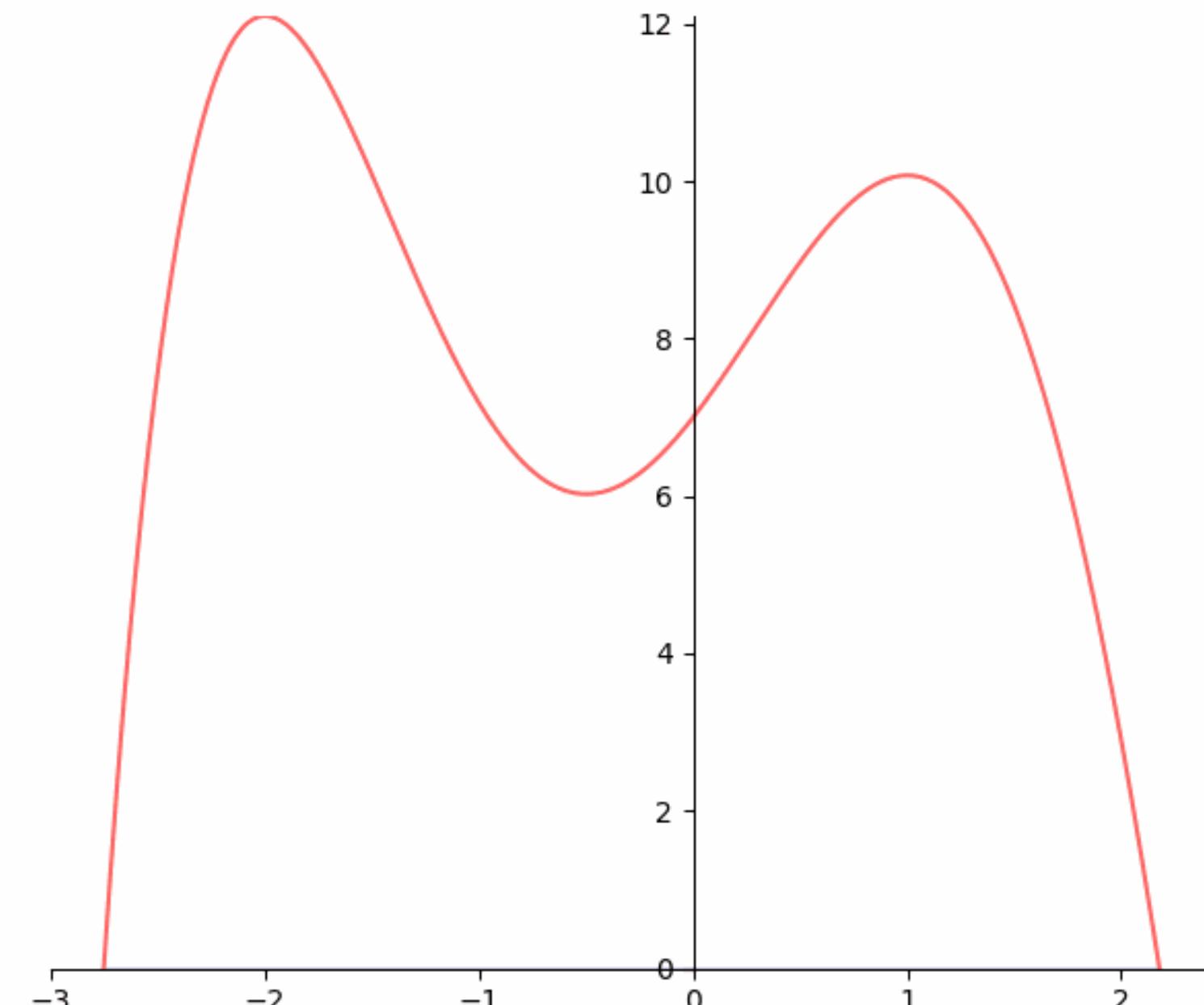
$$\int_{|x| \leq r} |f(x) - f_n(x)| dx \leq \text{Something small}$$

ℓ_1 approximation: proof by picture

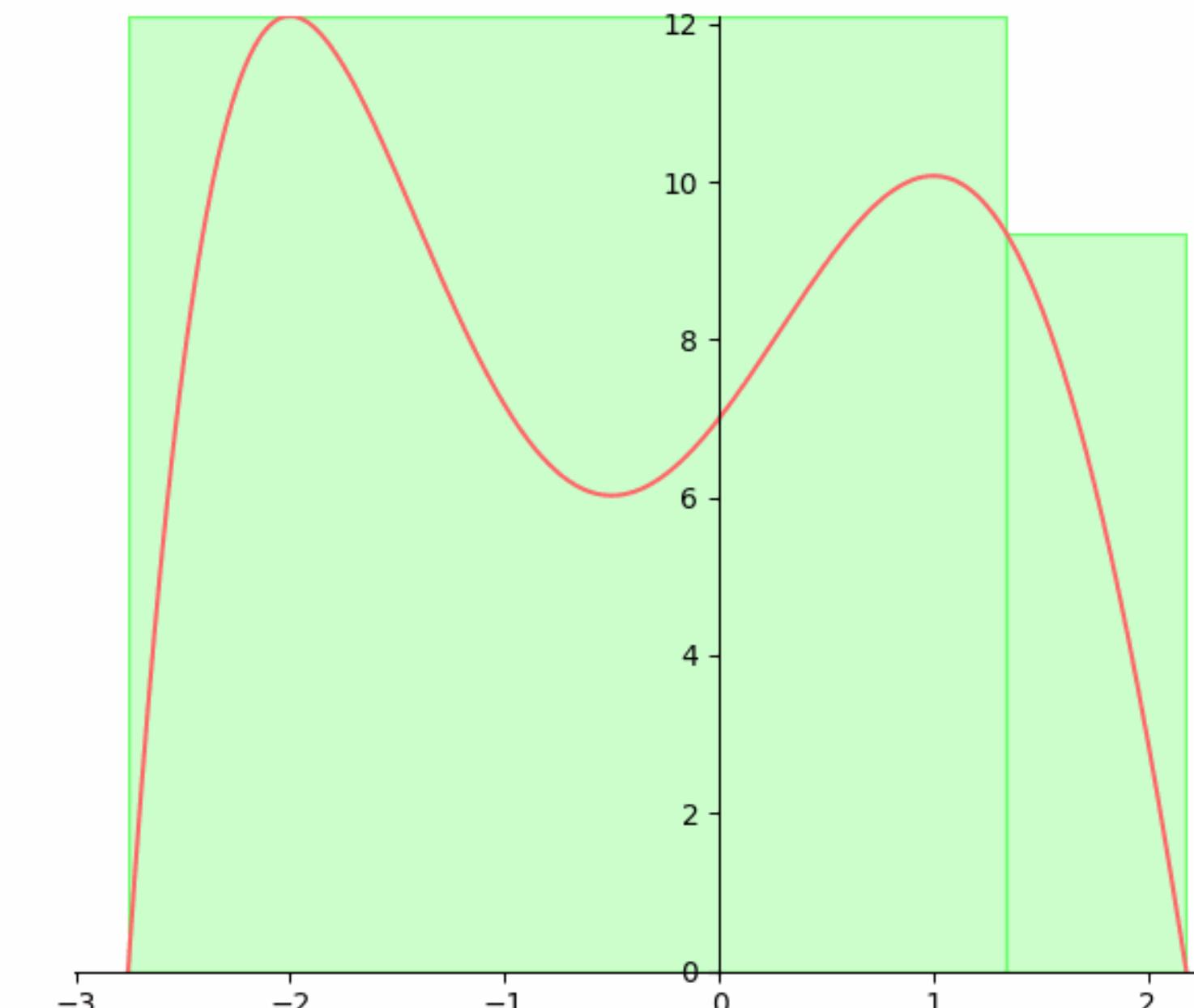
Consider a function $f: \mathbb{R} \rightarrow \mathbb{R}$ on a bounded domain



ℓ_1 approximation: approximation of the function by a sum of rectangle functions



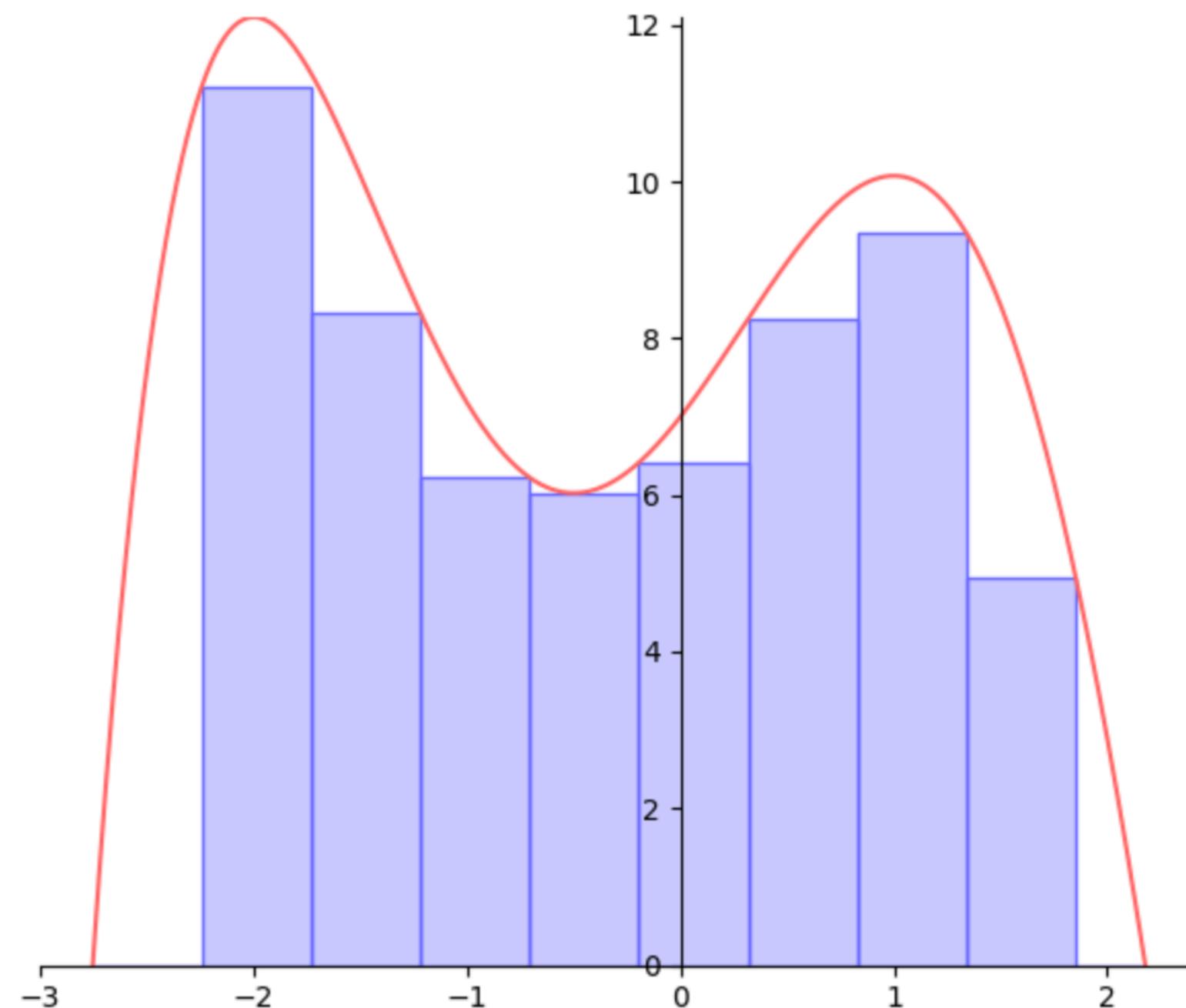
From below



From above

The function is Riemann integrable - its integral can be approximated to any desired accuracy using “lower” and “upper” sums of the area of rectangles

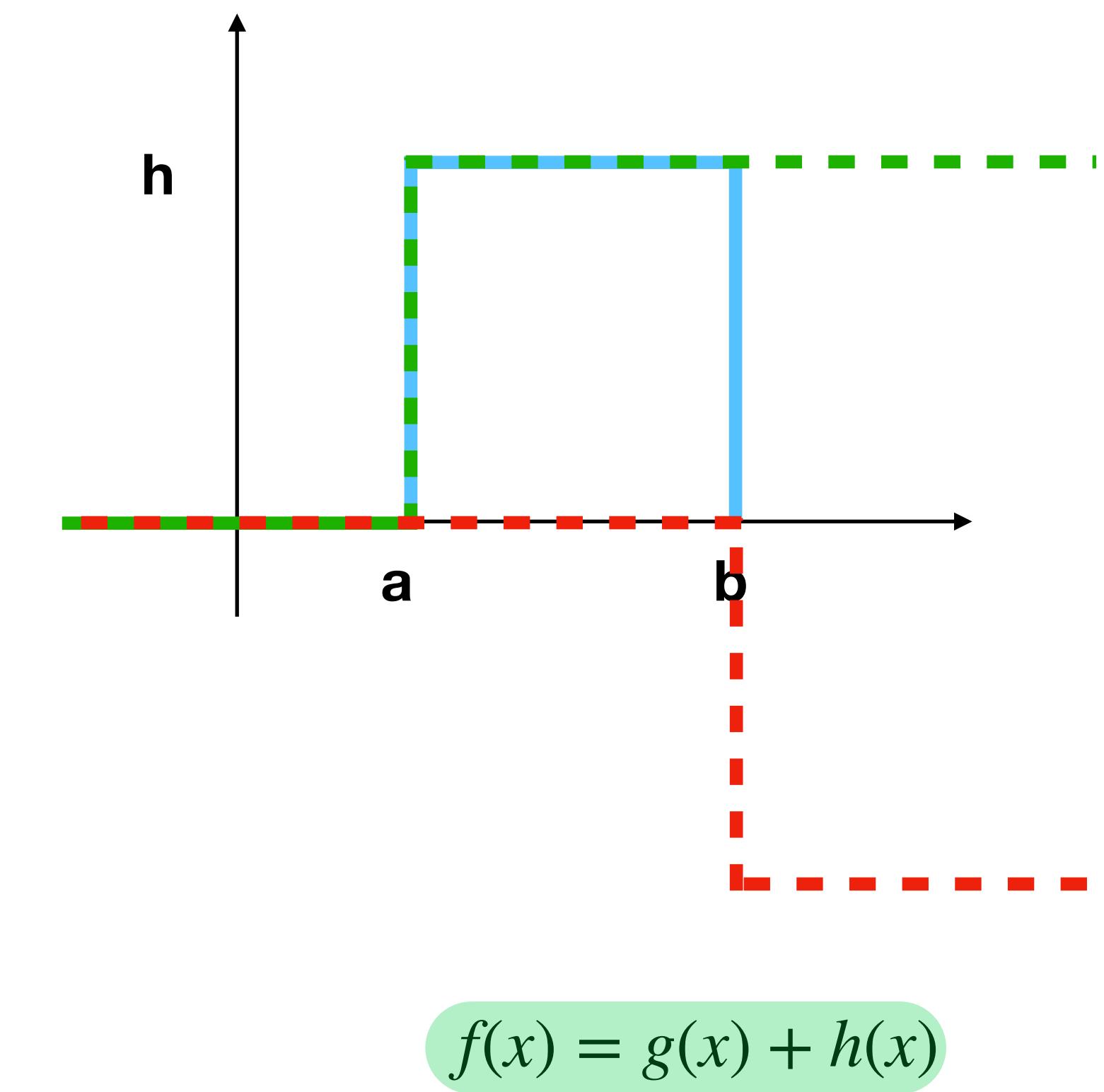
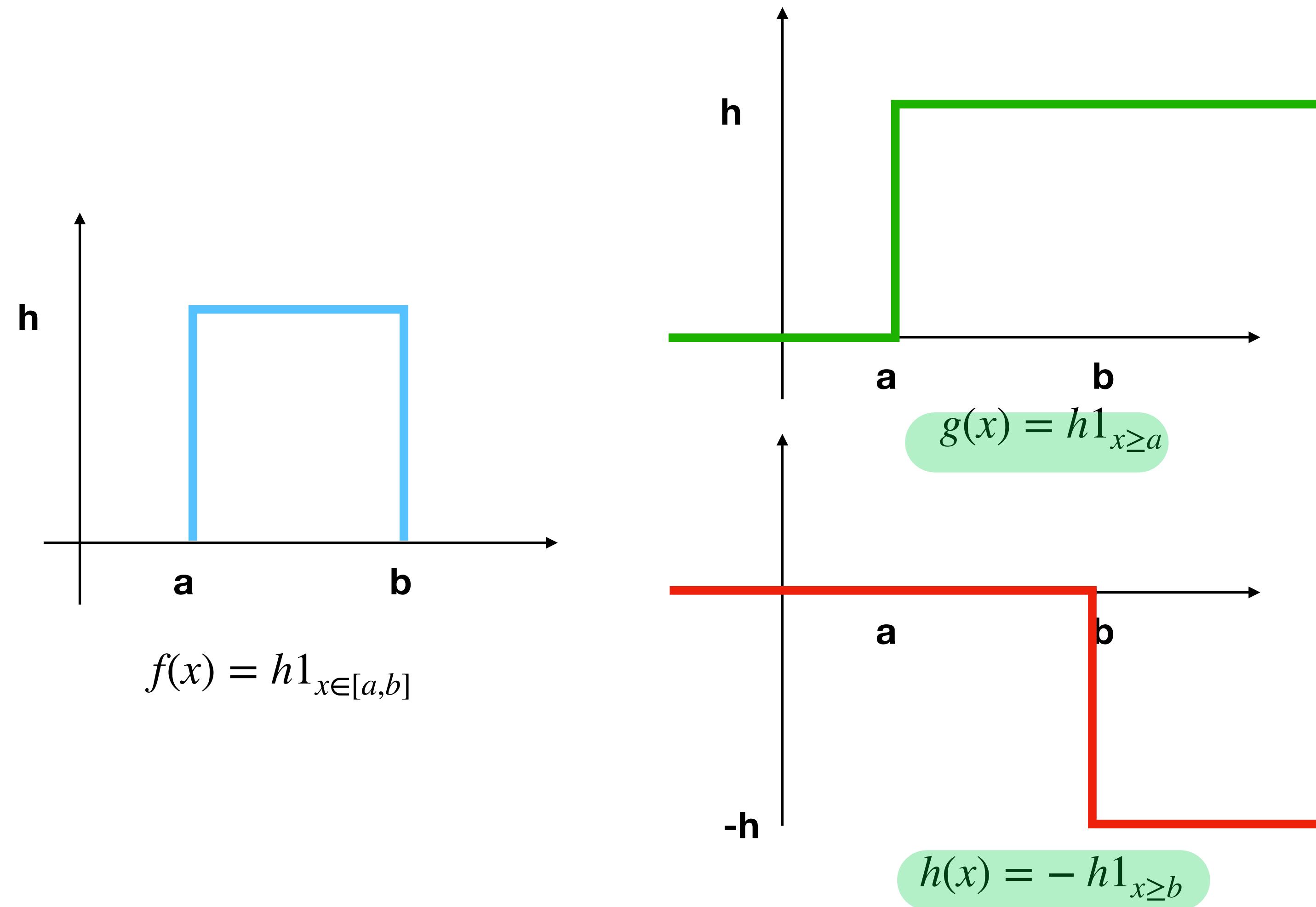
ℓ_1 approximation: approximation of the function by a sum of rectangle functions



Approximation in ℓ_1 -norm:

$$\begin{aligned}\int_{|x| \leq r} |f(x) - f_n(x)| dx &= \int_{|x| \leq r} (f(x) - f_n(x)) dx \\ &= \int_{|x| \leq r} f(x) dx - \int_{|x| \leq r} f_n(x) dx \\ &= \int_{|x| \leq r} f(x) - \sum_i \text{Area(Rectangle}_i)\end{aligned}$$

A rectangle function is equal to the sum of two step functions



Approximate a step function with a sigmoid

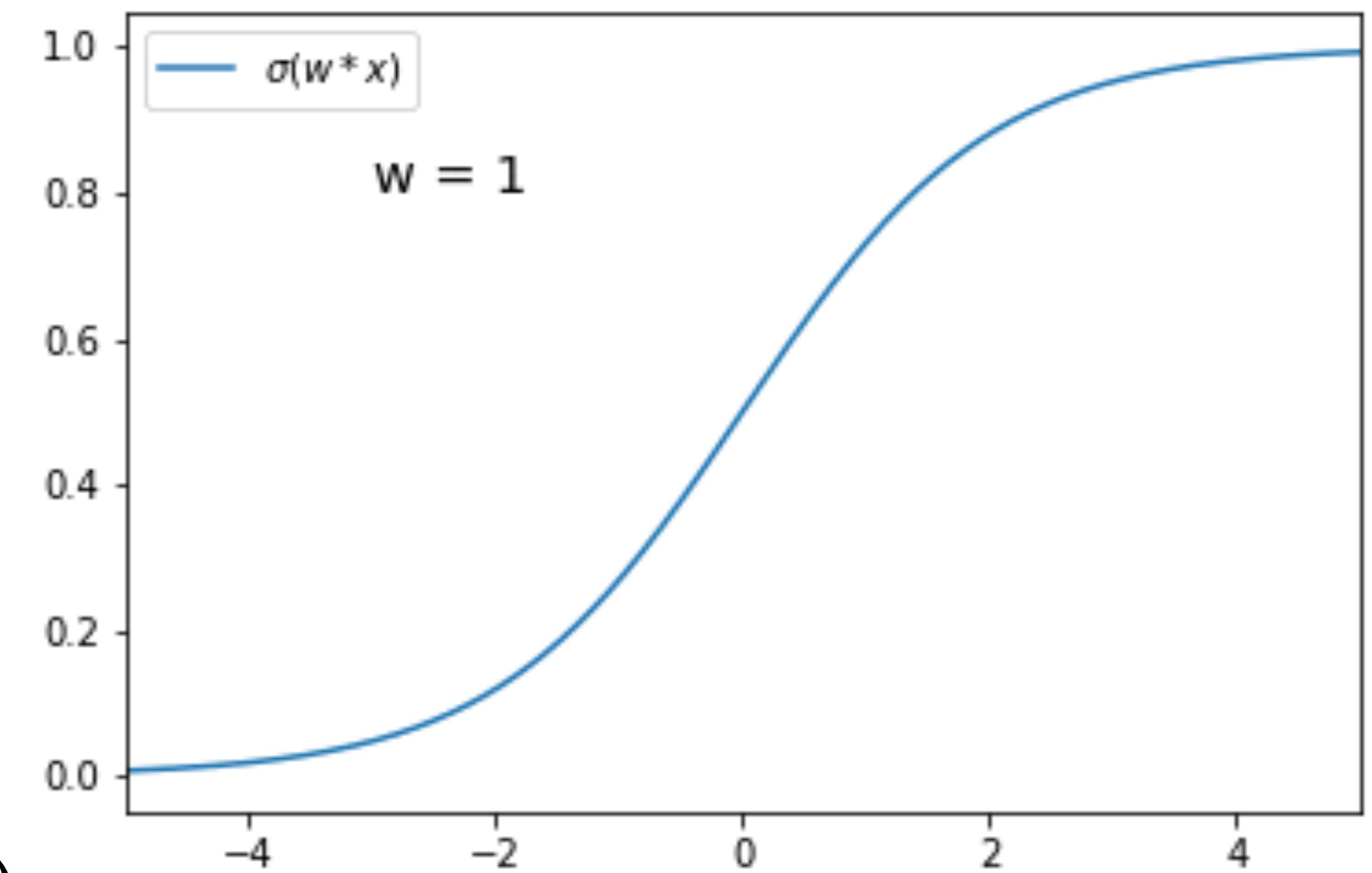
$$\tilde{\phi}(x) = \phi(w(x - b))$$

By setting:

- b : where the transition happens
- w : makes the transition steeper

Derivative: $\tilde{\phi}'(b) = w/4$

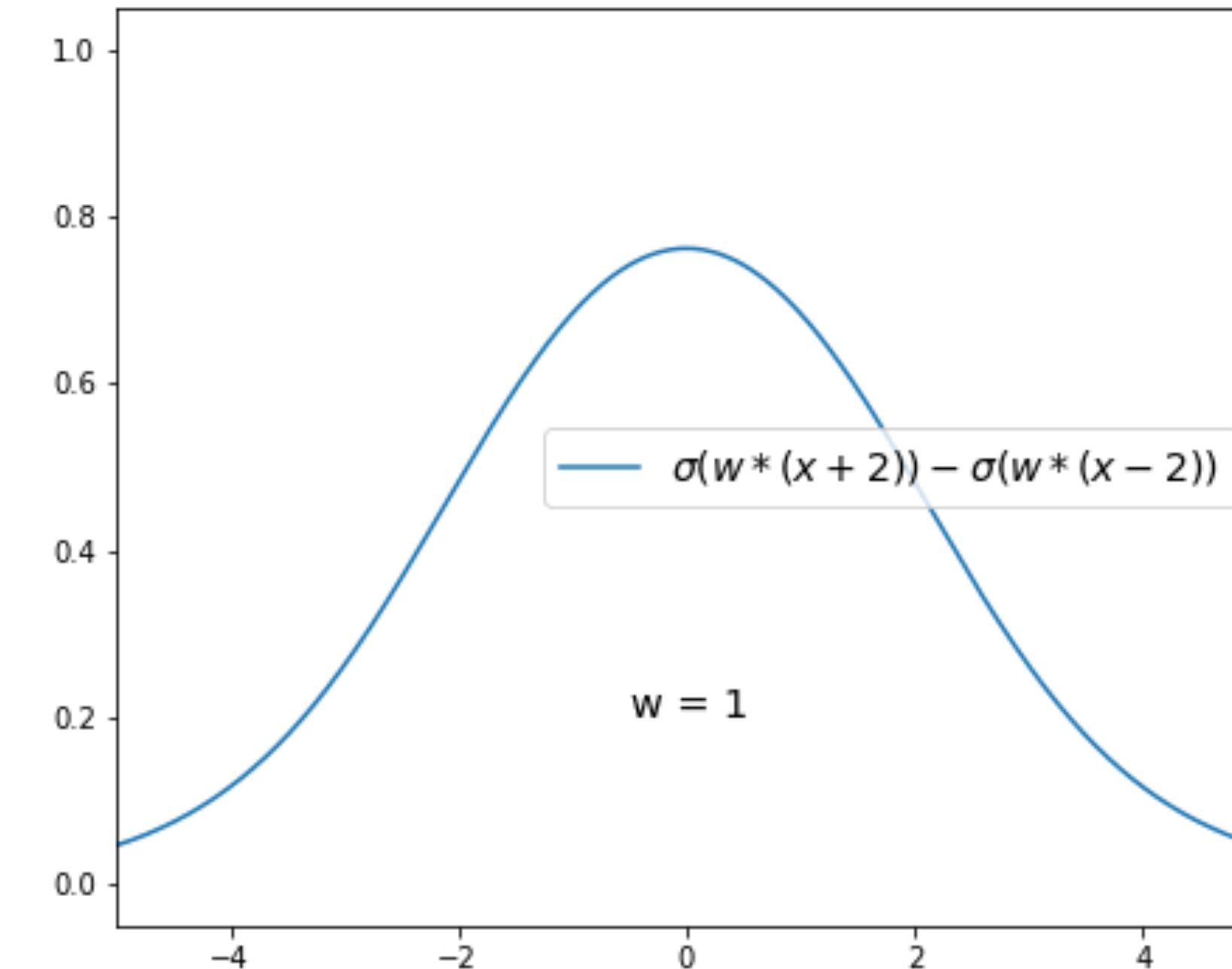
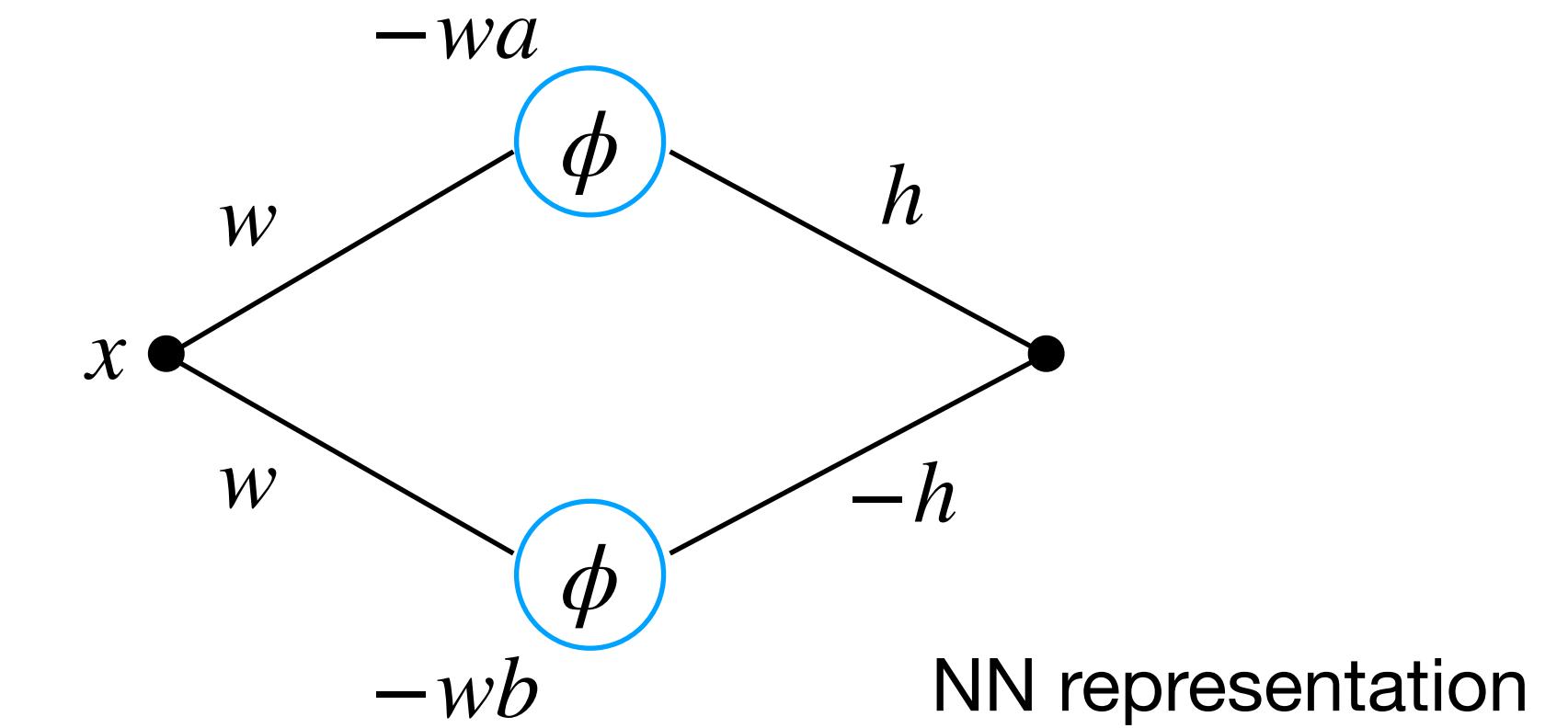
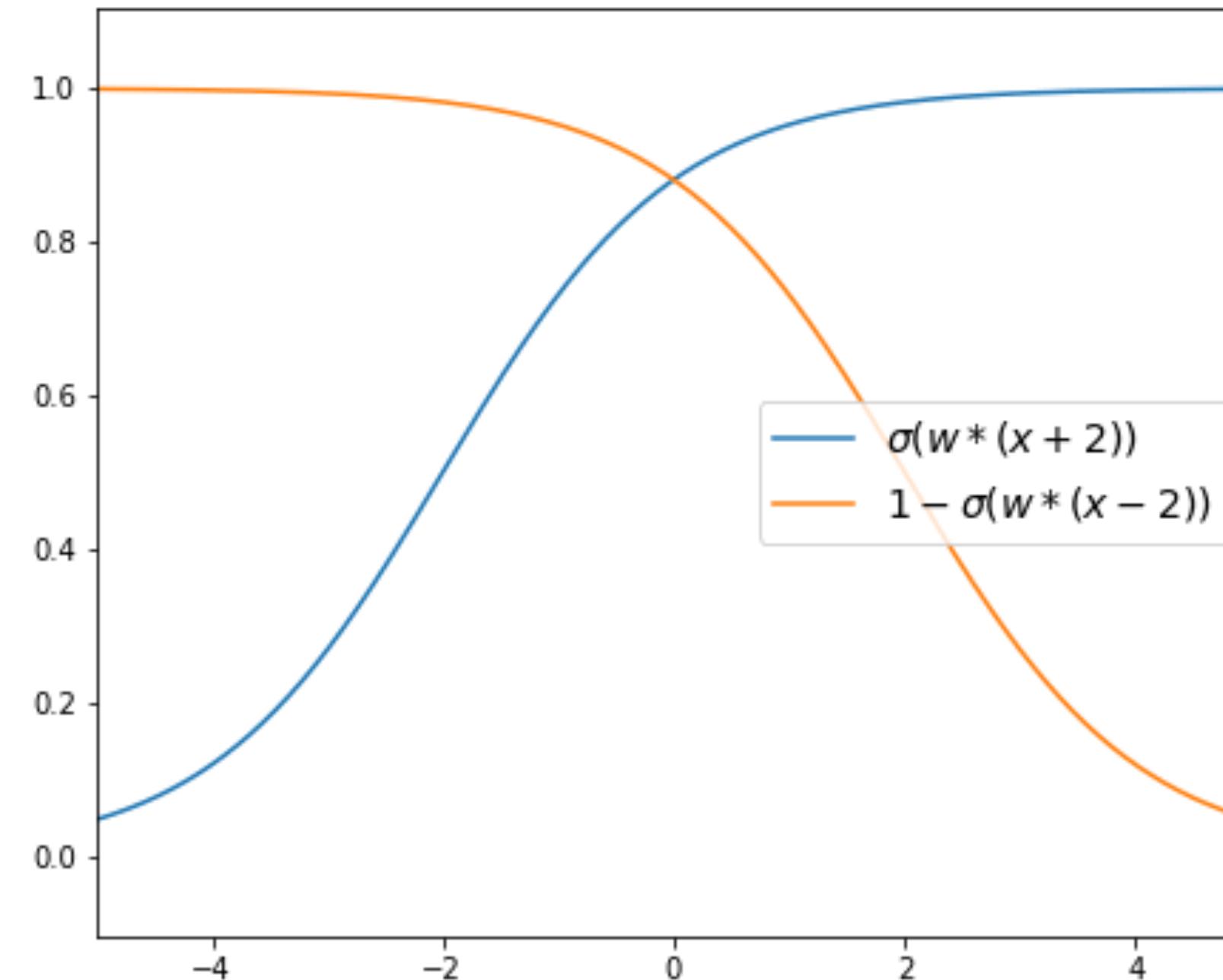
→ The width of the transition is $O(4/w)$



Approximation of the rectangle

$$h(\phi(w(x - a)) - \phi(w(x - b)))$$

$$\begin{aligned}a &= -2 \\b &= 2 \\h &= 1\end{aligned}$$

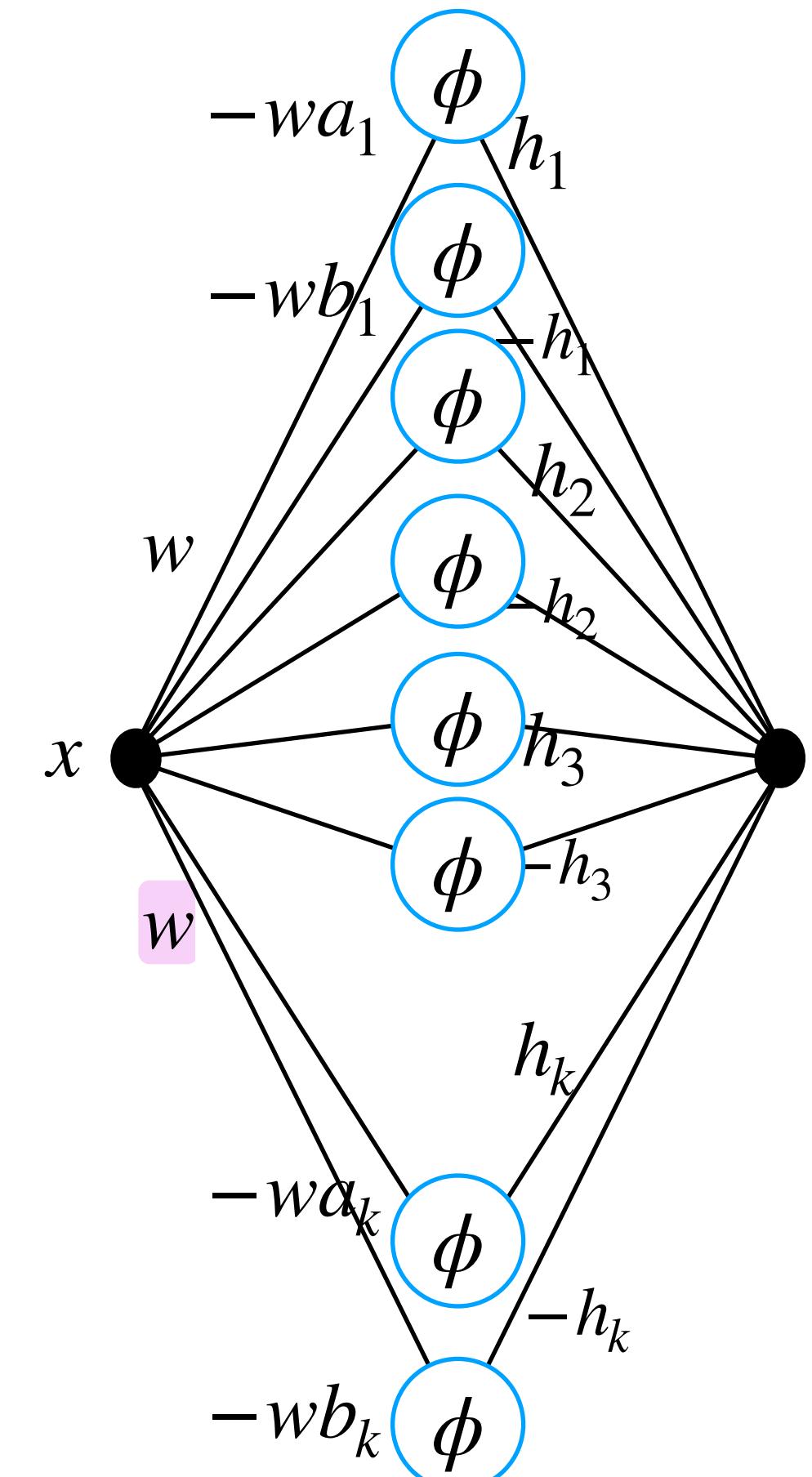


Conclusion in the 1D case

1. Approximate the function in the Riemann sense by a sum of k rectangles
2. Represent each rectangle using two nodes in the hidden layer of a neural network
3. Compute the sum of all nodes in the hidden layer (considering appropriate weights and signs) to get the final output
→ NN with one hidden layer containing $2k$ nodes for a Riemann sum with k rectangles

Remarks:

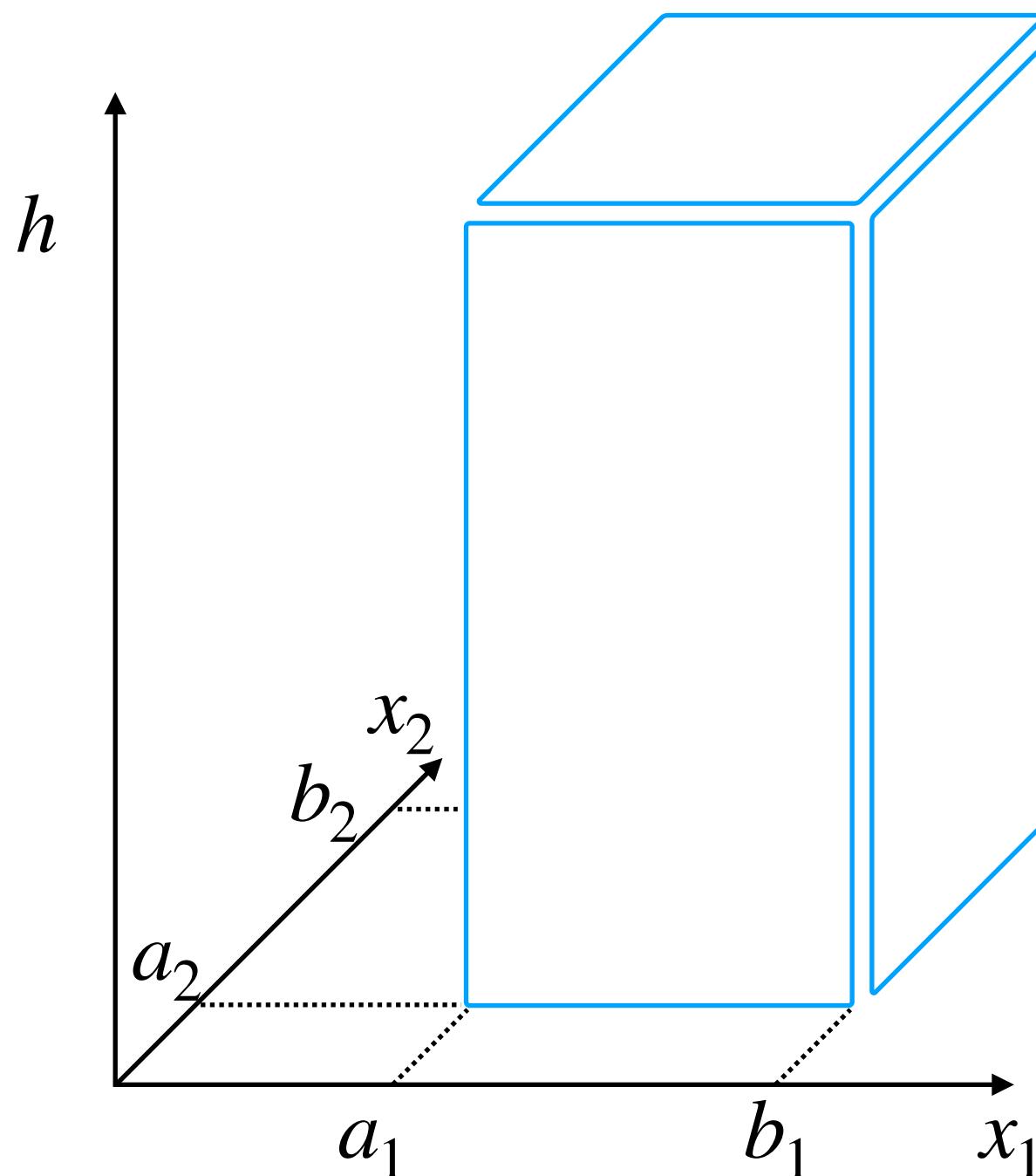
- The same intuition applies to any sigmoid-like function
- This is an intuitive explanation, not a quantitative one
- The weights w must be large



Larger dimension: $d = 2$

Same idea:

- Approximate the function by 2D rectangle functions
- Approximate a 2D rectangle function by sigmoids

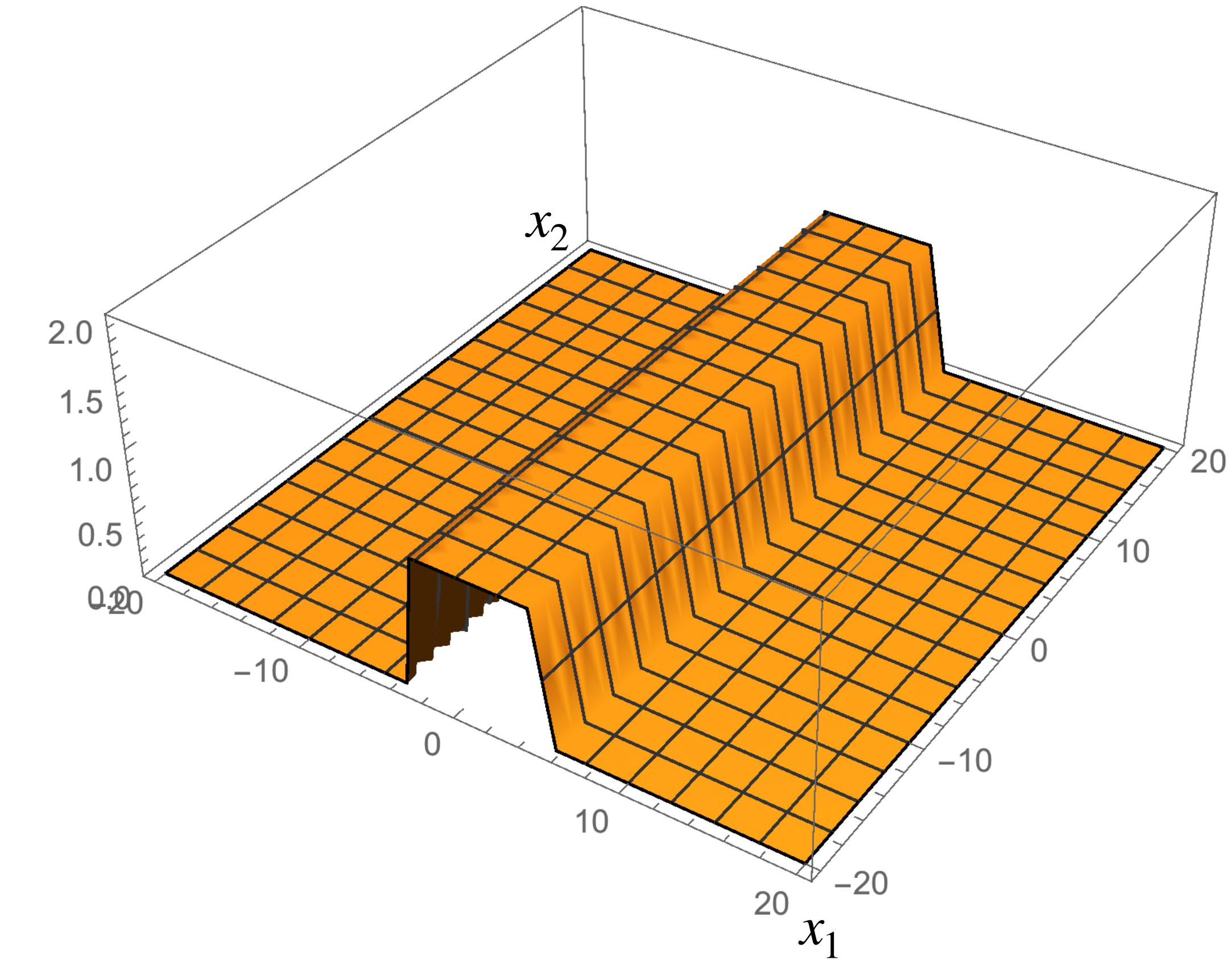
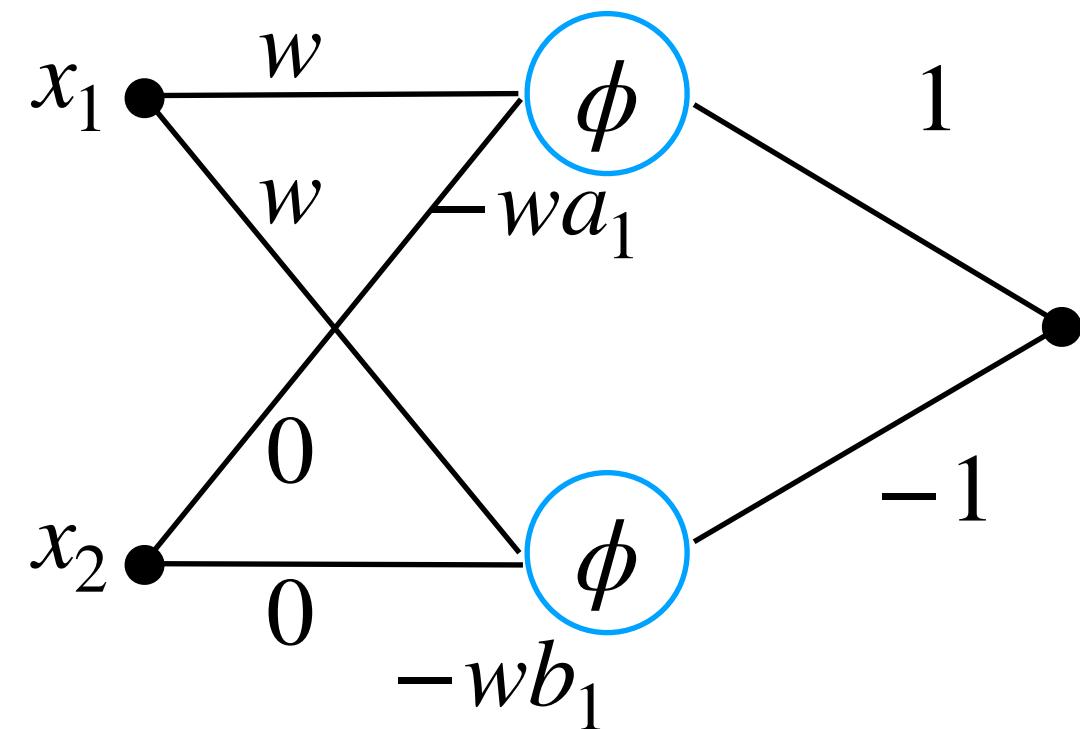


Two sigmoids can approximate an infinite rectangle function

$$(x_1, x_2) \mapsto \phi(w(x_1 - a_1)) - \phi(w(x_1 - b_1))$$

The rectangle:

- ranges from a_1 to b_1 in the x_1 direction
- is unbounded in the x_2 direction

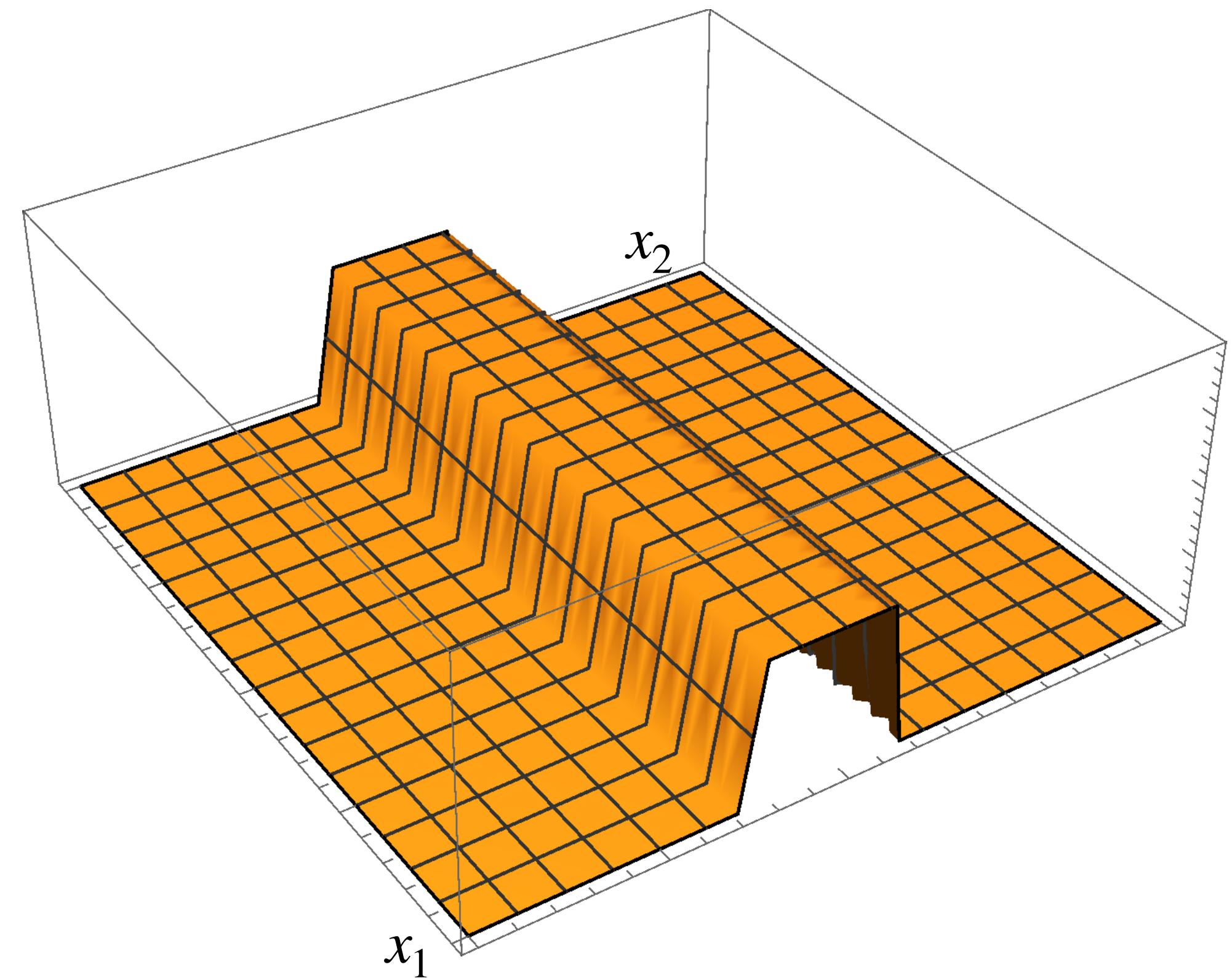
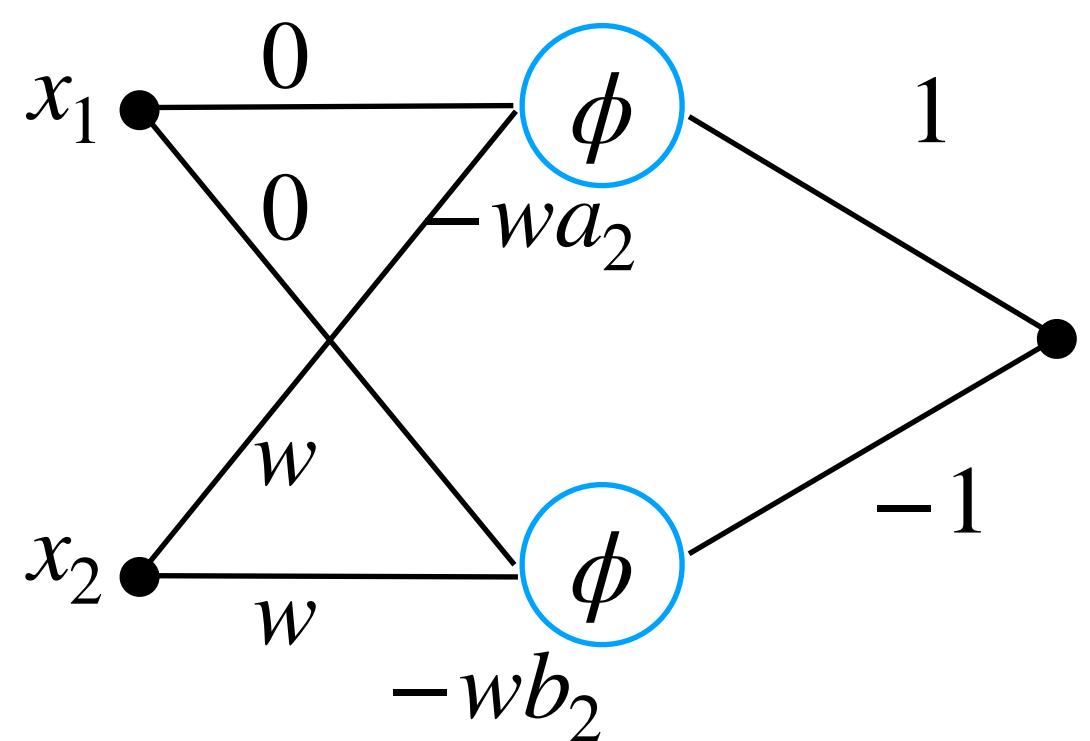


Two sigmoids can approximate an infinite rectangle function

$$(x_1, x_2) \mapsto \phi(w(x_2 - a_2)) - \phi(w(x_2 - b_2))$$

The rectangle:

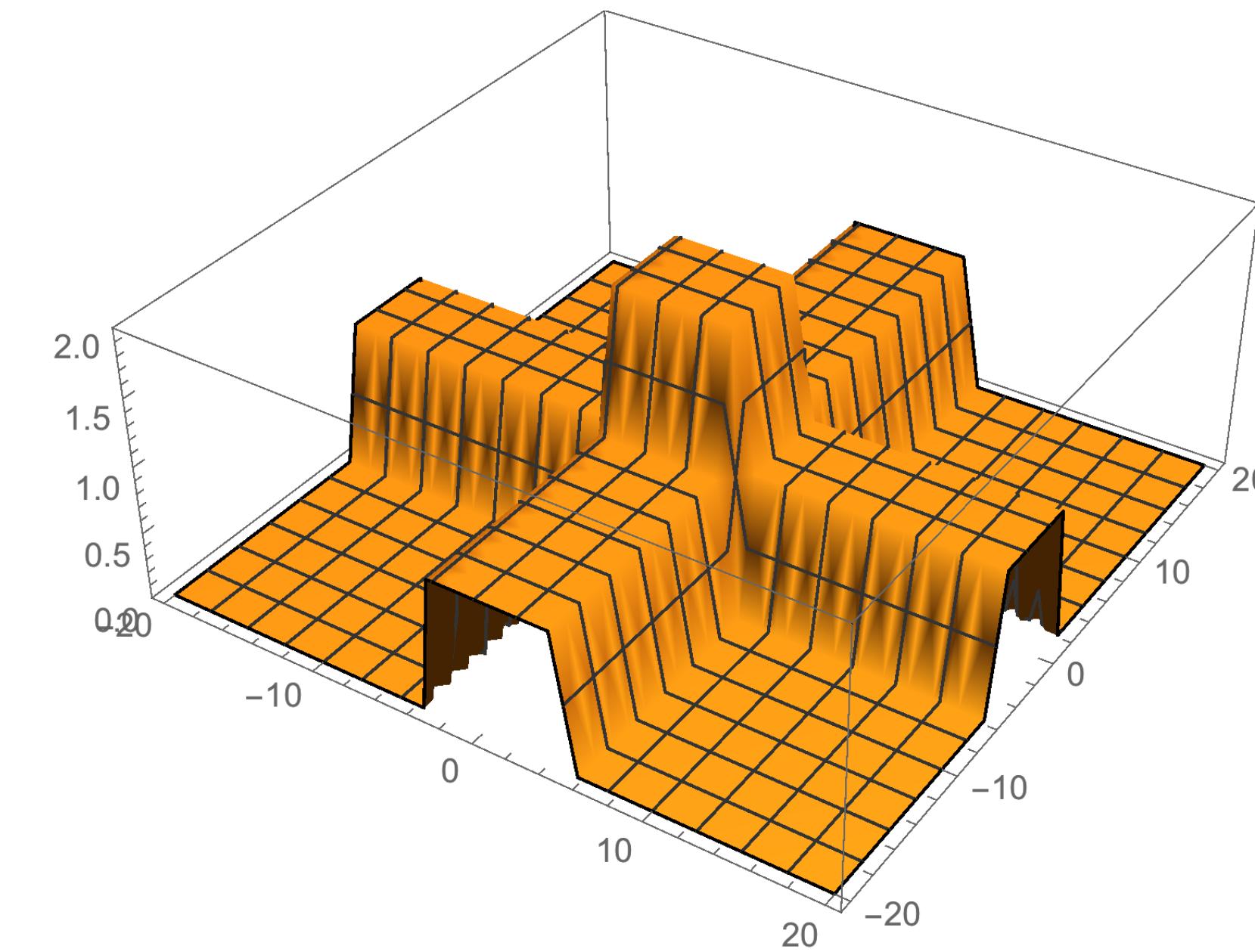
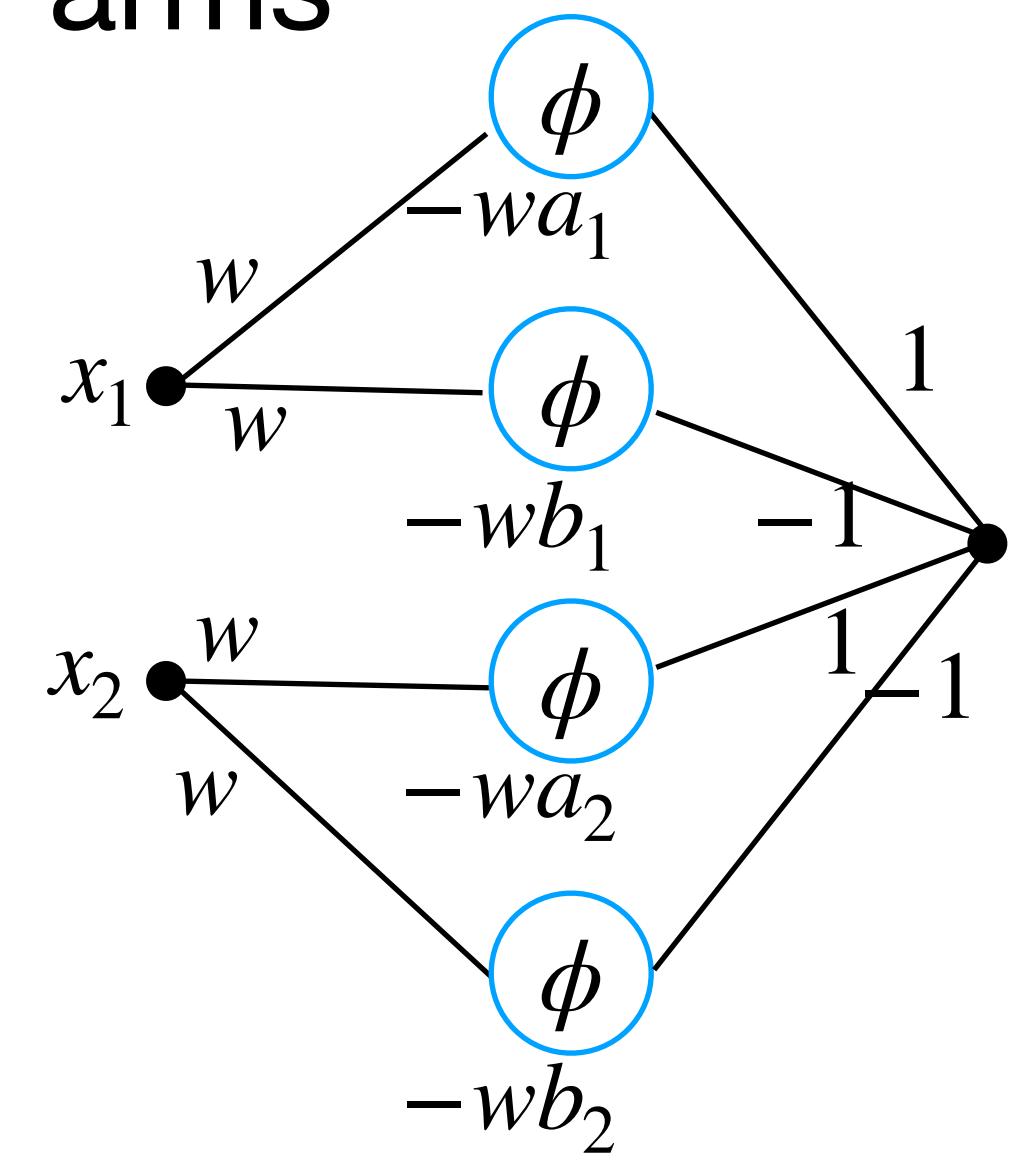
- ranges from a_2 to b_2 in the x_2 direction
- is unbounded in the x_1 direction



Four sigmoids approximate a cross

$$(x_1, x_2) \mapsto \phi(w(x_1 - a_1)) - \phi(w(x_1 - b_1)) + \phi(w(x_2 - a_2)) - \phi(w(x_2 - b_2))$$

- This approximation is close to our objective, with the exception of the two infinite “arms”



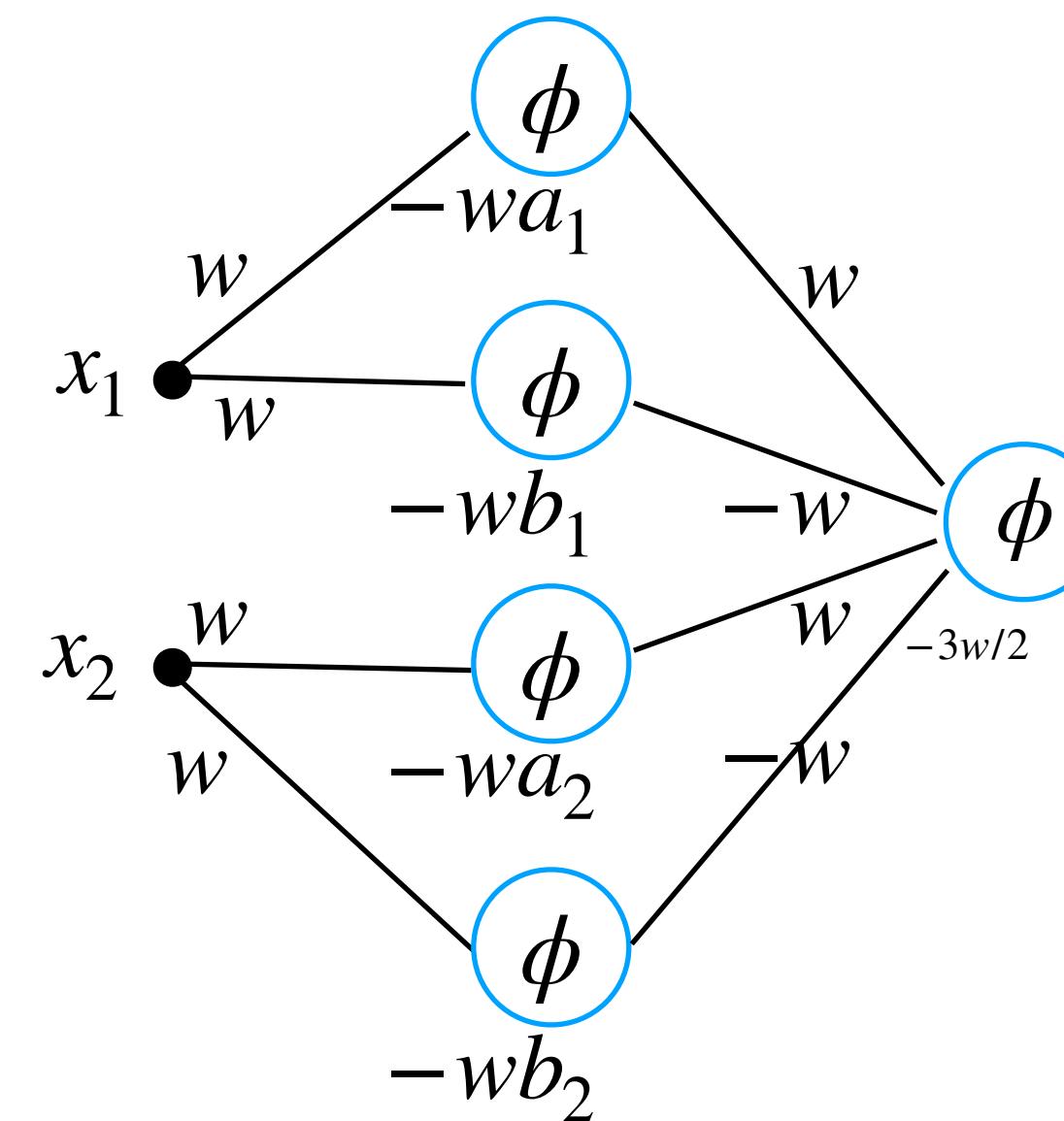
How can we eliminate the crossed arms?

Adding sigmoid in output to threshold unwanted infinite arms

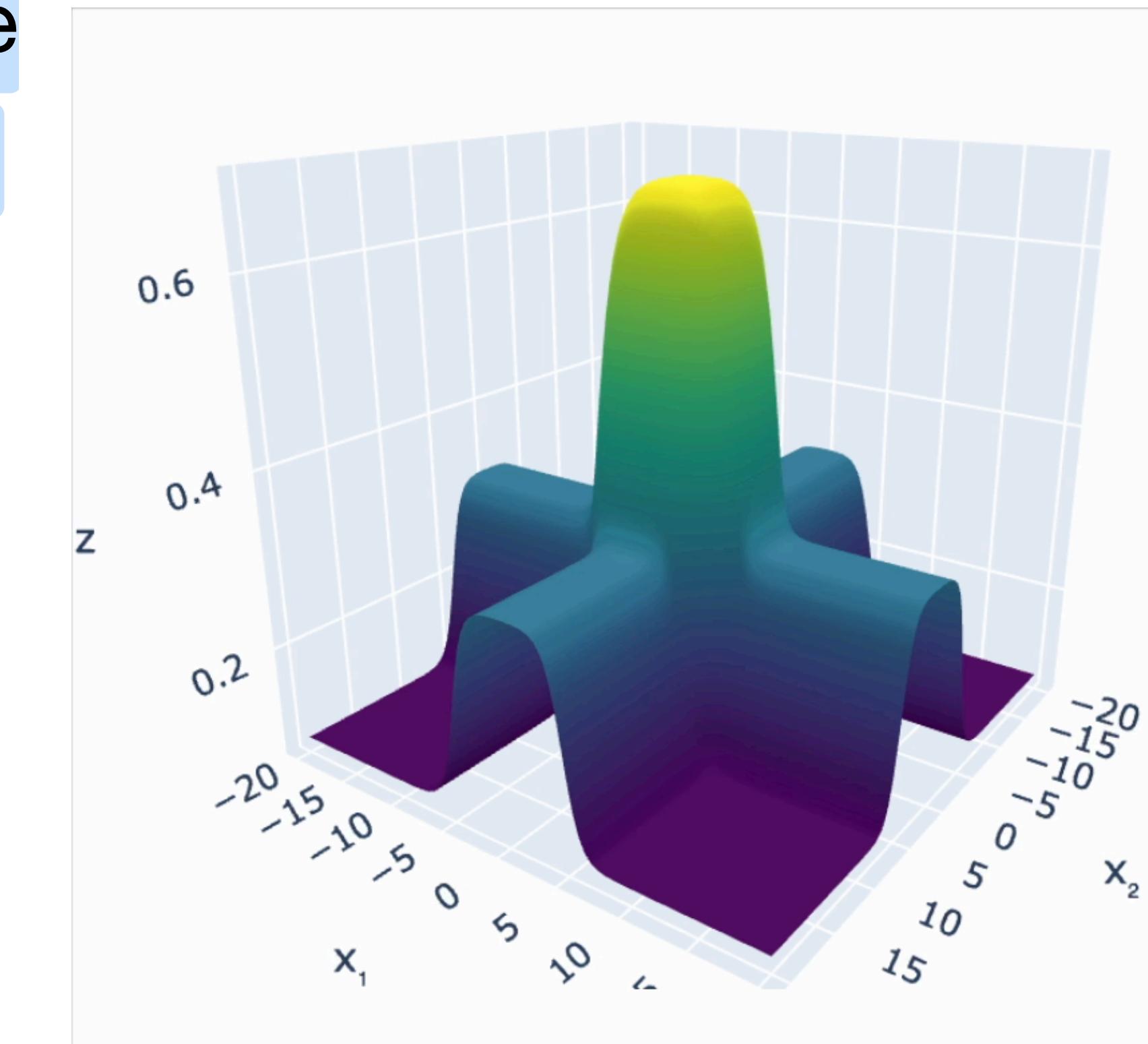
Thresholding the function will eliminate the arms

It is equivalent to composing it with $1_{y \geq c}$ for $c \in (1,2]$

- Approximate $1_{y \geq c}$ using a sigmoid with a large weight w and an appropriate bias (e.g., $3w/2$)

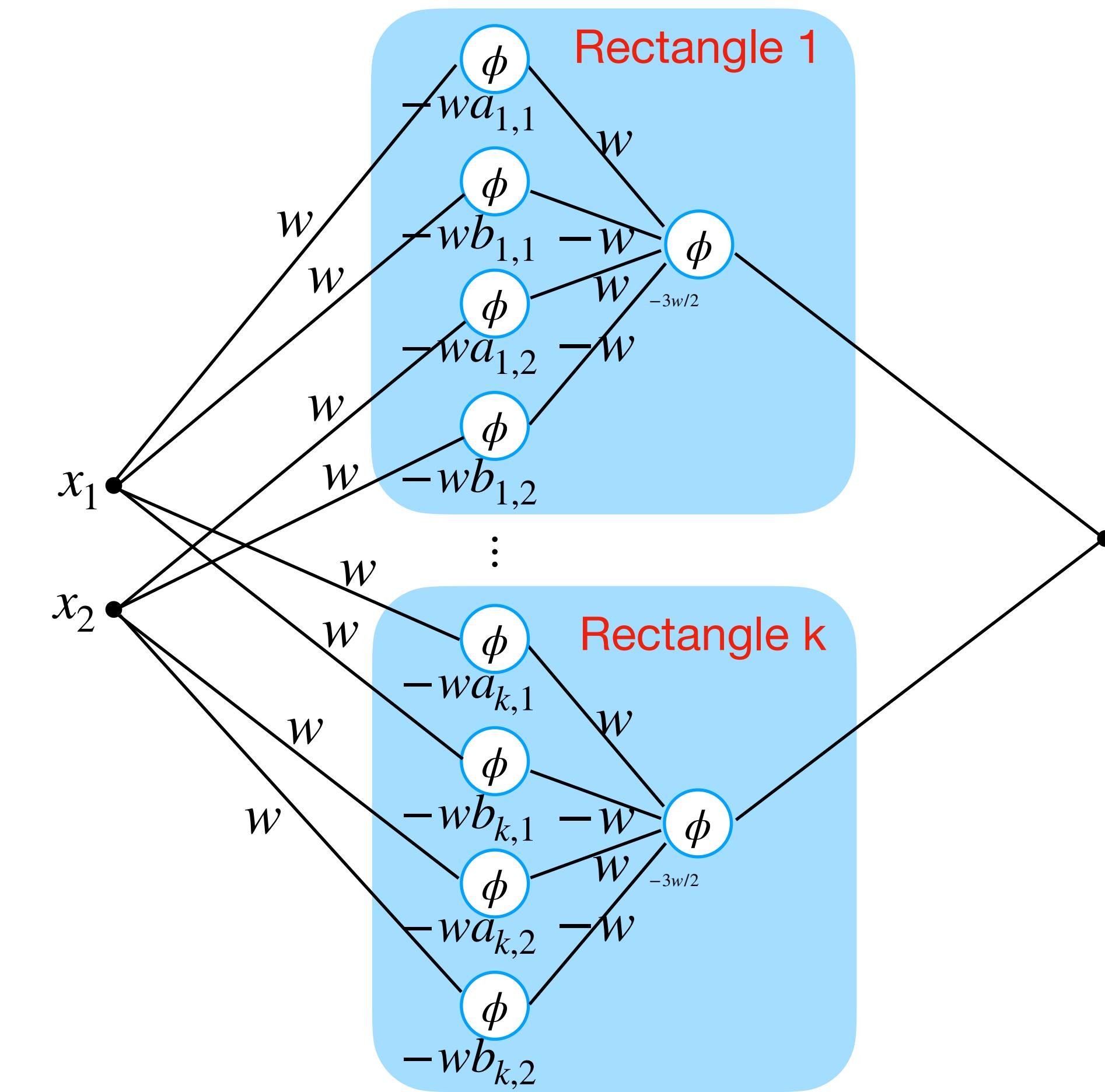


[Link to animation](#)

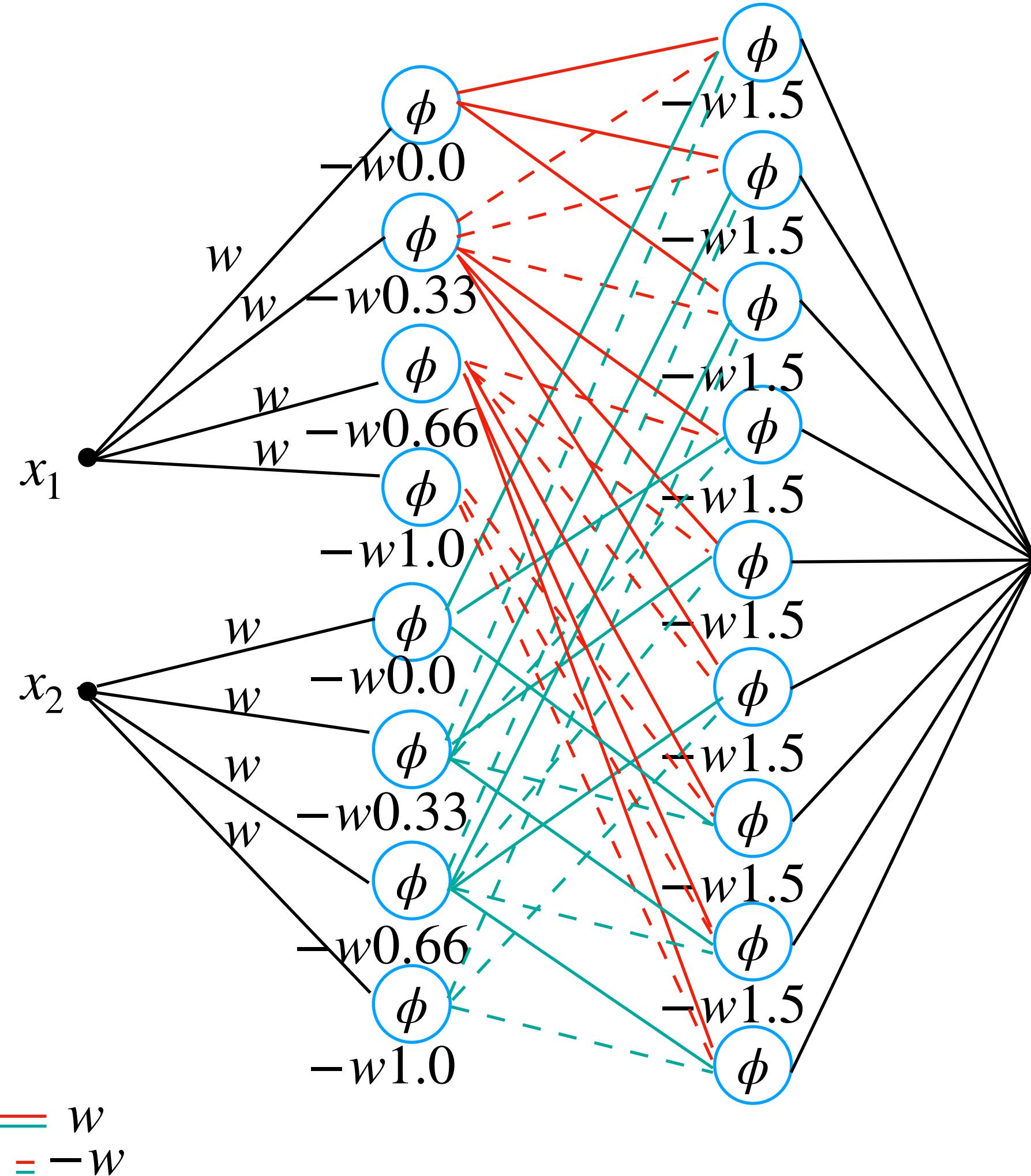


More rectangles with 2 hidden layers

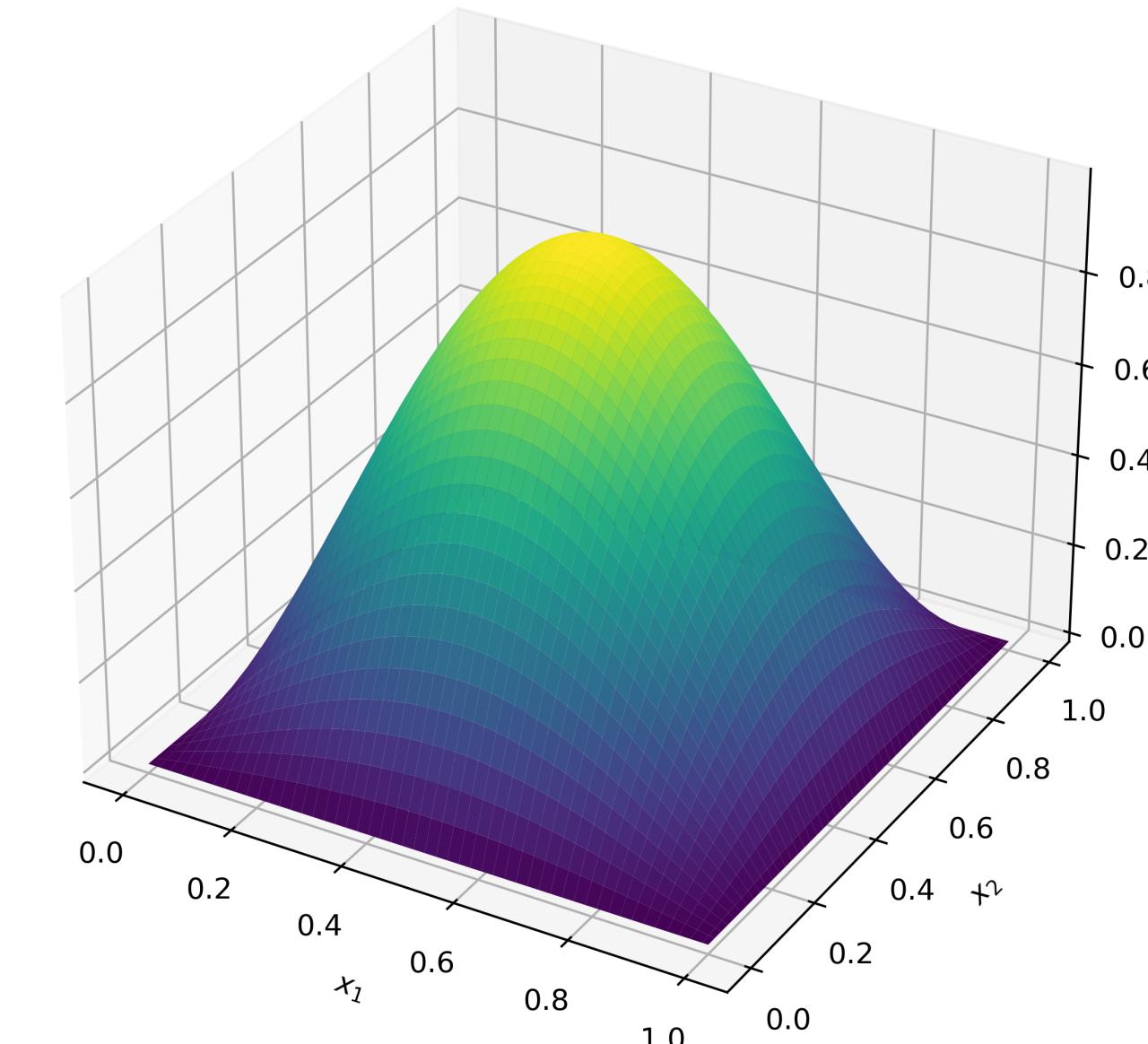
- To approximate a sufficiently smooth 2d function we need to add multiple rectangles
- We can implement multiple rectangles with 2 hidden layers



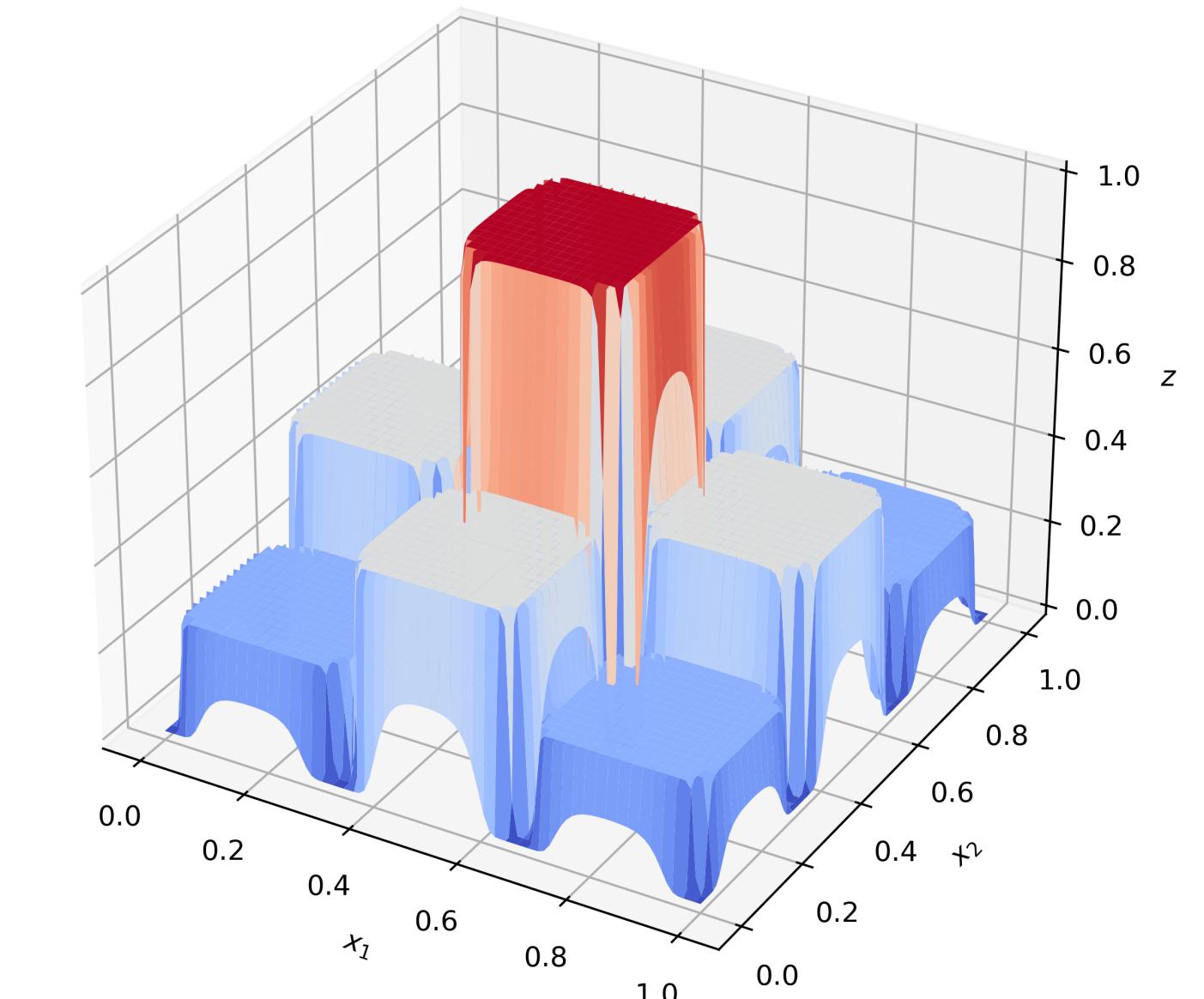
Example: approximate $f(x_1, x_2) = \sin(\pi x_1) \cdot \sin(\pi x_2)$ with 9 rectangles on $[0,1] \times [0,1]$



$$f(x_1, x_2) = \sin(\pi x_1) \cdot \sin(\pi x_2)$$



Neural Network Approximation



[Link to animation for
more rectangles](#)

ℓ_2 vs ℓ_∞ approximations

- **ℓ_2 -norm (average approx.):**
 - Measures the average deviation between the true function and the approximation across a region
 - If the function deviates in some points but compensates with smaller deviations elsewhere, it can still yield a low L_2
 - **ℓ_∞ -norm (pointwise approx.):**
 - Measures the maximum error at any point
 - Ensures uniformly small error across the domain allowing for precise control
- Barron's result was for ℓ_2 approximation. How can we get ℓ_∞ approximation?

ℓ_∞ approximation with piecewise linear functions

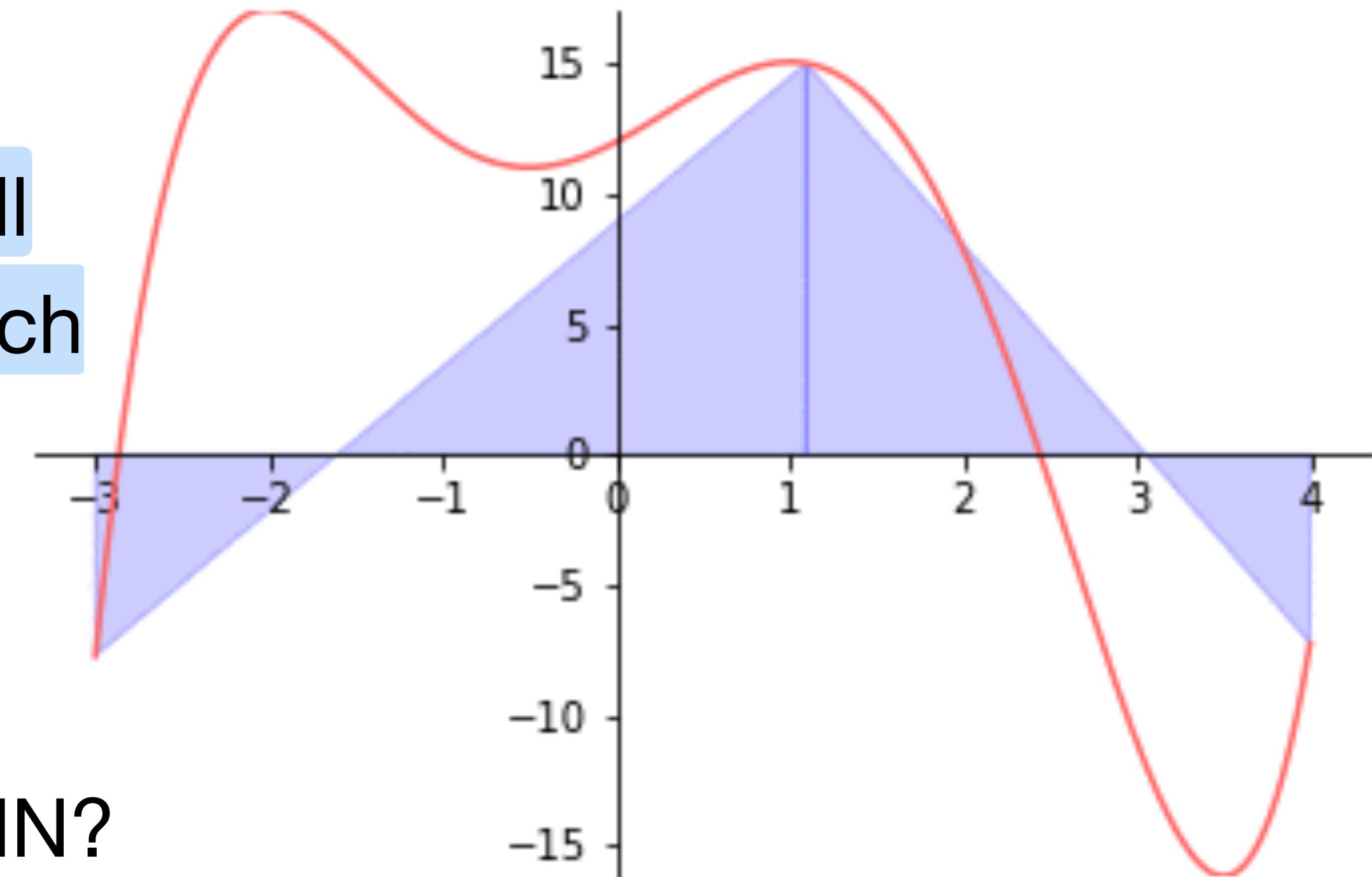
Def: piecewise linear (PWL) function:

$$q(x) = \sum_{i=1}^m (a_i x + b_i) 1_{r_{i-1} \leq x < r_i} \text{ with } a_i r_i + b_i = a_{i+1} r_i + b_{i+1}$$

ℓ_∞ -approximation result (Shektman, 1982):

Let f be a continuous function on $[c, d]$. For all $\varepsilon > 0$, it exists a piecewise linear function q such that:

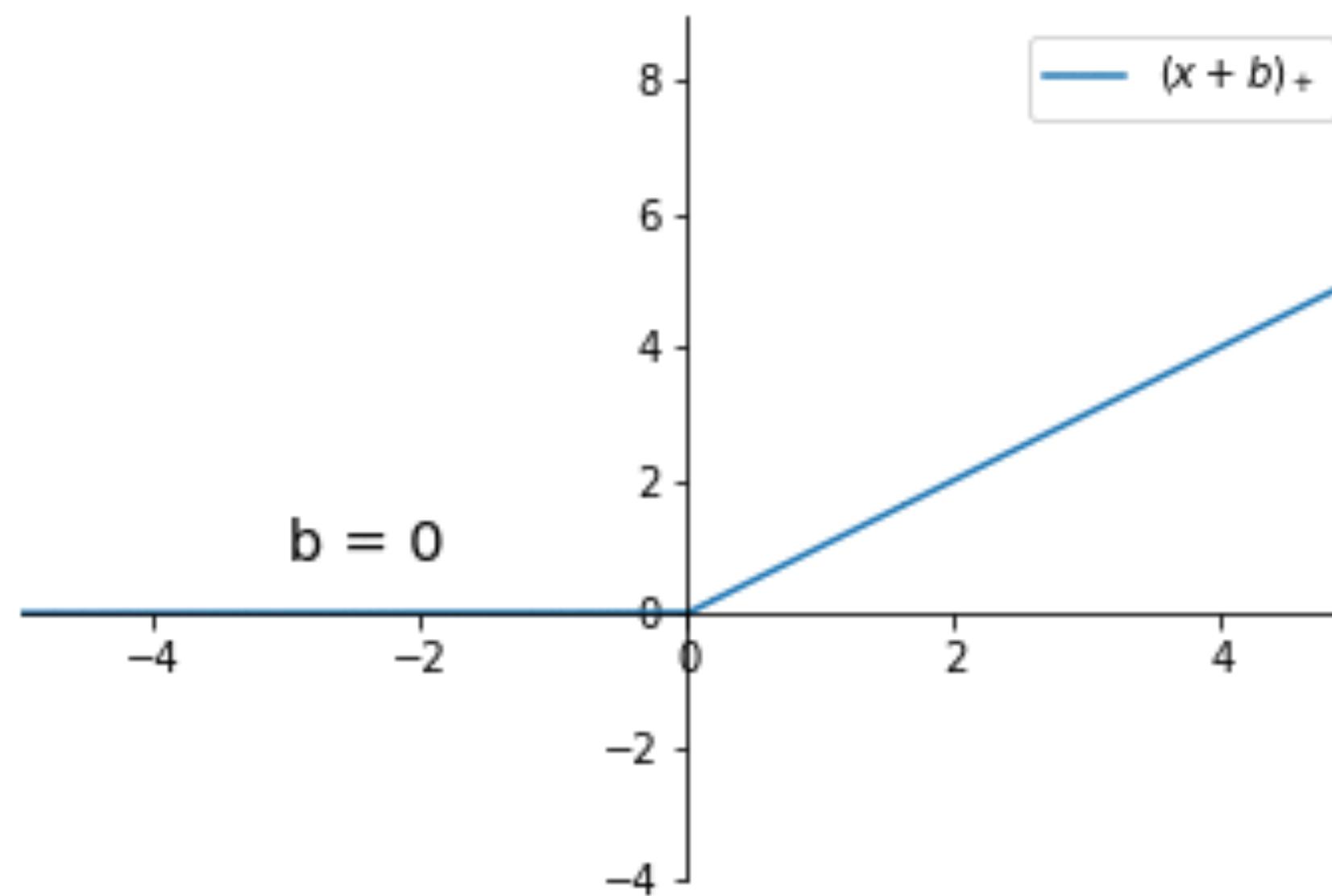
$$\sup_{x \in [c, d]} |f(x) - q(x)| \leq \varepsilon$$



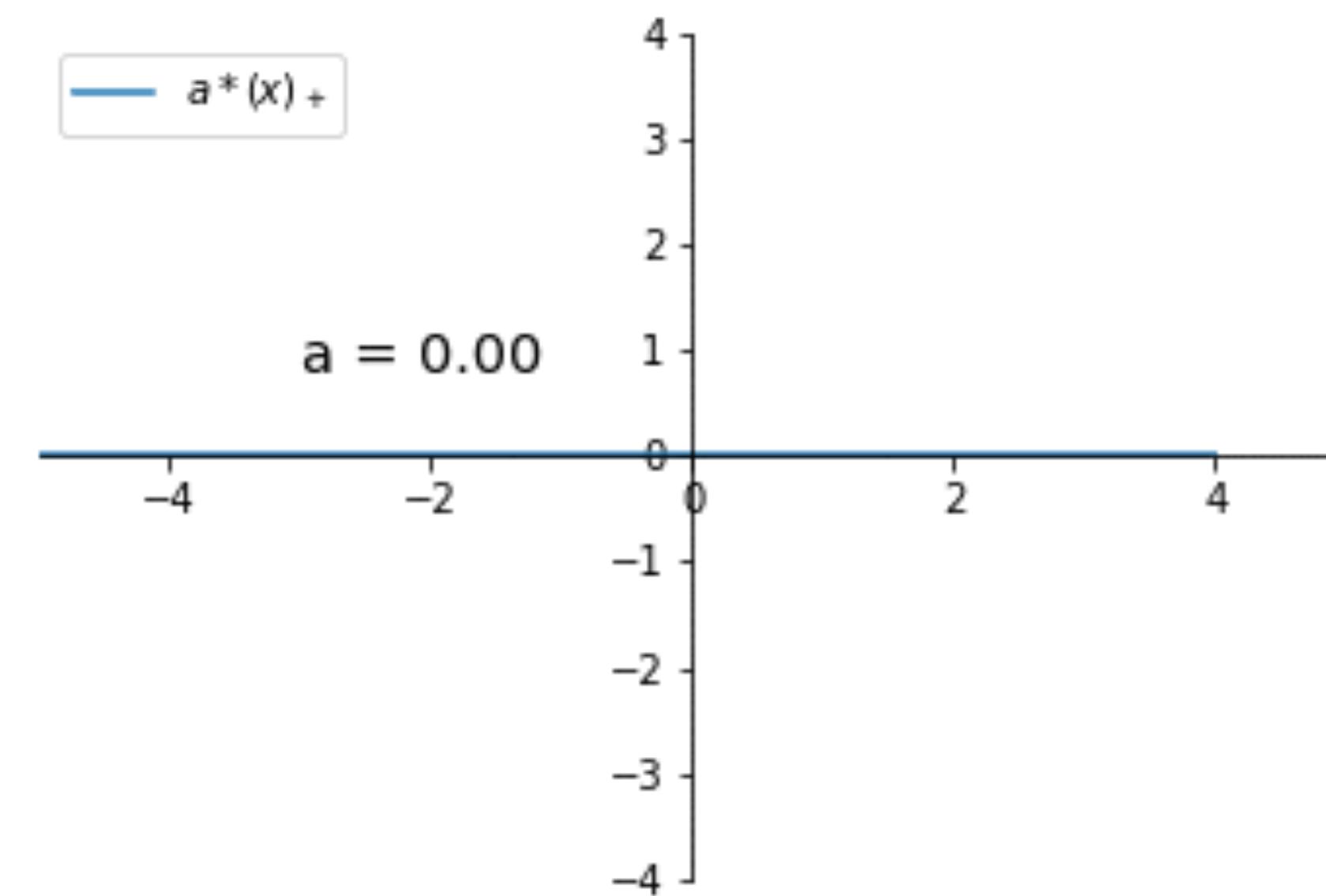
→ How to approximate PWL functions with a NN?

ReLU activation and PWL functions

- Role of the bias and weight in ReLU function $(ax + b)_+ = \max\{0, ax + b\}$:



The bias b determines the position of the kink



The weight a determines the slope

- A linear combination of ReLUs $\sum_{i=1}^m \tilde{a}_i(x - \tilde{b}_i)_+$ is a piecewise linear function

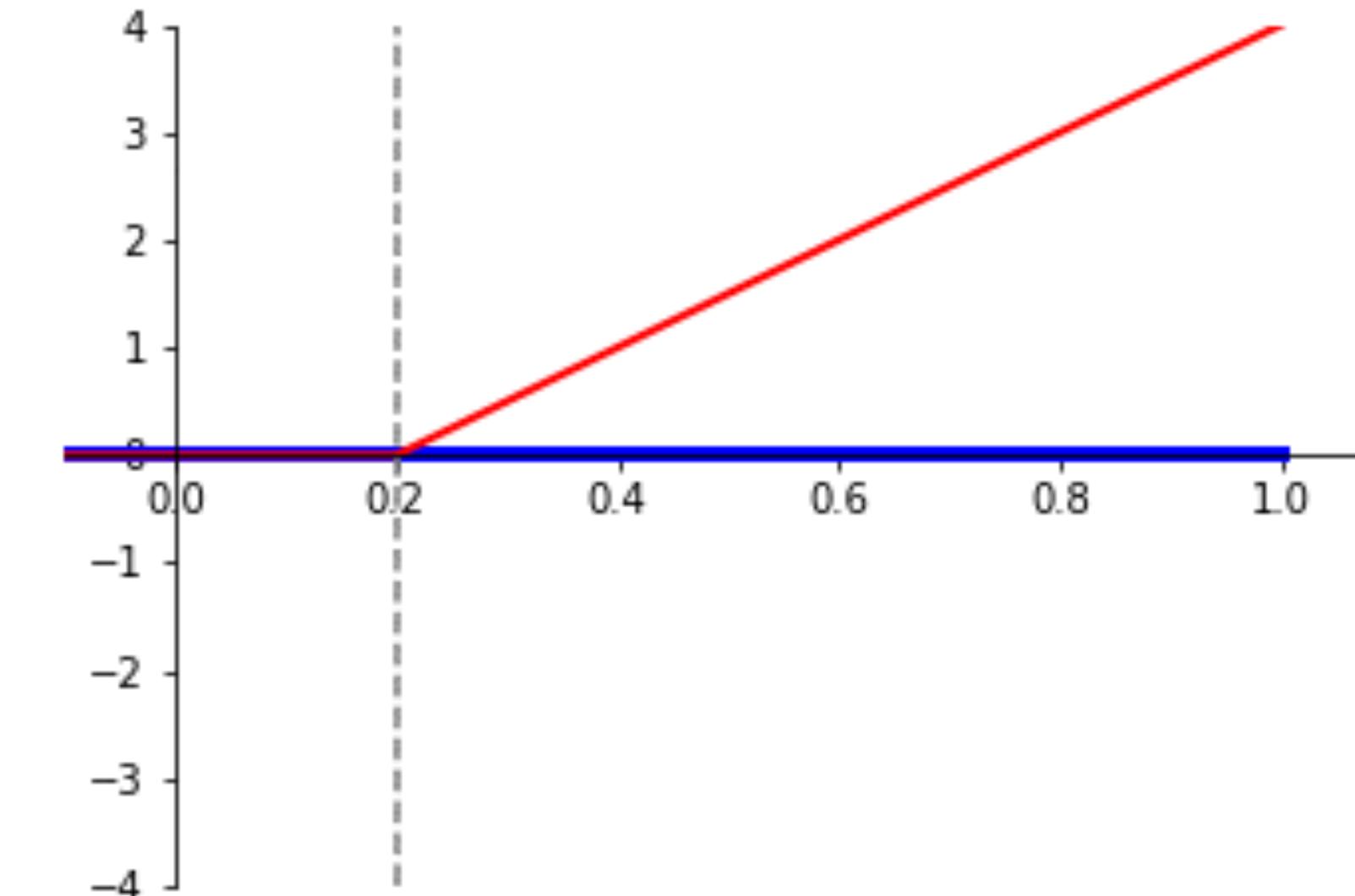
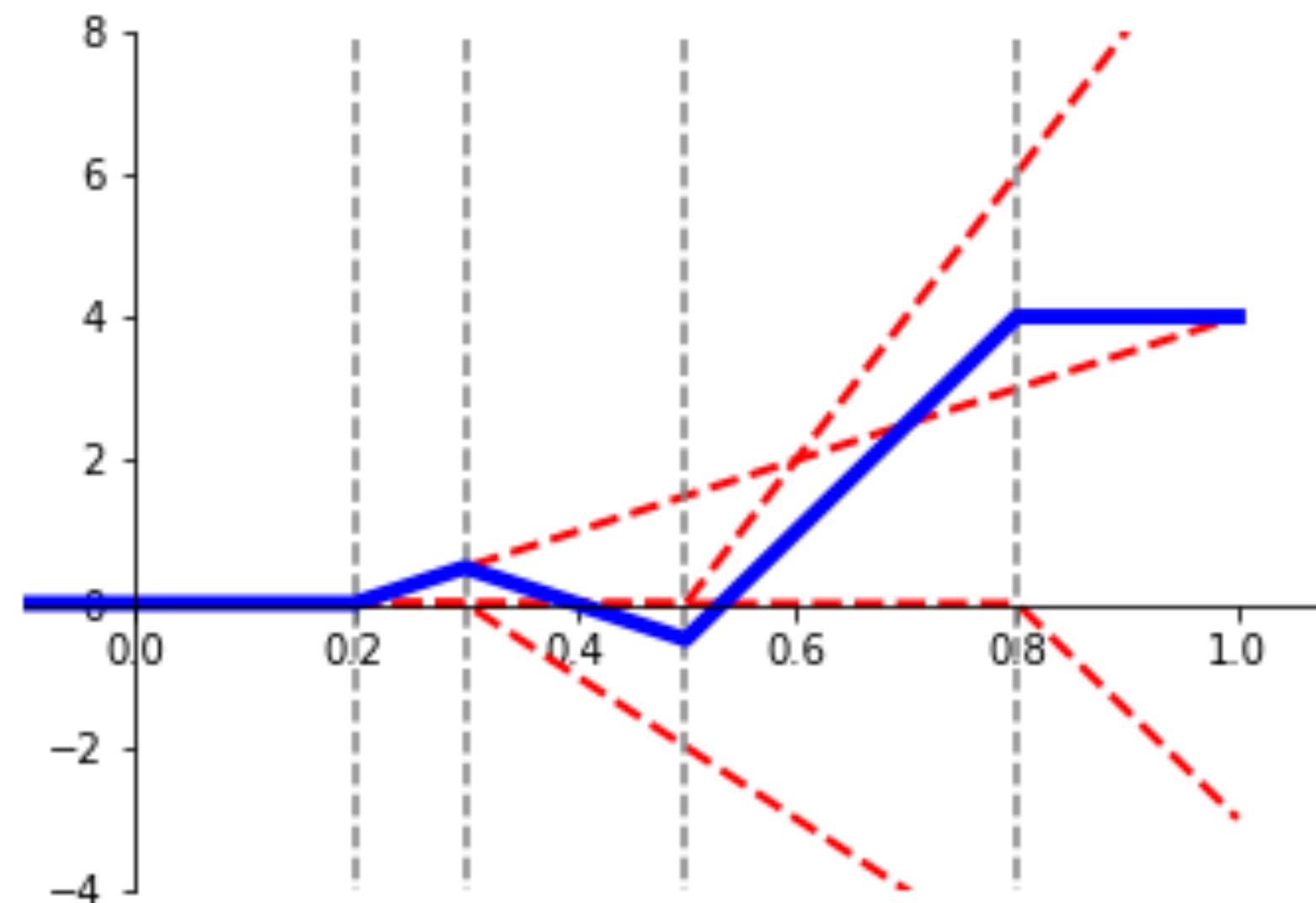
Piecewise linear functions can be written as combination of RELU

Claim 1: Any PWL q can be rewritten as:

$$q(x) = \tilde{a}_1 x + \tilde{b}_1 + \sum_{i=2}^m \tilde{a}_i (x - \tilde{b}_i)_+ \text{ where } \tilde{a}_1 = a_1, \tilde{b}_1 = b_1, a_i = \sum_{j=1}^i \tilde{a}_j \text{ and } \tilde{b}_i = r_{i-1}$$

Proof sketch:

- How do we get a new segment with slope a starting at $r > \max(\tilde{b}_i)$?
- Get the kink at r by setting $\tilde{b}_{i+1} = r$ and slope by additionally canceling existing slope i.e.
$$\tilde{a}_{i+1} = a - \sum_i \tilde{a}_i$$



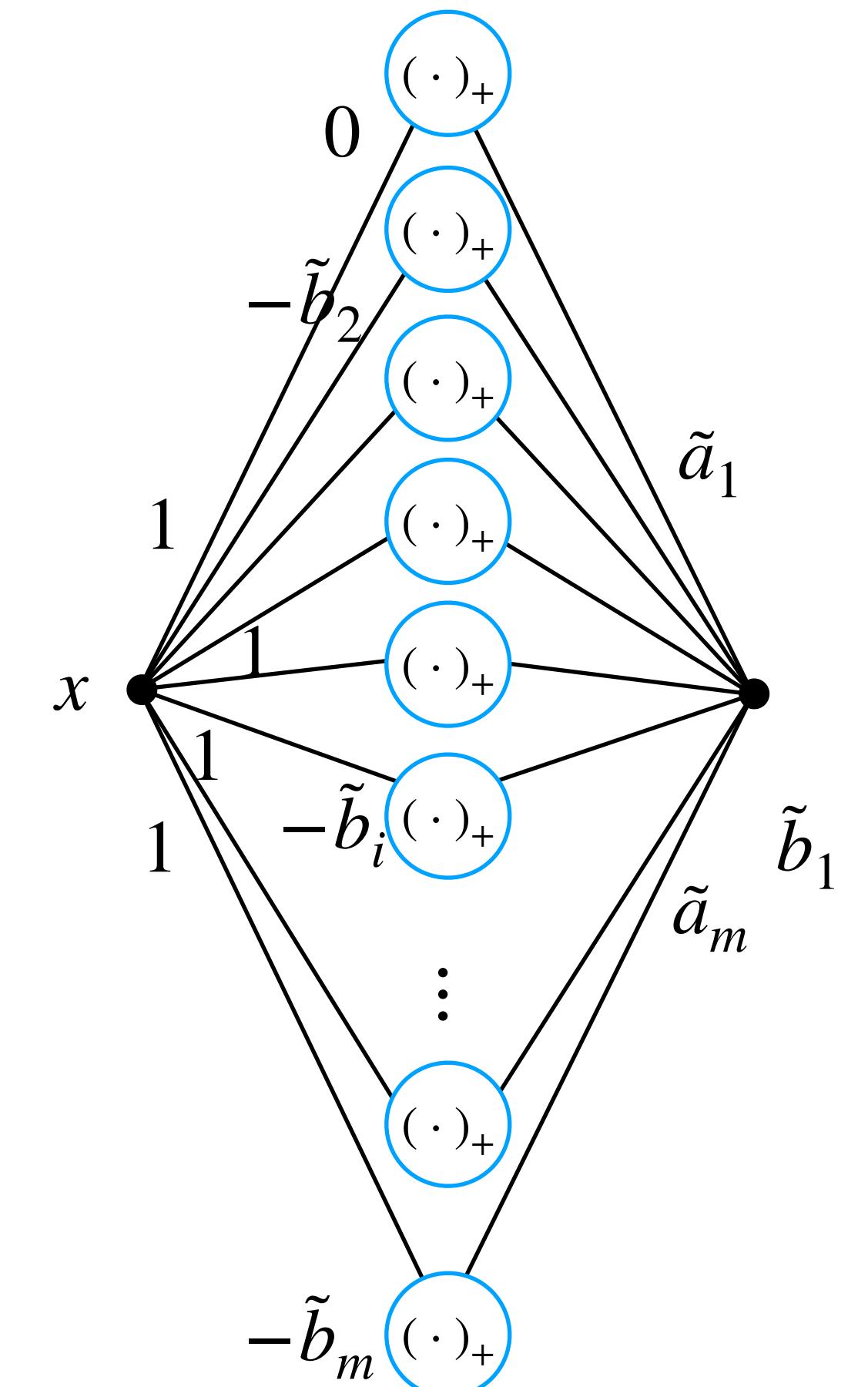
PWL as one-hidden layer NN with ReLU

Claim 2: q can be implemented as a one-hidden-layer NN with ReLU activation. Each term corresponds to one node:

- Bias $-\tilde{b}_i$
- Output weight \tilde{a}_i

The term $\tilde{a}_1x + \tilde{b}_1$ also corresponds to one node:

- Bias \tilde{b}_1 : bias of the output node
- Term $\tilde{a}_1x = \tilde{a}_1(x)_+$ since $x \in [0,1]$



Proof of the equivalent formulation

$$q(x) = \sum_{i=1}^m (a_i x + b_i) 1_{r_{i-1} \leq x < r_i} \quad r(x) = \tilde{a}_1 x + \tilde{b}_1 + \sum_{i=2}^m \tilde{a}_i (x - \tilde{b}_i)_+$$

$$\tilde{a}_1 = a_1, \tilde{b}_1 = b_1 \text{ and } a_i = \sum_{j=1}^i \tilde{a}_j \text{ and } \tilde{b}_i = r_{i-1}$$

- For $x \in [0, r_1]$
 $(\tilde{a}_1, \tilde{b}_1) = (a_1, b_1) \implies q(x) = a_1 x + b_1 = \tilde{a}_1 x + \tilde{b}_1 = r(x)$ because $\tilde{b}_2 = r_1$
- For $x \in [r_1, r_2]$,
$$\begin{aligned} r(x) &= \tilde{a}_1 x + \tilde{b}_1 + (a_2 - a_1)(x - r_1)_+ \\ &= a_1 x + b_1 + (a_2 - a_1)(x - r_1) = a_2 x + b_1 - (a_2 - a_1)r_1 \end{aligned}$$

 $r'(x) = a_2$ and $r(r_1) = q(r_1)$ as shown above
 $\implies r(x) = q(x)$ for $x \in [r_1, r_2]$

Proof by induction

Let's assume that $r(x) = q(x)$ for $x \in [0, r_{i-1}]$

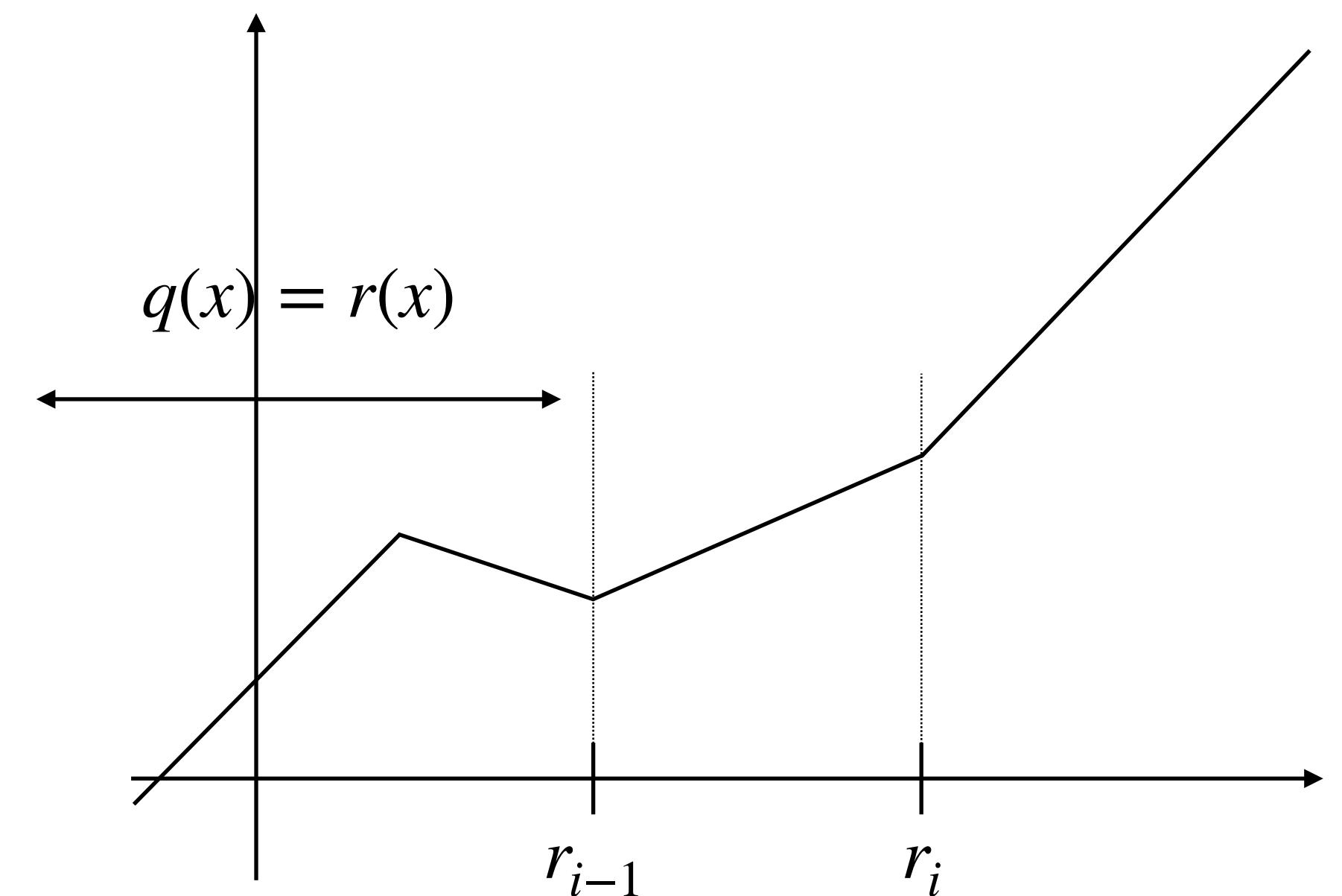
For $x \in [r_{i-1}, r_i]$

$$\begin{aligned} r(x) &= \tilde{a}_1 x + \tilde{b}_1 + \sum_{j=2}^m \tilde{a}_j (x - \tilde{b}_j)_+ \\ &= \tilde{a}_1 x + \tilde{b}_1 + \sum_{j=2}^i \tilde{a}_j (x - \tilde{b}_j) \\ &= \sum_{j=1}^i \tilde{a}_j x + \tilde{b}_1 - \sum_{j=2}^i \tilde{a}_j \tilde{b}_j \end{aligned}$$

Thus

- $r'(x) = \sum_{j=1}^i \tilde{a}_j = a_i$ good slope
- $r(r_{i-1}) = q(r_{i-1})$ good starting point

$$\implies r(x) = q(x) \text{ for } x \in [r_{i-1}, r_i]$$



Why: two affine functions with the same starting point and the same slope are equal

Recap

- Neural networks consist of linear layers stacked together with non-linearities
- A neural network can be seen as a **learned feature extractor** + a **linear predictor**
- Neural networks typically require large amounts of **data** and **compute** to learn good features
- Neural networks have very high representational power (i.e., the universal approximation result) in contrast to simple models like linear regression