A1, A2, 2 did it

B1: Rewrite equation for linear elastostatics

$$-\nabla\cdot\sigma = f \qquad \text{in } \Omega$$

$$\sigma = 2\mu\varepsilon(u) + \lambda(\nabla\cdot u)I \qquad \text{in } \Omega$$

$$u = 0 \qquad \text{on } \Gamma_D$$

$$\sigma\cdot n = g_N \qquad \text{on } \Gamma_N$$

$$\sigma = 2\mu\,\varepsilon(u) + \lambda(\nabla\cdot u)I$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad i,j = 1,2,3$$

$$\nabla\cdot u = \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

thus we have

$$\sigma_{11} = \mu\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) + \lambda\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)$$

$$\sigma_{22} = \mu\left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_2}\right) + \lambda\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)$$

$$\sigma_{33} = \mu\left(\frac{\partial u_3}{\partial x_3} + \frac{\partial u_3}{\partial x_3}\right) + \lambda\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)$$

$$\sigma_{12} = \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \quad \sigma_{13} = \mu\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)$$

$$\sigma_{21} = \mu\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) \quad \sigma_{23} = \mu\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)$$

$$\sigma_{31} = \mu\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) \quad \sigma_{32} = \mu\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right)$$

$$\nabla \cdot \sigma = \begin{bmatrix} \nabla \cdot \vec{\sigma_1} \\ \nabla \cdot \vec{\sigma_2} \\ \nabla \cdot \vec{\sigma_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_1}\sigma_{11} + \frac{\partial}{\partial x_2}\sigma_{12} + \frac{\partial}{\partial x_3}\sigma_{13} \\ \frac{\partial}{\partial x_1}\sigma_{21} + \frac{\partial}{\partial x_2}\sigma_{22} + \frac{\partial}{\partial x_3}\sigma_{23} \\ \frac{\partial}{\partial x_1}\sigma_{31} + \frac{\partial}{\partial x_2}\sigma_{32} + \frac{\partial}{\partial x_3}\sigma_{33} \end{bmatrix}$$

we donate $\frac{\partial}{\partial x_1}$ as ∂_1, $\frac{\partial}{\partial x_2}$ as ∂_2, $\frac{\partial}{\partial x_3}$ as ∂_3

thus

$$\nabla \cdot \sigma = \begin{bmatrix} \partial_1\sigma_{11} + \partial_2\sigma_{12} + \partial_3\sigma_{13} \\ \partial_1\sigma_{21} + \partial_2\sigma_{22} + \partial_3\sigma_{23} \\ \partial_1\sigma_{31} + \partial_2\sigma_{32} + \partial_3\sigma_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 2\mu\partial_{11}u_1 + \lambda(\partial_{11}u_1 + \partial_{12}u_2 + \partial_{13}u_3) + \mu(\partial_{12}u_1 + \partial_{11}u_2) \\ \qquad + \mu(\partial_{13}u_1 + \partial_{11}u_3) \\ \mu(\partial_{21}u_2 + \partial_{22}u_1) + 2\mu\partial_{22}u_2 + \lambda(\partial_{21}u_1 + \partial_{22}u_2 + \partial_{23}u_3) \\ \qquad + \mu(\partial_{23}u_2 + \partial_{22}u_3) \\ \mu(\partial_{31}u_3 + \partial_{33}u_1) + \mu(\partial_{32}u_3 + \partial_{33}u_2) \\ \qquad + 2\mu\partial_{33}u_3 + \lambda(\partial_{31}u_1 + \partial_{32}u_2 + \partial_{33}u_3) \end{bmatrix}$$

thus, we have

$$\boxed{\begin{aligned} &2\mu\partial_{11}u_1 + \lambda(\partial_{11}u_1 + \partial_{12}u_2 + \partial_{13}u_3) + \mu(\partial_{12}u_1 + \partial_{11}u_2) \\ &\quad + \mu(\partial_{13}u_1 + \partial_{11}u_3) = f_1 \\ &\mu(\partial_{21}u_2 + \partial_{22}u_1) + 2\mu\partial_{22}u_2 + \lambda(\partial_{21}u_1 + \partial_{22}u_2 + \partial_{23}u_3) \\ &\quad + \mu(\partial_{23}u_2 + \partial_{22}u_3) = f_2 \\ &\mu(\partial_{31}u_3 + \partial_{33}u_1) + \mu(\partial_{32}u_3 + \partial_{33}u_2) \\ &\quad + 2\mu\partial_{33}u_3 + \lambda(\partial_{31}u_1 + \partial_{32}u_2 + \partial_{33}u_3) = f_3 \end{aligned}}$$

with bondary condition

$$\vec{u}\big|_{\Gamma_D} = \vec{g}_D \Rightarrow \begin{cases} u_1 = g_{D1} \\ u_2 = g_{D2} \\ u_3 = g_{D3} \end{cases}$$

$$\sigma \cdot \vec{n}\big|_{\Gamma_N} = \vec{g}_N$$

$$\sigma \cdot \vec{n} = \begin{bmatrix} \sigma_{11} + \sigma_{12} + \sigma_{13} \\ \sigma_{21} + \sigma_{22} + \sigma_{23} \\ \sigma_{31} + \sigma_{32} + \sigma_{33} \end{bmatrix} = \begin{bmatrix} g_{n1} \\ g_{n2} \\ g_{n3} \end{bmatrix}$$

$$\boxed{\begin{aligned}
&2\mu \partial_1 u_1 + \lambda(\partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3) + \mu(\partial_1 u_2 + \partial_2 u_1) + \mu(\partial_1 u_3 + \partial_3 u_1) = g_{n1} \\
&\mu(\partial_1 u_2 + \partial_2 u_1) + 2\mu \partial_2 u_2 + \lambda(\partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3) + \mu(\partial_2 u_3 + \partial_3 u_2) = g_{n2} \\
&\mu(\partial_1 u_3 + \partial_3 u_1) + \mu(\partial_2 u_3 + \partial_3 u_2) + 2\mu \partial_3 u_3 + \lambda(\partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3) = g_{n3}
\end{aligned}}$$

B2: Euler-Bernoulli Beam:

$$(EI\,w'')'' = q(x) \qquad x \in (0, L)$$

$$w(0) = w_0 \,, w'(0) = w_1$$

$$(EIw'')(L) = w_2 \quad (EIw'')'(L) = w_3$$

Derive a weak form for this BVP

we have a test function v

$$v\,(EI\,w'')'' = q(x)\,v$$

$\Rightarrow$

$$\int_0^L v\,(EIw'')''\,dx = \int_0^L q(x)\,v\,dx$$

the LHS we have $v(EIw'')'' + v'(EIw'')' = \left[v(EIw'')'\right]'$

$$\Rightarrow \int_0^L v(EIw'')''\,dx =$$

$$v(L)(EIw'')'(L) - v(0)(EIw'')'(0) - \int_0^L v'(EIw'')'\,dx$$

$$= v(L)\,w_3 - v(0)\,(EIw'')'(0) - \int_0^L v'(EIw'')'\,dx$$

in this case we let $V(0)=0$ to elimate $(EIw'')(0)$

Now we have

$$\int_0^L v(EIw'')''dx = V(L)w_3 - \int_0^L v'(EIw'')'dx$$

$$= q(x)v$$

Next, we consider the term $\int_0^L v'(EIw'')'dx$

$$v'(EIw'')' + v''(EIw'') = [v'(EIw'')]'$$

as a result

$$\int_0^L v'(EIw'')'dx = v'(EIw'')\Big|_0^L - \int_0^L v''(EIw'')dx$$

$$= v'(L)(EIw'')(L) - v'(0)(EIw'')(0) - \int_0^L v''(EIw'')dx$$

Since $(EIw'')(L) = w_2$

we have

$$\Rightarrow v'(L)w_2 - v'(0)(EIw'')(0) - \int_0^L v''(EIw'')dx$$

we let $v'(0)=0$ to elimate the term $EIw''(0)$

Now, we have

$$\int_0^L v(EIw'')''dx = v(L)w_3 + \int_0^L v''(EIw'')dx - v'(L)w_2$$

$$\int_0^L v''(EIw'')dx + v(L)w_3 - v'(L)w_2 = \int_0^L q(x)V dx$$

just one weak equation? how many?

-2 with $V(0)=0 \quad v'(0)=0$

$w(0)=w_0 \quad w'(0)=w_1$

$$T = \{u \mid u \in H^2(0,L),\ u(0)=0,\ u'(0)=0\}$$

$$S = \{u \mid u \in H^2(0,L),\ u(0)=w_0,\ w'(0)=w_1\}$$

in this case $(EIw'')(L) = w_2$ & $(EIw'')'(L)=w_3$ are natural boundary conditions

$w(0)=w_0$ & $w'(0)=w_1$ are essential b.c.

C_1: $x \in (0,1)$

$$\begin{cases} -u''(x)+u'(x)+u(x) = (x+1)^2 + x - 2 - \sin(x) + 2\cos(x) \\ u(0) = 1 \\ u'(1) = 3-\sin 1 \end{cases}$$

we have the test function v

$$-vu'' + vu' + uv = v[(x+1)^2 + x - 2 - \sin(x) + 2\cos(x)]$$

$$\int_0^1 (-vu'' + vu' + uv)\,dx = \int_0^1 v[(x+1)^2 + x - 2 - \sin(x) + 2\cos(x)]\,dx$$

$$vu'' + v'u' = (vu')'$$

$$\Rightarrow \int_0^1 (v'u' + vu' + vu)\,dx - vu'\Big|_0^1$$

$$= \int_0^1 (v'u' + vu' + vu)\,dx - v(1)u'(1) + v(0)u'(0)$$

$$= \int_0^1 (v'u' + vu' + vu)\,dx - v(1)(3-\sin(1)) + v(0)u'(0)$$

Let $v(0)=0$ to elimate $u'(0)$

Now we have

$$\int_0^1 (v'u' + vu' + vu)\,dx - v(1)(3-\sin(1))$$

$$= \int_0^1 v\left[(x+1)^2 + x - 2 - \sin(x) + 2\cos(x)\right] dx$$

where is the weak form of the given BVP?

$u(0)=1$ is essential b.c.

-2

$u'(1) = 3 - \sin(1)$ is natural b.c.

(b) $$T_2 = \text{span}\{\sin(x), x\}$$

$$S_3 = \text{span}\{\cos(x), e^x, x\}$$

we have $\psi_1(x) = \sin(x)$, $\psi_2(x) = x$

$\phi_1(x) = \cos(x)$, $\phi_2(x) = e^x$, $\phi_3(x) = x$

$$u_{pg} = u_1\phi_1(x) + u_2\phi_2(x) + u_3\phi_3(x)$$

with the essential b.c.

$u_{pg}(0) = 1 \Rightarrow \boxed{\phi_1(0)u_1 + \phi_2(0)u_2 + \phi_3(0)u_3 = 1}$

$$\int_0^1 \Big[\psi_1'(x)\left(u_1\phi_1'(x) + u_2\phi_2'(x) + u_3\phi_3'(x)\right)$$

$$+ \psi_1(x)\left(u_1\phi_1'(x) + u_2\phi_2'(x) + u_3\phi_3'(x)\right)$$

$$+ \psi_1(x)\left(u_1\phi_1(x) + u_2\phi_2(x) + u_3\phi_3(x)\right)\Big] dx$$

$$= \psi_1(1)(3-\sin(1)) + \int_0^1 \psi_1(x)\left[(x+1)^2 + x - 2 - \sin(x) + 2\cos(x)\right]dx$$

$$\boxed{\begin{aligned} \Rightarrow & \int_0^1 \left[\psi_1'(x)\phi_1'(x) + \psi_1(x)\phi_1'(x) + \psi_1(x)\phi_1(x)\right]dx \cdot u_1 \\ & + \int_0^1 \left[\psi_1'(x)\phi_2'(x) + \psi_1(x)\phi_2'(x) + \psi_1(x)\phi_2(x)\right]dx \cdot u_2 \\ & + \int_0^1 \left[\psi_1'(x)\phi_3'(x) + \psi_1(x)\phi_3'(x) + \psi_1(x)\phi_3(x)\right]dx \cdot u_3 \\ & = \psi_1(1)(3-\sin(1)) + \int_0^1 \psi_1(x)\left[(x+1)^2 + x - 2 - \sin(x) + 2\cos(x)\right]dx \end{aligned}}$$

also

more digits

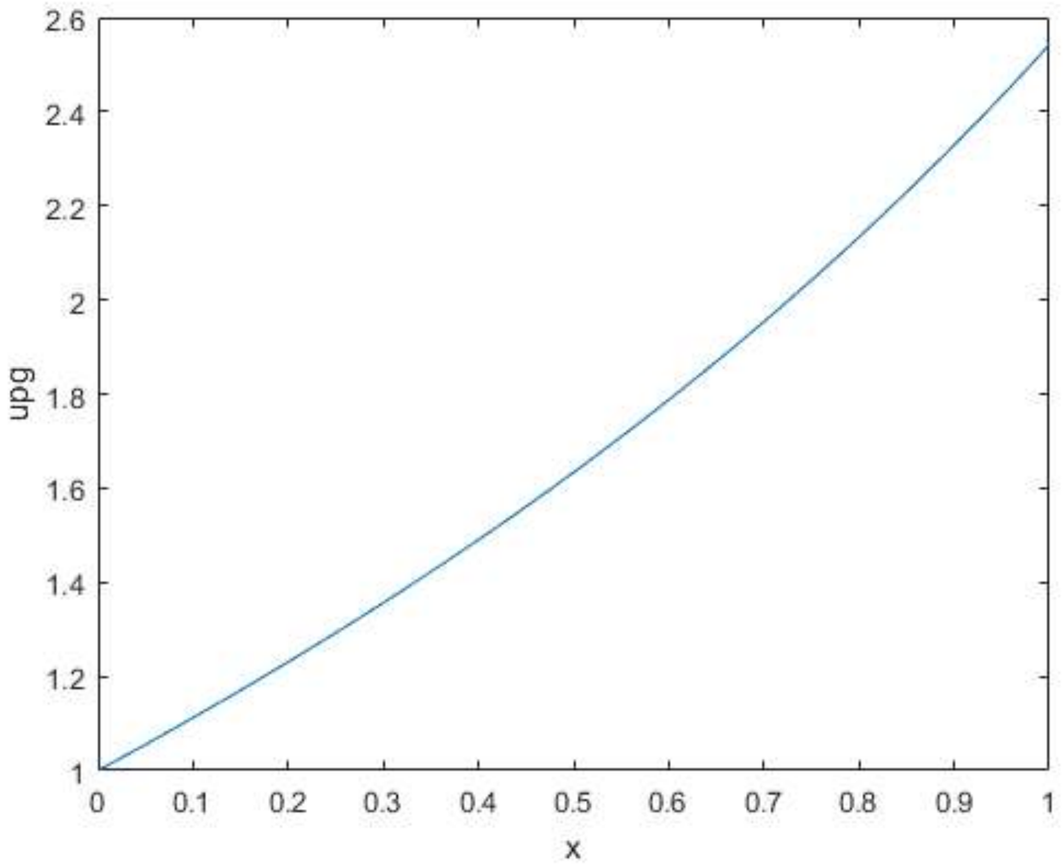
$$\boxed{\begin{aligned} \Rightarrow & \int_0^1 \left[\psi_2'(x)\phi_1'(x) + \psi_2(x)\phi_1'(x) + \psi_2(x)\phi_1(x)\right]dx \cdot u_1 \\ & + \int_0^1 \left[\psi_2'(x)\phi_2'(x) + \psi_2(x)\phi_2'(x) + \psi_2(x)\phi_2(x)\right]dx \cdot u_2 \\ & + \int_0^1 \left[\psi_2'(x)\phi_3'(x) + \psi_2(x)\phi_3'(x) + \psi_2(x)\phi_3(x)\right]dx \cdot u_3 \\ & = \psi_2(1)(3-\sin(1)) + \int_0^1 \psi_2(x)\left[(x+1)^2 + x - 2 - \sin(x) + 2\cos(x)\right]dx \end{aligned}}$$

$$u_{pg}\left(\frac{\pi}{5.7}\right) = 1.711$$

do not use screen shot to present programs

```
%Finite Element Method HW#1
clear;
clc;
format long
% test function
ph1 = @(x) sin(x);
ph2 = @(x) x;
ph1d = @(x) cos(x);
ph2d = @(x) 1;

%trial function
ps1 = @(x) cos(x);
ps2 = @(x) exp(x);
ps3 = @(x) x;
ps1d = @(x) -sin(x);
ps2d = @(x) exp(x);
ps3d = @(x) 1;

M = zeros (3,3);
M (1, :)=[ps1(0), ps2(0), ps3(0)];
tmp21 = @(x) ph1d(x).*ps1d(x)+ph1(x).*ps1d(x)+ph1(x).*ps1(x);
M(2,1) = integral (tmp21, 0, 1);
tmp22 = @(x) ph1d(x).*ps2d(x)+ph1(x).*ps2d(x)+ph1(x).*ps2(x);
M(2,2) = integral (tmp22, 0, 1);
tmp23 = @(x) ph1d(x).*ps3d(x)+ph1(x).*ps3d(x)+ph1(x).*ps3(x);
M(2,3) = integral (tmp23, 0, 1);
tmp31 = @(x) ph2d(x).*ps1d(x)+ph2(x).*ps1d(x)+ph2(x).*ps1(x);
M(3,1) = integral (tmp31, 0, 1);
tmp32 = @(x) ph2d(x).*ps2d(x)+ph2(x).*ps2d(x)+ph2(x).*ps2(x);
M(3,2) = integral (tmp32, 0, 1);
tmp33 = @(x) ph2d(x).*ps3d(x)+ph2(x).*ps3d(x)+ph2(x).*ps3(x);
M(3,3) = integral (tmp33, 0, 1);

f = @(x)  (x+1).*(x+1) + x-2 -sin(x)+ 2*cos(x);
tmpr1 = @(x) ph1(x).*f(x);
rhs1 = integral(tmpr1, 0, 1);
tmpr2 = @(x) ph2(x).*f(x);
rhs2 = integral(tmpr2, 0, 1);

rhs = [1; ph1(1)*(3-sin(1))+rhs1; ph2(1)*(3-sin(1))+rhs2];
vu = M\rhs;

u_pg = @(x) vu(1)*ps1(x)+vu(2)*ps2(x)+vu(3)*ps3(x);

u_pg(pi/5.7)

x=0:0.01:1;
plot(x, u_pg(x));
xlabel('x');
ylabel('upg');
```

```
ans =

   1.710995019301113
```



Published with MATLAB® R2015b