Notes for Applied PDEs

Wenge Huang

March 11, 2023

1 Lecture1

Let's consider the 2nd order PDEs. The most general form of a 2nd order PDE with two variables is:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \tag{1}$$

where $A, B, ...G$ are constants or given functions of x and y.

1.1 Types of PDEs

All linear PDEs in the form of Eq(1) can be classified into three types: ***hyperbolic, parabolic*** and ***elliptic***.

The hyperbolic type: e.g. the wave equation

$$\Delta(x, y) = B^2(x, y) - 4A(x, y)C(x, y) > 0 \tag{2}$$

The parabolic type: e.g. the heat equation

$$\Delta(x, y) = B^2(x, y) - 4A(x, y)C(x, y) = 0 \tag{3}$$

The elliptic type: e.g. the Laplace equation

$$\Delta(x, y) = B^2(x, y) - 4A(x, y)C(x, y) < 0 \tag{4}$$

1.2 Superposition principle

If u_1, u_2, ... u_n, are the solutions to the linear homogeneous PDE $Lu = 0$, and u_1, u_2, ... $u_n \in R$. Here L is a linear differential operator. Then the linear combination of $\sum_1^n c_i u_i$ is also the solution of the PDE.

Let S_h be the set of all solutions to the homogeneous problem

$$Lu = 0$$

Then we consider the inhomogeneous problem

$$Lu = f$$

The set of all solutions to this inhomogeneous problem is given by

$$S_i = \{u_i + u_h | u_h \in S_h\}$$

Here u_i is a particular solution to the inhomogeneous problem and S_i is the translation of S_h by u_i.

2 Lecture2

2.1 ODE

Let's consider the 2nd ODE first.

$$ax''(t) + bx'(t) + cx(t) = 0 \tag{5}$$

where a, b and $c \in R$ and $a \neq 0$.

We consider the characteristic equation first:

$$a\lambda^2 + b\lambda + c = 0 \tag{6}$$

with two solution λ_1 and λ_2.

Case I, when $\lambda_1 \neq \lambda_2$, we have the independent solutions:

$$\begin{cases} x_1(t) = e^{\lambda_1 t} \\ x_2(t) = e^{\lambda_2 t} \end{cases}$$

.

Case II, when have the same roots $\lambda_1 = \lambda_2$, we have the solution:

$$\begin{cases} x_1(t) = e^{\lambda_1 t} \\ x_2(t) = \boldsymbol{t} e^{\lambda_2 t} \end{cases}$$

.

Case III, we have complex conjugate pairs of roots $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$. The two independent solutions are:

$$\begin{cases} x_1(t) = e^{\alpha t} cos(\beta t) \\ x_2(t) = e^{\alpha t} sin(\beta t) \end{cases}$$

.

With initial conditions given, we'll search solution with a linear combination of the independent solutions

$$x(t) = C_1 x_1(t) + C_2 x_2(t) \tag{7}$$

where C_1 and $C_2 \in R$.

This is called the general solution of the homogeneous problem Eq(5).

Then for the inhomogeneous version,

$$ax''(t) + bx'(t) + cx(t) = f(t) \tag{8}$$

we need to use the **variable of parameter formula** to find a particular solution.

3 Heat equation

Let's consider the 1D heat equation on the interval $(0, L)$ and subject to some initial conditions(IBVP).

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} & x \in (0, L), t > 0 \\ u(0, t) = 0, u(L, t) = 0 & t > 0 \\ u(x, 0) = f(x) & 0 \leq x \leq L \end{cases} \tag{9}$$

3.1 Separation of variable

We are looking for non-trivial solutions. We assume:

$$u(x, t) = V(x)T(t) \qquad x \in (0, L), t > 0$$

Plugging it into the Eq(9), we can obtain

$$\frac{T'}{T} = \frac{V''}{V} = \beta \qquad \forall x \in (0, L), t > 0$$

Thus, we get

$$\begin{cases} T'(t) = \beta T(t) \\ V''(t) = \beta V(t) \end{cases}$$

we successfully transfer the PDE to ODEs.

Considering the boundary conditions $u(0, t) = 0, u(L, t) = 0, \quad t > 0$ we have $V(0) = 0$ and $V(L) = 0$.

$$\begin{cases} V''(t) = \beta V(t) & , x \in (0, L) \\ V(0) = 0 \\ V(L) = 0 \end{cases}$$

In this case, the characteristic equation is:

$$\lambda^2 - \beta = 0$$

we have three cases for β. And we can check that only when $\beta < 0$, the solution is non-trivial.

When $\beta > 0$ we have two distinguished real roots

$$\begin{cases} V_1(x) = e^{-\sqrt{\beta}x} \\ V_2(x) = e^{\sqrt{\beta}x} \end{cases}$$

Then $V(x) = C_1 V_1(x) + C_2 V_2(x)$ with the boundary conditions $V(0) = 0$ and $V(L) = 0$. As a result $C_1 = C_2 = 0$.

When $\beta = 0$, we have $\lambda_1 = \lambda_2 = 0$. Thus we can find that $V_1(x) = 1$ and $V_1(x) = x$. Considering the boundary conditions, the coefficients are also should be 0, *i.e.* $C_1 = C_2 = 0$. When $\beta < 0$, the solutions of the characteristic function is

$$\lambda = \pm\sqrt{-\beta}i$$

As a result, we find that $V_1(x) = cos(\sqrt{-\beta}x)$ and $V_2(x) = sin(\sqrt{-\beta}x)$. With the boundary condition given $V(0) = 0$ and $V(L) = 0$, we have that:

$$C_1 + 0 = 0$$

$$C_1 cos(\sqrt{-\beta}L) + C_2 sin(\sqrt{-\beta}L) = 0$$

For non-trivial solutions, we need to have $sin(\sqrt{-\beta}L) = 0$ and

$$\beta_n = -\left(\frac{n\pi}{L}\right)^2, \quad n \in N$$

Thus we find the eigenvalues and the eigenfunctions:

$$\begin{cases} \beta_n = -\left(\frac{n\pi}{L}\right)^2 & , n \in N \\ V_n(x) = sin(\frac{n\pi}{L}x) \end{cases} \tag{10}$$

Now for each β we have the solution for $T(t)$:

$$T'(t) = \beta_n T(t), \quad n \in N$$

$$T(t) = e^{-\left(\frac{n\pi}{L}\right)^2 t} \quad n \in N$$

We find the solution:

$$u_n(x,t) = V_n(x)T_n(t) = e^{-\left(\frac{n\pi}{L}\right)^2 t} sin(\frac{n\pi}{L}x) \quad n \in N \tag{11}$$

From the above calculation, we know that each u_n satisfies the following homogeneous BVP:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} & x \in (0, L), t > 0 \\ u(0,t) = 0, u(L,t) = 0 & t > 0 \end{cases} \tag{12}$$

We look for the solution of the form:

$$u(x,t) = \sum_{n=1}^{\infty} C_n u_n(x,t) \tag{13}$$

We need to utilize the initial conditions to find the coefficient. The initial condition is:

$$u(x,0) = f(x) \quad x \in [0, L]$$

With $t = 0$, we have $T(0) = 1$ and also:

$$\sum_{n=1}^{\infty} C_n sin(\frac{n\pi}{L}x) = f(x) \tag{14}$$

This is exactly the problem of finding the **Fourier sine expansion** of the given function f.

To find the coefficient C_n, we use the fact that V_n are orthogonal to each other in the sense that:

$$\int_0^L V_n(x)V_m(x)dx = \begin{cases} 0 & if \quad m \neq n \\ \frac{L}{2} & if \quad m = n \end{cases} \tag{15}$$

Now multiply both side of Eq(14) by V_m and integrate from 0 to L.

$$\int_0^L \sum_{n=1}^{\infty} C_n V_n(x)V_m(x)dx = \int_0^L f(x)V_m(x)dx$$

Assume we can switch $\int_0^L$ with $\sum_{n=1}^{\infty}$, we get

$$C_n = \frac{2}{L}\int_0^L f(x) sin(\frac{n\pi}{L}x)dx \quad n \in N$$

With known the coefficient, we finally obtain the complete solution of the IBVP Eq(9):

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{L}\int_0^L f(x') sin(\frac{n\pi}{L}x')dx' e^{-\left(\frac{n\pi}{L}\right)^2 t} sin(\frac{n\pi}{L}x) \quad n \in N, t > 0, x \in [0, L] \tag{16}$$

3.2 Source term

Now we are trying to solve the heat equation with the source term:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(x,t) & x \in (0,L), t > 0 \\ \\ u(0,t) = 0, u(L,t) = 0 & t > 0 \\ \\ u(x,0) = f(x) & 0 \leq x \leq L \end{cases} \tag{17}$$

First we recall the homogeneous BVP

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} & x \in (0,L), t > 0 \\ u(0,t) = 0, u(L,t) = 0 & t > 0 \end{cases}$$

We know the eigenvalues and eigenfunctions:

$$\begin{cases} \beta_n = -\left(\frac{n\pi}{L}\right)^2 \\ V_n(x) = sin(\frac{n\pi}{L}x) \end{cases} , n \in N$$

We are looking for the solution of the form:

$$u(x,t) = \sum_{n=1}^{\infty} \tilde{T}_n(t) V_n(x) \tag{18}$$

Plugging Eq(31) into the source term Eq(17), we get

$$\sum_{n=1}^{\infty} \frac{d}{dt}(\tilde{T}_n(t)) V_n(x) = \sum_{n=1}^{\infty} \tilde{T}_n(t) \frac{d^2}{dx^2}(V_n(x)) + g(x,t)$$

with knowing that

$$V''(x) = \beta V(x)$$

Now using the orthogonal property, we multiply both sides by V_m and integrate from 0 to L:

$$\sum_{n=1}^{\infty} \frac{d}{dt}(\tilde{T}_n(t)) \int_0^L V_n(x) V_m(x) dx = \sum_{n=1}^{\infty} \tilde{T}_n(t) \beta_n \int_0^L V_n(x) V_m(x) dx + \int_0^L g(x,t) V_m(x) dx$$

Using Eq(15)

$$\int_0^L V_n(x) V_m(x) dx = \begin{cases} 0 & if \quad m \neq n \\ \frac{L}{2} & if \quad m = n \end{cases}$$

We have

$$\frac{d}{dt}(\tilde{T}_m(t)) = \beta_m \tilde{T}_m(t) + \frac{2}{L} \int_0^L g(x,t) V_m(x) dx$$

we get an ODE

$$\frac{d}{dt}(\tilde{T}_n(t)) = \beta_n \tilde{T}_n(t) + h_n(t)$$

Then we consider the initial condition $u(x,0) = f(x)$:

$$u(x,0) = \sum_{n=1}^{\infty} \tilde{T}_n(0) V_n(x) = f(x)$$

Like before, we multiply both sides by V_m and integrate from 0 to L:

$$\sum_{n=1}^{\infty} \tilde{T}_n(0) \int_0^L V_n(x) V_m(x) dx = \int_0^L f(x) V_m(x) dx$$

$$\tilde{T}_n(0) = \frac{2}{L} \int_0^L f(x) V_n(x) dx = \omega_n$$

Now we have the IVP for $\tilde{T}_n(t)$:

$$\begin{cases} \frac{d}{dt}\tilde{T}_n(t) = \beta_n \tilde{T}_n(t) + h_n(t) \\ \tilde{T}_n(0) = \omega_n \end{cases} \tag{19}$$

where

$$h_n(t) = \frac{2}{L} \int_0^L g(x,t) V_n(x) dx$$

$$\omega_n = \frac{2}{L} \int_0^L f(x) V_n(x) dx$$

Now we need to solve the IVP problem with the variation of parameter formula:

$$\tilde{T}_n(t) = \omega_n e^{\beta_n t} + \int_0^t e^{\beta_n (t-s)} h_n(s) ds$$

Finally we find the complete solution of the heat equation with source term:

$$\begin{aligned} u(x,t) = &\sum_{n=1}^{\infty} \frac{2}{L} \int_0^L f(x') sin(\frac{n\pi}{L} x') dx' e^{-\left(\frac{n\pi}{L}\right)^2 t} sin(\frac{n\pi}{L} x) \\ &+ \sum_{n=1}^{\infty} \left[\int_0^t e^{-\left(\frac{n\pi}{L}\right)^2 (t-s)} \frac{2}{L} \int_0^L g(x,s) sin(\frac{n\pi}{L} x) ds \right] \end{aligned} \tag{20}$$

3.3 Non-homogeneous b.c.

Now we consider the fully nonhomogeneous problem, which means non-homogeneous source term in the equation (source term) and also in the boundary conditions.

Let's consider the following equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(x,t) & x \in (0,L), t > 0 \\ u(0,t) = u_1(t), u(L,t) = u_2(t) & t > 0 \\ u(x,0) = f(x) & 0 \leq x \leq L \end{cases} \tag{21}$$

here the boundary condition is called the non-homogeneous Dirichlet b.c.

Let's consider a new function $\theta(x,t)$:

$$\theta(x,t) = u(x,t) - w(x,t)$$

Let $w(0,t) = u_1(t)$ and $w(L,t) = u_2(t)$, we can have the new θ function to fit the homogeneous boundary conditions.

It is always a good choice to choose the linear relation. Thus, we can have:

$$w(x,t) = \frac{L-x}{L} u_1(t) + \frac{x}{L} u_2(t)$$

We can plug θ inside the function to get:

$$\frac{\partial \theta}{\partial t} + \frac{L-x}{L} u_1'(t) + \frac{x}{L} u_2'(t) = \frac{\partial^2 \theta}{\partial x^2} + g(x,t)$$

Thus, the problem is transferred back to the source term problem:

$$\begin{cases} \frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} + \tilde{g}(x,t) & x \in (0,L), t > 0 \\ \theta(0,t) = 0, \theta(L,t) = 0 & t > 0 \\ \theta(x,0) = \tilde{f}(x) & 0 \leq x \leq L \end{cases} \tag{22}$$

$$\tilde{g}(x,t) = g(x,t) - \frac{L-x}{L} u_1'(t) - \frac{x}{L} u_2'(t)$$

$$\tilde{f}(x,t) = f(x) - \frac{L-x}{L} u_1(0) - \frac{x}{L} u_2(0)$$

3.4 Wave equation

Now we consider the homogeneous wave equation:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} & x \in (0, L), t > 0 \\ \\ u(0,t) = 0, u(L,t) = 0 & t > 0 \\ \\ u(x,0) = \phi_1(x), \frac{\partial u}{\partial t} = \phi_2(x) & 0 \leq x \leq L \end{cases} \tag{23}$$

First, we consider the homogeneous BVP:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} & x \in (0, L), t > 0 \\ \\ u(0,t) = 0, u(L,t) = 0 & t > 0 \end{cases}$$

The homogeneous BVP wave equation has the following general solution set:

$$u_n(x,t) = V_n(x)T_n(x) = \sin(\frac{n\pi}{L}x)\left[a_n \cos(\frac{cn\pi}{L}t) + b_n \sin(\frac{cn\pi}{L}t)\right], n \in N$$

Now, back to the IBVP Eq(23), in order to satisfy the initial condition, we are looking for the solution in the form of:

$$u(x,t) = \sum_{n=1}^{\infty} \sin(\frac{n\pi}{L}x)\left[a_n \cos(\frac{cn\pi}{L}t) + b_n \sin(\frac{cn\pi}{L}t)\right] \tag{24}$$

We know that when $t = 0$, we have $\sin(\frac{cn\pi}{L}t) = 0$ and $\cos(\frac{cn\pi}{L}t) = 1$. Thus we have:

$$u(x,0) = \sum_{n=1}^{\infty} \sin(\frac{n\pi}{L}x) \cdot a_n = \phi_1(x) \tag{25}$$

$$u'(x,0) = \sum_{n=1}^{\infty} \sin(\frac{n\pi}{L}x)\left[\frac{cn\pi}{L}b_n\right] = \phi_2(x) \tag{26}$$

Recall the orthogonality we have:

$$\int_0^L \sin(\frac{n\pi}{L}x)\sin(\frac{m\pi}{L}x) = \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n \end{cases}$$

Thus, we can find that:

$$a_n = \frac{2}{L}\int_0^L \phi_1(x)\sin(\frac{n\pi}{L}x)dx$$

$$b_n = \frac{2}{cn\pi}\int_0^L \phi_2(x)\sin(\frac{n\pi}{L}x)dx$$

To summarize, the solution to the IBVP is$u(x,t)$:

$$\sum_{n=1}^{\infty}\sin(\frac{n\pi}{L}x)\left[\frac{2}{L}\int_0^L \phi_1(x')\sin(\frac{n\pi}{L}x')dx'\cos(\frac{cn\pi}{L}t) + \frac{2}{cn\pi}\int_0^L \phi_2(x')\sin(\frac{n\pi}{L}x')dx'\sin(\frac{cn\pi}{L}t)\right]$$

In the case with non-homogeneous B.C., we'll do the change of variable to make the boundary condition homogeneous as in the heat equation case. In the case with non-homogeneous source term, we need to find a particular solution for the inhomogeneous BVP using variation of parameter formula.

Boundary conditions determine the shape of the eigenfunctions.

Dirichlet boundary conditions (with rectangular domain in 1D) we have the **Fourier sine series**:

$$\left\{\sin(\frac{n\pi}{L}x)|n \in N\right\}$$

Neuman boundary conditions, we have **Fourier cosine series**:

$$\left\{\cos(\frac{n\pi}{L}x)|n \in N\right\}$$

For periodic boundary condtions, we will get both the sine and cosine eigenfunctions.

For non-rectangular domain, the eigenfunctions may not be trigonometric functions. **Can One Hear the Shape of a Drum?**

3.5 Problems in higher(spatial) dimentions

Now we consider a 2D(spatial) heat equation:

$$\begin{cases} u_t = u_{xx} + u_{yy} & 0 < x < 1, 0 < y < 1, t > 0 \\ \\ u(0,y,t) = 0, u_x(1,y,t) = 0 \\ u(x,0,t) = 0, u(x,1,t) = 0 & t > 0 \\ \\ u(x,y,0) = f(x,y) \end{cases} \tag{27}$$

note that we have a Neumann boundary condition.

we also do the separation of variable:

$$u(x,y,t) = X(x)Y(y)T(t)$$

plugging it into the 2D heat equation, we get:

$$X(x)Y(y)T'(t) = X''(x)Y(y)T(t) + X(x)Y''(y)T(t)$$

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \beta$$

Thus, we have:

$$\frac{T'(t)}{T(t)} = \beta \tag{28}$$

$$\frac{X''(x)}{X(x)} = \beta - \frac{Y''(y)}{Y(y)} = \mu$$

$$\begin{cases} X''(x) = \mu X(x) \\ Y''(y) = (\beta - \mu)Y(y) \end{cases} \tag{29}$$

Consider the boundary conditions:

$$u(0, y, t) = 0 \Rightarrow X(0) = 0$$

$$u_x(1, y, t) \Rightarrow X'(1) = 0$$

$$u(x, 0, t) = 0 \Rightarrow Y(0) = 0$$

$$u(x, 1, t) = 0 \Rightarrow Y(1) = 0$$

We end up with two eigenvalue problems.

$$\begin{cases} X''(x) = \mu X(x) \\ X(0) = 0, X'(1) = 0 \end{cases}$$

$$\begin{cases} Y''(y) = (\beta - \mu)Y(y) \\ Y(0) = 0, Y(1) = 0 \end{cases}$$

Then we have the eigenvalue and eigenfunction as such:

$$\mu_n = -(n\pi - \frac{\pi}{2})^2$$

$$X_n(x) = \sin((n\pi - \frac{\pi}{2})x), n \in N$$

then for each $\mu = \mu_n$, we have:

$$\begin{cases} Y''(y) = (\beta - \mu_n)Y(y) \\ Y(0) = Y(1) = 0 \end{cases}$$

The eigenvalue and eigenfunctions are:

$$\beta - \mu_n = -(m\pi)^2$$

$$Y_m(y) = \sin(m\pi y), m \in N$$

note that β depends on two indices m & n:

$$\beta_{m,n} = -(n\pi - \frac{\pi}{2})^2 - (m\pi)^2$$

Now back to Eq(28), we have the time term:

$$T'_{m,n}(t) = -[(n\pi - \frac{\pi}{2})^2 + (m\pi)^2]T_{m,n}(t)$$

$$T_{m,n}(t) = C_{m,n}e^{\beta_{m,n}t} = C_{m,n}e^{\ -[(n\pi - \frac{\pi}{2})^2 + (m\pi)^2]t} \tag{30}$$

Now for the BVP we have:

$$u(x,y,t) = \sum_{m,n=1}^{\infty} C_{m,n}e^{\ -[(n\pi - \frac{\pi}{2})^2 + (m\pi)^2]t} \sin((n\pi - \frac{\pi}{2})x)\sin(m\pi y)$$

At time $t = 0$, we have the initial condition

$$u(x,y,0) = f(x,y) = \sum_{m,n=1}^{\infty} C_{m,n}\sin((n\pi - \frac{\pi}{2})x)\sin(m\pi y)$$

To determine $C_{m,n}$ we it by $X_p(x)Y_q(y)$ for the orthogonality and integrate it:

$$\int_0^1\int_0^1 f(x,y)X_p(x)Y_q(y)dxdy = \sum_{m,n=1}^{\infty} C_{m,n}X_n(x)X_p(x)Y_m(y)Y_q(y)dxdy = \frac{1}{4}C_{p,q}$$

Thus we have:

$$C_{p,q} = 4\int_0^1\int_0^1 f(x,y)X_p(x)Y_q(y)dxdy$$

3.6 Laplace equation

Now let's consider the following equation:

$$\begin{cases} u_{xx} + u_{yy} = 0 & 0 < x < 1, 0 < y < 1, t > 0 \\ u(x,0) = 0, u(x,1) = x - x^2 & 0 \le x \le 1 \\ u(0,y) = 0, u(1,y) = 0 & 0 \le y \le 1 \end{cases} \tag{31}$$

Note that we no longer have the initial conditions, since there is no "time" dependence. The solutions to the Laplace equation $\Delta u = 0$ are the steady-state solutions to the heat equation.

Assuming that $u(x,y) = X(x)Y(y)$, we'll have:

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \beta$$

$$\begin{cases} X''(x) = \beta X(x) \\ Y''(y) = -\beta Y(y) \end{cases}$$

Note that we have three homogeneous boundary conditions:

$$\begin{array}{l} u(x,0) = 0 \Rightarrow Y(0) = 0 \\ u(0,y) = 0 \Rightarrow X(0) = 0 \\ u(1,y) = 0 \Rightarrow X(1) = 0 \\ u(x,1) = x - x^2 \end{array}$$

From above, we get:

$$\begin{cases} X''(x) = \beta X(x) \\ X(0) = 0, X(1) = 0 \end{cases}$$

with the eigenvalue and eigenfunction

$$\beta_n = -(n\pi)^2$$

$$X_n(x) = \sin(n\pi x), n \in N$$

For the y variable:

$$Y''(y) = (n\pi)^2 Y(y)$$

The eigenfunction is:

$$Y_n(y) = C_1 e^{n\pi y} + C_2 e^{-n\pi y}$$

Using the third boundary condition $Y(0) = 0$, we have $C_1 + C_2$:

$$Y_n(y) = C_1(e^{n\pi y} - e^{-n\pi y}) = 2C_1 \sinh(n\pi y)$$

Now with three homogeneous boundary conditions, we have the solution:

$$u_n(x,y) = \sum_{n=1}^{\infty} C_n \sin(n\pi x) \sinh(n\pi y), n \in N$$

Now we need to choose C_n to statisfy the remaining boundary condition $u(x,1) = x - x^2$.

$$\sum_{n=1}^{\infty} C_n \sin(n\pi x)\sinh(n\pi y) = x - x^2$$

multiply both side by $\sin(n\pi x)$ and integrate.

$$\frac{1}{2}C_n \sinh(n\pi) = \int_0^1 (x - x^2)\sin(n\pi x)dx$$

$$C_n = \frac{\int_0^1 (x - x^2)\sin(n\pi x)dx}{2\sinh(n\pi)}$$

Finally, we have:

$$u(x,y) = \sum_{n=1}^{\infty} \frac{\int_0^1 (x - x^2)\sin(n\pi x)dx}{2\sinh(n\pi)} \sin(n\pi x)\sinh(n\pi y)$$

Note that in the case we have more than one inhomogeneous boundary condition, we need to perform the **change of variables** to convert the equation to a problem with three homogeneous boundary conditions.

3.7 Inner product space

Note that the definition of a vector space gives an algebraic structure for vectors (i.e., addition and scalar multiplication). In the following, we shall build a geometric structure for vectors, such as length, distance, angle, etc.

First, we introduce the notation of *inner product*, which is a generalization of the dot product on the Euclidean space R^n.

Definition 5.1. Let V be a real vector space. Consider a function from $V \times V$ to R that maps any pair of $u, v \in V$ to a real number. The function is called an **inner product** donated by $\langle \cdot, \cdot \rangle$(i.e.,$\langle u, v \rangle$ is a real number) if for any $u, v, w \in V$ and $\alpha \in \mathrm{R}$ it satisfies:

$$\langle u, v \rangle = \langle v, u \rangle$$

$$\langle u + w, v \rangle = \langle u, v \rangle + \langle w, v \rangle$$

$$\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$$

$$\langle u, v \rangle \geq 0, \quad and \quad \langle u, v \rangle = 0 \quad if \quad u = 0$$

Let us define next the notation of *lengths*, which is called **norm** on the given vector space.

Definition 5.2. Consider a function form a vector space V to R. The function is called a **norm** denoted by $\|\cdot\|$ if it satisfies:

$$\|u\| \geq 0 \quad and \quad \|u\| = 0 \quad if \quad u = 0$$

$$\|\alpha u\| = |\alpha|\|u\|, \quad for \ all \ \alpha \in R$$

$$\|u + v\| \leq \|u\| + \|v\|$$

A vector space on which a norm is defined is then called a normed space or normed vector space.

Theorem 5.1. Given an inner product on a vector space V, the function defined by $\|u\| = \sqrt{\langle u, u\rangle}$ for all $u \in V$ is a norm.

Theorem 5.2. (Cauchy-Schwarz inequality). For any $u, v \in V$, it holds that:

$$|\langle u, v\rangle| \leq \|u\|\|v\| = \sqrt{\langle u, u\rangle}\sqrt{\langle v, v\rangle}$$

Example 5.1. in R^n the dot product is an inner product:

$$\langle u, v\rangle = u \cdot v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

The corresponding Euclidean norm is given by:

$$\|u\| = \sqrt{\langle u, u\rangle} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

Example 5.2. Let $\hat{L}^2(\Omega;\mathbb{R})$ be the subspace of $C(\Omega;\mathbb{R})$ defined by:

$$\hat{L}^2(\Omega;\mathbb{R}) = \{u \in C(\Omega;\mathbb{R}) | \int_\Omega |u(x)|^2 dx < +\infty\}$$

Define that

$$\langle f, g\rangle = \int_\Omega f(x)g(x)dx, \qquad \forall f, g \in \hat{L}^2(\Omega;\mathbb{R})$$

is an inner product on $\hat{L}^2(\Omega;\mathbb{R})$ and the corresponding norm is given by:

$$\|f\| = (\int_\Omega f(x)^2 dx)^{1/2}$$

$\hat{L}^2(\Omega;\mathbb{R})$ is a function space defined on a domain $\Omega \subset \mathbb{R}^n$ that is a subspace of the space of continuous functions $C(\Omega;\mathbb{R})$.

Specifically, $\hat{L}^2(\Omega;\mathbb{R})$ is the space of square-integrable functions on Ω, which means that a function $f : \Omega \to \mathbb{R}$ belongs to $\hat{L}^2(\Omega;\mathbb{R})$ if and only if its square is integrable on Ω, i.e., if the integral of $|f(x)|^2$ over Ω is finite. More formally, we define $\hat{L}^2(\Omega;\mathbb{R})$ as the set of functions $f : \Omega \to \mathbb{R}$ that satisfy

$$\int_\Omega |f(x)|^2 dx < \infty,$$

where dx denotes the integration measure on Ω.

The functions in $L^2(\Omega;\mathbb{R})$ are generally not required to be continuous, although they may have some continuity properties depending on the domain Ω.

The subspace $\hat{L}^2(\Omega;\mathbb{R})$ is a Hilbert space with respect to the inner product defined as

$$\langle f, g\rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx,$$

where $f, g \in \hat{L}^2(\Omega;\mathbb{R})$. This inner product induces a norm on $\hat{L}^2(\Omega;\mathbb{R})$ given by

$$\|f\|_{L^2(\Omega)} = \sqrt{\langle f, f\rangle_{L^2(\Omega)}}.$$

What's the difference between $L^2(\Omega;\mathbb{R})$ and $\hat{L}^2(\Omega;\mathbb{R})$?

$L^2(\Omega;\mathbb{R})$ and $\hat{L}^2(\Omega;\mathbb{R})$ are both function spaces that consist of square-integrable functions on a domain $\Omega \subseteq \mathbb{R}^n$. The main difference between them is in their function spaces.

$L^2(\Omega;\mathbb{R})$ is the space of all square-integrable functions on Ω, regardless of their regularity properties. In particular, functions in $L^2(\Omega;\mathbb{R})$ need not be continuous or have any other regularity properties. The norm on $L^2(\Omega;\mathbb{R})$ is given by

$$\|f\|_{L^2(\Omega)} = \sqrt{\int_\Omega |f(x)|^2 dx},$$

which measures the size of a function in terms of its L^2 norm, or equivalently, the square root of the integral of its square over Ω.

On the other hand, $\hat{L}^2(\Omega;\mathbb{R})$ is a subspace of $L^2(\Omega;\mathbb{R})$ consisting of square-integrable functions on Ω that are also continuous on Ω. That is, $\hat{L}^2(\Omega;\mathbb{R})$ consists of functions that are both square-integrable and continuous on Ω. The norm on $\hat{L}^2(\Omega;\mathbb{R})$ is also given by the L^2 norm, but with the restriction that we consider only continuous functions:

$$\|f\|_{\hat{L}^2(\Omega)} = \sqrt{\int_\Omega |f(x)|^2 dx}.$$

In summary, the main difference between $L^2(\Omega;\mathbb{R})$ and $\hat{L}^2(\Omega;\mathbb{R})$ is that the former contains all square-integrable functions on Ω, while the latter contains only those square-integrable functions that are also continuous on Ω.
Orthogonal bases. Let us first introduce the notion of *angle*. Recall the following property of the dot product on $\mathbb{R}^n$:

$$u \cdot v = \|u\|\|v\| \cos(\theta)$$

where *theta* $\in [0, \pi]$ is the angle formed by the two vectors u and v. This can be generalized to any inner product space in a natural way by defining:

$$\cos(\theta) = \frac{\langle u, v\rangle}{\|u\|\|v\|}$$

where u and v are two vectors in a given inner product space V. Note that it follows from the Cauchy-Schwarz inequality that the ratio $\frac{\langle u,v\rangle}{\|u\|\|v\|}$ is indeed between -1 and 1.
Definition 5.3 Let V be a vector space with an inner product $\langle \cdot, \cdot \rangle$. Two vector u and v in V are called orthogonal (or perpendicular) if:

$$\langle u, v\rangle = 0$$

A set of vectors $\{v_1, v_2, ...v_n$ in V are orthogonal if:

$$\langle v_i, v_j\rangle = 0 \quad for \quad any \quad i \neq j$$

Theorem 5.3 Let V be a finite-dimensional inner product space with dimension n; and $\mathbb{B} = \{e_1, e_2, ..., e_n\}$ be an orthonormal basis of V. Then for any $u \in V$, it holds that:

$$u = \langle u, e_1\rangle e_1 + \langle u, e_2\rangle e_2 + ... + \langle u, e_n\rangle e_n = \sum_{i=1}^{n} \langle u, e_i\rangle e_i$$

and

$$\|u\|^2 = |\langle u, e_1\rangle|^2 + |\langle u, e_2\rangle|^2 + ... + |\langle u, e_n\rangle|^2 = \sum_{i=1}^{n} |\langle u, e_n\rangle|^2$$

3.8 Fourier transform

Motivation: One can think of the Fourier transform as an analogue of the Fourier series for functions defined on $\mathbb{R}$ instead of $[-L, L]$.

Let's do a little calculation to see this is indeed the case. Remember for the heat equation on $[-L, L]$, if we have Dirichlet boundary condition we'll have the solution as Fourier sine series. If we have Neumann boundary condition, we'll have the solution as Fourier cosine series.

For heat equation on $[-L, L]$ with period boundary condition (symmetry boundary condition), the eigenvalue problem becomes:

$$\begin{cases} V'(x) = \beta V(x) \\ V(-L) = V(L) \\ V'(-L) = V'(L) \end{cases}$$

The eigenvalue and eigenfunctions are:

$$\beta_n = -\left(\frac{n\pi}{L}\right)^2, n = 0, 1, 2$$

$$V_0 \equiv 1$$

$$\begin{cases} V_n^c(x) = \cos\left(\frac{n\pi}{L}x\right) \\ V_n^s(x) = \sin\left(\frac{n\pi}{L}x\right) \end{cases}$$

These eigenfunctions form an orthogonal basis of

$$L^2([-L, L], \mathbb{R}) = \{f, [-L, L] \to \mathbb{R} | \int_{-L}^{L} f^2(x)dx < \infty\}$$

For each $f \in L^2([-L, L], \mathbb{R})$ it holds:

$$f(x) = \frac{\langle f, V_0\rangle}{\langle V_0, V_0\rangle} V_0 + \sum_{n=1}^{\infty} \frac{\langle f, V_n^c\rangle}{\langle V_n^c, V_n^c\rangle} V_n^c + \sum_{n=1}^{\infty} \frac{\langle f, V_n^s\rangle}{\langle V_n^s, V_n^s\rangle} V_n^s \tag{32}$$

this is based on the Theorem 5.3
Also we have the orthogonormal part:

$$\frac{\langle f, V_n^c\rangle}{\langle V_n^c, V_n^c\rangle} V_n^c = \langle f, \frac{V_n^c}{\sqrt{\langle V_n^c, V_n^c\rangle}}\rangle \frac{V_n^c}{\sqrt{\langle V_n^c, V_n^c\rangle}}$$

where the inner product is:

$$\langle f, V_n\rangle = \int_{-L}^{L} f(x)V_n(x)dx$$

Check that we have:

$$\langle V_n^i, V_n^i\rangle = \begin{cases} 2L, & n = 0 \\ L, & n \geq 1, \quad i = c, s \end{cases}$$

So that we can rewrite Eq(32) as

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(x)V_0(x)dx + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \cdot \cos\left(\frac{n\pi}{L}x\right) + \tag{33}$$

$$\frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx \cdot \sin\left(\frac{n\pi}{L}x\right)$$

The RHS is the Fourier series.

Fourier series, Fourier sine series, and Fourier cosine series are all methods for representing periodic functions in terms of trigonometric functions, but they differ in the types of functions they can represent.

A Fourier series represents a periodic function $f(x)$ with period 2π as a sum of trigonometric functions of the form $\cos(nx)$ and $\sin(nx)$, where n is an integer. That is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right],$$

where the coefficients a_0, a_n, and b_n are given by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx.$$

Note that the Fourier series can represent any periodic function with period 2π, whether it is even, odd, or neither.

A Fourier cosine series, on the other hand, represents a periodic function $f(x)$ with period 2π as a sum of cosine functions of the form $\cos(nx)$ only, that is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx),$$

where the coefficients a_0 and a_n are given by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx.$$

Note that in the Fourier cosine series, there are no sine terms. Therefore, a Fourier cosine series can only represent even periodic functions, i.e., functions that satisfy $f(-x) = f(x)$.

Finally, a Fourier sine series represents a periodic function $f(x)$ with period 2π as a sum of sine functions of the form $\sin(nx)$ only, that is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx),$$

where the coefficients b_n are given by

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx.$$

Note that in the Fourier sine series, there are no cosine terms. Therefore, a Fourier sine series can only represent odd periodic functions, i.e., functions that satisfy $f(-x) = -f(x)$.

In summary, the main difference between Fourier series, Fourier cosine series, and Fourier sine series is the type of functions they can represent: Fourier series can represent any periodic function, Fourier cosine series can only represent even periodic functions, and Fourier sine series can only represent odd periodic functions.

Using Euler's formula $e^{ix} = \cos(x) + i \sin(x)$, we can rewrite the Fourier series as:

$$f(x) = \frac{1}{2L} \sum_{n=-\infty}^{\infty} \int_{-L}^{L} f(s) e^{-i(\frac{n\pi(s-x)}{L})} ds \tag{34}$$

$$e^{-i(\frac{n\pi(s-x)}{L})} = e^{-i(\frac{n\pi s}{L})} \cdot e^{i(\frac{n\pi x}{L})} = [\cos(\frac{n\pi s}{L}) - i \sin(\frac{n\pi s}{L})] \cdot [\cos(\frac{n\pi x}{L}) + i \sin(\frac{n\pi x}{L})]$$

$$= \cos(\frac{n\pi s}{L}) \cos(\frac{n\pi x}{L}) + i \cos(\frac{n\pi s}{L}) \sin(\frac{n\pi x}{L}) - i \cos(\frac{n\pi x}{L}) \sin(\frac{n\pi s}{L}) + \sin(\frac{n\pi s}{L}) \sin(\frac{n\pi x}{L})$$

$$\int_{-L}^{L} f(s) e^{-i(\frac{n\pi(s-x)}{L})} ds = \int_{-L}^{L} f(s) [\cos(\frac{n\pi s}{L}) - i \sin(\frac{n\pi s}{L})] \cdot [\cos(\frac{n\pi x}{L}) + i \sin(\frac{n\pi x}{L})] ds$$

Using Euler's formula $e^{ix} = \cos(x) + i \sin(x)$, we can rewrite the cosine and sine terms as complex exponentials:

$$\cos\left(\frac{n\pi}{L} s\right) = \frac{1}{2}\left(e^{i\frac{n\pi}{L}s} + e^{-i\frac{n\pi}{L}s}\right), \quad \sin\left(\frac{n\pi}{L} s\right) = \frac{1}{2i}\left(e^{i\frac{n\pi}{L}s} - e^{-i\frac{n\pi}{L}s}\right)$$

Substituting these into the expression for $f(x)$, we have:

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(x)V_0 dx + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(s)\left(\frac{1}{2}\left(e^{i\frac{n\pi}{L}s} + e^{-i\frac{n\pi}{L}s}\right)\right) ds \cdot \left(\frac{1}{2}\left(e^{i\frac{n\pi}{L}x} + e^{-i\frac{n\pi}{L}x}\right)\right)$$
$$+ \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(s)\left(\frac{1}{2i}\left(e^{i\frac{n\pi}{L}s} - e^{-i\frac{n\pi}{L}s}\right)\right) ds \cdot \left(\frac{1}{2i}\left(e^{i\frac{n\pi}{L}x} - e^{-i\frac{n\pi}{L}x}\right)\right)$$

we can find that

$$\frac{1}{4L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(s)\left(\left(e^{i\frac{n\pi}{L}s} + e^{-i\frac{n\pi}{L}s}\right)\right) ds \cdot \left(\left(e^{i\frac{n\pi}{L}x} + e^{-i\frac{n\pi}{L}x}\right)\right) +$$

$$-\frac{1}{4L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(s)\left(\left(e^{i\frac{n\pi}{L}s} - e^{-i\frac{n\pi}{L}s}\right)\right) ds \cdot \left(\left(e^{i\frac{n\pi}{L}x} - e^{-i\frac{n\pi}{L}x}\right)\right)$$

$$= \frac{1}{2L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(s)\left(\left(e^{i\frac{n\pi}{L}s}\right)\right) ds \cdot \left(\left(e^{-i\frac{n\pi}{L}x}\right)\right) + \frac{1}{2L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(s)\left(\left(e^{-i\frac{n\pi}{L}s}\right)\right) ds \cdot \left(\left(e^{i\frac{n\pi}{L}x}\right)\right)$$

$$= \frac{1}{2L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(s)\left(\left(e^{i\frac{n\pi}{L}s}\right)\right) ds \cdot \left(\left(e^{-i\frac{n\pi}{L}x}\right)\right) + \frac{1}{2L}\sum_{-\infty}^{n=-1}\int_{-L}^{L} f(s)\left(\left(e^{i\frac{n\pi}{L}s}\right)\right) ds \cdot \left(\left(e^{-i\frac{n\pi}{L}x}\right)\right)$$

note that when $n = 0$

$$\frac{1}{2L}\int_{-L}^{L} f(x)V_0 dx = \left(\frac{1}{2L}\int_{-L}^{L} f(s)e^{-i\frac{n\pi}{L}(s-x)} ds\right) e^{i\frac{n\pi}{L}x} = \frac{1}{2L}\int_{-L}^{L} f(s)V_0 ds$$

Simplifying this expression, we get:

$$f(x) = \frac{1}{2L}\sum_{-\infty}^{\infty}\int_{-L}^{L} f(s)e^{-i(\frac{n\pi(s-x)}{L})} ds$$

What we get if $L \to \infty$?
To see what happens, we define:

$$\omega_n = \frac{n\pi}{L}, \quad n \in \mathbb{Z}$$

$$\omega_{n+1} - \omega_n = \frac{\pi}{L}$$

as $L \to \infty$ the difference $\frac{\pi}{L} \to 0$ and ω_n becomes continuous.

Thus, we can write the $\sum$ as $\int$:

$$f(x) = \frac{1}{2L}\sum_{-\infty}^{\infty}\int_{-L}^{L} f(s)e^{-i(\frac{n\pi(s-x)}{L})} ds \tag{35}$$

$$= \frac{1}{2\pi}\left(\frac{\pi}{L}\right)\sum_{-\infty}^{\infty}\int_{-L}^{L} f(s)e^{-i(\frac{n\pi(s-x)}{L})}ds$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s)e^{-i\omega(s-x)}dsd\omega$$

To summarize, we have:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}d\omega \qquad (36)$$

with

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s)e^{-i\omega s}ds \qquad (37)$$

which are the **Fourier transform** and **inverse Fourier transform**.

The Fourier transform of an **absolutely integrable function** (L^1 space) f: $\mathbb{R} \to \mathbb{R}$ is defined by:

$$F(\omega) = c(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s)e^{-i\omega s}ds$$

The inverse Fourier transfer if $\hat{f}$ is defined by:

$$\check{\hat{f}}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega$$

Note that $\hat{f}$ is called the frequence representation of f. The Fourier integral expresses f as a supperposition of the builf-block functions $\{e^{i\omega x}|\omega \in \mathbb{R}\}$. So $\{e^{i\omega x}|\omega \in \mathbb{R}\}$ plays the sam role as the eigenfunctions.

There are some properties of the Fourier transform:

(i)

$$(\alpha f \hat{+} \beta g)(\omega) = \alpha\hat{f}(\omega) + \beta\hat{g}(\omega)$$

Fourier transform is linear

(ii)

$$\mathbb{F}\left\{f(x-a)\right\} = e^{-iax}\hat{f}(\omega) = e^{-iax}\mathbb{F}\left\{f(x)\right\}$$

(iii)

$$\mathbb{F}\left\{e^{iax}f(x)\right\} = \hat{f}(\omega - a)$$

(iv)

$$\mathbb{F}\left\{f^{(n)}(x)\right\} = (i\omega)^n\hat{f}(\omega)$$

here the n means the number of derivative

(v)

$$\mathbb{F}\left\{(-ix)^n f(x)\right\} = \frac{d^n \hat{f}(x)}{d\omega^n}$$

This one is very important:

(vi)

$$\mathbb{F}^{-1}\left\{\hat{f}(\omega)\hat{g}(\omega)\right\} = \frac{1}{\sqrt{2\pi}}(f * g)(x)$$

(vii) Assume $f \in L^2(\mathbb{R};\mathbb{R})$ we have:

$$f = \check{\hat{f}}$$

Now we'll focus our anasys on property (vi). We have:

$$\mathbb{F}^{-1}\left\{\hat{f}(\omega)\hat{f}(\omega)\right\} = \frac{1}{\sqrt{2\pi}}(f*g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s)g(x-s)ds = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-s)g(s)ds$$

A side note about convolution: convolution can be regarded as a special weighted averaging process.

Now let's show the proof of the convolution. We know that:

$$\mathbb{F}\left\{f(x)\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

$$\begin{aligned}
\mathbb{F}\left\{(f * g)(x)\right\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (f * g)(x)e^{-i\omega x}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(s)g(x-s)ds\right)e^{-i\omega x}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} g(x-s)e^{-i\omega x}dx\right)f(s)ds \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} g(x-s)e^{-i\omega(x-s)}dx\right)f(s)e^{-i\omega s}ds \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-s)e^{-i\omega(x-s)}dx\sqrt{2\pi}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s)e^{-i\omega s}ds \\
&= \sqrt{2\pi}\hat{g}(\omega)\hat{f}(\omega)
\end{aligned}$$

Example: find the Fourier transform of the following equation with $a > 0$:

$$f(x) = e^{-ax^2}$$

solution

$$\mathbb{F}\left\{f(x)\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2}e^{-i\omega x}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a(x^2+\frac{i\omega}{a}x)}dx = \frac{1}{\sqrt{2\pi}}e^{-\frac{\omega^2}{4a}}\int_{-\infty}^{\infty} e^{-a(x+\frac{i\omega}{2a})^2}dx$$

using change of variable and the polar coordinates, we have:

$$\int_{-\infty}^{\infty} e^{-ay^2}dy = \sqrt{\frac{\pi}{a}}$$

Finally we find:

$$\mathbb{F}\left\{e^{-ax^2}\right\} = \frac{1}{\sqrt{2a}}e^{-\frac{\omega^2}{4a}}$$

Then we show how to use Fourier transform to solve PDEs. For the following initial value problems:

$$\begin{cases} u_t = u_{xx} \\ \\ u(x,0) = \phi(x) \end{cases} \tag{38}$$

Let's donate $\hat{u}(\omega,t)$ as the Fourier transform of $u(x,t)$ w.r.t x.

$$\hat{u}(\omega,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}dx$$

then we plug it into $u_t = u_{xx}$

$$\frac{\partial}{\partial t}\hat{u}(\omega,t) = \mathbb{F}\left\{u^{(2)}(x,t)\right\} = (i\omega)^2\hat{u}(\omega,t) = -\omega^2\hat{u}(\omega,t)$$

Now we get a new ODE

$$\begin{cases} \frac{\partial}{\partial t}\hat{u}(\omega,t) = -\omega^2\hat{u}(\omega,t) \\ \\ \hat{u}(\omega,t) = \hat{\phi}(\omega) \end{cases} \tag{39}$$

we can solve it and find that:

$$\hat{u}(\omega,t) = \hat{\phi}(\omega)e^{-\omega^2 t}$$

Now we need to find the inverse of the Fourier transform and get the solution.

$$\mathbb{F}^{-1}\left\{\hat{\phi}(\omega)e^{-\omega^2 t}\right\}$$

recall that in the example above, we have:

$$\mathbb{F}\left\{e^{-ax^2}\right\} = \frac{1}{\sqrt{2a}}e^{-\frac{\omega^2}{4a}}$$

with some scaling, we have:

$$\mathbb{F}\left\{\frac{1}{\sqrt{2t}}e^{-\frac{x^2}{4t}}\right\} = \mathbb{F}\left\{g(x,t)\right\} = e^{-\omega^2 t}$$

we have:

$$\begin{aligned}
u(x,t) &= \mathbb{F}^{-1}\left\{\hat{\phi}(\omega)\hat{g}(x,t)\right\}(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(s)g(x-s,t)ds \\
&= \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4t}}\phi(s)ds
\end{aligned}$$

We can define that:

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$$

$x \in \mathbb{R}, t > 0$ is called the fundamental solution of the 1D heat equation.

Then, we can use the convolution of the fundamental solution to express the solution:

$$u(x,t) = \int_{-\infty}^{\infty}\Phi(x-s)\phi(s)ds = \Phi(x,t) * \phi(x)$$

Next, we consider the Fourier sine and cosine transforms. They are analogous to the Fourier transform.
Consider an odd function $f : \mathbb{R} \to \mathbb{R}$ $(f(x) = -f(-x))$:

$$\begin{aligned}
\hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos(\omega x)dx + \frac{1}{2\pi}\int_{-\infty}^{\infty} if(x)\sin(-\omega x)dx \\
&= -\frac{2i}{\sqrt{2\pi}}\int_{0}^{\infty} f(x)\sin(\omega x)dx
\end{aligned}$$

$\omega \in \mathbb{R}$

since $f(x)\cos(\omega x)$ is $odd \cdot even = odd$ function and the integration from $-\infty$ to ∞ is 0.

Note that $\hat{f}(\omega)$ itself is also an odd function with respect to ω.

$$\hat{f}(\omega) = -\hat{f}(-\omega)$$

The inverse Fourier transform:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega = \frac{2i}{\sqrt{2\pi}}\int_{0}^{\infty}\hat{f}(\omega)\sin(\omega x)d\omega$$

Here are some properties of the Fourier sine transform. Assume $f :\to [0, \infty)\mathbb{R}$ is absolutely integrable, we define:

$$\mathbb{F}_S \left\{ f(x) \right\} = F_S(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(\omega x) dx$$

is called the Fourier sine transform of $f(x)$.

$$\mathbb{F}_S^{-1} \left\{ F_S(\omega) \right\} = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_S(\omega) \sin(\omega x) d\omega$$

Both the Fourier sine transform and inverse Fourier sine transform involve the multiplication by $\sin(\omega x)$.So, in both cases, the sine function $\sin(\omega x)$ serves as a weighting function, which weights the contribution of each frequency component to the overall transformation. Without this weighting function, the Fourier sine transform and inverse Fourier sine transform would not be well-defined.

Analogously, if the function f is an even function, we would get:

$$\hat{f}(\omega) = \frac{2i}{\sqrt{2\pi}} \int_0^\infty f(x) \cos(\omega x) dx$$

Assume $f :\to [0^+, \infty)\mathbb{R}$ is absolutely integrable, we define:

$$\mathbb{F}_C \left\{ f(x) \right\} = F_C(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(\omega x) dx$$

is called the Fourier cosine transform of $f(x)$.

$$\mathbb{F}_C^{-1} \left\{ F_C(\omega) \right\} = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_C(\omega) \cos(\omega x) d\omega$$

Here are the properties: **i**

$$\mathbb{F}_S \left\{ f'(x) \right\} = -\omega \mathbb{F}_C \left\{ f(x) \right\}$$

ii

$$\mathbb{F}_C \left\{ f'(x) \right\} = -\sqrt{\frac{2}{\pi}} f(0^+) + \omega \mathbb{F}_S \left\{ f(x) \right\}$$