Assignment 1

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February 13, 2023

Question 1: Find the Fourier sine expansion of the following function

$$f(x) = x^2, x \in [0, 1] \tag{1}$$

Solution: We assume

$$f(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) = x^2$$

The coefficient of $\sin(n\pi x)$ in Fourier sine series is:

$$\begin{aligned} B_n &= 2\int_0^1 x^2 \sin(n\pi x) dx \\ &= \frac{2(-1)^{n-1}}{n\pi} + \frac{4}{(n\pi)^3}((-1)^n - 1) \end{aligned}$$

Therefore, the Fourier sine expansion of the given function is

$$x^2 = \sum_{n=1}^{\infty} \left[\frac{2(-1)^{n-1}}{n\pi} + \frac{4}{(n\pi)^3}((-1)^n - 1) \right] \sin(n\pi x), x \in [0, 1]$$

Question 2: Consider the following IBVP (initial boundary value problem) for the 1D heat equation posed on the initerval [0, L] for some L > 0:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} & x \in (0, L), t > 0 \\ u_x(0, t) = 0, u_x(L, t) = 0 & t > 0 \\ u(x, 0) = x & 0 \leq x \leq L \end{cases} \tag{2}$$

Solution: We assume the solution is of form:

$$u(x, t) = V(x)T(t)$$

Then the equation is

$$T'(t)/T(t) = V''(x)/V(x) = \beta$$

where β is a constant. We solve V by eignefunctions.
(a) If $\beta > 0$,

$$V(x) = c_1 e^{-\sqrt{\beta}x} + c_2 e^{\sqrt{\beta}x}$$

$$V'(x) = -\sqrt{\beta} c_1 e^{-\sqrt{\beta}x} + \sqrt{\beta} c_2 e^{\sqrt{\beta}x}$$

By $V'(0) = V'(L) = 0$, we have $c_1 = c_2 = 0$. There is no non-trivial solutions.
(b)If $\beta = 0$

$$V(x) = c_1 + c_2 x$$

$$V'(x) = c_2$$

By $V'(0) = V'(L) = 0$, we have $c_2 = 0$. The eigenfunction is $V_0(x) = 1$.
(c) If $\beta < 0$

$$V(x) = c_1 \cos(\sqrt{-\beta}x) + c_2 \sin(\sqrt{-\beta}x)$$

$$V'(x) = -c_1\sqrt{-\beta}\sin(\sqrt{-\beta}x) + c_2\sqrt{-\beta}\cos(\sqrt{-\beta}x)$$

By $V'(0) = V'(L) = 0$, we have $c_2 = 0$, $\beta_n = -\left(\frac{n\pi}{L}\right)^2$, $n \in N$. The eigenfunction $V_n = \cos(\frac{n\pi}{L}x)$, $n \in N$. The corresponding $T_n(t) = e^{\beta_n t} = e^{-\left(\frac{n\pi}{L}\right)^2 t}$.

The solution can be written as:

$$u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n V_n(x) T_n(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{L}\right)^2 t} \cos(\frac{n\pi}{L}x)$$

The boundary value

$$x = u(x,0) = c_0 + \sum_{n=1}^{\infty} c_n \cos(\frac{n\pi}{L}x)$$

is exactly the Fourier cosine expansion of x on $[0,L]$, thus

$$c_0 = \frac{1}{L}\int_0^L x dx = \frac{L}{2}$$

$$c_n = \frac{2}{L}\int_0^L x\cos(\frac{n\pi}{L}x)dx, n \in N$$

Therefore, for $t > 0$, $x \in [0,L]$,

$$u(x,t) = \frac{L}{2} + \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L x\cos(\frac{n\pi}{L}x)dx\right) e^{-\left(\frac{n\pi}{L}\right)^2 t}\cos(\frac{n\pi}{L}x)$$