Notes for Applied PDEs

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1 Lecture1

Let's consider the 2nd order PDEs. The most general form of a 2nd order PDE with two variables is:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \tag{1}$$

where $A, B, ...G$ are constants or given functions of x and y.

1.1 Types of PDEs

All linear PDEs in the form of Eq(1) can be classified into three types: ***hyperbolic, parabolic*** and ***elliptic***.

The hyperbolic type: e.g. the wave equation

$$\Delta(x, y) = B^2(x, y) - 4A(x, y)C(x, y) > 0 \tag{2}$$

The parabolic type: e.g. the heat equation

$$\Delta(x, y) = B^2(x, y) - 4A(x, y)C(x, y) = 0 \tag{3}$$

The elliptic type: e.g. the Laplace equation

$$\Delta(x, y) = B^2(x, y) - 4A(x, y)C(x, y) < 0 \tag{4}$$

1.2 Superposition principle

If u_1, u_2, ... u_n, are the solutions to the linear homogeneous PDE $Lu = 0$, and u_1, u_2, ... $u_n \in R$. Here L is a linear differential operator. Then the linear combination of $\sum_1^n c_i u_i$ is also the solution of the PDE.

Let S_h be the set of all solutions to the homogeneous problem

$$Lu = 0$$

Then we consider the inhomogeneous problem

$$Lu = f$$

The set of all solutions to this inhomogeneous problem is given by

$$S_i = \{u_i + u_h | u_h \in S_h\}$$

Here u_i is a particular solution to the inhomogeneous problem and S_i is the translation of S_h by u_i.

2 Lecture2

2.1 ODE

Let's consider the 2nd ODE first.

$$ax''(t) + bx'(t) + cx(t) = 0 \tag{5}$$

where a, b and $c \in R$ and $a \neq 0$.

We consider the characteristic equation first:

$$a\lambda^2 + b\lambda + c = 0 \tag{6}$$

with two solution λ_1 and λ_2.

Case I, when $\lambda_1 \neq \lambda_2$, we have the independent solutions:

$$\begin{cases} x_1(t) = e^{\lambda_1 t} \\ x_2(t) = e^{\lambda_2 t} \end{cases}$$

.

Case II, when have the same roots $\lambda_1 = \lambda_2$, we have the solution:

$$\begin{cases} x_1(t) = e^{\lambda_1 t} \\ x_2(t) = \boldsymbol{t} e^{\lambda_2 t} \end{cases}$$

.

Case III, we have complex conjugate pairs of roots $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$. The two independent solutions are:

$$\begin{cases} x_1(t) = e^{\alpha t} cos(\beta t) \\ x_2(t) = e^{\alpha t} sin(\beta t) \end{cases}$$

.

With initial conditions given, we'll search solution with a linear combination of the independent solutions

$$x(t) = C_1 x_1(t) + C_2 x_2(t) \tag{7}$$

where C_1 and $C_2 \in R$.

This is called the general solution of the homogeneous problem Eq(5).

Then for the inhomogeneous version,

$$ax''(t) + bx'(t) + cx(t) = f(t) \tag{8}$$

we need to use the **variable of parameter formula** to find a particular solution.

3 Heat equation

Let's consider the 1D heat equation on the interval $(0, L)$ and subject to some initial conditions(IBVP).

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} & x \in (0, L), t > 0 \\ u(0, t) = 0, u(L, t) = 0 & t > 0 \\ u(x, 0) = f(x) & 0 \leq x \leq L \end{cases} \tag{9}$$

3.1 Separation of variable

We are looking for non-trivial solutions. We assume:

$$u(x, t) = V(x)T(t) \qquad x \in (0, L), t > 0$$

Plugging it into the Eq(9), we can obtain

$$\frac{T'}{T} = \frac{V''}{V} = \beta \qquad \forall x \in (0, L), t > 0$$

Thus, we get

$$\begin{cases} T'(t) = \beta T(t) \\ V''(t) = \beta V(t) \end{cases}$$

we successfully transfer the PDE to ODEs.

Considering the boundary conditions $u(0, t) = 0, u(L, t) = 0, \quad t > 0$ we have $V(0) = 0$ and $V(L) = 0$.

$$\begin{cases} V''(t) = \beta V(t) & , x \in (0, L) \\ V(0) = 0 \\ V(L) = 0 \end{cases}$$

In this case, the characteristic equation is:

$$\lambda^2 - \beta = 0$$

we have three cases for β. And we can check that only when $\beta < 0$, the solution is non-trivial.

When $\beta > 0$ we have two distinguished real roots

$$\begin{cases} V_1(x) = e^{-\sqrt{\beta}x} \\ V_2(x) = e^{\sqrt{\beta}x} \end{cases}$$

Then $V(x) = C_1V_1(x) + C_2V_2(x)$ with the boundary conditions $V(0) = 0$ and $V(L) = 0$. As a result $C_1 = C_2 = 0$.

When $\beta = 0$, we have $\lambda_1 = \lambda_2 = 0$. Thus we can find that $V_1(x) = 1$ and $V_1(x) = x$. Considering the boundary conditions, the coefficients are also should be 0, *i.e.* $C_1 = C_2 = 0$. When $\beta < 0$, the solutions of the characteristic function is

$$\lambda = \pm\sqrt{-\beta}i$$

As a result, we find that $V_1(x) = cos(\sqrt{-\beta}x)$ and $V_2(x) = sin(\sqrt{-\beta}x)$. With the boundary condition given $V(0) = 0$ and $V(L) = 0$, we have that:

$$C_1 + 0 = 0$$

$$C_1cos(\sqrt{-\beta}L) + C_2sin(\sqrt{-\beta}L) = 0$$

For non-trivial solutions, we need to have $sin(\sqrt{-\beta}L) = 0$ and

$$\beta_n = -\left(\frac{n\pi}{L}\right)^2, \quad n \in N$$

Thus we find the eigenvalues and the eigenfunctions:

$$\begin{cases} \beta_n = -\left(\frac{n\pi}{L}\right)^2 & , n \in N \\ V_n(x) = sin(\frac{n\pi}{L}x) \end{cases} \tag{10}$$

Now for each β we have the solution for $T(t)$:

$$T'(t) = \beta_n T(t), \quad n \in N$$

$$T(t) = e^{-\left(\frac{n\pi}{L}\right)^2 t} \quad n \in N$$

We find the solution:

$$u_n(x,t) = V_n(x)T_n(t) = e^{-\left(\frac{n\pi}{L}\right)^2 t} sin(\frac{n\pi}{L}x) \quad n \in N \tag{11}$$

From the above calculation, we know that each u_n satisfies the following homogeneous BVP:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} & x \in (0, L), t > 0 \\ u(0, t) = 0, u(L, t) = 0 & t > 0 \end{cases} \tag{12}$$

We look for the solution of the form:

$$u(x, t) = \sum_{n=1}^{\infty} C_n u_n(x, t) \tag{13}$$

We need to utilize the initial conditions to find the coefficient. The initial condition is:

$$u(x, 0) = f(x) \quad x \in [0, L]$$

With $t = 0$, we have $T(0) = 1$ and also:

$$\sum_{n=1}^{\infty} C_n sin(\frac{n\pi}{L} x) = f(x) \tag{14}$$

This is exactly the problem of finding the **Fourier sine expansion** of the given function f.

To find the coefficient C_n, we use the fact that V_n are orthogonal to each other in the sense that:

$$\int_0^L V_n(x) V_m(x) dx = \begin{cases} 0 & if \quad m \neq n \\ \frac{L}{2} & if \quad m = n \end{cases} \tag{15}$$

Now multiply both side of Eq(14) by V_m and integrate from 0 to L.

$$\int_0^L \sum_{n=1}^{\infty} C_n V_n(x) V_m(x) dx = \int_0^L f(x) V_m(x) dx$$

Assume we can switch $\int_0^L$ with $\sum_{n=1}^{\infty}$, we get

$$C_n = \frac{2}{L} \int_0^L f(x) sin(\frac{n\pi}{L} x) dx \quad n \in N$$

With known the coefficient, we finally obtain the complete solution of the IBVP Eq(9):

$$u(x, t) = \sum_{n=1}^{\infty} \frac{2}{L} \int_0^L f(x') sin(\frac{n\pi}{L} x') dx' e^{-\left(\frac{n\pi}{L}\right)^2 t} sin(\frac{n\pi}{L} x) \quad n \in N, t > 0, x \in [0, L] \tag{16}$$

3.2 Source term

Now we are trying to solve the heat equation with the source term:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(x,t) & x \in (0,L), t > 0 \\ u(0,t) = 0, u(L,t) = 0 & t > 0 \\ u(x,0) = f(x) & 0 \leq x \leq L \end{cases} \tag{17}$$

First we recall the homogeneous BVP

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} & x \in (0,L), t > 0 \\ u(0,t) = 0, u(L,t) = 0 & t > 0 \end{cases}$$

We know the eigenvalues and eigenfunctions:

$$\begin{cases} \beta_n = -\left(\frac{n\pi}{L}\right)^2 & , n \in N \\ V_n(x) = sin(\frac{n\pi}{L}x) \end{cases}$$

We are looking for the solution of the form:

$$u(x,t) = \sum_{n=1}^{\infty} \tilde{T}_n(t) V_n(x) \tag{18}$$

Plugging Eq(31) into the source term Eq(17), we get

$$\sum_{n=1}^{\infty} \frac{d}{dt}(\tilde{T}_n(t)) V_n(x) = \sum_{n=1}^{\infty} \tilde{T}_n(t) \frac{d^2}{dx^2}(V_n(x)) + g(x,t)$$

with knowing that

$$V''(x) = \beta V(x)$$

Now using the orthogonal property, we multiply both sides by V_m and integrate from 0 to L:

$$\sum_{n=1}^{\infty} \frac{d}{dt}(\tilde{T}_n(t)) \int_0^L V_n(x)V_m(x)dx = \sum_{n=1}^{\infty} \tilde{T}_n(t)\beta_n \int_0^L V_n(x)V_m(x)dx + \int_0^L g(x,t)V_m(x)dx$$

Using Eq(15)

$$\int_0^L V_n(x)V_m(x)dx = \begin{cases} 0 & if \quad m \neq n \\ \frac{L}{2} & if \quad m = n \end{cases}$$

We have

$$\frac{d}{dt}(\tilde{T}_m(t)) = \beta_m \tilde{T}_m(t) + \frac{2}{L}\int_0^L g(x,t)V_m(x)dx$$

we get an ODE

$$\frac{d}{dt}(\tilde{T}_n(t)) = \beta_n \tilde{T}_n(t) + h_n(t)$$

Then we consider the initial condition $u(x,0) = f(x)$:

$$u(x,0) = \sum_{n=1}^{\infty} \tilde{T}_n(0) V_n(x) = f(x)$$

Like before, we multiply both sides by V_m and integrate from 0 to L:

$$\sum_{n=1}^{\infty} \tilde{T}_n(0) \int_0^L V_n(x) V_m(x) dx = \int_0^L f(x) V_m(x) dx$$

$$\tilde{T}_n(0) = \frac{2}{L} \int_0^L f(x) V_n(x) dx = \omega_n$$

Now we have the IVP for $\tilde{T}_n(t)$:

$$\begin{cases} \frac{d}{dt}\tilde{T}_n(t) = \beta_n \tilde{T}_n(t) + h_n(t) \\ \tilde{T}_n(0) = \omega_n \end{cases} \tag{19}$$

where

$$h_n(t) = \frac{2}{L} \int_0^L g(x,t) V_n(x) dx$$

$$\omega_n = \frac{2}{L} \int_0^L f(x) V_n(x) dx$$

Now we need to solve the IVP problem with the variation of parameter formula:

$$\tilde{T}_n(t) = \omega_n e^{\beta_n t} + \int_0^t e^{\beta_n (t-s)} h_n(s) ds$$

Finally we find the complete solution of the heat equation with source term:

$$\begin{aligned} u(x,t) = &\sum_{n=1}^{\infty} \frac{2}{L} \int_0^L f(x') sin(\frac{n\pi}{L}x') dx' e^{-\left(\frac{n\pi}{L}\right)^2 t} sin(\frac{n\pi}{L}x) \\ &+ \sum_{n=1}^{\infty} \left[\int_0^t e^{-\left(\frac{n\pi}{L}\right)^2 (t-s)} \frac{2}{L} \int_0^L g(x,s) sin(\frac{n\pi}{L}x) ds \right] \end{aligned} \tag{20}$$

3.3 Non-homogeneous b.c.

Now we consider the fully nonhomogeneous problem, which means non-homogeneous source term in the equation (source term) and also in the boundary conditions.

Let's consider the following equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(x,t) & x \in (0,L), t > 0 \\ u(0,t) = u_1(t), u(L,t) = u_2(t) & t > 0 \\ u(x,0) = f(x) & 0 \leq x \leq L \end{cases} \tag{21}$$

here the boundary condition is called the non-homogeneous Dirichlet b.c.

Let's consider a new function $\theta(x,t)$:

$$\theta(x,t) = u(x,t) - w(x,t)$$

Let $w(0,t) = u_1(t)$ and $w(L,t) = u_2(t)$, we can have the new θ function to fit the homogeneous boundary conditions.

It is always a good choice to choose the linear relation. Thus, we can have:

$$w(x,t) = \frac{L-x}{L}u_1(t) + \frac{x}{L}u_2(t)$$

We can plug θ inside the function to get:

$$\frac{\partial \theta}{\partial t} + \frac{L-x}{L}u_1'(t) + \frac{x}{L}u_2'(t) = \frac{\partial^2 \theta}{\partial x^2} + g(x,t)$$

Thus, the problem is transferred back to the source term problem:

$$\begin{cases} \frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} + \tilde{g}(x,t) & x \in (0,L), t > 0 \\ \theta(0,t) = 0, \theta(L,t) = 0 & t > 0 \\ \theta(x,0) = \tilde{f}(x) & 0 \leq x \leq L \end{cases} \tag{22}$$

$$\tilde{g}(x,t) = g(x,t) - \frac{L-x}{L}u_1'(t) - \frac{x}{L}u_2'(t)$$

$$\tilde{f}(x,t) = f(x) - \frac{L-x}{L}u_1(0) - \frac{x}{L}u_2(0)$$

3.4 Wave equation

Now we consider the homogeneous wave equation:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} & x \in (0, L), t > 0 \\ u(0, t) = 0, u(L, t) = 0 & t > 0 \\ u(x, 0) = \phi_1(x), \frac{\partial u}{\partial t} = \phi_2(x) & 0 \leq x \leq L \end{cases} \tag{23}$$

First, we consider the homogeneous BVP:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} & x \in (0, L), t > 0 \\ u(0, t) = 0, u(L, t) = 0 & t > 0 \end{cases}$$

The homogeneous BVP wave equation has the following general solution set:

$$u_n(x, t) = V_n(x) T_n(x) = \sin(\frac{n\pi}{L} x) \left[a_n \cos(\frac{cn\pi}{L} t) + b_n \sin(\frac{cn\pi}{L} t) \right], n \in N$$

Now, back to the IBVP Eq(23), in order to satisfy the initial condition, we are looking for the solution in the form of:

$$u(x, t) = \sum_{n=1}^{\infty} \sin(\frac{n\pi}{L} x) \left[a_n \cos(\frac{cn\pi}{L} t) + b_n \sin(\frac{cn\pi}{L} t) \right] \tag{24}$$

We know that when $t = 0$, we have $\sin(\frac{cn\pi}{L} t) = 0$ and $\cos(\frac{cn\pi}{L} t) = 1$. Thus we have:

$$u(x, 0) = \sum_{n=1}^{\infty} \sin(\frac{n\pi}{L} x) \cdot a_n = \phi_1(x) \tag{25}$$

$$u'(x, 0) = \sum_{n=1}^{\infty} \sin(\frac{n\pi}{L} x) \left[\frac{cn\pi}{L} b_n \right] = \phi_2(x) \tag{26}$$

Recall the orthogonality we have:

$$\int_0^L \sin(\frac{n\pi}{L} x) \sin(\frac{m\pi}{L} x) = \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n \end{cases}$$

Thus, we can find that:

$$a_n = \frac{2}{L} \int_0^L \phi_1(x) \sin(\frac{n\pi}{L} x) dx$$

$$b_n = \frac{2}{cn\pi}\int_0^L \phi_2(x)\sin(\frac{n\pi}{L}x)dx$$

To summarize, the solution to the IBVP is$u(x,t)$:

$$\sum_{n=1}^{\infty}\sin(\frac{n\pi}{L}x)\left[\frac{2}{L}\int_0^L \phi_1(x')\sin(\frac{n\pi}{L}x')dx'\cos(\frac{cn\pi}{L}t) + \frac{2}{cn\pi}\int_0^L \phi_2(x')\sin(\frac{n\pi}{L}x')dx'\sin(\frac{cn\pi}{L}t)\right]$$

In the case with non-homogeneous B.C., we'll do the change of variable to make the boundary condition homogeneous as in the heat equation case. In the case with non-homogeneous source term, we need to find a particular solution for the inhomogeneous BVP using variation of parameter formula.

Boundary conditions determine the shape of the eigenfunctions.

Dirichlet boundary conditions (with rectangular domain in 1D) we have the **Fourier sine series**:

$$\left\{\sin(\frac{n\pi}{L}x)|n \in N\right\}$$

Neuman boundary conditions, we have **Fourier cosine series**:

$$\left\{\cos(\frac{n\pi}{L}x)|n \in N\right\}$$

For periodic boundary condtions, we will get both the sine and cosine eigenfunctions.

For non-rectangular domain, the eigenfunctions may not be trigonometric functions. **Can One Hear the Shape of a Drum?**

3.5 Problems in higher(spatial) dimentions

Now we consider a 2D(spatial) heat equation:

$$\begin{cases} u_t = u_{xx} + u_{yy} & 0 < x < 1, 0 < y < 1, t > 0 \\ \\ u(0,y,t) = 0, u_x(1,y,t) = 0 \\ u(x,0,t) = 0, u(x,1,t) = 0 & t > 0 \\ \\ u(x,y,0) = f(x,y) \end{cases} \tag{27}$$

we have one Neumann b.c. here.

we also do the separation of variable:

$$u(x,y,t) = X(x)Y(y)T(t)$$

plugging it into the 2D heat equation, we get:

$$X(x)Y(y)T'(t) = X''(x)Y(y)T(t) + X(x)Y''(y)T(t)$$

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \beta$$

Thus, we have:

$$\frac{T'(t)}{T(t)} = \beta \tag{28}$$

$$\frac{X''(x)}{X(x)} = \beta - \frac{Y''(y)}{Y(y)} = \mu$$

$$\begin{cases} X''(x) = \mu X(x) \\ Y''(y) = (\beta - \mu)Y(y) \end{cases} \tag{29}$$

Applying the boundary conditions, we can find:

$$u(0, y, t) = 0 \Rightarrow X(0) = 0$$

$$u_x(1, y, t) = 0 \Rightarrow X'(1) = 0$$

$$u(x, 0, t) = 0 \Rightarrow Y(0) = 0$$

$$u(x, 1, t) = 0 \Rightarrow Y(1) = 0$$

Thus we have the ODEs:

$$\begin{cases} X''(x) = \mu X(x) & 0 < x < 1 \\ X(0) = 0, X'(1) = 0 \end{cases} \tag{30}$$

For this ODE, we have the different eignevalues and eigenfunctions from the past heat equation:

$$\mu_n = -(n\pi - \frac{\pi}{2})^2 \tag{31}$$

$$X_n(x) = \sin[(n\pi - \frac{\pi}{2})x], n \in N \tag{32}$$

Then for each $\mu = \mu_n$, we have:

$$\begin{cases} Y''(y) = (\beta - \mu_n)Y(y) & 0 < y < 1 \\ Y(0) = 0, Y(1) = 0 \end{cases} \tag{33}$$

The eigenvalues and the eigenfucntions are:

$$\beta - \mu_n = -(m\pi)^2$$

$$Y_m(y) = \sin(m\pi y), m \in N$$

Note that β depends on two indices m & n(which are independent with each other)

$$\beta_{m,n} = -(n\pi - \frac{\pi}{2}) - (m\pi)^2$$

Now back to Eq(28) we get the

$$\frac{T'(t)}{T(t)} = \beta$$

$$T'_{m,n}(t) = -[(n\pi - \frac{\pi}{2}) + (m\pi)^2]T_{m,n}(t)$$

$$T_{m,n}(t) = C_{m,n}e^{-[(n\pi-\frac{\pi}{2})+(m\pi)^2]t} \tag{34}$$

For the BVP we have the solution:

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{m,n}e^{-[(n\pi-\frac{\pi}{2})+(m\pi)^2]t}\sin[(n\pi - \frac{\pi}{2})x]\sin(m\pi y) \tag{35}$$

Using the orthogonal property to determine $C_{m,n}$

$$\int_0^1\int_0^1 f(x,y)X_p(x)Y_q(y)dxdy = \sum_{m,n=1}^{\infty}\int_0^1\int_0^1 C_{m,n}X_n(x)X_p(x)Y_m(y)Y_q(y)dxdy$$

Recall that

$$\int_0^1 X_n(x)X_p(x)dx = \begin{cases} 0 & n \neq p \\ \frac{1}{2} & n = p \end{cases}$$

Thus, we can find that

$$C_{p,q} = 4\int_0^1\int_0^1 f(x,y)X_p(x)Y_q(y)dxdy$$

3.6 Laplace equation

In the section we do the separation of variable for the Laplace equation. Consider the following BVP:

$$\begin{cases} u_{xx} + u_{yy} = 0 & 0 < x < 1, 0 < y < 1 \\ u(x,0) = 0, u(x,1) = x - x^2 & 0 \leq x \leq 1 \\ u(0,y) = 0, u(1,y) = 0 & 0 \leq y \leq 1 \end{cases} \tag{36}$$

Note that we no longer have the initial condition since there is no "time" dependence.

The solutions to Laplace equation $\nabla u = 0$ are the steady state solutions to the heat equation. we have $u(x, y) = X(x)Y(y)$:

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(x)}{Y(x)} = \beta$$

With the boundary condition $u(0, y) = 0, u(1, y) = 0$, we have the ODE for x:

$$\begin{cases} X''(x) = \beta X(x) & 0 < x < 1 \\ X(0) = 0, X(1) = 0 \end{cases}$$

This is what we have solved before with the eigenvalues and eigenfunctions:

$$\beta_n = -(n\pi)^2$$

$$X_n(x) = \sin(n\pi x), n \in N$$

Also, we have the functions for $Y(y)$:

$$Y''(x) = (n\pi)^2 Y(x)$$

we can find that

$$Y_n(y) = C_1 e^{n\pi y} + C_2 e^{-n\pi y}$$

with one boundary condition that $Y(0) = 0$

We have the general solution as:

$$Y_n(y) = C_1(e^{n\pi y} - e^{-n\pi y}) = 2C_1 \sinh(n\pi y)$$