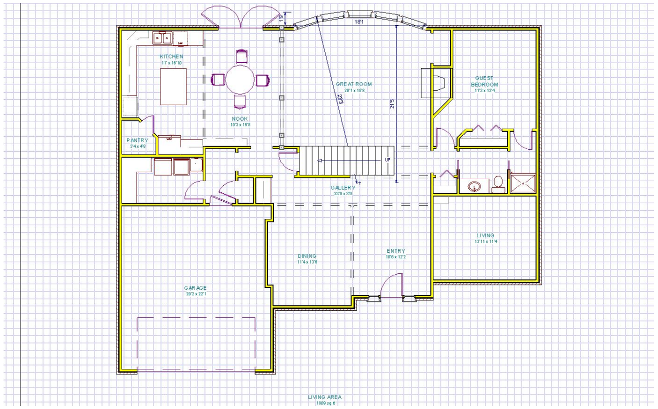
A VERY sketchy framework for finite element computations

Construction:



Finite Element Methods:

Mesh, finite element functions/space, quadrature rules, etc

Use a weak form and a FE space to set up a Finite Element Scheme/Equation

Solve the FE equation to obtain a finite element solution

Chapters 2, Finite Element Methods for 1D Problems

Examples presented in our previous lectures and exercises indicate the Galerkin method can be used as a computational tool to solve BVPs.

However, a few remaining issues including those listed below need to be resolved before we can adopt the Galerkin method as a general computational method for solving BVPs.

- How to choose the parameter m and n?
- How to choose/design the functions in test function space?
- How to choose/design the functions in the trial function set?
- It is desirable to design a Galerkin method that has "finite" features suitable for implementations on computers.

Finite element functions can address these issues systematically.

We note that functions in the test function space and the trial function set do not act individually. Instead, they work collectively for a Galerkin method. Similarly, finite element functions are usually developed collectively for the test and trial functions space/set in the following framework:

- Partition the solution domain Ω into finitely many sub-domains. This partition is often called a mesh of the solution domain and the sub-domains are called elements. For easy implementation, these sub-domains usually are simple in geometry, such as line segments for 1D problems and polygons like triangles and rectangles for 2D problems.
- The basis functions in the test and trial function space/set are defined as piecewise polynomials according to the mesh and the specific continuity determined by the problem or the method for solving this problem.

Remarks:

- Each polynomial can be describe by a finitely number of coefficient, and computations involving polynomials, either algebraic computations or calculus, are usually easy even for computers.
- However, piecewise polynomials are not polynomials. One of the main issues is that they do not have enough global smoothness and this leads to challenges in the theoretical analysis for finite element methods.

2.1, The mesh for a 1D domain: Consider a 1D domain $\Omega = (x_L, x_R) \subset \mathbb{R}$

Three essential components for a mesh for a domain $\Omega = (x_L, x_R)$:

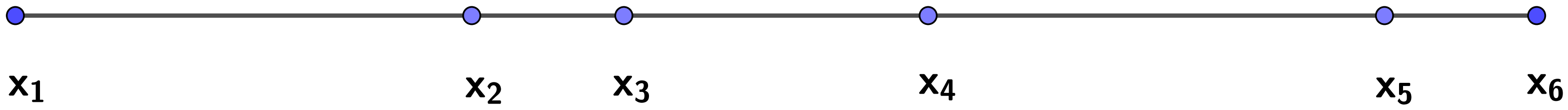
- Choose an integer n and introduce $n+1$ points/nodes $x_i, i = 1, 2, \cdots, n+1$ in Ω.
- Use all of these nodes to partition Ω into 1D simplexes:

$$K_p = [x_i, x_j], \quad 1 \le i, j \le n, \quad x_i < x_j, \;\; p = 1, 2, \cdots$$

called elements of this mesh. These simplexes need to be such that

$$\cup_p K_p = \overline{\Omega} = [x_L, x_R], \quad K_p \cap K_q = \emptyset \;\; \text{or} \;\; \exists i, 1 \le i \le n, \;\; K_p \cap K_q = \{x_i\}$$

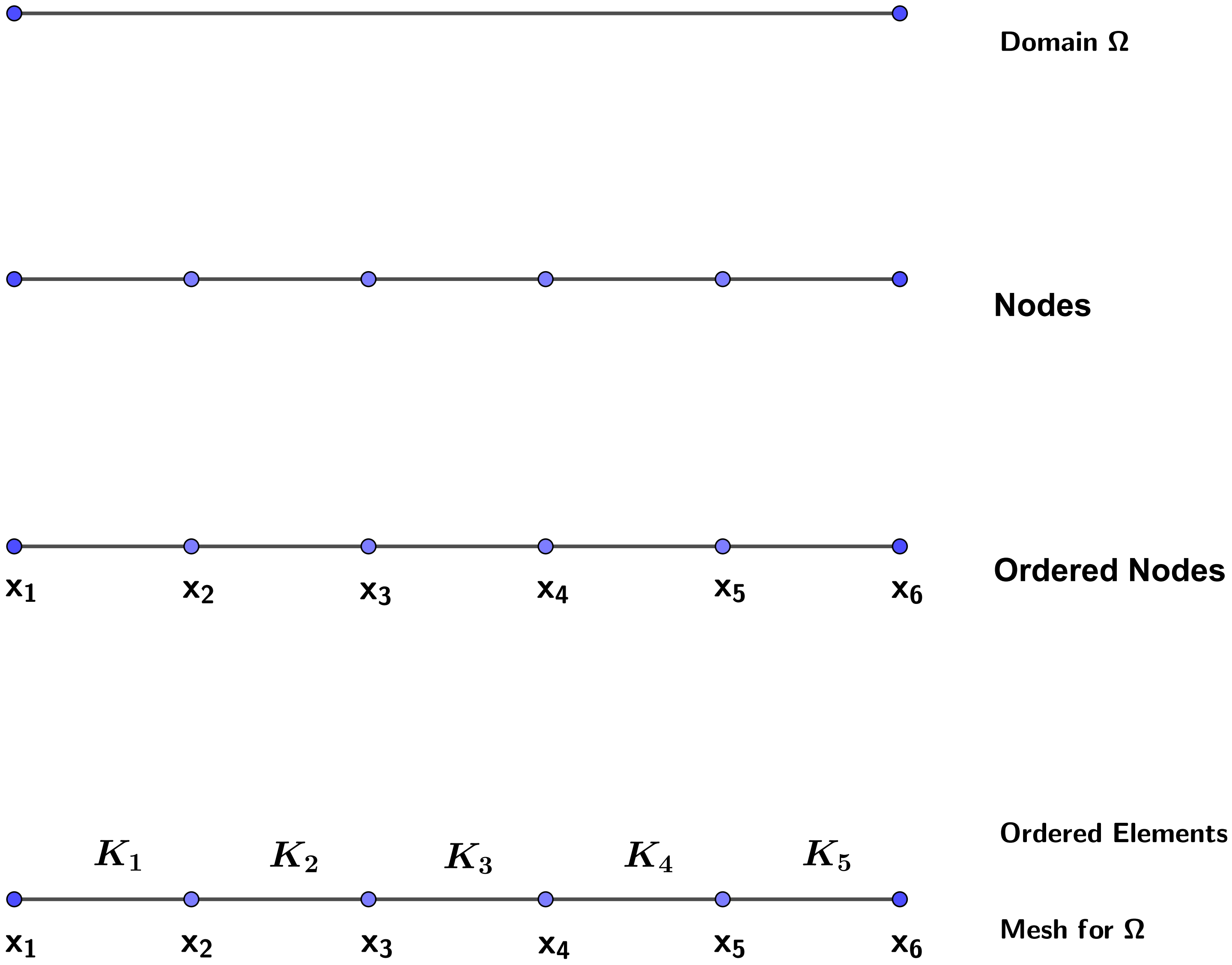
Not allowed: $K_1 = [x_1, x_3]$, $K_2 = [x_2, x_3]$, $K_3 = [x_3, x_4]$, $K_4 = [x_4, x_5]$, $K_5 = [x_5, x_6]$



Also, x_i and x_{i+1} in element $K_p = [x_i, x_j]$ are called the vertices of the element/simplex K_p.

- When $K_p \cap K_q = \{x_i\}$, then x_i is a common vertex of elements K_p and K_q and it is also called an edge of the mesh.

Due to the 1D geometry of $\Omega = (x_l, x_r)$, a mesh with the above features can be constructed as follows:



Domain Ω

Nodes

x_1 x_2 x_3 x_4 x_5 x_6

Ordered Nodes

Ordered Elements

K_1 K_2 K_3 K_4 K_5

x_1 x_2 x_3 x_4 x_5 x_6

Mesh for Ω

A summary:

$$\begin{aligned} \text{Nodes:} \quad & x_l = x_1 < x_2 < x_3 < \cdots < x_n < x_{n+1} = x_r \\ \text{Elements:} \quad & K_p = [x_p, x_{p+1}], \ \ p = 1, 2, \cdots, n \\ \text{Mesh size:} \quad & h = \max_{1 \le p \le n} |K_p| = max_{1 \le p \le n} |x_{p+1} - x_p| \end{aligned}$$

Three sets to describe a mesh:

- The set of nodes $\mathcal{N}_h = \{x_i\}_{i=1}^{n+1}$.
- The set of elements $\mathcal{T}_h = \left\{K_i\right\}_{i=1}^{n}$ which are formed with nodes.
- The set of edges $\mathcal{E}_h$ which are intersections of elements

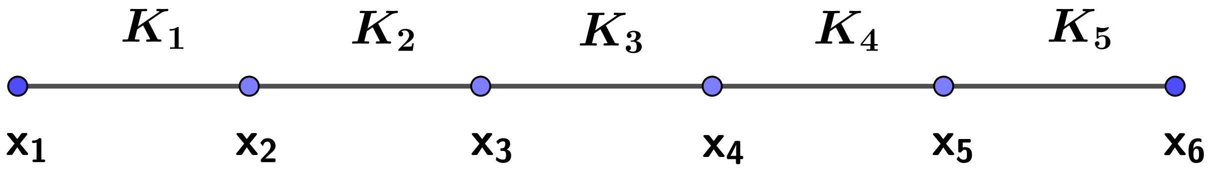
From now on, we often simply call $\mathcal{T}_h = \left\{K_i\right\}_{i=1}^{n}$ a mesh of for a domain Ω.

A mesh $\mathcal{T}_h$ in a computer can be described by two essential matrices/arrays:

- The nodal coordinate matrix p. It is of the size $1 \times n$ containing the coordinates of the nodes $x_i, i = 1, 2, \cdots, n$.
- The connectivity matrix t for the mesh nodes. We index all the nodes. Put these indices in an integer array t of the size:
  (num of vertices forming each element) $\times$ (num. of elements)
  such that its p-th column contains the indices of the vertices forming element K_p with a chosen order.

Matrix/Array t has $|\mathcal{T}_h|$ columns it tells us which mesh nodes are used to form which element.

Recall:



We can implement a 1D uniform mesh generator in Matlab as follows:

```
function mesh = mesh_generator_1D(domain, n)
x_l = domain(1); x_r = domain(2);
p = x_l:(x_r - x_l)/n:x_r;

t = zeros(2,n); t(:,1) = [1;2];
for i=2:n
    t(:,i) = t(:,i-1) + 1;
end
mesh = struct('p', p, 't', t);
```

- `domain = [x_l, x_r]` defines the solution domain $\Omega = (x_l, x_r)$.
- `n` is the integer specifying the $n + 1$ uniformly distributed nodes generated in Ω.
- `mesh` is a structure array with two basic fields p and t. The matrix p can be extracted by `mesh.p` and t can be extracted by `mesh.t`. The less important field is for the set of edge which can be added later or by extending this program.

Example: The 1D mesh generator can be used to generate a mesh as follows:

```
domain = [0, 1]; n = 5;
mesh = mesh_generator_1D(domain, n);
```

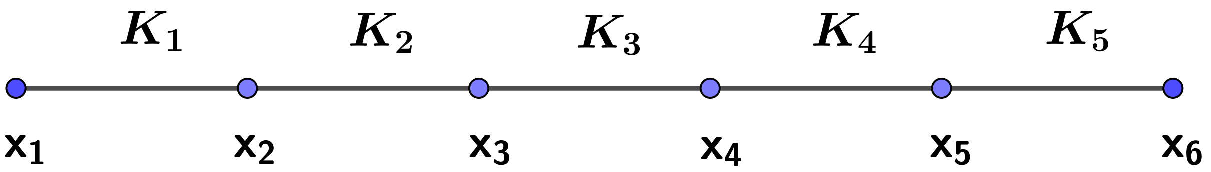
which contains the nodal coordinate matrix/array `mesh.p`:

```
0      0.2000      0.4000      0.6000      0.8000      1.0000
```

and the connectivity matrix/arrya `mesh.t`:

```
1       2       3       4       5
2       3       4       5       6
```

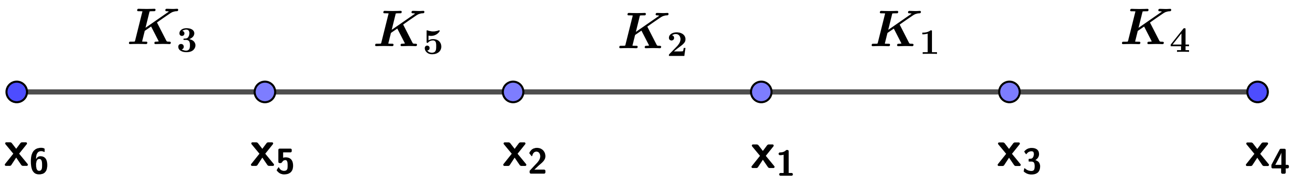
These arrays define a mesh in $\Omega = (0, 1)$ with the following geometry:



These two arrays contain rich information about the mesh. For example, the total number of nodes is `length(mesh.p)`, the total number of elements is `length(mesh.t)`. The coordinate of the 4-th node x_4 in the mesh can be extracted by `mesh.p(4)`. The indices for the vertices/nodes forming the 4-th elements are given by `mesh.t(:,4)`. We can further find the coordinates of the nodes for the 4-th element by `mesh.p(mesh.t(:,4))`.

Remark: The nodes and elements have to be indexed, but how to order them is not important.

For example, if we order the nodes and elements in the mesh as follows:



Then, mehs.p should be:

```
0.6000 0.4000 0.8000 1.0000 0.2000 0
```

and mehs.t should be:

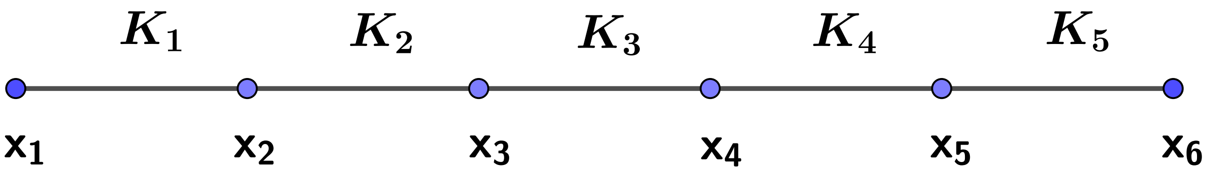
```
1 2 6 3 5
3 1 5 4 2
```

Even though these matrices are different from those in the previous example, their usages in finite element computations will be the same.

However, it is desirable for the node and connectivity matrices to have a simpler and more natural order, IF POSSIBLE.

Remark: The structure `mesh` can be easily enhance later to include other useful information about a mesh such as the edges, points, etc. For example, recall that an edge in a mesh is a geometric object where two elements meet.

For a 1D mesh , its edges are the nodes, and each edge has two neighbor elements, one is on the left and other is on the right:



Instead of programming edges explicitly, we can introduce an edge matrix/array e to describe the edges in this mesh together with the nodal coordinate matrix p:

$$\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
0 & 1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 & 0
\end{array}$$

In this matrix/array, the i-th column contains information for the i-th nodes: the first number is the index of the point for i-th edge, the second and the third numbers are the indices of the elements on the left and right of the i-th edge, respectively. This edge matrix can be used to retrieve information associated with a particular edge.

Here we order the edges points from the left to the right.

We can revise the `mesh_generator_1D.m` to include a new field for the edges as follows:

```
function mesh = mesh_generator_1D(domain, n)
x_l = domain(1); x_r = domain(2);
p = x_l:(x_r - x_l)/n:x_r;

t = zeros(2,n); t(:,1) = [1;2];
for i=2:n
    t(:,i) = t(:,i-1) + 1;
end
e = zeros(3,n+1); e(1,:) = 1:n+1; e(2,:) = 0:n; e(3,:) = [1:n,0];
mesh = struct(’p’, p, ’t’, t, ’e’, e);
```

Example: For the mesh generated by `mesh = mesh_generator_1D([0,1], 5)`, `mesh.e(:, 4)` produces the following array:

```
4
3
4
```

which states that this edge is formed by the 4-th node x_4, the element on the left is K_3 and the element on the right is K_4.

There are two ways to extend a structure with new fields.

- Modify the Matlab function such that it can produce all the new fields together with the original ones.
- When the new fields needs extra inputs, then the above approach requires to change the interface of the Matlab function which might further require change other related codes. To avoid this situation, we can develop a new function for these new fields and extend the structure by Matlab's `setfield` command.

Example: Assume we have produced the structure `mesh` but generate the array `e` for the edges by another program. Then we can insert `e` into `mesh` by the Matlab command:

```
mesh = setfield(mesh, 'E', E);
```

2.2, 1D finite element spaces:

One key feature of finite element methods is literally “finite”, i.e., all the involve computations are carried out through finite procedures over all elements in a mesh, element by element repeatedly, and finite element functions over the whole solution domain Ω are rarely programmed or used directely except for mathematical descriptions and analysis of the involved computation procedures.

We now discuss finite element functions for solving differential equations, i.e., we discuss the construction of the trial and test function spaces consisting of finite element functions.

In particular, we consider a finite element function of degree p which is a piecewise polynomial defined on a mesh plus other features required by the weak form used in a finite element method for solving a particular differential equation.

Since the restriction of a standard finite element function of degree p to each element of a mesh is a polynomial of degree p, we will start from the local finite element space defined on each element.

2.2.1, Local p-th degree Lagrange shape functions on each element: Consider a typical element $K = [x_i, x_{i+1}]$ in a mesh $\mathcal{T}_h$. First, we introduce $p+1$ equally spaced local nodes in K as follows

$$x_i = t_{K,1} < t_{K,2} < \cdots < t_{K,p} < t_{K,p+1} = x_{i+1}$$

Then, for each local node $t_{K,j}$, we introduce a polynomial called a shape function as follows

$$L_{K,j}^p(x) = \prod_{k=1,k\neq j}^{p+1} \frac{x - t_{K,k}}{t_{K,j} - t_{K,k}}, \quad j = 1, 2, \cdots, p, p+1 \tag{9}$$

- $L_{K,j}^p(x), 1 \leq j \leq p+1$ are called Lagrange cardinal functions which are polynomials of degree exactly p determined by the local nodes $t_1, t_2, \cdots, t_{p+1}$ in the element K.
- $$L_{K,j}^p(t_{K,i}) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$
- Let $w(x) = \prod_{k=1}^{p+1}(x - t_{K,k})$, then

$$L_{K,j}^p(x) = \frac{w(x)}{(x - t_{K,j})w'(t_{K,j})}, \quad j = 1, 2, \cdots, p+1$$

Lagrange shape functions on a reference element: Let $\hat{K} = [-1, 1]$. Then element element $K = [x_i, x_{i+1}]$ in a mesh $\mathcal{T}_h$ is homeomorphic to the reference element $\hat{K} = [-1, 1]$ by the following affine mapping:

$$t : K \to \hat{K}, \quad t(x) = \frac{2}{x_{i+1} - x_i}(x - x_i) - 1, \quad \forall x \in K \tag{10}$$

Let $t_1 = -1 < t_2 < \cdots < t_p < t_{p+1} = 1$ be equally spaced nodes in $\hat{K} = [-1, 1]$ and let

$$\hat{L}_j^p(t) = \prod_{k=1, k\neq j}^{p+1} \frac{t - t_k}{t_j - t_k}, \quad j = 1, 2, \cdots, p, p+1 \tag{11}$$

Then, we have

$$L_{K,j}^p(x) = \hat{L}_j^p(t(x)) = \hat{L}_j^p\left(\frac{2}{x_{i+1} - x_i}(x - x_i) - 1\right), \quad j = 1, 2, \cdots, p, p+1 \tag{12}$$

and

$$\frac{d^r L_{K,j}^p(x)}{dx^r} = \left(\frac{2}{x_{i+1} - x_i}\right)^r \frac{d^r \hat{L}_j^p(t(x))}{dt^r}, \quad j = 1, 2, \cdots, p, p+1 \tag{13}$$

Implementation of shape functions on the reference element:

First, we can use a symbolic software such as Mathematica to prepare formulas for the shape functions on the reference element:

```
ClearAll
(* Ld lists the BI-th p-th degree Lagrange poly and its derivatives *)
p = 2;
(*degree*)
BI = 1; (*base index,BI<=p+1*)
nodes = Table[-1 + (i - 1)*(2/p), {i, 1, p + 1}];
L[t_] = Product[(t - nodes[[k]])/(nodes[[BI]] - nodes[[k]]), {k, 1,
     BI - 1}]*
   Product[(t - nodes[[k]])/(nodes[[BI]] - nodes[[k]]), {k, BI + 1,
     p + 1}];
"The BI-th shape function and its derivatives"
Ld = Simplify[Expand[Table[D[L[t], {t, i}], {i, 0, p}]]]
```

This will produce the 1st shape function of degree 2 and its derivatives on the reference element:

```
{1/2 (-1+t)t,-(1/2)+t,1}
```

An implementation for the 1D p-th degree shape function on the reference element:

```
function f = shape_fun_1D_Lagrange_ref(t, shape_index, d_index, degree)

if degree == 0
    if shape_index == 1
        if d_index == 0
            f = ones(size(t));
        elseif d_index > 0
            f = zeros(size(t));
        end
    else
        disp([’shape_index = ’, int2str(shape_index)])
        disp(’This program is not for this value of shape_index’)
        disp(’Terminate this program by Ctrl+C’)
        pause
    end
```

Recall function interface:

```
function f = shape_fun_1D_Lagrange_ref(t, shape_index, d_index, degree)
```

Continued:

```
elseif degree == 1
    if shape_index == 1
        if d_index == 0
            f = (1-t)/2;
        elseif d_index == 1
            f = (-1/2)*ones(size(t));
        elseif d_index > 1
            f = zeros(size(t));
        end
    elseif shape_index == 2
        if d_index == 0
            f = (1+t)/2;
        elseif d_index == 1
            f = (1/2)*ones(size(t));
        elseif d_index > 1
            f = zeros(size(t));
        end
    else
        disp(['shape_index = ', int2str(shape_index)])
        disp('This program is not for this value of shape_index')
        disp('Terminate this program by Ctrl+C')
        pause
    end
```

Recall function interface:

```
function f = shape_fun_1D_Lagrange_ref(t, shape_index, d_index, degree)
```

Continued:

```
elseif degree == 2
    if shape_index == 1
        if d_index == 0
            f = -(1/2)*(1-t).*t;
        elseif d_index == 1
            f = -1/2+t;
        elseif d_index == 2
            f = ones(size(t));
        elseif d_index > 2
            f = zeros(size(t));
        end
    elseif shape_index == 2
        if d_index == 0
            f = 1-t.^2;
        elseif d_index == 1
            f = -2*t;
        elseif d_index == 2
            f = -2*ones(size(t));
        elseif d_index > 2
            f = zeros(size(t));
        end
    elseif shape_index == 3
        if d_index == 0
            f = (1/2)*t.*(1+t);
        elseif d_index == 1
            f = 1/2 + t;
        elseif d_index == 2
            f = ones(size(t));
        elseif d_index > 2
            f = zeros(size(t));
        end
    else
        disp([’shape_index = ’, int2str(shape_index)])
        disp(’This program is not for this value of shape_index’)
        disp(’Terminate this program by Ctrl+C’)
        pause
    end
elseif degree > 3
    disp([’degree = ’, int2str(degree)])
    disp(’This program is not for this value of degree’)
    disp(’Terminate this program by Ctrl+C’)
    pause
end
```

An implementation of the 1D local shape functions: Since both the finite element functions and their derivatives are used to solve partial differential equations, we should implement the local shape function such that it can generate both the function and its derivative values according to the demand.

For

$$\frac{d^r L_{K,j}^p(x)}{dx^r} = \left(\frac{2}{x_{i+1} - x_i}\right)^r \frac{d^r \hat{L}_j^p(t(x))}{dt^r}, \quad 1 \le j \le p+1, \quad 0 \le r \le p \tag{13}$$

we can implement it with a Matlab function in the following format:

```
function f = shape_fun_1D_Lagrange(x, elem, degree, ...
                                           shape_index, d_index)
t = (2/(elem(2)-elem(1)))*(x-elem(1)) - 1;
f = ((2/(elem(2)-elem(1)))^d_index)*shape_fun_1D_Lagrange_ref(t, ...
                     shape_index, d_index, degree);
```

- x is a real array where we would like to evaluate the shape function or its derivatives.
- elem provides the coordinates for the vertices of the element K. Note elem for element K_i in a mesh is elem = mesh.p(mesh.t(:,i)).
- shape_index specifies which local shape function is evaluated.
- d_index specifies which derivative of the shape function is evaluated.

We can use this shape function as follows

```
elem = [0, 1]; degree = 1; shape_index = 1; d_index = 0;
x = elem(1):0.001:elem(2);
y1 = shape_fun_1D_Lagrange(x, elem, degree, shape_index, d_index);
plot(x,y1)
```

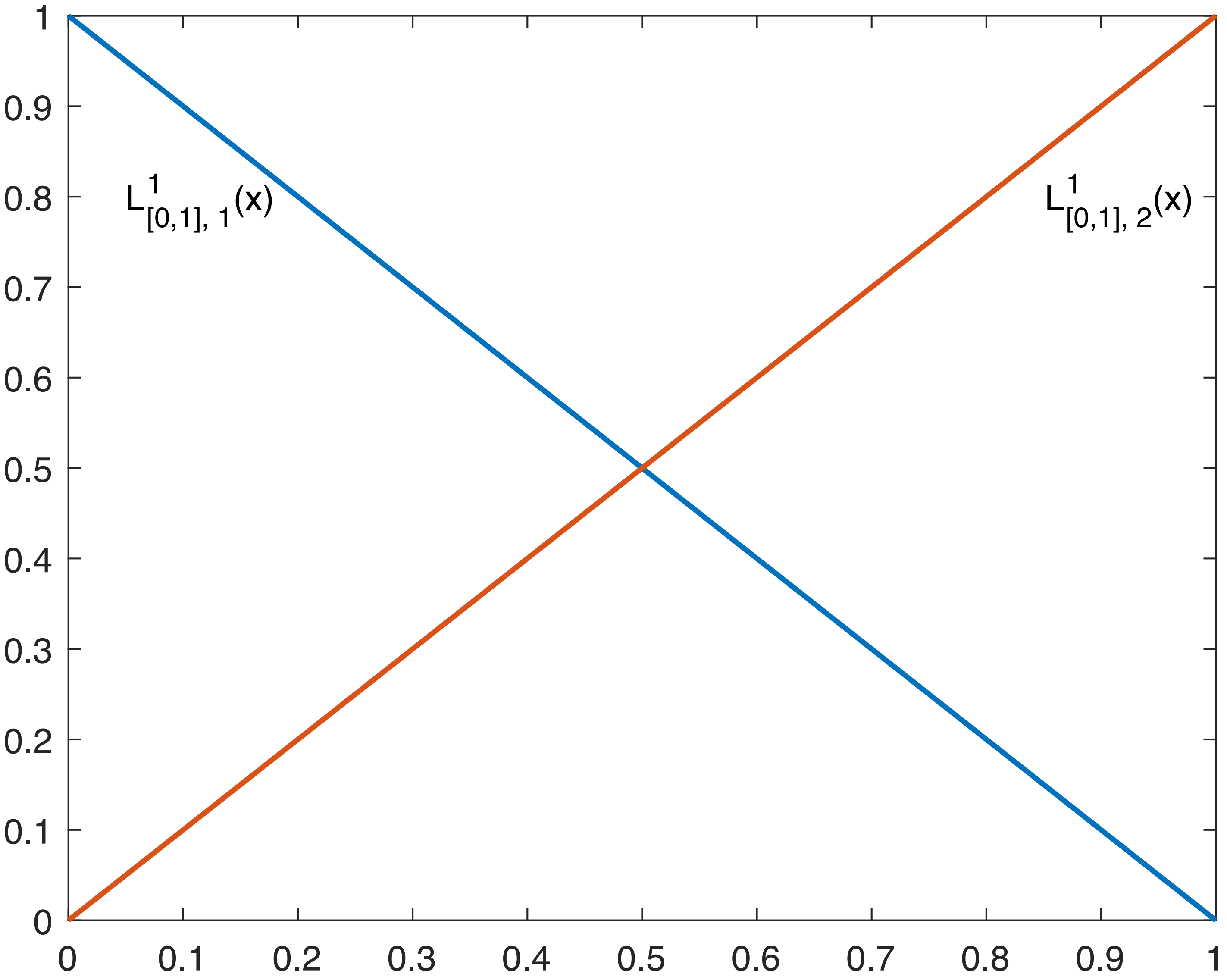
or

```
elem = [0, 1]; degree = 1; d_index = 0;
x = linspace(elem(1), elem(2), 200);
y1 = shape_fun_1D_Lagrange(x, elem, degree, 1, d_index);
y2 = shape_fun_1D_Lagrange(x, elem, degree, 2, d_index);
plot(x, y1, x, y2)
```

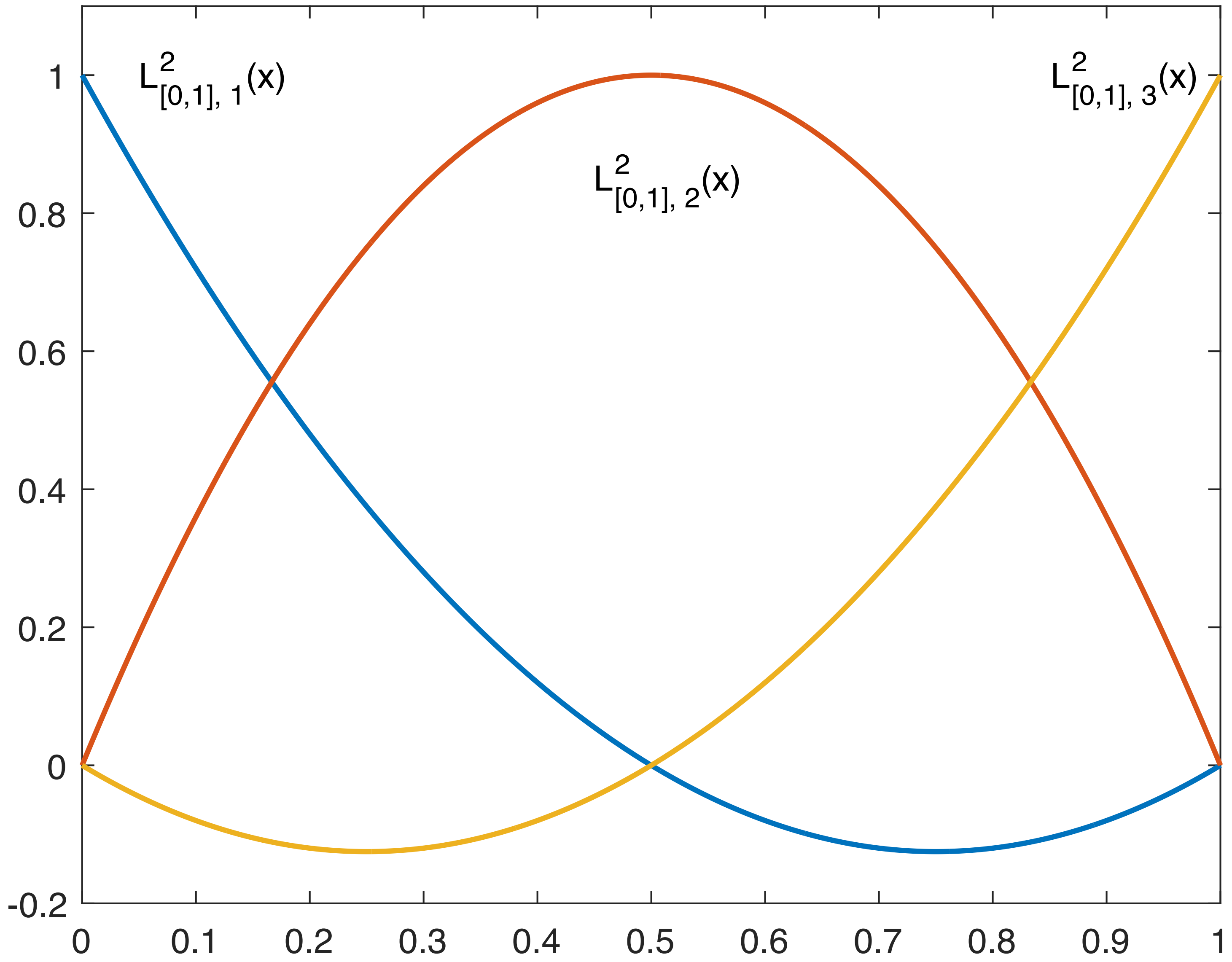
We can also use `shape_fun_1D_Lagrange` as functions:

```
elem = [0, 1]; degree = 1; d_index = 0;
x = linspace(elem(1), elem(2), 200);
shfun1 = @(x) shape_fun_1D_Lagrange(x, elem, degree, 1, d_index);
shfun2 = @(x) shape_fun_1D_Lagrange(x, elem, degree, 2, d_index);
plot(x, shfun1(x), x, shfun2(x))
```

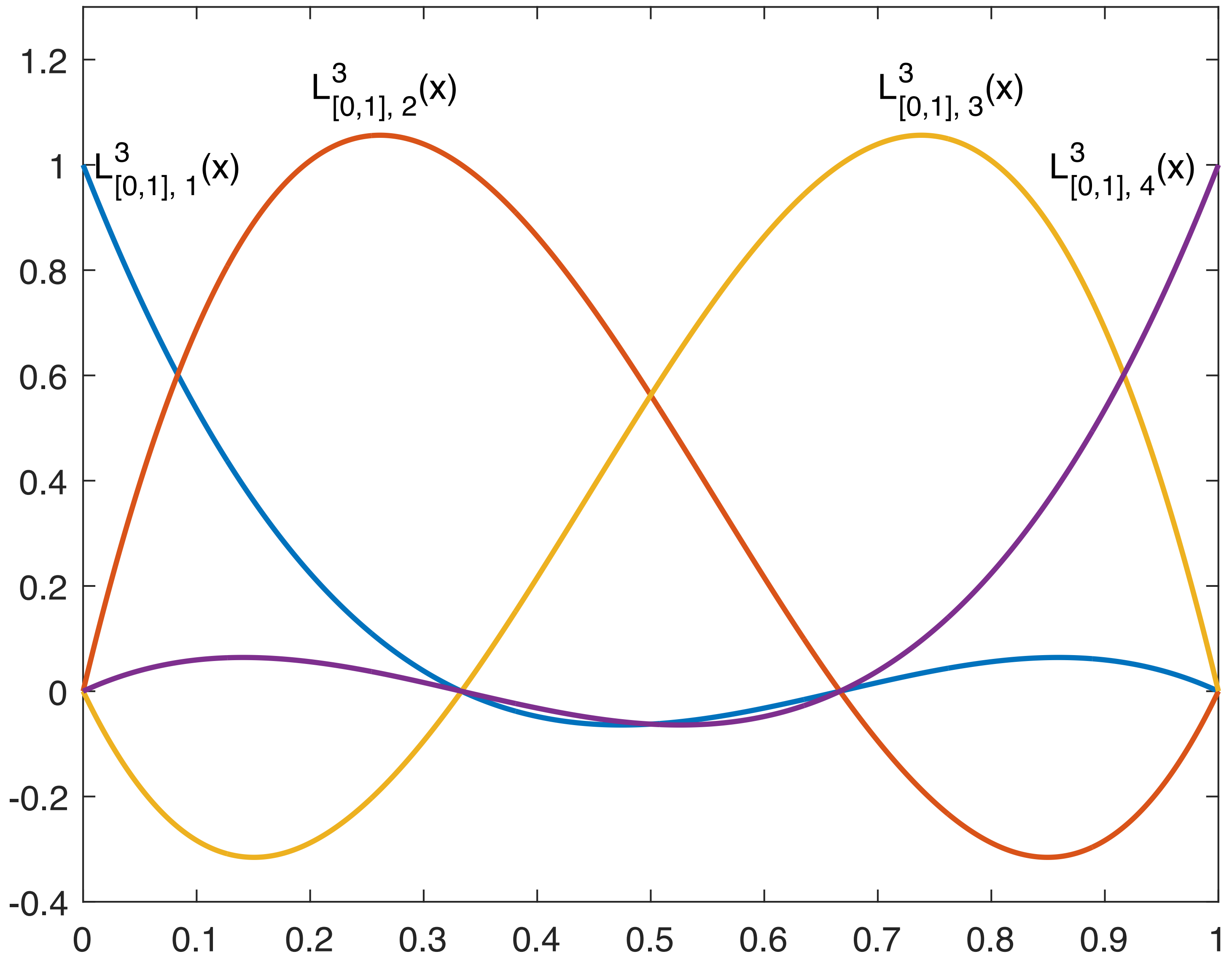
All the 1st degree shape function over $[0, 1]$:



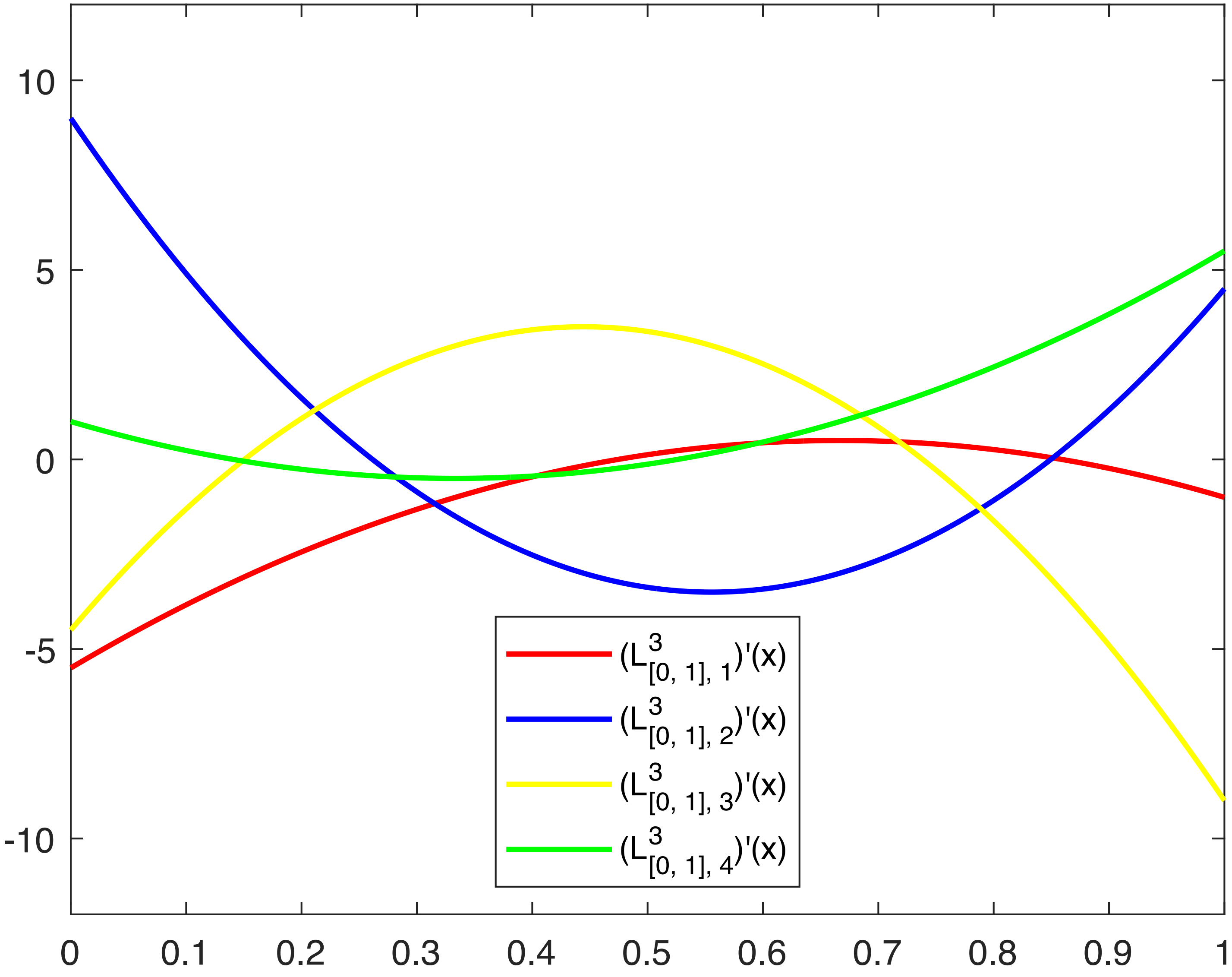
All the 2nd degree shape functions over $[0, 1]$:



All the 3rd degree shape functions over $[0, 1]$:



1st derivative of 3rd degree shape functions over $[0, 1]$:



The local p-th degree FE space on an element: for every element $K \in \mathcal{T}_h$,

$$V_h^p(K) = span\{L_{K,j}^p(x), \quad j = 1, 2, \cdots, p, p+1\} = \Pi_p \tag{14}$$

where Π_p is the space of polynomials of degree p or less.

Recall that we have implemented the p-th degree shape functions on an element in the following Matlab function:

```
function f = shape_fun_1D_Lagrange(x, elem, degree, ...
                                   shape_index, d_index)
```

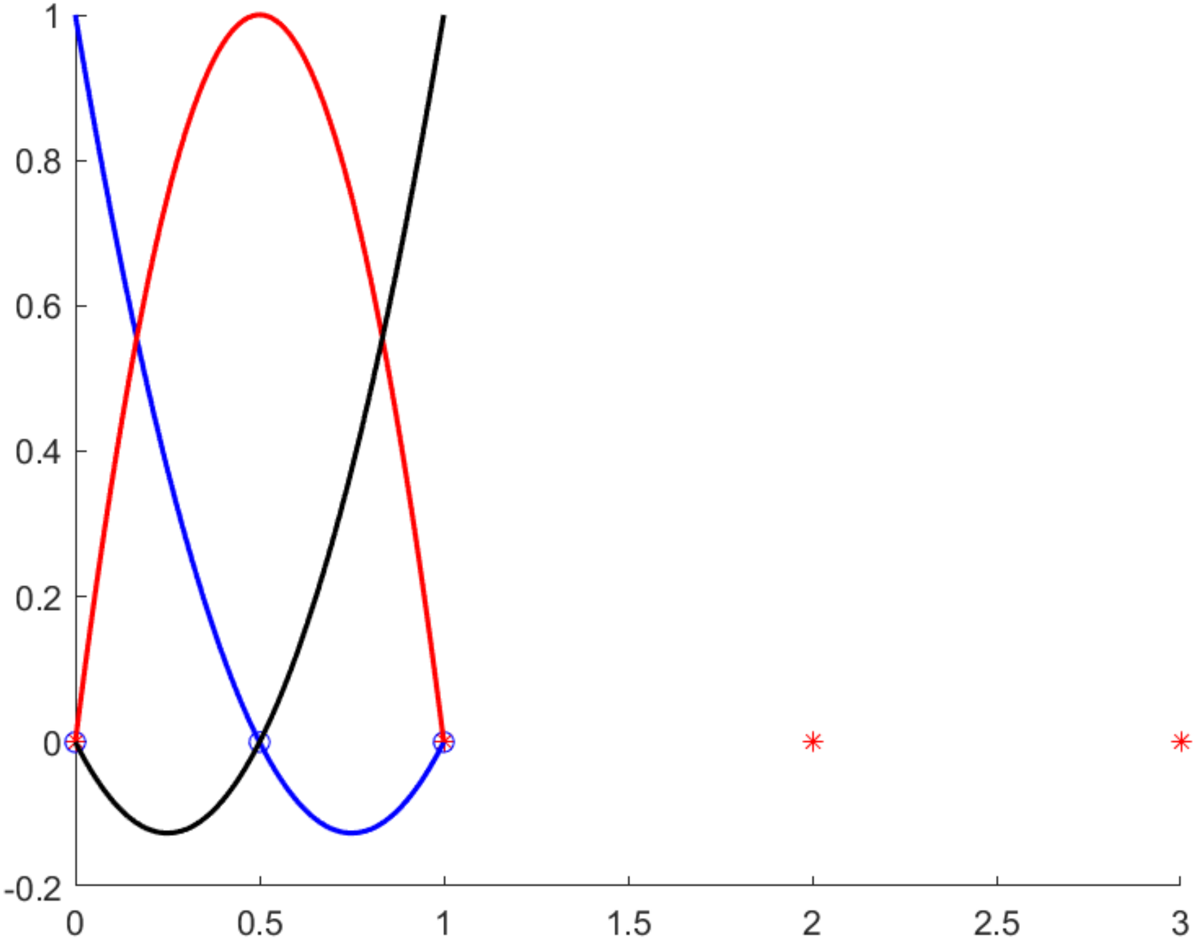
to generate the `shape_index`-th shape function with the specified degree.

The following script plots all the 2nd degree shape functions in the 1st element of a mesh for $\Omega = [0, 3]$:

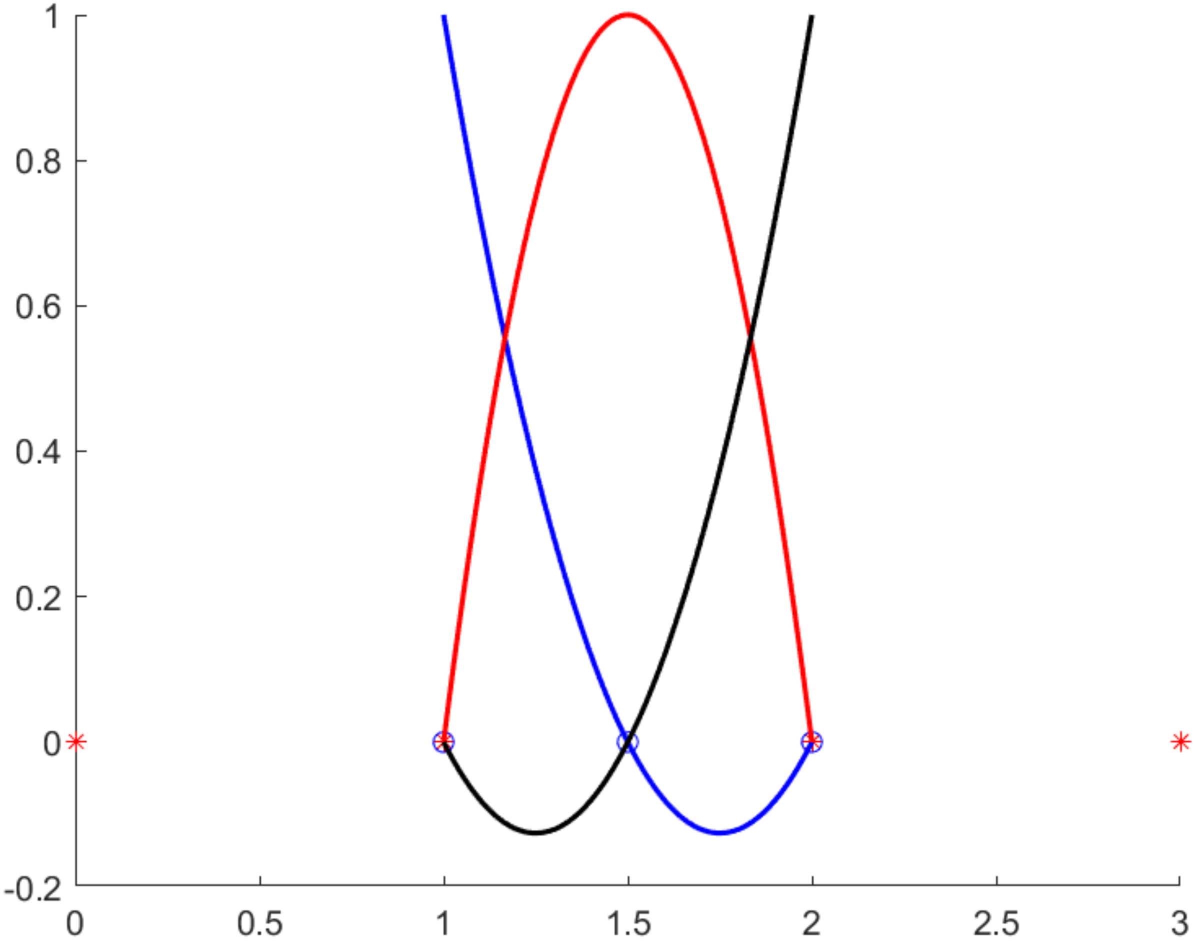
```
domain = [0, 3]; n = 3; degree = 2;
mesh = mesh_generator_1D(domain, n);
k = 1; elem = mesh.p(:, mesh.t(:, k)); % k-th element
figure(k); clf; hold on; axis([domain(1), domain(2), -0.2, 1])
plot(mesh.p, zeros(size(mesh.p)), ’r*’) % plot the mesh points
x = elem(1):0.001:elem(2);
shape_index = 1; d_index = 0;
y1 = shape_fun_1D_Lagrange(x, elem, degree, shape_index, d_index);
shape_index = 2;
y2 = shape_fun_1D_Lagrange(x, elem, degree, shape_index, d_index);
shape_index = 3;
y3 = shape_fun_1D_Lagrange(x, elem, degree, shape_index, d_index);
plot(x, y1, ’b’, x, y2, ’r’, x, y3, ’k’, ’LineWidth’, 1.5)
```

By choosing different element by the value of k in the script above, we visualize the 2nd degree shape functions on other elements.

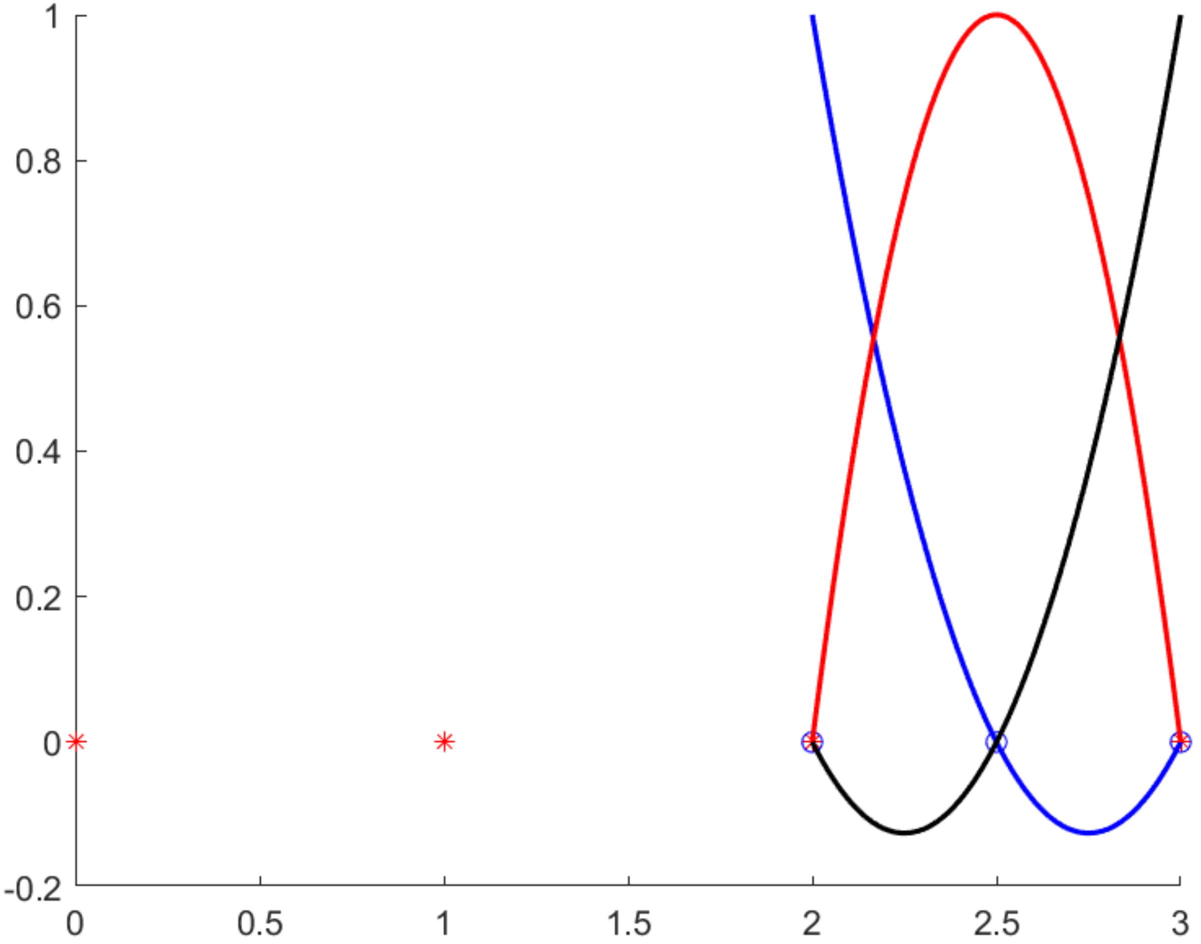
2nd degree shape functions in the 1st element:



2nd degree shape functions in the 2nd element:



2nd degree shape functions in the 3rd element:



Shape functions are important PARTS of finite element functions.

2.2.2, The global p-th degree finite element spaces:

Finite element functions over the solution domain Ω should be developed with the following considerations:

- Use shape functions locally on elements. This is problem independent.
- Use features for the trial and test functions specified by the weak form of the PDE problems.
- Use features for the finite element functions demanded by the chosen finite element methods.
- Others ...

Here, let us first consider 1D continuous Lagrange type finite element functions for solving the prototype 1D BVP problem. Recall the BVP: find u such that

$$-(au')' + cu = f, \ x \in \Omega = (0,1), \tag{1}$$

$$a(0)u'(0) = g_0, \tag{2}$$

$$u(1) = g_1, \tag{3}$$

A Weak Form for this BVP: find $u \in \mathcal{S}$ such that

$$\int_0^1 av'u'dx + \int_0^1 cvudx = \int_0^1 vfdx - v(0)g_0, \ \forall v \in \mathcal{T}, \tag{5}$$

where

$$\mathcal{T} = \{w \mid w \in H^1(0,1), w(1) = 0\}. \tag{7}$$

$$\mathcal{S} = \{w \mid w \in H^1(0,1), w(1) = g_1\}. \tag{8}$$

The Galerkin or Petrov-Galerkin requires

$$\mathcal{T}_h = span\{\phi_1(x), \phi_2(x), \cdots, \phi_n(x)\} \subset \mathcal{T} \subset H^1(0,1)$$

and we now discuss how to construct $\phi_i, 1 \le i \le n$ as finite element functions

Theorem 1.1

Assume that $u(x)$ is a piecewise polynomial according to a mesh $\mathcal{T}_h$ of $\Omega \subset \mathbb{R}^1$. Then $u \in H^1(\Omega)$ if and only $u \in C^0(\overline{\Omega})$.

C^0 (i.e., continuous) p-th degree Lagrange type FE space:

- Form a mesh $\mathcal{T}_h = \{K_j\}$ of Ω described by nodes $x_i, 1 \le i \le n+1$.
- Form the set of finite element nodes for the p-th degree finite element space by collecting local nodes from all elements of $\mathcal{T}_h$, denote this set by $\mathcal{N}_{h,dof}$.
- Index finite element nodes. Each of these indices corresponds to a finite element basis function defined on the whole Ω; hence, we call each of these indices a global degree of freedom. We put all the global degrees of freedom into an array T_g of size $(p+1) \times |\mathcal{T}_h|$ such that its j-th column contains indices of the finite element nodes in element K_j ordered according to local nodes on that element. We call T_g the connectivity matrix for the global degrees of freedom.
- For each $\tilde{x}_j \in \mathcal{N}_{h,dof}$, we define a piecewise polynomial $\phi_j(x)$ such that:

$$\phi_j|_K \in V_h^p(K), \ \forall K \in \mathcal{T}_h \tag{15}$$

$$\phi_j(\tilde{x}_i) = \delta_{ij}, \ \forall \tilde{x}_i \in \mathcal{N}_{h,dof} \qquad \text{(Lagrange property)} \tag{16}$$

  we can show that $\phi_j \in C^0(\Omega)$???

- The p-th degree C^0 Lagrange type FE space is the following space

$$V_h^p(\Omega) = span\{\phi_i, \ \forall \tilde{x}_i \in \mathcal{N}_{h,dof}\} \tag{17}$$

  Each ϕ_j is called a global basis function for the finite element space $V_h^p(\Omega)$. These finite element basis functions are linearly independent according to the property specified in (16), and $dim(V_h^p(\Omega)) = |\mathcal{N}_{h,dof}|$.

Recall the finite element space $V_h^p(\Omega) = span\{\phi_i,\ \forall \tilde{x}_i \in \mathcal{N}_{h,dof}\}$.

Every finite element function $u_h(x)$ in $V_h^p(\Omega)$ can be written as

$$u_h(x) = \sum_{j=1}^{\left|\mathcal{N}_{h,dof}\right|} u_j \phi_j(x) \tag{18}$$

which means a finite element function $u_h(x)$ is mathematically determined by the array $\vec{u} = (u_1, u_2, \cdots, u_{\left|\mathcal{N}_{h,dof}\right|})^t$.

For Lagrange type finite element functions, we note

$$u_h(\tilde{x}_i) = \sum_{j=1}^{\left|\mathcal{N}_{h,dof}\right|} u_j \phi_j(\tilde{x}_i) = u_i, \quad i = 1, 2, \cdots |\mathcal{N}_{h,dof}| \tag{19}$$

Hence the coefficient u_i in a finite element function $u_h(x) = \sum_{j=1}^{\left|\mathcal{N}_{h,dof}\right|} u_j \phi_j(x)$ is its value at the finite element node $\tilde{x}_i$.

Implementation of global finite element basis functions:

Recall the mathematical definition of the nodal FE basis functions

$$\phi_j \in C^0(\Omega), \quad \phi_j|_K \in V_h^p(K), \ \forall K \in \mathcal{T}_h \tag{15}$$

$$\phi_j(\tilde{x}_i) = \delta_{ij}, \ \forall \tilde{x}_i \in \mathcal{N}_{h,dof} \qquad \text{(Lagrange property)} \tag{16}$$

These p-th degree C^0 finite element basis functions are determined/programmed with

```
function mesh = mesh_generator_1D(domain, n)
function f = shape_fun_1D_Lagrange(x, elem, degree, ...
                                   shape_index, d_index)
```

and the connectivity array T_{dof} for the global degrees of freedom. We emphasize that finite element functions are rarely used globally. Computations involving finite element functions are generally carried out element by element without direct usage of the global finite element basis functions, we even do not have to program the global basis functions.

To facilitate the implementation of this idea, we need to develop a program for creating the following arrays:

For $\mathcal{N}_{h,dof}$: a matrix containing coordinates for all finite element nodes of the finite element space.

T_{dof}: the connectivity matrix for the global degrees of freedom. This matrix is of the size (number of FE nodes in each element) × (number of elements)

The k-th column of T_{dof} contains the indices of the finite element nodes inside k-th element.

We can implement these matrices with a Matlab function as follows:

```
fem = fem_generator_Lagrange_1D(mesh, degree)
```

`fem` is a structure for describing finite element basis functions.

`fem.p` provides coordinates for points in $\mathcal{N}_{h,dof}$.

`fem.t` is the the connectivity matrix T_{dof} for the global degrees of freedom.

`fem.degree` is the degree of polynomials used for the finite element space.

Here, we assume that the data structure `mesh` has been prepared by

```
function mesh = mesh_generator_1D(domain, n)
function fem = fem_generator_Lagrange_1D(mesh, degree)
n_elem = size(mesh.t,2);
t = zeros(degree+1,n_elem); t(:,1) = (1:degree+1)’;
for k=2:n_elem
    t(:,k) = t(:,k-1)+degree;
end

p = zeros(1, (degree+1) + (n_elem - 1)*degree);
k = 1;
elem = mesh.p(mesh.t(:,k));
h = (elem(2) - elem(1))/degree;
p(1:degree+1) = elem(1):h:elem(2);
pnt_count = degree + 1;
for k=2:n_elem
    elem = mesh.p(mesh.t(:,k));
    h = (elem(2) - elem(1))/degree;
    p(pnt_count+1:pnt_count+degree) = elem(1)+h:h:elem(2);
    pnt_count = pnt_count+degree;
end

fem = struct(’p’, p, ’t’, t, ’degree’, degree);
```

Example: For the Lagrange type C^0 FE of degree 3 defined on a mesh with 4 element on $\Omega = (2, 3)$, we have following arrays/structures:

```
domain = [2, 3]; n = 4; mesh = mesh_generator_1D(domain, n);
degree = 3; fem = fem_generator_Lagrange_1D(mesh, degree);
```

Then typing `mesh.p`, `mesh.t` in Matlab will show the following arrays:

```
2.0000    2.2500    2.5000    2.7500    3.0000


1     2     3     4
2     3     4     5
```

But typing `fem.p`, and `fem.t` will show

```
Columns 1 through 11
2.0000    2.0833    2.1667    2.2500    2.3333    2.4167    2.5000    2.5833
Columns 9 through 13
2.6667    2.7500    2.8333    2.9167    3.0000


     1     4     7    10
     2     5     8    11
     3     6     9    12
     4     7    10    13
```

Remark: The command `k = 3; fem.t(:, k)` will display which finite element nodes are inside element k.

A summary:

In mathematical description of finite element methods, we need the p-th degree C^0 finite element space:

$$V_h^p(\Omega) = span\{\phi_i,\ \forall \tilde{x}_i \in \mathcal{N}_{h,dof}\} \tag{17}$$

with

$$\phi_j \in C^0(\Omega),\quad \phi_j|_K \in V_h^p(K),\ \forall K \in \mathcal{T}_h \tag{15}$$

$$\phi_j(\tilde{x}_i) = \delta_{ij},\ \forall \tilde{x}_i \in \mathcal{N}_{h,dof} \qquad \text{(Lagrange property)} \tag{16}$$

Claim: Any computations involving the p-th degree C^0 finite element basis functions $\phi_i,\ \forall \tilde{x}_i \in \mathcal{N}_{h,dof}$ can be done by using the following functions:

```
(1): function mesh = mesh_generator_1D(domain, n)
(2): function fem = fem_generator_Lagrange_1D(mesh, degree)
(3): function f = shape_fun_1D_Lagrange(x, elem, degree, ...
                              shape_index, d_index)
```

In other words, we have implemented the p-th degree C^0 finite element space defined by (17).

We proceed to demonstrate how to use functions in the p-th degree C^0 finite element space.

Example: Consider the C^0 linear and quadratic basis functions defined on a uniform mesh $\mathcal{T}_h$ of the domain $\Omega = (2, 3)$ with 5 nodes.

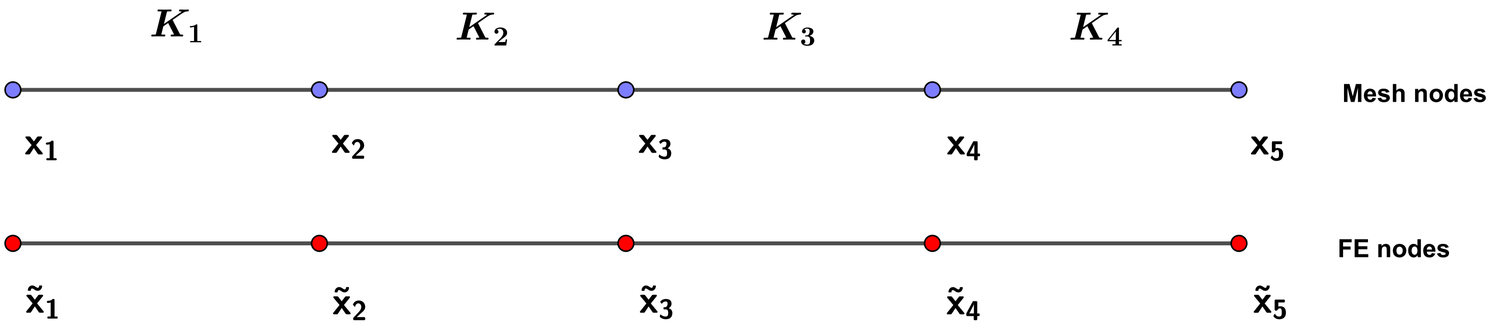
Mesh: $x_1 = 2, x_2 = 2.25, x_3 = 2.5, x_4 = 2.75, x_5 = 3$

$$\mathcal{N}_h = \{x_1, x_2, x_3, x_4, x_5\}, \ \mathcal{T}_h = \{[x_1, x_2], [x_2, x_3], [x_3, x_4], [x_4, x_5]\}$$

Linear finite element basis functions: Note that there are two local nodes for linear polynomials on each element, we can simply let

$$\mathcal{N}_{h,dof} = \{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5\} = \{x_1, x_2, x_3, x_4, x_5\} = \mathcal{N}_h$$

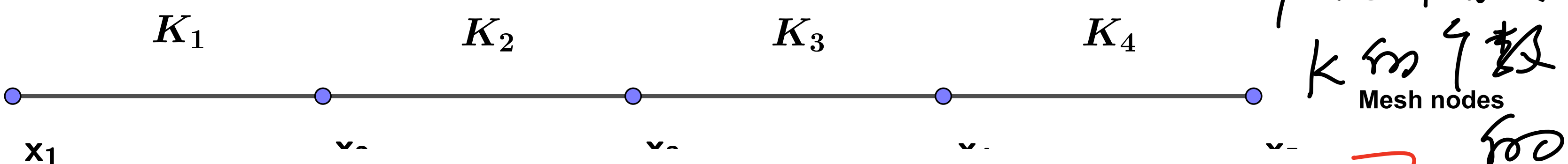
and there are 5 linear basis functions.



The relationship between linear finite element basis functions and elements in the mesh can be described by the connectivity matrix for the global degrees of freedom:

$$T_{dof} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix}$$

Let us discuss how a finite element basis function is described by the local shape functions on elements of a mesh. Recall the mesh:



中的个数是由
K的个数决定
的

$$\phi_j \in C^0(\Omega), \quad \phi_j|_K \in V_h^p(K), \ \forall K \in \mathcal{T}_h \tag{15}$$

$$\phi_j(\tilde{x}_i) = \delta_{ij}, \ \forall \tilde{x}_i \in \mathcal{N}_{h,dof} \qquad \text{(Lagrange property)} \tag{16}$$

Finite element nodes

For $\phi_1(x)$, the basis function associated with $\tilde{x}_1$: By (15) and (16),

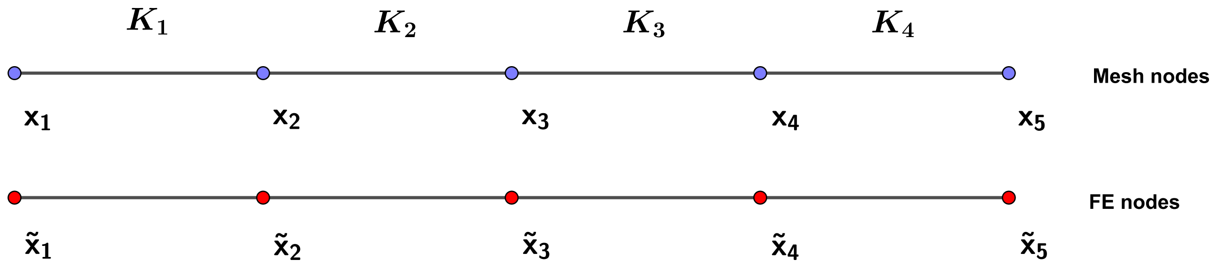
$$\phi_1(x) \in C^0([0,1]), \phi_1|_K \in \Pi_1, \forall K \in \mathcal{T}_h, \ \phi_1(\tilde{x}_1) = 1, \ \phi_1(\tilde{x}_j) = 0, j \neq 1$$

On the 1st element $K_1 = [x_1, x_2]$, $\phi_1(x)$ is a linear polynomial with $\phi_1(x_1) = 1, \phi_1(x_2) = 0$; hence, $\phi_1(x) = L^1_{K_1,1}(x)$ for $x \in K_1$. Applying similar arguments we can see that $\phi_1(x) = 0$ for $x \in K_j, j \neq 1$. Thus,

$$\phi_1(x) = \begin{cases} L^1_{K_1,1}(x), & x \in K_1 = [x_1, x_2] \\ 0, & x \in K_i, i = 2, 3, 4 \end{cases}$$

and it is easy to verify that $\phi_1(x)$ defined by the formula above is a continuous (C^0) piecewise linear polynomial such that $\phi_1(\tilde{x}_j) = \delta_{1j}, 1 \leq j \leq 5$.

Recall the mesh and the connectivity array for the degrees of freedom:



$$T_{dof} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix}$$

In fact, the formula for the global finite element basis function $\phi_1(x)$ can be derived from the mesh and T_{dof} as follows: Since the index of the basis function $\phi_1(x)$ appears in the 1st column only, the piecewise polynomial $\phi_1(x)$ should be a zero polynomial on all elements except for the 1st element.

Also, since the index of the basis function $\phi_1(x)$ appears in the 1st column and 1st row, we know that the piecewise polynomial $\phi_1(x)$ is the polynomial (shape function) $L^1_{K_1,1}(x)$ on 1st element. Hence,

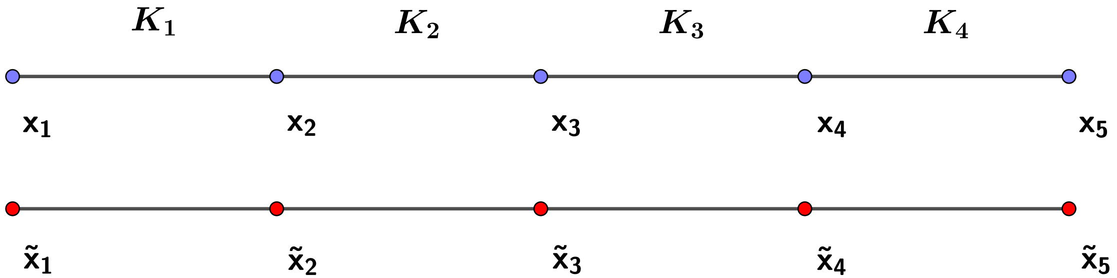
$$\phi_1(x) = \begin{cases} L^1_{K_1,1}(x), & x \in K_1 = [x_1, x_2] \\ 0, & x \in K_i, i = 2, 3, 4 \end{cases}$$

The finite element basis function $\phi_1(x)$ mathematically defined by (15) and (16) is actually defined/implemented by:

(1) a mesh.

(2) finite element nodes and T_{dof}

(3) the shape functions on elements



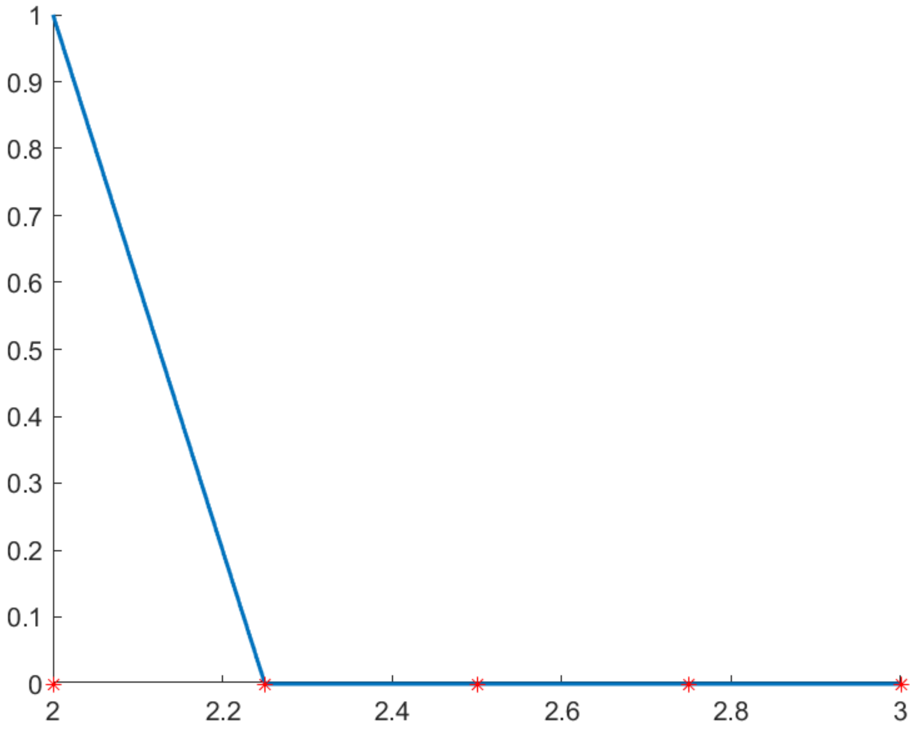
Mesh nodes

FE nodes

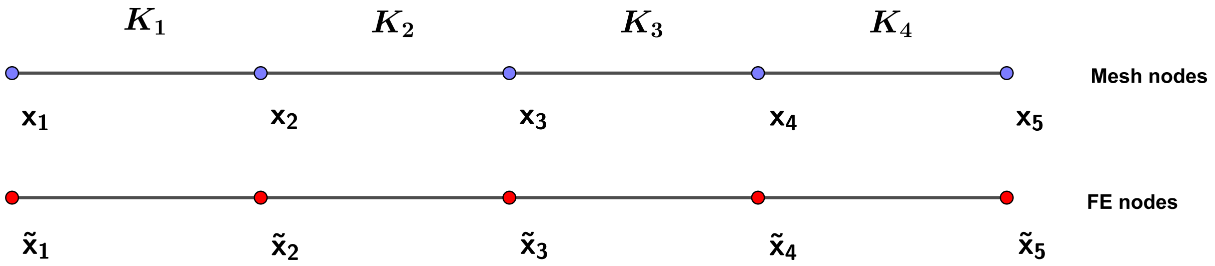
$$T_{dof} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix}$$

Basis function $\phi_1(x)$:

$$T_{dof} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix}, \qquad \phi_1(x) = \begin{cases} L^1_{K_1,1}(x), & x \in K_1 = [x_1, x_2] \\ 0, & x \in K_i, i = 2, 3, 4 \end{cases}$$



Recall the mesh and the connectivity array for the degrees of freedom:



$$T_{dof} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix}$$

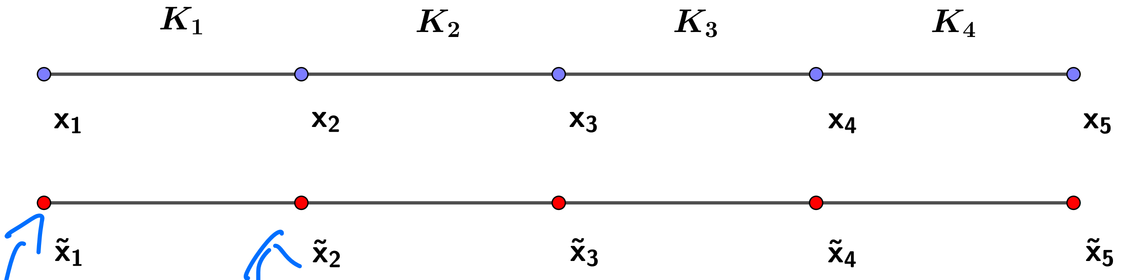
For $\phi_3(x)$, the basis function associated with $\tilde{x}_3$:

Since the index of the basis function $\phi_3(x)$ appears in the 2nd and 3rd columns only, the piecewise polynomial $\phi_3(x)$ should be a zero polynomial on all elements except for the 2nd and 3rd elements.

Also, since the index of the basis function $\phi_3(x)$ appears in the 2nd column and 2nd row, we know that the piecewise polynomial $\phi_3(x)$ is the polynomial (shape function) $L^1_{K_2,2}(x)$ on 2nd element. Similarly, $\phi_3(x)$ is the polynomial (shape function) $L^1_{K_3,1}(x)$ on 3rd element. Hence

$$\phi_3(x) = \begin{cases} L^1_{K_2,2}(x), & x \in K_2 = [x_2, x_3] \\ L^1_{K_3,1}(x), & x \in K_3 = [x_3, x_4] \\ 0, & x \in K_i, i \neq 2, 3 \end{cases}$$

Again, the finite element basis function $\phi_3(x)$ mathematically defined by (15) and (16) is defined by the mesh, T_{dof}, and the shape functions on elements.



Mesh nodes

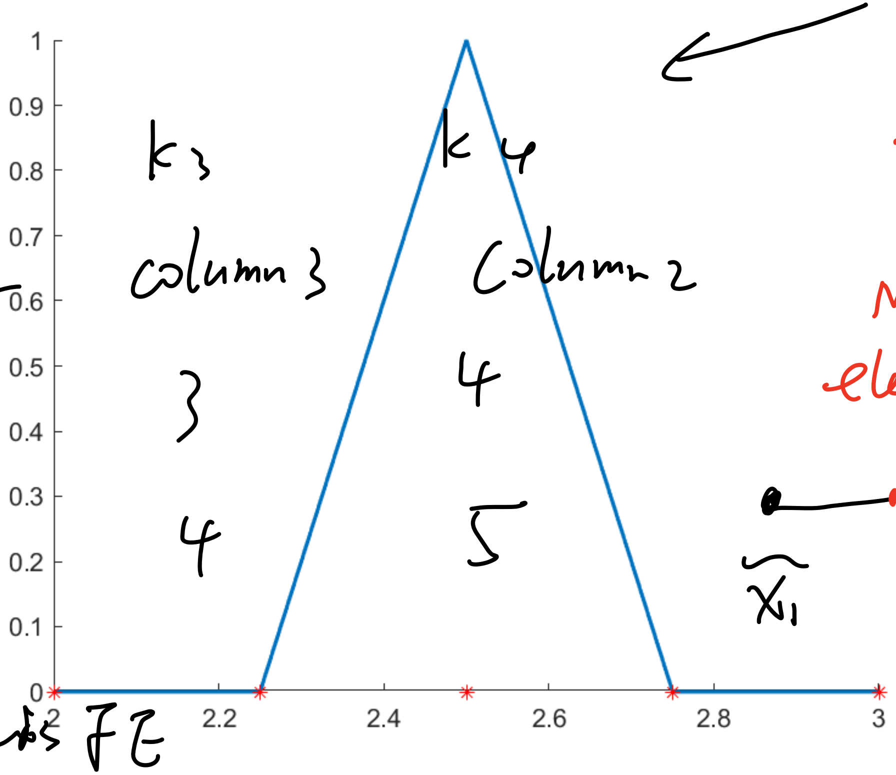
FE nodes

$$T_{dof} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix}$$

Basis function $\phi_3(x)$:

$$T_{dof} = \begin{bmatrix} 1 & 2 & \color{red}{3} & 4 \\ 2 & \color{red}{3} & 4 & 5 \end{bmatrix}, \qquad \phi_3(x) = \begin{cases} L^1_{K_2,2}(x), & x \in K_2 = [x_2, x_3] \\ L^1_{K_3,1}(x), & x \in K_3 = [x_3, x_4] \\ 0, & x \in K_i, i \neq 2, 3 \end{cases}$$

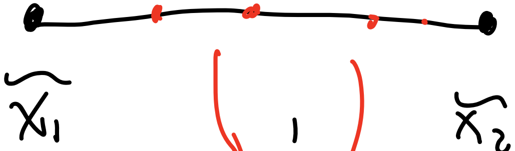
We assume the finite element is ordered.



K_1	K_2	K_3	K_4
column 1	column 2	column 3	Column 2
1	2	3	4
2	3	4	5

means the node in this FE

if there are nodes inside the element



1
⋮
2

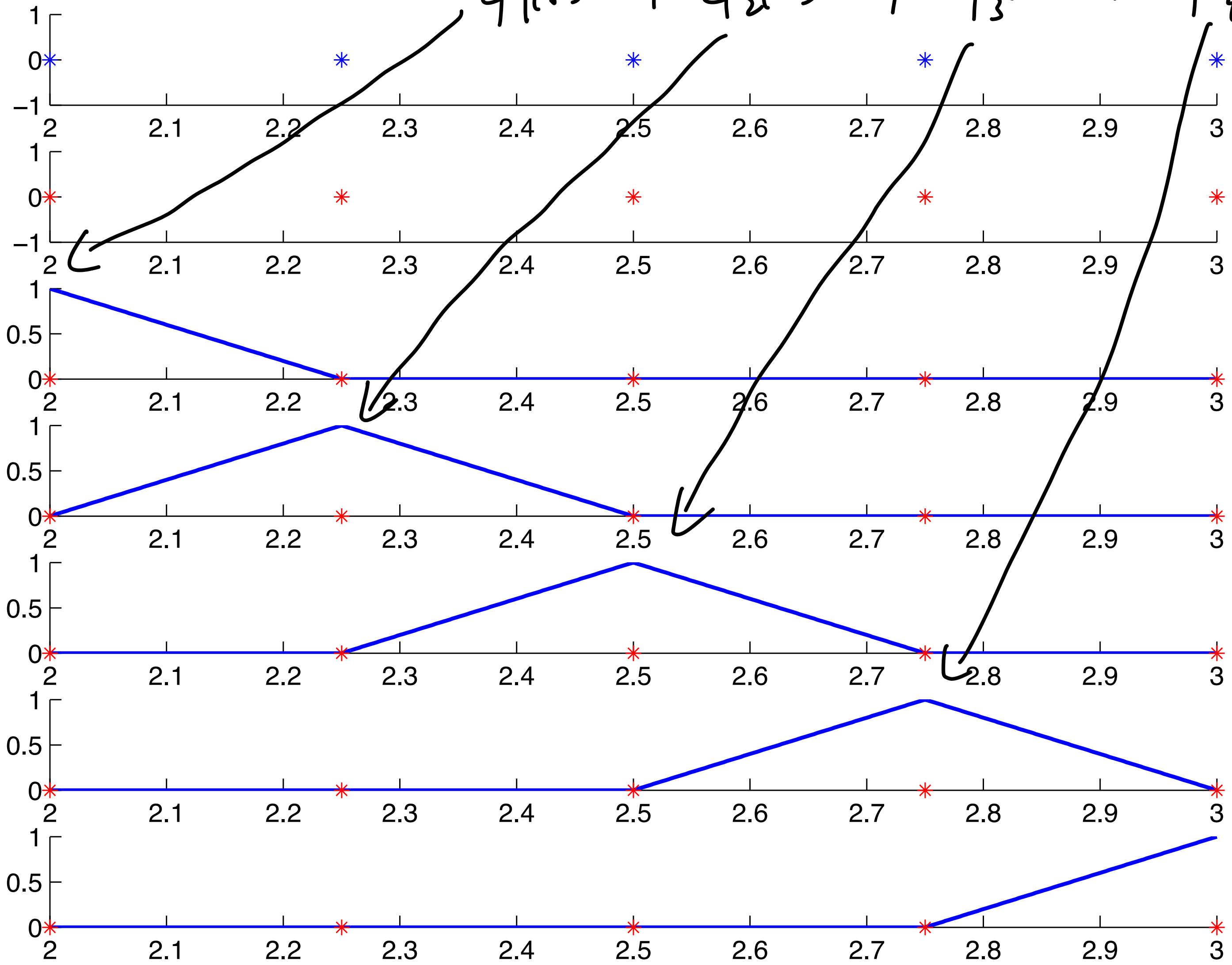
then the column would be

The linear finite element basis functions at other nodes can be described by the local shape function similarly. Putting them together, we have the following plot for all the 5 linear finite element basis functions defined on the mesh:

$$T_{dof} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix}, \qquad \phi_1(x),\ \phi_2(x),\ \phi_3(x),\ \phi_4(x),\ \phi_5(x)$$

$\phi_5(5)=1$

$\phi_1(1)=1$ $\phi_2(2)=1$ $\phi_3(3)=1$ $\phi_4(4)=1$



$P=2$

Quadratic finite element basis functions: Recall that the mesh is characterized by $x_1 = 2, x_2 = 2.25, x_3 = 2.5, x_4 = 2.75, x_5 = 3,$

$$\mathcal{N}_h = \underbrace{\{x_1, x_2, x_3, x_4, x_5\}}_{\text{Mesh node}}, \quad \mathcal{T}_h = \{[x_1, x_2], [x_2, x_3], [x_3, x_4], [x_4, x_5]\}$$

Since there are 3 local nodes for quadratic polynomials on each element, we have 9 nodes for quadratic finite element functions:

finite element node

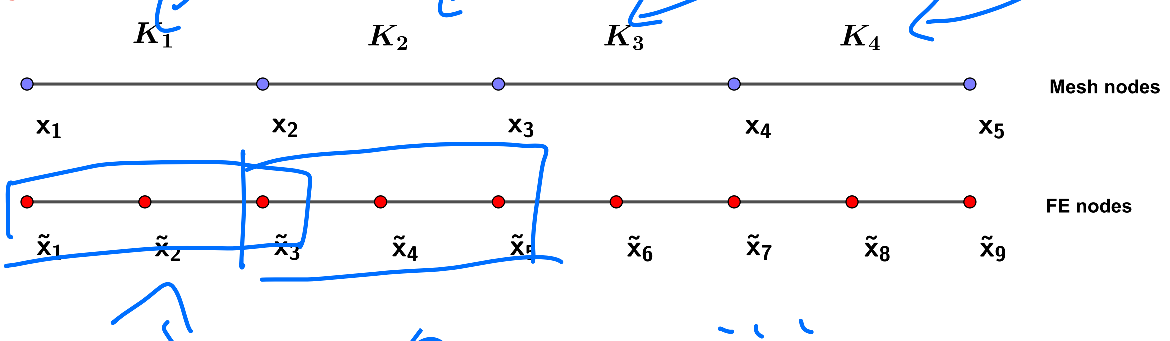
there should be p+1 nodes in every element

$$\mathcal{N}_{h,dof} = \underline{\{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7, \tilde{x}_8, \tilde{x}_9\}} \supseteq \mathcal{N}_h$$

Hence, there are 9 quadratic finite element basis functions. Without loss of generality, let

$$\tilde{x}_1 = x_1, \tilde{x}_3 = x_2, \tilde{x}_5 = x_3, \tilde{x}_7 = x_4, \tilde{x}_9 = x_5$$

$$\tilde{x}_2 = (\tilde{x}_1 + \tilde{x}_3)/2, \tilde{x}_4 = (\tilde{x}_3 + \tilde{x}_5)/2, \tilde{x}_6 = (\tilde{x}_5 + \tilde{x}_7)/2, \tilde{x}_8 = (\tilde{x}_7 + \tilde{x}_9)/2$$



The connectivity matrix for the degrees of freedom:

$$T_{dof} = \begin{array}{cccc} K_1 & K_2 & K_3 & K_4 \\ \hline 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \\ 3 & 5 & 7 & 9 \end{array}$$

Remark: Again, how to order the finite element node $\tilde{x}_i$s does not mater much, but we need to order them to form the matrix T_{dof}.

Recall

$$T_{dof} = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \\ 3 & 5 & 7 & 9 \end{bmatrix}$$

For $\phi_1(x)$, the quadratic basis function associated with $\tilde{x}_1$: Since the index of $\phi_1(x)$ appears in the first column only, this piecewise polynomial should be the zero polynomial on all elements except for the 1st element. Since the index of $\phi_1(x)$ appears in the first column and 1st row, piecewise polynomial $\phi_1(x)$ should be the polynomial (shape function) $L^2_{K1,1}(x)$. Hence,

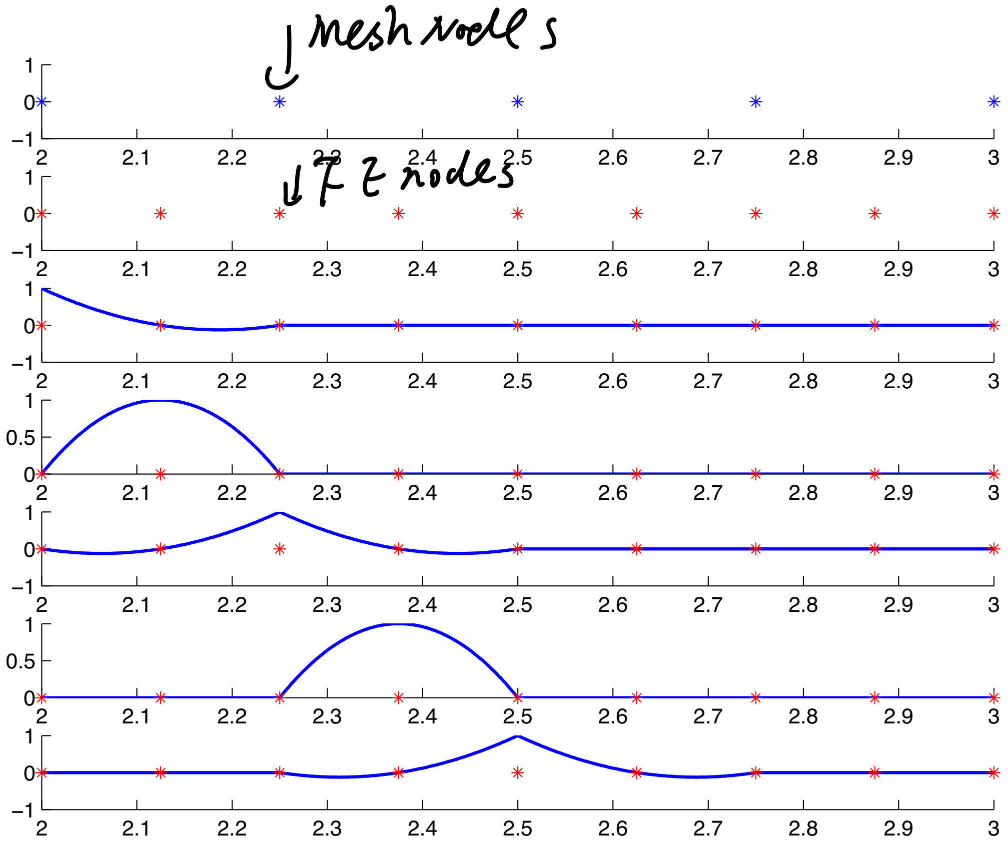
$$\phi_1(x) = \begin{cases} L^2_{K_1,1}(x), & x \in K_1 \\ 0, & x \in K_j, j \neq 1 \end{cases}$$

Similarly,

$$\phi_2(x) = \begin{cases} L^2_{K_1,2}(x), & x \in K_1 \\ 0, & x \in K_j, j \neq 1 \end{cases}$$

but

$$\phi_3(x) = \begin{cases} L^2_{K_1,3}(x), & x \in K_1 \\ L^2_{K_2,1}(x), & x \in K_2 \\ 0, & x \in K_j, j \neq 1, 2 \end{cases}$$



$$T_{dof} = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \\ 3 & 5 & 7 & 9 \end{bmatrix}$$

first, let's see $\phi_1(x)$, which has index 1

index 1 only shows up in the first column, which is actually the first element K_1

so, $\phi_1(x)$ on K_2, K_3, K_4 are all zero

number of ϕ is dete by K

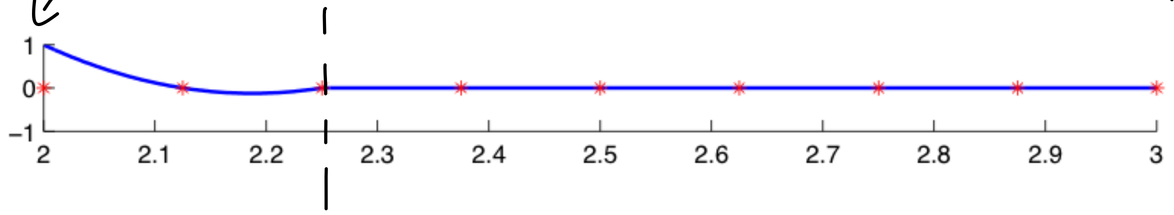
$$\phi_j \in C^0(\Omega), \quad \phi_j|_K \in V_h^p(K), \ \forall K \in \mathcal{T}_h \tag{15}$$

$$\phi_j(\tilde{x}_i) = \delta_{ij}, \ \forall \tilde{x}_i \in \mathcal{N}_{h,dof} \qquad \text{(Lagrange property)} \tag{16}$$

Finite element nodes

$\phi_1(\tilde{x}_1) = 1$, others $= 0$

should be a quadratic (3 points)

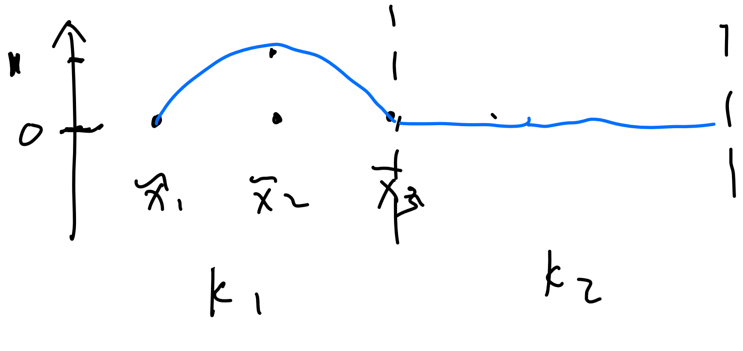


then look at $\phi_2(x)$, which is also only in column 1 (K_1)

$\phi_2(\tilde{x}_2) = 1$,

confine between K_1 and K_2

Smooth in K_1

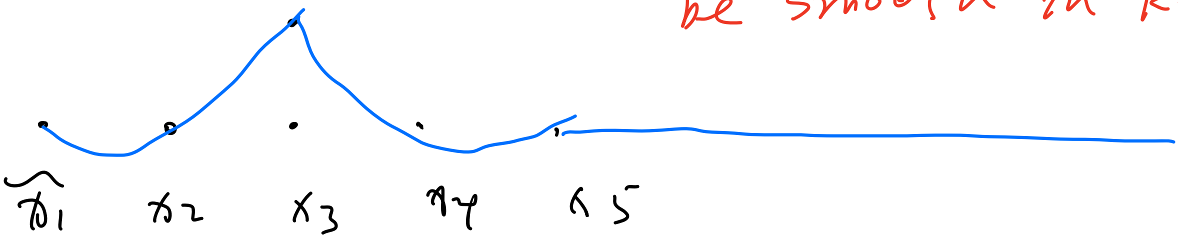


$$T_{dof} = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \\ 3 & 5 & 7 & 9 \end{bmatrix}$$

finally, we comes to $\phi_3(x)$

which shows up in K_1 and K_2

$\phi_3(\tilde{x}_3) = 1$ which means $\phi_3(x)$ should be smooth in k_1 and k_2



Tdof is generated by MATLAB codes fem

Remarks:

- In finite element computations, the two structures `mesh` and `fem` and the program for shape functions on each element are sufficient in general for describing finite element functions defined on the mesh.

- The word “finite” in finite element method is very special. Finite element methods are computational methods, programming of finite element methods is guided by the “finite” spirit. For example, the finite element functions just introduced have several finite flavors. A finite element function on the whole domain is actually described by polynomials locally on elements of a mesh. Here, consider “globally on the whole domain” as everywhere and unlimited, but “locally on an element” hints limited and finite. Also, a polynomial is determined by a finite number of coefficients. As another example, for our prototype 1D BVP, seeking a weak solution is to find a function from the space $H^1(\Omega)$ which is of infinite dimension, but computing an approximation to the weak solution by the Petrove-Galerkin method or the Galerkin method is to find a function from a finite dimensional space. Finiteness is a necessary feature of finite element computations to be carried out on computers which are ultimately finite devices.

2.3, Evaluation of finite element functions: Consider the task: given $x \in \Omega$, compute the value of a finite element function

$$u_h(x) \in V_h^p(\Omega) = span\{\phi_i, \ \forall \tilde{x}_i \in \mathcal{N}_{h,dof}\}$$

By definition, $u_h(x) = \sum_{j=1}^{|\mathcal{N}_{h,dof}|} u_j \phi_j(x)$.

In computations, a finite element function $u_h(x) = \sum_{j=1}^{|\mathcal{N}_{h,dof}|} u_j \phi_j(x)$ on a domain Ω is described by the following 4 basic components:

```
(1): function mesh = mesh_generator_1D(domain, n)
(2): function fem = fem_generator_Lagrange_1D(mesh, degree)
(3): function f = shape_fun_1D_Lagrange(x, elem, degree, ...
                                  shape_index, d_index)
```

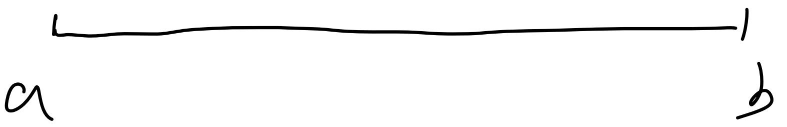
(4): $\vec{u} = (u_j)_{j=1}^{|\mathcal{N}_{h,dof}|}$ for the coefficients of the finite element functions $u_h(x)$.

We will explain that computing the value of a finite element function $u_h(x)$ for a specific x does not require an explicit implementation for the global finite element basis functions even though $u_h(x)$ is mathematically defined by them.

funtion "mesh" determines the number of finite elements , the total domain (a, b)

e.g. $\Omega = (a, b)$

$K_1 \quad K_2 \quad \cdots \quad K_m$ → determinet the n



$n+1$ = Mesh node #

then, with degree = p given

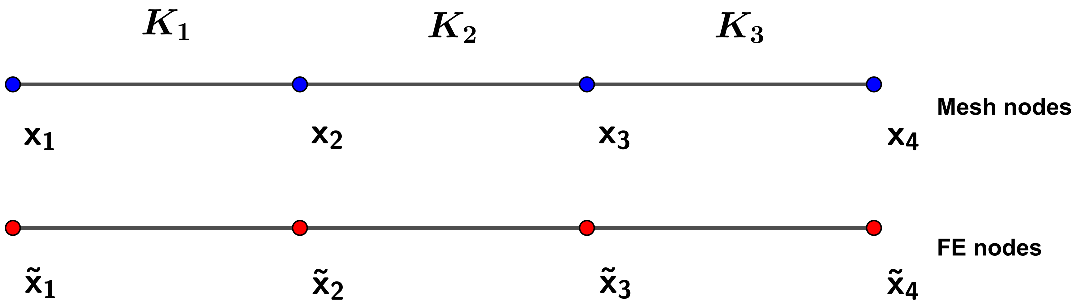
funtion fem deter the Finite element Nodes

eg. K_1 ↙ P

finite , we use funtion "f"

to obtain all the functions. ϕ

Example: Consider the evaluation of $u_h \in V_h^1(\Omega)$ formed on the following mesh of $\Omega = (2,3)$:



$$T_{dof} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

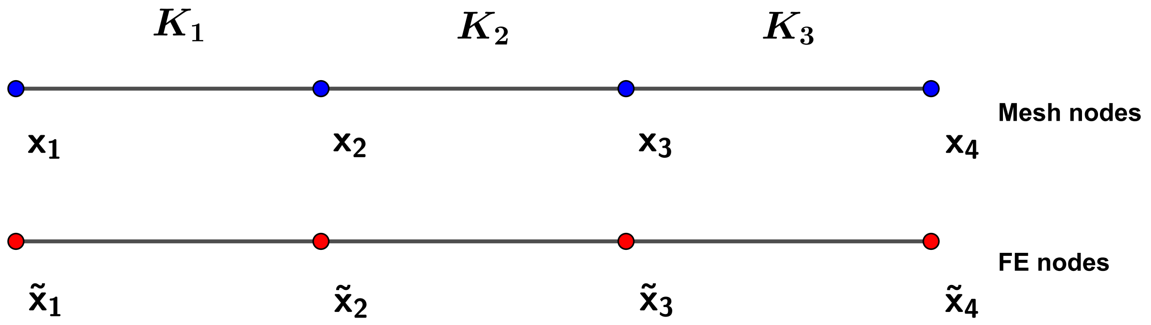
The global basis functions for $V_h^1(\Omega) = span\{\phi_1, \phi_2, \phi_3, \phi_4\}$ are:

$$\phi_1(x) = \begin{cases} L^1_{K_1,1}(x), & x \in K_1, \\ 0, & \text{otherwise} \end{cases}, \quad \phi_2(x) = \begin{cases} L^1_{K_1,2}(x), & x \in K_1, \\ L^1_{K_2,1}(x), & x \in K_2, \\ 0, & \text{otherwise} \end{cases}$$

$$\phi_3(x) = \begin{cases} L^1_{K_2,2}(x), & x \in K_2, \\ L^1_{K_3,1}(x), & x \in K_3, \\ 0, & \text{otherwise} \end{cases}, \quad \phi_4(x) = \begin{cases} 0, & \text{otherwise} \\ L^1_{K_3,2}(x), & x \in K_3, \end{cases}$$

For $x \in K_1$, mathematically, we have

$$u_h(x) \quad = \quad \sum_{i=1}^{4} u_i\phi_i(x) = u_1\phi_1(x) + u_i\phi_2(x) = u_1 L^1_{K_1,1}(x) + u_2 L^1_{K_1,2}(x)$$



$$T_{dof} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

Recall: For $x \in K_1$, mathematically, we have

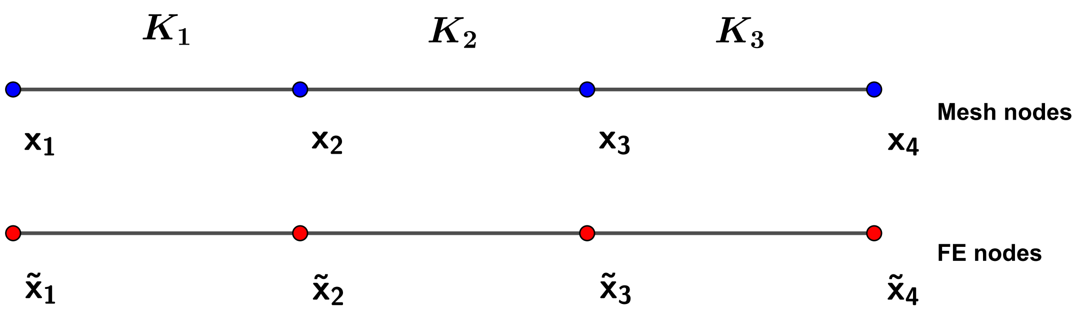
$$u_h(x) = \sum_{i=1}^{4} u_i\phi_i(x) = u_1\phi_1(x) + u_i\phi_2(x) = u_1 L_{K_1,1}^1(x) + u_2 L_{K_1,2}^1(x)$$

However, since $x \in K_1$, by

$$T_{dof} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

we use u_1, $L_{K_1,1}^1$ and u_2, $L_{K_1,2}^1$ for evaluating $u_h(x)$, and we do not even need to know how the global finite element basis functions $\phi_i, i = 1, 2, 3, 4$ are defined at all.

Recall:



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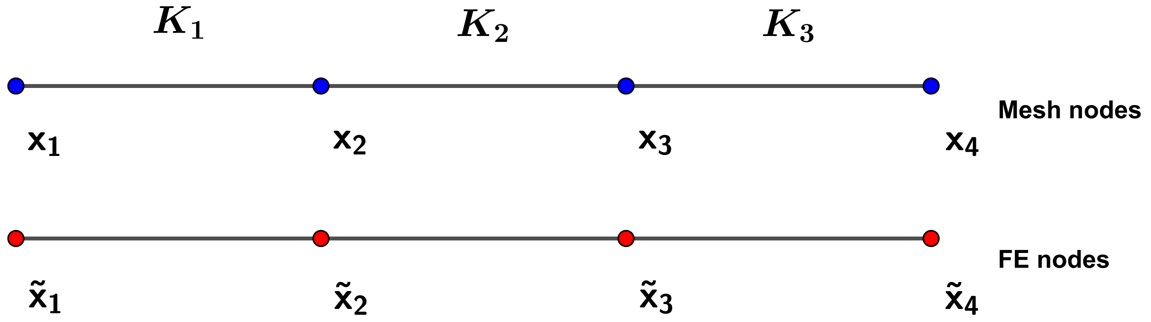
For $x \in K_2$, mathematically, we have

$$u_h(x) \;=\; \sum_{i=1}^{4} u_i \phi_i(x) = u_2 \phi_2(x) + u_3 \phi_3(x) = u_2 L^1_{K_2,1}(x) + u_3 L^1_{K_2,2}(x)$$



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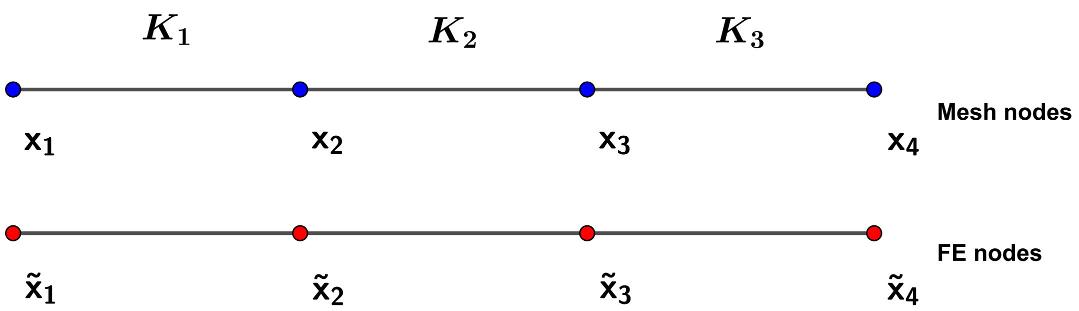

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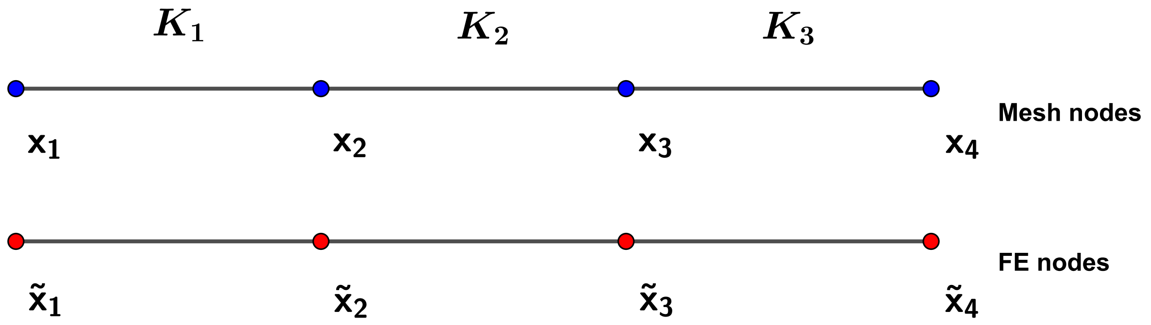
$$u_h(x) = \sum_{i=1}^{4} u_i \phi_i(x) = u_2\phi_2(x) + u_3\phi_3(x) = u_3 L^1_{K_3,1}(x) + u_4 L^1_{K_3,2}(x)$$



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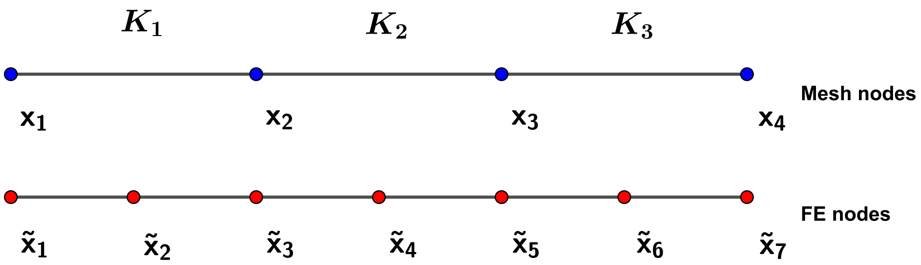
Example: Consider the evaluation of $u_h \in V_h^2(\Omega)$ formed on the following mesh of $\Omega = (2,3)$.



$$T_{dof} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$$

The global basis functions are:

$$\phi_1(x) = \begin{cases} L^2_{K_1,1}(x), & x \in K_1, \\ 0, & \text{otherwise} \end{cases} \qquad \phi_2(x) = \begin{cases} L^2_{K_1,2}(x), & x \in K_1, \\ 0, & \text{otherwise} \end{cases}$$

$$\phi_3(x) = \begin{cases} L^2_{K_1,3}(x), & x \in K_1, \\ L^2_{K_2,1}(x), & x \in K_2, \\ 0, & \text{otherwise} \end{cases} \qquad \phi_4(x) = \begin{cases} L^2_{K_2,2}(x), & x \in K_2, \\ 0, & \text{otherwise} \end{cases}$$

$$\phi_5(x) = \begin{cases} L^2_{K_2,3}(x), & x \in K_2, \\ L^2_{K_3,1}(x), & x \in K_3, \\ 0, & \text{otherwise} \end{cases} \qquad \phi_6(x) = \begin{cases} L^2_{K_3,2}(x), & x \in K_3, \\ 0, & \text{otherwise} \end{cases}$$

$$\phi_7(x) = \begin{cases} 0, & \text{otherwise} \\ L^2_{K_3,3}(x), & x \in K_3, \end{cases}$$

Recall

For $x \in K_1$:
$$\begin{aligned} u_h(x) &= \sum_{i=1}^{7} u_i \phi_i(x) = u_1\phi_1(x) + u_2\phi_2(x) + u_2\phi_3(x) \\ &= u_1 L^2_{K_1,1}(x) + u_2 L^2_{K_1,2}(x) + u_3 L^2_{K_1,3}(x) \end{aligned}$$

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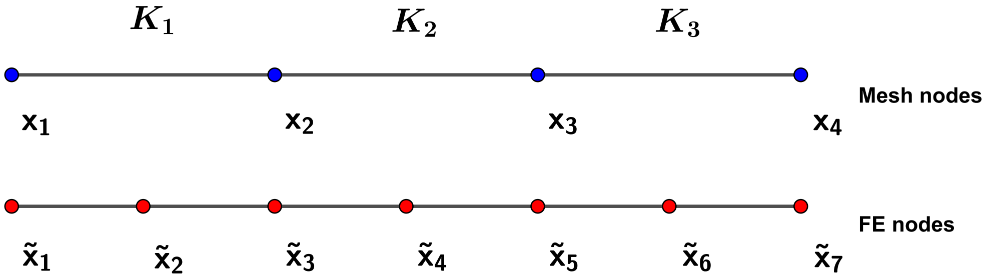

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Recall

For $x \in K_2$: $$\begin{aligned} u_h(x) &= \sum_{i=1}^{7} u_i\phi_i(x) = u_3\phi_3(x) + u_4\phi_4(x) + u_5\phi_5(x) \\ &= u_3 L^2_{K_2,1}(x) + u_4 L^2_{K_2,2}(x) + u_5 L^2_{K_2,3}(x) \end{aligned}$$

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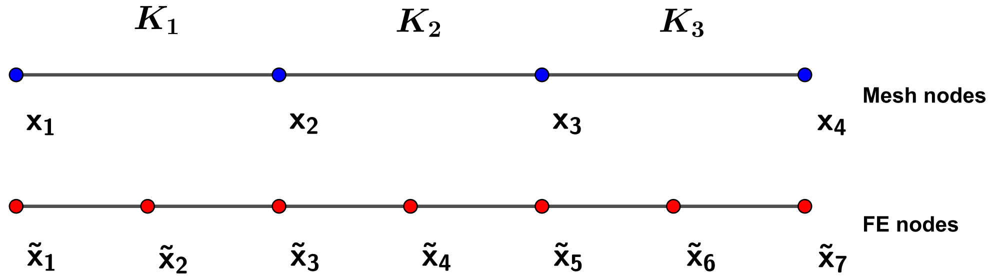

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Recall

For $x \in K_3$:

$$\begin{aligned} u_h(x) &= \sum_{i=1}^{7} u_i \phi_i(x) = u_5\phi_5(x) + u_6\phi_6(x) + u_7\phi_7(x) \\ &= u_5 L^2_{K_3,1}(x) + u_6 L^2_{K_3,2}(x) + u_7 L^2_{K_3,3}(x). \end{aligned}$$

$$T_{dof} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$$