In previous chapters, we have developed finite element tools and we have applied them to solve elliptic BVPs. However, those tools can be readily used to solve other types of partial differential equations.

In this chapter, for demonstrating this feature, we consider how to use those finite element tools to solve a boundary value problem for a system of partial equations. In particular, we will consider the system of PDEs modeling a Stokes flow.

5.1, A Stokes Boundary Value Problem: Consider a domain $\Omega \subset \mathbf{R}^2$ occupied by a steady state Stokes fluid. Let $\mathbf{u}(x, y) = (u_1(x, y), u_2(x, y))^t$ be flow velocity and let $p(x, y)$ be the pressure at a point $(x, y) \in \Omega$. Then, under certain assumptions/simplifications, $\mathbf{u}(X)$ and $p(X)$ are determined by the Stokes system:

$$
\begin{aligned}
-\nabla \cdot (\nu \nabla \mathbf{u}) + \frac{1}{\rho} \nabla p &= \mathbf{f} \text{ in } \Omega & (1) \\
\nabla \cdot \mathbf{u} &= 0 \text{ in } \Omega & (2) \\
\mathbf{u} &= \mathbf{g}_D \text{ on } \partial\Omega & (3)
\end{aligned}
$$

where ρ is the fluid density, $\nu = \mu/\rho$ is the Kinematic viscosity, μ is the (dynamic/sheer) viscosity, $\mathbf{f}(x, y) = (f_1(x, y), f_2(x, y))^t$ is the body force, $\mathbf{g}_D(x, y) = (g_{D,1}(x, y), g_{D,2}(x, y))^t$ is the specified velocity on the boundary.

For simplicity, we assume that ρ is a constant function.

Component form of the Stokes system: Counting the components in the unknowns, we can see that the Stokes system consists of three scalar partial differential equations about three scalar unknowns $u_1(X), u_2(X)$ and $p(X)$. Recall

$$-\nabla \cdot (\nu \nabla \mathbf{u}) + \frac{1}{\rho}\nabla p = \mathbf{f} \text{ in } \Omega \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \text{ in } \Omega \tag{2}$$

These partial differential equations can be written as:

$$-\nabla \cdot (\nu \nabla u_1) + \frac{1}{\rho} p_x = f_1 \tag{4}$$

$$-\nabla \cdot (\nu \nabla u_2) + \frac{1}{\rho} p_y = f_2 \tag{5}$$

$$u_{1,x} + u_{2,y} = 0 \tag{6}$$

Since p appears in the Stokes system only through its gradient, p is determined up to an arbitrary constant. The following extra condition is often used to overcome this uncertainty:

$$\int_\Omega p dX = 0. \tag{7}$$

A weak form of the Stokes system: We derive a weak form for the Stokes BVP described by (1)-(2) and (3) by its component equations (4)-(6).

Formally, multiplying a suitably smooth function v_1 to (4), integrating over Ω and applying the divergence theorem, we have

$$
\begin{aligned}
& -v_1\nabla\cdot(\nu\nabla u_1)+\frac{1}{\rho}v_1p_x=v_1f_1 \\
& -\int_\Omega v_1\left((\nu u_{1,x})_x+(\nu u_{1,y})_y\right)dX+\frac{1}{\rho}\int_\Omega v_1p_xdX=\int_\Omega v_1f_1dX \\
& -\left(-\int_\Omega(\nu v_{1,x}u_{1,x}+\nu v_{1,y}u_{1,y})dX+\int_{\partial\Omega}v_1(\nu\nabla u_1\cdot\mathbf{n})ds\right) \\
& +\frac{1}{\rho}\left(-\int_\Omega v_{1,x}pdX+\int_{\partial\Omega}v_1pn_xds\right)=\int_\Omega v_1f_1dX
\end{aligned}
$$

Since $\nabla u_1\cdot\mathbf{n}$ is not provided on $\partial\Omega$, the above equation suggests to choose $v_1\in H_0^1(\Omega)$ and this leads to the first weak equation for the Stokes system:

$$\int_\Omega(\nu v_{1,x}u_{1,x}+\nu v_{1,y}u_{1,y})dX-\int_\Omega\frac{1}{\rho}v_{1,x}pdX=\int_\Omega v_1f_1dX,\ \forall v_1\in H_0^1(\Omega) \tag{8}$$

Recall

$$\int_\Omega (\nu v_{1,x} u_{1,x} + \nu v_{1,y} u_{1,y}) dX - \int_\Omega \frac{1}{\rho} v_{1,x} p dX = \int_\Omega v_1 f_1 dX, \ \forall v_1 \in H_0^1(\Omega) \tag{8}$$

Similarly, from (5) and using $v_2 \in H_0^1(\Omega)$, we have

$$\int_\Omega (\nu v_{2,x} u_{2,x} + \nu v_{2,y} u_{2,y}) dX - \int_\Omega \frac{1}{\rho} v_{2,y} p dX = \int_\Omega v_2 f_2 dX, \ \forall v_2 \in H_0^1(\Omega) \tag{9}$$

We note that these weak equations also suggest to look for the unknown $u_1(x, y)$ and $u_2(x, y)$ from $H^1(\Omega)$. Consequently, this choice will yield $\nabla \cdot \mathbf{u} \in L^2(\Omega)$. Hence, from last equation (6) in the Stokes system, we can choose $q \in L_0^2(\Omega)$ as the test function to derive its weak form:

$$\begin{aligned} q(u_{1,x} + u_{2,y}) &= 0 \\ -\int_\Omega \left(\frac{1}{\rho} q u_{1,x} + \frac{1}{\rho} q u_{2,y} \right) dX &= 0, \ \ \forall q \in L_0^2(\Omega) \end{aligned} \tag{10}$$

where $-\frac{1}{\rho}$ is inserted to make the weak system symmetric and

$$L_0^2(\Omega) = \left\{ v \in L^2(\Omega) \mid \int_\Omega v dX = 0 \right\}$$

Recall:

The Stokes BVP:

$$-\nabla \cdot (\nu \nabla u_1) + \frac{1}{\rho} p_x = f_1 \tag{4}$$

$$-\nabla \cdot (\nu \nabla u_2) + \frac{1}{\rho} p_y = f_2 \tag{5}$$

$$u_{1,x} + u_{2,y} = 0 \tag{6}$$

$$(u_1, u_2)^t = \mathbf{u} = \mathbf{g}_D = (g_{D,1}, g_{D,2})^t \qquad \text{on } \partial\Omega \tag{3}$$

A weak formulation of the Stokes BVP: find $u_1, u_2 \in H^1(\Omega), p \in L_0^2(\Omega)$ such that $(u_1, u_2)^t|_{\partial\Omega} = \mathbf{g}_D$ and

$$\int_\Omega (\nu v_{1,x} u_{1,x} + \nu v_{1,y} u_{1,y}) dX - \int_\Omega \frac{1}{\rho} v_{1,x} p dX = \int_\Omega v_1 f_1 dX, \ \ \forall v_1 \in H_0^1(\Omega) \tag{8}$$

$$\int_\Omega (\nu v_{2,x} u_{2,x} + \nu v_{2,y} u_{2,y}) dX - \int_\Omega \frac{1}{\rho} v_{2,y} p dX = \int_\Omega v_2 f_2 dX, \ \ \forall v_2 \in H_0^1(\Omega) \tag{9}$$

$$-\int_\Omega \left(\frac{1}{\rho} q u_{1,x} + \frac{1}{\rho} q u_{2,y} \right) dX = 0, \ \ \forall q \in L_0^2(\Omega) \tag{10}$$

5.2, A finite element discretization for the Stokes BVP:

To describe the finite element method, we first introduce a mesh $\mathcal{T}_h$ on Ω, then introduce 3 C^0 finite element spaces

$$\begin{aligned}
\text{for } u_k(X): &\qquad V_h^{p_{u_k}}(\Omega) = V_h^{p_u}(\Omega) = span\left\{\phi_j^u,\ X_j \in \mathcal{N}_{h,dof}^u\right\}, \quad k = 1, 2 \\
\text{for } p(X): &\qquad V_h^{p_p}(\Omega) = span\left\{\phi_j^p,\ X_j \in \mathcal{N}_{h,dof}^p\right\}
\end{aligned}$$

where we follow a standard practice to use one finite element space for both $u_1(X)$ and $u_2(X)$, $\mathcal{N}_{h,dof}^u$ is the set of nodes of $V_h^{p_u}(\Omega)$ and $\mathcal{N}_{h,dof}^p$ and the set of nodes of $V_h^{p_p}(\Omega)$.

Because of the given boundary conditions for $u_1(X)$ and $u_2(X)$, we have the following sets of nodes for test functions in the FE equations:

$$\mathcal{N}_{h,u}^u = \{X \in \mathcal{N}_{h,dof}^u \mid X \notin \partial\Omega_D\}, \quad \mathcal{N}_{h,u}^p = \mathcal{N}_{h,dof}^p$$

The setup above leads to the following test finite element spaces:

$$\begin{aligned}
\mathbf{T}_h^{u_k} &= span\{\phi_j^u \in V_h^{p_u}(\Omega), \quad \forall X_j \in \mathcal{N}_{h,u}^u\} \subset V_h^{p_u}(\Omega), \quad k = 1, 2 \\
\mathbf{T}_h^p &= span\{\phi_j^p \in V_h^{p_p}(\Omega), \quad \forall X_j \in \mathcal{N}_{h,u}^p\} = V_h^{p_p}(\Omega)
\end{aligned}$$

and solution finite element sets:

$$\begin{aligned}
&\mathbf{S}_h^{u_k} = \left\{w_h \in V_h^{p_u}(\Omega) \mid w_h(X) = g_{D,k}(X) \ \ \forall X \in \mathcal{N}_{h,dof}^u \cap \partial\Omega_D\right\}, \quad k = 1, 2 \\
&\mathbf{S}_h^p = \mathbf{T}_h^p
\end{aligned}$$

Recall:

The test finite element spaces:

$$\mathbf{T}_h^{u_k} = span\{\phi_j^u \in V_h^{p_u}(\Omega), \quad \forall X_j \in \mathcal{N}_{h,u}^u\} \subset V_h^{p_u}(\Omega), \quad k = 1, 2$$
$$\mathbf{T}_h^{p} = span\{\phi_j^p \in V_h^{p_p}(\Omega), \quad \forall X_j \in \mathcal{N}_{h,u}^p\} = V_h^{p_p}(\Omega)$$

and solution finite element sets:

$$\mathbf{S}_h^{u_k} = \left\{ w_h \in V_h^{p_u}(\Omega) \mid w_h(X) = g_{D,k}(X) \ \ \forall X \in \mathcal{N}_{h,dof}^u \cap \partial\Omega_D \right\}, \quad k = 1, 2$$
$$\mathbf{S}_h^p = \mathbf{T}_h^p$$

Then, we need to compute three finite element functions:

$$u_{k,h}(X) = \sum_{\hat{j}=1}^{\left|\mathcal{N}_{h,dof}^u\right|} u_{k,\hat{j}}\phi_{\hat{j}}^u(X) = \sum_{X_{\hat{j}} \in \mathcal{N}_{h,u}^u} {\color{red}u_{k,\hat{j}}}\phi_{\hat{j}}^u(X) + \sum_{X_{\hat{j}} \notin \mathcal{N}_{h,u}^u} {\color{blue}u_{k,\hat{j}}}\phi_{\hat{j}}^u(X), \quad k = 1, 2$$

$$p_h(X) = \sum_{\hat{j}=1}^{\left|\mathcal{N}_{h,dof}^p\right|} p_{\hat{j}}\phi_{\hat{j}}^p(X) = \sum_{\hat{j}=1}^{\left|\mathcal{N}_{h,u}^p\right|} {\color{red}p_{\hat{j}}}\phi_{\hat{j}}^p(X)$$

or we need to compute three vectors:

$$\mathbf{u}_1 = \left({\color{red}u_{1,\hat{j}}}\right)_{\hat{j} \in \mathcal{N}_{h,u}^u}, \quad \mathbf{u}_2 = \left({\color{red}u_{2,\hat{j}}}\right)_{\hat{j} \in \mathcal{N}_{h,u}^u}, \quad \mathbf{p} = \left({\color{red}p_{\hat{j}}}\right)_{\hat{j} \in \mathcal{N}_{h,u}^p}$$

Recall the weak form of the Stokes BVP: find $u_1, u_2 \in H^1(\Omega), p \in L_0^2(\Omega)$ such that $(u_1, u_2)^t|_{\partial\Omega} = \mathbf{g}_D$ and

$$\int_\Omega (\nu v_{1,x} u_{1,x} + \nu v_{1,y} u_{1,y}) dX - \int_\Omega \frac{1}{\rho} v_{1,x} p dX = \int_\Omega v_1 f_1 dX, \ \ \forall v_1 \in H_0^1(\Omega) \tag{8}$$

$$\int_\Omega (\nu v_{2,x} u_{2,x} + \nu v_{2,y} u_{2,y}) dX - \int_\Omega \frac{1}{\rho} v_{2,y} p dX = \int_\Omega v_2 f_2 dX, \ \ \forall v_2 \in H_0^1(\Omega) \tag{9}$$

$$-\int_\Omega \left(\frac{1}{\rho} q u_{1,x} + \frac{1}{\rho} q u_{2,y} \right) dX = 0, \ \ \forall q \in L_0^2(\Omega) \tag{10}$$

From the weak equation (8) above, we have the following finite element equation for $u_{1,h}(X)$ and $p_h(X)$: $(M_{h,xx}^{u,u} + M_{h,yy}^{u,u})\mathbf{u}_1 - M_{h,x0}^{u,p}\mathbf{p} = \mathbf{f}_{1,u,h} + \mathbf{bc}_{u_1,eM_{h,xx}} + \mathbf{bc}_{u_1,eM_{h,yy}}$ where

$$\begin{aligned}
M_{h,ss}^{u,u} &= \left(\int_\Omega \nu \frac{\partial \phi_{\hat{i}}^u(X)}{\partial s} \frac{\partial \phi_{\hat{j}}^u(X)}{\partial s} dX \right)_{\hat{i},\ \hat{j} \in \mathcal{N}_{h,u}^u}, \quad s = x, y \\
M_{h,s0}^{u,p} &= \left(\int_\Omega \frac{1}{\rho} \frac{\partial \phi_{\hat{i}}^u(X)}{\partial s} \phi_{\hat{j}}^p(X) dX \right)_{\hat{i} \in \mathcal{N}_{h,u}^u,\ \hat{j} \in \mathcal{N}_{h,u}^p}, \quad s = x, y \\
\mathbf{f}_{k,u,h} &= \left(\int_\Omega \phi_{\hat{i}}^u(X) f_k(X) dX \right)_{\hat{i} \in \mathcal{N}_{h,u}^u}, \quad k = 1, 2 \\
\mathbf{bc}_{u_k,eM_{h,ss}} &= -\left(\sum_{\hat{j} \notin \mathcal{N}_{h,u}^u} \left(\int_\Omega \nu \frac{\partial \phi_{\hat{i}}^u(X)}{\partial s} \frac{\partial \phi_{\hat{j}}^u(X)}{\partial s} dX \right) u_{k,\hat{j}} \right)_{\hat{i} \in \mathcal{N}_{h,u}^u}, \quad k = 1, 2, \ s = x, y
\end{aligned}$$

From the weak equation

$$\int_\Omega (\nu v_{2,x} u_{2,x} + \nu v_{2,y} u_{2,y}) dX - \int_\Omega \frac{1}{\rho} v_{2,y} p dX = \int_\Omega v_2 f_2 dX, \ \ \forall v_2 \in H_0^1(\Omega) \tag{9}$$

we have the following finite element equation for $u_{2,h}(X)$ and $p_h(X)$:

$$\left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right)\mathbf{u}_2 - M_{h,y0}^{u,p}\mathbf{p} = \mathbf{f}_{2,u,h} + \mathbf{bc}_{u_2,eM_{h,xx}} + \mathbf{bc}_{u_2,eM_{h,yy}}$$

From the weak equation

$$-\int_\Omega \left(\frac{1}{\rho} q u_{1,x} + \frac{1}{\rho} q u_{2,y}\right) dX = 0, \ \ \forall q \in L_0^2(\Omega) \tag{10}$$

we have the following finite element equation for $u_{1,h}(X)$ and $u_{2,h}(X)$:

$$-M_{h,0x}^{p,u}\mathbf{u}_1 - M_{h,0y}^{p,u}\mathbf{u}_2 = \mathbf{bc}_{u_1,eM_{h,0x}} + \mathbf{bc}_{u_2,eM_{h,0y}}$$

where $M_{h,0s}^{p,u} = \left(M_{h,s0}^{p,u}\right)'$, $\ \ s = x, y$ and

$$\mathbf{bc}_{u_k,eM_{h,0s}} = \left(\sum_{\hat{j} \notin \mathcal{N}_{h,u}^u} \left(\int_\Omega \frac{1}{\rho} \phi_{\hat{i}}^p(X) \frac{\partial \phi_{\hat{j}}^u(X)}{\partial s} dX\right) u_{k,\hat{j}}\right)_{\hat{i} \in \mathcal{N}_{h,u}^p}, \ \ k = 1, 2, \ s = x, y$$

Weak form of the Stokes BVP: find $u_1, u_2 \in H^1(\Omega), p \in L_0^2(\Omega)$ such that $(u_1, u_2)^t|_{\partial\Omega} = \mathbf{g}_D$ and

$$\int_\Omega (\nu v_{1,x} u_{1,x} + \nu v_{1,y} u_{1,y}) dX - \int_\Omega \frac{1}{\rho} v_{1,x} p dX = \int_\Omega v_1 f_1 dX, \;\; \forall v_1 \in H_0^1(\Omega) \tag{8}$$

$$\int_\Omega (\nu v_{2,x} u_{2,x} + \nu v_{2,y} u_{2,y}) dX - \int_\Omega \frac{1}{\rho} v_{2,y} p dX = \int_\Omega v_2 f_2 dX, \;\; \forall v_2 \in H_0^1(\Omega) \tag{9}$$

$$-\int_\Omega \left(\frac{1}{\rho} q u_{1,x} + \frac{1}{\rho} q u_{2,y} \right) dX = 0, \;\; \forall q \in L_0^2(\Omega) \tag{10}$$

FE equations:

$$\begin{aligned}
&\left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right)\mathbf{u}_1 - M_{h,x0}^{u,p}\mathbf{p} = \mathbf{f}_{1,u,h} + \mathbf{bc}_{u_1,eM_{h,xx}} + \mathbf{bc}_{u_1,eM_{h,yy}} \\
&\left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right)\mathbf{u}_2 - M_{h,y0}^{u,p}\mathbf{p} = \mathbf{f}_{2,u,h} + \mathbf{bc}_{u_2,eM_{h,xx}} + \mathbf{bc}_{u_2,eM_{h,yy}} \\
&-M_{h,0x}^{p,u}\mathbf{u}_1 - M_{h,0y}^{p,u}\mathbf{u}_2 = \mathbf{bc}_{u_1,eM_{h,0x}} + \mathbf{bc}_{u_2,eM_{h,0y}}
\end{aligned}$$

Matrix form of the FE equations:

$$\begin{bmatrix} \left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right) & \mathbf{0} & -M_{h,x0}^{u,p} \\ \mathbf{0} & \left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right) & -M_{h,y0}^{u,p} \\ -M_{h,0x}^{p,u} & -M_{h,0y}^{p,u} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{p} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{f}_{1,u,h} \\ \mathbf{f}_{2,u,h} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{bc}_{u_1,eM_{h,xx}} + \mathbf{bc}_{u_1,eM_{h,yy}} \\ \mathbf{bc}_{u_2,eM_{h,xx}} + \mathbf{bc}_{u_2,eM_{h,yy}} \\ \mathbf{bc}_{u_1,eM_{h,0x}} + \mathbf{bc}_{u_2,eM_{h,0y}} \end{bmatrix} \tag{11}$$

Finite element approximation to $p(X)$ with zero mean, version 1:

Read Section 12.2.10.1 of BK2 for more details.

The algebraic system (11) is the finite element discretization of the Stokes system without any consideration on the zero mean requirement for the pressure $p(x, y)$. Hence, this finite element discretization has defects, one of which is that its matrix is singular so that solving is (11) is difficult if not impossible.

We need to discretize the zero mean condition: $\int_\Omega p(x, y) dX = 0$:

$$\int_\Omega p_h(X) dX = 0, \quad \int_\Omega \sum_{j=1}^{|\mathcal{N}_{h,dof}^p|} p_j \phi_j^p(X) dX = 0, \quad \sum_{j=1}^{|\mathcal{N}_{h,dof}^p|} p_j \int_\Omega \phi_j^p(X) dX = 0$$

$$\mathbf{1}_p^t \mathbf{p} = 0, \quad \text{with} \quad \mathbf{1}_p = \left(\int_\Omega \phi_j^p(X) dX \right)_{j=1}^{|\mathcal{N}_{h,dof}^p|} \tag{12}$$

We note that vector $\mathbf{1}_p$ can be generated the vector assembler:

```
function b = FE_vec_2D_Lagrange_tri(f_name, mesh, fem, ...
                                    d_x, d_y, n_qp)
```

```
with f_name = @(x, y) 1
```

Recall: FE equations without the zero mean consideration:

$$\begin{bmatrix} \left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right) & \mathbf{0} & -M_{h,x0}^{u,p} \\ \mathbf{0} & \left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right) & -M_{h,y0}^{u,p} \\ -M_{h,0x}^{p,u} & -M_{h,0y}^{p,u} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{1,u,h} \\ \mathbf{f}_{2,u,h} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{bc}_{u_1,eM_{h,xx}} + \mathbf{bc}_{u_1,eM_{h,yy}} \\ \mathbf{bc}_{u_2,eM_{h,xx}} + \mathbf{bc}_{u_2,eM_{h,yy}} \\ \mathbf{bc}_{u_1,eM_{h,0x}} + \mathbf{bc}_{u_2,eM_{h,0y}} \end{bmatrix} \tag{11}$$

Note the zero mean condition requires that $\mathbf{1}_p^t \mathbf{p} = 0$ which is a linear equation about $\mathbf{p}$. Putting it together with the finite element system (11) requires us to introduce an extra unknown γ to make the new algebraic system consistent in which the number of unknowns and the number of equations are the same, and this extra unknown is called a Lagrange multiplier.

FE system equations with zero mean:

$$\begin{bmatrix} \left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right) & \mathbf{0} & -M_{h,x0}^{u,p} & \mathbf{0} \\ \mathbf{0} & \left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right) & -M_{h,y0}^{u,p} & \mathbf{0} \\ -M_{h,0x}^{p,u} & -M_{h,0y}^{p,u} & \mathbf{0} & \mathbf{1_p} \\ \mathbf{0} & \mathbf{0} & \mathbf{1_p^t} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{p} \\ \gamma \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{1,u,h} \\ \mathbf{f}_{2,u,h} \\ \mathbf{0} \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{bc}_{u_1,eM_{h,xx}} + \mathbf{bc}_{u_1,eM_{h,yy}} \\ \mathbf{bc}_{u_2,eM_{h,xx}} + \mathbf{bc}_{u_2,eM_{h,yy}} \\ \mathbf{bc}_{u_1,eM_{h,0x}} + \mathbf{bc}_{u_2,eM_{h,0y}} \\ 0 \end{bmatrix} \tag{13}$$

Example: We now present a sample Matlab script to illustrate implementation issues for solving the BVP of the Stokes system. Consider the BVP: find $u(x, y)$ and $p(x, y)$ such that

$$
\begin{aligned}
&-\triangle \mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in } \Omega = (0,1)\times(0,1), \\
&\nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega, \quad \mathbf{u} = \mathbf{g}_D, \quad \text{on } \partial\Omega,
\end{aligned}
$$

with $\quad \mathbf{f}(x, y) = \begin{pmatrix} x(3xy + \cos(y)) \\ x^3 - 3y^2 + \cos(x) - \sin(y) \end{pmatrix}, \quad \mathbf{g}_D(x, y) = \begin{pmatrix} x\cos(y) \\ \cos(x) - \sin(y) \end{pmatrix}$

Prepare parameters, the solution domain, mesh and fem:

```
nu = @(x, y) 1; rhoinv = @(x, y) 1;
f1_name = @(x, y) x.*(3*x.*y + cos(y));
f2_name = @(x, y) x.^3 - 3*y.^2 + cos(x) - sin(y);
g1_D = @(x, y) x.*cos(y); g2_D = @(x, y) cos(x) - sin(y);

xmin = 0; xmax = 1; ymin = 0; ymax = 1; nx = 20; ny = 20;
mesh = mesh_generator_2D_tri(xmin, xmax, ymin, ymax, nx, ny);
[e, M_e, M_ne] = edges_in_mesh(mesh, 1, 1);
mesh.e = e; mesh.M_e = M_e; mesh.M_ne = M_ne;

degree_u = 1; degree_p = 1; % Are these good choices????
fem_p = fem_generator_Lagrange_2D_tri(mesh, degree_p);
fem_u = fem_generator_Lagrange_2D_tri(mesh, degree_u);
```

Prepare dof_u's and boundary conditions

```
B_name = @(x, y) x.*(x-xmax).*y.*(y-ymax);
B_D_name_u = @(x, y) x.*(x-xmax).*y.*(y-ymax);
B_D_name_p = @(x, y) ones(size(x)); % no essential BC for p
[dof_u_u, nt_u] = FE_dof_u_2d_tri(fem_u, B_name, B_D_name_u);
[dof_u_p, nt_p] = FE_dof_u_2d_tri(fem_p, B_name, B_D_name_p);

u1_e = zeros(length(fem_u.p), 1); % ess BC vector for u1
u2_e = zeros(length(fem_u.p), 1); % ess BC vector for u2
I = find(nt_u == 0);
u1_e(I) = g1_D(fem_u.p(1, I), fem_u.p(2, I));
u2_e(I) = g2_D(fem_u.p(1, I), fem_u.p(2, I));
```

Recall FE equations:

$$\left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right)\mathbf{u}_1 - M_{h,x0}^{u,p}\mathbf{p} = \mathbf{f}_{1,u,h} + \mathbf{bc}_{u_1,eM_{h,xx}} + \mathbf{bc}_{u_1,eM_{h,yy}}$$
$$\left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right)\mathbf{u}_2 - M_{h,y0}^{u,p}\mathbf{p} = \mathbf{f}_{2,u,h} + \mathbf{bc}_{u_2,eM_{h,xx}} + \mathbf{bc}_{u_2,eM_{h,yy}}$$
$$-M_{h,0x}^{p,u}\mathbf{u}_1 - M_{h,0y}^{p,u}\mathbf{u}_2 = \mathbf{bc}_{u_1,eM_{h,0x}} + \mathbf{bc}_{u_2,eM_{h,0y}}$$

Prepare matrices and associated vectors for boundary conditions:

```
d_x1 = 1; d_y1 = 0; d_x2 = 1; d_y2 = 0; n_qp = 7;
Mhuu_hxx =  FE_matrix_2D_Lagrange_tri(nu, mesh, ...
    fem_u, d_x1, d_y1, fem_u, d_x2, d_y2, n_qp);
d_x1 = 0; d_y1 = 1; d_x2 = 0; d_y2 = 1;
Mhuu_hyy =  FE_matrix_2D_Lagrange_tri(nu, mesh, ...
    fem_u, d_x1, d_y1, fem_u, d_x2, d_y2, n_qp);
Muu_hxx = Mhuu_hxx(dof_u_u, dof_u_u);
tmp = -Mhuu_hxx*u1_e; bc_u1_eMhxx = tmp(dof_u_u);
tmp = -Mhuu_hxx*u2_e; bc_u2_eMhxx = tmp(dof_u_u);
Muu_hyy = Mhuu_hyy(dof_u_u, dof_u_u);
tmp = -Mhuu_hyy*u1_e; bc_u1_eMhyy = tmp(dof_u_u);
tmp = -Mhuu_hyy*u2_e; bc_u2_eMhyy = tmp(dof_u_u);
```

Recall FE equations:

$$\left(M^{u,u}_{h,xx} + M^{u,u}_{h,yy}\right)\mathbf{u}_1 - M^{u,p}_{h,x0}\mathbf{p} = \mathbf{f}_{1,u,h} + \mathbf{bc}_{u_1,eM_{h,xx}} + \mathbf{bc}_{u_1,eM_{h,yy}}$$
$$\left(M^{u,u}_{h,xx} + M^{u,u}_{h,yy}\right)\mathbf{u}_2 - M^{u,p}_{h,y0}\mathbf{p} = \mathbf{f}_{2,u,h} + \mathbf{bc}_{u_2,eM_{h,xx}} + \mathbf{bc}_{u_2,eM_{h,yy}}$$
$$-M^{p,u}_{h,0x}\mathbf{u}_1 - M^{p,u}_{h,0y}\mathbf{u}_2 = \mathbf{bc}_{u_1,eM_{h,0x}} + \mathbf{bc}_{u_2,eM_{h,0y}}$$

Prepare matrices and associated vectors for boundary conditions (continued):

```
d_x1 = 1; d_y1 = 0; d_x2 = 0; d_y2 = 0;
Mhup_hx0 = FE_matrix_2D_Lagrange_tri(rhoinv, mesh, ...
        fem_u, d_x1, d_y1, ...
        fem_p, d_x2, d_y2, n_qp);
Mup_hx0 = Mhup_hx0(dof_u_u, dof_u_p);
d_x1 = 0; d_y1 = 1; d_x2 = 0; d_y2 = 0;
Mhup_hy0 = FE_matrix_2D_Lagrange_tri(rhoinv, mesh, ...
        fem_u, d_x1, d_y1, ...
        fem_p, d_x2, d_y2, n_qp);
Mup_hy0 = Mhup_hy0(dof_u_u, dof_u_p);

Mhpu_h0x = Mhup_hx0'; Mpu_h0x = Mhpu_h0x(dof_u_p, dof_u_u);
tmp = -Mhpu_h0x*u1_e; bc_u1_eMh0x = tmp(dof_u_p);
Mhpu_h0y = Mhup_hy0'; Mpu_h0y = Mhpu_h0y(dof_u_p, dof_u_u);
tmp = -Mhpu_h0y*u2_e; bc_u2_eMh0y = tmp(dof_u_p);
```

Recall FE equations:

$$
\begin{aligned}
&\left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right)\mathbf{u}_1 - M_{h,x0}^{u,p}\mathbf{p} = \mathbf{f}_{1,u,h} + \mathbf{bc}_{u_1,eM_{h,xx}} + \mathbf{bc}_{u_1,eM_{h,yy}} \\
&\left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right)\mathbf{u}_2 - M_{h,y0}^{u,p}\mathbf{p} = \mathbf{f}_{2,u,h} + \mathbf{bc}_{u_2,eM_{h,xx}} + \mathbf{bc}_{u_2,eM_{h,yy}} \\
&-M_{h,0x}^{p,u}\mathbf{u}_1 - M_{h,0y}^{p,u}\mathbf{u}_2 = \mathbf{bc}_{u_1,eM_{h,0x}} + \mathbf{bc}_{u_2,eM_{h,0y}}
\end{aligned}
$$

Prepare vectors associated with the body force term:

```
d_x = 0; d_y = 0;
f1h_h =  FE_vec_2D_Lagrange_tri(f1_name, mesh, fem_u, ...
            d_x, d_y, n_qp);
f1_h = f1h_h(dof_u_u);
f2h_h =  FE_vec_2D_Lagrange_tri(f2_name, mesh, fem_u, ...
            d_x, d_y, n_qp);
f2_h = f2h_h(dof_u_u);
```

Recall the matrix: $\begin{bmatrix} \left(M^{u,u}_{h,xx} + M^{u,u}_{h,yy}\right) & \mathbf{0} & -M^{u,p}_{h,x0} & \mathbf{0} \\ \mathbf{0} & \left(M^{u,u}_{h,xx} + M^{u,u}_{h,yy}\right) & -M^{u,p}_{h,y0} & \mathbf{0} \\ -M^{p,u}_{h,0x} & -M^{p,u}_{h,0y} & \mathbf{0} & \mathbf{1_p} \\ \mathbf{0^t} & \mathbf{0^t} & \mathbf{1_p^t} & \mathbf{0} \end{bmatrix}$

Form the matrix for the FE system:

```
nu = 1; rho = 1;
SP0_u = sparse(length(dof_u_u), length(dof_u_u));
SP0_p = sparse(length(dof_u_p), length(dof_u_p));

M_tmp = [(Muu_hxx + Muu_hyy), SP0_u, -Mup_hx0; ...
        SP0_u, (Muu_hxx + Muu_hyy), -Mup_hy0; ...
        -Mpu_h0x, -Mpu_h0y, SP0_p ];

f_name = @(x, y) 1;
fh_h =  FE_vec_2D_Lagrange_tri(f_name, mesh, ...
            fem_p, 0, 0, n_qp);
one_vec = fh_h(dof_u_p);

tmp_vec = [zeros(2*length(dof_u_u), 1); one_vec];

M = [M_tmp, tmp_vec;
     tmp_vec', 0];
```

Recall FE system equations with zero mean:

$$\begin{bmatrix} \left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right) & \mathbf{0} & -M_{h,x0}^{u,p} & \mathbf{0} \\ \mathbf{0} & \left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right) & -M_{h,y0}^{u,p} & \mathbf{0} \\ -M_{h,0x}^{p,u} & -M_{h,0y}^{p,u} & \mathbf{0} & \mathbf{1_p} \\ \mathbf{0} & \mathbf{0} & \mathbf{1_p^t} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{p} \\ \gamma \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{f}_{1,u,h} \\ \mathbf{f}_{2,u,h} \\ \mathbf{0} \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{bc}_{u_1,eM_{h,xx}} + \mathbf{bc}_{u_1,eM_{h,yy}} \\ \mathbf{bc}_{u_2,eM_{h,xx}} + \mathbf{bc}_{u_2,eM_{h,yy}} \\ \mathbf{bc}_{u_1,eM_{h,0x}} + \mathbf{bc}_{u_2,eM_{h,0y}} \\ 0 \end{bmatrix} \tag{13}$$

Form the vector in the FE system:

```
rhstmp1 = [f1_h; ...
           f2_h; ...
           zeros(length(dof_u_p), 1); ...
           0];
rhstmp2 = [bc_u1_eMhxx + bc_u1_eMhyy; ...
           bc_u2_eMhxx + bc_u2_eMhyy; ...
           -bc_u1_eMh0x - bc_u2_eMh0y; ...
           0];
rhs = rhstmp1 + rhstmp2;
```

Solve the FE system:

```
u1u2p = M\rhs;
```

However, Matlab fails to produce a solution for the FE system! It shows the following error message:

```
Warning: Matrix is singular to working precision.
```

If we check the size and rank of the matrix, we obtain the following data:

```
>> rank(full(M))
ans = 1157
>> size(M)
ans = 1164        1164
```

This rank deficiency means the matrix is singular. WHY???

Something wrong with our codes? No!!! Did we make any mistakes in deriving the finite element system? No!!!

5.3, Brezzi Theorem and Inf-Sup/LBB Condition

Read Sec. 6.1.1, 6.2 in BK1, Sec. 12.2.4 in BK2 for more details.

Recall the weak formulation of the Stokes Problem: find $u_1, u_1 \in H^1(\Omega), p \in L_0^2(\Omega)$ such that $(u_1, u_2)^t|_{\partial\Omega} = \mathbf{g}_D$ and

$$\int_\Omega (\nu v_{1,x} u_{1,x} + \nu v_{1,y} u_{1,y}) dX - \int_\Omega \frac{1}{\rho} v_{1,x} p dX = \int_\Omega v_1 f_1 dX, \ \ \forall v_1 \in H_0^1(\Omega) \tag{8}$$

$$\int_\Omega (\nu v_{2,x} u_{2,x} + \nu v_{2,y} u_{2,y}) dX - \int_\Omega \frac{1}{\rho} v_{2,y} p dX = \int_\Omega v_2 f_2 dX, \ \ \forall v_2 \in H_0^1(\Omega) \tag{9}$$

$$-\int_\Omega \left(\frac{1}{\rho} q u_{1,x} + \frac{1}{\rho} q u_{2,y} \right) dX = 0, \ \ \forall q \in L_0^2(\Omega) \tag{10}$$

A more compact weak form: find $\mathbf{u} = (u_1, u_2) \in (H^1(\Omega))^2$ and $p \in L_0^2(\Omega)$ such that $\mathbf{u}|_{\partial\Omega} = \mathbf{g}_D$ and

$$a(\mathbf{v}, \mathbf{u}) + b(\mathbf{v}, p) = l(\mathbf{v}), \ \ \mathbf{v} = (v_1, v_2) \in (H_0^1(\Omega))^2 = V \tag{14}$$

$$b(\mathbf{u}, q) = 0, \qquad q \in L_0^2(\Omega) = Q \tag{15}$$

where

$$a(\mathbf{v}, \mathbf{u}) = \int_\Omega (\nu v_{1,x} u_{1,x} + \nu v_{1,y} u_{1,y} + \nu v_{2,x} u_{2,x} + \nu v_{2,y} u_{2,y}) dX, \ \ \forall \mathbf{u}, \mathbf{v} \in V \tag{16}$$

$$b(\mathbf{v}, q) = -\int_\Omega \frac{1}{\rho} q (v_{1,x} + v_{2,y}) dX, \ \ \forall \mathbf{v} \in (H^1(\Omega))^2, \ q \in Q \tag{17}$$

$$l(\mathbf{v}) = \int_\Omega v_1 f_1 dX + \int_\Omega v_2 f_2 dX, \ \ \forall \mathbf{v} \in (H^1(\Omega))^2 \tag{18}$$

Recall the weak problem: find $\mathbf{u} = (u_1, u_2) \in (H^1(\Omega))^2$ and $p \in L_0^2(\Omega)$ such that $\mathbf{u}|_{\partial\Omega} = \mathbf{g}_D$ and

$$a(\mathbf{v}, \mathbf{u}) + b(\mathbf{v}, p) = l(\mathbf{v}), \quad \mathbf{v} \in (H_0^1(\Omega))^2 = V \tag{14}$$

$$b(\mathbf{u}, q) = 0, \qquad q \in L_0^1(\Omega) = Q \tag{15}$$

where
$$a(\mathbf{v}, \mathbf{u}) = \int_\Omega (\nu v_{1,x} u_{1,x} + \nu v_{1,y} u_{1,y} + \nu v_{2,x} u_{2,x} + \nu v_{2,y} u_{2,y}) dX, \quad \forall \mathbf{u}, \mathbf{v} \in V \tag{16}$$

$$b(\mathbf{v}, q) = -\int_\Omega \frac{1}{\rho} q(v_{1,x} + v_{2,y}) dX, \quad \forall \mathbf{v} \in (H^1(\Omega))^2, \ q \in Q \tag{17}$$

$$l(\mathbf{v}) = \int_\Omega v_1 f_1 dX + \int_\Omega v_2 f_2 dX, \quad \forall \mathbf{v} \in (H^1(\Omega))^2 \tag{18}$$

Now, let $\tilde{V} = V \times Q$ and for $\mathbf{U} = (\mathbf{u}, p), \mathbf{V} = (\mathbf{v}, q) \in \tilde{V}$, we let

$$B(\mathbf{V}, \mathbf{U}) = a(\mathbf{v}, \mathbf{u}) + b(\mathbf{v}, p) + b(\mathbf{u}, q), \quad L(\mathbf{V}) = l(\mathbf{v})$$

An alternative weak form: find $\mathbf{U} \in \tilde{V}$ such that

$$B(\mathbf{V}, \mathbf{U}) = L(\mathbf{V}), \quad \forall\ \mathbf{V} \in \tilde{V}$$

We can show that $B(\cdot, \cdot)$ is a continuous bilinear form on $\tilde{V} \times \tilde{V}$ and $L(\cdot)$ is continuous linear form on $\tilde{V}$. However, $B(\cdot, \cdot)$ is not coercive on $\tilde{V}$ (???). Hence, we cannot apply the L-M theorem for the existence and uniqueness of this weak problem.

Does the weak Stokes problem have a solution?

The weak formulation above leads to the following abstract Saddle Point Problem: find $\mathbf{u} \in V$ and $p \in Q$ such that

$$a(\mathbf{v}, \mathbf{u}) + b(\mathbf{v}, p) = l(\mathbf{v}), \quad \mathbf{v} \in V \tag{19}$$

$$b(\mathbf{u}, q) = 0, \qquad q \in Q \tag{20}$$

Here, V and Q be two Hilbert spaces, and let $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ be continuous bilinear forms on $V \times V$ and $V \times Q$, respectively, and $l(\cdot)$ is a continuous linear form on V.

The kernel space of $b(\cdot,\cdot)$: $Z = \{\mathbf{v} \in V \mid b(\mathbf{v}, q) = 0, \ \forall q \in Q\}$.

Theorem 5.1

(Brezzi, Theorem 6.4 in BK1, Theorem 12.2 in BK2) If $a(\cdot,\cdot)$ is coercive on Z and if $b(\cdot,\cdot)$ satisfies the Inf-Sup/BB/LBB condition: there exist a constant $C_{inf\_sup} > 0$ such that

$$C_{inf\_sup} \|q\|_Q \leq \sup_{\mathbf{v} \in V} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_V} \quad \forall q \in Q \tag{21}$$

then the saddle point problem described by (19) and (20) has a unique solution $(\mathbf{u}, p) \in V \times Q$.

Recall the Inf-Sup/BB/LBB condition:

$$C_{inf\_sup} \|q\|_Q \leq \sup_{\mathbf{v} \in V} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_V} \quad \forall q \in Q \tag{21}$$

Inf-Sup: The inequality (21) is called the inf-sup condition because it can also be written as

$$C_{inf\_sup} \leq \inf_{q \in Q} \sup_{\mathbf{v} \in V} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_V \|q\|_Q} \tag{22}$$

To apply the Brezzi theorem to the weak Stokes problem described by (14) and (15), we can first show that the related bilinear form $a(\cdot,\cdot)$ defined by (16) is continuous and is coercive on Z. We can also show that the related bilinear form $b(\cdot,\cdot)$ defined by (17) can satisfy the inf-sup condition, i.e., there exists a constant $C_{inf\_sup} > 0$ such that the bilinear form $b(\cdot,\cdot)$ defined by (17) satisfies the following inequality:

$$C_{inf\_sup} \|q\|_{L^2(\Omega)} \le \sup_{\mathbf{v}\in(H_0^1(\Omega))^2} \frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_{(H^1(\Omega))^2}} \quad \forall q \in L_0^2(\Omega). \tag{23}$$

Furthermore, we can extend $\mathbf{g}_D$ from $\partial\Omega$ to Ω and write $\mathbf{u} = \mathbf{g}_d + \mathbf{u}_0$ with $\mathbf{u}_0 \in (H_0^1(\Omega))^2$. Then, the weak problem for $\mathbf{u}$ described by (14) and (15) is reduced to a weak problem for u_0: find $\mathbf{u}_0 \in (H_0^1(\Omega))^2$ and $p \in L_0^2(\Omega)$ such that

$$a(\mathbf{v},\mathbf{u}_0) + b(\mathbf{v},p) = l(\mathbf{v}) - a(\mathbf{v},\mathbf{g}_d), \quad \mathbf{v} \in (H_0^1(\Omega))^2 \tag{24}$$

$$b(\mathbf{u}_0,q) = 0, \quad q \in L_0^2(\Omega) \tag{25}$$

Then applying the Brezzi theorem to this modified weak Stokes problem, we know that it has a unique solution; hence, the original weak Stokes problem (14) and (15) has a unique solution.

Furthermore, it turns out that the Stokes BVP requires a careful choice of the finite element space pair $V_h^{p_u}(\Omega)$-$V_h^{p_p}(\Omega)$; otherwise, the matrix in the finite element system cannot be guaranteed to be nonsingular. A sufficient condition for a good pair of finite element spaces is such that the bilinear forms satisfy the conditions requires by the Brezzi theorem on these finite element spaces. For example, the corresponding Ladyženskaja-Babuška-Brezzi (LBB or the discrete inf-sup) condition is: there exists a constant $\tilde{C}_{inf\_sup} > 0$ such that

$$\tilde{C}_{inf\_sup} \|q_h\|_{L^2(\Omega)} \leq \sup_{\mathbf{v}_h \in (\mathbf{T}_h^u)^2} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{(H^1(\Omega))^2}}, \ \forall q_h \in L_0^2(\Omega) \cap \mathbf{T}_h^p \tag{26}$$

where $(\mathbf{T}_h^u)^2 \subset \left(V_h^{p_u}(\Omega)\right)^2$.

Extensive research on the finite element methods for the Stokes BVP has produced finite element pairs that can satisfy the discrete inf-sup condition.

The Taylor-Hood finite element method: Use a C^0 finite element space pair $V_h^{p_u}(\Omega)$-$V_h^{p_p}(\Omega)$ such that

$$p_u = p_p + 1, \quad \text{with} \quad p_p \geq 1$$

We note that we have used a finite element space pair $V_h^{p_u}(\Omega) = V_h^1(\Omega)$ and $V_h^{p_p}(\Omega) = V_h^1(\Omega)$ which is not in the Taylor-Hood configuration.

Example: Use the Taylor-Hood finite element method to solve the following Stokes BVP: find $u(x, y)$ and $p(x, y)$ such that

$$-\triangle \mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in } \Omega = (0,1) \times (0,1),$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega, \quad \mathbf{u} = \mathbf{g}_D, \quad \text{on } \partial\Omega,$$

with $$\mathbf{f}(x,y) = \begin{pmatrix} x(3xy + \cos(y)) \\ x^3 - 3y^2 + \cos(x) - \sin(y) \end{pmatrix}, \quad \mathbf{g}_D(x,y) = \begin{pmatrix} x\cos(y) \\ \cos(x) - \sin(y) \end{pmatrix}$$

We use exactly the same Matlab script except `degree_u = 2 degree_p = 1`

Prepare the solution domain, mesh and fem:

```
nu = @(x, y) 1; rhoinv = @(x, y) 1;
f1_name = @(x, y) x.*(3*x.*y + cos(y));
f2_name = @(x, y) x.^3 - 3*y.^2 + cos(x) - sin(y);
g1_D = @(x, y) x.*cos(y); g2_D = @(x, y) cos(x) - sin(y);

xmin = 0; xmax = 1; ymin = 0; ymax = 1; nx = 20; ny = 20;
mesh = mesh_generator_2D_tri(xmin, xmax, ymin, ymax, nx, ny);
[e, M_e, M_ne] = edges_in_mesh(mesh, 1, 1);
mesh.e = e; mesh.M_e = M_e; mesh.M_ne = M_ne;


degree_u = 2; degree_p = 1;

fem_p = fem_generator_Lagrange_2D_tri(mesh, degree_p);
fem_u = fem_generator_Lagrange_2D_tri(mesh, degree_u);
```

Prepare dof_u and boundary conditions

```
B_name = @(x, y) x.*(x-xmax).*y.*(y-ymax);
B_D_name_u = @(x, y) x.*(x-xmax).*y.*(y-ymax);
B_D_name_p = @(x, y) ones(size(x)); % no essential BC for p
[dof_u_u, nt_u] = FE_dof_u_2d_tri(fem_u, B_name, B_D_name_u);
[dof_u_p, nt_p] = FE_dof_u_2d_tri(fem_p, B_name, B_D_name_p);

u1_e = zeros(length(fem_u.p), 1); % ess BC vector for u1
u2_e = zeros(length(fem_u.p), 1); % ess BC vector for u2
I = find(nt_u == 0);
u1_e(I) = g1_D(fem_u.p(1, I), fem_u.p(2, I));
u2_e(I) = g2_D(fem_u.p(1, I), fem_u.p(2, I));
```

Prepare matrices and associated vectors for boundary conditions: Recall FE equations:

$$
\begin{aligned}
&\left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right)\mathbf{u}_1 - M_{h,x0}^{u,p}\mathbf{p} = \mathbf{f}_{1,u,h} + \mathbf{bc}_{u_1,eM_{h,xx}} + \mathbf{bc}_{u_1,eM_{h,yy}} \\
&\left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right)\mathbf{u}_2 - M_{h,y0}^{u,p}\mathbf{p} = \mathbf{f}_{2,u,h} + \mathbf{bc}_{u_2,eM_{h,xx}} + \mathbf{bc}_{u_2,eM_{h,yy}} \\
&-M_{h,0x}^{p,u}\mathbf{u}_1 - M_{h,0y}^{p,u}\mathbf{u}_2 = \mathbf{bc}_{u_1,eM_{h,0x}} + \mathbf{bc}_{u_2,eM_{h,0y}}
\end{aligned}
$$

Prepare matrices and associated vectors for boundary conditions:

```
d_x1 = 1; d_y1 = 0; d_x2 = 1; d_y2 = 0; n_qp = 7;
Mhuu_hxx = FE_matrix_2D_Lagrange_tri(nu, mesh, ...
    fem_u, d_x1, d_y1, fem_u, d_x2, d_y2, n_qp);
d_x1 = 0; d_y1 = 1; d_x2 = 0; d_y2 = 1;
Mhuu_hyy = FE_matrix_2D_Lagrange_tri(nu, mesh, ...
    fem_u, d_x1, d_y1, fem_u, d_x2, d_y2, n_qp);

Muu_hxx = Mhuu_hxx(dof_u_u, dof_u_u);
tmp = -Mhuu_hxx*u1_e; bc_u1_eMhxx = tmp(dof_u_u);
tmp = -Mhuu_hxx*u2_e; bc_u2_eMhxx = tmp(dof_u_u);
Muu_hyy = Mhuu_hyy(dof_u_u, dof_u_u);
tmp = -Mhuu_hyy*u1_e; bc_u1_eMhyy = tmp(dof_u_u);
tmp = -Mhuu_hyy*u2_e; bc_u2_eMhyy = tmp(dof_u_u);
```

Recall FE equations:

$$
\begin{aligned}
&\left(M_{h,xx}^{u,u}+M_{h,yy}^{u,u}\right)\mathbf{u}_1 - M_{h,x0}^{u,p}\mathbf{p} = \mathbf{f}_{1,u,h} + \mathbf{bc}_{u_1,eM_{h,xx}} + \mathbf{bc}_{u_1,eM_{h,yy}}\\
&\left(M_{h,xx}^{u,u}+M_{h,yy}^{u,u}\right)\mathbf{u}_2 - M_{h,y0}^{u,p}\mathbf{p} = \mathbf{f}_{2,u,h} + \mathbf{bc}_{u_2,eM_{h,xx}} + \mathbf{bc}_{u_2,eM_{h,yy}}\\
&-M_{h,0x}^{p,u}\mathbf{u}_1 - M_{h,0y}^{p,u}\mathbf{u}_2 = \mathbf{bc}_{u_1,eM_{h,0x}} + \mathbf{bc}_{u_2,eM_{h,0y}}
\end{aligned}
$$

Prepare matrices and associated vectors for boundary conditions (continued):

```
d_x1 = 1; d_y1 = 0; d_x2 = 0; d_y2 = 0;
Mhup_hx0 = FE_matrix_2D_Lagrange_tri(rhoinv, mesh, ...
        fem_u, d_x1, d_y1, ...
        fem_p, d_x2, d_y2, n_qp);
Mup_hx0 = Mhup_hx0(dof_u_u, dof_u_p);
d_x1 = 0; d_y1 = 1; d_x2 = 0; d_y2 = 0;
Mhup_hy0 = FE_matrix_2D_Lagrange_tri(rhoinv, mesh, ...
        fem_u, d_x1, d_y1, ...
        fem_p, d_x2, d_y2, n_qp);
Mup_hy0 = Mhup_hy0(dof_u_u, dof_u_p);

Mhpu_h0x = Mhup_hx0'; Mpu_h0x = Mhpu_h0x(dof_u_p, dof_u_u);
tmp = -Mhpu_h0x*u1_e; bc_u1_eMh0x = tmp(dof_u_p);
Mhpu_h0y = Mhup_hy0'; Mpu_h0y = Mhpu_h0y(dof_u_p, dof_u_u);
tmp = -Mhpu_h0y*u2_e; bc_u2_eMh0y = tmp(dof_u_p);
```

Recall FE equations:

$$\begin{aligned}
&\left(M_{h,xx}^{u,u}+M_{h,yy}^{u,u}\right)\mathbf{u}_1-M_{h,x0}^{u,p}\mathbf{p}=\mathbf{f}_{1,u,h}+\mathbf{bc}_{u_1,eM_{h,xx}}+\mathbf{bc}_{u_1,eM_{h,yy}}\\
&\left(M_{h,xx}^{u,u}+M_{h,yy}^{u,u}\right)\mathbf{u}_2-M_{h,y0}^{u,p}\mathbf{p}=\mathbf{f}_{2,u,h}+\mathbf{bc}_{u_2,eM_{h,xx}}+\mathbf{bc}_{u_2,eM_{h,yy}}\\
&-M_{h,0x}^{p,u}\mathbf{u}_1-M_{h,0y}^{p,u}\mathbf{u}_2=\mathbf{bc}_{u_1,eM_{h,0x}}+\mathbf{bc}_{u_2,eM_{h,0y}}
\end{aligned}$$

Prepare vectors associated with the forcing term:

```
d_x = 0; d_y = 0;
f1h_h = FE_vec_2D_Lagrange_tri(f1_name, mesh, fem_u, ...
            d_x, d_y, n_qp);
f1_h = f1h_h(dof_u_u);
f2h_h = FE_vec_2D_Lagrange_tri(f2_name, mesh, fem_u, ...
            d_x, d_y, n_qp);
f2_h = f2h_h(dof_u_u);
```

Recall the matrix: $$\begin{bmatrix} \left(M^{u,u}_{h,xx} + M^{u,u}_{h,yy}\right) & \mathbf{0} & -M^{u,p}_{h,x0} & \mathbf{0} \\ \mathbf{0} & \left(M^{u,u}_{h,xx} + M^{u,u}_{h,yy}\right) & -M^{u,p}_{h,y0} & \mathbf{0} \\ -M^{p,u}_{h,0x} & -M^{p,u}_{h,0y} & \mathbf{0} & \mathbf{1_p} \\ \mathbf{0^t} & \mathbf{0^t} & \mathbf{1_p^t} & \mathbf{0} \end{bmatrix}$$

Form the matrix for the FE system:

```
SP0_u = sparse(length(dof_u_u), length(dof_u_u));
SP0_p = sparse(length(dof_u_p), length(dof_u_p));

M_tmp = [(Muu_hxx + Muu_hyy), SP0_u, -Mup_hx0; ...
        SP0_u, (Muu_hxx + Muu_hyy), -Mup_hy0; ...
        -Mpu_h0x, -Mpu_h0y, SP0_p ];

f_name = @(x, y) 1;
fh_h = FE_vec_2D_Lagrange_tri(f_name, mesh, ...
            fem_p, 0, 0, n_qp);
one_vec = fh_h(dof_u_p);

tmp_vec = [zeros(2*length(dof_u_u), 1); one_vec];

M = [M_tmp, tmp_vec;
     tmp_vec', 0];
```

Recall FE system equations with zero mean:

$$\begin{bmatrix} \left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right) & \mathbf{0} & -M_{h,x0}^{u,p} & \mathbf{0} \\ \mathbf{0} & \left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right) & -M_{h,y0}^{u,p} & \mathbf{0} \\ -M_{h,0x}^{p,u} & -M_{h,0y}^{p,u} & \mathbf{0} & \mathbf{1_p} \\ \mathbf{0} & \mathbf{0} & \mathbf{1_p^t} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{p} \\ \gamma \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{f}_{1,u,h} \\ \mathbf{f}_{2,u,h} \\ \mathbf{0} \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{bc}_{u_1,eM_{h,xx}} + \mathbf{bc}_{u_1,eM_{h,yy}} \\ \mathbf{bc}_{u_2,eM_{h,xx}} + \mathbf{bc}_{u_2,eM_{h,yy}} \\ \mathbf{bc}_{u_1,eM_{h,0x}} + \mathbf{bc}_{u_2,eM_{h,0y}} \\ 0 \end{bmatrix} \tag{13}$$

Form the vector in the FE system:

```
rhstmp1 = [f1_h; ...
           f2_h; ...
           zeros(length(dof_u_p), 1); ...
           0];
rhstmp2 = [bc_u1_eMhxx + bc_u1_eMhyy; ...
           bc_u2_eMhxx + bc_u2_eMhyy; ...
           -bc_u1_eMh0x - bc_u2_eMh0y; ...
           0];
rhs = rhstmp1 + rhstmp2;
```

Recall FE system equations with zero mean:

$$
\begin{bmatrix}
\left(M_{h,xx}^{u,u}+M_{h,yy}^{u,u}\right) & \mathbf{0} & -M_{h,x0}^{u,p} & \mathbf{0}\\
\mathbf{0} & \left(M_{h,xx}^{u,u}+M_{h,yy}^{u,u}\right) & -M_{h,y0}^{u,p} & \mathbf{0}\\
-M_{h,0x}^{p,u} & -M_{h,0y}^{p,u} & \mathbf{0} & \mathbf{1_p}\\
\mathbf{0} & \mathbf{0} & \mathbf{1_p^t} & \mathbf{0}
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_1\\ \mathbf{u}_2\\ \mathbf{p}\\ \gamma
\end{bmatrix}
$$

$$
=\begin{bmatrix}
\mathbf{f}_{1,u,h}\\ \mathbf{f}_{2,u,h}\\ \mathbf{0}\\ 0
\end{bmatrix}
+\begin{bmatrix}
\mathbf{bc}_{u_1,eM_{h,xx}}+\mathbf{bc}_{u_1,eM_{h,yy}}\\
\mathbf{bc}_{u_2,eM_{h,xx}}+\mathbf{bc}_{u_2,eM_{h,yy}}\\
\mathbf{bc}_{u_1,eM_{h,0x}}+\mathbf{bc}_{u_2,eM_{h,0y}}\\
0
\end{bmatrix}
\tag{13}
$$

Solve the FE system: `u1u2p = M\rhs;`

Form the finite element solution for u1(X), u2(X), and p(X):

```
u1 = u1u2p(1:length(dof_u_u));
u1_fe = u1_e; u1_fe(dof_u_u) = u1;
u2 = u1u2p(length(dof_u_u) + 1:(2*length(dof_u_u)));
u2_fe = u2_e; u2_fe(dof_u_u) = u2;
p = u1u2p((2*length(dof_u_u))+1:(end-1));
p_fe = p;
```

We can validate numerical result by the exact solution functions to this Stokes problem:

$$u_1(x,y) = x\cos(y),\ u_2(x,y) = \cos(x) - \sin(y),\ p(x,y) = x^3y - y^3 + 1/8$$

Find an approximation to $u_1(\pi/4, \pi/6)$:

```
tu1 = @(x, y) x.*cos(y); tu2 = @(x, y) cos(x) - sin(y);
xs = pi/4; ys = pi/6; Xs = [xs; ys];
for k = 1:length(mesh.t)
    elem = mesh.p(:, mesh.t(:, k));
    Tri = coord_X2Tri(elem, Xs);
    tmp = min(Tri);
    if tmp > 2*eps % tri-coordinates of X^* are positive
        ks = k; break;
    end
end
u1_fe_loc = u1_fe(fem_u.t(:, ks));
u1 = FE_evaluation_2D_Lagrange_tri(xs, ys, u1_fe_loc, elem, ...
    degree_u, 0, 0);
[u1, tu1(xs, ys)]
```

which produces `0.680174992926960` $\approx u_1(\pi/4, \pi/6) = 0.680174761587832$.

A visualization of the FE solution: Make a plot for the velocity field

```
figure(1); clf;
quiver(fem_u.p(1, :), fem_u.p(2, :), u1_fe', u2_fe', ...
       'Color', [1, 0, 0])
axis equal
axis([-0.1, 1.1, -0.1, 1.1])

figure(2); clf;
tu1 = @(x, y) x.*cos(y); tu2 = @(x, y) cos(x)-sin(y);
tu1_fe = tu1(fem_u.p(1, :), fem_u.p(2, :));
tu2_fe = tu2(fem_u.p(1, :), fem_u.p(2, :));
quiver(fem_u.p(1, :), fem_u.p(2, :), tu1_fe, tu2_fe, ...
       'Color', [1, 0, 0])
axis equal
axis([-0.1, 1.1, -0.1, 1.1])
```

Remark: Choosing finite element spaces and choosing a suitable finite element scheme are two of the key parts for solving a complicated PDE problem by finite element methods.

We can also use the finite element solution to plot the solution functions $u_1(x, y), u_2(x, y)$ and $p(x, y)$. However, as mentioned before, the plotting programs `trisuf` and `trimesh` given in Matlab are not for higher degree finite element functions. For example, since $u_1(x, y)$ are approximated by a second degree finite element function, a direct plotting of this finite element function does not look very pleasant:

```
figure(1); clf;
trisurf(fem_u.t', fem_u.p(1, :), fem_u.p(2, :), u1_fe, ...
      'facecolor','interp', 'EdgeColor','none');
```

However, because fem generated by

```
fem = fem_generator_Lagrange_2D_tri(mesh, degree_p);
```

inherits the nodes from the mesh, we extract values of u1_fe at the mesh nodes for a better looking plot as follows:

```
figure(2); clf;
np = size(mesh.p, 2);
u1_fe_meshpoints = u1_fe(1:np); % extract usable values for plotting
trisurf(mesh.t', mesh.p(1, :), mesh.p(2, :), u1_fe_meshpoints, ...
      'facecolor','interp', 'EdgeColor','none');
figure(3); clf;  tu1 = @(x, y) x.*cos(y);
tu1_fe = tu1(mesh.p(1, :), mesh.p(2, :));
trisurf(mesh.t', mesh.p(1, :), mesh.p(2, :), tu1_fe, ...
      'facecolor','interp', 'EdgeColor','none');
```

Remark: We can also use pdesurf from Matlab's PDEtool for plotting a higher degree FE solution.

The same approach can be applied to plotting u2_fe:

```
figure(4); clf;
np = size(mesh.p, 2);
% extract values usable for plotting with mesh
u2_fe_meshpoints = u2_fe(1:np);
trisurf(mesh.t', mesh.p(1, :), mesh.p(2, :), u2_fe_meshpoints, ...
      'facecolor','interp', 'EdgeColor','none');
AZ = -1.49e+02; EL = 19.42; view([AZ, EL]); %set viewing angle
figure(5); clf;  tu2 = @(x, y) cos(x)-sin(y);
tu2_fe = tu2(mesh.p(1, :), mesh.p(2, :));
trisurf(mesh.t', mesh.p(1, :), mesh.p(2, :), tu2_fe, ...
      'facecolor','interp', 'EdgeColor','none');
view([AZ, EL]); % use the same viewing angle
```

Since we compute the pressure $p(x, y)$ by a first degree finite element function, we can just plot it directly:

```
figure(6); clf;
trisurf(fem_p.t', fem_p.p(1, :), fem_p.p(2, :), p_fe, ...
      'facecolor','interp', 'EdgeColor','none');
figure(7); clf; tp = @(x, y) y.*x.^3 - y.^3 + 1/8;
tp_fe = tp(fem_p.p(1, :), fem_p.p(2, :));
trisurf(mesh.t', mesh.p(1, :), mesh.p(2, :), tp_fe, ...
      'facecolor','interp', 'EdgeColor','none');
```

Since we know the exact solution for this Stokes problem, we can assess the errors in its finite element solution as follow:

```
tu1 = @(x, y) x.*cos(y);
tu1_x = @(x, y) cos(y); tu1_y = @(x, y) -x.*sin(y);
err = FE_error_2D_Lagrange_tri(tu1, u1_fe, mesh, fem_u, ...
        0, 0, n_qp); errL2u1 = sqrt(err);
err_dx = FE_error_2D_Lagrange_tri(tu1_x, u1_fe, mesh, fem_u, ...
        1, 0, n_qp);
err_dy = FE_error_2D_Lagrange_tri(tu1_y, u1_fe, mesh, fem_u, ...
        0, 1, n_qp); errH1u1 = sqrt(err_dx + err_dy);
fprintf('errL2u1 = %.16e, errH1u1 = %.16e\n', errL2u1, errH1u1)
```

which produce

```
errL2u1 = 8.3713999229531202e-07, errH1u1 = 1.6779774145806993e-04
```

Similarly, for $u_{2,h}(x, y)$, we have

```
tu2 = @(x, y) cos(x)-sin(y);
tu2_x = @(x, y) -sin(x); tu2_y = @(x, y) -cos(y);
err = FE_error_2D_Lagrange_tri(tu2, u2_fe, mesh, fem_u, ...
        0, 0, n_qp); errL2u2 = sqrt(err);
err_dx = FE_error_2D_Lagrange_tri(tu2_x, u2_fe, mesh, fem_u, ...
        1, 0, n_qp);
err_dy = FE_error_2D_Lagrange_tri(tu2_y, u2_fe, mesh, fem_u, ...
        0, 1, n_qp); errH1u2 = sqrt(err_dx + err_dy);
fprintf('errL2u2 = %.16e, errH1u2 = %.16e\n', errL2u2, errH1u2)
```

which produces

```
errL2u2 = 6.5009634194649261e-07, errH1u2 = 9.3835605594606280e-05
```

For the pressure $p(x, y)$, we have

```
tp = @(x, y) y.*x.^3 - y.^3 + 1/8;
tp_x = @(x, y) 3*y.*x.^2; tp_y = @(x, y) x.^3 - 3*y.^2;
err = FE_error_2D_Lagrange_tri(tp, p_fe, mesh, fem_p, ...
        0, 0, n_qp); errL2p = sqrt(err);
err_dx = FE_error_2D_Lagrange_tri(tp_x, p_fe, mesh, fem_p, ...
        1, 0, n_qp);
err_dy = FE_error_2D_Lagrange_tri(tp_y, p_fe, mesh, fem_p, ...
        0, 1, n_qp); errH1p = sqrt(err_dx + err_dy);
fprintf('errL2p = %.16e, errH1p = %.16e\n', errL2p, errH1p)
```

which produces

```
errL2p  = 4.1600839995461830e-04, errH1p = 6.7197745732421724e-02
```

A summary:

The Stokes BVP:

$$-\nabla \cdot (\nu \nabla u_1) + \frac{1}{\rho} p_x = f_1 \tag{4}$$

$$-\nabla \cdot (\nu \nabla u_2) + \frac{1}{\rho} p_y = f_2 \tag{5}$$

$$u_{1,x} + u_{2,y} = 0 \tag{6}$$

$$(u_1, u_2)^t = \mathbf{u} = \mathbf{g}_D = (g_{D,1}, g_{D,2})^t \quad \text{ on } \partial\Omega \tag{3}$$

Weak form of the Stokes BVP: find $u_1, u_2 \in H^1(\Omega), p \in L_0^2(\Omega)$ such that $(u_1, u_2)^t|_{\partial\Omega} = \mathbf{g}_D$ and

$$\int_\Omega (\nu v_{1,x} u_{1,x} + \nu v_{1,y} u_{1,y}) dX - \int_\Omega \frac{1}{\rho} v_{1,x} p dX = \int_\Omega v_1 f_1 dX, \ \ \forall v_1 \in H_0^1(\Omega) \tag{8}$$

$$\int_\Omega (\nu v_{2,x} u_{2,x} + \nu v_{2,y} u_{2,y}) dX - \int_\Omega \frac{1}{\rho} v_{2,y} p dX = \int_\Omega v_2 f_2 dX, \ \ \forall v_2 \in H_0^1(\Omega) \tag{9}$$

$$-\int_\Omega \left(\frac{1}{\rho} q u_{1,x} + \frac{1}{\rho} q u_{2,y} \right) dX = 0, \ \ \forall q \in L_0^2(\Omega) \tag{10}$$

FE equations:

$$\begin{aligned}
&\left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right)\mathbf{u}_1 - M_{h,x0}^{u,p}\mathbf{p} = \mathbf{f}_{1,u,h} + \mathbf{bc}_{u_1,eM_{h,xx}} + \mathbf{bc}_{u_1,eM_{h,yy}} \\
&\left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right)\mathbf{u}_2 - M_{h,y0}^{u,p}\mathbf{p} = \mathbf{f}_{2,u,h} + \mathbf{bc}_{u_2,eM_{h,xx}} + \mathbf{bc}_{u_2,eM_{h,yy}} \\
&-M_{h,0x}^{p,u}\mathbf{u}_1 - M_{h,0y}^{p,u}\mathbf{u}_2 = \mathbf{bc}_{u_1,eM_{h,0x}} + \mathbf{bc}_{u_2,eM_{h,0y}}
\end{aligned}$$

Finite element approximation to p(x,y) with zero mean, version 2:

Recall the FE system for the Stokes BVP:

$$\begin{bmatrix} \left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right) & \mathbf{0} & -M_{h,x0}^{u,p} \\ \mathbf{0} & \left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right) & -M_{h,y0}^{u,p} \\ -M_{h,0x}^{p,u} & -M_{h,0y}^{p,u} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{p} \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{f}_{1,u,h} \\ \mathbf{f}_{2,u,h} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{bc}_{u_1,eM_{h,xx}} + \mathbf{bc}_{u_1,eM_{h,yy}} \\ \mathbf{bc}_{u_2,eM_{h,xx}} + \mathbf{bc}_{u_2,eM_{h,yy}} \\ \mathbf{bc}_{u_1,eM_{h,0x}} + \mathbf{bc}_{u_2,eM_{h,0y}} \end{bmatrix} \tag{11}$$

The matrix in this linear system is singular even if the finite element space pair $V_h^u(\Omega)$-$V_h^p(\Omega)$ are chosen as specified by the Taylor-Hood scheme because this linear system does not incorporate the condition $\int_\Omega p(X)dX = 0$.

Since this zero average condition for the pressure $p(X)$ provide ONE scalar constraint, it can be shown that the matrix in (11) is rank-1 deficient.

Note that there are $\left|\mathcal{N}_{h,u}^u\right| + \left|\mathcal{N}_{h,u}^u\right| + \left|\mathcal{N}_{h,u}^p\right|$ equations for the same number of unknowns in (11). The rank-1 deficient property suggests that we can try to use the first $\left|\mathcal{N}_{h,u}^u\right| + \left|\mathcal{N}_{h,u}^u\right| + \left|\mathcal{N}_{h,u}^p\right| - 1$ equations to solve for the first $\left|\mathcal{N}_{h,u}^u\right| + \left|\mathcal{N}_{h,u}^u\right| + \left|\mathcal{N}_{h,u}^p\right| - 1$ unknowns. Then, we can determine the last entry of $\mathbf{p}$ by $\int_\Omega p_h(X)dX = 0$.

Specifically, we solve for $\mathbf{u}_1, \mathbf{u}_2$ and $\tilde{\mathbf{p}} = \left(\tilde{p}_{\hat{j}}\right)_{1 \le \hat{j} \le \left|\mathcal{N}_{h,u}^{p}\right| - 1}$ from a linear system obtained as follows:

$$
\begin{bmatrix} \left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right) & \mathbf{0} & -M_{h,x0}^{u,p} \\ \mathbf{0} & \left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right) & -M_{h,y0}^{u,p} \\ -M_{h,0x}^{p,u} & -M_{h,0y}^{p,u} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{p} \end{bmatrix}
= \begin{bmatrix} \mathbf{f}_{1,u,h} \\ \mathbf{f}_{2,u,h} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{bc}_{u_1, eM_{h,xx}} + \mathbf{bc}_{u_1, eM_{h,yy}} \\ \mathbf{bc}_{u_2, eM_{h,xx}} + \mathbf{bc}_{u_2, eM_{h,yy}} \\ \mathbf{bc}_{u_1, eM_{h,0x}} + \mathbf{bc}_{u_2, eM_{h,0y}} \end{bmatrix} \tag{11}
$$

$$\downarrow$$

$$
\tilde{M} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \tilde{\mathbf{p}} \end{bmatrix} = \tilde{\mathbf{f}}_h \tag{27}
$$

where the matrix $\tilde{M}$ is formed by deleting the last row and the last column from the matrix in (11), the vector $\tilde{\mathbf{f}}_h$ formed by deleting the last row from the vector in the right hand side of (11), and $\tilde{\mathbf{p}}$ is a sub-vector of $\mathbf{p}$ by discarding its last entry.

Now, let $\tilde{p}_h(X)$ be the finite element function whose coefficients are entries of the vector $[\tilde{\mathbf{p}}; 0]$:

$$\tilde{p}_h(X) = \sum_{\hat{j}=1}^{\left|\mathcal{N}_{h,u}^{p}\right|-1} \tilde{p}_{\hat{j}}\phi_{\hat{j}}^{p}(X) + 0\phi_{\left|\mathcal{N}_{h,u}^{p}\right|}^{p}(X) = \sum_{\hat{j}=1}^{\left|\mathcal{N}_{h,dof}^{p}\right|-1} \tilde{p}_{\hat{j}}\phi_{\hat{j}}^{p}(X)$$

Then, we introduce another vector:

$$\mathbf{p} = (p_{\hat{j}})_{\hat{j}=1}^{\left|\mathcal{N}_{h,dof}^{p}\right|} = \begin{bmatrix} \tilde{\mathbf{p}} \\ 0 \end{bmatrix} - \frac{1}{|\Omega|}\int_{\Omega} \tilde{p}_h(X)dX \tag{28}$$

which can be used to form another finite element function:

$$p_h(X) = \sum_{\hat{j}=1}^{\left|\mathcal{N}_{h,dof}^{p}\right|} p_{\hat{j}}\phi_{\hat{j}}^{p}(X) = \tilde{p}_h(X) - \frac{1}{|\Omega|}\int_{\Omega} \tilde{p}_h(X)dX \quad (???) \tag{29}$$

which satisfies the zero mean condition and we can use $p_h(X)$ as a finite element approximation to the pressure $p(X)$. (Prove this ???)

The constant correction in (28) can be easily computed as follows:

$$\frac{1}{|\Omega|}\int_{\Omega} \tilde{p}_h(X)dX = \frac{1}{|\Omega|}\mathbf{1}_p^t \begin{bmatrix} \tilde{\mathbf{p}} \\ 0 \end{bmatrix} \quad (???) \tag{30}$$

where the vector $\mathbf{1}_p$ can prepared by our vector assembler according to formula (12).

Example: Use the Taylor-Hood finite element method to solve the following Stokes BVP: find $u(x,y)$ and $p(x,y)$ such that

$$
\begin{aligned}
&-\triangle \mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in } \Omega = (0,1)\times(0,1) \\
&\nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega, \quad \mathbf{u} = \mathbf{g}_D, \quad \text{on } \partial\Omega
\end{aligned}
$$

with $\quad \mathbf{f}(x,y) = \begin{pmatrix} x(3xy + \cos(y)) \\ x^3 - 3y^2 + \cos(x) - \sin(y) \end{pmatrix}, \quad \mathbf{g}_D(x,y) = \begin{pmatrix} x\cos(y) \\ \cos(x) - \sin(y) \end{pmatrix}$

We will use the 2nd approach to treat the zero mean pressure condition.

Prepare the solution domain, mesh and fem:

```
nu = @(x, y) 1; rhoinv = @(x, y) 1;
f1_name = @(x, y) x.*(3*x.*y + cos(y));
f2_name = @(x, y) x.^3 - 3*y.^2 + cos(x) - sin(y);
g1_D = @(x, y) x.*cos(y); g2_D = @(x, y) cos(x) - sin(y);

xmin = 0; xmax = 1; ymin = 0; ymax = 1; nx = 20; ny = 20;
mesh = mesh_generator_2D_tri(xmin, xmax, ymin, ymax, nx, ny);
[e, M_e, M_ne] = edges_in_mesh(mesh, 1, 1);
mesh.e = e; mesh.M_e = M_e; mesh.M_ne = M_ne;
degree_u = 2; degree_p = 1; % A Taylor-Hood pair
fem_p = fem_generator_Lagrange_2D_tri(mesh, degree_p);
fem_u = fem_generator_Lagrange_2D_tri(mesh, degree_u);
```

Prepare dof_u and boundary conditions

```
B_name = @(x, y) x.*(x-xmax).*y.*(y-ymax);
B_D_name_u = @(x, y) x.*(x-xmax).*y.*(y-ymax);
B_D_name_p = @(x, y) ones(size(x)); % no essential BC for p
[dof_u_u, nt_u] = FE_dof_u_2d_tri(fem_u, B_name, B_D_name_u);
[dof_u_p, nt_p] = FE_dof_u_2d_tri(fem_p, B_name, B_D_name_p);

u1_e = zeros(length(fem_u.p), 1); % ess BC vector for u1
u2_e = zeros(length(fem_u.p), 1); % ess BC vector for u2
I = find(nt_u == 0);
u1_e(I) = g1_D(fem_u.p(1, I), fem_u.p(2, I));
u2_e(I) = g2_D(fem_u.p(1, I), fem_u.p(2, I));
```

Prepare matrices and associated vectors for boundary conditions:

Recall FE equations:

$$
\begin{aligned}
&\left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right)\mathbf{u}_1 - M_{h,x0}^{u,p}\mathbf{p} = \mathbf{f}_{1,u,h} + \mathbf{bc}_{u_1,eM_{h,xx}} + \mathbf{bc}_{u_1,eM_{h,yy}} \\
&\left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right)\mathbf{u}_2 - M_{h,y0}^{u,p}\mathbf{p} = \mathbf{f}_{2,u,h} + \mathbf{bc}_{u_2,eM_{h,xx}} + \mathbf{bc}_{u_2,eM_{h,yy}} \\
&-M_{h,0x}^{p,u}\mathbf{u}_1 - M_{h,0y}^{p,u}\mathbf{u}_2 = \mathbf{bc}_{u_1,eM_{h,0x}} + \mathbf{bc}_{u_2,eM_{h,0y}}
\end{aligned}
$$

Prepare matrices and associated vectors for boundary conditions:

```
d_x1 = 1; d_y1 = 0; d_x2 = 1; d_y2 = 0; n_qp = 7;
Mhuu_hxx =  FE_matrix_2D_Lagrange_tri(nu, mesh, ...
    fem_u, d_x1, d_y1, fem_u, d_x2, d_y2, n_qp);
d_x1 = 0; d_y1 = 1; d_x2 = 0; d_y2 = 1;
Mhuu_hyy =  FE_matrix_2D_Lagrange_tri(nu, mesh, ...
    fem_u, d_x1, d_y1, fem_u, d_x2, d_y2, n_qp);
Muu_hxx = Mhuu_hxx(dof_u_u, dof_u_u);
tmp = -Mhuu_hxx*u1_e; bc_u1_eMhxx = tmp(dof_u_u);
tmp = -Mhuu_hxx*u2_e; bc_u2_eMhxx = tmp(dof_u_u);
Muu_hyy = Mhuu_hyy(dof_u_u, dof_u_u);
tmp = -Mhuu_hyy*u1_e; bc_u1_eMhyy = tmp(dof_u_u);
tmp = -Mhuu_hyy*u2_e; bc_u2_eMhyy = tmp(dof_u_u);
```

Recall FE equations:

$$\left(M^{u,u}_{h,xx} + M^{u,u}_{h,yy}\right)\mathbf{u}_1 - M^{u,p}_{h,x0}\mathbf{p} = \mathbf{f}_{1,u,h} + \mathbf{bc}_{u_1,eM_{h,xx}} + \mathbf{bc}_{u_1,eM_{h,yy}}$$
$$\left(M^{u,u}_{h,xx} + M^{u,u}_{h,yy}\right)\mathbf{u}_2 - M^{u,p}_{h,y0}\mathbf{p} = \mathbf{f}_{2,u,h} + \mathbf{bc}_{u_2,eM_{h,xx}} + \mathbf{bc}_{u_2,eM_{h,yy}}$$
$$-M^{p,u}_{h,0x}\mathbf{u}_1 - M^{p,u}_{h,0y}\mathbf{u}_2 = \mathbf{bc}_{u_1,eM_{h,0x}} + \mathbf{bc}_{u_2,eM_{h,0y}}$$

Prepare matrices and associated vectors for boundary conditions (continued):

```
d_x1 = 1; d_y1 = 0; d_x2 = 0; d_y2 = 0;
Mhup_hx0 = FE_matrix_2D_Lagrange_tri(rhoinv, mesh, ...
        fem_u, d_x1, d_y1, ...
        fem_p, d_x2, d_y2, n_qp);
Mup_hx0 = Mhup_hx0(dof_u_u, dof_u_p);
d_x1 = 0; d_y1 = 1; d_x2 = 0; d_y2 = 0;
Mhup_hy0 = FE_matrix_2D_Lagrange_tri(rhoinv, mesh, ...
        fem_u, d_x1, d_y1, ...
        fem_p, d_x2, d_y2, n_qp);
Mup_hy0 = Mhup_hy0(dof_u_u, dof_u_p);

Mhpu_h0x = Mhup_hx0'; Mpu_h0x = Mhpu_h0x(dof_u_p, dof_u_u);
tmp = -Mhpu_h0x*u1_e; bc_u1_eMh0x = tmp(dof_u_p);
Mhpu_h0y = Mhup_hy0'; Mpu_h0y = Mhpu_h0y(dof_u_p, dof_u_u);
tmp = -Mhpu_h0y*u2_e; bc_u2_eMh0y = tmp(dof_u_p);
```

Recall FE equations:

$$\left(M^{u,u}_{h,xx} + M^{u,u}_{h,yy}\right)\mathbf{u}_1 - M^{u,p}_{h,x0}\mathbf{p} = \mathbf{f}_{1,u,h} + \mathbf{bc}_{u_1,eM_{h,xx}} + \mathbf{bc}_{u_1,eM_{h,yy}}$$
$$\left(M^{u,u}_{h,xx} + M^{u,u}_{h,yy}\right)\mathbf{u}_2 - M^{u,p}_{h,y0}\mathbf{p} = \mathbf{f}_{2,u,h} + \mathbf{bc}_{u_2,eM_{h,xx}} + \mathbf{bc}_{u_2,eM_{h,yy}}$$
$$-M^{p,u}_{h,0x}\mathbf{u}_1 - M^{p,u}_{h,0y}\mathbf{u}_2 = \mathbf{bc}_{u_1,eM_{h,0x}} + \mathbf{bc}_{u_2,eM_{h,0y}}$$

Prepare vectors associated with the forcing term:

```
d_x = 0; d_y = 0;
f1h_h =  FE_vec_2D_Lagrange_tri(f1_name, mesh, fem_u, ...
            d_x, d_y, n_qp);
f1_h = f1h_h(dof_u_u);
f2h_h =  FE_vec_2D_Lagrange_tri(f2_name, mesh, fem_u, ...
            d_x, d_y, n_qp);
f2_h = f2h_h(dof_u_u);
```

Recall the FE system for the Stokes BVP:

$$\begin{bmatrix} \left(M_{h,xx}^{u,u}+M_{h,yy}^{u,u}\right) & \mathbf{0} & -M_{h,x0}^{u,p} \\ \mathbf{0} & \left(M_{h,xx}^{u,u}+M_{h,yy}^{u,u}\right) & -M_{h,y0}^{u,p} \\ -M_{h,0x}^{p,u} & -M_{h,0y}^{p,u} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{p} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{f}_{1,u,h} \\ \mathbf{f}_{2,u,h} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{bc}_{u_1,eM_{h,xx}} + \mathbf{bc}_{u_1,eM_{h,yy}} \\ \mathbf{bc}_{u_2,eM_{h,xx}} + \mathbf{bc}_{u_2,eM_{h,yy}} \\ \mathbf{bc}_{u_1,eM_{h,0x}} + \mathbf{bc}_{u_2,eM_{h,0y}} \end{bmatrix} \tag{11}$$

Form the matrix and vector for the FE system in (11):

```
SP0_u = sparse(length(dof_u_u), length(dof_u_u));
SP0_p = sparse(length(dof_u_p), length(dof_u_p));
M = [(Muu_hxx + Muu_hyy), SP0_u, -Mup_hx0; ...
       SP0_u, (Muu_hxx + Muu_hyy), -Mup_hy0; ...
       -Mpu_h0x, -Mpu_h0y, SP0_p ];
rhs = [f1_h + bc_u1_eMhxx + bc_u1_eMhyy; ...
      f2_h + bc_u2_eMhxx + bc_u2_eMhyy; ...
      -bc_u1_eMh0x - bc_u2_eMh0y];
```

$$
\begin{bmatrix} \left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right) & \mathbf{0} & -M_{h,x0}^{u,p} \\ \mathbf{0} & \left(M_{h,xx}^{u,u} + M_{h,yy}^{u,u}\right) & -M_{h,y0}^{u,p} \\ -M_{h,0x}^{p,u} & -M_{h,0y}^{p,u} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{p} \end{bmatrix}
$$

$$
= \begin{bmatrix} \mathbf{f}_{1,u,h} \\ \mathbf{f}_{2,u,h} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{bc}_{u_1,eM_{h,xx}} + \mathbf{bc}_{u_1,eM_{h,yy}} \\ \mathbf{bc}_{u_2,eM_{h,xx}} + \mathbf{bc}_{u_2,eM_{h,yy}} \\ \mathbf{bc}_{u_1,eM_{h,0x}} + \mathbf{bc}_{u_2,eM_{h,0y}} \end{bmatrix} \tag{11}
$$

$$
\downarrow
$$

$$
\tilde{M} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \tilde{\mathbf{p}} \end{bmatrix} = \tilde{\mathbf{f}}_h \tag{27}
$$

```
% solve the system with one less unknown
M_t = M(1:end-1, 1:end-1);
rhs_t = rhs(1:end-1);
u1u2p_tmp = M_t\rhs_t;
```

```
% form FE solutions
u1 = u1u2p_tmp(1:length(dof_u_u));
u1_fe = u1_e; u1_fe(dof_u_u) = u1;
u2 = u1u2p_tmp(length(dof_u_u) + 1:(2*length(dof_u_u)));
u2_fe = u2_e; u2_fe(dof_u_u) = u2;
% pt is for the FE function not having a zero mean
pt = [u1u2p_tmp((2*length(dof_u_u))+1:end); 0];
```

Recall

$$\mathbf{p} = (p_{\hat{j}})_{\hat{j}=1}^{|\mathcal{N}_{h,dof}^{p}|} = \begin{bmatrix} \tilde{\mathbf{p}} \\ 0 \end{bmatrix} - \frac{1}{|\Omega|}\int_{\Omega} \tilde{p}_h(X)dX \tag{28}$$

$$\frac{1}{|\Omega|}\int_{\Omega} \tilde{p}_h(X)dX = \frac{1}{|\Omega|}\mathbf{1}_p^t \begin{bmatrix} \tilde{\mathbf{p}} \\ 0 \end{bmatrix} \tag{30}$$

Construct the zero mean FE pressure: by (30) and (28)

```
f_name = @(x, y) 1;
d_x = 0; d_y = 0;
fh_h = FE_vec_2D_Lagrange_tri(f_name, mesh, fem_p, ...
            d_x, d_y, n_qp);
one_vec = fh_h(dof_u_p);
domain_v = (xmax - xmin)*(ymax - ymin); % domain volume
p_fe = pt - (1/domain_v)*(one_vec'*pt);
```

Remark: The FE system (27) is easier to solve than (13) because the matrix in (27) has less nonzero fill-ins than the matrix in (13).

The exact solution for the Stokes problem in this example is

$$u_1(X) = x\cos(y), \quad u_2(X) = \cos(x) - \sin(y), \quad p(X) = y(x^3 - y^2) + 1/8$$

which can be used to check the convergence of the finite element.

Visualizations of the FE solution: Compare the pressure with its FE approximation

```
tu1 = @(x, y) x.*cos(y); tu2 = @(x, y) cos(y) - sin(y);
tp = @(x, y) x.^3.*y - y.^3 - 1/8;
figure(1); clf; % plot the FE solution for pressure
trisurf(fem_p.t', fem_p.p(1, :), fem_p.p(2, :), p_fe, ...
          'facecolor','interp', 'EdgeColor','none');
p_fun_val = tp(fem_p.p(1, :), fem_p.p(2, :));
figure(2); clf; % plot the exact solution for pressure
trisurf(fem_p.t', fem_p.p(1, :), fem_p.p(2, :), p_fun_val, ...
          'facecolor','interp', 'EdgeColor','none');
```

Compare the velocity with its FE approximation

```
figure(3); clf; % plot the FE solution to u1(x, y)
trisurf(fem_u.t', fem_u.p(1, :), fem_u.p(2, :), u1_fe, ...
          'facecolor','interp', 'EdgeColor','none');
u1_fun_val = tu1(fem_u.p(1, :), fem_u.p(2, :));
figure(4); clf; % plot the exact solution to u1(x, y)
trisurf(fem_u.t', fem_u.p(1, :), fem_u.p(2, :), u1_fun_val, ...
          'facecolor','interp', 'EdgeColor','none');
```

We note that these plots do not look good. This is because of the limitation of the visualization program `trisurf` which cannot handle higher degree finite element functions well.

Better visualization programs such as `pdesurf` are needed/available.

```
figure(3); clf;
pdesurf(fem_u.p, fem_u.t, u1_fe)
u1_fun = @(x, y) x.*cos(y);
u1_fun_val = tu1(fem_u.p(1, :), fem_u.p(2, :));
figure(4); clf;
pdesurf(fem_u.p, fem_u.t, u1_fun_val')
```

Of course, because we know the exact solution in this example, we can assess the accuracy of our finite element solution as follows:

```
tp = @(x, y) y.*x.^3 - y.^3 + 1/8; tp_dx = @(x, y) 3*y.*x.^2;
tp_dy = @(x, y) x.^3 - 3*y.^2;
err_p_L2_20 = sqrt(FE_error_2D_Lagrange_tri(tp, p_fe, mesh, ...
    fem_p, 0, 0, n_qp))
err_H1_x = FE_error_2D_Lagrange_tri(tp_dx, p_fe, mesh, ...
    fem_p, 1, 0, n_qp);
err_H1_y = FE_error_2D_Lagrange_tri(tp_dy, p_fe, mesh, ...
    fem_p, 0, 1, n_qp);
err_p_H1_20 = sqrt(err_H1_x + err_H1_y)
```

which produces the error in the FE solution for the pressure:

```
err_p_L2_20 = 4.160101817901451e-02
err_p_H1_20 = 6.719841672972562e-02
```

For the FE solution for the velocity:

```
tu1 = @(x, y) x.*cos(y);
tu1_dx = @(x, y) cos(y); tu1_dy = @(x, y) -x.*sin(y);
err_u1_L2_20 = sqrt(FE_error_2D_Lagrange_tri(tu1, u1_fe, mesh, ...
    fem_u, 0, 0, n_qp))
err_H1_x = FE_error_2D_Lagrange_tri(tu1_dx, u1_fe, mesh, ...
    fem_u, 1, 0, n_qp);
err_H1_y = FE_error_2D_Lagrange_tri(tu1_dy, u1_fe, mesh, ...
    fem_u, 0, 1, n_qp);
err_u1_H1_20 = sqrt(err_H1_x + err_H1_y)
```

which produces

```
err_u1_L2_20 = 8.370818300982527e-07
err_u1_H1_20 = 1.677971283242487e-04
```