CS5800-03 Algorithms

Spring 2020, Silicon Valley Campus

Introduction

- Goals: learn the basic concepts for designing and analyzing algorithms
- Instructional Staff:
  - Instructor: Anurag Bhardwaj
  - TA: Vishal Annamaneni, Zijun Wan, Gongzhan Xie
- Evaluation:
  - Midterm exam: 30%
  - Final exam: 30%
  - Individual assignments: 40%
- Suggested book: Introduction to Algorithms, Third Edition,
  by Cormen, Leiserson, Rivest, Stein from MIT Press

What is an Algorithm?

A Sequence of Instructions to Perform a Task

Formal Definition

- A process or set of rules to be followed in calculations or other problem-solving operations, especially by a computer: *a basic* ***algorithm for*** *division*
- What is important in an algorithm?
  1. Its correctness: it does what is suppose to do
  2. How fast it is: its time performance
  3. How many resources it uses:
     - Main memory
     - Secondary memory
     - Communication
     - Others ….
- Example: What is an algorithm?
- There can be many algorithms for the same problem, but ….

Etymology

- The word algorithm comes from a famous mathematician
  - Muḥammad ibn Mūsā **al'Khwārizmī** (~780-840 AC)
  - He worked in the House of Wisdom of the Caliphate of Baghdad
- Main contributions:
  - *The Compendious Book on Calculation by Completion and Balancing* (~820)
    (Arab: *al-Kitāb al-mukhtaṣar fī ḥisāb* ***al-jabr*** *wal-muqābala*)
  - *The Book of Addition and Subtraction According to the Hindu Calculation*
  - Astronomical tables of *Siddhanta* (includes trigonometric tables, 820)
  - *Book of the Description of the Earth* (833)

Historical Context: Caliphate of Baghdad


Georgia
Batumi
Tbilisi
Trabzon
Rize
Kars
Erzurum
Armenia
Azerbaijan
Izberbash
Избербаш
Derbent
Дербент
Ganja
Baku
Caspian Sea
Nukus
Dashoguz
Urgench
Khiva
Uzbekistan
Kaplankyr Reserve
Darvaza
Navoi
Bukhara
Turkmenabat
Turkmenistan
Ashgabat
Mary
Van
Diyarbakır
Siirt
Batman
Tabriz
Ardabil
Rasht
Golestan National Park
Gorgan
Sari
Amol
Mosul
Erbil
Zanjan
Tehran
Mashhad
Sabzevar
Neyshabur
Shahroud
Sulaymaniyah
Kirkuk
Hamedan
Qom
Torbat Heydariyeh
Herat
Afgha
Kashan
Baghdad
Iraq
Karbala
Isfahan
Birjand
Anar Dara
Farah
Najaf
Arar
Iran
Ahvaz
Zabol
Basrah
Kerman
Rafha
Shiraz
Bam
Zahedan
Kuwait

Example: Euclid’s Algorithm

- This algorithm finds the greatest common divisor (GCD) of two numbers
  - Replace the largest number by the difference of the largest with the smallest
  - Stop when the two numbers are equal
  - That number is the GCD
- This is probably the oldest attributed algorithm (300 BC)
- Can be done faster?
  - Yes. Replace the largest by the reminder of dividing the largest with the smallest
  - Steps are proportional to 5 times the number of digits of the smallest number
  - That is $5 \times \lceil \log_{10}(\text{smallest number}) \rceil$ (Gabriel Lamé, 1844)
- Can be done faster?

Comparing Functions

- Compare the two running time functions in ms of algorithms F & G:
  - $f(n) = n^2 + 1$
  - $g(n) = n \log_{10} n + 1000\, n + 9999$
- Value of the functions for $n = 100$:
  - $f(100) = 10{,}001 \cong 10$ sec. vs $g(100) = 110{,}199 \cong 110$ sec.
  - So can we say that F is better than G?
- What about $n = 10{,}000$?
  - $f(10{,}000) = 100{,}100{,}101 \cong 100{,}000$ sec. $\cong 27.7$ hours
  - $g(10{,}000) = 10{,}049{,}999 \cong 10{,}000$ sec. $\cong 2.7$ hours
- We need to focus in the dominant term (rate of growth)

Asymptotic Notations

Mathematical Tools To Compare the Efficiency of Different Algorithms

Motivation

- A mathematical tool (framework) used for describing algorithm’s efficiency, considering the two abstractions we saw:
  - Constant factors are not important.
  - The most dominating term matters (growth rate)
- Fairly theoretical, mostly for understanding/gaining insights in analysis of algorithms
  - Formal definitions (big-Oh, big-Omega, ...)
- Note, in real world, constant factors and non-dominating terms could matter (sometimes seriously)
- Let’s begin the theoretical journey!

O-Notation (Big-Oh)

- We say (define):
  - A function $f(n)$ is in big-Oh of $g(n)$ (denoted $O(g(n))$) iff (if and only if) there exist positive constants c and n_0 such that $0 \leq f(n) \leq cg(n)$ for all n that's greater than or equal to n_0.
- In other words,
  - $O\big(g(n)\big)$ is a set of all functions (call any such function $x(n)$, to avoid confusion with $f(n)$) where there exist positive constants c and n_0 such that $0 \leq x(n) \leq cg(n)$ for all n that's greater than or equal to n_0.
  - And $f(n)$ is a member of the set $O(g(n))$ (is in the set).
  - Proper notation would be: $f(n) \in O(g(n))$, but we abuse $=$, and write $f(n) = O(g(n))$ most of the time.
- What does all this mean?
  - Study the worked examples in the following slides

Big-Oh Notation Proof Examples

Time To Prove Big-Oh Notations Formally

Is $\frac{1}{2}n^2 - 3\mathrm{n} = O(n^2)$?

A function $f(n)$ is in big-Oh of $g(n)$ (denoted $O(g(n))$) iff (if and only if) there exist positive constants c and n_0 such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$.

- If you think it is, and need to prove it, you must find c and n_0 such that
$$0 \leq \frac{1}{2}n^2 - 3\mathrm{n} \leq cn^2$$
for all $n \geq n_0$
- This is an easy case, after trying a few possible c and n_0 with some intuitions:
  - As long as c is at least $\frac{1}{2}$, the inequality holds for any non-negative n!
  - Thus, we can simply let $c = \frac{1}{2}, n_0 = 6$, which satisfies the definition (the inequality above) of big-Oh.
  - In fact, there are infinitely many valid c and n_0 that can be used for the proof.

Is $100n^2 + 123\text{n} = O(n^2)$?

A function $f(n)$ is in big-Oh of $g(n)$ (denoted $O(g(n))$) iff (if and only if) there exist positive constants c and n_0 such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$.

- If you think it is, and need to prove it, you must find c and n_0 such that
$$0 \leq 100n^2 + 123n \leq cn^2$$
for all $n \geq n_0$
- This is another easy case, after trying a few possible c and n_0 with some intuitions:
  - You soon realize that c must be greater than 100. Let $c = 101$.
  - Then for what values of n does $100n^2 + 123n \leq 101n^2$?
    - Just solve the inequality, and you get $n \geq 123$, which gives us 123 for n_0.

Is $100n^2 + 123\mathrm{n} = O(n^3)$?

- If you think it is, and need to prove it, you must find c and n_0 such that
$$0 \leq 100n^2 + 123n \leq cn^3$$
for all $n \geq n_0$
- This is another easy case, after trying a few possible c and n_0 with some intuitions:
  - c doesn't have to be very big here, because we have n^3 that grows fast. Let $c = 1$.
  - Then for what values of n is $100n^2 + 123n \leq n^3$?
    - No need to solve the inequality precisely. Just divide both sides by n^2, which gives us $n \geq 100 + \frac{123}{n}$, where $100 + \frac{123}{n} \leq 223$, and this (223) is our n_0.

Is $\frac{1}{200}n^2 = O(n)$?

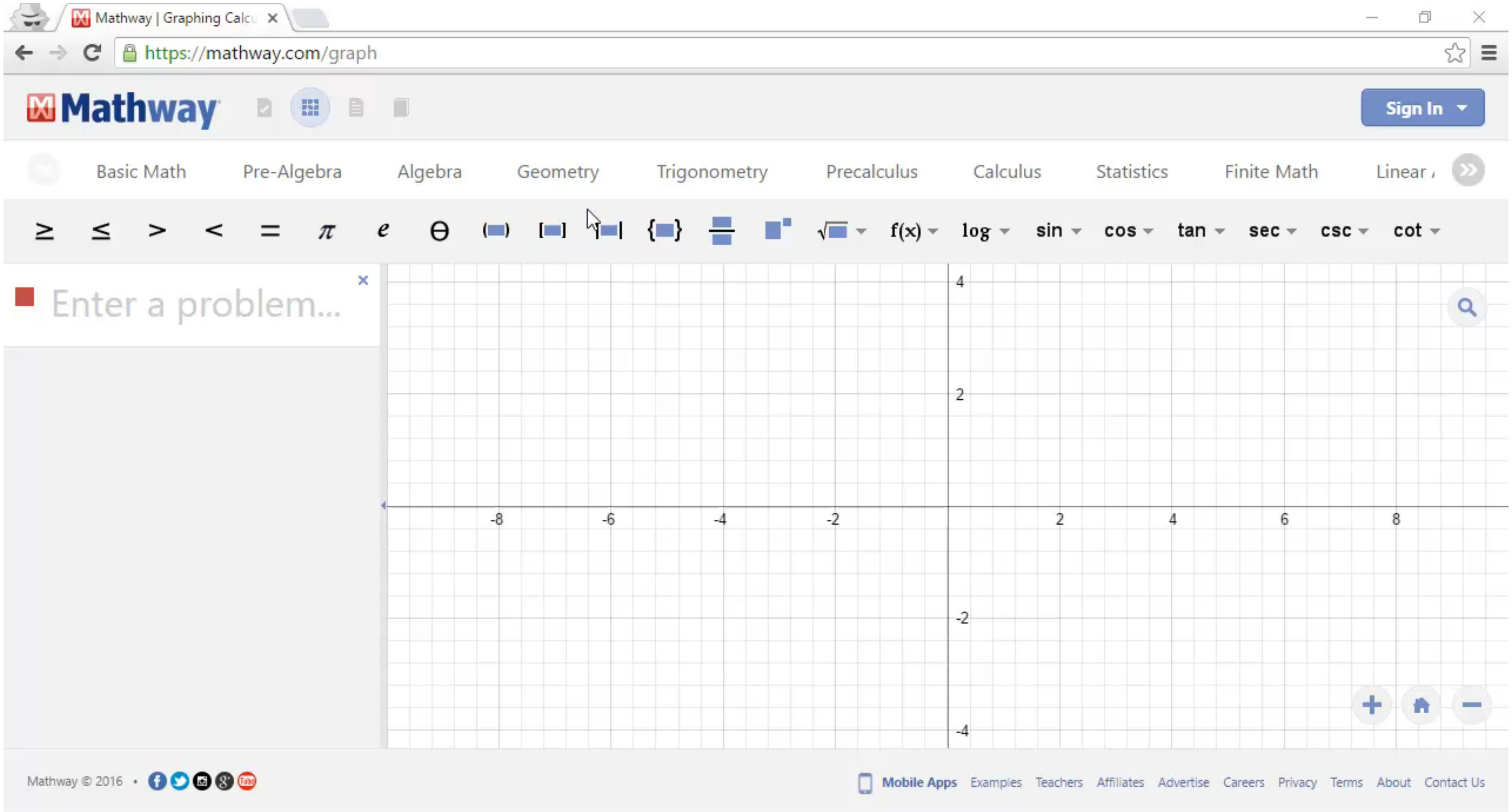
- If you think it is, and need to prove it, you must find c and n_0 such that

$$0 \leq \frac{1}{200}n^2 \leq cn$$

  for all $n \geq n_0$
- However, you soon realize that no matter what values of c and n_0 you choose, there'll be always some n that makes $\frac{1}{200}n^2 > cn$
  - Easy, just pick $n = 200c + 1$ for any c. Doesn't matter what n_0 is (in this case). This is called counterexample.
- Therefore, $\frac{1}{200}n^2 = O(n)$ is not true!

Intuition of Big-Oh:

Why $\frac{1}{200}n^2 \neq O(n)$?



Lesson From Visualization

Rate of growth of a higher order term (n^2) can't be overcome by a constant factor c multiplied to a lower order term (n), no matter how big c can get.

More Intuitions of Big-Oh

- Lower order term (e.g., $123n$ in $100n^2 + 123n$) can be always ignored by setting n_0 sufficiently large.
  - Highest order term is the "dominating" term.

- Rule of thumb: Pick the highest order term only, drop the constant factor, and that's your best big-Oh notation for a given function. E.g.,
  - $\frac{1}{2}n^2 - 3\mathrm{n}$: Pick the highest order term only ($\frac{1}{2}n^2$) and drop the constant factor ($\frac{1}{2}$), which gives $O(n^2)$.
  - $100n^2 + 123n$: Same, pick $100n^2$, drop 100, giving $O(n^2)$.

Bounding Properties of Asymptotic Notations

Useful Intuitions On Asymptotic Notations

Big-Oh Is Upper Bound

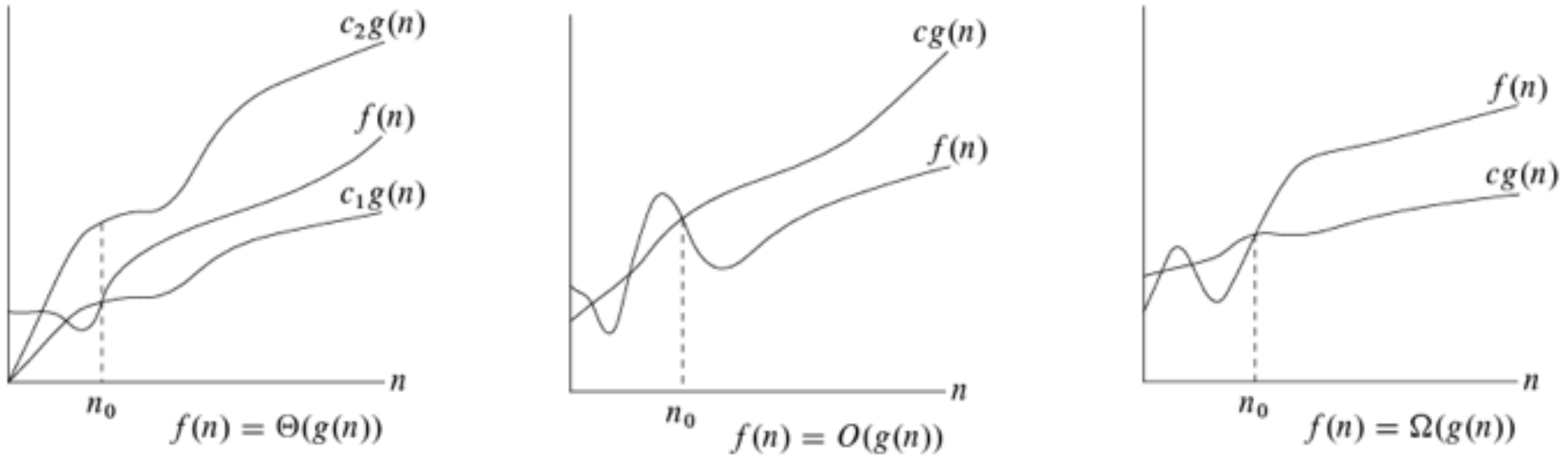
- If $f(n)$ is in $O(n^2)$, then $f(n)$ is also in $O(n^3)$, $O(n^4)$, … .
  - E.g., $f(n) = \frac{1}{200}n^2 + 123n$.
- But, $f(n)$ is not in $O(n)$ for the above example!
- The function in $O(\,)$ for a given function $f(n)$ is an “asymptotic” upper bound.
- There are many upper bounds, and we’d prefer to find the “tight” upper bound.
- In the above example, it’s $O(n^2)$.
  - Because it’s tighter than all other $O(n^k)$ for any $k > 2$.
  - And also because $f(n)$ cannot be in $O(n)$.

Big-Omega Notation: Asymptotic Lower Bound

- Not as frequently mentioned as big-Oh, but still important notation for asymptotic lower bound
    - E.g., algorithm A takes at least $\Omega(n)$ time for input size n
- Exactly the same patterned definition, just the different inequality:
    - A function $f(n)$ is in $\Omega(g(n))$ iff there exist positive constants c and n_0 such that $0 \leq \mathrm{cg}(\mathrm{n}) \leq f(n)$ for all n that's greater than or equal to n_0.
- Exercise: Prove that $n^4 + 12n^3 - 34n^2 + 56n + 78$
    - Is in $\Omega(n^4)$, in $\Omega(n^3)$, in $\Omega(n^2)$, in $\Omega(n^1)$, and in $\Omega(1)$.
    - But not in $\Omega(n^5)$.

Theta Notation: Asymptotic Tight Bound

- If $f(n) = \mathrm{O}(\mathrm{g}(\mathrm{n}))$ and also $f(n) = \Omega(g(n))$, we say $f(n) = \Theta(g(n))$. In other words,
  - A function $f(n)$ is in $\Theta(g(n))$ iff there exist positive constants c_1, c_2 and n_0 such that $0 \leq \mathrm{c}_1\mathrm{g}(\mathrm{n}) \leq f(n) \leq c_2 g(n)$ for all n that's greater than or equal to n_0.
- Graphic examples of Θ, O, and Ω (CLRS Fig. 3.1)



Small-oh/omega Notations: Non-tight Bounds

- $2n^2 = O(n^2)$ is asymptotically tight, but $2n = O(n^2)$ is not tight.
- Use o-notation to denote an upper bound that is not asymptotically tight. Define:
  - A function $f(n)$ is in $o(g(n))$ iff for any positive constant $c > 0$, there exists a constant $n_0 > 0$ such that $0 \leq f(\mathrm{n}) < \mathrm{c}g(n)$ for all n that's greater than or equal to n_0.
- E.g.: $2n = o(n^2)$, but $2n^2 \neq o(n^2)$.
- Similar definition for $\mathrm{f(n)} = \omega(g(n))$.

Useful Relationships of Asymptotic Notations

Different Asymptotic Notations Are Related

Properties of Asymptotic Notations

- $f(n) = o(g(n))$ iff $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.
- $f(n) = \omega(g(n))$ iff $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$.
- Transitivity
  - $f(n) = X(g(n))$ and $g(n) = X(h(n))$ implies $f(n) = X(h(n))$. (X can be any of $O, \Omega, \Theta, o, \omega$.
- Reflexivity: $f(n) = X(f(n))$ for $X = O, \Omega, \Theta$.
- Symmetry: $f(n) = \Theta(g(n))$ iff $g(n) = \Theta(f(n))$.
- Transpose symmetry:
  - $f(n) = O(g(n))$ iff $g(n) = \Omega(f(n))$.
  - $f(n) = o(g(n))$ iff $g(n) = \omega(f(n))$.

Analogy with Numeric Inequalities

- $f(n) = O(g(n))$ is like $a \leq b$.
- $f(n) = \Omega(g(n))$ is like $a \geq b$.
- $f(n) = \Theta(g(n))$ is like $a = b$.
- $f(n) = o(g(n))$ is like $a < b$.
- $f(n) = \omega(g(n))$ is like $a > b$.

Asymptotic Notations and Algorithm Analysis

How Asymptotic Notations Are Used In Algorithm Analysis

Lesson Objectives

- Analyze simple algorithms for the asymptotic notations of their running time performance (time complexities)
- Prove formally that the derived asymptotic notations are correct.

Selection Sort Analysis

- Selection sort works in two steps:
  - Find the index of the minimum of array values
  - Place the minimum in the correct position
    - And repeat this process on the smaller array
- Analysis of selection sort:
  - Time for FIND_MIN_INDEX: $T_1(n) = \Theta(n)$
  - Time for INS_SORT: $T_2(n) = T_1(n) + c + T_2(n-1)$
    - This is a recurrence!
    - Solving recurrence, get $T_2(n) = \Theta(n^2)$
- The same is derived with more detail in the following slides

Selection Sort

- Visit http://visualgo.net/sorting
  - Click SEL, then Sort to see how selection sort algorithm works
- Note sub-problem decomposition:
  - For array A[i,n]:
    - Find index of minimum of array values in index i to n. Call it j.
    - Swap A[i] and A[j].
    - Repeat the above for subarray A[i+1,n].
    - Of course if i=n, nothing to do, so just stop.

FindMinIndex(A, s)

- Input: Array A[1,n], starting index s.
- Output: Index j (between s and n) of the minimum of A[s,n]
- E.g., if A={55,88,33,44,99} and s=2, return 3 (the index of 33, which is the minimum of A[2,5])
- Steps:
  - indexOfMinimumSoFar = s
  - for i = s to n do
    - if A[i] < A[indexOfMinimumSoFar]:
      - indexOfMinimumSoFar = i
  - return indexOfMinimumSoFar

SelSort(A)

- Let m=FindMinIndex(A,1). Then swap A[1] and A[m].
- Let m=FindMinIndex(A,2). Then swap A[2] and A[m].
- …
- Steps:
  - for s = 1 to n:
    - m = FindMinIndex(A,s)
    - tmp = A[m]; A[m] = A[s]; A[s] = tmp;   // Swap A[s] and A[m]

SelSort(A) Running Time Analysis

- Count number of operations executed
  - for s = 1 to n: // n times of 1 comparison and 1 increment, plus:
    - m = FindMinIndex(A, s)   // # FindMin(A,1) + # FindMin(A,2) + ... (not n times)
    - tmp = A[m]; A[m] = A[s]; A[s] = tmp;   // 3
- Therefore, T_SelSort(n) = 5n + $\sum_{s=1}^{n} \# \mathit{FindMinIndex}(A, s)$
- # FindMinIndex(A,s):
  - indexOfMinimumSoFar = s   // 1
  - for i = s to n: // n-s+1 times of1 comparison and 1 increment, plus:
    - if A[i] < A[indexOfMinimumSoFar]:   // 1
      - indexOfMinimumSoFar = i   // 0 ~ 1 (1 if condition is true, 0 otherwise)
  - return indexOfMinimumSoFar   // 1
- # FindMinIndex(A,s) = 2(n-s+1)+2+(0~1)*(n-s+1) $\geq 2(n-s)+c_1, \leq 3(n-s)+c_2 \Rightarrow \Theta(n-s)$

Final T_SelSort(n): $\Theta(n^2)$

- T_SelSort(n) = 5n + $\sum_{s=1}^{n} \# \, FindMinIndex(A, s)$
  $$\geq 5n + 2\{(n-1) + (n-2) + \cdots + 1\} + c_1 n$$
  $$= 2\frac{n(n-1)}{2} + 5\mathrm{n} + \mathrm{c}_1\mathrm{n}$$
  $$= n^2 + c_2 n$$
- T_SelSort(n) = 5n + $\sum_{s=1}^{n} \# \, FindMinIndex(A, s)$
  $$\leq 5n + 3\{(n-1) + (n-2) + \cdots + 1\} + c_1 n$$
  $$= 3\frac{n(n-1)}{2} + 5\mathrm{n} + \mathrm{c}_1\mathrm{n}$$
  $$= \frac{3}{2}n^2 + c_3 n$$
- These 2 inequalities fit the definition of $\Theta(n^2)$!