hw2

May 8, 2024

[ ]:
```python
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline
```

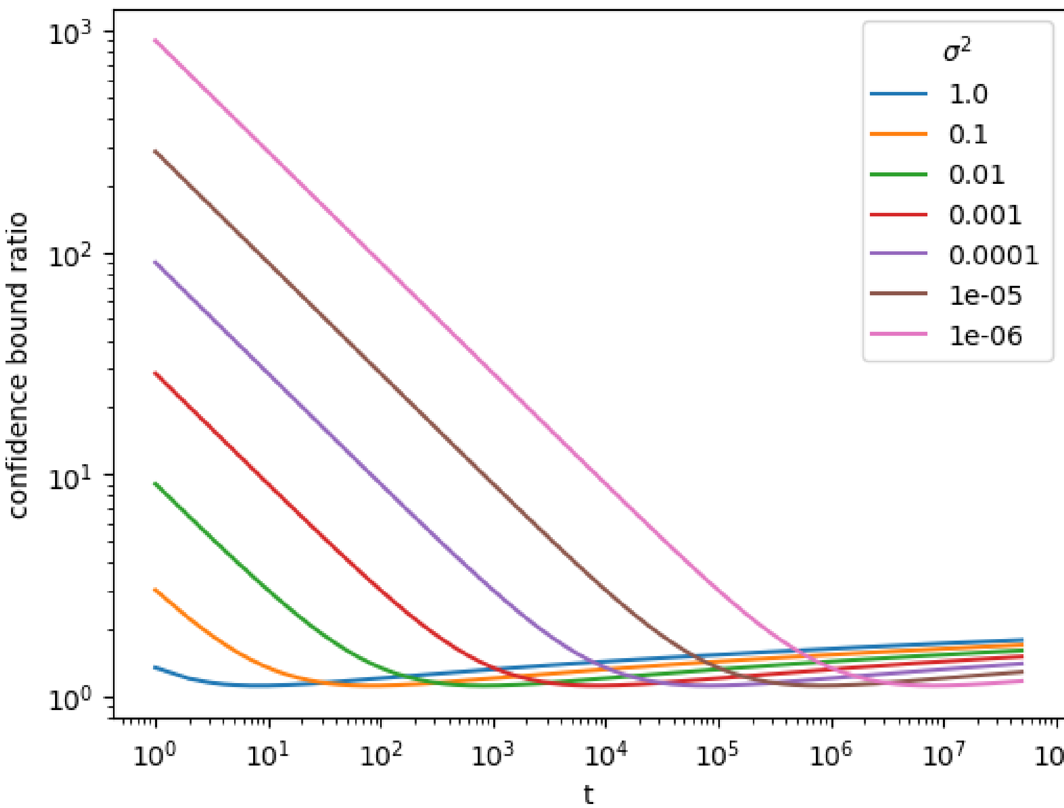
0.1 Ex 4.1

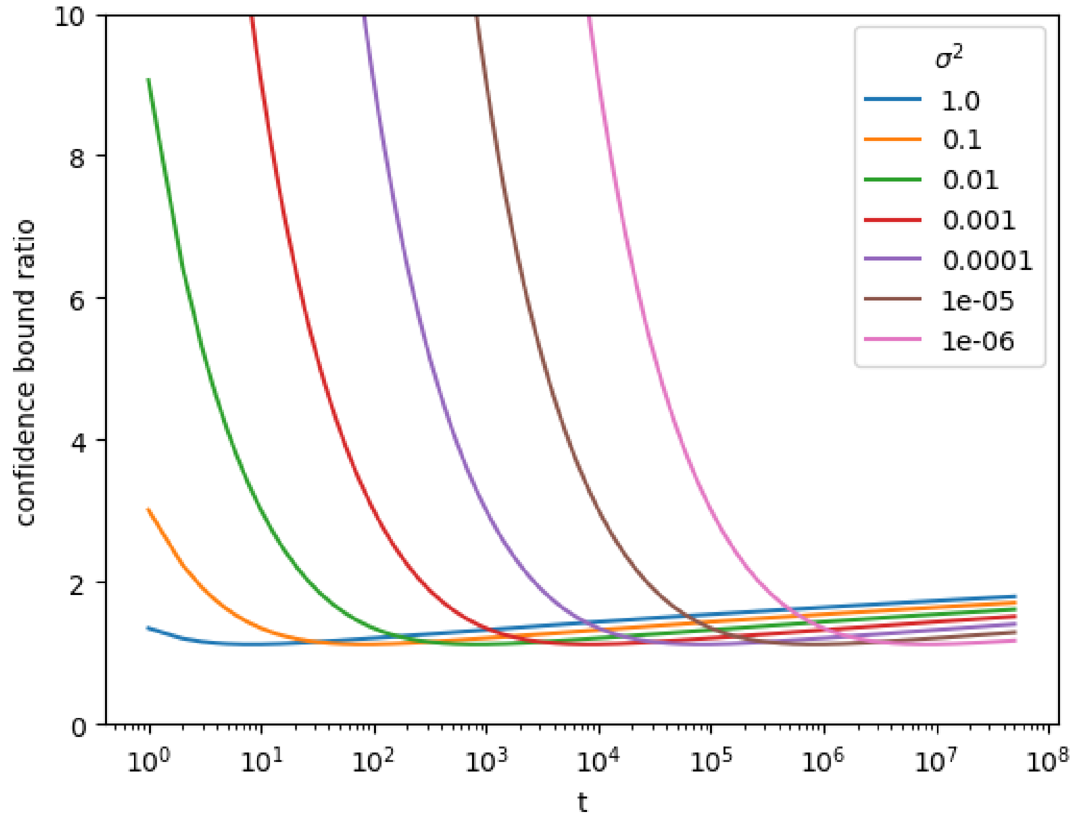
[ ]:
```python
def conf_bound_ratio(T, sigma_sqr, delta):
    t = np.arange(1, T + 1)
    conf_bound_1 = np.sqrt(1 + 1 / (t * sigma_sqr)) * np.sqrt((2 * np.log(1 /
 ↪delta) + np.log(t * sigma_sqr + 1)) / t)
    conf_bound_2 = np.sqrt((2 * np.log(2 / delta)) / t)
    return conf_bound_1 / conf_bound_2
```

[ ]:
```python
delta = 0.05
sigma_sqrs = 1 / (10 ** np.arange(7))
T = 5e7

fig_1, ax_1 = plt.subplots()
fig_2, ax_2 = plt.subplots()
for i in range(len(sigma_sqrs)):
    result = conf_bound_ratio(T, sigma_sqrs[i], delta)
    print(rf"The optimal t for sigma^2 = {sigma_sqrs[i]} is {np.argmin(result)
 ↪+ 1}")
    ax_1.loglog(np.arange(1, T + 1), result, label = sigma_sqrs[i])
    ax_2.semilogx(np.arange(1, T + 1), result, label = sigma_sqrs[i])

ax_1.set_xlabel("t")
ax_1.set_ylabel("confidence bound ratio")
ax_1.legend(title = r"$\sigma^2$")
fig_1.show()

ax_2.set_xlabel("t")
ax_2.set_ylim(0, 10)
ax_2.set_ylabel("confidence bound ratio")
ax_2.legend(title = r"$\sigma^2$")
fig_2.show()
```

```
The optimal t for sigma^2 = 1.0 is 8
The optimal t for sigma^2 = 0.1 is 82
The optimal t for sigma^2 = 0.01 is 821
The optimal t for sigma^2 = 0.001 is 8212
The optimal t for sigma^2 = 0.0001 is 82120
The optimal t for sigma^2 = 1e-05 is 821197
The optimal t for sigma^2 = 1e-06 is 8211968
```

```
C:\Users\Wenhao\AppData\Local\Temp\ipykernel_39988\3795402273.py:16:
UserWarning: FigureCanvasAgg is non-interactive, and thus cannot be shown
  fig_1.show()
C:\Users\Wenhao\AppData\Local\Temp\ipykernel_39988\3795402273.py:22:
UserWarning: FigureCanvasAgg is non-interactive, and thus cannot be shown
  fig_2.show()
```





[ ]:
```
sigma_sqr = 1e-4
t = 82120
delta = 0.05
bound = np.sqrt(1 + 1 / (t * sigma_sqr)) * np.sqrt((2 * (np.log(1 / delta)) +
  np.log(t * sigma_sqr + 1)) / t)
bound
```

[ ]: 0.010591360280542013

[ ]:
```
sigma_sqr = 1e-5
t = 821197
delta = 0.05
bound = np.sqrt(1 + 1 / (t * sigma_sqr)) * np.sqrt((2 * (np.log(1 / delta)) +
  np.log(t * sigma_sqr + 1)) / t)
bound
```

[ ]: 0.003349288318393692

[ ]:
```
bound
```

[ ]: 0.003349288318393692

0.2 4.2

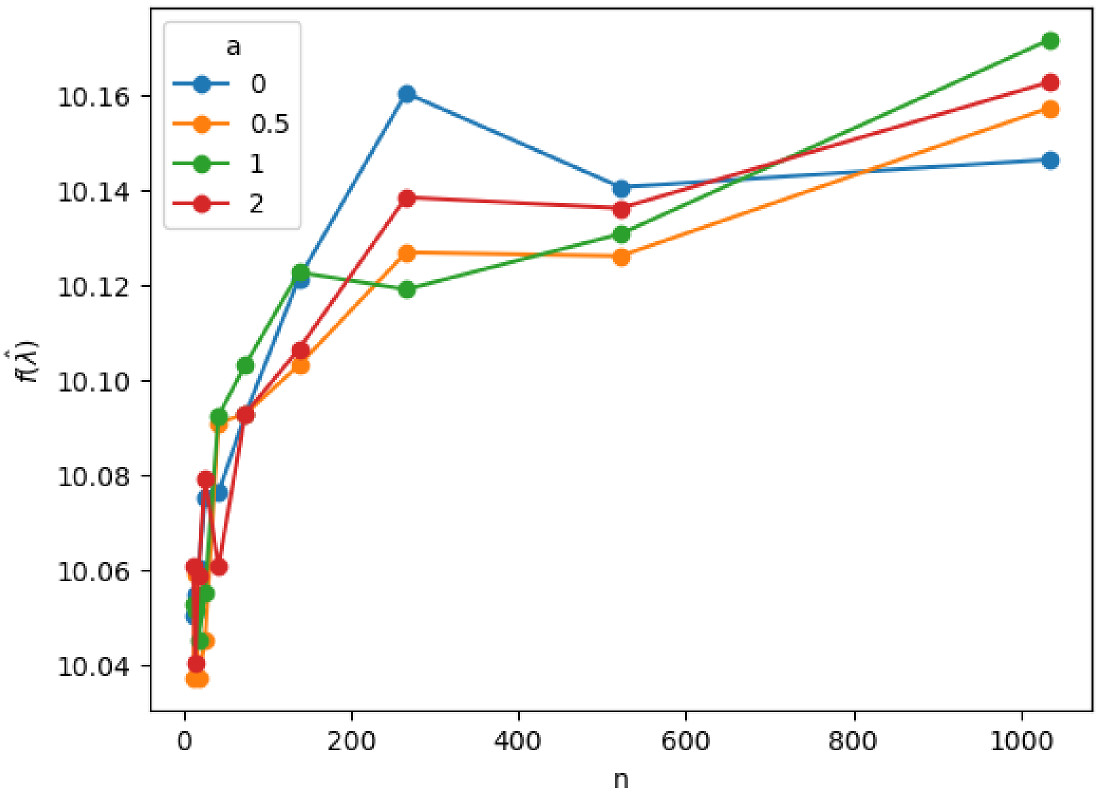
[ ]:
```
def f_design_obj(A, X):
    A_inv = np.linalg.inv(A)
    return np.max(np.diag(X @ A_inv @ X.T))

def g_design_obj(A):
    return np.linalg.det(A)

def greedy(N, n, d, X):
    I = np.zeros(N)
    I[:2 * d] = rng.integers(n, size = 2 * d)

    for t in np.arange(2 * d, N):
        design_objs = np.zeros(n)
        for k in np.arange(n):
            A = np.outer(X[k,:], X[k,:])
            X_I = X[I[:t].astype(int), :]
            A += X_I.T @ X_I
            design_objs[k] = g_design_obj(A)
        I[t] = np.argmax(design_objs)

    return I
```

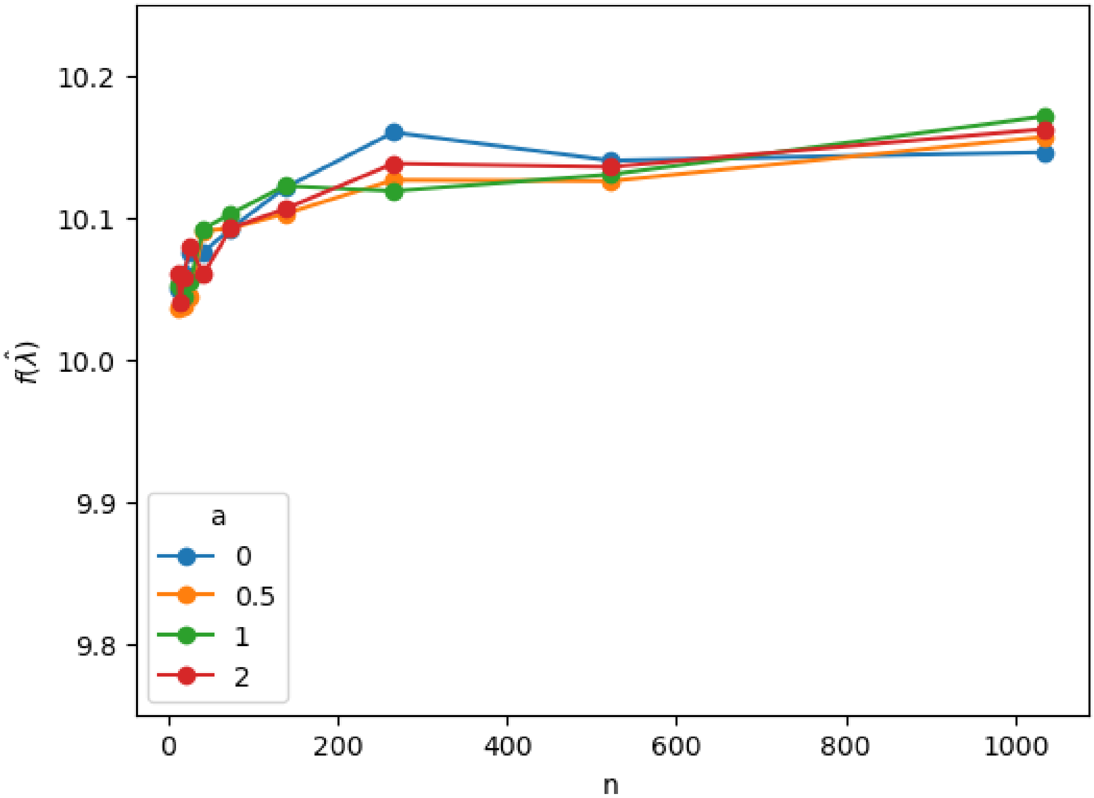
[ ]:
```
n_range = 10 + 2 ** np.arange(1, 11)
d = 10
N = 1000

rng = np.random.default_rng(541)

fig_1, ax_1 = plt.subplots()
fig_2, ax_2 = plt.subplots()

# a = 0
a = 0
cov_mat = np.diag(1 / (np.arange(1, d + 1) ** a))
final_objs_1 = np.zeros(len(n_range))
for i in range(len(n_range)):
    X = rng.multivariate_normal(np.zeros(d), cov_mat, size=n_range[i])
    I = greedy(N, n_range[i], d, X)
    final_objs_1[i] = f_design_obj(X[I.astype(int), :].T @ X[I.astype(int), :] /
 ↪ N, X)

ax_1.plot(n_range, final_objs_1, "o-", label=a)
ax_2.plot(n_range, final_objs_1, "o-", label=a)
```

```python
# a = 0.5
a = 0.5
cov_mat = np.diag(1 / (np.arange(1, d + 1) ** a))
final_objs_2 = np.zeros(len(n_range))
for i in range(len(n_range)):
    X = rng.multivariate_normal(np.zeros(d), cov_mat, size=n_range[i])
    I = greedy(N, n_range[i], d, X)
    final_objs_2[i] = f_design_obj(X[I.astype(int), :].T @ X[I.astype(int), :] /
    ↪ N, X)

ax_1.plot(n_range, final_objs_2, "o-", label=a)
ax_2.plot(n_range, final_objs_2, "o-", label=a)

# a = 1
a = 1
cov_mat = np.diag(1 / (np.arange(1, d + 1) ** a))
final_objs_3 = np.zeros(len(n_range))
for i in range(len(n_range)):
    X = rng.multivariate_normal(np.zeros(d), cov_mat, size=n_range[i])
    I = greedy(N, n_range[i], d, X)
    final_objs_3[i] = f_design_obj(X[I.astype(int), :].T @ X[I.astype(int), :] /
    ↪ N, X)

ax_1.plot(n_range, final_objs_3, "o-", label=a)
ax_2.plot(n_range, final_objs_3, "o-", label=a)

# a = 2
a = 2
cov_mat = np.diag(1 / (np.arange(1, d + 1) ** a))
final_objs_4 = np.zeros(len(n_range))
for i in range(len(n_range)):
    X = rng.multivariate_normal(np.zeros(d), cov_mat, size=n_range[i])
    I = greedy(N, n_range[i], d, X)
    final_objs_4[i] = f_design_obj(X[I.astype(int), :].T @ X[I.astype(int), :] /
    ↪ N, X)

ax_1.plot(n_range, final_objs_4, "o-", label=a)
ax_2.plot(n_range, final_objs_4, "o-", label=a)

ax_1.legend(title="a")
ax_1.set_xlabel("n")
ax_1.set_ylabel(r"$f(\hat{\lambda})$")
fig_1.show()

ax_2.legend(title="a")
ax_2.set_xlabel("n")
```

```
ax_2.set_ylabel(r"$f(\hat{\lambda})$")
ax_2.set_ylim(9.75, 10.25)
fig_2.show()
```

```
C:\Users\Wenhao\AppData\Local\Temp\ipykernel_39988\3946703167.py:61:
UserWarning: FigureCanvasAgg is non-interactive, and thus cannot be shown
  fig_1.show()
C:\Users\Wenhao\AppData\Local\Temp\ipykernel_39988\3946703167.py:67:
UserWarning: FigureCanvasAgg is non-interactive, and thus cannot be shown
  fig_2.show()
```





0.3 4.3

```
[ ]: import numpy as np
np.random.seed(541)
n = 300
X = np.concatenate((np.linspace(0, 1, 50), 0.25 + 0.01 * np.random.randn(250)),␣
 ↪0)
X = np.sort(X)

K = np.zeros ((n, n))
for i in range(n):
    for j in range(n):
        K[i, j] = 1 + min(X[i], X[j])

e, v = np.linalg.eigh(K) # eigenvalues are increasing in order
d = 30
Phi = np.real(v @ np.diag(np.sqrt(np.abs(e))))[:, (n - d)::]

def f(x):
    return -x ** 2 + x * np.cos(8 * x) + np.sin(15 * x)
```

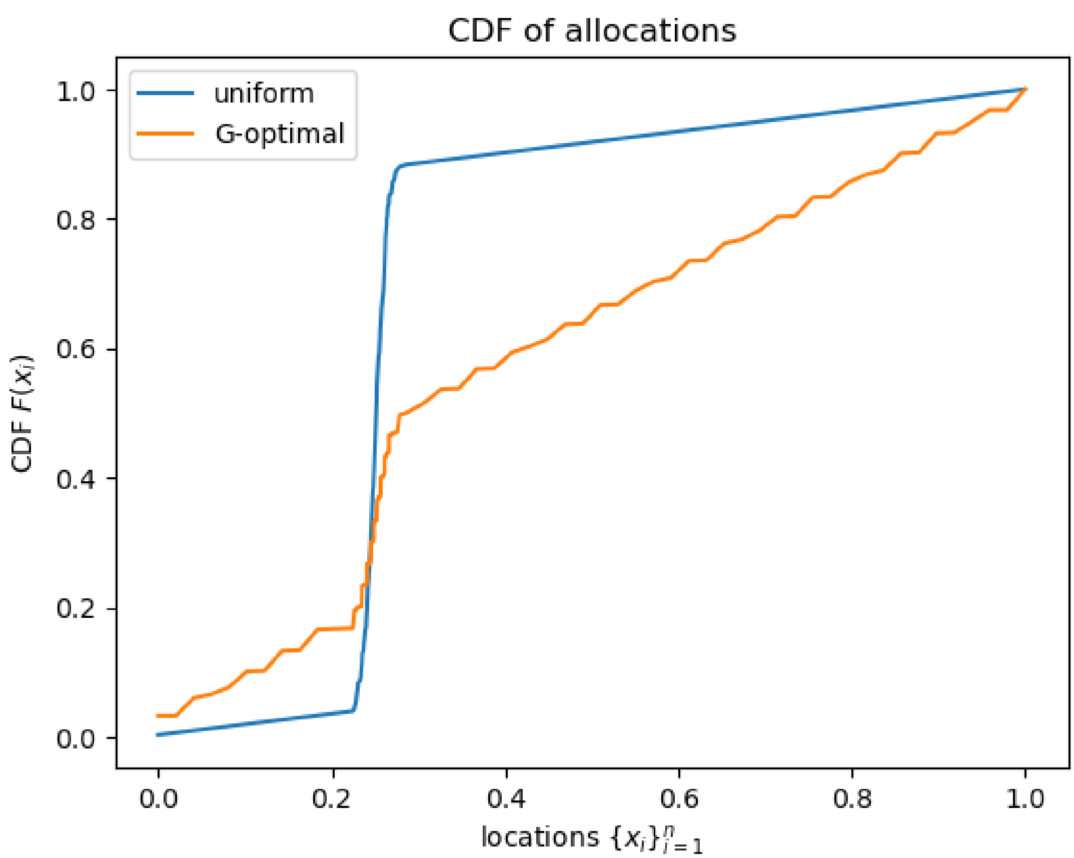
```python
f_star = f(X)

theta = np.linalg.lstsq(Phi, f_star, rcond = None)[0]
f_hat = Phi @ theta
```

[ ]:
```python
def G_optimal(N, n, d, X):
    def g_design_obj(A):
        return np.linalg.det(A)

    I = np.zeros(N)
    I[:2 * d] = np.random.randint(n, size = 2 * d)

    for t in np.arange(2 * d, N):
        design_objs = np.zeros(n)
        for k in np.arange(n):
            A = np.outer(X[k,:], X[k,:])
            X_I = X[I[:t].astype(int), :]
            A += X_I.T @ X_I
            design_objs[k] = g_design_obj(A)
        I[t] = np.argmax(design_objs)

    lbda = np.zeros(n)
    for i in I:
        lbda[int(i)] += 1
    lbda /= N

    return lbda

def observe(idx):
    return f(X[idx]) + np.random.randn(len(idx))

def sample_and_estimate(X, lbda, tau):
    n, d = X.shape
    reg = 1e-6 # we can add a bit of regularization to avoid divide by 0
    idx = np.random.choice(np.arange(n), size = tau , p = lbda)
    y = observe(idx)

    XtX = X[idx].T @ X[idx]
    XtY = X[idx].T @ y

    theta = np.linalg.lstsq (XtX + reg * np.eye(d), XtY, rcond = None)[0]
    return Phi @ theta, XtX

T = 1000

lbda_G = G_optimal(1000, n, 30, Phi)
f_G_Phi, A = sample_and_estimate(Phi, lbda_G, T)
```

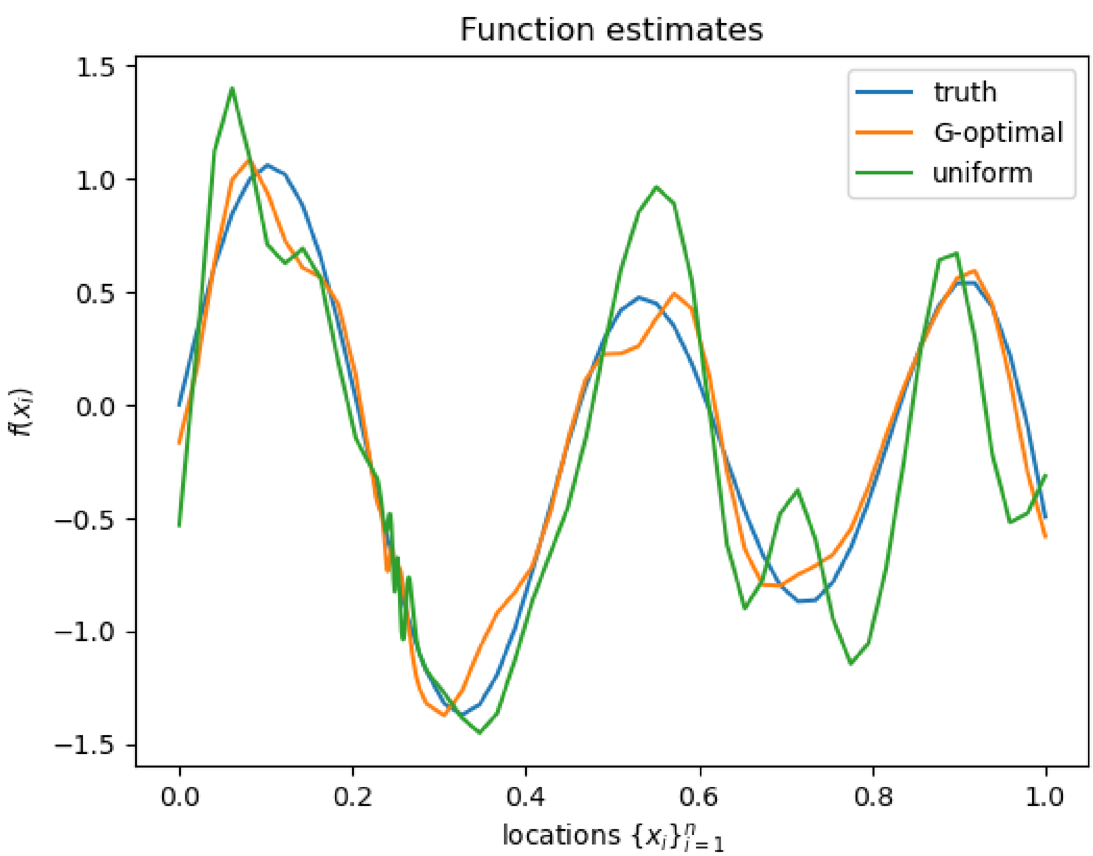
```
conf_G = np.sqrt(np.sum(Phi @ np.linalg.inv(A) * Phi, axis = 1))

lbda_unif = np.ones(n) / n
f_unif_Phi, A = sample_and_estimate(Phi, lbda_unif , T)
conf_unif = np.sqrt(np.sum(Phi @ np.linalg.inv(A) * Phi, axis = 1))
```

[ ]:
```
from scipy.stats import uniform

def ecdf(a):
    x, counts = np.unique(a, return_counts=True)
    cusum = np.cumsum(counts)
    return x, cusum / cusum[-1]
```

[ ]:
```
fig, ax = plt.subplots()
ax.plot(X, ecdf(X)[1], label = "uniform")
ax.plot(X, np.cumsum(lbda_G), label = "G-optimal")
ax.legend()
ax.set_xlabel(r"locations $\{x_i\}_{i=1}^n$")
ax.set_ylabel(r"CDF $F(x_i)$")
ax.set_title("CDF of allocations")
plt.show()
```

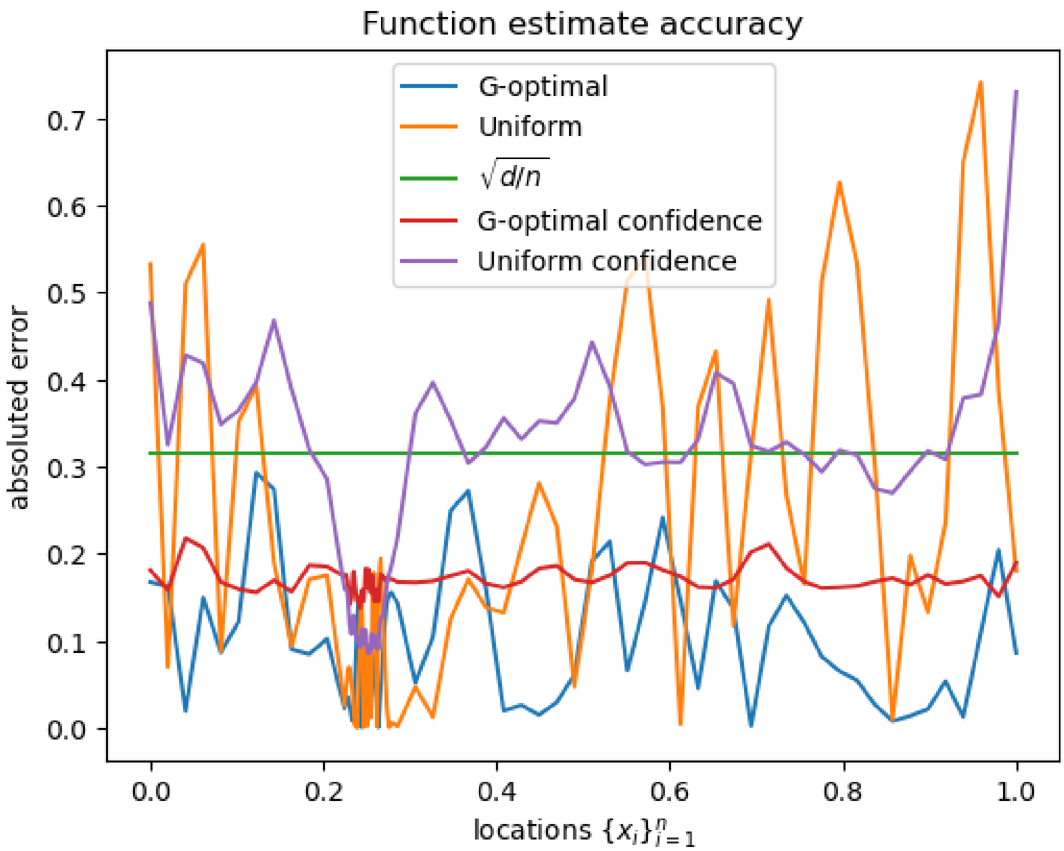


[ ]:
```
fig, ax = plt.subplots()
ax.plot(X, f_star, label = "truth")
ax.plot(X, f_G_Phi, label = "G-optimal")
ax.plot(X, f_unif_Phi, label = "uniform")
ax.legend()
ax.set_xlabel(r"locations $\{x_i\}_{i=1}^n$")
ax.set_ylabel(r"$f(x_i)$")
ax.set_title("Function estimates")
plt.show()
```



[ ]:
```
fig, ax = plt.subplots()
ax.plot(X, np.abs(f_G_Phi - f_star), label = "G-optimal")
ax.plot(X, np.abs(f_unif_Phi - f_star), label = "Uniform")
ax.plot(X, np.ones_like(X) * np.sqrt(d / n), label = r"$\sqrt{d/n}$")
ax.plot(X, conf_G, label = "G-optimal confidence")
ax.plot(X, conf_unif, label = "Uniform confidence")
ax.legend()
```

```
ax.set_xlabel(r"locations $\{x_i\}_{i=1}^n$")
ax.set_ylabel(r"absoluted error")
ax.set_title("Function estimate accuracy")
plt.show()
```



0.4 4.4

```
[ ]: def generate_data():
         def f(x):
             return -x ** 2 + x * np.cos(8 * x) + np.sin(15 * x)

         n = 300
         X = np.concatenate((np.linspace(0, 1, 50), 0.25 + 0.01 * np.random.
      ↪randn(250)), 0)
         X = np.sort(X)

         K = np.zeros ((n, n))
         for i in range(n):
             for j in range(n):
```

```
            K[i, j] = 1 + min(X[i], X[j])

    e, v = np.linalg.eigh(K) # eigenvalues are increasing in order
    d = 30
    Phi = np.real(v @ np.diag(np.sqrt(np.abs(e))))[:, (n - d)::]

    f_star = f(X)

    return X, Phi, f_star
```

[ ]:
```
def Elimination(T, X, Phi):
    def observe(idx):
        return f(X[idx]) + np.random.randn(len(idx))

    tau = 100
    delta = 1 / T
    gam = 1
    U = 1
    N = 1000

    V_k = gam * np.eye(d)
    S_k = 0
    n, D = Phi.shape
    active_idx = np.arange(n).tolist()

    y_t = np.zeros(T)

    for k in np.arange(1, int(np.floor(T / tau) + 1)):
    # k = 1

        # n, d = X_k.shape
        n = len(active_idx)
        X_active = Phi[active_idx]

        # first line
        lam_k = G_optimal(N, n, 30, X_active)

        # second line
        idxs = np.random.choice(active_idx, size = tau, replace=True, p = lam_k)

        # third line
        xs = Phi[idxs]
        ys = observe(idxs)
        y_t[(k-1) * tau: k*tau] = ys

        # fourth line
        V_k += xs.T @ xs
```

```
        S_k += (xs.T * ys).T.sum(axis = 0)
        theta_k = np.linalg.solve(V_k, S_k)

        # fifth line
        beta_k = np.sqrt(gam) * U + np.sqrt(2 * np.log(1 / delta) + np.log(np.
↪linalg.det(V_k) / gam ** D))

        # sixth line
        x_k = X_active[np.argmax(X_active @ theta_k)]

        # last line
        # active_idx = [
        #     idx for idx in active_idx if
        #     np.dot(x_k - Phi[idx], theta_k) <
        #     (beta_k * np.sqrt(np.dot(x_k - Phi[idx], np.linalg.solve(V_k, x_k␣
↪- Phi[idx]))))
        # ]
        temp = []
        for idx in active_idx:
            diff = x_k - Phi[idx]
            lhs = np.dot(diff, theta_k)
            rhs = beta_k * np.sqrt(np.dot(diff, np.linalg.solve(V_k, diff)))
            # rhs = beta_k * np.dot(diff, np.linalg.solve(V_k, diff))
            if lhs <= rhs:
                temp.append(idx)
        active_idx = temp

    return y_t
```

[ ]:
```
def UCB(T, X, Phi):
    def observe(idx):
        return f(X[idx]) + np.random.randn(len(idx))

    delta = 1 / T
    gam = 1
    U = 1

    V_t = gam * np.eye(d)
    S_t = np.zeros(d)
    n, D = Phi.shape

    y_t = np.zeros(T)

    for t in range(T):
        beta_t = np.sqrt(gam) * U + np.sqrt(2 * np.log(1 / delta) + np.log(np.
↪linalg.det(V_t) / gam ** D))
        theta_t = np.linalg.solve(V_t, S_t)
```

```
        max_idx = np.argmax(Phi @ theta_t + beta_t * np.sqrt(np.diag(Phi @ np.
↪linalg.inv(V_t) @ Phi.T)))
        x_t = Phi[max_idx]
        y_t[t] = observe(np.array([max_idx]))[0]
        V_t += np.outer(x_t, x_t)
        S_t += x_t * y_t[t]

    return y_t
```

[ ]:
```
def Thompson(T, X, Phi):
    def observe(idx):
        return f(X[idx]) + np.random.randn(len(idx))

    gam = 1
    V_t = gam * np.eye(d)
    S_t = np.zeros(d)
    n, D = Phi.shape

    y_t = np.zeros(T)

    for t in range(T):
        V_t_inv = np.linalg.inv(V_t)
        theta_t = V_t_inv @ S_t
        theta_t_sample = np.random.multivariate_normal(theta_t, V_t_inv)
        max_idx = np.argmax(Phi @ theta_t_sample)
        x_t = Phi[max_idx]
        y_t[t] = observe(np.array([max_idx]))[0]
        V_t += np.outer(x_t, x_t)
        S_t += x_t * y_t[t]

    return y_t
```

[ ]:
```
np.random.seed(541)

trial_1 = generate_data()
Elimination_regret_1 = np.cumsum(trial_1[2].max() - Elimination(40000,␣
↪trial_1[0], trial_1[1]))
UCB_regret_1 = np.cumsum(trial_1[2].max() - UCB(40000, trial_1[0], trial_1[1]))
Thompson_regret_1 = np.cumsum(trial_1[2].max() - Thompson(40000, trial_1[0],␣
↪trial_1[1]))


trial_2 = generate_data()
Elimination_regret_2 = np.cumsum(trial_2[2].max() - Elimination(40000,␣
↪trial_2[0], trial_2[1]))
UCB_regret_2 = np.cumsum(trial_2[2].max() - UCB(40000, trial_2[0], trial_2[1]))
```

```
Thompson_regret_2 = np.cumsum(trial_2[2].max() - Thompson(40000, trial_2[0],␣
 ↪trial_2[1]))

trial_3 = generate_data()
Elimination_regret_3 = np.cumsum(trial_3[2].max() - Elimination(40000,␣
 ↪trial_3[0], trial_3[1]))
UCB_regret_3 = np.cumsum(trial_3[2].max() - UCB(40000, trial_3[0], trial_3[1]))
Thompson_regret_3 = np.cumsum(trial_3[2].max() - Thompson(40000, trial_3[0],␣
 ↪trial_3[1]))

trial_4 = generate_data()
Elimination_regret_4 = np.cumsum(trial_4[2].max() - Elimination(40000,␣
 ↪trial_4[0], trial_4[1]))
UCB_regret_4 = np.cumsum(trial_4[2].max() - UCB(40000, trial_4[0], trial_4[1]))
Thompson_regret_4 = np.cumsum(trial_4[2].max() - Thompson(40000, trial_4[0],␣
 ↪trial_4[1]))

trial_5 = generate_data()
Elimination_regret_5 = np.cumsum(trial_5[2].max() - Elimination(40000,␣
 ↪trial_5[0], trial_5[1]))
UCB_regret_5 = np.cumsum(trial_5[2].max() - UCB(40000, trial_5[0], trial_5[1]))
Thompson_regret_5 = np.cumsum(trial_5[2].max() - Thompson(40000, trial_5[0],␣
 ↪trial_5[1]))
```

[ ]:
```
fig, ax = plt.subplots()
ax.plot(np.arange(T), Elimination_regret_1, color = "green", alpha = 0.3,␣
 ↪linewidth=0.7)
ax.plot(np.arange(T), Elimination_regret_2, color = "green", alpha = 0.3,␣
 ↪linewidth=0.7)
ax.plot(np.arange(T), Elimination_regret_3, color = "green", alpha = 0.3,␣
 ↪linewidth=0.7)
ax.plot(np.arange(T), Elimination_regret_4, color = "green", alpha = 0.3,␣
 ↪linewidth=0.7)
ax.plot(np.arange(T), Elimination_regret_5, color = "green", alpha = 0.3,␣
 ↪linewidth=0.7)
ax.plot(np.arange(T),
        (Elimination_regret_1 + Elimination_regret_2 + Elimination_regret_3 + +␣
 ↪Elimination_regret_4 + + Elimination_regret_5) / 5,
        color = "green", label = "Elimination (averaged)")

ax.plot(np.arange(T), UCB_regret_1, color = "red", alpha = 0.3, linewidth=0.7)
ax.plot(np.arange(T), UCB_regret_2, color = "red", alpha = 0.3, linewidth=0.7)
ax.plot(np.arange(T), UCB_regret_3, color = "red", alpha = 0.3, linewidth=0.7)
ax.plot(np.arange(T), UCB_regret_4, color = "red", alpha = 0.3, linewidth=0.7)
ax.plot(np.arange(T), UCB_regret_5, color = "red", alpha = 0.3, linewidth=0.7)
ax.plot(np.arange(T),
```

```
        (UCB_regret_1 + UCB_regret_2 + UCB_regret_3 + + UCB_regret_4 + +␣
↪UCB_regret_5) / 5,
        color = "red", label = "UCB (averaged)")

ax.plot(np.arange(T), Thompson_regret_1, color = "blue", alpha = 0.3,␣
↪linewidth=0.7)
ax.plot(np.arange(T), Thompson_regret_2, color = "blue", alpha = 0.3,␣
↪linewidth=0.7)
ax.plot(np.arange(T), Thompson_regret_3, color = "blue", alpha = 0.3,␣
↪linewidth=0.7)
ax.plot(np.arange(T), Thompson_regret_4, color = "blue", alpha = 0.3,␣
↪linewidth=0.7)
ax.plot(np.arange(T), Thompson_regret_5, color = "blue", alpha = 0.3,␣
↪linewidth=0.7)
ax.plot(np.arange(T),
        (Thompson_regret_1 + Thompson_regret_2 + Thompson_regret_3 + +␣
↪Thompson_regret_4 + + Thompson_regret_5) / 5,
        color = "blue", label = "Thompson (averaged)")

ax.set_xlabel("Time")
ax.set_ylabel("Regret")
ax.legend()
plt.show()
```