Math 104 Lec 004 (M-Th 4-6 pm)

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(private posts)

Office hours: Tu 9-10 pm, Th 1-3 pm PDT
(tentative) (same Zoom ID)

Homework

- 7 graded HWs, due Fridays, 11:59 pm
  via Gradescope
- lowest HW dropped
- HW 1 due this Friday, June 25.

Exams

Midterm: Thursday, July 15

Final: Thursday, August 12.

Grading

better of

30% HW
30% MT
40% F

OR

30% HW
70% F

Textbook:

Elementary Analysis: The Theory of Calculus
(Ross) (2nd edition)

(free - SpringerLink)
use Berkeley VPN.

Recommended:
Principles of Math. Analysis
- Pugh - Rudin

Motivating the study of real numbers

- Begin with the natural numbers

  $\mathbb{N} = \{1, 2, 3, \ldots\}$ can add, mult.

- To be able to freely subtract, expand to integers

  $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

- To be able to freely divide, expand to rationals

  $\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z},\ n \neq 0 \right\}$.

Remark: not quite rigorous to identify rationals with pairs of integers; need the notion of equivalence classes.

For this class, mainly just care about properties of $\mathbb{Q}$ as an algebraic system.

Specifically, $\mathbb{Q}$ satisfies the properties of an ordered field.

Ordered field a set with operations + and × satisfying some properties:

properties of a field.

<u>Addition</u>:

$a+(b+c) = (a+b)+c$ <u>associativity</u>

$a+b = b+a$ <u>commutativity</u>

$a+0 = a$ <u>identity</u>.

for any element a, there exists $-a$ such that $a+(-a)=0$. <u>inverse</u>.

<u>Multiplication</u>:

$a(bc) = (ab)c$ associativity

$ab = ba$ commutativity.

$a\cdot 1 = a$ identity

for any $a\neq 0$, there exists a^{-1} such that $aa^{-1}=1$. <u>inverse</u>.

<u>Distributivity</u>

$a(b+c) = ab+ac$

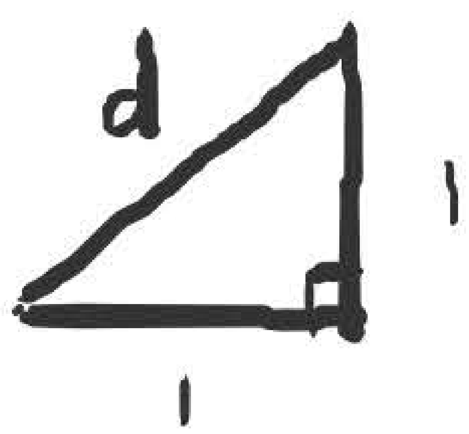
Order structure:

1. Given a and b, either $a \le b$ or $b \le a$ (or both)
2. If $a \le b$ and $b \le a$, then $a = b$.
3. If $a \le b$ and $b \le c$, then $a \le c$. (transitivity)
4. If $a \le b$, then $a + c \le b + c$ for any c.
5. If $a \le b$ and $c \ge 0$, then $ac \le bc$.

<u>Definition</u>: <u>$a < b$</u> means that $a \le b$ but $a \ne b$.

$\mathbb{Q}$ seems very good! But there are still some issues.

- there are "gaps".

e.g. 

$d = \sqrt{2}$ is not rational!

- $\mathbb{Q}$ do not satisfy the LUBP.

<u>least upper bound property</u>: An ordered set S has the <u>LUBP</u> if every nonempty subset $A \subseteq S$ that has an upper bound has a least upper bound.

e.g. $\{1, 2, 3, 5\}$

10 is an upper bound.
8 is an upper bound.
5 is the <u>least</u> upper bound.

<u>Ex</u>. $A = \{ q \in \mathbb{Q} : \underbrace{q^2 < 2}_{q < \sqrt{2} = 1.41421356237...} \}$

some upper bounds: $1.5,\ 1.42,\ 1.415,\ 1.4143,\ \ldots$

– see rigorous proof later.

Exercise: Prove that $\sqrt{2}$ is not rational.

Proof: (Proof by contradiction)

Suppose $\sqrt{2}$ is rational. $\sqrt{2} = \frac{m}{n}$, $\gcd(m,n) = 1$.

$$2 = \frac{m^2}{n^2}$$

$$2n^2 = m^2$$

$\Rightarrow m^2$ is even, so m is even, say $m = 2k$ for some $k \in \mathbb{Z}$.

$$2n^2 = (2k)^2$$

$$2n^2 = 4k^2$$

$$n^2 = 2k^2$$

$\Rightarrow \gcd(m,n) \geq 2$.

$\Rightarrow n^2$ is even, so n is even. Contradiction.

Can also think about $\sqrt{2}$ as a solution to the polynomial equation $x^2 - 2 = 0$. "algebraic number"

There are also nonalgebraic irrational numbers: π, e

<u>The Real numbers</u> $\mathbb{R}$

- There is a rigorous, abstract construction of $\mathbb{R}$ via <u>dedekind cuts</u> - treatment in textbook (optional).
- For our purposes, we will accept $\mathbb{R}$ as the set of numbers that can be identified on the real number line.

Observation: $\mathbb{R}$ is an ordered field.