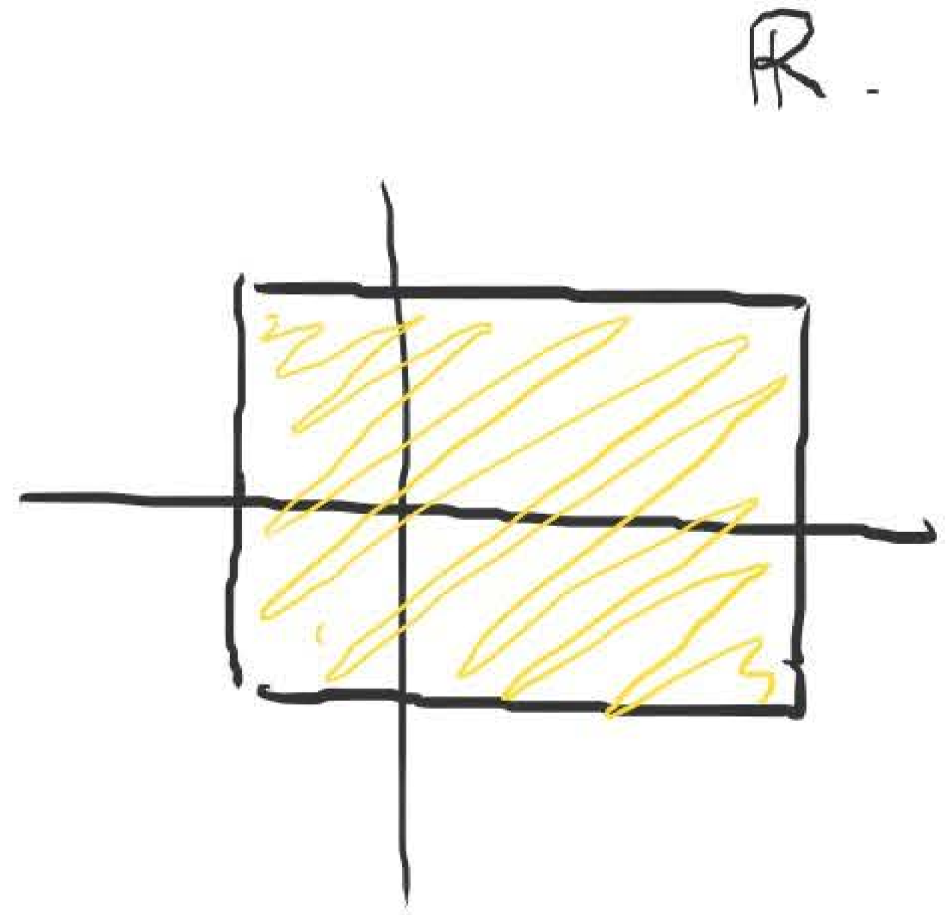
Tuesday, July 13



Recall: We proved that k-cells are compact in $\mathbb{R}^k$.

$$[a_1,b_1]\times[a_2,b_2]\times\dots\times[a_k,b_k].$$

Theorem: A set $E\subseteq\mathbb{R}^k$ is compact if and only if it is closed and bounded.

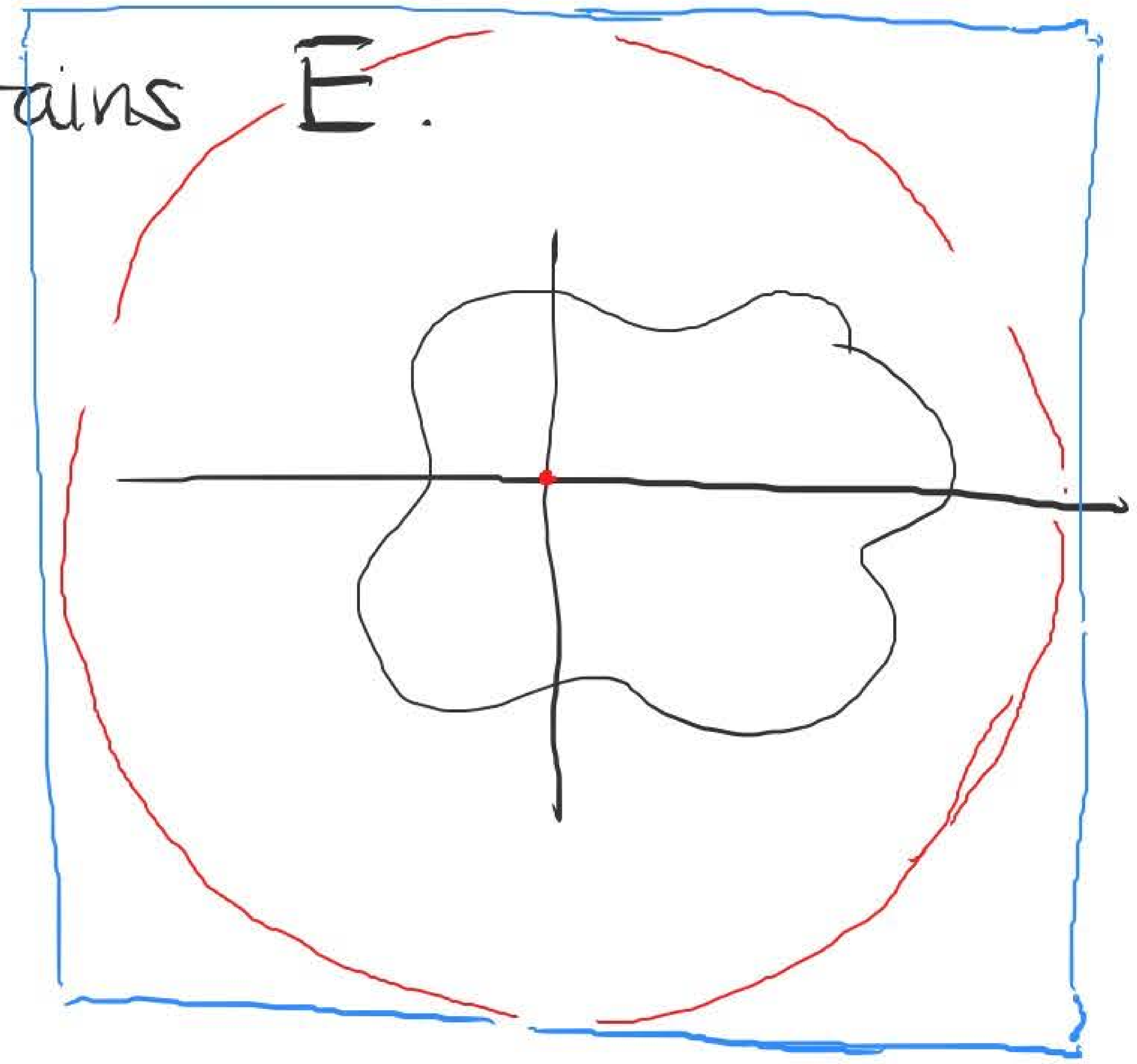
Proof: $\Rightarrow$ | True in any metric. Know already that compact sets are always closed.

Let $x\in X$ (metric space). $\{B_n(x)\}_{n\in\mathbb{N}}$ is an open cover of E.

Since E is compact, this open cover has a finite subcover $B_{n_1}(x),\dots,B_{n_\ell}(x)$. Then $\bigcup_{j=1}^{\ell} B_{n_j}(x) = B_{\max(n_1,\dots,n_\ell)}(x)$ and $E\subseteq B_{\max(n_1,\dots,n_k)}(x)$. Therefore E is bounded.

$\Leftarrow$ Suppose $E \subseteq \mathbb{R}^k$ is closed and bounded.

There exists a k-cell which contains E.



Since this k-cell is compact and E is a closed subset of this compact, therefore E is compact.

Math 104 Worksheet 8

UC Berkeley, Summer 2021

Monday, July 12

On Worksheet 8, we showed that in a metric space (X, d), if E is a compact set then every sequence in E has a convergent subsequence (whose limit lies in E). This worksheet guides a proof of the converse.

In general, Conv $\Rightarrow$ Cauchy.

Suppose (x_n) conv. to x. Let $\varepsilon > 0$. Since $x_n \to x$, there exists $N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow d(x_n, x) < \frac{\varepsilon}{2}$. Then $m, n \geq N$,

$$d(x_m, x_n) \leq \underbrace{d(x_m, x)}_{<\varepsilon/2} + \underbrace{d(x, x_n)}_{<\varepsilon/2} < \varepsilon.$$

Lemma 1. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of E. If every sequence in E has a convergent subsequence whose limit is in E, then there exists $\varepsilon > 0$ such that for every $x \in E$, there exists $\alpha \in A$ such that $B_\varepsilon(x) \subseteq U_\alpha$.

different than saying "for every $x \in E$, there exists U_α s.t. there exists $\varepsilon > 0$: $B_\varepsilon(x) \subseteq U_\alpha$".

Proof. (Contrapositive) Suppose that for any $\varepsilon > 0$, there exists $x \in E$ such that $B_\varepsilon(x) \not\subseteq U_\alpha$ for all $\alpha \in A$. Then for each $n \in \mathbb{N}$, there exists x_n such that $V_n := B_{1/n}(x_n) \not\subseteq U_\alpha$ for all $\alpha \in A$.

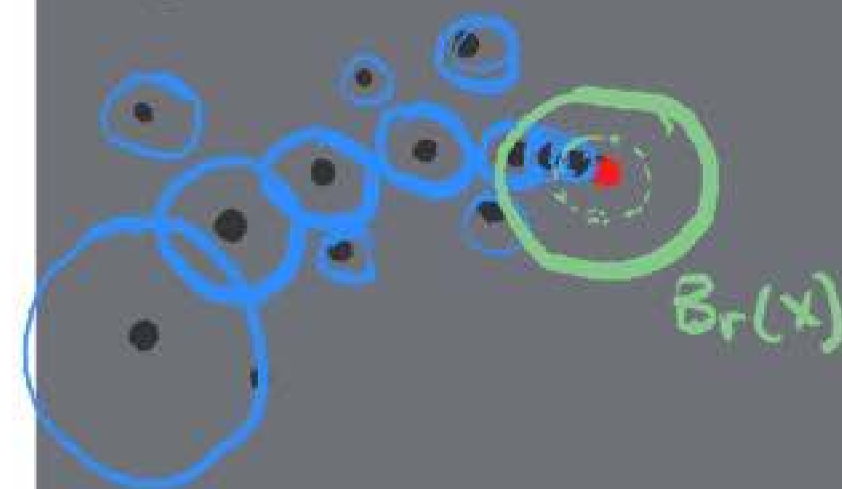
Claim: (x_n) does not have a convergent subsequence. to an element of E.

Exercise 1. Prove the claim by contradiction. (*Hint:* If (x_n) did have a convergent subsequence, to some $x \in E$. then the limit x would be in U_α for some α. Since U_α is open, there is an open ball around x that fits inside U_α. Show that some V_n fits inside that ball.)

Suppose (x_n) has a convergent subsequence (x_{n_k}) to $x \in E$. Then $x \in U_\alpha$ for some α. There exists $r > 0$ s.t. $B_r(x) \subseteq U_\alpha$. There exists $K \in \mathbb{N}$: $k \geq K \Rightarrow x_{n_k} \in B_{\frac{r}{2}}(x)$ ... Pick x_{n_K} s.t.

$$V_{n_K} = B_{\frac{1}{n_K}}(x_{n_K}) \subseteq B_r(x) \subseteq U_\alpha.$$

contradiction.



Lemma 2. If every sequence in E has a convergent subsequence whose limit is in E, then for any $\varepsilon > 0$ there exists a finite collection $x_1, \ldots, x_n$ of points in E such that $E \subseteq \bigcup_{i=1}^n B_\varepsilon(x_i)$.

Proof. (Contrapositive) Suppose that for some $\varepsilon > 0$, E cannot be covered by finitely many open balls of radius ε.

Exercise 2. Construct (inductively) a sequence (x_n) in E such that $d(x_m, x_n) \geq \varepsilon$ for any $m \neq n$. Explain why this sequence has no convergent subsequence.

Let $x_1 \in E$. Having already chosen $x_1, x_2, \ldots, x_{k-1}$, choose $x_k \in E$ but $x_k \notin \bigcup_{i=1}^{k-1} B_\varepsilon(x_i)$. $d(x_k, x_i) \geq \varepsilon$ for $i = 1, \ldots, k-1$.

Then for (x_n), $d(x_m, x_n) \geq \varepsilon$ for $m \neq n$. Any subsequence of (x_n) has the same property $\Rightarrow$ not Cauchy $\Rightarrow$ not convergent.

Theorem. E is compact if and only if every sequence in E has a convergent subsequence whose limit is in E.

Proof. (The forward direction has already been proven.) Suppose every sequence in E has a convergent subsequence whose limit is in E. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of E.

Exercise 3. Use Lemma 1 and Lemma 2 to show that $\{U_\alpha\}_{\alpha \in A}$ has a finite subcover.

By Lemma 1, there exists $\varepsilon > 0$ such that for every $x \in E$, $B_\varepsilon(x) \subseteq U_{\alpha_x}$ for some α_x. There exists $x_1, \ldots, x_n \in E$ s.t. $E \subseteq \bigcup_{i=1}^n B_\varepsilon(x_i) \subseteq \bigcup_{i=1}^n U_{\alpha_{x_i}}$ ← finite subcover.

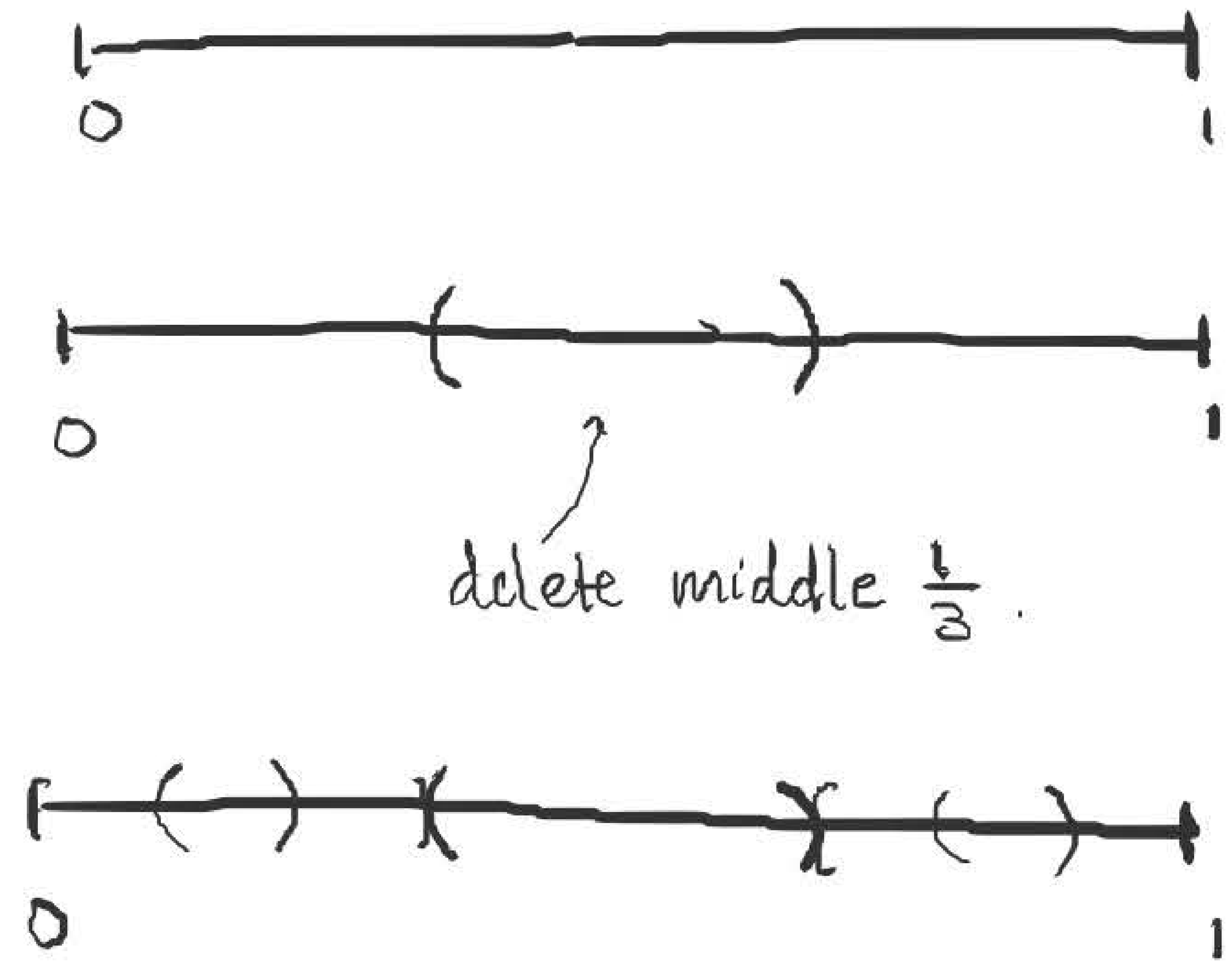
Cantor set

Define $C = \bigcap_{i=0}^{\infty} C_i$.

2^0 intervals C_0 — $[0,1]$

2^1 interval C_1 — $[0,\frac{1}{3}] \cup [\frac{2}{3},1]$.

2^2 interval. C_2 — $[0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1]$.



⋮

C_n union of 2^n closed intervals, each of length $(\frac{1}{3})^n$.

Series of real numbers.

Summation notation:

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \dots + a_n .$$

Infinite series:

$$\sum_{k=m}^{\infty} a_k \stackrel{\text{def}}{=} \lim_{n\to\infty} \underbrace{\sum_{k=m}^{n} a_k}_{\text{sequence}} .$$

$$(a_m,\ a_m+a_{m+1},\ a_m+a_{m+1}+a_{m+2},\ \dots).$$

$(s_1, s_2, s_3, \dots)$ where

$$s_n = \sum_{k=m}^{m+n-1} a_k .$$

"n^{th} partial sum of the series $\sum_{k=m}^{\infty} a_k$"

Def: An infinite series $\sum_{n=m}^{\infty} a_n$ is said to converge if the sequence of partial sums (s_n) converges, in which case we define $\sum_{n=m}^{\infty} a_n = \lim_{n\to\infty} s_n$.

- An infinite series is said to diverge if it does not converge.
- If $\lim s_n = \infty$ or $-\infty$, then we say the series diverges to ∞ (or $-\infty$).

Example: If $a_n \geq 0$ for all n, then $\sum_{k=m}^{\infty} a_n$ either converges or diverges to ∞.

Def: $\sum_{k=m}^{\infty} a_k$ converges absolutely if $\sum_{k=m}^{\infty} |a_k|$ converges.

Ex. $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$ converges, but not absolutely. (to be proven later).

$(1-r)(1+r+r^2+\dots+r^n)$

$= 1 + \cancel{r} + \cancel{r^2} + \dots + \cancel{r^n} - \cancel{r} - \cancel{r^2} - \dots - r^{n+1} = 1 - r^{n+1}$.

Ex. Geometric series.

$$\sum_{k=0}^{n} r^k = 1 + r + r^2 + \dots + r^n = \frac{1-r^{n+1}}{1-r} \quad (r \neq 1)$$

If $|r|<1$, then $\sum_{k=0}^{\infty} r^k = \lim_{n\to\infty} \frac{1-r^{n+1}}{1-r} = \frac{1}{1-r}$. since $r^{n+1} \xrightarrow[|r|<1]{} 0$.

Def: A series $\sum a_n$ satisfies the Cauchy criterion if its sequence of partial sums is Cauchy.

shorthand for infinite series

Meaning: (s_n) Cauchy: for any $\varepsilon>0$, there exists $N \in \mathbb{N}$: $n \geq m \geq N \Rightarrow |s_n - s_m| < \varepsilon$.

$\sum_{k=1}^{n} a_k$ $\sum_{k=1}^{m} a_k$.

there exists $N \in \mathbb{N}$ such that

$$n \geq m \geq N \Rightarrow \left| \sum_{k=m}^{n} a_k \right| < \varepsilon.$$

Theorem: A series converges if and only if it satisfies the Cauchy criterion.

$s_n - s_m = \sum_{k=m+1}^{n} a_k$.

Corollary: If a series $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

(follows from Cauchy criterion with $m=n$).