Math 104 Worksheet 16

UC Berkeley, Summer 2021

Tuesday, August 3

Lemma. Let f be defined on an open interval I containing x. If f attains its maximum (or minimum) at x and f is differentiable at x, then $f'(x) = 0$.

1. Prove the preceding lemma.
(*Hint:* Suppose that f attains its maximum at x. Argue by contradiction: show that if $f'(x) > 0$, then there exists $y \in I$ such that $f(y) > f(x)$, and analogously for $f'(x) < 0$.)

Rolle's Theorem. Suppose f is continuous on $[a, b]$ and differentiable on (a, b), and that $f(a) = f(b)$. Then there exists $x \in (a, b)$ such that $f'(x) = 0$.

2. Prove Rolle's Theorem.
(*Hint*: f is a continuous function on the compact set $[a, b]$, so it attains its maximum and minimum in the closed interval. Consider cases depending on whether or not the max/min occurs at the endpoints of the interval.)

3. (a) Prove that if f is a differentiable function on (a, b) with bounded derivative (i.e. there exists $M > 0$ such that $f'(x) \leq M$ for all $x \in (a, b)$), then f is uniformly continuous on (a, b).

Proof. Let $\varepsilon > 0$. Let M be such that $|f'(x)| \leq M$ for every $x \in (a, b)$. Let $\delta = \varepsilon/M$.
(Show that for $x, y \in (a, b)$, if $|x - y| < \delta$ then $|f(x) - f(y)| \leq \varepsilon$. Hint: mean value theorem.)

(b) Show that the converse does not hold in general by finding an example of a uniformly continuous function on an interval whose derivative is not bounded.

4. Generalized Mean Value Theorem. Suppose f and g are continuous on $[a, b]$ and differentiable on (a, b). Prove that there exists $x \in (a, b)$ such that

$$f'(x)(g(b) - g(a)) = g'(x)(f(b) - f(a)).$$

Note: Using the function $g(x) = x$ gives us the classic mean value theorem.
(*Hint:* Recall that in the proof of the classic mean value theorem, we defined a function $h(x) = (f(b) - f(a))x - (b - a)f(x)$.)