Monday, August 2

Warm-Up: Prove that if $f: X \to Y$ is continuous at x_0 and $g: Y \to Z$ is continuous at $f(x_0)$ then $g \circ f$ is continuous at x_0.

Proof:

(1) $x_n \to x_0$. Then $f(x_n) \to f(x_0)$.

Then $\underbrace{g(f(x_n))}_{g\circ f\,(x_n)} \to \underbrace{g(f(x_0))}_{g\circ f\,(x_0)}$.

(2) Let $\varepsilon > 0$. There exists $\gamma > 0$ s.t. $|y - f(x_0)| < \gamma \Rightarrow |g(y) - g(f(x_0))| < \varepsilon$.

There exists $\delta > 0$ s.t.

$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \gamma$.

Then $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \gamma \Rightarrow |g\circ f(x) - g\circ f(x_0)| < \varepsilon$.

Math 104 Worksheet 15

UC Berkeley, Summer 2021

Thursday, July 29

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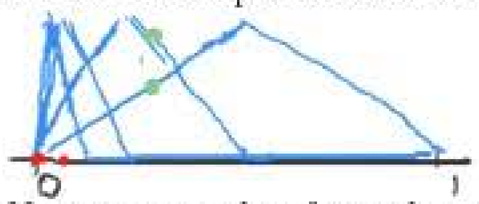
1. Let (f_n) be a sequence of continuous functions on $[a,b]$ which converge pointwise to 0, i.e. $f_n(x) \to 0$ for each $x \in [a,b]$.

(a) Find an example to show that (f_n) does not necessarily converge uniformly to 0.



(b) Now suppose that for each $x \in [a,b]$, the sequence $(f_n(x))$ is nonincreasing, i.e. $f_{n+1}(x) \leq f_n(x)$ for each $n \in \mathbb{N}$. Prove that $f_n \to 0$ uniformly by following the outline below.

Proof. (Contradiction) Suppose that (f_n) does not converge uniformly to 0. Then there exists $\varepsilon > 0$ such that for each $N \in \mathbb{N}$, there exists $n \geq N$: $|f_n(x)| \geq \varepsilon$ for some x.

Then there exists a subsequence (f_{n_k}) of (f_n) such that for each $k \in \mathbb{N}$, there exists $x_k \in [a,b]$ such that $|f_{n_k}(x_k)| \geq \varepsilon$.

Now $(x_k)_{k\in\mathbb{N}}$ is a sequence in $[a,b]$, so by Bolzano-Weierstrass there exists a subsequence (x_{k_j}) of (x_k) such that $x_{k_j} \to x^*$ for some $x^* \in [a,b]$. Fix $p \in \mathbb{N}$. Since $(f_{n_k}(x))$ is nonincreasing for each $x \in [a,b]$, for $j > p$ we have the inequality

$$f_{n_{k_p}}(x_{k_j}) \geq f_{n_{k_j}}(x_{k_j}) \geq \varepsilon.$$

(Complete the proof by using continuity of $f_{n_{k_p}}$, followed by convergence of (f_n) to find a contradiction.)

Since $f_{n_{k_p}}$ is continuous,

$$\underbrace{f_{n_{k_p}}(x_{k_j})}_{\geq \varepsilon} \longrightarrow f_{n_{k_p}}(x^*) \geq \varepsilon$$

true for any $p \in \mathbb{N}$. Contradiction to $f_n \to 0$ pointwise.

(c) Apply part (b) to prove **Dini's Theorem**: If (f_n) is a sequence of continuous functions on $[a,b]$ such that $(f_n(x))$ is nondecreasing for each $x \in [a,b]$ and $f_n \to f$ pointwise for some continuous function f, then $f_n \to f$ uniformly on $[a,b]$.

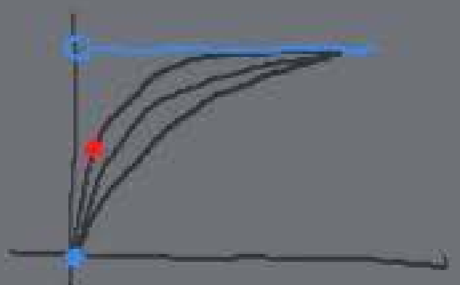
Consider $g_n = f - f_n$. By above, $g_n \to 0$ uniformly.
$\Rightarrow f_n \to f$ uniform.

(d) Find an example to show that the conclusion in part (c) does not necessarily hold if f is not assumed to be continuous.

$$f_n(x) = x^{\frac{1}{n}} \longrightarrow f(x) = \begin{cases} 0 & x = 0 \\ 1 & x \in (0,1] \end{cases}$$

not uniform.

$f_n(x) = x^n$



2. Abel's Theorem

Lemma. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence 1 and the series converges at $x = 1$, then f is continuous on $[0, 1]$.

You may use the preceding lemma without proof (yet) for the following exercises.

(a) Use the lemma to show that if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R with $0 < R < \infty$ and the series converges at $x = R$, then f is continuous at R.
(*Hint:* Consider the function $g(x) = f(Rx)$.)

$$g(x) = f(Rx) = \sum_{n=0}^{\infty} a_n (Rx)^n = \sum_{n=0}^{\infty} a_n R^n x^n$$

$g(x)$ converges at $x=1$ $\xRightarrow{\text{by Lemma.}}$ g is continuous at $x = 1$.

$$f(x) = g\left(\underbrace{\frac{x}{R}}\right) \quad \text{continuous at } x = R.$$

continuous at $x=R$; g continuous at 1.

(b) Use the result of part (a) to show that if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R with $0 < R < \infty$ and the series converges at $x = -R$, then f is continuous at $x = -R$.
(*Hint:* Consider the function $h(x) = f(-x)$.)

$$h(x) = f(-x) = \sum_{n=0}^{\infty} a_n (-x)^n = \sum_{n=0}^{\infty} (-1)^n a_n x^n.$$

$h(x)$ converges at $x = R$ $\xRightarrow{\text{by (a)}}$ h is continuous at $x = R$.

$$f(x) = h(\overline{-x}) \quad \text{continuous at } -R.$$

continuous at $-R$; h is continuous R.

Lemma: $f(x) = \sum a_n x^n$ radius of convergence 1, converges at $x=1$.

f is continuous on $[0,1]$.

Proof: $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Let $A = \sum_{n=0}^{\infty} a_n$ $(= f(1))$.

Consider $g = f - A$: $g(x) = (a_0 - A) + \sum_{n=1}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$

$b_0 = a_0 - A$
$b_k = a_k$ for all $k \geq 1$.

$g(1) = 0$.

Let $g_n(x) = \sum_{k=0}^{n} b_k x^k$, $s_n = g_n(1) = \sum_{k=0}^{n} b_k$.

$f_n(x) = \sum_{\cancel{k}=0}^{n} a_k x^n$

Observe: $b_n = s_n - s_{n-1}$ and $s_n \to 0$.

Know: $g_n \to g$ pointwise on $[0,1]$,

each g_n is continuous.

Goal: Show that $g_n \to g$ uniformly on $[0,1]$.

To do this, we show that (g_n) is uniformly Cauchy on $[0,1]$.

For $m < n$:

$$\underline{g_n(x) - g_m(x)} = \sum_{k=m+1}^{n} \overbrace{b_k}^{S_k - S_{k-1}} x^k = \sum_{k=m+1}^{n} (S_k - S_{k-1})\, x^k$$

$$= \underbrace{\sum_{k=m+1}^{n} S_k x^k}_{S_n x^n - S_m x^m + \boxed{\sum_{k=m}^{n-1} S_k x^k}} - \underbrace{\sum_{k=m+1}^{n} S_{k-1} x^k}_{x \overbrace{\boxed{\sum_{k=m}^{n-1} S_k x^k}}^{S_m x^m + \dots + S_{n-1} x^{n-1}}}$$

$$= S_n x^n - S_m x^m + (1-x) \sum_{k=m}^{n-1} S_k x^k .$$

Let $\varepsilon > 0$.

- Since $S_n \to 0$: there exists N: $n \geq N \Rightarrow |S_n| < \frac{\varepsilon}{3}$.
- For $n \geq m \geq N$,

$$\left| (1-x) \sum_{k=m}^{n-1} S_k x^k \right| \underset{\text{for } x \in [0,1)}{\overset{=0 \text{ for } x=1}{\leq}} (1-x) \sum_{k=m}^{n-1} \underbrace{|S_k|}_{< \varepsilon/3} x^k < \frac{\varepsilon}{3} \cancel{(1-x)} \underbrace{\sum_{k=m}^{n-1} x^k}_{\leq \sum_{k=0}^{\infty} x^k = \frac{1}{\cancel{1-x}}} \leq \frac{\varepsilon}{3} .$$

- For $n \geq m \geq N$ and $x \in [0,1]$,

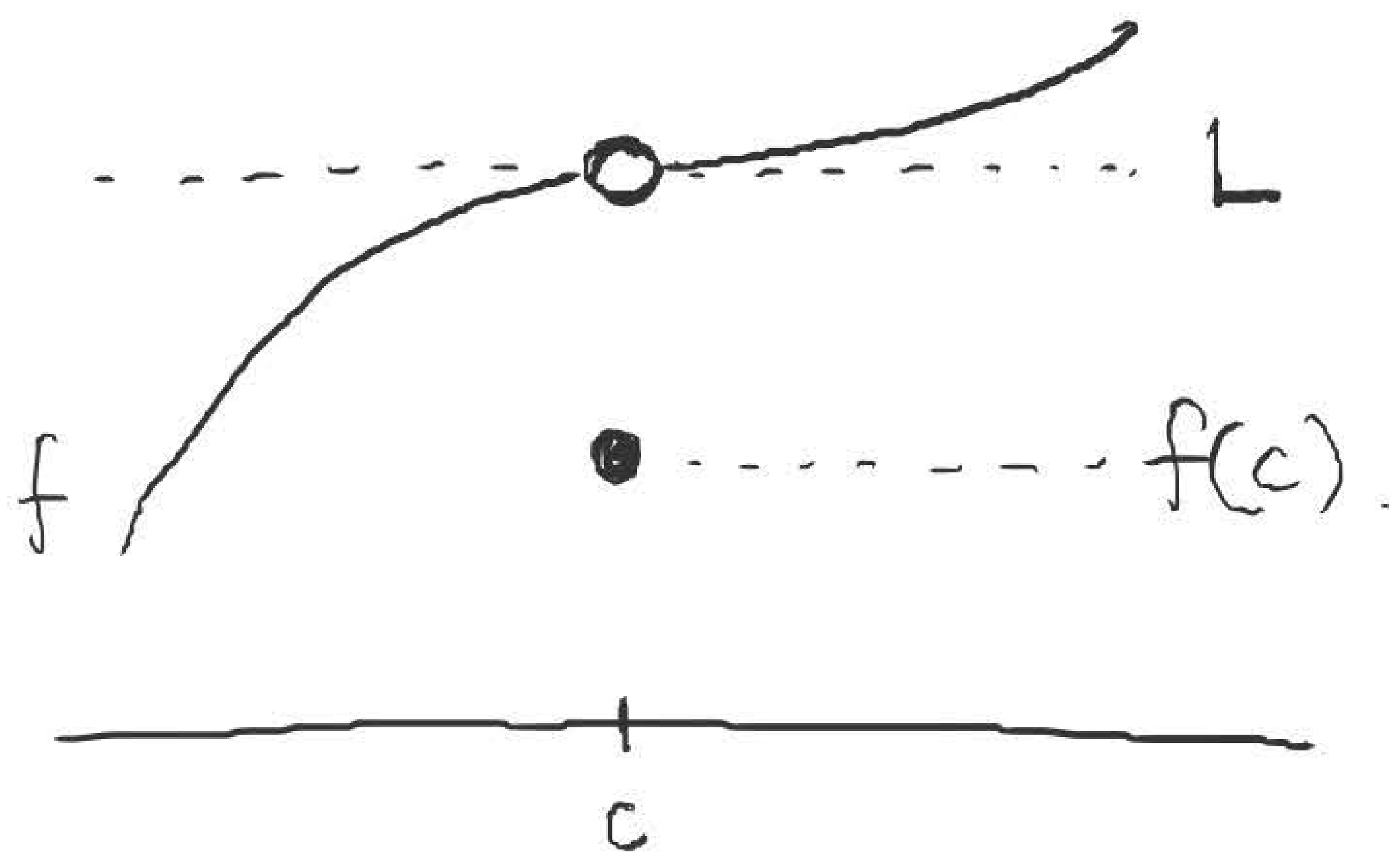
$$|g_n(x) - g_m(x)| \leq \underbrace{|S_n| x^n}_{< \frac{\varepsilon}{3}} + \underbrace{|S_m| x^m}_{< \frac{\varepsilon}{3}} + \underbrace{\left| (1-x) \sum_{k=m}^{n-1} S_k x^k \right|}_{< \frac{\varepsilon}{3}} < \varepsilon .$$

$g_n \to g$ uniformly.
$f_n \to f$ uniformly on $[0,1]$.

Limits of functions

<u>Def</u>: $\lim_{x \to c} f(x) = L$ means that for every sequence $(x_n) \subseteq \mathrm{dom}(f) \setminus \{c\}$ such that $x_n \to c$, we have $f(x_n) \to L$.

$\in \mathbb{R} \cup \{-\infty, \infty\}$.



Want $\lim_{x \to c} f(x) = L$.

Also have ε-δ definition:
For any $\varepsilon > 0$, there exists $\delta > 0$ such that
$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

Observe: f is continuous at c if and only $\lim_{x \to c} f(x) = f(c)$.

<u>Def</u>: $\lim_{x \to c^-} f(x) = L$ means that there exists $a < c$ such that $(a, c) \subseteq \mathrm{dom} f$ and for any sequence $(x_n) \subseteq (a, c)$ such that $x_n \to c$, we have $f(x_n) \to L$.

Similarly define $\lim_{x \to c^+} f(x) = L$.

Theorem: $\lim_{x \to c} f(x)$ exists if and only if $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$, in which case, all equal.

Differentiation :

Def: Let f be a real-valued function defined on an open interval containing a point x.

Let $\varphi_x(y) = \dfrac{f(y) - f(x)}{y - x}$ defined on $\mathrm{dom}(f) \setminus \{x\}$.

difference quotient.

Say f is differentiable at x if $\lim_{y \to x} \varphi_x(y)$ exists and is finite (i.e. some real number), in which case we define the derivative of f at x as $f'(x) = \lim_{y \to x} \varphi_x(y)$

$$= \lim_{y \to x} \frac{f(y) - f(x)}{y - x}.$$

- differentiable on a set E : diff at every $x \in E$.
- differentiable : diff at every $x \in \mathrm{dom}(f)$.
- Can consider f' as a function, $\mathrm{dom}(f') = \{ x \in \mathrm{dom}(f) ; \lim_{y \to x} \varphi_y(x) \text{ exists, finite} \} \subseteq \mathrm{dom}(f)$.

$\underline{\text{Ex}}$. $f(x) = x^2$, $x = 2$.

$$\lim_{y \to 2} \frac{f(y) - f(2)}{y-2} = \lim_{y \to 2} \frac{y^2 - 4}{y-2} = \lim_{y \to 2} \frac{\cancel{(y-2)}(y+2)}{\cancel{y-2}} = \lim_{y \to 2} (y+2) = 4 .$$

$$f'(2) = 4 .$$

More generally,

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = \lim_{y \to x} \frac{y^2 - x^2}{y - x} = \lim_{y \to x} (y + x) = 2x .$$

$f'(x) = 2x$ for all $x \in \mathbb{R}$.