Math 104 Homework 5 Solutions

UC Berkeley, Summer 2021

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1. Let (X, d_X) and (Y, d_Y) be two metric spaces. Let $X \times Y := \{(x, y) : x \in X, y \in Y\}$ and define the function $d : (X \times Y) \times (X \times Y) \to \mathbb{R}$ by

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.$$

(a) Show that d defines a metric on $X \times Y$.
(b) Show that E is a compact set in X and F is a compact set in Y, then $E \times F$ is compact in $X \times Y$.

Solution.
(a) Positive definiteness and symmetry are trivial. For the triangle inequality, we have

$$\begin{aligned} d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)) &= \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} + \max\{d_X(x_2, x_3), d_Y(y_2, y_3)\} \\ &\geq \max\{d_X(x_1, x_2) + d_X(x_2, x_3), d_Y(y_1, y_2) + d_Y(y_2, y_3)\} \\ &\geq \max\{d_X(x_1, x_3), d_Y(y_1, y_3)\} \\ &= d((x_1, y_1), (x_3, y_3)). \end{aligned}$$

(b) Let (x_n, y_n) be a sequence in $E \times F$. Then (x_n) has a convergent subsequence to some $x_0 \in E$, and there is a further subsequence such that (y_{n_k}) converges to some $y_0 \in F$. Then $d((x_{n_k}, y_{n_k}), (x_0, y_0)) = \max\{d_X(x_{n_k}, x_0), d_Y(y_{n_k}, y_0)\} \to 0$ as $k \to \infty$.

2. Prove that if $\sum a_n$ is a convergent series of nonzero terms then $\sum \frac{1}{a_n}$ diverges.

Solution. If $\sum a_n$ converges then $a_n \to 0$, so $\frac{1}{|a_n|} \to \infty$ and hence $\sum \frac{1}{a_n}$ diverges.

3. (Ross 14.8) Show that if $\sum a_n$ and $\sum b_n$ are two convergent series of nonnegative real numbers, then $\sum \sqrt{a_n b_n}$ converges. (Hint: Show that $\sqrt{a_n b_n} \leq a_n + b_n$ for all n.)

Solution. For $x, y \geq 0$, we have $xy \leq x^2 + 2xy + y^2 = (x + y)^2$, so $\sqrt{xy} \leq x + y$. Since $\sum a_n$ and $\sum b_n$ converge, so does $\sum (a_n + b_n)$, and by the comparison test, $\sum \sqrt{a_n b_n}$ converges as well.

4. (Ross 14.14) Let (a_n) be a sequence of real numbers such that $\liminf |a_n| = 0$. Prove that there exists a subsequence (a_{n_k}) of (a_n) such that $\sum_{k=1}^{\infty} a_{n_k}$ converges.

Solution. Since $\liminf |a_n| = 0$, we can construct a subsequence $(|a_{n_k}|)$ of $(|a_n|)$ such that $|a_{n_k}| \leq 2^{-k}$. Then by the comparison test, $\sum |a_{n_k}|$ converges, so the series $\sum a_{n_k}$ is absolutely convergent and thus convergent.

5. Give an example of a convergent series $\sum a_n$ for which $\sum a_n^2$ diverges.

Solution. $a_n = \frac{(-1)^n}{\sqrt{n}}$

6. (Ross 15.7) (a) Prove that if (a_n) is a nonincreasing sequence of real numbers and if $\sum a_n$ converges, then $\lim n a_n = 0$. (Hint: Consider $|a_N + a_{N+1} + \ldots + a_n|$ for suitable N.) Note that this gives an alternative proof that $\sum \frac{1}{n}$ diverges.

Solution. First observe that if (a_n) is nonincreasing and $\sum a_n$ converges, then a_n must be nonnegative for all n. Let $\varepsilon > 0$. By the Cauchy criterion, there exists N such that for all $n \geq N$, $a_N + a_{N+1} + \ldots + a_n < \varepsilon$. Then for $n \geq N$, $(n - N + 1)a_n \leq a_N + a_{N+1} + \ldots + a_n < \varepsilon$, so the sequence $((n-N+1)a_n)_{n \geq N}$ converges to 0. Since $(-N+1)a_n \to 0$, it follows that $na_n \to 0$ as well.

7. Determine whether each of the following series converges or diverges and prove it.

$$\text{(a)} \sum_{n=1}^{\infty} \frac{n}{(n+1)!} \quad \text{(b)} \sum_{n=1}^{\infty} \frac{a^n}{n!} \quad (a \in \mathbb{R}) \quad \text{(c)} \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}} \quad \text{(d)} \sum_{n=2}^{\infty} \frac{1}{n \log n}$$

Solution.
(a) converges by comparison test (or other)
(b) converges by ratio test (or other)
(c) converges by alternating series test
(d) diverges by integral test

8. (Ross 17.5) (a) Prove that for any $n \in \mathbb{N}$ the function $f(x) = x^n$ is continuous.
(b) Prove that every polynomial function $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ is continuous.

Solution. (a) Since $g(x) = x$ is continuous and the product of continuous functions is continuous, it follows that $f(x) = g(x)^n = x^n$ is continuous.
(b) This follows from part (a) and the fact that constant multiples and sums of continuous functions are continuous.

9. (a) Prove that the function

$$f(x) = \begin{cases} 1 \text{ if } x \geq 0, \\ 0 \text{ if } x < 0 \end{cases}$$

is discontinuous at 0.
(b) Prove that the function

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is discontinuous at 0.

Solution.
(a) The sequence $x_n = -\frac{1}{n}$ converges to 0, but $f(-\frac{1}{n})$ does not converge to $f(0)$.
(b) The sequence $x_n = \frac{1}{\frac{\pi}{2} + 2n\pi}$ converges to 0, but $f(x_n)$ does not converges to $f(0)$.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q}, \\ 0 \text{ if } x \notin \mathbb{Q}. \end{cases}$$

Show that f is discontinuous at every $r \in \mathbb{R}$.

Solution. If $r \in \mathbb{Q}$, then by denseness of the irrational numbers in $\mathbb{R}$ there exists a sequence (r_n) of irrational numbers converging to r, so $f(r_n)$ does not converge to $f(r)$. Likewise, if $r \in \mathbb{R} \setminus \mathbb{Q}$, by denseness of the rational numbers in $\mathbb{R}$ there exists a sequence (r_n) of rational numbers converging to r, so $f(r_n)$ does not converge to $f(r)$.