Math 104 Worksheet 15

UC Berkeley, Summer 2021
Thursday, July 29

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1. Let (f_n) be a sequence of continuous functions on $[a, b]$ which converge pointwise to 0, i.e. $f_n(x) \to 0$ for each $x \in [a, b]$.

(a) Find an example to show that (f_n) does not necessarily converge uniformly to 0.

(b) Now suppose that for each $x \in [a, b]$, the sequence $(f_n(x))$ is nonincreasing, i.e. $f_{n+1}(x) \leq f_n(x)$ for each $n \in \mathbb{N}$. Prove that $f_n \to 0$ uniformly by following the outline below.

Proof. (Contradiction) Suppose that (f_n) does not converge uniformly to 0. Then there exists $\varepsilon > 0$ such that for each $N \in \mathbb{N}$,

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Then there exists a subsequence (f_{n_k}) of (f_n) such that for each $k \in \mathbb{N}$, there exists $x_k \in [a, b]$ such that

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Now $(x_k)_{k \in \mathbb{N}}$ is a sequence in $[a, b]$, so by Bolzano-Weierstrass there exists a subsequence (x_{k_j}) of (x_k) such that $x_{k_j} \to x^*$ for some $x^* \in [a, b]$. Fix $p \in \mathbb{N}$. Since $(f_{n_k}(x))$ is nonincreasing for each $x \in [a, b]$, for $j > p$ we have the inequality

$$f_{n_{k_p}}(x_{k_j}) \geq \underline{\qquad\qquad} \geq \underline{\qquad}.$$

(Complete the proof by using continuity of $f_{n_{k_p}}$, followed by convergence of (f_n) to find a contradiction.)

$\square$

(c) Apply part (b) to prove **Dini's Theorem**: If (f_n) is a sequence of continuous functions on $[a, b]$ such that $(f_n(x))$ is nondecreasing for each $x \in [a, b]$ and $f_n \to f$ pointwise for some continuous function f, then $f_n \to f$ uniformly on $[a, b]$.

(d) Find an example to show that the conclusion in part (c) does not necessarily hold if f is not assumed to be continuous.

2. Abel's Theorem

Lemma. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence 1 and the series converges at $x = 1$, then f is continuous on $[0, 1]$.

You may use the preceding lemma without proof (yet) for the following exercises.

(a) Use the lemma to show that if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R with $0 < R < \infty$ and the series converges at $x = R$, then f is continuous at R.
(*Hint:* Consider the function $g(x) = f(Rx)$.)

(b) Use the result of part (a) to show that if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R with $0 < R < \infty$ and the series converges at $x = -R$, then f is continuous at $x = -R$.
(*Hint:* Consider the function $h(x) = f(-x)$.)