Alternating series test

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

If $a_1 \geq a_2 \geq \ldots \geq 0$ and $\lim a_n = 0$, then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Proof:

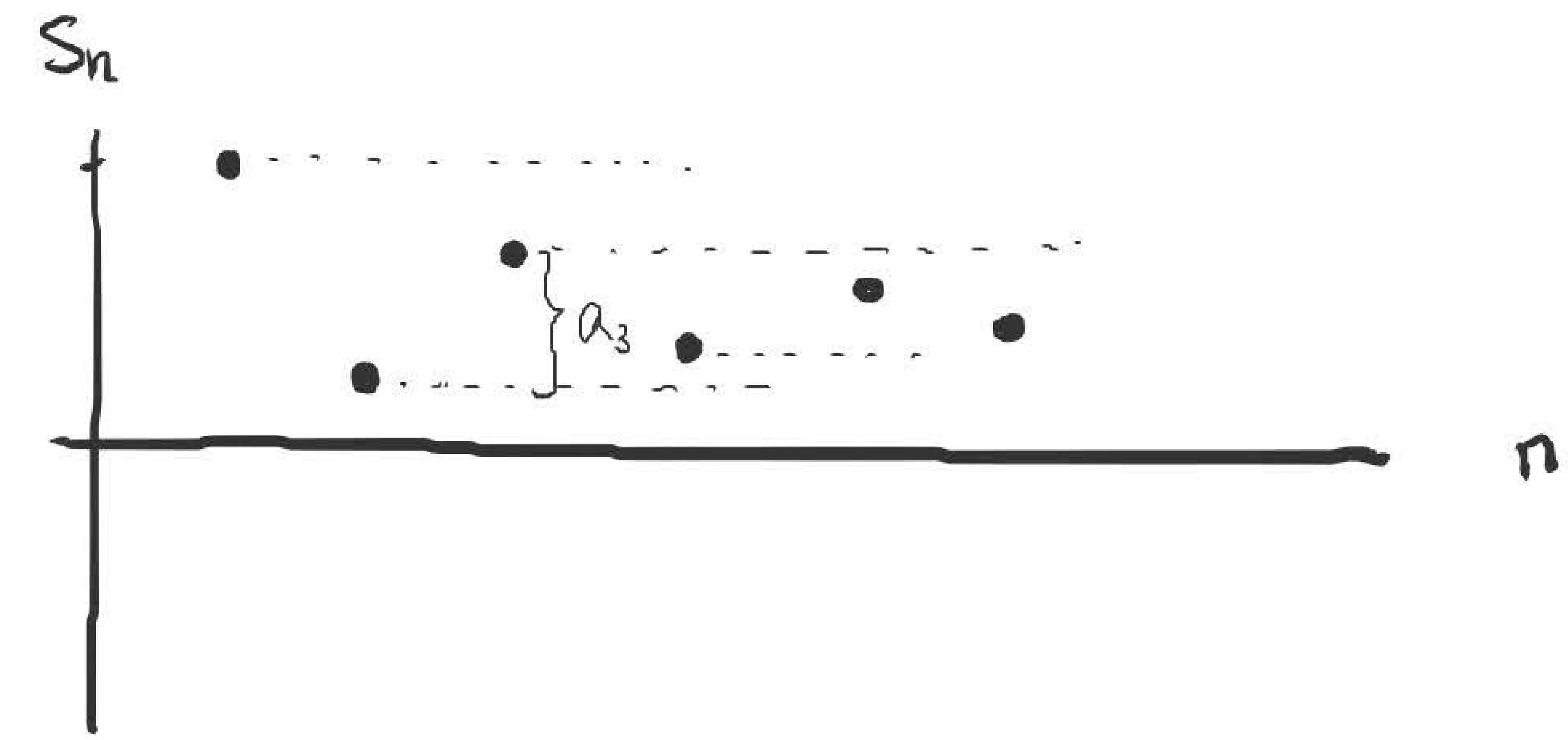
Partial sums:

$S_1 = a_1$

$S_2 = a_1 - a_2$

$S_3 = a_1 - a_2 + a_3$

$S_4 = a_1 - a_2 + a_3 - a_4$



$a_n \geq 0$,

Ex: $a_n \to 0$, $\sum (-1)^{n+1} a_n$ diverges.

$(a_n) = \left(2, 1, 2(\tfrac{1}{2}), \tfrac{1}{2}, 2(\tfrac{1}{3}), \tfrac{1}{3}, \ldots\right)$

$\sum (-1)^{n+1} a_n = \underbrace{2-1}_{1} + \underbrace{2(\tfrac{1}{2}) - \tfrac{1}{2}}_{\frac{1}{2}} + \underbrace{2(\tfrac{1}{3}) - \tfrac{1}{3}}_{\frac{1}{3}}$

(S_{2n+1}) nonincreasing, bounded $\Rightarrow$ converges

(S_{2n}) nondecreasing, bounded $\Rightarrow$ converges.

$S_{2n+1} \geq S_{2n}$

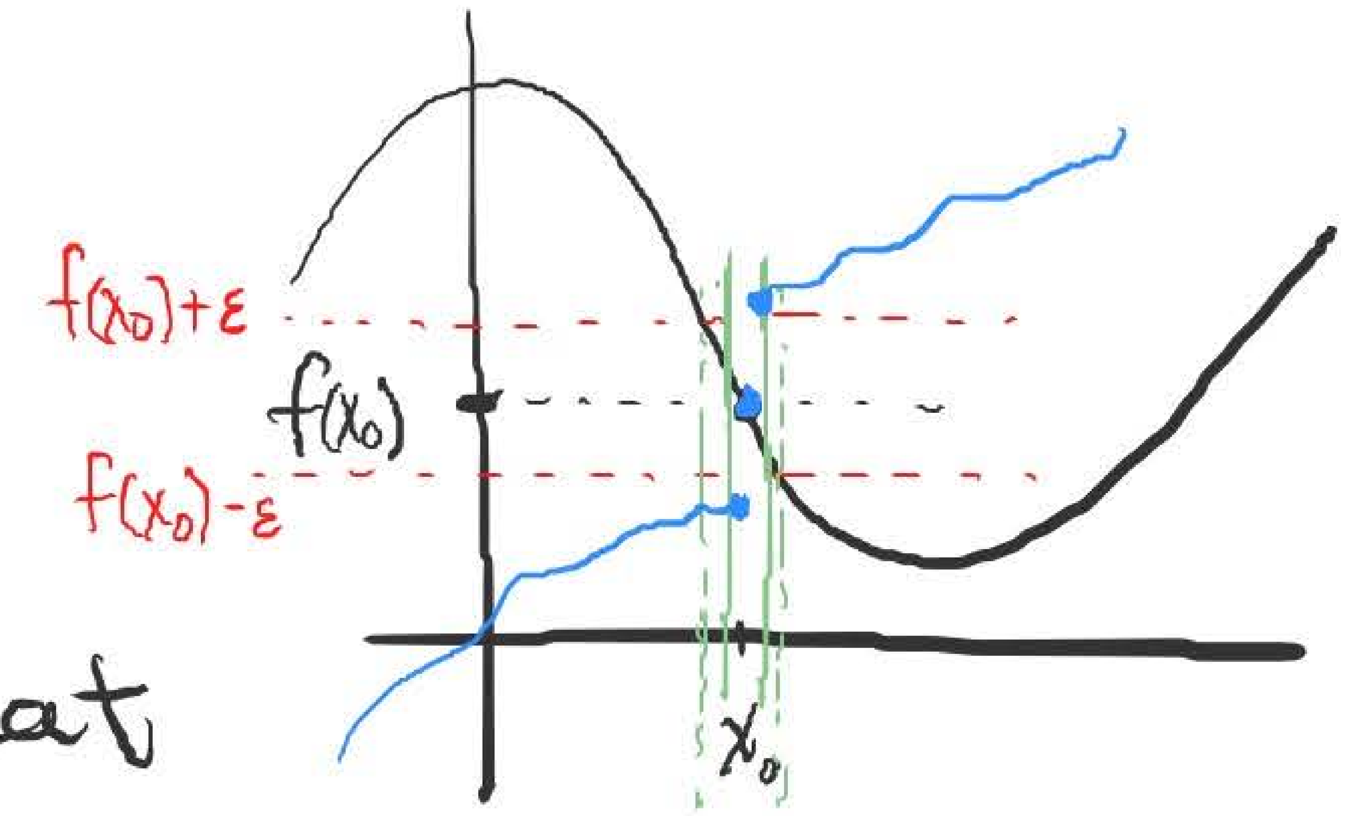
$$\lim \big(\underbrace{S_{2n+1} - S_{2n}}_{= a_{2n+1}}\big) = \lim a_{2n+1} = 0 \Rightarrow \lim S_{2n} = \lim S_{2n+1} = L$$

$\Rightarrow \lim S_n$ exists, $= L$.

$\lim S_{2n+1} - \lim S_{2n}$

Continuity: Let $S \subseteq \mathbb{R}$. Consider $f: S \to \mathbb{R}$.

dom(f).



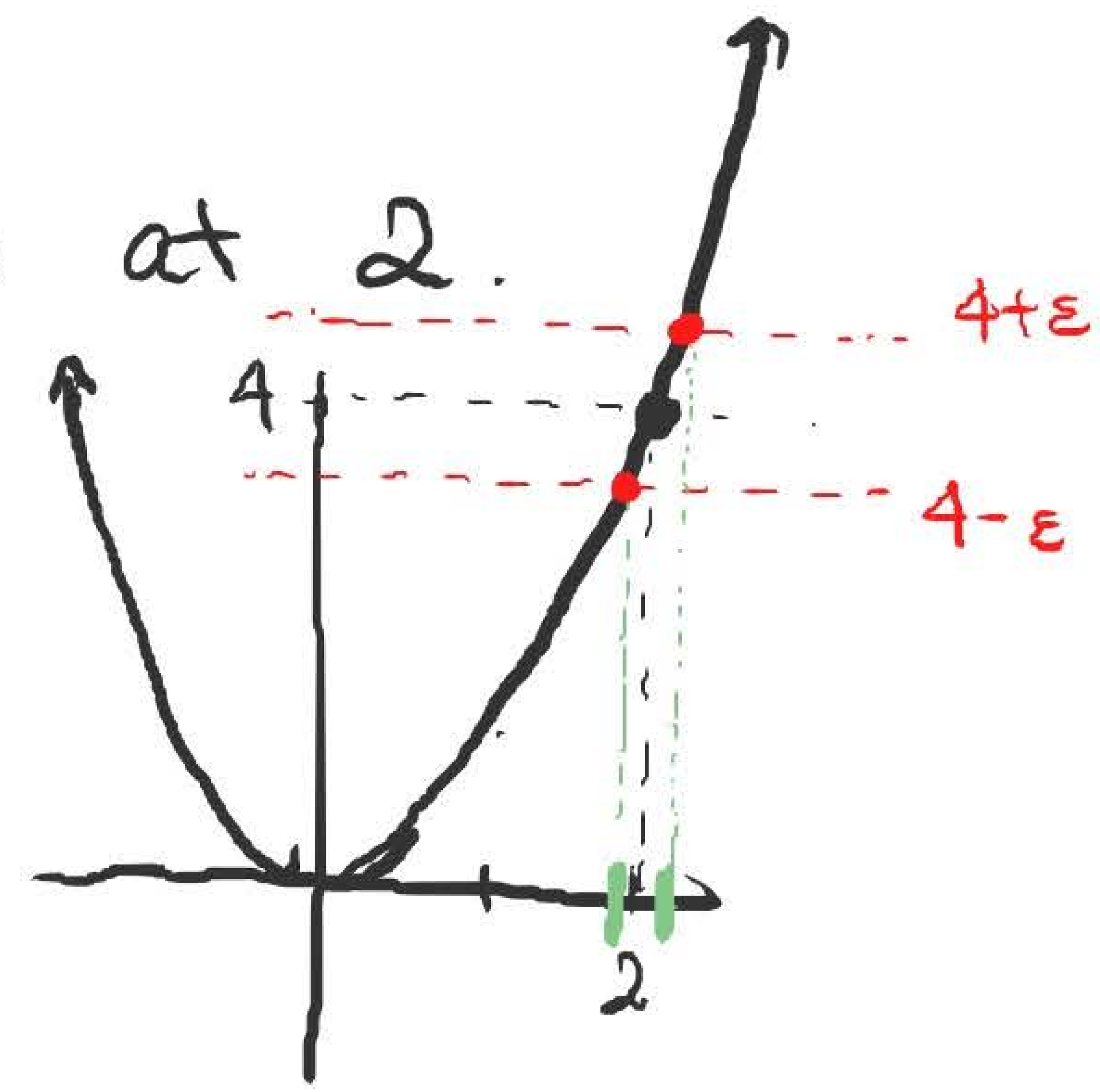
Def: f is continuous at $x_0 \in \mathrm{dom}(f)$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x \in \mathrm{dom}(f) \text{ and } |x - x_0| < \delta \text{ implies } |f(x) - f(x_0)| < \varepsilon.$$

Ex: Show that $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is continuous at 2.

Let $\varepsilon > 0$. $\delta = \min\left(2 - \sqrt{4-\varepsilon}, \sqrt{4+\varepsilon} - 2\right)$.

Then if $|x-2| < \delta$, then $|f(x) - 4| < \varepsilon$.



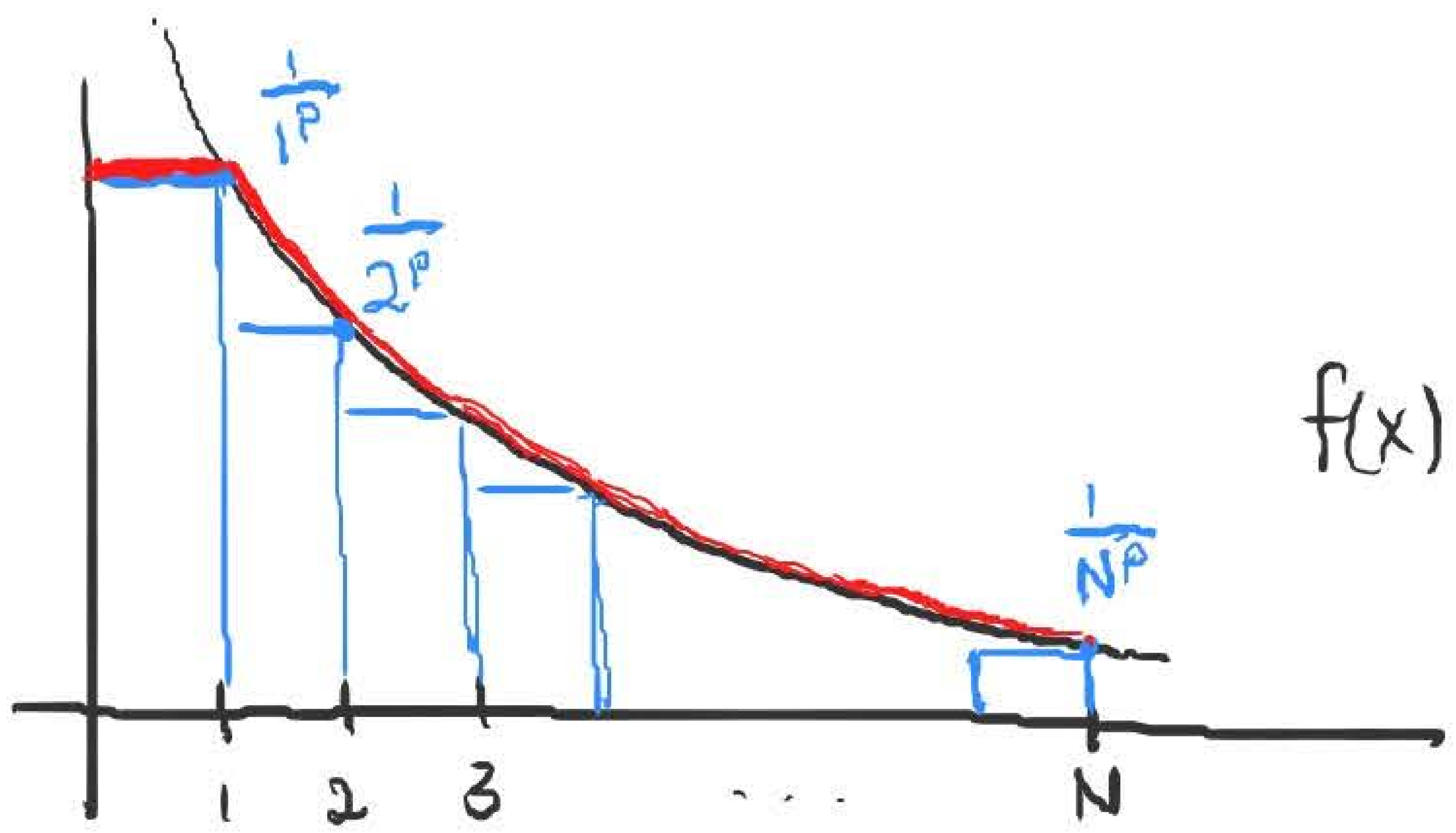
Def: f is continuous if it is continuous at every point in its domain.

Exercise: Show that any function $f: \mathbb{Z} \to \mathbb{R}$ is continuous.

Recall:

p-series: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\begin{cases} \text{diverges if } p=1. & \text{(Harmonic series).} \\ \text{diverges if } p<1. & \\ \text{converges if } p>1 & \text{(proof to follow).} \end{cases}$

Use calculus:



$f(x) = \frac{1}{x^p}$ $(p>1)$

"Integral test"

not on final exam.

sum of the areas of $\Box$'s:

$$\sum_{n=1}^{N} \frac{1}{n^p} < 1 + \int_1^N \underbrace{\frac{1}{x^p}}_{x^{-p}} dx = 1 + \left[\frac{-1}{p-1} x^{-p+1}\right]_1^N = 1 + \frac{1}{p-1}\overbrace{\left[1 - \frac{1}{\underbrace{N^{p-1}}_{>1}}\right]}^{\text{between 0 and 1.}}$$

$$< 1 + \frac{1}{p-1} = \frac{p}{p-1}.$$

$\Rightarrow \frac{p}{p-1}$ is an upper bound for the sequence of partial sums of $\sum_{n=1}^{\infty} \frac{1}{n^p}$. $\therefore$ the series converges.

Exercise 2. Determine whether or not the following series converges:

$$\sum_{n=1}^{\infty} \underbrace{\left(\frac{2}{(-1)^n - 3}\right)^n}_{a_n}.$$

$(a_n) = \left(-\frac{1}{2}, 1, -\frac{1}{8}, 1, -\frac{1}{32}, 1, \ldots\right)$

$a_n \not\to 0$.

$\therefore$ diverges.

Exercise 3. Consider the series $\sum a_n$ with $a_n = 2^{(-1)^n - n}$, so $\sum a_n = 2 + \frac{1}{4} + \frac{1}{2} + \frac{1}{16} + \frac{1}{8} + \frac{1}{64} + \ldots$

(a) Use the comparison test to show that $\sum a_n$ converges.

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 1.$

$2^{1-n} \geq 2^{(-1)^n - n}$.

compare to $\displaystyle\sum_{n=1}^{\infty} 2^{1-n} = \sum_{n=1}^{\infty} 2 \cdot \left(\frac{1}{2}\right)^n = 2 \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

$$2\left(\underbrace{\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n}_{\frac{1}{1-\frac{1}{2}}} - 1\right) = 2.$$

(b) Show that the ratio test gives no information.

$$\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} \dfrac{2^{-1-(n+1)}}{2^{1-n}} = \dfrac{2^{-2-n}}{2^{1-n}} = \dfrac{1}{8} & n \text{ even} \\[2ex] \dfrac{2^{1-(n+1)}}{2^{-1-n}} = \dfrac{2^{-n}}{2^{-1-n}} = 2 & n \text{ odd} \end{cases}$$

$\liminf \left|\frac{a_{n+1}}{a_n}\right| < 1$

$\limsup \left|\frac{a_{n+1}}{a_n}\right| > 1$.

(c) Use the root test to show that $\sum a_n$ converges.

$$|a_n|^{\frac{1}{n}} = \left(2^{(-1)^n - n}\right)^{\frac{1}{n}} = 2^{\frac{(-1)^n}{n} - 1} = \boxed{\frac{1}{2}}\, 2^{\frac{(-1)^n}{n}} \longrightarrow \frac{1}{2} < 1.$$

$2^{\frac{1}{n}} \longrightarrow 1$

$2^{-\frac{1}{n}} = \left(\frac{1}{2}\right)^{\frac{1}{n}} \longrightarrow 1.$

$\therefore$ root test shows convergence

(d) Can you find a series for which the ratio test proves convergence or divergence, but the root test gives no information?

No.

if ratio test works: either

$\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow \limsup |a_n|^{\frac{1}{n}} < 1$.

$\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1 \Rightarrow \limsup |a_n|^{\frac{1}{n}} > 1$.

$(n!)^{\frac{1}{n}} \xrightarrow{?} \infty$

$= \left(n(n-1)(n-2)\cdots 2 \cdot 1\right)^{\frac{1}{n}}$

Exercise 4. The series in Exercise 3 is an example of a series for which the root test proves convergence but the ratio test gives no information. Find an example of a series for which the root test proves **divergence** but the ratio test gives no information.

$\sum b_n$, $b_n = 2^{(-1)^n + n}$

$\left(\left|\frac{b_{n+1}}{b_n}\right|\right) = \left(2, \frac{1}{8}, 2, \frac{1}{8}, \ldots\right)$

$|a_n|^{\frac{1}{n}} = 2 \cdot 2^{\frac{(-1)^n}{n}} \to 2 > 1.$

root test shows divergence