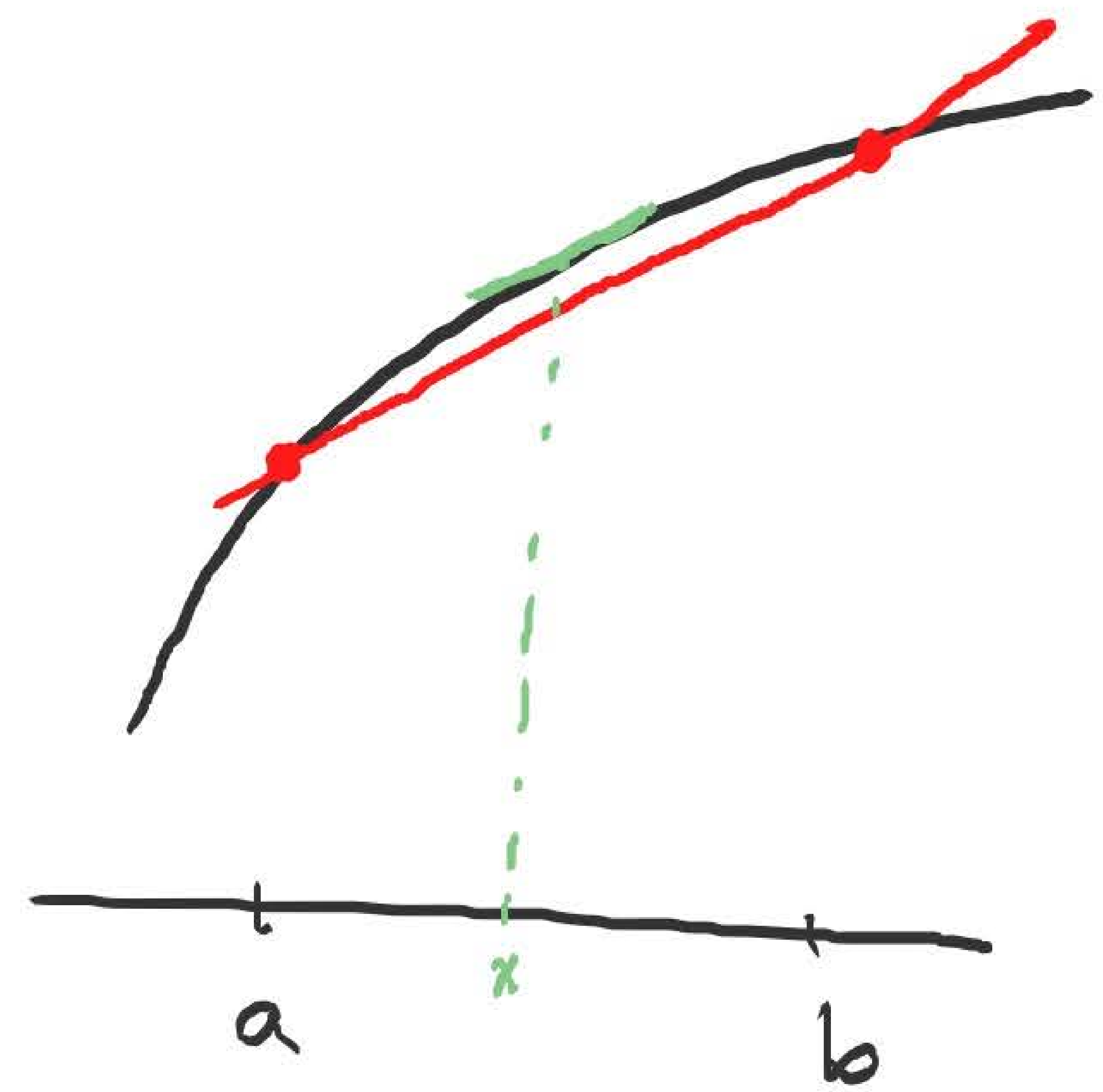
Wednesday, August 4



Recall: mean value theorem (MVT):

f cont. on $[a,b]$, differentiable on (a,b), then there exists $x \in (a,b)$ such that

$$f'(x) = \frac{f(b)-f(a)}{b-a}.$$

Corollary: If f is diff. on (a,b), and $f'(x)=0$ for all $x \in (a,b)$, then f is constant on $(a,b]$.

Proof: contrapositive.

Corollary: If f, g are diff. on (a,b) and $f'(x) = g'(x)$ for all $x \in (a,b)$, then $f = g + C$ for some $C \in \mathbb{R}$.

Proof: $(f-g)'(x) = 0$ for all x.

By previous Corollary, $f - g = C$, so $f = g + C$.

Corollary:

(i) If $f'(x) > 0$ for all $x \in (a,b)$, then f is strictly increasing.

(ii) If $f'(x) < 0$ ... decreasing.

Proof: (i) Let $x_1, x_2 \in (a,b)$, $x_1 < x_2$.

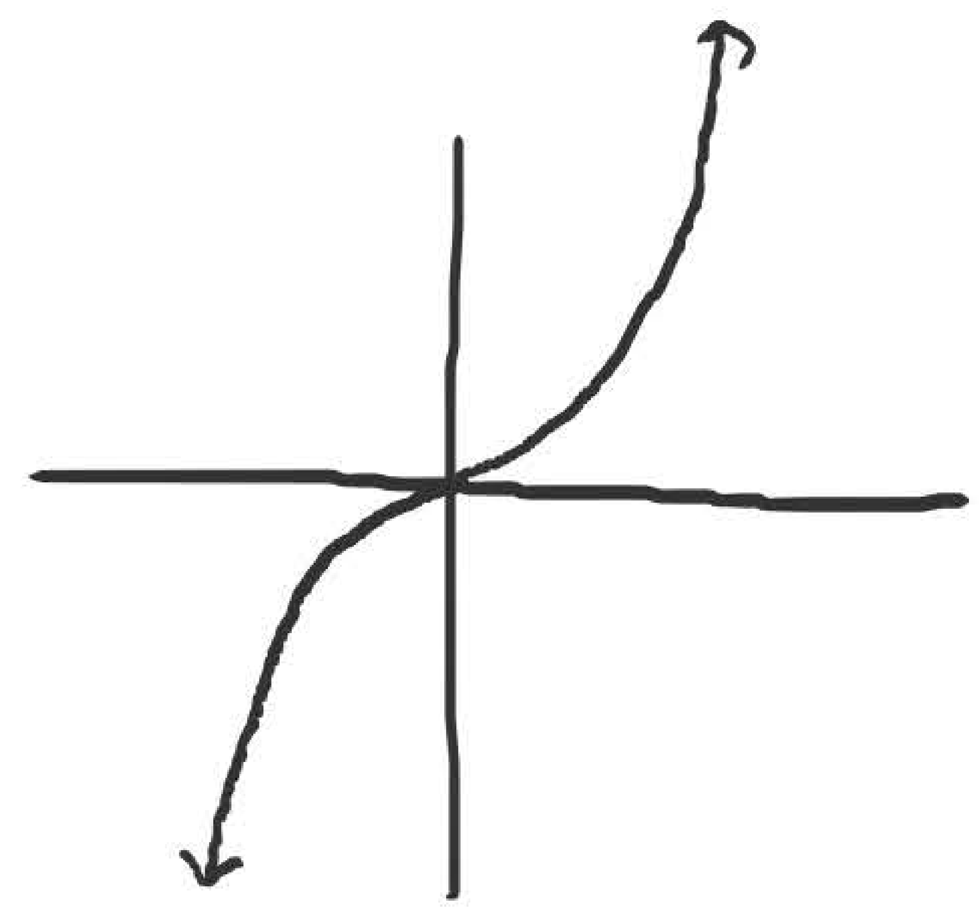
$$\frac{f(x_2)-f(x_1)}{\underbrace{x_2 - x_1}_{\text{positive}}} = f'(c) > 0$$

for some $c \in (x_1, x_2) \subseteq (a,b)$

$\Rightarrow f(x_2) - f(x_1) > 0$.

$f(x_2) > f(x_1)$.

Counterexample: $f(x) = x^3$.



- strictly increasing on $\mathbb{R}$
- $f'(0) = 0$.

  not the case that $f'(x) > 0$ for all $x \in \mathbb{R}$.

3. (a) Prove that if f is a differentiable function on (a, b) with bounded derivative (i.e. there exists $M > 0$ such that $|f'(x)| \leq M$ for all $x \in (a, b)$), then f is uniformly continuous on (a, b).

Proof. Let $\varepsilon > 0$. Let M be such that $|f'(x)| \leq M$ for every $x \in (a, b)$. Let $\delta = \varepsilon/M$. (Show that for $x, y \in (a, b)$, if $|x - y| < \delta$ then $|f(x) - f(y)| \leq \varepsilon$. Hint: mean value theorem.)

$$|f(x) - f(y)| = |f'(c)||x-y| \quad \text{for some } c \in (a,b) \text{ by MVT}$$
$$\leq M \underbrace{|x-y|}_{< \delta = \varepsilon/M}$$
$$< \varepsilon.$$

(b) Show that the converse does not hold in general by finding an example of a uniformly continuous function on an interval whose derivative is not bounded.

$$f(x) = \sqrt{x} \quad \text{on} \quad (0,1).$$

$$f'(x) = \frac{1}{2\sqrt{x}} \quad \text{not bounded on } (0,1).$$

4. Generalized Mean Value Theorem. Suppose f and g are continuous on $[a, b]$ and differentiable on (a, b). Prove that there exists $x \in (a, b)$ such that

$$f'(x)(g(b) - g(a)) = g'(x)(f(b) - f(a)).$$

Note: Using the function $g(x) = x$ gives us the classic mean value theorem.
(*Hint:* Recall that in the proof of the classic mean value theorem, we defined a function $h(x) = (f(b) - f(a))x - (b - a)f(x)$.)

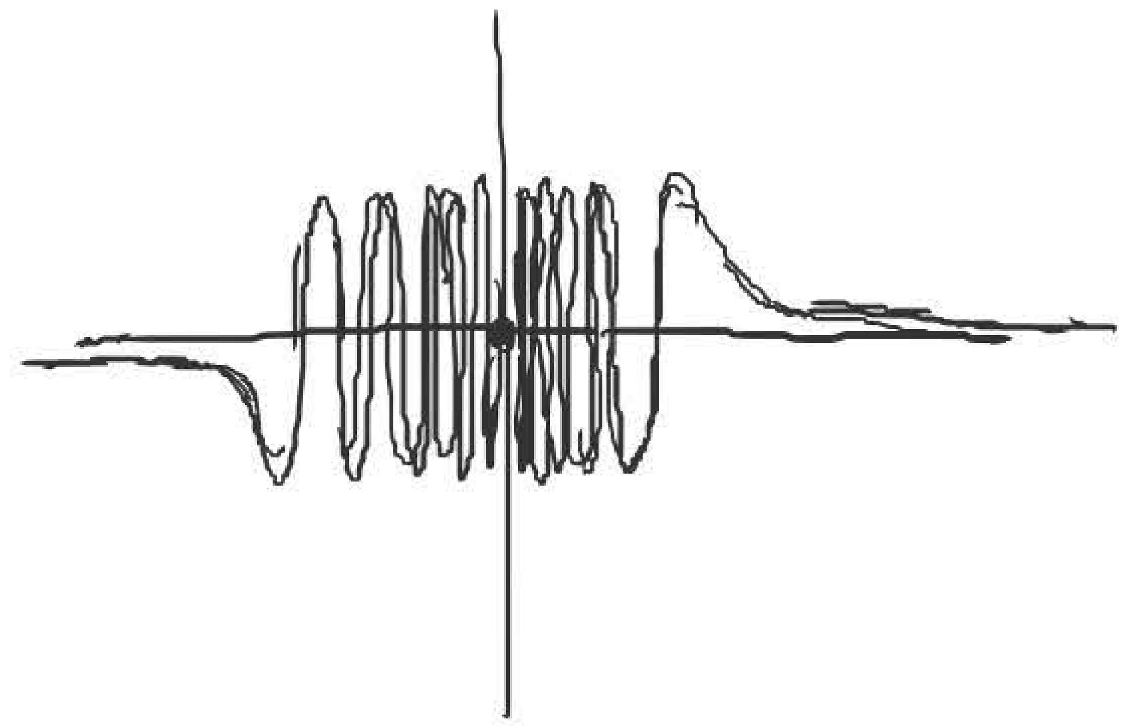
$$h(x) = (f(b) - f(a))\, g(x) - (g(b) - g(a))\, f(x) \qquad \Rightarrow \exists x \in (a,b): \ h'(x) = 0.$$

$$h(a) = (f(b) - f(a))\, g(a) - (g(b) - g(a))\, f(a).$$
$$= f(b)g(a) - \cancel{f(a)g(a)} - g(b)f(a) + \cancel{g(a)f(a)}$$
$$h(b) = (\cancel{f(b)} - f(a))\, g(b) - (\cancel{g(b)} - g(a))\, f(b)$$
$$= h(a)$$

$$(f(b) - f(a))\, g'(x) - (g(b) - g(a))\, f'(x) = 0.$$

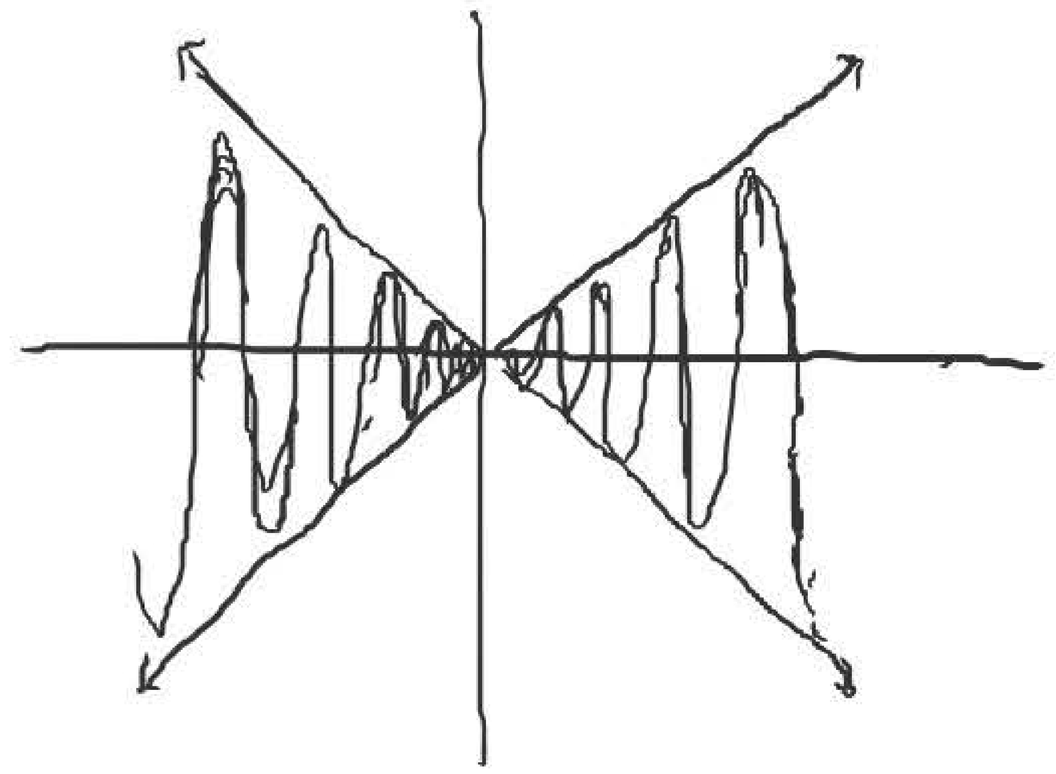
Examples

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$



not continuous at 0.

$$g(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$



$$-|x| \leq x\sin\left(\tfrac{1}{x}\right) \leq |x|$$

$$\lim_{x \to 0} g(x) = 0 = g(0).$$

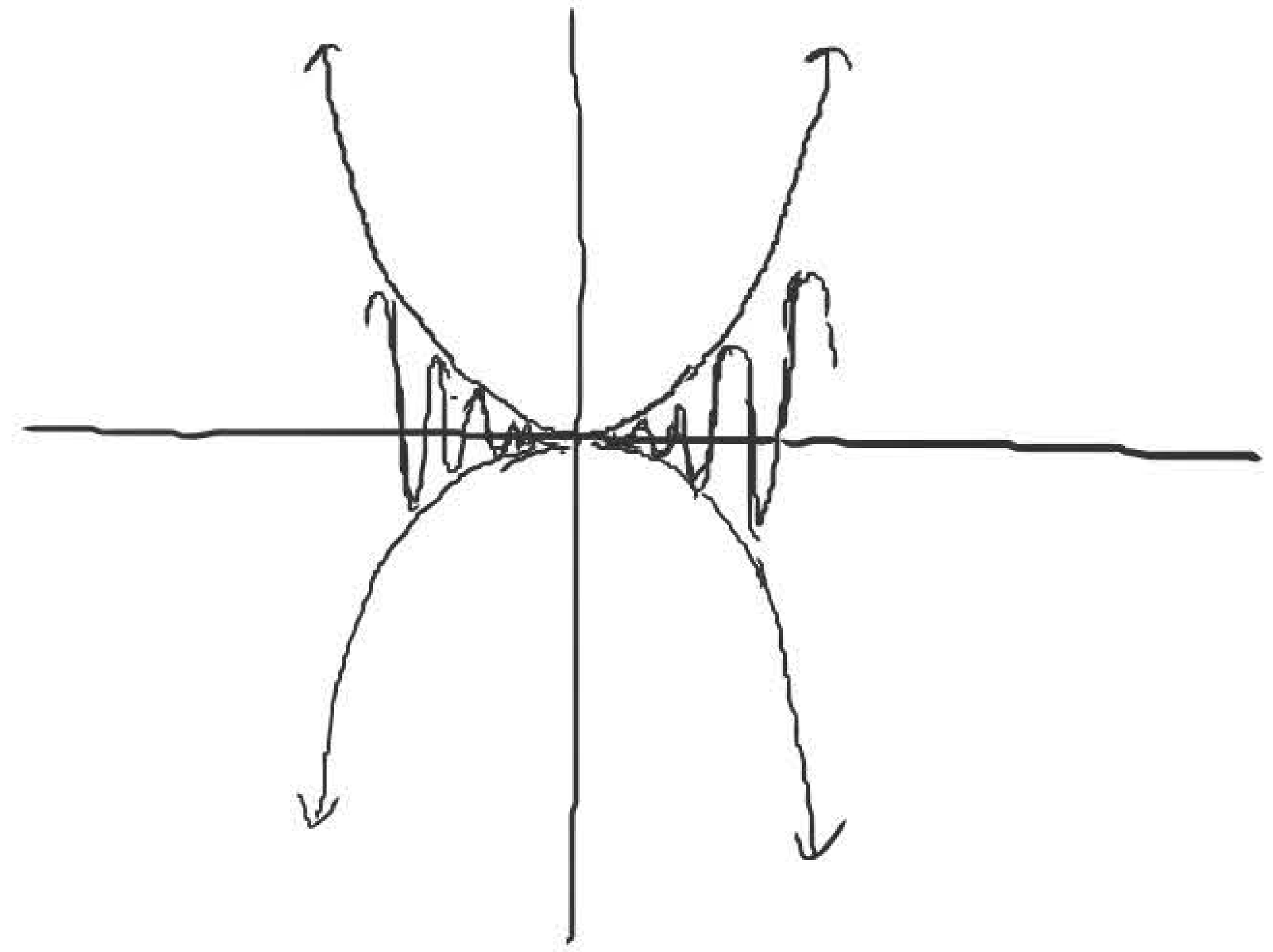
continuous at 0 (therefore on all of $\mathbb{R}$)

Is g differentiable at 0?

$$\lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{x\sin\left(\frac{1}{x}\right)}{x} = \lim_{x \to 0} \sin\left(\frac{1}{x}\right) \text{ does not exist.}$$

g is NOT differentiable at 0.

$$h(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$



h is cont at 0.

$$\lim_{x \to 0} \frac{h(x) - h(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{x}$$

$$= \lim_{x \to 0} \underbrace{x \sin\left(\frac{1}{x}\right)}_{g(x)} = 0.$$

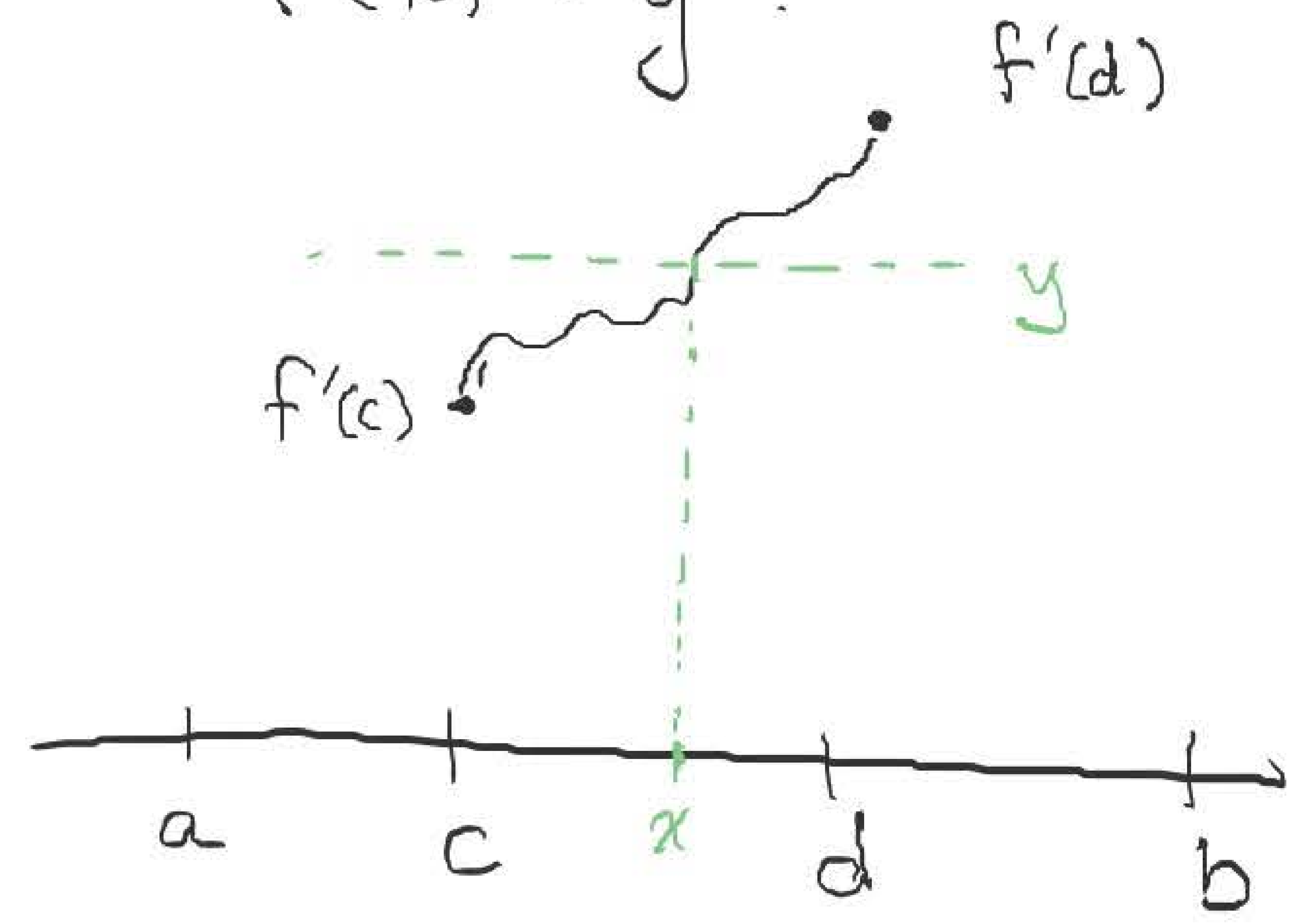
$\therefore \; h'(0) = 0.$

<u>Theorem</u>: Intermediate value theorem for derivatives.

Let f be a differentiable function on (a,b).

If $a < c < d < b$ and y is between $f'(c)$ and $f'(d)$,

then there exists $x \in (c,d)$ such that $f'(x) = y$.



Assume WLOG that $f'(c) < y < f'(d)$.

<u>Proof</u>: Let $g(x) = f(x) - yx$.

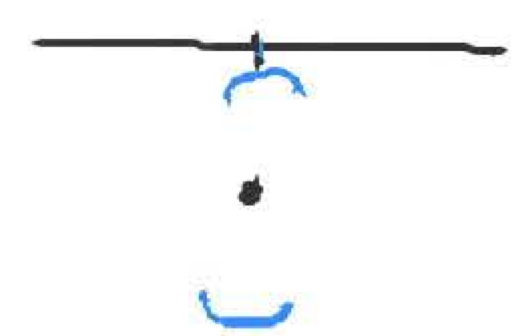
g is cont. on $[c,d] \Rightarrow g$ attains its minimum at some $x \in [c,d]$.

<u>Recall</u>: If a function h attains its min or max at $x_0 \in (a,b)$ and h is differentiable on (a,b), then $h'(x_0) = 0$.

Want to show that $x \in (c,d)$.

$$g'(x) = f'(x) - y \Rightarrow g'(c) = f'(c) - y < 0.$$
$$g'(d) = f'(d) - y > 0.$$

$$g'(c) < 0 < g'(d).$$



$\lim\limits_{s \to c} \dfrac{g(s) - g(c)}{s - c} = g'(c) < 0 \Rightarrow$ there exists $s > c$ such that $\dfrac{g(s) - g(c)}{s - c} < 0 \Rightarrow g(s) < g(c)$.

$\lim\limits_{t \to d} \dfrac{g(t) - g(d)}{t - d} = g'(d) > 0 \Rightarrow$ there exists $t < d$ such that $\dfrac{g(t) - g(d)}{t - d} > 0 \Rightarrow g(t) < g(d)$.

$\Rightarrow \quad x \neq c \quad$ and $\quad x \neq d, \quad$ so $\quad x \in (c,d).$

$\Rightarrow \quad g'(x) = 0.$

$g(x) = f(x) - yx$

$\Rightarrow \quad f'(x) - y = 0$

$g'(x) = f'(x) - y.$

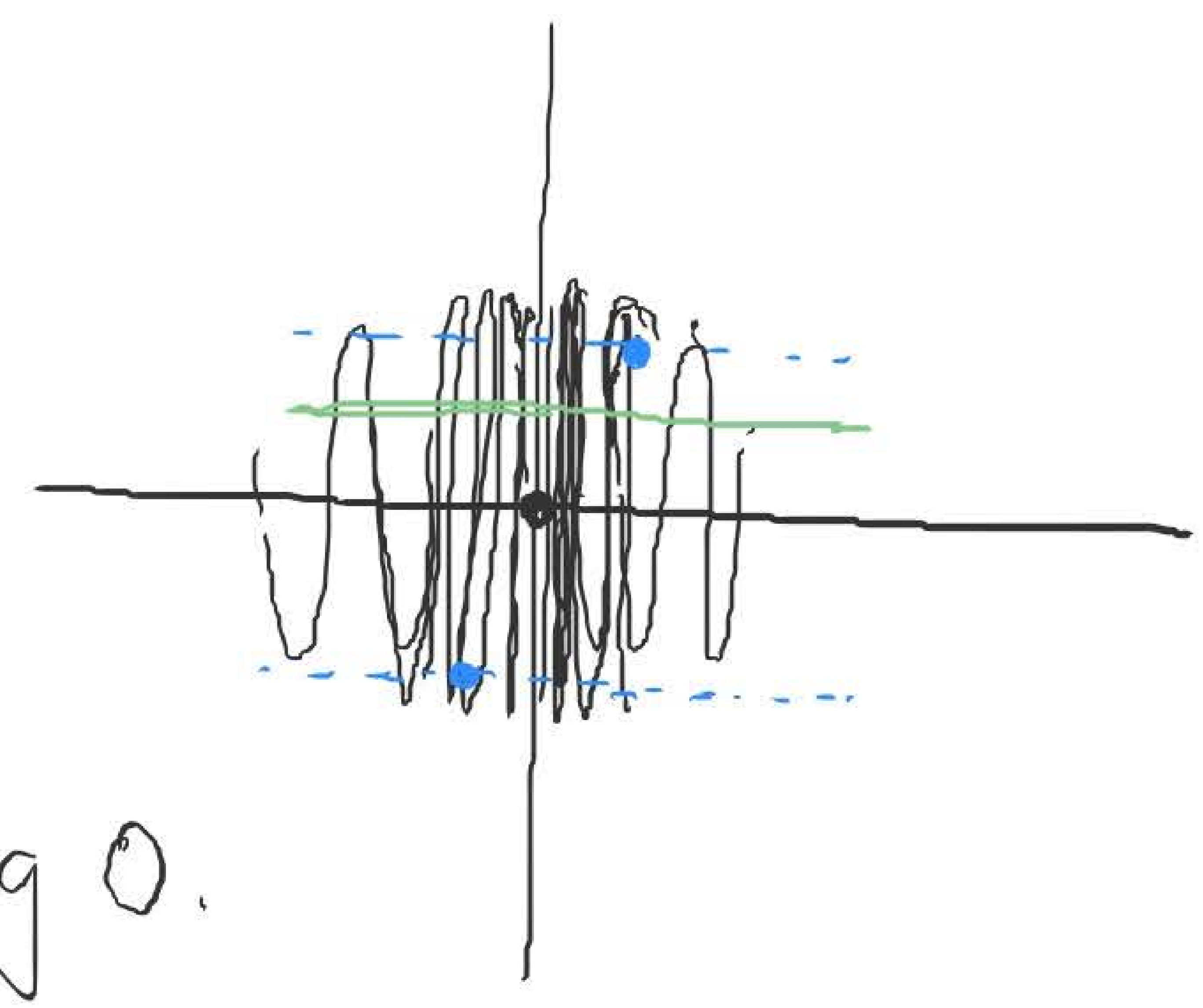
$\Rightarrow \quad f'(x) = y.$

Ex. $h(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is differentiable on $\mathbb{R}$.

$$h'(x) = \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

h' is not continuous at 0.

but h' satisfies the intermediate value property on any interval containing 0.



L' Hospital's rule   (L' Hôpital ?)

Suppose f, g are differentiable on (a,b) and $g'(x) \neq 0$ for all $x \in (a,b)$, where $-\infty \leq a < b \leq \infty$. Let $s \in \{a, b\}$.

If $\displaystyle\lim_{x \to s} \frac{f'(x)}{g'(x)} = L$ $(-\infty \leq L \leq \infty)$. and either

(i) $\displaystyle\lim_{x \to s} f(x) = \lim_{x \to s} g(x) = 0$; or

(ii) $\displaystyle\lim_{x \to s} g(x) = \infty$ or $-\infty$.

Then $\boxed{\displaystyle\lim_{x \to s} \frac{f(x)}{g(x)}} = L$.

Examples

- $\lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = 0$

- $\lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{2x}{e^x} = 0$

  $\cdots \lim_{x \to \infty} \frac{x^n}{e^x} = 0.$

- $\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \log x}$

  $\lim_{x \to 0^+} x \log x = \lim_{x \to 0^+} \frac{\log x}{\frac{1}{x}}$

  $= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} -x = 0$

  $y_n = x_n \log x_n \to 0.$ by L'Hospital.

  $\therefore \lim_{x \to 0^+} e^{x \log x} = e^{\lim_{x \to 0^+} x \log x} = e^0 = 1.$

  $f(x) = e^x$ is continuous (at 0): if $x_n \to 0$, then $e^{x_n} \to e^0 = 1.$

- $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$

  not logically sound.

  To prove that if $f(x) = \sin x$, then $f'(x) = \cos x$, we use the fact that $\lim_{x \to 0} \frac{\sin x}{x} = 1.$

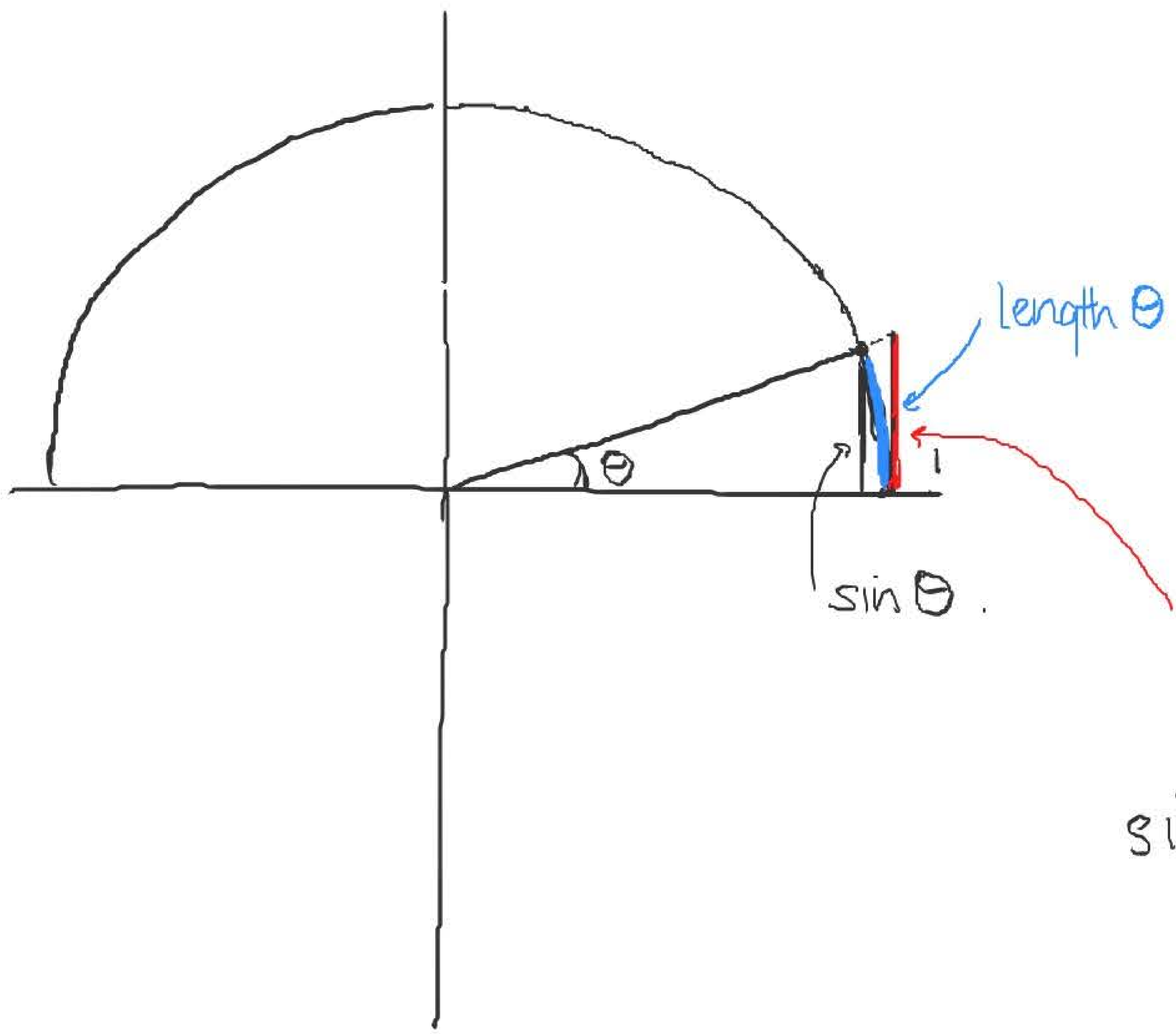
  $\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}.$

$f(x) = C^x$

$f'(x) = C^x \cdot \ln C.$

$g(x) = x^c$

$g'(c) = c x^{c-1}.$



$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta}.$$

$\tan\theta = y.$

$$\sin\theta \le \theta \le \tan\theta.$$

$$1 \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}.$$

$\downarrow \qquad\qquad \downarrow \qquad$ as $\theta \to 0$.

$1 \qquad\qquad 1$

Math 104 Worksheet 17

UC Berkeley, Summer 2021

Wednesday, August 4

Consider the function

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Exercise 1. Show that $f'(0) = 0$. (*Hint:* Consider the left and right limits separately.)

Exercise 2. Show by induction that for $x > 0$, $f^{(n)}(x)$ has the form

$$q_n\left(\frac{1}{x}\right)e^{-1/x}$$

where $q_n(t)$ is a polynomial in t.

Exercise 3. Show by induction that $f^{(n)}(0) = 0$ for all n.
(Therefore, $T^{f,0}(x) \equiv 0$, so $f(x) \neq T^{f,0}(x)$ for all $x > 0$.)