MATH 104 Cheat Sheet

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This document is a collection of all mentioned definitions, theorems, and corollaries from *Elementary Analysis* by Kenneth A. Ross or Theodore Zhu's lectures of MATH 104 Summer 2021.

Contents

Chapter 1

Introduction

1.1 The Set $\mathbb{N}$ of Natural Numbers

We denote the set $\{1, 2, 3, \dots\}$ of all *positive integers* by $\mathbb{N}$. Each positive integer n has a successor, namely $n + 1$. The following is 5 properties of $\mathbb{N}$:

N1. 1 belongs to $\mathbb{N}$.

N2. If $n \in \mathbb{N}$, then its successor $n + 1 \in \mathbb{N}$.

N3. 1 is not the successor of any element in $\mathbb{N}$.

N4. If n and m in $\mathbb{N}$ have the same successor, then $n = m$.

N5. A subset of $\mathbb{N}$ which contains 1, and which contains $n+1$ whenever it contains n, must equal $\mathbb{N}$.

Axiom **N5** is the basis of mathematical induction, which asserts all the statements $P_1, P_2, P_3, \dots$ are true provided

(I_1) P_1 is true,

(I_2) P_{n+1} is true whenever P_n is true.

1.2 The Set $\mathbb{Q}$ of Rational Numbers

Definition 1.2.1. A number is called an *algebraic number* if it satisfies a polynomial equation

$$c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0$$

where the coefficients $c_0, c_1, \ldots, c_n$ are integers, $c_n \neq 0$ and $n \geq 1$.

Rational numbers are always algebraic numbers. If $r = \frac{m}{n}$ is a rational number [$m, n \in \mathbb{Z}$ and $n \neq 0$], then it satisfies the equation $nx - m = 0$.

Theorem 1.2.2 (Rational Zeros Theorem)**.** *Suppose $c_0, c_1, \ldots, c_n$ are integers and r is a rational number satisfying the polynomial equation*

$$c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0 \tag{1}$$

where $n \geq 1$, $c_n \neq 0$ and $c_0 \neq 0$. Let $r = \frac{c}{d}$ where c, d are integers having no common factors and $d \neq 0$. Then $c \mid c_0$ and $d \mid c_n$.

In other words, the only rational candidates for solutions of (1) have the form $\frac{c}{d}$ where c divides c_0 and d divides c_n.

Proof. We are given

$$c_n\left(\frac{c}{d}\right)^n + c_{n-1}\left(\frac{c}{d}\right)^{n-1} + \cdots + c_1\left(\frac{c}{d}\right) + c_0 = 0$$

Multiply both sides by d^n and obtain

$$c_n c^n + c_{n-1}c^{n-1}d + c_{n-2}c^{n-2}d^2 + \cdots + c_2 c^2 d^{n-2} + c_1 c d^{n-1} + c_0 d^n = 0$$

Solve for $c_0 d^n$ and obtain

$$c_0 d^n = -c[c_n c^{n-1} + c_{n-1}c^{n-2}d + \cdots + c_2 c d^{n-2} + c_1 d^{n-1}]$$

Since c and d^n have no common factors, c divides c_0. Do the same thing and solve for $c_n c^n$ and we will see d divides c_n. ■

Corollary 1.2.2.1. *Consider the polynomial equation*

$$x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0$$

where the coefficients $c_0, c_1, \ldots, c_{n-1}$ are integers and $c_0 \neq 0$. Any rational solution of this equation must be an integer that divides c_0.

Proof. By the Rational Zeros Theorem 1.2.2, the denominator of r must divide the coefficient of x^n, which is 1. Thus r is an integer dividing c_0. ■

1.3 The Set $\mathbb{R}$ of Real Numbers

The set $\mathbb{Q}$ of Rational numbers also have the following properties for addition and multiplication:

A1. $a+(b+c)=(a+b)+c$ for all a,b,c.

A2. $a+b=b+a$ for all a,b.

A3. $a+0=a$ for all a.

A4. For each a, there is an element $-a$ such that $a+(-a)=0$.

M1. $a(bc)=(ab)c$ for all a,b,c.

M2. $ab=ba$ for all a,b.

M3. $a\cdot 1=a$ for all a.

M4. For each $a\neq 0$, there is an element a^{-1} such that $aa^{-1}=1$.

DL $a(b+c)=ab+ac$ for all a,b,c.

The set $\mathbb{Q}$ also has an order structure $\leq$ satisfying

O1. Given a and b, either $a\leq b$ or $b\leq a$.

O2. If $a\leq b$ and $b\leq a$, then $a=b$.

O3. If $a\leq b$ and $b\leq c$, then $a\leq c$.

O4. If $a\leq b$, then $a+c\leq b+c$.

O5. If $a\leq b$ and $0\leq c$, then $ac\leq bc$.

Theorem 1.3.1. *The following are consequences of the field properties:*

(i) $a+c=b+c \implies a=b$;

(ii) $a\cdot 0=0$ *for all* a;

(iii) $(-a)b=-ab$ *for all* a,b;

(iv) $(-a)(-b)=ab$ *for all* a,b;

(v) $(ac=bc)\wedge(c\neq 0) \implies a=b$;

(vi) $ab=0 \implies (a=0)\vee(b=0)$ *for* $a,b,c\in\mathbb{R}$.

for $a,c,c\in\mathbb{R}$.

Theorem 1.3.2. *The following are consequences of the properties of an ordered field:*

(i) $a \leq b \implies -b \leq -a$*;*

(ii) $(a \leq b) \wedge (c \leq 0) \implies bc \leq ac$*;*

(iii) $(0 \leq a) \wedge (0 \leq b) \implies 0 \leq ab$*;*

(iv) $0 \leq a^2$ *for all* a*;*

(v) $0 < 1$*;*

(vi) $0 < a \implies 0 < a^{-1}$*;*

(vii) $0 < a < b \implies 0 < b^{-1} < a^{-1}$*;*

for $a, c, c \in \mathbb{R}$*.*

Note that $a < b$ can be represented as $(a \leq b) \wedge (a < b)$.

Definition 1.3.3. We define

$$|a| = a \text{ if } a \geq 0 \text{ and } |a| = -a \text{ if } a \leq 0$$

An useful fact: $|a| \leq b \iff -b \leq a \leq b$.

Definition 1.3.4. For numbers a and b we define $\operatorname{dist}(a, b) = |a - b|$; $\operatorname{dist}(a, b)$ represents the *distance between* a *and* b.

Theorem 1.3.5.

(i) $|a| \geq 0$ *for all* $a \in \mathbb{R}$*.*

(ii) $|ab| = |a| \cdot |b|$ *for all* $a, b \in \mathbb{R}$*.*

(iii) $|a + b| \leq |a| + |b|$ *for all* $a, b \in \mathbb{R}$*.*

Corollary 1.3.5.1. $dist(a, c) \leq dist(a, b) + dist(b, c)$ *for all* $a, b, c \in \mathbb{R}$*. This is equivalent to* $|a - c| \leq |b - c| + |b - c|$*.*

Theorem 1.3.6 (Triangle Inequality)**.** $|a + b| \leq |a| + |b|$ *for all* a, b*.*

Corollary 1.3.6.1 (Reverse Triangular Inequality). $\big||a|-|b|\big| \leq |a-b|$ *for all* $a, b \in \mathbb{R}$.

Here is one of the most important techniques in real analysis.

(a) If $a \leq b + \epsilon$ for any $\epsilon > 0$, then $a \leq b$.

(b) If $a \geq b - \epsilon$ for any $\epsilon > 0$, then $a \geq b$.

(c) If $|a - b| < \epsilon$ for any $\epsilon > 0$, then $|a - b| = 0$.

1.4 The Completeness Axiom

The completeness axiom for $\mathbb{R}$ ensure us $\mathbb{R}$ has no "gaps".

Definition 1.4.1. Let S be a nonempty subset of $\mathbb{R}$.

(a) If S contains a largest element s_0 [that is, $s_0 \in S$ and $\forall s \in S,\ s \leq s_0$], then we call s_0 the *maximum* of S and write $s_0 = \max S$.

(b) If S contains a smallest element s_0 [that is, $s_0 \in S$ and $\forall s \in S,\ s \geq s_0$], then we call s_0 the *minimum* of S and write $s_0 = \min S$.

Open intervals like $(a, b) = \{x \in \mathbb{R} : a < x \leq b\}$ have no minimum or maximum since the endpoints a and b is not in the interval.

Definition 1.4.2. Let S be a nonempty subset of $\mathbb{R}$.

(a) If a real number M satisfies $s \leq M$ for all $s \in S$, then M is called an *upper bound* of S and the set S is said to be *bounded above*.

(b) If a real number m satisfies $m \leq s$ for all $s \in S$, then m is called an *lower bound* of S and the set S is said to be *bounded below*.

(c) The set S is said to be *bounded* if it is bounded above and bounded below. Thus S is bounded if there exist real numbers m and M such that $S \subseteq [m, M]$.

The maximum of a set is always an upper bound for the set. Likewise, the minimum of a set is always a lower bound for the set.

Definition 1.4.3. Least Upper Bound Property (LUBP)
An ordered set S has the LUBP if every nonempty subset $\mathcal{A} \subset S$ that has an upper bound has a least upper bound in S.

Note that the set $\mathbb{Q}$ of rational number does not satisfy the LUBP but $\mathbb{R}$ does. e.g. $(A) = \{q \in \mathbb{Q} : q^2 < 2\}$.

Definition 1.4.4. Let S be a nonempty subset of $\mathbb{R}$.

(a) If S is bounded above and S has a least upper bound, then we will call it the *supremum* of S and denote it by $\sup S$.

(b) If S is bounded below and S has a greatest lower bound, then we will call it the *infimum* of S and denote it by $\inf S$.

If S is bounded above, then $M = \sup S$ if and only if (i) $s \leq M$ for all $s \in S$, and (ii) whenever $M_1 < M$, there exists $s_1 \in S$ such that $s_1 > M_1$. Or for each $\epsilon > 0$, there exists

$s \in S$ such that $s > \sup S - \epsilon$.

Note that for a positive set $S = \{s : s > 0\}$, its infimum is not always positive. Example: $\{\frac{1}{n} : n \in \mathbb{N}\}$. Each element is positive but the infimum is 0.

Here are some basic facts:

- If a set S has finitely many elements, then $\max S$ exists.
- If $\max S$ exists, then $\sup S = \max S$.
- For any set $S \neq \varnothing$, $\inf S \leq \sup S$

Theorem 1.4.5 (Completeness Axiom). *Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. In other words, $\sup S$ exists and is a real number.*

Note that the completeness axiom does not hold for $\mathbb{Q}$.

Corollary 1.4.5.1. *Every nonempty subset S of $\mathbb{R}$ that is bounded below has a greatest lower bound. In other words, $\inf S$ exists and is a real number.*

Theorem 1.4.6 (Archimedean Property). *If $a > 0$ and $b > 0$, then for some positive integer n, we have $na > b$.*

Corollary 1.4.6.1. *(Set $a = 1$). For any $b > 0$, there exists $n \in \mathbb{N}$ such that $n > b$*

Corollary 1.4.6.2. *(Set $b = 1$). For any $a > 0$, there exists $n \in \mathbb{N}$ such that $na > 1 \implies \frac{1}{n} < a$.*

Lemma 1.4.7. *If $x, y \in \mathbb{R}$ such that $y - x > 1$, then there exists $m \in \mathbb{Z}$ such that $x < m < y$.*

Theorem 1.4.8 (Denseness of $\mathbb{Q}$). *If $a, b \in \mathbb{R}$ and $a < b$, then there is a rational $r \in \mathbb{Q}$ such that $a < r < b$.*

1.5 The Symbols $+\infty$ and $-\infty$

The symbols $+\infty$ and $-\infty$ are extremely useful even though they are **not** real numbers. So for each real number a, $-\infty < a < \infty$. If a set S is not bounded above, we define $\sup S = +\infty$. Likewise, if S is not bounded below, then we define $\inf S = -\infty$.

We can extend real numbers to $\mathbb{R} \cup \{-\infty, \infty\}$. Notice that this is not a **field**, so it does not satisfy all field properties.

For emphasis, we recapitulate:

Let S be any nonempty subset of $\mathbb{R}$. The *symbols* $\sup S$ and $\inf S$ always make sense. If S is not bounded above, then $\sup S$ is a *real* number; otherwise $\sup S = +\infty$. If S is bounded below, then $\inf S$ is a *real* number; otherwise $\inf S = -\infty$. Moreover, we have $\inf S \leq \sup S$.

Chapter 2

Sequences

2.1 Limits of Sequences

Definition 2.1.1. A sequence (s_n) of real numbers is said to **converge** to the real number s provided that

$$\forall \epsilon > 0,\ \exists N,\ n > N \Rightarrow |s_n - s| < \epsilon.$$

If (s_n) converges to s, we write $\lim_{n\to\infty} s_n = s$ or $s_n \to s$. s is the *limit* of the sequence (s_n). A sequence that does not converge (i.e. it has no *limit*) is said to *diverge*.
Notice that in the definition, instead of simple ϵ, we can also use some other complicated forms with some extra constants like $M\epsilon$, $\frac{\epsilon}{c}$, $a^2\epsilon$ and so on.

Intuitively, the definition means that no matter how small you pick $\epsilon > 0$, **eventually** the sequence will stay within ϵ of s at some point (the threshold N) and forever after.

Theorem 2.1.2. *The limit of a sequence* (s_n) *is unique. i.e.* $(\lim s_n = s) \wedge (lims_n = t) \Rightarrow s = t$.

Theorem 2.1.3.

- *If* $s_n \geq a$ *for all but finitely many* n*, then* $\lim s_n \geq a$.
- *If* $s_n \leq b$ *for all but finitely many* n*, then* $\lim s_n \leq b$.

Theorem 2.1.4 (Squeeze Lemma)**.** *If* $a_n \leq s_n \leq b_n$ *for all* n *and* $\lim a_n = \lim b_n = s$*, then* $\lim s_n = s$.

2.2 A Discussion about Proofs

This section gives several examples of proofs with some discussion using the definition of the limit of a sequence.

Example. Prove $\lim \frac{1}{n^2} = 0$.

Discussion. According to the definition of the limit, we need to consider an $\epsilon > 0$ such that $|\frac{1}{n^2} - 0| < \epsilon$ for $n >$ some$N$. $|\frac{1}{n^2} - 0| < \epsilon$ implies that $\frac{1}{\epsilon} < n^2$or$\frac{1}{\sqrt{\epsilon}} < n$. Thus we can suppose $N = \frac{1}{\sqrt{\epsilon}}$ and check if we reverse our reasoning into proof, it still makes sense.

Example. Prove $\lim \frac{3n+1}{7n-4} = \frac{3}{7}$

Discussion. Just like the last example, we can start from the definition 2.1.1 to get a suitable N.

Proof. Let $\epsilon > 0$ and $N = \frac{19}{49\epsilon} + \frac{4}{7}$, then

$$\begin{aligned} n > N \Rightarrow 7n &> \frac{19}{7\epsilon} + 4 \\ &\Rightarrow \frac{19}{7(7n-4)} < \epsilon \\ &\Rightarrow \frac{3n+1}{7n-4} - \frac{3}{7} < \epsilon \\ &\Rightarrow \left| \frac{3n+1}{7n-4} - \frac{3}{7} \right| < \epsilon \qquad \text{since } n > 0 \end{aligned}$$

This proofs $\lim \frac{3n+1}{7n-4} = \frac{3}{7}$ according to the definition of the limit 2.1.1. ∎

Example. Prove $\lim \frac{4n^3+3n}{n^3-6} = 4$

Discussion. Since $\frac{4n^3+3n}{n^3-6} - 4 = \frac{3n+24}{n^3-6}$, when $n > 1$, we can find an upper bound for $\frac{3n+24}{n^3-6}$ so that the bound $< \epsilon \Rightarrow \left|\frac{3n+24}{n^3-6}\right| < \epsilon$. Finding an upper bound for a fraction is equivalent to finding a upper bound for its numerator and a lower bound for its denominator. We know $3n + 24 \leq 27n$ for $n > 1$. Also we note $n^3 - 6 \geq \frac{n^3}{2} \Rightarrow n > 2$. Thus we can have $\frac{3n+24}{n^3-6} < \frac{27n}{n^3/2} < \epsilon \Rightarrow n > \sqrt{\frac{54}{\epsilon}}$, provided $n > 2$.

Proof. Let $\epsilon > 0$ and $N = \max\{2, \sqrt{\frac{54}{\epsilon}}\}$, then

$$
\begin{aligned}
n > N &\Rightarrow (n > \sqrt{\frac{54}{\epsilon}}) \wedge (n > 2) \\
&\Rightarrow (\frac{27n}{n^3/2} < \epsilon) \wedge (\frac{n^3}{2} \leq n^3 - 6) \wedge (27n \geq 3n + 24) \\
&\Rightarrow \frac{3n+24}{n^3-6} < \frac{27n}{n^3/2} < \epsilon \\
&\Rightarrow \left| \frac{4n^3+3n}{n^3-6} - 4 \right| < \epsilon
\end{aligned}
$$

This proofs $\lim \frac{4n^3+3n}{n^3-6} = 4$ according to the definition of the limit 2.1.1. ∎

Example. Show that $a_n = (-1)^n$ does not converge.

Discussion. Assume $\lim(-1)^n = a$, and we can see that no matter what a is, either 1 or -1 is at least 1 from a, so it means $|(-1)^n - a| < 1$ will not hold for all large n.

Proof. Suppose $\lim(-1)^n = a$ and $\epsilon = 1$. By 2.1.1, $|(-1)^n - a| < 1 \Rightarrow (|1-a| < 1) \wedge (|-1-a| < 1)$. Now by **??**, $2 = |1 - a + a - (-1)| \leq |1-a| + |a-(-1)| < 1 + 1 = 2$, which is a contradiction. ∎

Example. Let (s_n) be a sequence of nonnegative real numbers and suppose $s = \lim s_n$. Note $s \geq 0$. Prove $\lim \sqrt{s_n} = \sqrt{s}$

Proof. There are two cases.

1. $s > 0$: Let $\epsilon > 0$. $\lim s_n = s \Rightarrow (\exists N,\ n > N \Rightarrow |s_n - s| < \sqrt{s}\epsilon)$. $n > N$ also implies

$$|\sqrt{s_n} - \sqrt{s}| = \frac{(\sqrt{s_n} - \sqrt{s})(\sqrt{s_n} + \sqrt{s})}{\sqrt{s_n} + \sqrt{s}} = \frac{|s_n - s|}{\sqrt{s_n} + \sqrt{s}} \leq \frac{|s_n - s|}{\sqrt{s}} < \frac{\sqrt{s}\epsilon}{\sqrt{s}} = \epsilon$$

2. $s = 0$: EXERCISE 8.3

∎

Example. Let (s_n) be a convergent sequence of real numbers such that $s_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim s_n = s \neq 0$. Prove $\inf\{|s_n| : n \in \mathbb{N}\} > 0$

Proof. Let $\epsilon = \frac{|s|}{2}$. Since $\lim s_n = s$,

$$n > N \Rightarrow |s_n - s| < \frac{|s|}{2} \Rightarrow |s_n| \geq \frac{|s|}{2}$$

The last implication is because otherwise

$$|s| = |s - s_n + s_n| \leq |s - s_n| + |s_n| < \frac{|s|}{2} + \frac{|s|}{2} = |s|$$

which is a contradiction. Now if we set $m = \min\{\frac{|s|}{2}, |s_1|, |s_2|, \ldots, |s_N|\}$, then clearly we have $m > 0$ since and $|s_n| \geq m$ for all $n \in \mathbb{N}$. Thus $\inf\{|s_n| : n \in \mathbb{N}\} \geq m > 0$**WHY???** ■

2.3 Limit Theorems for Sequences

Definition 2.3.1. A sequence (s_n) is said to be *bounded* if $\exists M,\ \forall n,$ such that $|s_n| \leq M$

Theorem 2.3.2. *Convergent sequences are bounded.*

Remark. In other words, unbounded sequences are not convergent.

Theorem 2.3.3. *If the sequence (s_n) converges to s and $k \in \mathbb{R}$, then (ks_n) converges to ks. i.e.* $\lim(ks_n) = k \cdot \lim s_n$.

Theorem 2.3.4. *If (s_n) and (t_n) converge to s and t, then $(s_n + t_n)$ converges to $s + t$. That is,*

$$\lim(s_n + t_n) = \lim s_n + \lim t_n.$$

Theorem 2.3.5. *If (s_n) and (t_n) converge to s and t, then $(s_n t_n)$ converges to st. That is,*

$$\lim(s_n t_n) = (\lim s_n)(\lim t_n)$$

Lemma 2.3.6. *If $(s_n) \to s \neq 0$ and $s_n \neq 0$ and for all n, then* $\inf\{|s_n| : n \in \mathbb{N}\} > 0$.

Lemma 2.3.7. *If (s_n) converges to s, $s_n \neq 0$ for all n, and $s \neq 0$, then $(1/s_n)$ converges to $1/s$.*

Theorem 2.3.8. *Suppose (s_n) and (t_n) converge to s and t. If $s \neq 0$ and $s_n \neq 0$ for all n, then (t_n/s_n) converges to t/s.*

Theorem 2.3.9.

(a) $\lim_{n\to\infty}(\frac{1}{n^p}) = 0$ *for* $p > 0$.

(b) $\lim_{n\to\infty} a^n = 0$ *if* $|a| < 1$.

(c) $\lim(n^{1/n}) = 1$.

(d) $\lim_{n\to\infty} a^{1/n} = 1$ *for* $a > 0$.

Definition 2.3.10. For a (s_n), we write $\lim s_n = +\infty$ provided for each $M > 0$ there is a number N wuch that $n > N \Rightarrow s_n > M$. Similarly, we write $\lim s_n = -\infty$ provided for each $M < 0$ there is a number N wuch that $n > N \Rightarrow s_n < M$.

This implies that if $\lim s_n > -\infty$, $\exists T,\ \forall n, s_n > T$. $\lim s_n < \infty$, $\exists T,\ \forall n, s_n < T$. Be careful that we say $\lim s_n = +\infty$ as (s_n) **diverges** to ∞, **not converge** to ∞.

Theorem 2.3.11. *Let* $\lim s_n = +\infty$ *and* $\lim t_n > 0$. *Then* $\lim s_n t_n = +\infty$.

Theorem 2.3.12. *For a* (s_n) *of* positive *real numbers, we have* $\lim s_n = +\infty$ *if and only if* $\lim(\frac{1}{s_n}) = 0$.

Theorem 2.3.13. *Assume all* $s_n \neq 0$ *and that the limit* $L = \lim \left|\frac{s_{n+1}}{s_n}\right|$ *exists.*

(a) If $L < 1$, *then* $\lim s_n = 0$.

(b) If $L > 1$, *then* $\lim |s_n| = +\infty$.

2.4 Monotone Sequences and Cauchy Sequence

Definition 2.4.1. (s_n) is called an *increasing sequence (or nondecreasing)* if $\forall n,\ s_n \leq s_{n+1}$ and $s_n \leq s_m$ whenever $n < m$. Similarly, (s_n) is called an *decreasing sequence (or nonincreasing)* if $\forall n,\ s_n \geq s_{n+1}$. An increasing or decreasing sequence is called *monotone* or *monotonic* sequence.

Theorem 2.4.2. *All bounded monotone sequences converge.*

Remark. From the proof procedure above, we can see that bounded monotone sequences **converge to its infimum or supremum**.

Theorem 2.4.3.

(i) If (s_n) is an unbounded increasing sequence, then $\lim s_n = +\infty$.

(ii) If (s_n) is an unbounded decreasing sequence, then $\lim s_n = -\infty$.

Corollary 2.4.3.1. *If (s_n) is monotone, then $\lim s_n$ is always meaningful. i.e. $\lim s_n = s,\ +\infty,$ or $-\infty$.*

Suppose (s_n) is bounded. Define $u_n = \inf\{s_m : m \geq n\}$ and $v_n = \sup s_m : m \geq n$. Then observe that (u_n) is nondecreasing and (v_n) is nonincreasing since as n increases, the set has fewer elements. i.e. we have fewer choices for infimum and supremum. In general, if $A \subseteq B$, then $\inf A \geq \inf B$ and $\sup A \leq \sup B$.

Definition 2.4.4. Let (s_n) be a sequence in $\mathbb{R}$, define

- $\limsup s_n = \lim\limits_{N\to\infty} \sup\{s_n : n > N\}$
- $\liminf s_n = \lim\limits_{N\to\infty} \inf\{s_n : n > N\}$

If (s_n) is not bounded above. $\sup\{s_n : n > N\} = +\infty$ for all N and we decree $\limsup s_n = +\infty$. Likewise, if (s_n) is not bounded below. $\inf\{s_n : n > N\} = -\infty$ for all N and we decree $\liminf s_n = -\infty$.

Notice that $\limsup s_n$ need not equal to $\sup\{s_n : n > N\}$, but $\limsup s_n \leq \sup\{s_n : n > N\}$.

Remark. Since v_n and u_n are monotone, $\lim v_n = \limsup s_n$ and $\lim u_n = \liminf s_n$ always exist.

Theorem 2.4.5. *Let (s_n) be a sequence in $\mathbb{R}$.*

(i) If $\lim s_n$ is defined, then $\liminf s_n = \lim s_n = \limsup s_n$.

(ii) If $\liminf s_n = \limsup s_n$, then $\lim s_n$ is defined and $\lim s_n = \liminf s_n = \limsup s_n$.

Definition 2.4.6. A (s_n) is called a *Cauchy sequence* if

$$\forall \epsilon > 0,\ \exists N \text{ such that } m, n > N \Rightarrow |s_n - s_m| < \epsilon$$

Lemma 2.4.7. *Convergent sequences are Cauchy sequences.*

Lemma 2.4.8. *Cauchy sequences are bounded.*

Theorem 2.4.9. *A sequence in $\mathbb{R}$ is a convergent sequence if and only if it is a Cauchy sequence.*

2.5 Subsequences

Definition 2.5.1. Suppose $(s_n)_{n\in\mathbb{N}}$ is a sequence. A *subsequence* of this sequence is $(t_k)_{k\in\mathbb{N}}$ where for each k there is a positive integer n_k such that

$$n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$$

and

$$t_k = s_{n_k}.$$

Thus (t_k) is just a selection of some [possibly all] of the s_n's taken in order.

For the subset $\{n_1, n_2, \dots\}$ there is a natural function σ given by $\sigma(k) = n_k$ for $k \in \mathbb{N}$. The function σ "selects" an infinite subset of $\mathbb{N}$ in order. Then the subsequence of s corresponding to σ is simply the composite function $t = s \circ \sigma$. That is

$$t_k = t(k) = s \circ \sigma(k) = s(\sigma(k)) = s(n_k) = s_{n_k} \quad \text{for} \quad k \in N.$$

Notice that σ needs to be an *increasing* function.

Recall that the set $\mathbb{Q}$ of rational numbers is *countable*: there is a bijection from $\mathbb{N}$ to $\mathbb{Q}$. Therefore we have a sequence $(q_n) = (q_1, q_2, q_3, \dots)$ such that $\{q_n : n \in \mathbb{N}\} = \mathbb{Q}$. Then we have the following proposition:

Theorem 2.5.2. *Let (q_n) be an enumeration of $\mathbb{Q}$. Then for any $a \in \mathbb{R}$, there exists a subsequence (q_{n_k}) of (q_n) such that $q_{n_k} \to a$.*

Theorem 2.5.3. *Let (s_n) be a sequence in $\mathbb{R}$.*

(i) If t is in $\mathbb{R}$ then there is a subsequence of (s_n) converging to t if and only if the set $\{n \in \mathbb{N} : |s_n - t| < \epsilon\}$ is infinite *for all $\epsilon > 0$.*

(ii) If (s_n) is unbounded above, it has a subsequence with limit $+\infty$.

(iii) If (s_n) is unbounded below, it has a subsequence with limit $-\infty$.

In each case, the subsequence can be taken to be monotonic.

Theorem 2.5.4. *If (s_n) in $\mathbb{R}$ converges, then every subsequence converges to the same limit. If there are two subsequences of (s_n) with different limits, (s_n) does not converge.*

Theorem 2.5.5. *Every sequence (s_n) in $\mathbb{R}$ has a monotonic subsequence.*

Theorem 2.5.6 (Bolzano-Weierstrass Theorem)**.** *Every bounded sequence in* $\mathbb{R}$ *has a convergent subsequence.*

Definition 2.5.7. Let (s_n) be a sequence in $\mathbb{R}$. A *subsequential limit* is any real number or symbol $+\infty$ or $-\infty$ that is the limit of some subsequence of (s_n).

Theorem 2.5.8. *Let* (s_n) *be any sequence. There exists a monotonic subsequence whose limit is* $\limsup s_n$*, and there exists a monotonic subsequence whose limit is* $\liminf s_n$*.*

Theorem 2.5.9. *Let* (s_n) *be any sequence in* $\mathbb{R}$*, and let* S *denote the set of subsequential limits of* (s_n)*.*

(i) S *is nonempty.*

(ii) $\sup S = \limsup s_n$ *and* $\inf S = \liminf s_n$*.*

(iii) $\lim s_n$ *exists if and only if* S *has exactly one element, namely* $\lim s_n$*.*

(iv) $\limsup s_n \in S$ *and* $\liminf s_n \in S$*.*

Theorem 2.5.10. *Let* S *denote the set of subsequential limits of a sequence* (s_n)*. Suppose* (t_n)*is a sequence in* $S \cap \mathbb{R}$ *and that* $t = \lim t_n$*. Then* t *belongs to* S*.*

2.6 lim sup's and lim inf's

Theorem 2.6.1. *If* (s_n) *converges to a positive real number* s *and* (t_n) *is any sequence, then*

$$\limsup s_n t_n = s \cdot \limsup t_n.$$

Here we allow the conventions $s \cdot (+\infty) = +\infty$ *and* $s \cdot (-\infty) = -\infty$ *for* $s > 0$.

Theorem 2.6.2. *Let* (s_n) *be any sequence of nonzero real numbers. Then we have*

$$\liminf \left| \frac{s_{n+1}}{s_n} \right| \leq \liminf |s_n|^{1/n} \leq \limsup |s_n|^{1/n} \leq \limsup \left| \frac{s_{n+1}}{s_n} \right|$$

Corollary 2.6.2.1. *If* $\lim \left| \frac{s_{n+1}}{s_n} \right|$ *exists [and equals* L*], then* $\lim |s_n|^{1/n}$ *exists [and equals* L*].*

2.7 Some Topological Concepts in Metric Spaces

Definition 2.7.1. Let X be a set, and suppose d is a function $d : X \times X \to [0, \infty]$ defined for all pairs (x, y) of elements from X satisfying

1. $d(x, x) = 0$ for all $x \in S$ and $d(x, y) > 0$ for distinct $x, y \in X$. (Positive Definiteness)
2. $d(x, y) = d(y, x)$ for all $x, y \in X$. (Symmetry)
3. $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$. (Triangle Inequality)

Such a function d is called a *distance function* or a *metric* on X. A *metric space* X is a set X together with a metric on it.

Remark. The positive definiteness can be also expressed as $\forall x, y \in X \; d(x, y) \ge 0$ and $d(x, y) = 0 \iff x = y$. The distance function cannot be $+\infty$.

Example. Discrete metric space is defined as

$$\text{For any set } X \text{ with metric or distance function as } \begin{cases} 1 & x \neq y \\ 0 & x = y. \end{cases}$$

Notice that all sets in discrete metric space are both open and closed.

Definition 2.7.2 (Convergence)**.** A sequence (x_n) in a metric space (X, d) converges to x in X if $\lim_{n\to\infty} d(s_n, s) = 0$.

Remark. In other words, a sequence (x_n) converges to x if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \ge N \implies d(x_n, x) < \epsilon$.

Definition 2.7.3 (Cauchy)**.** A sequence (x_n) in X is a *Cauchy* if for any $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$m, n \ge \implies d(x_m, x_n) < \epsilon.$$

Definition 2.7.4 (Complete)**.** A metric space (X, d) is *complete* if every Cauchy sequence in X converges.

Remark. Every convergent sequence (x_n) in X is Cauchy.

Definition 2.7.5 (Open Ball)**.** Let (X, d) be a metrc space. For $x \in X$ and $r > 0$, the open ball of radius r centered at x is the set

$$B_r(x) = \{y \in X : d(y, x) < r\}$$

Definition 2.7.6 (Interior Point)**.** Let (X, d) be a metric space. Let E be a subset of X. An element $x \in E$ is *interior* to E if for some $r > 0$ we have

$$B_r(x) \subseteq E$$

We write E° for the set of points in E that are interior to E.

Remark.

- The relationship between E and X may affect whether a point in E is interior to E. For example, for $E = [0, 1] \subset [-1, 2] = X$, 0 is not interior to $[0, 1]$. However if $E = [0, 1] \subset [0, 1] = X$, then 0 is interior to 0 since there is not point in X beyond the left of 0.
- E° is open.
- $E = E^\circ$ if and only if E is open.
- If F is an open set such that $F \subseteq E$, then $F \subseteq E^\circ$.

Definition 2.7.7 (Open Set)**.** A set $E \subseteq X$ is *open* if every point $x \in E$ is an interior point of E. i.e., if $E = E^\circ$.

Remark.

- A set being open does **not** mean it is **not** closed. e.g. $[0, 1)$ is neither open nor closed.

Example.

- $(a, b), (a, \infty), (-\infty, a)$ are open sets.
- In $\mathbb{R}$, $\mathbb{Q}$ is *not* open since $B_r(q)$ may contain irrational numbers in $\mathbb{R}$ so $B_r(q) \nsubseteq \mathbb{Q}$.
- In any metric space (X, d), X and $\mathbb{Q}$ are open trivially.

Theorem 2.7.8 (Open ball is open)**.** *Let (X, d) be a metric space. Given $x \in X$ and $r > 0$, $B_r(x)$ is an open set in X.*

Proof. Consider arbitrary $y \in B_r(x)$ and let $s = r - d(x, y)$. It is easy to show that $B_x(y) \subseteq B_r(x)$. Thus y is an interior point of $B_r(x)$. Since y is arbitrary, by the definition $B_r(x)$ is open. ■

Theorem 2.7.9 (Union and intersection of open sets)**.** *Let (X, d) be a metric space.*

(i) If $\{\mathcal{U}_\alpha\}_{\alpha\in\mathcal{A}}$ is any collection of open sets in X, then $\bigcup_{\alpha\in\mathcal{A}}\mathcal{U}_\alpha$ is open. i.e. the union of any *collection of open sets is open.*

(ii) If $\{\mathcal{U}_1, \dots, \mathcal{U}_n\}$ is a finite collection of open sets in X, then $\bigcap_{i=1}^{n}\mathcal{U}_i$ is open.

Proof.

(i) Consider $x \in \bigcup_{\alpha\in\mathcal{A}}\mathcal{U}_\alpha$, then $\exists\beta \in \mathcal{A}$ such that $x \in \mathcal{U}_\beta$. Since $\mathcal{U}_\beta$ is open, $\exists r > 0$ such that $B_r(x) \subseteq \mathcal{U}_\beta \subseteq \bigcup_{\alpha\in\mathcal{A}}\mathcal{U}_\alpha$. Thus x is interior to $\bigcup_{\alpha\in\mathcal{A}}\mathcal{U}_\alpha$, completing the proof.

(ii) Consider $x \in \bigcap_{i=1}^{n}\mathcal{U}_i$. Since $x \in \mathcal{U}_i$ for $i = 1, \dots, n$, $\exists r_i > 0$ such that $B_{r_i}(x) \subseteq \mathcal{U}_i$. Take $r = \min\{r_1, \dots, r_n\}$, then clearly $B_r(x) \subseteq \bigcap_{i=1}^{n}\mathcal{U}_i$.

■

Remark. The examples for infinite collection in (ii) is $\bigcap_{n=1}^{\infty}(1 - \frac{1}{n}, 1 + \frac{1}{n}) = \{1\}$. Since 1 is not an interior point of $\{1\}$, $\{1\}$ is not open.

Definition 2.7.10 (Complement)**.** For a set $E \subseteq X$, the *complement* of E is the set $E^C = X\backslash E = \{x \in X : x \notin E\}$.

Definition 2.7.11 (Limit Point)**.** For a set $E \subseteq X$, a point $x \in X$ is a *limit point* of E if for any $r > 0$, we have that $(B_r(x)\backslash\{x\}) \cap E \neq \varnothing$.
E' denotes the set of all limit points of E.

Remark.

- In other words, for any radius $r > 0$, no matter how small is r, there is some element of E which sits in $B_r(x)$ other than x itself.
- If $E \subseteq F$, then $E' \subseteq F'$.
- $(E \cup F)' = E' \cup F'$.

Example.

- In $\mathbb{R}$, the set of limit points of $(0, 1)$ is $[0, 1]$.
- In $\mathbb{R}$, the only limit point of $\{\frac{1}{n} : n \in \mathbb{N}\}$ is 0.
- In $\mathbb{R}$, the set of limit point of $\mathbb{Q}$ is $\mathbb{R}$.

Theorem 2.7.12. *A point x is a limit point of a set $E \subseteq X$ if and only if $x = \lim x_n$ for some sequence x_n of points in $E\backslash\{x\}$.*

Proof. See homework 3.7. ■

Definition 2.7.13 (Isolated Point)**.** For a set $E \subseteq X$, $x \in E$ is called an *isolated point* if x is not a limit point of E

Remark. In other words, x is an isolated point or not a limit point of E if there exists a radius r such that $B_r(x)$ does not contain any element of E except x itself.

Example.

- In $\mathbb{R}$, every integer is an isolated point of $\mathbb{Z}$.
- In $\mathbb{R}$, the set Q has no isolated point.
- In $\mathbb{R}$, every element of $\{\frac{1}{n} : n \in \mathbb{N}\}$ is an isolated point.

Definition 2.7.14 (Closed Set)**.** A set is *closed* if $E' \subseteq E$.

Definition 2.7.15 (Closed Set)**.** Let (X, d) be a metric space. A subset E of X is *closed* if its complement E^{C} is an open set.

Remark.

- The above two definitions are equivalent.
- In other words, E contains all of its limit points, or every limit point of E is in E.
- In any metric space (X, d), X and $\varnothing$ are closed.
- A set being closed does **not** mean it is **not** open. e.g. $[0, 1)$ is neither open nor closed.

Example.

- In $\mathbb{R}$, $[0, 1]$ is closed. $[a, \infty), (-\infty, a]$ are closed.
- In $\mathbb{R}$, the set $\{\frac{1}{n} : n \in \mathbb{N}\}$ is not closed, but $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is closed.
- In any metric space, X and $\varnothing$ are closed.
- All finite sets do not have limit point, so they are trivially closed.

Theorem 2.7.16. *A set $E \subseteq \mathbb{R}$ is closed if and only if every Cauchy sequence contained in E has a limit that is also an element of E.*

Theorem 2.7.17 (The set of limit points is closed). *Let (X, d) be a metric space. Let $E \subseteq X$, then E', (the set of limit points of E), is closed.*

Proof. We need to show for any limit point x of E', x is in E'. Since x is a limit point of E', $\forall r > 0$, $(B_r(x) \backslash \{x\}) \cap E' \neq \varnothing$. i.e. there exists $y \in E'$ such that $y \neq x$ and $y \in B_r(x)$. Take $s = \min\{r - d(x,y), d(x,y)\}$. Since $y \in E'$, $(B_s(y) \backslash \{y\}) \cap E \neq \varnothing$. i.e. $\exists z \in (B_s(y) \backslash \{y\}) \cap E \neq \varnothing$.

Now since $s < r - d(x,y)$, $d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + (r - d(x,y)) = r \implies z \in B_r(x)$. Also since $s < d(x,y)$, $z \neq x$. Thus $z \in (B_r(x) \backslash \{x\}) \cap E \implies (B_r(x) \backslash \{x\}) \cap E \neq \varnothing$, which implies x is a limit point of E. i.e. $x \in E'$, completing the proof. ■

Theorem 2.7.18 (Union and intersection of closed sets).

(i) If $\{\mathcal{E}_\alpha\}_{\alpha \in \mathcal{A}}$ is any collection of closed set, then $\bigcap_{\alpha \in \mathcal{A}} \mathcal{E}_\alpha$ is closed.

(ii) If $\{\mathcal{E}_1, \ldots, \mathcal{E}_n\}$ is a finite collection of closed sets in X, then $\bigcup_{i=1}^n \mathcal{E}_i$ is closed.

Proof.

(i) Observe that $\left(\bigcap_{\alpha \in \mathcal{A}} \mathcal{E}_\alpha\right)^{\mathsf{C}} = \bigcup_{\alpha \in \mathcal{A}} \mathcal{E}_\alpha^{\mathsf{C}}$. Since $\mathcal{E}_\alpha$ is closed, $\mathcal{E}_\alpha^{\mathsf{C}}$ is open. By 2.7.9, the union of open sets $\bigcup_{\alpha \in \mathcal{A}} \mathcal{E}_\alpha^{\mathsf{C}}$ is open, completing the proof.

(ii) Observe that $(\bigcup_{i=1}^n \mathcal{E}_i)^{\mathsf{C}} = \bigcap_{i=1}^n \mathcal{E}_i^{\mathsf{C}}$. Since $\mathcal{E}_i$ is closed, $\mathcal{E}_i^{\mathsf{C}}$ is open. By 2.7.9, the intersection of finite open sets $\bigcap_{i=1}^n \mathcal{E}_i^{\mathsf{C}}$ is open, completing the proof.

■

Remark. $\bigcup_{x \in (0,1)} \{x\} = (0,1)$ is an example to the union of infinite closed sets is open in (ii).

The proof above uses one of DeMorgan's Laws for sets.

DeMorgan's Laws for sets

Suppose a metric space (X, d) and let $\forall\alpha \in \mathcal{A}\ U_\alpha \in X$. Then $\bigcap_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha^{\mathsf{C}} = \left(\bigcup_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha\right)^{\mathsf{C}}$.

Proof. We want to show both directions.

$\subseteq$: Consider $u \in \bigcap_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha^{\mathsf{C}}$, then we have

$$\begin{aligned} \forall\alpha \in \mathcal{A}\ u \in \mathcal{U}_\alpha^{\mathsf{C}} &\implies \forall\alpha \in \mathcal{A}\ u \notin \mathcal{U}_\alpha && (1)\\ &\implies u \notin \bigcup_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha && (2)\\ &\implies u \in \left(\bigcup_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha\right)^{\mathsf{C}}. && (3) \end{aligned}$$

(1) $\implies$ (2) because

$$(\neg(u \in \mathcal{U}_1)) \wedge (\neg(u \in \mathcal{U}_2)) \wedge \cdots = \neg((u \in \mathcal{U}_1 \vee (u \in \mathcal{U}_2) \vee \cdots) = \neg\left(u \in \bigcup \mathcal{U}_i\right)$$

Thus $\bigcap_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha^{\mathsf{C}} \subseteq \left(\bigcup_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha\right)^{\mathsf{C}}$.

$\supseteq$: Consider $u \in \left(\bigcup_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha\right)^{\mathsf{C}}$, then we have

$$\begin{aligned} u \notin \bigcup_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha &\implies \forall\alpha \in \mathcal{A}\ u \notin \mathcal{U}_\alpha\\ &\implies \forall\alpha \in \mathcal{A}\ u \in \mathcal{U}_\alpha^{\mathsf{C}}\\ &\implies u \in \bigcap_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha^{\mathsf{C}} \end{aligned}$$

Thus $\bigcap_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha^{\mathsf{C}} \supseteq \left(\bigcup_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha\right)^{\mathsf{C}}$, and hence $\bigcap_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha^{\mathsf{C}} = \left(\bigcup_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha\right)^{\mathsf{C}}$.

■

Definition 2.7.19 (Bounded Set)**.** A set $E \subseteq X$ is bounded if for some $x \in X$ and $M > 0$ such that $d(x, y) \leq M$ for all $y \in E$.

Remark.

- In $\mathbb{R}^k$, $X \subseteq \mathbb{R}^k$ is bounded if there exists $M > 0$ such that $\forall\mathbf{x} \in X\ d(\mathbf{x}, \mathbf{0}) = \sqrt{x_1^2 + \cdots + x_k^2} \leq M$.
- Finite union of bounded sets is bounded.

- Intersection of bounded sets is bounded.

- Contained in some open ball.

Theorem 2.7.20. *In R, any closed and bounded sets always have maximum and minimum.*

Definition 2.7.21 (Closure)**.** The *closure* of E in X is $\bar{E} = E \cup E'$.

Remark.

- $\bar{E}$ is the intersection of all closed sets containing E.

- $\bar{E}$ is closed.

- $E = \bar{E}$ if and only if E is closed.

- If F is a closed set such that $E \subseteq F$, then $\bar{E} \subseteq F$.

- The union of closures of finite sets is equal to the closure of unions of the sets. i.e. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Theorem 2.7.22. *For any $E \subseteq X$, its closure $\bar{E} = E \cup E'$ is closed and is the smallest closed set containing A.*

Definition 2.7.23 (Dense Set)**.** A set $E \subseteq X$ is *dense* in X if $\bar{E} = X$.

Example.

- $\mathbb{Q}$ is dense in $\mathbb{R}$.
- In any metric space (X, d), X is dense in X.

Definition 2.7.24 (Dense Set)**.** A set $E \subseteq X$ is dense in X if and only if for any $x \in X$ and $r > 0$.

$$B_r(x) \cap E \neq \varnothing.$$

Lemma 2.7.25.

- *A sequence* $(\mathbf{x}^{(n)})$ *in* $\mathbb{R}^k$ *converges to* $\mathbf{x} = (x_1, \ldots, x_k)$ *if and only if for each* $j = 1, 2 \ldots, k$, *the sequence* $(x_j^{(n)})$ *converges in* $\mathbb{R}$.
- *A sequence* $(\mathbf{x}^{(n)})$ *in* $\mathbb{R}^k$ *is a Cauchy sequence if and only if each sequence* $(x_j^{(n)})$ *is a Cauchy sequence in* $\mathbb{R}$.

Proof. First observe for $\mathbf{x},\mathbf{y} \in \mathbb{R}^k$ and $j = 1, \ldots, k$

$$|x_j - y_j| = \sqrt{(x_j - y_j)^2} \leq \sqrt{(x_1 - y_1)^2 + \cdots + (x_k - y_k)^2} = d(\mathbf{x},\mathbf{y}) \leq \sqrt{k} \max\{|x_j - y_j| : j = 1, \ldots, k\} \tag{1}$$

First assertion:

$\Longrightarrow$: Given that $(\mathbf{x}^{(n)})$ converges to $\mathbf{x}$. For each *epsilon* > 0 there exists $N \in \mathbb{N}$ such that $n \geq N \implies d(\mathbf{x}^{(n)}, \mathbf{x}) < \epsilon$. Then by (1) for $j = 1, \ldots, k$

$$n \geq N \implies |x_j^{(n)} - x_j| \leq d(\mathbf{x}^{(n)}, \mathbf{x}) < \epsilon,$$

so $x_j^{(n)} \to x_j$.

$\Longleftarrow$: For $j = 1, \ldots, k$, $\forall \epsilon > 0$, there exists $N_j \in \mathbb{N}$ such that

$$n \geq N_j \implies |x_j^{(n)} - x_j| < \frac{\epsilon}{\sqrt{k}}.$$

Take $N = \max\{N_1, \ldots, N_k\}$, then by (1) we have

$$n \geq N \implies d(\mathbf{x}^{(n)}, \mathbf{x}) \leq \sqrt{k} \max\{|x_j - y_j| : j = 1, \ldots, k\} < \sqrt{k} \cdot \frac{\epsilon}{\sqrt{k}} = \epsilon.$$

Thus $(\mathbf{x}^{(n)}) \to \mathbf{x}$

Second assertion:

$\Rightarrow$: Suppose $(\mathbf{x}^{(n)})$ is a Cauchy sequence, from the definition we know

$$m, n > N \Rightarrow d(\mathbf{x}^{(m)}, \mathbf{x}^{(n)}) < \epsilon$$

From (1) we see

$$m, n > N \Rightarrow |x_j^{(m)} - x_j^{(n)}| < \epsilon$$

so $(x_j^{(n)})$ is a Cauchy sequence.

$\Leftarrow$: Suppose $(x_j^{(n)})$ is a Cauchy sequence, then for $j = 1, \dots, k$

$$m, n > N_j \Rightarrow |x_j^{(m)} - x_j^{(n)}| < \frac{\epsilon}{\sqrt{k}}.$$

If $N = \max\{N_1, N_2, \dots, N_k\}$, then by (1)

$$m, n > N \Rightarrow d(\mathbf{x}^{(m)}, \mathbf{y}^{(n)}) < \epsilon$$

i.e. $(\mathbf{x}^{(n)})$ is a Cauchy sequence.

■

Theorem 2.7.26. *Euclidean k-space $\mathbb{R}^k$ is complete.*

Proof. Consider a Cauchy sequence $(\mathbf{x}^{(n)})$ in $\mathbb{R}^k$. By 2.7.25, each $(x_j^{(n)})$ is a Cauchy sequence. By 2.4.9 each $(x_j^{(n)})$ converges. Thus by 2.7.25 $(\mathbf{x}^{(n)})$ converges. ■

Theorem 2.7.27 (Bolzano-Weierstrass in $\mathbb{R}^k$)**.** *Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.*

Proof. Since $(\mathbf{x}^{(n)})$ is bounded, then each $(x_j^{(n)})$ is bounded in $\mathbb{R}$. By 2.5.6, we could replace $(\mathbf{x}^{(n)})$ by one of its subsequence, say $(\bar{\mathbf{x}}^{(n)})$, whose $(x_1^{(n)})$ converges. By 2.5.6 again, we may replace $(\mathbf{x}^{(n)})$ by a subsequence of $(\mathbf{x}^{(n)})$ such that both $(x_1^{(n)})$ and $(x_2^{(n)})$ converge. $(x_1^{(n)})$ still converges because 2.5.4. Repeating this argument by k times, we obtain a new sequence $(\mathbf{x}^{(n)})$ where each $(x_j^{(n)})$ converges, $j = 1, \dots, k$, which is a subsequence of the original sequence, and it converges by 2.7.25. ■

Remark. In any general metric space (X, d), it is not true that any bounded sequence has a convergent subsequence. E.g. $(\mathbb{Q}, d)$ and infinite discrete metric space

Theorem 2.7.28. *Let E be a subset of a metric space (S, d).*

1. *E is closed $\iff$ $E = E^-$.*
2. *E is closed $\iff$ E contains the limit of every convergent sequence of points in E.*
3. *An element is in E^- $\iff$ it is the limit of some sequence of points in E.*
4. *A point is in the boundary of E $\iff$ it belongs to the closure of both E and its complement.*

Compactness

Definition 2.7.29 (Open Cover)**.** Let (X, d) be a metric space and $E \subseteq X$. An open cover of E is a collection of open sets $\{\mathcal{U}_\alpha\}_{\alpha \in \mathcal{A}}$ such that $E \subseteq \bigcup_{\alpha \in \mathcal{A}} \mathcal{U}_\alpha$. An open cover is finite if it contains finitely many sets.

Definition 2.7.30 (Subcover)**.** A subcover of an open cover $\{\mathcal{U}_\alpha\}_{\alpha \in \mathcal{A}}$ of E is an *open cover* $\{\mathcal{U}_\alpha\}_{\alpha \in \mathcal{B}}$ such that $\mathcal{B} \subseteq \mathcal{A}$.

Definition 2.7.31 (Compact Set)**.** A set $E \subseteq X$ is compact if every open cover of E has a *finite* subcover.

Example.

- Every finite set is compact.
- Infinite discrete metric space is not compact.
- $\mathbb{R}$ is not compact: $\{(-n, n)\}_{n \in \mathbb{N}}$ is an open cover of $\mathbb{R}$ but does not have a finite subcover.
- $(0, 1)$ is not compact: $\{(0, r)\}_{r \in (0,1)}$ is an open cover of $(0, 1)$ but does not have a finite subcover.
- Closed interval in R is compact.

Theorem 2.7.32. *Compact sets are closed in any metric space.*

Proof. Let $E \subseteq X$ be compact. To show E is closed, we can show E^C is open. Consider $x \in E^\mathsf{C}$. For each $y \in E$, let $r_y := \frac{1}{2} d(x, y)$. Clearly $\{B_{r_y}(y)\}_{y \in E}$ is an open cover of E because each point in E is a center of an open ball. By the assumption, E is compact, so there is a finite subcover $\{B_{r_{y_1}}(y_1), \ldots, B_{r_{y_n}}(y_n)\}$ such that $E \subseteq \bigcup_{i=1}^n B_{r_{y_i}}(y_i)$.

Now take $r = \min\{r_{y_1}, \ldots, r_{y_n}\}$, and hence $B_r(x) \cap (\bigcup_{i=1}^n B_{r_{y_i}}(y_i)) = \varnothing$. Since $E \subseteq \bigcup_{i=1}^n B_{r_{y_i}}(y_i)$, $B_r(x) \cap E = \varnothing \implies B_r(x) \subseteq E^\mathsf{C}$. Thus x is an interior point of E^C, completing the proof. ∎

Remark. Non-closed sets are not compact in any metric space. Notice open set does not mean non-closed.

Theorem 2.7.33. *Closed subsets of compact sets are compact.*

Proof. See worksheet 7. ∎

Corollary 2.7.33.1. *If $\{K_\alpha\}_{\alpha\in A}$ is a collection of compact sets, then $\bigcap_{\alpha\in\mathcal{A}} K_\alpha$ is compact.*

Proof. Since compact sets are closed, $\bigcap_{\alpha\in\mathcal{A}} K_\alpha$ is the intersection of closed sets, which is also closed. Since $\bigcap_{\alpha\in\mathcal{A}} K_\alpha$ is a subset of compact sets U_α, it is compact. ∎

Remark. Finite union of compact sets in X is compact.

Theorem 2.7.34. *Every sequence in a compact set has a convergent subsequence.*

Proof. See worksheet 7. ∎

Theorem 2.7.35 (Compact Set)**.** *A set $E \subseteq X$ is compact if and only if every sequence in E has a convergent subsequence converging to a point in E.*

Theorem 2.7.36 (Nested Compact Sets Property)**.** *Let (F_n) be a sequence of closed, bounded, nonempty sets in $\mathbb{R}^k$ such that $F_1 \supseteq F_2 \supseteq \cdots$, then $F = \bigcap_{n=1}^{\infty} F_n \neq \varnothing$ and F is closed and bounded.*

Theorem 2.7.37. *Suppose $\{E_\alpha\}_{\alpha\in\mathcal{A}}$ is a collection of compact sets such that $\bigcap_{\alpha\in\mathcal{B}} E_\alpha \neq \varnothing$ for any finite $\mathcal{B} \subseteq \mathcal{A}$. Then $\bigcap_{\alpha\in\mathcal{A}} E_\alpha \neq \varnothing$.*

Definition 2.7.38 (K-cell)**.** A K-cell is a subset of $\mathbb{R}^k$ of the form $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k]$.

Theorem 2.7.39. *Every k-cell F in $\mathbb{R}^k$ is compact.*

Proof. TODO ∎

Theorem 2.7.40. *A subset E of $\mathbb{R}^k$ is compact if and only if it is closed and bounded.*

Proof. TODO ∎

Remark. The forward direction is true in any metric space.

Characterization of compact sets

(1) and (2) are equivalent in any metric space. Forward direction of (3) is true in any metric space. All of three are equivalent in $\mathbb{R}^k$.

1. Every open cover of E has a finite subcover.
2. A set $E \subseteq X$ is compact if and only if every sequence in E has a convergent subsequence converging to a point in E.
3. A subset E of $\mathbb{R}^k$ is compact if and only if it is closed and bounded.

Cantor Set

Definition 2.7.41 (Cantor Set)**.** Let $\mathcal{C}_0$ be $[0,1]$. Then define $\mathcal{C}_1$ as the union of 2^1 interval $[0,\frac{1}{3}]\cup[\frac{2}{3},1]$. Each time delete the middle $\frac{1}{3}$ of intervals. Thus $\mathcal{C}_2$ is the union of 2^2 intervals which is $[0,\frac{1}{9}]\cup[\frac{2}{9},\frac{1}{3}]\cup[\frac{2}{3},\frac{7}{9}]\cup[\frac{8}{9},1]$.
In short, C_n is the union of 2^n disjoint closed intervals of which length is $(\frac{1}{3})^n$. Then define Cantor Set

$$\mathcal{C} = \bigcap_{i=0}^{\infty} \mathcal{C}_i.$$

Theorem 2.7.42. *Here are some facts/properties about the Cantor set $\mathcal{C}$:*

- *$\mathcal{C}$ is compact.*
- *$\mathcal{C}$ does not contain any intervals.*
- *$\mathcal{C}$ does not have any interior points.*
- *Every point in $\mathcal{C}$ is a limit point of $\mathcal{C}$.*
- *Every point in $\mathcal{C}$ is a limit point of $\mathcal{C}^{\mathsf{C}}$.*

2.8 Series

For an infinite series $\sum_{n=m}^{\infty} a_n$, we say it *converge* provided the sequence (s_n) of partial sums

$$s_n = a_m + a_{m+1} + \cdots + a_n = \sum_{k=m}^{n} a_k$$

also converges to a real number S. i.e.

$$\sum_{n=m}^{\infty} a_n = S \quad \text{means} \quad \lim s_n = S \quad \text{or} \quad \lim_{n\to\infty} \left(\sum_{k=m}^{n} a_k \right) = S$$

A series that does not converge is said to *diverge*, so $\sum_{n=m}^{\infty} a_n$ *diverge* to $+\infty$, $\sum_{n=m}^{\infty} a_n = +\infty$, provided $\lim s_n = +\infty$. Similar for diverging to $-\infty$.

If the terms in $\sum a_n$ are all nonnegative, then the corresponding partial sums (s_n) form an increasing sequence, so $\sum a_n$ either converges or diverges to $+\infty$ by 2.4.2 and 2.4.3. In particular, $\sum |a_n|$ is meaningful for any (s_n) whatever. The series $\sum a_n$ is said to *converge absolutely* or to be *absolutely convergent* if $\sum |a_n|$ converges.

We use $\sum a_n$ to represent $\sum_{n=m}^{\infty} a_n$

Example (Geometric Series)**.** A series of the form $\sum_{n=0}^{\infty} ar^n$ for constants a and r is called a geometric series. For $r \neq 1$, the partial sums s_n are given by

$$\sum_{k=0}^{n} ar^k = a\frac{1-r^{n+1}}{1-r}.$$

Furthermore, if $|r| < 1$, then $\lim_{n\to\infty} r^{n+1} = 0$ and

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

If $a \neq 0$ and $|r| \geq 1$, then (ar^n) does not converge to 0, so $\sum ar^n$ diverges.

Example.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges if and only if} \quad p > 1$$

If $p \leq 1$, $\sum 1/n^p = +\infty$

Definition 2.8.1. We say a series $\sum a_n$ satisfies the *Cauchy criterion* if its sequence (s_n) of partial sums is a Cauchy sequence which is:

$$\forall \epsilon > 0,\ \exists N,\ m, n > N \Rightarrow |s_n - s_m| < \epsilon \tag{1}$$

which is equivalent to

$$\forall \epsilon > 0,\ \exists N,\ n \geq m > N \Rightarrow |s_n - s_{m-1}| < \epsilon. \tag{2}$$

Since $s_n - s_{m-1} = \sum_{k=m}^{n} a_k$, we can write (2) as

$$\forall \epsilon > 0,\ \exists N,\ n \geq m > N \Rightarrow \left| \sum_{k=m}^{n} a_k \right| < \epsilon \tag{3}$$

Theorem 2.8.2. *A series converges $\iff$ it satisfies the Cauchy criterion.*

Proof. By 2.4.9, we know its partial sum converges, so the series also converges. ∎

Corollary 2.8.2.1. *If a series $\sum a_n$ converges, then* $\lim a_n = 0$

Proof. By setting $n = m$ in the condition of 2.8.1, we get

$$(\forall \epsilon > 0,\ \exists N,\ n > N \Rightarrow |a_n| < \epsilon) \Rightarrow \lim a_n = 0$$

∎

A useful contrapositive of this corollary is "If $\lim a_n \neq 0$, then $\sum a_n$ does not converge."

Theorem 2.8.3 (Comparison Test)**.** *Let $\sum a_n$ be a series where $a_n \geq 0$ for all n.*

(i) If $\sum a_n$ converges and $|b_n| \leq a_n$ for all n, then $\sum b_n$ converges.

(ii) If $\sum a_n = +\infty$ and $b_n \geq a_n$ for all n, then $\sum b_n = +\infty$

Proof.

(i) For $n \geq m$ we have

$$\left| \sum_{k=m}^{n} b_k \right| \leq \sum_{k=m}^{n} |b_k| \leq \sum_{k=m}^{n} a_k \tag{1}$$

Since $\sum a_n$ converges, it satisfies 2.8.1(1). Then from (1) we can see $\sum b_n$ also satisfies the Cauchy criterion in 2.8.1(3), and hence $\sum b_n$ converges.

(ii) Since $b_n \geq a_n$ for all n, obviously we have $\sum_{k=m}^{n} b_k \geq \sum_{k=m}^{n} a_k$. Since $\lim \sum_{k=m}^{n} b_k = +\infty$, $\lim \sum_{k=m}^{n} a_k = +\infty$.

∎

Corollary 2.8.3.1. *Absolutely convergent series are convergent.*

Proof. Suppose $\sum b_n$ is absolutely convergent. This means $\sum a_n$ converges where $a_n = |b_n|$ for all n. Then $|b_n| \leq a_n$ and $\sum b_n$ converges trivially from 2.8.3. ■

Theorem 2.8.4 (Root Test)**.** *Let $\sum a_n$ be a series and let $\alpha = \limsup |a_n|^{1/n}$. The series $\sum a_n$*

(i) converges absolutely if $\alpha < 1$

(ii) diverges if $\alpha > 1$

(iii) Otherwise the test does not provide any useful information.

Proof. (i) Suppose $\alpha < 1$, and select $\epsilon > 0$ so that $\alpha + \epsilon < 1$. Then

$$\alpha - \epsilon < \sup\{|a_n|^{1/n} : n > N\} < \alpha + \epsilon$$

so

$$|a_n| < (a+\epsilon)^n \quad \text{for} \quad n > N.$$

Since $0 < \alpha + \epsilon < 1$, $\sum_{n=N+1}^{\infty}(\alpha+\epsilon)^n$ converges and 2.8.3(i) tells $\sum_{n=N+1}^{\infty} a_n$ converges. Then clearly $\sum a_n$ converges.

(ii) If $\alpha > 1$, then there is a subsequence of $|a_n|^{1/n}$ has limit $\alpha > 1$ by 2.5.8. This means $|a_n| > 1$ for infinitely many choices of n. In particular, (a_n) cannot possibly converge to 0, so $\sum a_n$ cannot converge by the contrapositive of 2.8.2.1.

(iii) Example: $\sum \frac{1}{n}$ diverges but $\sum \frac{1}{n^2}$ converges.

■

Theorem 2.8.5 (Ratio Test)**.** *A series $\sum a_n$ of nonzero terms*

(i) converges absolutely if $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$,

(ii) diverges if $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$.

(iii) Otherwise $\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$ and the test gives no information.

Proof. let $\alpha = \limsup |a_n|^{1/n}$. By **??** we have

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq \alpha \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|.$$

(i) If $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\alpha < 1$ and the series converges by 2.8.4.

(ii) If $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\alpha > 1$ and the series diverges by 2.8.4.

(iii) If $\alpha = 1$, then same reasoning as the proof in 2.8.4(iii).

■

If the terms a^n are nonzero and if $\lim \left|\frac{a_{n+1}}{a_n}\right| = 1$, then $\alpha = \limsup |a_n|^{1/n} = 1$ by 2.6.2.1, so neither the Ratio Test nor the Root Test gives information about the convergence of $\sum a_n$.

Chapter 3

Useful Tricks

1. Here is one of the most important techniques in real analysis.

   (a) If $a \leq b + \epsilon$ for any $\epsilon > 0$, then $a \leq b$.

   (b) If $a \geq b - \epsilon$ for any $\epsilon > 0$, then $a \geq b$.

   (c) If $|a - b| < \epsilon$ for any $\epsilon > 0$, then $|a - b| = 0$.

2. Let S be a bounded nonempty subset of $\mathbb{R}$ and suppose $\sup S \notin S$. Then there is a (strictly) increasing sequence (s_n) of points in S such that $\lim s_n = \sup S$.

3. A point x is a limit point of a set $E \subseteq X$ if and only if $x = \lim x_n$ for some sequence x_n of points in $E \backslash \{x\}$.

4. Let (s_n) be a convergent sequence.

   - If $s_n \geq a$ for all but finitely many n, then $\lim s_n \geq a$.
   - If $s_n \leq b$ for all but finitely many n, then $\lim s_n \leq b$.

5. (Squeeze Theorem) If $a_n \leq s_n \leq b_n$ for all n and $\lim a_n = \lim b_n = s$, then $\lim s_n = s$.

6. Assume all $s_n \neq 0$ and that the limit $L = \lim \left| \frac{s_{n+1}}{s_n} \right|$ exists.

   (a) If $L < 1$, then $\lim s_n = 0$.

   (b) If $L > 1$, then $\lim |s_n| = +\infty$.

7. The set $\mathbb{Q}$ of rational number can be listed as a sequence (r_n). Given any real number a there exists a subsequence (r_{n_k}) of (r_n) converging to a.

8. Given two **convergent** sequences (s_n) and (t_n). If there exists $N \in \mathbb{N}$ such that $s_n \leq t_n$ for all $n \geq N$, then $\lim s_n \leq \lim t_n$.

9. In general, if $A \subseteq B$, then $\inf A \geq \inf B$ and $\sup A \leq \sup B$.