Math 104 Midterm Exam (Compact Version)

UC Berkeley, Summer 2020

Thursday, July 16, 12:00pm - 2:00pm PDT

Problem 1. Short answers. No justification required for examples. (2 points each)

(a) Please copy verbatim the following text, followed by your signature. This MUST be handwritten UNLESS you are writing your entire exam electronically.

"As a member of the UC Berkeley community, I will act with honesty, integrity, and respect for others during this exam. The work that I will upload is my own work. I will not collaborate with or contact anyone during the exam, search online for problem solutions, or otherwise violate the instructions for this examination."

(b) Give an example of a sequence (s_n) of real numbers such that $\limsup s_n = \infty$, $\liminf s_n = -\infty$, and the sequence $(\bar{s}_n)$ defined by $\bar{s}_n := \frac{s_1 + \ldots + s_n}{n}$ converges.

(c) In the metric space $\mathbb{R}$ with standard Euclidean metric, give an example of an infinite set S of rational numbers such that S is a closed and bounded subset of $\mathbb{R}$.

(d) In the metric space $\mathbb{R}^2$ with standard Euclidean metric, let

$$E := [0,1] \times (0,1) = \{(x,y) : 0 \leq x \leq 1,\ 0 < y < 1\} \subseteq \mathbb{R}^2.$$

Give an example of an open cover of E which has no finite subcover.

(e) We proved in class that in a metric space (X,d), a set E is compact if and only if every sequence in E has a convergent subsequence whose limit lies in E. Explain carefully, in **no more than four sentences**, how this implies that closed and bounded subsets in $\mathbb{R}^k$ are compact.

Problem 2. (5 points) Let (s_n) and (t_n) be two sequences of real numbers such that $s_n \leq 2020 \leq t_n$ and $t_n - s_n \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. Prove that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = 2020$.

Problem 3. (5 points) Let (s_n) be a bounded sequence of real numbers. Suppose $\alpha \in \mathbb{R}$ has the property that for any $\beta > \alpha$, there exists $N \in \mathbb{N}$ such that $s_n < \beta$ for all $n \geq N$. Prove that $\limsup s_n \leq \alpha$.

Problem 4. (5 points) Let S be a subset of the rational numbers. Suppose that S is dense in the metric space $\mathbb{Q}$ (with the usual distance function). Prove that S is dense in $\mathbb{R}$. At the beginning of your proof, state the definition of **dense** that you will use in your proof.

Problem 5. (5 points) Let (X,d) be a metric space. In class, we proved that if a set $E \subseteq X$ is compact, then it is closed. (We did this by showing that E^c is open.) Re-prove this result by showing that if E is compact, then every limit point of E is contained in E.
(Hint: The proof I have in mind uses one of our alternate characterizations of compact sets.)