Math 104 Homework 8 Partial Solutions

UC Berkeley, Summer 2021

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2. Let f be a function defined on an open interval I containing x_0 which is twice differentiable on I such that f'' is continuous at x_0. Prove that for any $\varepsilon > 0$, there exists $h > 0$ such that

$$\left|\frac{f(x_0+h)-f(x_0-h)}{2h} - f'(x_0)\right| < h \cdot \varepsilon.$$

(Hint: Use Taylor's theorem about x_0.)

Solution. Let $\varepsilon > 0$. Since f'' is continuous at x_0, there exists $h > 0$ such that $|f''(x) - f''(x_0)| < 2\varepsilon$ for all $x \in I$ with $|x-x_0| < h$. By Taylor's theorem, there exists $\alpha_1 \in (x_0, x_0+h)$ and $\alpha_2 \in (x_0-h, x_0)$ such that

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(\alpha_1), \qquad f(x_0-h) = f(x_0) - hf'(x_0) + \frac{h^2}{2}f''(\alpha_2).$$

Then

$$\frac{f(x_0+h)-f(x_0-h)}{2h} = f'(x_0) + \frac{h}{4}(f''(\alpha_1) - f''(\alpha_2)),$$

so

$$\left|\frac{f(x_0+h)-f(x_0-h)}{2h} - f'(x_0)\right| = \frac{h}{4}\left|f''(\alpha_1)-f''(\alpha_2)\right| \le \frac{h}{4}\left(|f''(\alpha_1)-f''(x_0)|+|f''(x_0)-f''(\alpha_2)|\right) < h\cdot\varepsilon.$$

3. (Ross 32.8) Let f be an integrable function on $[a,b]$. Prove that f is integrable on any closed subinterval $[c,d] \subseteq [a,b]$.

Solution. Let $\varepsilon > 0$. Let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ be a partition of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$. Let $P^* = P \cup \{c,d\}$, and let $Q = \{t \in P^* : c \le t \le d\}$ be the partition of $[c,d]$ induced by P^*. Then $U(f,Q) - L(f,Q) \le U(f,P^*) - L(f,P^*) < \varepsilon$.

4. (a) Let $S = \{x_1, \ldots, x_n\}$ be a finite subset of $[a,b]$. Let f be the function

$$f(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases}$$

Prove that f is integrable and $\int_a^b f = 0$.

(b) Let g be an integrable function on $[a,b]$, and suppose h is a function on $[a,b]$ such that $h(x) = g(x)$ for all but finitely many points x in $[a,b]$. Prove that h is integrable and $\int_a^b h = \int_a^b g$.

Solution. (a) Let $\varepsilon > 0$. Let P be a partition of $[a,b]$ with $\text{mesh}(P) < \frac{\varepsilon}{2n}$. Then $U(f,P) - L(f,P) < 2n \cdot \frac{\varepsilon}{2n} = \varepsilon$, so f is integrable. Since $L(f,P) = 0$ for any partition P of $[a,b]$, it follows that $L(f) = 0$, so $\int_a^b f = 0$.
(b) Apply similar idea as part (a) to $h - g$.

5. Show that the function

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is integrable on $[0,1]$.

Solution. Let $\varepsilon > 0$. Since f is continuous on $[\frac{\varepsilon}{4}, 1]$, there exists a partition $P = \{\frac{\varepsilon}{4} = t_0 < t_1 < \ldots < t_n = 1\}$ such that $U(f,P) - L(f,P) < \frac{\varepsilon}{2}$. Let P^* be the partition of $[0,1]$ given by $\{0\} \cup P$. Then $U(f,P^*) - L(f,P^*) = \left(M(f,[0,\frac{\varepsilon}{4}]) - m(f,[0,\frac{\varepsilon}{4}])\right) \cdot \frac{\varepsilon}{4} + U(f,P) - L(f,P) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

8. (Ross 33.9) Let (f_n) be a sequence of integrable functions on $[a,b]$ such that $f_n \to f$ uniformly on $[a,b]$. Prove that f is integrable and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

Solution. Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that $|f_N(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in [a,b]$. Let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ be a partition of $[a,b]$ such that $U(f_N,P) - L(f_N,P) < \frac{\varepsilon}{3}$. Then $U(f,P) \leq U(f_N,P) + \frac{\varepsilon}{3}$ and $L(f,P) \geq L(f_N,P) - \frac{\varepsilon}{3}$ (requires more justification). Then

$$U(f,P) - L(f,P) \leq U(f_N,P) + \frac{\varepsilon}{3} - \left(L(f_N,P) - \frac{\varepsilon}{3}\right) < \varepsilon.$$

9. A sequence (f_n) of integrable functions on $[a,b]$ is said to **converge in** $L^1([a,b])$ to an integrable function f if $\lim_{n\to\infty} \int_a^b |f_n - f| = 0$. Find an example of a sequence of integrable functions (f_n) on $[0,1]$ and an integrable function f on $[0,1]$ such that $f_n \to f$ pointwise and $\lim_{n\to 0} \int_0^1 f_n = \int_0^1 f$, but f_n does not converge to f in $L^1([0,1])$.

Solution. $f_n = n\chi_{(0,\frac{1}{n})} - n\chi_{(1-\frac{1}{n},1)}$, $f \equiv 0$.