MATH 104 Notes

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Contents

1 Introduction **5**

1.1 The Set $\mathbb{N}$ of Natural Numbers . . . 6
1.2 The Set $\mathbb{Q}$ of Rational Numbers . . . 7
1.3 The Set $\mathbb{R}$ of Real Numbers . . . 8
1.4 The Completeness Axiom . . . 11
1.5 The Symbols $+\infty$ and $-\infty$ . . . 14

2 Sequences **15**

2.1 Limits of Sequences . . . 16
2.2 A Discussion about Proofs . . . 17
2.3 Limit Theorems for Sequences . . . 20
2.4 Monotone Sequences and Cauchy Sequence . . . 25
2.5 Subsequences . . . 29
2.6 lim sup's and lim inf's . . . 34
2.7 Some Topological Concepts in Metric Spaces . . . 36
2.8 Series . . . 48
2.9 Alternating Series and Integral Tests . . . 52

3 Continuity **55**

3.1 Continuous Functions . . . 56
3.2 Properties of Continuous Functions . . . 59
3.3 Uniform Continuity . . . 62
3.4 Continuity in Metric Space . . . 64
3.5 Limits of Functions . . . 67

4 Sequences and Series of Functions **71**

4.1 Power Series . . . 72
4.2 Uniform Convergence . . . 74
4.3 More on Uniform Convergence . . . 76
4.4 Differentiation and Integration of Power Series . . . 81

5 Differentiation **83**

5.1 Basic Properties of the Derivative . . . 84
5.2 The Mean Value Theorem . . . 87
5.3 L'Hospital's Rule . . . 89

6 Useful Tricks **91**

Chapter 1

Introduction

1.1 The Set $\mathbb{N}$ of Natural Numbers

We denote the set $\{1, 2, 3, \ldots\}$ of all *positive integers* by $\mathbb{N}$. Each positive integer n has a successor, namely $n + 1$. The following is 5 properties of $\mathbb{N}$:

N1. 1 belongs to $\mathbb{N}$.

N2. If $n \in \mathbb{N}$, then its successor $n + 1 \in \mathbb{N}$.

N3. 1 is not the successor of any element in $\mathbb{N}$.

N4. If n and m in $\mathbb{N}$ have the same successor, then $n = m$.

N5. A subset of $\mathbb{N}$ which contains 1, and which contains $n+1$ whenever it contains n, must equal $\mathbb{N}$.

Axiom **N5** is the basis of mathematical induction, which asserts all the statements $P_1, P_2, P_3, \ldots$ are true provided

(I_1) P_1 is true,

(I_2) P_{n+1} is true whenever P_n is true.

1.2 The Set $\mathbb{Q}$ of Rational Numbers

Definition 1.2.1. A number is called an *algebraic number* if it satisfies a polynomial equation

$$c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0$$

where the coefficients $c_0, c_1, \ldots, c_n$ are integers, $c_n \neq 0$ and $n \geq 1$.

Rational numbers are always algebraic numbers. If $r = \frac{m}{n}$ is a rational number [$m, n \in \mathbb{Z}$ and $n \neq 0$], then it satisfies the equation $nx - m = 0$.

Theorem 1.2.2 (Rational Zeros Theorem)**.** *Suppose $c_0, c_1, \ldots, c_n$ are integers and r is a rational number satisfying the polynomial equation*

$$c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0 \tag{1}$$

where $n \geq 1$, $c_n \neq 0$ and $c_0 \neq 0$. Let $r = \frac{c}{d}$ where c, d are integers having no common factors and $d \neq 0$. Then $c \mid c_0$ and $d \mid c_n$.

In other words, the only rational candidates for solutions of (1) have the form $\frac{c}{d}$ where c divides c_0 and d divides c_n.

Proof. We are given

$$c_n \left(\frac{c}{d}\right)^n + c_{n-1}\left(\frac{c}{d}\right)^{n-1} + \cdots + c_1\left(\frac{c}{d}\right) + c_0 = 0$$

Multiply both sides by d^n and obtain

$$c_n c^n + c_{n-1}c^{n-1}d + c_{n-2}c^{n-2}d^2 + \cdots + c_2 c^2 d^{n-2} + c_1 c d^{n-1} + c_0 d^n = 0$$

Solve for $c_0 d^n$ and obtain

$$c_0 d^n = -c[c_n c^{n-1} + c_{n-1}c^{n-2}d + \cdots + c_2 c d^{n-2} + c_1 d^{n-1}]$$

Since c and d^n have no common factors, c divides c_0. Do the same thing and solve for $c_n c^n$ and we will see d divides c_n. ■

Corollary 1.2.2.1. *Consider the polynomial equation*

$$x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0$$

where the coefficients $c_0, c_1, \ldots, c_{n-1}$ are integers and $c_0 \neq 0$. Any rational solution of this equation must be an integer that divides c_0.

Proof. By the Rational Zeros Theorem 1.2.2, the denominator of r must divide the coefficient of x^n, which is 1. Thus r is an integer dividing c_0. ■

1.3 The Set $\mathbb{R}$ of Real Numbers

The set $\mathbb{Q}$ of Rational numbers also have the following properties for addition and multiplication:

A1. $a + (b + c) = (a + b) + c$ for all a, b, c.

A2. $a + b = b + a$ for all a, b.

A3. $a + 0 = a$ for all a.

A4. For each a, there is an element $-a$ such that $a + (-a) = 0$.

M1. $a(bc) = (ab)c$ for all a, b, c.

M2. $ab = ba$ for all a, b.

M3. $a \cdot 1 = a$ for all a.

M4. For each $a \neq 0$, there is an element a^{-1} such that $aa^{-1} = 1$.

DL $a(b + c) = ab + ac$ for all a, b, c.

The set $\mathbb{Q}$ also has an order structure $\leq$ satisfying

O1. Given a and b, either $a \leq b$ or $b \leq a$.

O2. If $a \leq b$ and $b \leq a$, then $a = b$.

O3. If $a \leq b$ and $b \leq c$, then $a \leq c$.

O4. If $a \leq b$, then $a + c \leq b + c$.

O5. If $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

Theorem 1.3.1. *The following are consequences of the field properties:*

(i) $a + c = b + c \implies a = b$;

(ii) $a \cdot 0 = 0$ *for all* a;

(iii) $(-a)b = -ab$ *for all* a, b;

(iv) $(-a)(-b) = ab$ *for all* a, b;

(v) $(ac = bc) \wedge (c \neq 0) \implies a = b$;

(vi) $ab = 0 \implies (a = 0) \vee (b = 0)$ *for* $a, b, c \in \mathbb{R}$.

for $a, c, c \in \mathbb{R}$.

Theorem 1.3.2. *The following are consequences of the properties of an ordered field:*

(i) $a \leq b \implies -b \leq -a$;

(ii) $(a \leq b) \wedge (c \leq 0) \implies bc \leq ac$;

(iii) $(0 \leq a) \wedge (0 \leq b) \implies 0 \leq ab$;

(iv) $0 \leq a^2$ *for all* a;

(v) $0 < 1$;

(vi) $0 < a \implies 0 < a^{-1}$;

(vii) $0 < a < b \implies 0 < b^{-1} < a^{-1}$;

for $a, c, c \in \mathbb{R}$.

Note that $a < b$ can be represented as $(a \leq b) \wedge (a < b)$.

Definition 1.3.3. We define

$$|a| = a \text{ if } a \geq 0 \text{ and } |a| = -a \text{ if } a \leq 0$$

An useful fact: $|a| \leq b \iff -b \leq a \leq b$.

Definition 1.3.4. For numbers a and b we define $\operatorname{dist}(a, b) = |a - b|$; $\operatorname{dist}(a, b)$ represents the *distance between* a *and* b.

Theorem 1.3.5.

(i) $|a| \geq 0$ *for all* $a \in \mathbb{R}$.

(ii) $|ab| = |a| \cdot |b|$ *for all* $a, b \in \mathbb{R}$.

(iii) $|a + b| \leq |a| + |b|$ *for all* $a, b \in \mathbb{R}$.

Corollary 1.3.5.1. *dist*$(a, c) \leq$ *dist*$(a, b)+$ *dist*(b, c) *for all* $a, b, c \in \mathbb{R}$. *This is equivalent to* $|a - c| \leq |b - c| + |b - c|$.

Theorem 1.3.6 (Triangle Inequality)**.** $|a + b| \leq |a| + |b|$ *for all* a, b.

Corollary 1.3.6.1 (Reverse Triangular Inequality). *$\big||a|-|b|\big| \leq |a-b|$ for all $a, b \in \mathbb{R}$.*

Here is one of the most important techniques in real analysis.

(a) If $a \leq b + \epsilon$ for any $\epsilon > 0$, then $a \leq b$.

(b) If $a \geq b - \epsilon$ for any $\epsilon > 0$, then $a \geq b$.

(c) If $|a - b| < \epsilon$ for any $\epsilon > 0$, then $|a - b| = 0$.

Proof. The proof for two cases is similar, so I will only show (a) here. Suppose that $a > b$. Let $\epsilon = (a - b)/2 > 0$. Then $a > b + \epsilon$, so the statement that $a \leq b + \epsilon$ for any $\epsilon > 0$ is not true. ∎

1.4 The Completeness Axiom

The completeness axiom for $\mathbb{R}$ ensure us $\mathbb{R}$ has no ”gaps”.

Definition 1.4.1. Let S be a nonempty subset of $\mathbb{R}$.

(a) If S contains a largest element s_0 [that is, $s_0 \in S$ and $\forall s \in S,\ s \leq s_0$], then we call s_0 the *maximum* of S and write $s_0 = \max S$.

(b) If S contains a smallest element s_0 [that is, $s_0 \in S$ and $\forall s \in S,\ s \geq s_0$], then we call s_0 the *minimum* of S and write $s_0 = \min S$.

Open intervals like $(a, b) = \{x \in \mathbb{R} : a < x \leq b\}$ have no minimum or maximum since the endpoints a and b is not in the interval.

Definition 1.4.2. Let S be a nonempty subset of $\mathbb{R}$.

(a) If a real number M satisfies $s \leq M$ for all $s \in S$, then M is called an *upper bound* of S and the set S is said to be *bounded above*.

(b) If a real number m satisfies $m \leq s$ for all $s \in S$, then m is called an *lower bound* of S and the set S is said to be *bounded below*.

(c) The set S is said to be *bounded* if it is bounded above and bounded below. Thus S is bounded if there exist real numbers m and M such that $S \subseteq [m, M]$.

The maximum of a set is always an upper bound for the set. Likewise, the minimum of a set is always a lower bound for the set.

Definition 1.4.3. Least Upper Bound Property (LUBP)
An ordered set S has the LUBP if every nonempty subset $\mathcal{A} \subset S$ that has an upper bound has a least upper bound in S.

Note that the set $\mathbb{Q}$ of rational number does not satisfy the LUBP but $\mathbb{R}$ does. e.g. $(A) = \{q \in \mathbb{Q} : q^2 < 2\}$.

Definition 1.4.4. Let S be a nonempty subset of $\mathbb{R}$.

(a) If S is bounded above and S has a least upper bound, then we will call it the *supremum* of S and denote it by $\sup S$.

(b) If S is bounded below and S has a greatest lower bound, then we will call it the *infimum* of S and denote it by $\inf S$.

If S is bounded above, then $M = \sup S$ if and only if (i) $s \leq M$ for all $s \in S$, and (ii) whenever $M_1 < M$, there exists $s_1 \in S$ such that $s_1 > M_1$. Or for each $\epsilon > 0$, there exists

$s \in S$ such that $s > \sup S - \epsilon$.

Note that for a positive set $S = \{s : s > 0\}$, its infimum is not always positive. Example: $\{\frac{1}{n} : n \in \mathbb{N}\}$. Each element is positive but the infimum is 0.

Here are some basic facts:

- If a set S has finitely many elements, then $\max S$ exists.
- If $\max S$ exists, then $\sup S = \max S$.
- For any set $S \neq \varnothing$, $\inf S \leq \sup S$

Theorem 1.4.5 (Completeness Axiom). *Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. In other words, $\sup S$ exists and is a real number.*

Note that the completeness axiom does not hold for $\mathbb{Q}$.

Corollary 1.4.5.1. *Every nonempty subset S of $\mathbb{R}$ that is bounded below has a greatest lower bound. In other words, $\inf S$ exists and is a real number.*

Theorem 1.4.6 (Archimedean Property). *If $a > 0$ and $b > 0$, then for some positive integer n, we have $na > b$.*

Corollary 1.4.6.1. *(Set $a = 1$). For any $b > 0$, there exists $n \in \mathbb{N}$ such that $n > b$*

Corollary 1.4.6.2. *(Set $b = 1$). For any $a > 0$, there exists $n \in \mathbb{N}$ such that $na > 1 \implies \frac{1}{n} < a$.*

Lemma 1.4.7. *If $x, y \in \mathbb{R}$ such that $y - x > 1$, then there exists $m \in \mathbb{Z}$ such that $x < m < y$.*

Proof.

Case 1: $x \geq 0$. Let $S = \{n \in \mathbb{Z}_+ \, n \leq x\}$. By the corollary of Archimedean property 1.4.6 (set $a = 1$), S has finitely many elements, so $k = \max S$ exists. Then we have

$$x < k + 1 \leq x + 1 < y$$

where $k + 1$ is an integer.

Case 2: $x < 0$. Then $-x > 0$. By the corollary of Archimedean property 1.4.6 (set $a = 1$), there exists $N \in \mathbb{N}$ such that $N > -x$. Consider $x + N > 0$ and $(y + N) - (x + N) > 1$. By Case 1, there exists $m \in \mathbb{Z}$ such that $x + N < m < y + N$. Then $x < m - N < y$ where $m - N$ is an integer.

■

Theorem 1.4.8 (Denseness of $\mathbb{Q}$)**.** *If* $a, b \in \mathbb{R}$ *and* $a < b$*, then there is a rational* $r \in \mathbb{Q}$ *such that* $a < r < b$*.*

Proof. By Archimean property 1.4.6, there exists $n \in \mathbb{N}$ such that $n(b-a) > 1$. i.e. $nb - na > 1$. By 1.4.7, there exists an integer $m \in \mathbb{Z}$ between na and nb. Thus $na < m < nb \implies a < \frac{m}{n} < b$. ∎

1.5 The Symbols $+\infty$ and $-\infty$

The symbols $+\infty$ and $-\infty$ are extremely useful even though they are **not** real numbers. So for each real number a, $-\infty < a < \infty$. If a set S is not bounded above, we define $\sup S = +\infty$. Likewise, if S is not bounded below, then we define $\inf S = -\infty$.

We can extend real numbers to $\mathbb{R} \cup \{-\infty, \infty\}$. Notice that this is not a **field**, so it does not satisfy all field properties.

For emphasis, we recapitulate:

Let S be any nonempty subset of $\mathbb{R}$. The *symbols* $\sup S$ and $\inf S$ always make sense. If S is not bounded above, then $\sup S$ is a *real* number; otherwise $\sup S = +\infty$. If S is bounded below, then $\inf S$ is a *real* number; otherwise $\inf S = -\infty$. Moreover, we have $\inf S \leq \sup S$.

Chapter 2

Sequences

2.1 Limits of Sequences

A *sequence* is a function whose domain is $\{n \in \mathbf{Z} : n \geq m,\ m \text{ is usually } 1 \text{ or } 0\}$. We usually denote a sequence by s and its value at n by s_n. $(s_n)_{n=m}^{\infty} = (s_m, s_{m+1}, \dots)$. $(s_n)_{n \in N}$ represents the sequence with $m = 1$.

Example.

- $(s_n)_{n\in N}$ where $s_n = \frac{1}{n^2}$ is the sequence $(1, \frac{1}{4}, \frac{1}{9}, \dots)$
- $(a_n)_{n=0}^{\infty}$ where $a_n = (-1)^n$ is the sequence $(1, -1, 1, -1, 1, \dots)$

The "limits" of a sequence is a real number that the values s_n are close to for large values of n.

Definition 2.1.1. A sequence (s_n) of real numbers is said to **converge** to the real number s provided that

$$\forall \epsilon > 0,\ \exists N,\ n > N \Rightarrow |s_n - s| < \epsilon.$$

If (s_n) converges to s, we write $\lim_{n\to\infty} s_n = s$ or $s_n \to s$. s is the *limit* of the sequence (s_n). A sequence that does not converge (i.e. it has no *limit*) is said to *diverge*. Notice that in the definition, instead of simple ϵ, we can also use some other complicated forms with some extra constants like $M\epsilon$, $\frac{\epsilon}{c}$, $a^2\epsilon$ and so on.

Intuitively, the definition means that no matter how small you pick $\epsilon > 0$, **eventually** the sequence will stay within ϵ of s at some point (the threshold N) and forever after.

Theorem 2.1.2. *The limit of a sequence (s_n) is unique. i.e.* $(\lim s_n = s) \wedge (lims_n = t) \Rightarrow s = t$.

Proof. By the definition of limit, we have

$$n > N_1 \Rightarrow |s_n - s| < \frac{\epsilon}{2}$$
$$n > N_2 \Rightarrow |s_n - t| < \frac{\epsilon}{2}$$

For $n > \max\{N_1, N_2\}$, by Triangular Inequality **??**,

$$|s - t| = |(s - s_n) + (s_n - t)| \leq |s - s_n| + |s_n - t| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

This shows $\forall \epsilon > 0,\ |s - t| < \epsilon \Rightarrow |s - t| = 0 \Rightarrow s = t$ ∎

Theorem 2.1.3. *Let (s_n) be a convergent sequence.*

- *If $s_n \geq a$ for all but finitely many n, then $\lim s_n \geq a$.*
- *If $s_n \leq b$ for all but finitely many n, then $\lim s_n \leq b$.*

Remark. Notice that a_n and b_n need to converge so that the theorem can work.

Theorem 2.1.4 (Squeeze Lemma)**.** *If $a_n \leq s_n \leq b_n$ for all n and $\lim a_n = \lim b_n = s$, then $\lim s_n = s$.*

Remark. Notice that a_n and b_n need to converge so that the theorem can work.

2.2 A Discussion about Proofs

This section gives several examples of proofs with some discussion using the definition of the limit of a sequence.

Example. Prove $\lim \frac{1}{n^2} = 0$.

Discussion. According to the definition of the limit, we need to consider an $\epsilon > 0$ such that $|\frac{1}{n^2} - 0| < \epsilon$ for $n >$ some$N$. $|\frac{1}{n^2} - 0| < \epsilon$ implies that $\frac{1}{\epsilon} < n^2$or$\frac{1}{\sqrt{\epsilon}} < n$. Thus we can suppose $N = \frac{1}{\sqrt{\epsilon}}$ and check if we reverse our reasoning into proof, it still makes sense.

Proof. Let $\epsilon > 0$ and $N = \frac{1}{\sqrt{\epsilon}}$, then

$$\begin{aligned} n > N &\Rightarrow \epsilon > \frac{1}{n^2} \\ &\Rightarrow \frac{1}{n^2} - 0 < \epsilon - 0 \\ &\Rightarrow \left| \frac{1}{n^2} - 0 \right| < \epsilon \end{aligned}$$

This proofs $\lim \frac{1}{n^2} = 0$ according to the definition of the limit 2.1.1. ■

Example. Prove $\lim \frac{3n+1}{7n-4} = \frac{3}{7}$

Discussion. Just like the last example, we can start from the definition 2.1.1 to get a suitable N.

Proof. Let $\epsilon > 0$ and $N = \frac{19}{49\epsilon} + \frac{4}{7}$, then

$$\begin{aligned} n > N &\Rightarrow 7n > \frac{19}{7\epsilon} + 4 \\ &\Rightarrow \frac{19}{7(7n-4)} < \epsilon \\ &\Rightarrow \frac{3n+1}{7n-4} - \frac{3}{7} < \epsilon \\ &\Rightarrow \left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < \epsilon \qquad \text{since } n > 0 \end{aligned}$$

This proofs $\lim \frac{3n+1}{7n-4} = \frac{3}{7}$ according to the definition of the limit 2.1.1. ∎

Example. Prove $\lim \frac{4n^3+3n}{n^3-6} = 4$

Discussion. Since $\frac{4n^3+3n}{n^3-6} - 4 = \frac{3n+24}{n^3-6}$, when $n > 1$, we can find an upper bound for $\frac{3n+24}{n^3-6}$ so that the bound $< \epsilon \Rightarrow \left|\frac{3n+24}{n^3-6}\right| < \epsilon$. Finding an upper bound for a fraction is equivalent to finding a upper bound for its numerator and a lower bound for its denominator. We know $3n + 24 \leq 27n$ for $n > 1$. Also we note $n^3 - 6 \geq \frac{n^3}{2} \Rightarrow n > 2$. Thus we can have $\frac{3n+24}{n^3-6} < \frac{27n}{n^3/2} < \epsilon \Rightarrow n > \sqrt{\frac{54}{\epsilon}}$, provided $n > 2$.

Proof. Let $\epsilon > 0$ and $N = \max\{2, \sqrt{\frac{54}{\epsilon}}\}$, then

$$\begin{aligned} n > N &\Rightarrow (n > \sqrt{\frac{54}{\epsilon}}) \wedge (n > 2) \\ &\Rightarrow (\frac{27n}{n^3/2} < \epsilon) \wedge (\frac{n^3}{2} \leq n^3 - 6) \wedge (27n \geq 3n + 24) \\ &\Rightarrow \frac{3n+24}{n^3-6} < \frac{27n}{n^3/2} < \epsilon \\ &\Rightarrow \left|\frac{4n^3+3n}{n^3-6} - 4\right| < \epsilon \end{aligned}$$

This proofs $\lim \frac{4n^3+3n}{n^3-6} = 4$ according to the definition of the limit 2.1.1. ∎

Example. Show that $a_n = (-1)^n$ does not converge.

Discussion. Assume $\lim(-1)^n = a$, and we can see that no matter what a is, either 1 or -1 is at least 1 from a, so it means $|(-1)^n - a| < 1$ will not hold for all large n.

Proof. Suppose $\lim(-1)^n = a$ and $\epsilon = 1$. By 2.1.1, $|(-1)^n - a| < 1 \Rightarrow (|1 - a| < 1) \wedge (|-1 - a| < 1)$. Now by **??**, $2 = |1 - a + a - (-1)| \leq |1 - a| + |a - (-1)| < 1 + 1 = 2$, which is a contradiction. ∎

Example. Let (s_n) be a sequence of nonnegative real numbers and suppose $s = \lim s_n$. Note $s \geq 0$. Prove $\lim \sqrt{s_n} = \sqrt{s}$

Proof. There are two cases.

1. $s > 0$: Let $\epsilon > 0$. $\lim s_n = s \Rightarrow (\exists N,\ n > N \Rightarrow |s_n - s| < \sqrt{s}\epsilon)$. $n > N$ also implies

$$|\sqrt{s_n} - \sqrt{s}| = \frac{(\sqrt{s_n} - \sqrt{s})(\sqrt{s_n} + \sqrt{s})}{\sqrt{s_n} + \sqrt{s}} = \frac{|s_n - s|}{\sqrt{s_n} + \sqrt{s}} \leq \frac{|s_n - s|}{\sqrt{s}} < \frac{\sqrt{s}\epsilon}{\sqrt{s}} = \epsilon$$

2. $s = 0$: EXERCISE 8.3

$\blacksquare$

Example. Let (s_n) be a convergent sequence of real numbers such that $s_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim s_n = s \neq 0$. Prove $\inf\{|s_n| : n \in \mathbb{N}\} > 0$

Proof. Let $\epsilon = \frac{|s|}{2}$. Since $\lim s_n = s$,

$$n > N \Rightarrow |s_n - s| < \frac{|s|}{2} \Rightarrow |s_n| \geq \frac{|s|}{2}$$

The last implication is because otherwise

$$|s| = |s - s_n + s_n| \leq |s - s_n| + |s_n| < \frac{|s|}{2} + \frac{|s|}{2} = |s|$$

which is a contradiction. Now if we set $m = \min\{\frac{|s|}{2}, |s_1|, |s_2|, \ldots, |s_N|\}$, then clearly we have $m > 0$ since and $|s_n| \geq m$ for all $n \in \mathbb{N}$. Thus $\inf\{|s_n| : n \in \mathbb{N}\} \geq m > 0$**WHY???** $\blacksquare$

2.3 Limit Theorems for Sequences

Definition 2.3.1. A sequence (s_n) is said to be *bounded* if $\exists M,\ \forall n,$ such that $|s_n| \leq M$

Theorem 2.3.2. *Convergent sequences are bounded.*

Proof. Let (s_n) be a convergent sequence and $\lim s_n = s$, then select $\epsilon = 1$ and we have

$$n > N \Rightarrow |s_n - s| < 1$$

From the reverse triangular inequality 1.3.6.1, $|s_n| - |s| \leq |s_n - s| < 1 \Rightarrow |s_n| < |s| + 1$ when $n > N$. Thus define $M = \max\{|s| + 1, |s_1|, |s_2|, \ldots, |s_N|\}$, then $|s_n| < M$ for all n. ∎

Remark. In other words, unbounded sequences are not convergent.

Theorem 2.3.3. *If the sequence (s_n) converges to s and $k \in \mathbb{R}$, then (ks_n) converges to ks. i.e.* $\lim(ks_n) = k \cdot \lim s_n$.

Proof. Assume $k \neq 0$ and let $\frac{\epsilon}{|k|}$, then there exists N such that

$$n > N \Rightarrow |s_n - s| < \frac{\epsilon}{|k|} \Rightarrow |ks_n - ks| < \epsilon$$

∎

Theorem 2.3.4. *If (s_n) and (t_n) converge to s and t, then $(s_n + t_n)$ converges to $s + t$. That is,*

$$\lim(s_n + t_n) = \lim s_n + \lim t_n.$$

Proof. From 2.1.1, we know

$$n > N_1 \Rightarrow |s_n - s| < \frac{\epsilon}{2}$$
$$n > N_2 \Rightarrow |t_n - t| < \frac{\epsilon}{2}$$

Thus, let $N = \max\{N_1, N_2\}$,

$$n > N \Rightarrow |s_n + t_n - (s + t)| \leq |s_n - s| + |t_n - t| < \epsilon$$

∎

Theorem 2.3.5. *If (s_n) and (t_n) converge to s and t, then $(s_n t_n)$ converges to st. That is,*

$$\lim(s_n t_n) = (\lim s_n)(\lim t_n)$$

Proof. Let $\epsilon > 0$. By 2.3.2, $|s_n| \leq M$ for some $M > 0$. From 2.1.1, we have

$$n > N_1 \Rightarrow |s_n - s| < \frac{\epsilon}{2(|t|+1)}$$
$$n > N_2 \Rightarrow |t_n - t| < \frac{\epsilon}{2M}$$

Thus, let $N = \max\{N_1, N_2\}$,

$$\begin{aligned} n > N \Rightarrow |s_n t_n - st| &= |s_n t_n - s_n t + s_n t - st| \\ &\leq |s_n t_n - s_n t| + |s_n t - st| \\ &= |s_n| \cdot |t_n - t| + |t| \cdot |s_n - s| \\ &\leq M \cdot \frac{\epsilon}{2M} + |t| \cdot \frac{\epsilon}{2(|t|+1)} \\ &= \epsilon \end{aligned}$$

■

Lemma 2.3.6. *If $(s_n) \to s \neq 0$ and $s_n \neq 0$ and for all n, then* $\inf\{|s_n| : n \in \mathbb{N}\} > 0$.

Proof. Since $(s_n) \to s$, select $\epsilon = \frac{|s|}{2}$ and we have $n \geq N \implies |s_n - s| < \frac{|s|}{2}$, which implies $|s_n| > \frac{|s|}{2}$. Thus select $m = \min\{s_1, \ldots, s_N, \frac{|s|}{2}\}$, and then $|s_n| \geq m$ for all n. Since $m > 0$ and m is a lower bound of $(|s_n|)$, $\inf\{|s_n| : n \in \mathbb{N}\} \geq m > 0$. ■

Lemma 2.3.7. *If (s_n) converges to s, $s_n \neq 0$ for all n, and $s \neq 0$, then $(1/s_n)$ converges to $1/s$.*

Proof. Let $\epsilon > 0$. Since there exists $m > 0$ such that $|s_n| \geq m$ for all n. By 2.1.1, we have

$$\begin{aligned} n > N &\Rightarrow |s - s_n| < \epsilon \cdot m|s| \\ &\Rightarrow \left|\frac{1}{s_n} - \frac{1}{s}\right| = \frac{|s - s_n|}{|s_n s|} \leq \frac{|s - s_n|}{m|s|} < \epsilon. \end{aligned}$$

■

Theorem 2.3.8. *Suppose (s_n) and (t_n) converge to s and t. If $s \neq 0$ and $s_n \neq 0$ for all n, then (t_n/s_n) converges to t/s.*

Proof. By 2.3.7, $(1/s_n)$ converges to $1/s$, so

$$\lim \frac{t_n}{s_n} = \lim \frac{1}{s_n} \cdot \lim t_n = \frac{1}{s} \cdot t = \frac{t}{s}$$

by 2.3.5. ■

Theorem 2.3.9.

(a) $\lim_{n\to\infty}(\frac{1}{n^p}) = 0$ *for* $p > 0$.

(b) $\lim_{n\to\infty} a^n = 0$ *if* $|a| < 1$.

(c) $\lim(n^{1/n}) = 1$.

(d) $\lim_{n\to\infty} a^{1/n} = 1$ *for* $a > 0$.

Proof.

(a) Let $N = (\frac{1}{\epsilon})^{1/p}$ and the rest is easy.

(b) If $a = 0$ then it's obvious. Otherwise, since $|a| < 1$ we can write $|a| = \frac{1}{1+b}$ where $b > 0$. Since $(1+b)^n \geq 1 + nb > nb$,

$$|a^n - 0| = |a^n| = \frac{1}{(1+b)^n} < \frac{1}{nb}$$

. Then let $N = \frac{1}{\epsilon b}$ and finish the proof.

(c) Let $s_n = (n^{1/n}) - 1$ and note $s_n \geq 0$ for all n. By 2.3.4, we only need to show $\lim s_n = 0$. $1 + s_n = (n^{1/n}) \Rightarrow n = (1+s_n)^n$. For $n \geq 2$, the binomial expansion tells

$$n = (1+s_n)^n \geq 1 + ns_n + \frac{1}{2}n(n-1)s_n^2 > \frac{1}{2}n(n-1)s_n^2.$$

This implies $s_n < \sqrt{\frac{2}{n-1}}$ for $n \geq 2$. Now we can suppose $N = \frac{\epsilon}{\epsilon - 2}$ to finish the proof.

(d) If $a \geq 1$, then for $n \geq a$ we have $1 \leq a^{1/n} \leq n^{1/n}$. Since $\lim n^{1/n} = 1$, by Squeeze Theorem we have $\lim a^{1/n} = 1$. Now if $0 < a < 1$, then $\frac{1}{a} > 1$, so $\lim(\frac{1}{a})^{1/n} = 1$ from above. By 2.3.7, $\lim a^{1/n} = 1$.

■

Definition 2.3.10. For a (s_n), we write $\lim s_n = +\infty$ provided for each $M > 0$ there is a number N wuch that $n > N \Rightarrow s_n > M$. Similarly, we write $\lim s_n = -\infty$ provided for each $M < 0$ there is a number N wuch that $n > N \Rightarrow s_n < M$.

This implies that if $\lim s_n > -\infty$, $\exists T$, $\forall n, s_n > T$. $\lim s_n < \infty$, $\exists T$, $\forall n, s_n < T$. Be careful that we say $\lim s_n = +\infty$ as (s_n) **diverges** to ∞, **not converge** to ∞.

Example. Prove that $\lim(\sqrt{n} + 7) = +\infty$.

Proof. Let $M > 0$ and let $N = (M-7)^2$. Then $n > N \Rightarrow \sqrt{n} + 7 > M$. ■

Example. Prove $\lim \frac{n^2+3}{n+1} = +\infty$

Discussion. We want to find a simpler lower bound for $\frac{n^2+3}{n+1} = +\infty$.

Proof. Let N=2M. Then

$$\frac{n^2+3}{n+1} > \frac{n^2}{2n} = \frac{1}{2}n > M.$$

∎

Theorem 2.3.11. *Let* $\lim s_n = +\infty$ *and* $\lim t_n > 0$*. Then* $\lim s_n t_n = +\infty$*.*

Proof. Let $M > 0$ and select an m so that $0 < m < \lim t_n$. It is clear that there exists N_1 so that

$$n > N_1 \Rightarrow t_n > m$$

Since $\lim s_n = +\infty$, there exists N_2 so that

$$n > N_2 \Rightarrow s_n > \frac{M}{m}$$

Thus $n > \max\{N_1, N_2\} \Rightarrow s_n t_n > \frac{M}{m} \cdot m = M$. ∎

Theorem 2.3.12. *For a* (s_n) *of* positive *real numbers, we have* $\lim s_n = +\infty$ *if and only if* $\lim(\frac{1}{s_n}) = 0$*.*

Proof. We need to show it in both directions.

$\Rightarrow$: Let $\epsilon > 0$ and $M = \frac{1}{\epsilon}$. Since $\lim s_n = +\infty$, $n > N \Rightarrow s_n > M = \frac{1}{\epsilon}$. Therefore, $n > N \Rightarrow \left|\frac{1}{s_n} - 0\right| < \epsilon$.

$\Leftarrow$: Let $M > 0$ and $\epsilon = \frac{1}{M}$, then $n > N \Rightarrow \left|\frac{1}{s_n} - 0\right| < \epsilon = \frac{1}{M}$. Since $s_n > 0$, we have

$$n > N \Rightarrow 0 < \frac{1}{s_n} < \frac{1}{M} \Rightarrow s_n > M.$$

∎

Theorem 2.3.13. *Assume all* $s_n \neq 0$ *and that the limit* $L = \lim \left|\frac{s_{n+1}}{s_n}\right|$ *exists.*

(a) If $L < 1$*, then* $\lim s_n = 0$*.*

(b) If $L > 1$*, then* $\lim |s_n| = +\infty$*.*

Proof. See exercise 9.12 and HW2 Q8. ∎

Theorem 2.3.14. *Given two **convergent** sequences* (s_n) *and* t_n*. If there exists* $N \in \mathbb{N}$ *such that* $s_n \leq t_n$ *for all* $n \geq N_0$*, then* $\lim s_n \leq \lim t_n$*.*

Proof. See textbook exercise 9.9(c) ■

2.4 Monotone Sequences and Cauchy Sequence

Definition 2.4.1. (s_n) is called an *increasing sequence (or nondecreasing)* if $\forall n,\ s_n \leq s_{n+1}$ and $s_n \leq s_m$ whenever $n < m$. Similarly, (s_n) is called an *decreasing sequence (or nonincreasing)* if $\forall n,\ s_n \geq s_{n+1}$. An increasing or decreasing sequence is called *monotone* or *monotonic* sequence.

Theorem 2.4.2. *All bounded monotone sequences converge.*

Proof. Let (s_n) be a bounded increasing sequence, $S = \{s_n : n \in \mathbb{N}\}$. We can say $u = \sup S$ since (s_n) is bounded by 1.4.5. Since $u - \epsilon < u$, there exists N such that $s_N > u - \epsilon \Rightarrow \forall n > N,\ s_n > u - \epsilon$. Since u is the supremum, $u - \epsilon < s_n \leq u \Rightarrow |s_n - u| < \epsilon$. The proof for decreasing sequence is in exercise 10.2. ■

Remark. From the proof procedure above, we can see that bounded monotone sequences **converge to its infimum or supremum**.

Discussion of Decimals

Notice that real numbers are simply decimal expansions. For a decimal expansion like $K.d_1d_2d_3d_4\cdots$, we can define a sequence by

$$s_n = K + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}$$

where K is an nonnegative integer and each $d_j \in \{0,1,2,3,4,5,6,7,8,9\}$. When trying to $\lim s_n$, the formula of geometric series could help:

$$\lim_{n\to\infty} a(1 + r + r^2 + \cdots + r^n) = \frac{a}{1-r} \quad \text{for} \quad |r| < 1;$$

There are two important reversible facts:

1. Different decimal expansions can represent the same real number.
2. Every nonnegative real number has at least one decimal expansion

Theorem 2.4.3.

(i) If (s_n) is an unbounded increasing sequence, then $\lim s_n = +\infty$.

(ii) If (s_n) is an unbounded decreasing sequence, then $\lim s_n = -\infty$.

Proof.

(i) Let $M > 0$. Since $\{s_n : n \in \mathbb{N}\}$ is unbounded and bounded by s_1, it must be unbounded above. Thus there must be some $N \in \mathbb{N}$ so that $s_N > M$. Since s_n is increasing, we have $n > N \Rightarrow s_n \geq s_N > M$, so $\lim s_n = +\infty$.

(ii) Exercise 10.5

■

Theorem 2.4.4.

- *If* (s_n) *is a bounded and nonincreasing sequence, then* $\lim s_n = \inf\{s_n : n \in \mathbb{N}\}$.
- *If* (s_n) *is a bounded and nondecreasing sequence, then* $\lim s_n = \sup\{s_n : n \in \mathbb{N}\}$.

Corollary 2.4.4.1. *If* (s_n) *is monotone, then* $\lim s_n$ *is always meaningful. i.e.* $\lim s_n = s,\ +\infty,\ or\ -\infty$.

Suppose (s_n) is bounded. Define $u_n = \inf\{s_m : m \geq n\}$ and $v_n = \sup s_m : m \geq n$. Then observe that (u_n) is nondecreasing and (v_n) is nonincreasing since as n increases, the set has fewer elements. i.e. we have fewer choices for infimum and supremum. In general, if $A \subseteq B$, then $\inf A \geq \inf B$ and $\sup A \leq \sup B$.

Theorem 2.4.5. *Let S be a bounded nonempty subset of $\mathbb{R}$ and suppose $\sup S \notin S$. Then there is a (strictly) increasing sequence (s_n) of points in S such that $\lim s_n = \sup S$.*

Proof. See Homework 3.10 ■

Definition 2.4.6. Let (s_n) be a sequence in $\mathbb{R}$, define

- $\limsup s_n = \lim\limits_{N \to \infty} \sup\{s_n : n > N\}$
- $\liminf s_n = \lim\limits_{N \to \infty} \inf\{s_n : n > N\}$

If (s_n) is not bounded above. $\sup\{s_n : n > N\} = +\infty$ for all N and we decree $\limsup s_n = +\infty$. Likewise, if (s_n) is not bounded below. $\inf\{s_n : n > N\} = -\infty$ for all N and we decree $\liminf s_n = -\infty$.

Notice that $\limsup s_n$ need not equal to $\sup\{s_n : n > N\}$, but $\limsup s_n \leq \sup\{s_n : n > N\}$

Remark. Since v_n and u_n are monotone, $\lim v_n = \limsup s_n$ and $\lim u_n = \liminf s_n$ always exist.

Theorem 2.4.7. *Let (s_n) be a sequence in $\mathbb{R}$.*

(i) If $\lim s_n$ is defined, then $\liminf s_n = \lim s_n = \limsup s_n$.

(ii) If $\liminf s_n = \limsup s_n$, then $\lim s_n$ is defined and $\lim s_n = \liminf s_n = \limsup s_n$.

Proof. We use the notation $u_N = \inf\{s_n : n > N\}$, $v_N = \sup\{s_n : n > N\}$, $u = \lim u_N = \liminf s_n$ and $v = \lim v_N = \limsup s_n$.

(i)
- $\lim s_n = +\infty$: Let $M > 0$ then there is a N so that $n > N \Rightarrow s_n > M$. Then $u_N = \inf\{s_n : n > N\} \geq M$. This means $m > N \Rightarrow u_m \geq M \Rightarrow \lim u_N = \liminf s_n = +\infty$.
- $\lim s_n = -\infty$: It is similar to the previous proof.
- $\lim s_n = s$: Let $\epsilon > 0$ then $n > N \Rightarrow |s_n - s| < \epsilon$, so

$$v_N = \sup\{s_n : n > N\} \leq s + \epsilon.$$

  Also $m > N \Rightarrow v_m \leq s + \epsilon$ since v_n is nonincreasing, so

$$\limsup s_n = \lim v_m \leq s + \epsilon \Rightarrow \limsup s_n \leq s = \lim s_n.$$

  A similar argument shows $\lim s_n \leq \liminf s_n$. Since $\liminf s_n \leq \limsup s_n$ we get

$$\liminf s_n = \lim s_n = \limsup s_n$$

(ii)
- If $\liminf s_n = \limsup s_n = s$, then we have

$$|s - \sup s_n : n > N_0| < \epsilon$$

  which implies $\sup\{s_n : n > N_0\} < s + \epsilon \Rightarrow \forall n > N_0,\ s_n < s + \epsilon$. Similarly, we have

$$|s - \inf s_n : n > N_1| < \epsilon$$

  which implies $\inf\{s_n : n > N_1\} > s - \epsilon \Rightarrow \forall n > N_1,\ s_n > s - \epsilon$. Therefore,

$$\forall n > \max\{N_0, N_1\},\ s - \epsilon < s_n < s + \epsilon \Rightarrow |s_n - s| < \epsilon$$

- If $\liminf s_n = \limsup s_n = +\infty$, then

$$\liminf s_n = +\infty \Rightarrow \forall M > 0,\ \inf\{s_n : n > N_0\} > M \Rightarrow n > N_0,\ s_n > M.$$

- If $\liminf s_n = \limsup s_n = -\infty$, then

$$\limsup s_n = -\infty \Rightarrow \forall M < 0,\ \sup\{s_n : n > N_0\} < M \Rightarrow n > N_0,\ s_n < M.$$

■

Definition 2.4.8. A (s_n) is called a *Cauchy sequence* if

$$\forall \epsilon > 0,\ \exists N \text{ such that } m, n > N \Rightarrow |s_n - s_m| < \epsilon$$

Lemma 2.4.9. *Convergent sequences are Cauchy sequences.*

Proof. Suppose $\lim s_n = s$. Let $\epsilon > 0$ then

$$\begin{aligned} n, m > N &\Rightarrow |s_n - s| < \frac{\epsilon}{2} \text{ and } |s_m - s| < \frac{\epsilon}{2} \\ &\Rightarrow |s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s - s_m| < \epsilon. \end{aligned}$$

■

Lemma 2.4.10. *Cauchy sequences are bounded.*

Proof. By 2.4.8 and set $\epsilon = 1$ we have

$$m, n > N \Rightarrow |s_n - s_m| < 1$$

In particular $n > N \Rightarrow |s_n| - |s_{N+1}| \leq |s_n - s_{N+1}| < 1$. Let $M = \max\{|s_{N+1}| + 1, |s_1|, |s_2|, \ldots, |s_N|\}$ then $|s_n| \leq M$. ■

Theorem 2.4.11. *A sequence is a convergent sequence if and only if it is a Cauchy sequence.*

Proof. We've already proved the right direction so we only need to proved the left direction by showing $\liminf s_n = \limsup s_n$ from 2.4.7. Let $\epsilon > 0$ and since (s_n) is a Cauchy sequence, there exists N such that

$$m, n < N \Rightarrow |s_n - s_m| < \epsilon \Rightarrow \forall m > N,\ v_N = \sup\{s_n : n > N\} \leq s_m + \epsilon$$

Now $v_N - \epsilon$ becomes a lower bound for $\{s_m : m > N\}$ so $v_N - \epsilon \leq \inf\{s_m : m > N\} = u_N$. Thus

$$\limsup s_n \leq v_N \leq u_N + \epsilon \leq \liminf s_n + \epsilon$$

Since this is true for all $\epsilon > 0$, $\limsup s_n$ cannot be greater than $\liminf s_n$ (imagine ϵ is extremely small, then $\limsup s_n > \liminf s_n + \epsilon$). Thus we have $\liminf s_n \geq \limsup s_n$. $\liminf s_n \leq \limsup s_n$ is obviously true, so we have the equality. ■

2.5 Subsequences

Definition 2.5.1. Suppose $(s_n)_{n\in\mathbb{N}}$ is a sequence. A *subsequence* of this sequence is $(t_k)_{k\in\mathbb{N}}$ where for each k there is a positive integer n_k such that

$$n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$$

and

$$t_k = s_{n_k}.$$

Thus (t_k) is just a selection of some [possibly all] of the s_n's taken in order.

For the subset $\{n_1, n_2, \dots\}$ there is a natural function σ given by $\sigma(k) = n_k$ for $k \in \mathbb{N}$. The function σ "selects" an infinite subset of $\mathbb{N}$ in order. Then the subsequence of s corresponding to σ is simply the composite function $t = s \circ \sigma$. That is

$$t_k = t(k) = s \circ \sigma(k) = s(\sigma(k)) = s(n_k) = s_{n_k} \quad \text{for} \quad k \in N.$$

Notice that σ needs to be an *increasing* function.

Recall that the set $\mathbb{Q}$ of rational numbers is *countable*: there is a bijection from $\mathbb{N}$ to $\mathbb{Q}$. Therefore we have a sequence $(q_n) = (q_1, q_2, q_3, \dots)$ such that $\{q_n : n \in \mathbb{N}\} = \mathbb{Q}$. Then we have the following proposition:

Theorem 2.5.2. *Let (q_n) be an enumeration of $\mathbb{Q}$. Then for any $a \in \mathbb{R}$, there exists a subsequence (q_{n_k}) of (q_n) such that $q_{n_k} \to a$.*

Proof. First there exists $r_1 \in \mathbb{Q}$ such that $a - 1 < r_1 < a + 1$ by the denseness of $\mathbb{Q}$. Since (q_n) is an enumeration of $\mathbb{Q}$, there exists $n_1 \in \mathbb{N}$ such that $q_{n_1} = r_1$.

Given that we've already constructed $n_1, \dots, n_k$ such that $a - \frac{1}{j} < q_{n_j} < a + \frac{1}{j}$ for $j = 1, \dots, k$. Since there are infinitely many rational numbers between $a - \frac{1}{k+1}$ and $a + \frac{1}{k+1}$, and only finite many of them have been selected as $q_{n_1}, \dots, q_{n_k}$, we are able to find $n_{k+1} > n_k$ such that $a - \frac{1}{k+1} < q_{n_{k+1}} < a + \frac{1}{k+1}$.

Now we have (q_{n_k}) such that $a - \frac{1}{k} < q_{n_k} < a + \frac{1}{k}$ for each $k \in \mathbb{N}$. Thus by Squeeze Lemma, $\lim_k q_{n_k} = a$. ∎

Theorem 2.5.3. *Let (s_n) be a sequence.*

(i) If t is in $\mathbb{R}$ then there is a subsequence of (s_n) converging to t if and only if the set $\{n \in \mathbb{N} : |s_n - t| < \epsilon\}$ is infinite *for all $\epsilon > 0$.*

(ii) If (s_n) is unbounded above, it has a subsequence with limit $+\infty$.

(iii) If (s_n) is unbounded below, it has a subsequence with limit $-\infty$.

In each case, the subsequence can be taken to be monotonic.

Proof. The forward implications are easy to check. Let's check these backward implications:

(i) First suppose $\{n \in \mathbb{N} : s_n = t\}$ is infinite. Then we can simply create a subsequence $(s_{n_k})_{k\in\mathbb{N}}$ such that $s_{n_k} = t$ for all k.

Otherwise, suppose $\{n \in \mathbb{N} : s_n = t\}$ is finite. Then

$$\{n \in \mathbb{N} : 0 < |s_n - t| < \epsilon\} \quad \text{is infinite for all} \quad \epsilon > 0.$$

Since these sets are equal to

$$\{n \in \mathbb{N} : t - \epsilon < s_n < t\} \cup \{n \in \mathbb{N} : t < s_n < t + \epsilon\}$$

and these sets get smaller as $\epsilon \to 0$ we have

$$\{n \in \mathbb{N} : t - \epsilon < s_n < t\} \quad \text{is infinite for all} \quad \epsilon > 0 \tag{1}$$

or

$$\{n \in \mathbb{N} : t < s_n < t + \epsilon\} \quad \text{is infinite for all} \quad \epsilon > 0 \tag{2}$$

otherwise for sufficiently small $\epsilon > 0$ the sets in both (1) and (2) would be finite.

Assume (1) holds, now we want to construct a $(s_{n_k})_{k\in\mathbb{N}}$ satisfying

$$t - 1 < s_{n_1} < t \quad \text{and} \quad \max\{s_{n_{k-1}}, t - \frac{1}{k}\} \le s_{n_k} < t \quad \text{for} \quad k \ge 2 \tag{3}$$

Assume $n_1, \ldots, n_{k-1}$ have been selected satisfying (3) and show how to select n_k. This is called "inductive definition" or "definition by induction". A subsequence satisfying (3) is a monotone increasing sequence and by Squeeze Formula $\lim_k s_{n_k} = t$. Here is the construction: By (1) we can select n_1 such that $t - 1 < s_{n_1} < t$. Suppose we've selected $n_1, \ldots, n_{k-1}$ so that $n_1 < n_2, \cdots < n_{k-1}$ and

$$\max\{s_{n_{j-1}}, t - \frac{1}{j}\} \le s_{n_j} < t \quad \text{for} \quad j = 2, \ldots, k-1 \tag{4}$$

By using (1) with $\epsilon = max\{s_{n_{k-1}}, t - \frac{1}{k}\}$, we can select $n_k > n_{k-1}$ satisfying (4) for $j = k$, so (3) also holds for k.

(ii) Given $n_1 = 1$ and $n_1 < \cdots < n_{k-1}$, select $n_k > n_{k-1}$ so that $s_{n_k} > \max\{s_{n_{k-1}}, k\}$. This is possible since (s_n) is unbounded above. Then the subsequence will be monotonically unbounded above thereby have limit $+\infty$.

■

Theorem 2.5.4. *If (s_n) converges, then every subsequence converges to the same limit.*

Proof. Let (s_{n_k}) denote a subsequence of (s_n). Note that $n_k \ge k$ for all k. Let $s = \lim s_n$ and $\epsilon > 0$. There exists N so that $n > N \Rightarrow |s_n - s| < \epsilon$. Since $n_k \ge k > N$, $|s_{n_k} - s| < \epsilon$. Thus

$$\lim_{k\to\infty} s_{n_k} = s.$$

■

In the other way, if there are two subsequences of (s_n) with different limits, (s_n) does not converge.

Theorem 2.5.5. *Every sequence* (s_n) *has a monotonic subsequence.*

Proof. Define n-th term is *dominant* if it is greater than every term which follows it

$$s_m < s_n \quad \text{for all} \quad m > n \tag{1}$$

- Case 1: Suppose there are infinitely many dominant terms, then we can easily construct a monotone decreasing subsequence.
- Case 2: Suppose there are only finitely many dominant terms. Select n_1 so that s_{n_1} is beyond all the dominant terms of the sequence. Then

  $$\text{given } N \geq n_1 \text{ there exists } m > N \text{ such that} \quad s_m \geq s_N. \tag{2}$$

  Suppose $n_1, \ldots, n_{k-1}$ have been selected so that

  $$n_1 < n_2 < \cdots < n_{k-1} \tag{3}$$

  and

  $$s_{n_1} \leq \cdots \leq s_{n_{l-1}} \tag{4}$$

  Apply (2) with $N = n_{k-1}$ we can select $n_k > n_{k-1}$ such that $s_{n_k} \geq s_{n_{k-1}}$. Then the procedure continues by induction and we obtain an increasing subsequence.

■

Theorem 2.5.6 (Bolzano-Weierstrass Theorem)**.** *Every bounded sequence has a convergent subsequence.*

Proof. If (s_n) is a bounded sequence, it has a bounded monotonic subsequence which converges. ■

Definition 2.5.7. Let (s_n) be a sequence in $\mathbb{R}$. A *subsequential limit* is any real number or symbol $+\infty$ or $-\infty$ that is the limit of some subsequence of (s_n).

Theorem 2.5.8. *Let* (s_n) *be any sequence. There exists a monotonic subsequence whose limit is* $\limsup s_n$*, and there exists a monotonic subsequence whose limit is* $\liminf s_n$*.*

Proof. If (s_n) is not bounded above, then by 2.5.3(ii) there is a monotonic subsequence with limit $+\infty = \limsup s_n$. The proof for not bounded below is similar.

Now if (s_n) is bounded above, then let $t = \limsup s_n$, and consider $\epsilon > 0$. There exists N_0 so that

$$\sup\{s_n : n > N\} < t + \epsilon \quad \text{for} \quad N \geq N_0.$$

In particular, $s_n < t + \epsilon$ for all $n > N_0$. We now claim

$$\{n \in \mathbb{N} : t - \epsilon < s_n < t + \epsilon\} \quad \text{is infinite.} \tag{1}$$

Otherwise, there exists $N_1 > N_0$ so that $s_n \leq t - \epsilon$ for $n > N_1$(WHY???). Then $\sup\{s_n : n > N\} \leq t - \epsilon$ for $N \geq N_1$, so that $\limsup s_n < t$, a contradiction. Since (1) holds true for all $\epsilon > 0$, 2.5.3(i) shows that there is a monotonic subsequence verges to $t = \limsup s_n$. ■

Theorem 2.5.9. *Let (s_n) be any sequence in $\mathbb{R}$, and let S denote the set of subsequential limits of (s_n).*

(i) S is nonempty.

(ii) $\sup S = \limsup s_n$ and $\inf S = \liminf s_n$.

(iii) $\lim s_n$ exists if and only if S has exactly one element, namely $\lim s_n$.

(iv) $\limsup s_n \in S$ and $\liminf s_n \in S$.

Proof.

(i) By the last theorem.

(ii) Consider any limit t os a subsequence (s_{n_k}) of (s_n). By 2.4.7 $t = \liminf_k s_{n_k} = \limsup_k s_{n_k}$. Since $n_k > k$ for all k, we have $\{s_{n_k} : k > N\} \subseteq \{s_n : n > N\}$ for each $N \in \mathbb{N}$. Therefore

$$\liminf_n s_n \leq \liminf_k s_{n_k} = t = \limsup_k s_{n_k} \leq \limsup_n s_n$$

The inequality below holds true for all t in S, so

$$\liminf s_n \leq \inf S \leq \sup S \leq \limsup s_n$$

By the last theorem we know both $\liminf s_n$ and $\limsup s_n$ is in S, so (ii) holds.

(iii) This is simply a reformulation of 2.4.7.

(iv) This is from 2.5.8

■

Theorem 2.5.10. *Let S denote the set of subsequential limits of a sequence (s_n). Suppose $(t_n)$is a sequence in $S \cap \mathbb{R}$ and that $t = \lim t_n$. Then t belongs to S.*

Proof. Suppose t is finite, then some t_N is in $(t - \epsilon, t + \epsilon)$. Let $\delta = \min\{t + \epsilon - t_N, t_N - t + \epsilon\}$, so that

$$(t_N - \delta, t_N + \delta) \subseteq (t - \epsilon, t + \epsilon)$$

Since t_N is a subsequential limit, the set $\{n \in \mathbb{N} : s_n \in (t_N - \delta, t_N + \delta)\}$ is infinite, so the set $\{n \in \mathbb{N} : s_n \in (t - \epsilon, t + \epsilon)\}$ is also infinite. Thus by 2.4.7 t itself is a subsequential limit of (s_n).

If $t = +\infty$, then clearly the sequence (s_n) is unbounded above, so a subsequence of (s_n) has limit $+\infty$ by 2.4.7. Thus $+\infty$ is also in S. A similar argument applies if $t = -\infty$. ■

2.6 lim sup's and lim inf's

Theorem 2.6.1. *If (s_n) converges to a positive real number s and (t_n) is any sequence, then*

$$\limsup s_n t_n = s \cdot \limsup t_n.$$

Here we allow the conventions $s \cdot (+\infty) = +\infty$ and $s \cdot (-\infty) = -\infty$ for $s > 0$.

Proof. We first want to show

$$\limsup s_n t_n \geq s \cdot \limsup t_n. \tag{1}$$

We have three cases. Let $\beta = \limsup t_n$.

1. Suppose β is finite. By **??**, there exists a subsequence (t_{n_k}) of (t_n) such that $\lim_{k\to\infty} t_{n_k} = \beta$. We also have $\lim_{k\to\infty} s_{n_k} = s$ by 2.5.4, so $\lim_{k\to\infty} s_{n_k} t_{n_k} = s\beta$ thus $(s_{n_k} t_{n_k})$ is a subsequence of $(s_n t_n)$ converging to $s\beta$, and therefore $s\beta \leq \limsup s_n t_n$ by 2.5.9. Thus (1) holds.

2. Suppose $\beta = +\infty$. Then there exists a subsequence (t_{n_k}) of (t_n) converging to $+\infty$. Since $\lim_{k\to\infty} s_{n_k} = s > 0$, $\lim_{k\to\infty} s_{n_k} t_{n_k} = +\infty$. Hence $\limsup s_n t_n = +\infty$. Thus (1) holds.

3. Suppose $\beta = -\infty$. Then the right-hand side of (1) is equal to $-\infty$. Hence (1) is obviously true.

To show $\limsup s_n t_n \leq s \cdot \limsup t_n$, we may ignore the first few terms of (s_n) and assume all $s_n \neq 0$. Then we can write $\lim \frac{1}{s_n} = \frac{1}{s}$. Now we apply (1) with s_n replaced by $\frac{1}{s_n}$ and t_n replaced by $s_n t_n$:

$$\limsup t_n = \limsup(\frac{1}{s_n})(s_n t_n) \geq (\frac{1}{s}) \limsup s_n t_n,$$

which is

$$\limsup s_n t_n \leq s \cdot \limsup t_n$$

Therefore we have $\limsup s_n t_n = s \cdot \limsup t_n$. ■

Theorem 2.6.2. *Let (s_n) be any sequence of nonzero real numbers. Then we have*

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| \leq \liminf |s_n|^{1/n} \leq \limsup |s_n|^{1/n} \leq \limsup \left|\frac{s_{n+1}}{s_n}\right|$$

Proof. The middle inequality is obvious. The first and third inequalities have similar proofs. We will prove the third inequality as below:

Let $\alpha = \limsup |s_n|^{1/n}$ and $L = \limsup \left|\frac{s_{n+1}}{s_n}\right|$. Assume $L < +\infty$. To prove $\alpha \leq L$ it suffices to show

$$a \leq L_1 \quad \text{for any} \quad L_1 > L \tag{1}$$

because if $\exists L_1 > L,\ \alpha > L_1$, then $\alpha > L_1 > L \Rightarrow \alpha > L$ Since

$$L = \limsup \left|\frac{s_{n+1}}{s_n}\right| = \lim_{N\to\infty} \sup\left\{\left|\frac{s_{n+1}}{s_n}\right| : n > N\right\} < L_1$$

there exists a positive integer N such that

$$\sup\left\{\left|\frac{s_{n+1}}{s_n}\right| : n \geq N\right\} < N_1$$

Thus

$$\left|\frac{s_{n+1}}{s_n}\right| < L_1 \quad \text{for} \quad n \geq N \tag{2}$$

Now for $n > N$ we can write

$$|s_n| = \left|\frac{s_n}{s_{n-1}}\right| \cdot \frac{s_{n-1}}{s_{n-2}} \cdots \left|\frac{s_{N+1}}{s_N}\right| \cdot |s_N|.$$

Apply (2) we see that

$$\begin{aligned} |s_n| &< L_1^{n-N}|s_N| \quad \text{for} \quad n > N \\ |s_n| &< L_1^n a \quad \text{for} \quad n > N. \qquad \text{for } a = L_1^{-N}|s_N| \\ |s_n|^{1/n} &< L_1 a^{1/n} \quad \text{for} \quad n > N \end{aligned}$$

Since $\lim_{n\to\infty} a^{1/n} = 1$ we conclude $\alpha = \limsup |s_n|^{1/n} \leq L_1$ ■

Corollary 2.6.2.1. *If* $\lim \left|\frac{s_{n+1}}{s_n}\right|$ *exists [and equals L], then* $\lim |s_n|^{1/n}$ *exists [and equals L].*

Proof. If $\lim \left|\frac{s_{n+1}}{s_n}\right| = L$, then all four values in the last theorem are equal to L. Hence $\lim |s_n|^{1/n} = L$ by 2.4.7. ■

2.7 Some Topological Concepts in Metric Spaces

Definition 2.7.1. Let X be a set, and suppose d is a function $d : X \times X \to [0, \infty]$ defined for all pairs (x, y) of elements from X satisfying

1. $d(x, x) = 0$ for all $x \in S$ and $d(x, y) > 0$ for distinct $x, y \in X$. (Positive Definiteness)
2. $d(x, y) = d(y, x)$ for all $x, y \in X$. (Symmetry)
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$. (Triangle Inequality)

Such a function d is called a *distance function* or a *metric* on X. A *metric space* X is a set X together with a metric on it.

Remark. The positive definiteness can be also expressed as $\forall x, y \in X$ $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$. The distance function cannot be $+\infty$.

Example. Discrete metric space is defined as

$$\text{For any set } X \text{ with metric or distance function as } \begin{cases} 1 & x \neq y \\ 0 & x = y. \end{cases}$$

Notice that all sets in discrete metric space are both open and closed.

Definition 2.7.2 (Convergence)**.** A sequence (x_n) in a metric space (X, d) converges to x in X if $\lim_{n\to\infty} d(s_n, s) = 0$.

Remark. In other words, a sequence (x_n) converges to x if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N \implies d(x_n, x) < \epsilon$.

Definition 2.7.3 (Cauchy)**.** A sequence (x_n) in X is a *Cauchy* if for any $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$m, n \geq \implies d(x_m, x_n) < \epsilon.$$

Definition 2.7.4 (Complete)**.** A metric space (X, d) is *complete* if every Cauchy sequence in X converges.

Remark. Every convergent sequence (x_n) in X is Cauchy.

Definition 2.7.5 (Open Ball)**.** Let (X, d) be a metrc space. For $x \in X$ and $r > 0$, the open ball of radius r centered at x is the set

$$B_r(x) = \{y \in X : d(y, x) < r\}$$

Definition 2.7.6 (Interior Point)**.** Let (X, d) be a metric space. Let E be a subset of X. An element $x \in E$ is *interior* to E if for some $r > 0$ we have

$$B_r(x) \subseteq E$$

We write E° for the set of points in E that are interior to E.

Remark.

- The relationship between E and X may affect whether a point in E is interior to E. For example, for $E = [0, 1] \subset [-1, 2] = X$, 0 is not interior to $[0, 1]$. However if $E = [0, 1] \subset [0, 1] = X$, then 0 is interior to 0 since there is not point in X beyond the left of 0.
- E° is open.
- $E = E^\circ$ if and only if E is open.
- If F is an open set such that $F \subseteq E$, then $F \subseteq E^\circ$.

Definition 2.7.7 (Open Set)**.** A set $E \subseteq X$ is *open* if every point $x \in E$ is an interior point of E. i.e., if $E = E^\circ$.

Remark.

- A set being open does **not** mean it is **not** closed. e.g. $[0, 1)$ is neither open nor closed.

Example.

- $(a, b), (a, \infty), (-\infty, a)$ are open sets.
- In $\mathbb{R}$, $\mathbb{Q}$ is *not* open since $B_r(q)$ may contain irrational numbers in $\mathbb{R}$ so $B_r(q) \nsubseteq \mathbb{Q}$.
- In any metric space (X, d), X and $\mathbb{Q}$ are open trivially.

Theorem 2.7.8 (Open ball is open)**.** *Let (X, d) be a metric space. Given $x \in X$ and $r > 0$, $B_r(x)$ is an open set in X.*

Proof. Consider arbitrary $y \in B_r(x)$ and let $s = r - d(x, y)$. It is easy to show that $B_x(y) \subseteq B_r(x)$. Thus y is an interior point of $B_r(x)$. Since y is arbitrary, by the definition $B_r(x)$ is open. ∎

Theorem 2.7.9 (Union and intersection of open sets)**.** *Let (X, d) be a metric space.*

(i) If $\{\mathcal{U}_\alpha\}_{\alpha \in \mathcal{A}}$ is any collection of open sets in X, then $\bigcup_{\alpha \in \mathcal{A}} \mathcal{U}_\alpha$ is open. i.e. the union of any *collection of open sets is open.*

(ii) If $\{\mathcal{U}_1, \ldots, \mathcal{U}_n\}$ is a finite collection of open sets in X, then $\bigcap_{i=1}^n \mathcal{U}_i$ is open.

Proof.

(i) Consider $x \in \bigcup_{\alpha \in \mathcal{A}} \mathcal{U}_\alpha$, then $\exists \beta \in \mathcal{A}$ such that $x \in \mathcal{U}_\beta$. Since $\mathcal{U}_\beta$ is open, $\exists r > 0$ such that $B_r(x) \subseteq \mathcal{U}_\beta \subseteq \bigcup_{\alpha \in \mathcal{A}} \mathcal{U}_\alpha$. Thus x is interior to $\bigcup_{\alpha \in \mathcal{A}} \mathcal{U}_\alpha$, completing the proof.

(ii) Consider $x \in \bigcap_{i=1}^n \mathcal{U}_i$. Since $x \in \mathcal{U}_i$ for $i = 1, \ldots, n$, $\exists r_i > 0$ such that $B_{r_i}(x) \subseteq \mathcal{U}_i$. Take $r = \min\{r_1, \ldots, r_n\}$, then clearly $B_r(x) \subseteq \bigcap_{i=1}^n \mathcal{U}_i$.

■

Remark. The examples for infinite collection in (ii) is $\bigcap_{n=1}^\infty (1 - \frac{1}{n}, 1 + \frac{1}{n}) = \{1\}$. Since 1 is not an interior point of $\{1\}$, $\{1\}$ is not open.

Definition 2.7.10 (Complement)**.** For a set $E \subseteq X$, the *complement* of E is the set $E^C = X \backslash E = \{x \in X : x \notin E\}$.

Definition 2.7.11 (Limit Point)**.** For a set $E \subseteq X$, a point $x \in X$ is a *limit point* of E if for any $r > 0$, we have that $(B_r(x) \backslash \{x\}) \cap E \neq \varnothing$.
E' denotes the set of all limit points of E.

Remark.

- In other words, for any radius $r > 0$, no matter how small is r, there is some element of E which sits in $B_r(x)$ other than x itself.
- If $E \subseteq F$, then $E' \subseteq F'$.
- $(E \cup F)' = E' \cup F'$.

Example.

- In $\mathbb{R}$, the set of limit points of $(0, 1)$ is $[0, 1]$.
- In $\mathbb{R}$, the only limit point of $\{\frac{1}{n} : n \in \mathbb{N}\}$ is 0.
- In $\mathbb{R}$, the set of limit point of $\mathbb{Q}$ is $\mathbb{R}$.

Theorem 2.7.12. *A point x is a limit point of a set $E \subseteq X$ if and only if $x = \lim x_n$ for some sequence x_n of points in $E \backslash \{x\}$.*

Proof. See homework 3.7. ∎

Definition 2.7.13 (Isolated Point)**.** For a set $E \subseteq X$, $x \in E$ is called an *isolated point* if x is not a limit point of E

Remark. In other words, x is an isolated point or not a limit point of E if there exists a radius r such that $B_r(x)$ does not contain any element of E except x itself.

Example.

- In $\mathbb{R}$, every integer is an isolated point of $\mathbb{Z}$.
- In $\mathbb{R}$, the set Q has no isolated point.
- In $\mathbb{R}$, every element of $\{\frac{1}{n} : n \in \mathbb{N}\}$ is an isolated point.

Definition 2.7.14 (Closed Set)**.** A set is *closed* if $E' \subseteq E$.

Definition 2.7.15 (Closed Set)**.** Let (X, d) be a metric space. A subset E of X is *closed* if its complement $E^\complement$ is an open set.

Remark.

- The above two definitions are equivalent.
- In other words, E contains all of its limit points, or every limit point of E is in E.
- In any metric space (X, d), X and $\varnothing$ are closed.
- A set being closed does **not** mean it is **not** open. e.g. $[0, 1)$ is neither open nor closed.

Example.
- In $\mathbb{R}$, $[0, 1]$ is closed. $[a, \infty), (-\infty, a]$ are closed.
- In $\mathbb{R}$, the set $\{\frac{1}{n} : n \in \mathbb{N}\}$ is not closed, but $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is closed.
- In any metric space, X and $\varnothing$ are closed.
- All finite sets do not have limit point, so they are trivially closed.

Theorem 2.7.16. *A set $E \subseteq \mathbb{R}$ is closed if and only if every Cauchy sequence contained in E has a limit that is also an element of E.*

Theorem 2.7.17 (The set of limit points is closed)**.** *Let (X, d) be a metric space. Let $E \subseteq X$, then E', (the set of limit points of E), is closed.*

Proof. We need to show for any limit point x of E', x is in E'. Since x is a limit point of E', $\forall r > 0$, $(B_r(x) \backslash \{x\}) \cap E' \neq \varnothing$. i.e. there exists $y \in E'$ such that $y \neq x$ and $y \in B_r(x)$. Take $s = \min\{r - d(x, y), d(x, y)\}$. Since $y \in E'$, $(B_s(y) \backslash \{y\}) \cap E \neq \varnothing$. i.e. $\exists z \in (B_s(y) \backslash \{y\}) \cap E \neq \varnothing$.

Now since $s < r - d(x, y)$, $d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + (r - d(x, y)) = r \implies z \in B_r(x)$. Also since $s < d(x, y)$, $z \neq x$. Thus $z \in (B_r(x) \backslash \{x\}) \cap E \implies (B_r(x) \backslash \{x\}) \cap E \neq \varnothing$, which implies x is a limit point of E. i.e. $x \in E'$, completing the proof. ■

Theorem 2.7.18 (Union and intersection of closed sets)**.**

(i) If $\{\mathcal{E}_\alpha\}_{\alpha \in \mathcal{A}}$ is any collection of closed set, then $\bigcap_{\alpha \in \mathcal{A}} \mathcal{E}_\alpha$ is closed.

(ii) If $\{\mathcal{E}_1, \ldots, \mathcal{E}_n\}$ is a finite collection of closed sets in X, then $\bigcup_{i=1}^{n} \mathcal{E}_i$ is closed.

Proof.

(i) Observe that $\left(\bigcap_{\alpha \in \mathcal{A}} \mathcal{E}_\alpha\right)^{\mathsf{C}} = \bigcup_{\alpha \in \mathcal{A}} \mathcal{E}_\alpha^{\mathsf{C}}$. Since $\mathcal{E}_\alpha$ is closed, $\mathcal{E}_\alpha^{\mathsf{C}}$ is open. By 2.7.9, the union of open sets $\bigcup_{\alpha \in \mathcal{A}} \mathcal{E}_\alpha^{\mathsf{C}}$ is open, completing the proof.

(ii) Observe that $(\bigcup_{i=1}^{n} \mathcal{E}_i)^{\mathsf{C}} = \bigcap_{i=1}^{n} \mathcal{E}_i^{\mathsf{C}}$. Since $\mathcal{E}_i$ is closed, $\mathcal{E}_i^{\mathsf{C}}$ is open. By 2.7.9, the intersection of finite open sets $\bigcap_{i=1}^{n} \mathcal{E}_i^{\mathsf{C}}$ is open, completing the proof.

■

Remark. $\bigcup_{x \in (0,1)} \{x\} = (0, 1)$ is an example to the union of infinite closed sets is open in (ii).

The proof above uses one of DeMorgan's Laws for sets.

DeMorgan's Laws for sets

Suppose a metric space (X, d) and let $\forall\alpha \in \mathcal{A}\ U_\alpha \in X$. Then $\bigcap_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha^\complement = \left(\bigcup_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha\right)^\complement$.

Proof. We want to show both directions.

$\subseteq$: Consider $u \in \bigcap_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha^\complement$, then we have

$$\begin{aligned} \forall\alpha \in \mathcal{A}\ u \in \mathcal{U}_\alpha^\complement &\implies \forall\alpha \in \mathcal{A}\ u \notin \mathcal{U}_\alpha && (1)\\ &\implies u \notin \bigcup_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha && (2)\\ &\implies u \in \left(\bigcup_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha\right)^\complement. && (3) \end{aligned}$$

(1) $\implies$ (2) because

$$(\neg (u \in \mathcal{U}_1)) \wedge (\neg (u \in \mathcal{U}_2)) \wedge \cdots = \neg ((u \in \mathcal{U}_1 \vee (u \in \mathcal{U}_2) \vee \cdots) = \neg \left(u \in \bigcup \mathcal{U}_i\right)$$

Thus $\bigcap_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha^\complement \subseteq \left(\bigcup_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha\right)^\complement$.

$\supseteq$: Consider $u \in \left(\bigcup_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha\right)^\complement$, then we have

$$\begin{aligned} u \notin \bigcup_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha &\implies \forall\alpha \in \mathcal{A}\ u \notin \mathcal{U}_\alpha\\ &\implies \forall\alpha \in \mathcal{A}\ u \in \mathcal{U}_\alpha^\complement\\ &\implies u \in \bigcap_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha^\complement \end{aligned}$$

Thus $\bigcap_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha^\complement \supseteq \left(\bigcup_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha\right)^\complement$, and hence $\bigcap_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha^\complement = \left(\bigcup_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha\right)^\complement$.

■

Definition 2.7.19 (Bounded Set)**.** A set $E \subseteq X$ is bounded if for some $x \in X$ and $M > 0$ such that $d(x, y) \leq M$ for all $y \in E$.

Remark.

- In $\mathbb{R}^k$, $X \subseteq \mathbb{R}^k$ is bounded if there exists $M > 0$ such that $\forall \mathbf{x} \in X\ d(\mathbf{x}, \mathbf{0}) = \sqrt{x_1^2 + \cdots + x_k^2} \leq M$.
- Finite union of bounded sets is bounded.

- Intersection of bounded sets is bounded.

- Contained in some open ball.

Theorem 2.7.20. *In R, any closed and bounded sets always have maximum and minimum.*

Definition 2.7.21 (Closure)**.** The *closure* of E in X is $\bar{E} = E \cup E'$.

Remark.

- $\bar{E}$ is the intersection of all closed sets containing E.

- $\bar{E}$ is closed.

- $E = \bar{E}$ if and only if E is closed.

- If F is a closed set such that $E \subseteq F$, then $\bar{E} \subseteq F$.

- The union of closures of finite sets is equal to the closure of unions of the sets. i.e. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Theorem 2.7.22. *For any $E \subseteq X$, its closure $\bar{E} = E \cup E'$ is closed and is the smallest closed set containing A.*

Definition 2.7.23 (Dense Set)**.** A set $E \subseteq X$ is *dense* in X if $\bar{E} = X$.

Example.

- $\mathbb{Q}$ is dense in $\mathbb{R}$.
- In any metric space (X, d), X is dense in X.

Definition 2.7.24 (Dense Set)**.** A set $E \subseteq X$ is dense in X if and only if for any $x \in X$ and $r > 0$.

$$B_r(x) \cap E \neq \varnothing.$$

Lemma 2.7.25.

- *A sequence* $(\mathbf{x}^{(n)})$ *in* $\mathbb{R}^k$ *converges to* $\mathbf{x} = (x_1, \ldots, x_k)$ *if and only if for each* $j = 1, 2 \ldots, k$*, the sequence* $(x_j^{(n)})$ *converges in* $\mathbb{R}$*.*
- *A sequence* $(\mathbf{x}^{(n)})$ *in* $\mathbb{R}^k$ *is a Cauchy sequence if and only if each sequence* $(x_j^{(n)})$ *is a Cauchy sequence in* $\mathbb{R}$*.*

Proof. First observe for $\mathbf{x},\mathbf{y} \in \mathbb{R}^k$ and $j = 1, \ldots, k$

$$|x_j - y_j| = \sqrt{(x_j - y_j)^2} \leq \sqrt{(x_1 - y_1)^2 + \cdots + (x_k - y_k)^2} = d(\mathbf{x},\mathbf{y}) \leq \sqrt{k}\max\{|x_j - y_j| : j = 1, \ldots, k\} \tag{1}$$

First assertion:

$\Longrightarrow$: Given that $(\mathbf{x}^{(n)})$ converges to $\mathbf{x}$. For each *epsilon* > 0 there exists $N \in \mathbb{N}$ such that $n \geq N \implies d(\mathbf{x}^{(n)}, \mathbf{x}) < \epsilon$. Then by (1) for $j = 1, \ldots, k$

$$n \geq N \implies |x_j^{(n)} - x_j| \leq d(\mathbf{x}^{(n)}, \mathbf{x}) < \epsilon,$$

so $x_j^{(n)} \to x_j$.

$\Longleftarrow$: For $j = 1, \ldots, k$, $\forall \epsilon > 0$, there exists $N_j \in \mathbb{N}$ such that

$$n \geq N_j \implies |x_j^{(n)} - x_j| < \frac{\epsilon}{\sqrt{k}}.$$

Take $N = \max\{N_1, \ldots, N_k\}$, then by (1) we have

$$n \geq N \implies d(\mathbf{x}^{(n)}, \mathbf{x}) \leq \sqrt{k}\max\{|x_j - y_j| : j = 1, \ldots, k\} < \sqrt{k} \cdot \frac{\epsilon}{\sqrt{k}} = \epsilon.$$

Thus $(\mathbf{x}^{(n)}) \to \mathbf{x}$

Second assertion:

$\Rightarrow$: Suppose $(\mathbf{x}^{(n)})$ is a Cauchy sequence, from the definition we know

$$m, n > N \Rightarrow d(\mathbf{x}^{(m)}, \mathbf{x}^{(n)}) < \epsilon$$

From (1) we see

$$m, n > N \Rightarrow |x_j^{(m)} - x_j^{(n)}| < \epsilon$$

so $(x_j^{(n)})$ is a Cauchy sequence.

$\Leftarrow$: Suppose $(x_j^{(n)})$ is a Cauchy sequence, then for $j = 1, \dots, k$

$$m, n > N_j \Rightarrow |x_j^{(m)} - x_j^{(n)}| < \frac{\epsilon}{\sqrt{k}}.$$

If $N = \max\{N_1, N_2, \dots, N_k\}$, then by (1)

$$m, n > N \Rightarrow d(\mathbf{x}^{(m)}, \mathbf{y}^{(n)}) < \epsilon$$

i.e. $(\mathbf{x}^{(n)})$ is a Cauchy sequence.

■

Theorem 2.7.26. *Euclidean k-space $\mathbb{R}^k$ is complete.*

Proof. Consider a Cauchy sequence $(\mathbf{x}^{(n)})$ in $\mathbb{R}^k$. By 2.7.25, each $(x_j^{(n)})$ is a Cauchy sequence. By 2.4.11 each $(x_j^{(n)})$ converges. Thus by 2.7.25 $(\mathbf{x}^{(n)})$ converges. ■

Theorem 2.7.27 (Bolzano-Weierstrass in $\mathbb{R}^k$)**.** *Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.*

Proof. Since $(\mathbf{x}^{(n)})$ is bounded, then each $(x_j^{(n)})$ is bounded in $\mathbb{R}$. By 2.5.6, we could replace $(\mathbf{x}^{(n)})$ by one of its subsequence, say $(\bar{\mathbf{x}}^{(n)})$, whose $(x_1^{(n)})$ converges. By 2.5.6 again, we may replace $(\mathbf{x}^{(n)})$ by a subsequence of $(\mathbf{x}^{(n)})$ such that both $(x_1^{(n)})$ and $(x_2^{(n)})$ converge. $(x_1^{(n)})$ still converges because 2.5.4. Repeating this argument by k times, we obtain a new sequence $(\mathbf{x}^{(n)})$ where each $(x_j^{(n)})$ converges, $j = 1, \dots, k$, which is a subsequence of the original sequence, and it converges by 2.7.25. ■

Remark. In any general metric space (X, d), it is not true that any bounded sequence has a convergent subsequence. E.g. $(\mathbb{Q}, d)$ and infinite discrete metric space

Theorem 2.7.28. *Let E be a subset of a metric space (S, d).*

1. *E is closed $\iff$ $E = E^-$.*

2. *E is closed $\iff$ E contains the limit of every convergent sequence of points in E.*

3. *An element is in E^- $\iff$ it is the limit of some sequence of points in E.*

4. *A point is in the boundary of E $\iff$ it belongs to the closure of both E and its complement.*

Compactness

Definition 2.7.29 (Open Cover)**.** Let (X, d) be a metric space and $E \subseteq X$. An open cover of E is a collection of open sets $\{\mathcal{U}_\alpha\}_{\alpha \in \mathcal{A}}$ such that $E \subseteq \bigcup_{\alpha \in \mathcal{A}} \mathcal{U}_\alpha$. An open cover is finite if it contains finitely many sets.

Definition 2.7.30 (Subcover)**.** A subcover of an open cover $\{\mathcal{U}_\alpha\}_{\alpha \in \mathcal{A}}$ of E is an *open cover* $\{\mathcal{U}_\alpha\}_{\alpha \in \mathcal{B}}$ such that $\mathcal{B} \subseteq \mathcal{A}$.

Definition 2.7.31 (Compact Set)**.** A set $E \subseteq X$ is compact if every open cover of E has a *finite* subcover.

Example.

- Every finite set is compact.
- Infinite discrete metric space is not compact.
- $\mathbb{R}$ is not compact: $\{(-n, n)\}_{n \in \mathbb{N}}$ is an open cover of $\mathbb{R}$ but does not have a finite subcover.
- $(0, 1)$ is not compact: $\{(0, r)\}_{r \in (0,1)}$ is an open cover of $(0, 1)$ but does not have a finite subcover.
- Closed interval in R is compact.

Theorem 2.7.32. *Compact sets are closed in any metric space.*

Proof. Let $E \subseteq X$ be compact. To show E is closed, we can show E^{C} is open. Consider $x \in E^{\mathsf{C}}$. For each $y \in E$, let $r_y := \frac{1}{2} d(x, y)$. Clearly $\{B_{r_y}(y)\}_{y \in E}$ is an open cover of E because each point in E is a center of an open ball. By the assumption, E is compact, so there is a finite subcover $\{B_{r_{y_1}}(y_1), \ldots, B_{r_{y_n}}(y_n)\}$ such that $E \subseteq \bigcup_{i=1}^n B_{r_{y_i}}(y_i)$.

Now take $r = \min\{r_{y_1}, \ldots, r_{y_n}\}$, and hence $B_r(x) \cap (\bigcup_{i=1}^n B_{r_{y_i}}(y_i)) = \varnothing$. Since $E \subseteq \bigcup_{i=1}^n B_{r_{y_i}}(y_i)$, $B_r(x) \cap E = \varnothing \implies B_r(x) \subseteq E^{\mathsf{C}}$. Thus x is an interior point of E^{C}, completing the proof. ∎

Remark. Non-closed sets are not compact in any metric space. Notice open set does not mean non-closed.

Theorem 2.7.33. *Closed subsets of compact sets are compact.*

Proof. See worksheet 7. ∎

Corollary 2.7.33.1. *If $\{K_\alpha\}_{\alpha \in A}$ is a collection of compact sets, then $\bigcap_{\alpha \in \mathcal{A}} K_\alpha$ is compact.*

Proof. Since compact sets are closed, $\bigcap_{\alpha \in \mathcal{A}} K_\alpha$ is the intersection of closed sets, which is also closed. Since $\bigcap_{\alpha \in \mathcal{A}} K_\alpha$ is a subset of compact sets U_α, it is compact. ■

Remark. Finite union of compact sets in X is compact.

Theorem 2.7.34. *Every sequence in a compact set has a convergent subsequence.*

Proof. See worksheet 7. ■

Theorem 2.7.35 (Compact Set)**.** *A set $E \subseteq X$ is compact if and only if every sequence in E has a convergent subsequence converging to a point in E.*

Theorem 2.7.36 (Nested Compact Sets Property)**.** *Let (F_n) be a sequence of closed, bounded, nonempty sets in $\mathbb{R}^k$ such that $F_1 \supseteq F_2 \supseteq \cdots$, then $F = \bigcap_{n=1}^{\infty} F_n \neq \varnothing$ and F is closed and bounded.*

Theorem 2.7.37. *Suppose $\{E_\alpha\}_{\alpha \in \mathcal{A}}$ is a collection of compact sets such that $\bigcap_{\alpha \in \mathcal{B}} E_\alpha \neq \varnothing$ for any finite $\mathcal{B} \subseteq \mathcal{A}$. Then $\bigcap_{\alpha \in \mathcal{A}} E_\alpha \neq \varnothing$.*

Definition 2.7.38 (K-cell)**.** A K-cell is a subset of $\mathbb{R}^k$ of the form $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k]$.

Theorem 2.7.39. *Every k-cell F in $\mathbb{R}^k$ is compact.*

Proof. TODO ■

Theorem 2.7.40. *A subset E of $\mathbb{R}^k$ is compact if and only if it is closed and bounded.*

Proof. TODO ■

Remark. The forward direction is true in any metric space.

Characterization of compact sets

(1) and (2) are equivalent in any metric space. Forward direction of (3) is true in any metric space. All of three are equivalent in $\mathbb{R}^k$.

1. Every open cover of E has a finite subcover.
2. A set $E \subseteq X$ is compact if and only if every sequence in E has a convergent subsequence converging to a point in E.
3. A subset E of $\mathbb{R}^k$ is compact if and only if it is closed and bounded.

Cantor Set

Definition 2.7.41 (Cantor Set)**.** Let $\mathcal{C}_0$ be $[0,1]$. Then define $\mathcal{C}_1$ as the union of 2^1 interval $[0,\frac{1}{3}] \cup [\frac{2}{3},1]$. Each time delete the middle $\frac{1}{3}$ of intervals. Thus $\mathcal{C}_2$ is the union of 2^2 intervals which is $[0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1]$.
In short, C_n is the union of 2^n disjoint closed intervals of which length is $(\frac{1}{3})^n$. Then define Cantor Set

$$\mathcal{C} = \bigcap_{i=0}^{\infty} \mathcal{C}_i.$$

Theorem 2.7.42. *Here are some facts/properties about the Cantor set $\mathcal{C}$:*

- *$\mathcal{C}$ is compact.*
- *$\mathcal{C}$ does not contain any intervals.*
- *$\mathcal{C}$ does not have any interior points.*
- *Every point in $\mathcal{C}$ is a limit point of $\mathcal{C}$.*
- *Every point in $\mathcal{C}$ is a limit point of $\mathcal{C}^{\mathsf{C}}$.*

2.8 Series

For an infinite series $\sum_{n=m}^{\infty} a_n$, we say it *converge* provided the sequence (s_n) of partial sums

$$s_n = a_m + a_{m+1} + \cdots + a_n = \sum_{k=m}^{n} a_k$$

also converges to a real number S. i.e.

$$\sum_{n=m}^{\infty} a_n = S \quad \text{means} \quad \lim s_n = S \quad \text{or} \quad \lim_{n\to\infty} \left(\sum_{k=m}^{n} a_k \right) = S$$

A series that does not converge is said to *diverge*, so $\sum_{n=m}^{\infty} a_n$ *diverge* to $+\infty$, $\sum_{n=m}^{\infty} a_n = +\infty$, provided $\lim s_n = +\infty$. Similar for diverging to $-\infty$.

If the terms in $\sum a_n$ are all nonnegative, then the corresponding partial sums (s_n) form an increasing sequence, so $\sum a_n$ either converges or diverges to $+\infty$ by 2.4.2 and 2.4.3. In particular, $\sum |a_n|$ is meaningful for any (s_n) whatever. The series $\sum a_n$ is said to *converge absolutely* or to be *absolutely convergent* if $\sum |a_n|$ converges.

We use $\sum a_n$ to represent $\sum_{n=m}^{\infty} a_n$

Example (Geometric Series)**.** A series of the form $\sum_{n=0}^{\infty} ar^n$ for constants a and r is called a geometric series. For $r \neq 1$, the partial sums s_n are given by

$$\sum_{k=0}^{n} ar^k = a\frac{1 - r^{n+1}}{1 - r}.$$

Furthermore, if $|r| < 1$, then $\lim_{n\to\infty} r^{n+1} = 0$ and

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}$$

If $a \neq 0$ and $|r| \geq 1$, then (ar^n) does not converge to 0, so $\sum ar^n$ diverges.

Example.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges if and only if} \quad p > 1$$

If $p \leq 1$, $\sum 1/n^p = +\infty$

Definition 2.8.1. We say a series $\sum a_n$ satisfies the *Cauchy criterion* if its sequence (s_n) of partial sums is a Cauchy sequence which is:

$$\forall \epsilon > 0,\ \exists N,\ m, n > N \Rightarrow |s_n - s_m| < \epsilon \tag{1}$$

which is equivalent to

$$\forall \epsilon > 0,\ \exists N,\ n \geq m > N \Rightarrow |s_n - s_{m-1}| < \epsilon. \tag{2}$$

Since $s_n - s_{m-1} = \sum_{k=m}^{n} a_k$, we can write (2) as

$$\forall \epsilon > 0,\ \exists N,\ n \geq m > N \Rightarrow \left| \sum_{k=m}^{n} a_k \right| < \epsilon \tag{3}$$

Theorem 2.8.2. *A series converges* $\iff$ *it satisfies the Cauchy criterion.*

Proof. By 2.4.11, we know its partial sum converges, so the series also converges. ■

Corollary 2.8.2.1. *If a series* $\sum a_n$ *converges, then* $\lim a_n = 0$

Proof. By setting $n = m$ in the condition of 2.8.1, we get

$$(\forall \epsilon > 0,\ \exists N,\ n > N \Rightarrow |a_n| < \epsilon) \Rightarrow \lim a_n = 0$$

■

Remark. If $\lim a_n \neq 0$, then $\sum a_n$ does not converge.

A useful contrapositive of this corollary is "If $\lim a_n \neq 0$, then $\sum a_n$ does not converge."

Theorem 2.8.3 (Comparison Test)**.** *Let* $\sum a_n$ *be a series where* $a_n \geq 0$ *for all* n.

(i) If $\sum a_n$ *converges and* $|b_n| \leq a_n$ *for all* n*, then* $\sum b_n$ *converges.*

(ii) If $\sum a_n = +\infty$ *and* $b_n \geq a_n$ *for all* n*, then* $\sum b_n = +\infty$

Proof.

(i) For $n \geq m$ we have
$$\left| \sum_{k=m}^{n} b_k \right| \leq \sum_{k=m}^{n} |b_k| \leq \sum_{k=m}^{n} a_k \tag{1}$$
Since $\sum a_n$ converges, it satisfies 2.8.1(1). Then from (1) we can see $\sum b_n$ also satisfies the Cauchy criterion in 2.8.1(3), and hence $\sum b_n$ converges.

(ii) Since $b_n \geq a_n$ for all n, obviously we have $\sum_{k=m}^n b_k \geq \sum_{k=m}^n a_k$. Since $\lim \sum_{k=m}^n b_k = +\infty$, $\lim \sum_{k=m}^n a_k = +\infty$.

■

Corollary 2.8.3.1. *Absolutely convergent series are convergent.*

Proof. Suppose $\sum b_n$ is absolutely convergent. This means $\sum a_n$ converges where $a_n = |b_n|$ for all n. Then $|b_n| \leq a_n$ and $\sum b_n$ converges trivially from 2.8.3. ■

Theorem 2.8.4 (Root Test). *Let $\sum a_n$ be a series and let $\alpha = \limsup |a_n|^{1/n}$. The series $\sum a_n$*

(i) converges absolutely if $\alpha < 1$

(ii) diverges if $\alpha > 1$

(iii) Otherwise the test does not provide any useful information.

Proof. (i) Suppose $\alpha < 1$, and select $\epsilon > 0$ so that $\alpha + \epsilon < 1$. Then

$$\alpha - \epsilon < \sup\{|a_n|^{1/n} : n > N\} < \alpha + \epsilon$$

so

$$|a_n| < (a + \epsilon)^n \quad \text{for} \quad n > N.$$

Since $0 < \alpha + \epsilon < 1$, $\sum_{n=N+1}^\infty (\alpha + \epsilon)^n$ converges and 2.8.3(i) tells $\sum_{n=N+1}^\infty a_n$ converges. Then clearly $\sum a_n$ converges.

(ii) If $\alpha > 1$, then for each $\alpha - \epsilon > 1$, there exists $N \in \mathbb{N}$ such that $\sup\{|a_n|^{\frac{1}{n}} : n \geq N\} > 1$, i.e., $\exists N_1 \geq N$ such that $|a_{N_1}|^{\frac{1}{N_1}}| > 1 \implies |a_{N_1}| > 1$ since 1 is smaller than the supremum. This means $|a_n| > 1$ for infinitely many choices of n. In particular, (a_n) cannot possibly converge to 0, so $\sum a_n$ cannot converge by the contrapositive of 2.8.2.1.

(iii) Example: $\sum \frac{1}{n}$ diverges but $\sum \frac{1}{n^2}$ converges.

■

Theorem 2.8.5 (Ratio Test). *A series $\sum a_n$ of nonzero terms*

(i) converges absolutely if $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$,

(ii) diverges if $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$.

(iii) Otherwise $\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$ *and the test gives no information.*

Proof. let $\alpha = \limsup |a_n|^{1/n}$. By **??** we have

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq \alpha \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|.$$

(i) If $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\alpha < 1$ and the series converges by 2.8.4.

(ii) If $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\alpha > 1$ and the series diverges by 2.8.4.

(iii) If $\alpha = 1$, then same reasoning as the proof in 2.8.4(iii).

■

If the terms a^n are nonzero and if $\lim \left|\frac{a_{n+1}}{a_n}\right| = 1$, then $\alpha = \limsup |a_n|^{1/n} = 1$ by 2.6.2.1, so neither the Ratio Test nor the Root Test gives information about the convergence of $\sum a_n$.

2.9 Alternating Series and Integral Tests

Sometimes we can try to check convergence or divergence of series by comparing the partial sums with familiar integrals. By drawing the function a^n and the of rectangles corresponding to the series on a same picture and comparing the areas under the function and the sum of areas of these rectangles, we may get the information about the convergence of the series. For example, if all rectangles are below the function and the integral of the function is finite, then the series converge.

Theorem 2.9.1. $\sum \frac{1}{n^p}$ *converges* $\iff$ $p > 1$.

Proof. By drawing the function $\frac{1}{n^p}$ and the of rectangles corresponding to the series on a same picture, we can get

$$\sum_{k=1}^{n} \frac{1}{k^p} \leq 1 + \int_1^n \frac{1}{x^p} dx = 1 + \frac{1}{p-1}\left(1 - \frac{1}{n^{p-1}}\right) < 1 + \frac{1}{p-1} = \frac{p}{p-1}$$

Thus $\sum_{n=1}^{\infty} \frac{1}{n^p} \leq \frac{p}{p-1} < +\infty$

Suppose $0 < p \leq 1$. Then $\frac{1}{n^p} \geq \frac{1}{n}$ for all n, so $\sum \frac{1}{n^p}$ diverges when $\sum \frac{1}{n}$ diverges by 2.8.3. ∎

Theorem 2.9.2. *Here are the conditions under which an integral test is advisable:*

(a) All comparison, root, and ratio tests do not apply.

(b) The terms a_n of the series are nonnegative.

(c) There is a nice decreasing function f on $[1, \infty)$ such that $f(n) = a_n$ for all n.

(d) The integral of f is easy to calculate or estimate.

If $\lim_{n\to\infty} \int_1^n f(x)dx = +\infty$, then the series diverges. If $\lim_{n\to\infty} \int_1^n f(x)dx < +\infty$, then the series will converge.

Theorem 2.9.3 (Alternating Series Theorem)**.** *If $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots \geq 0$ and $\lim a_n = 0$, then the alternating series $\sum(-1)^{n+1}a_n$ converges. Moreover, the partial sums $s_n = \sum_{k=1}^{n}(-1)^{k+1}a_k$ satisfy $|s - s_n| \leq a_n$ for all n.*

Proof. To prove the series converge we need to show the partial sum (s_n) also converges. Note that the subsequence (s_{2n}) is increasing (accumulative sum of positive a_n) and the subsequence (s_{2n-1}) is decreasing (accumulative sum of negative a_n). We claim

$$s_{2m} \leq s_{2n+1} \quad \text{for all} \quad m, n \in \mathbb{N} \tag{2}$$

Since $s_{2n+1} - s_{2n} = a_{2n+1} \geq 0$, we have $s_{2n} \leq s_{2n+1}$ for all n. Thus if $m \leq n$ in (1) then (1) holds because $s_{2m} \leq s_{2n} \leq s_{2n+1}$, when (s_{2n}) is increasing. If $m \geq n$ in (1), then (1) also holds because $s_{2n+1} \geq s_{2m+1} \geq s_{2m}$ when (s_{2n+1}) is decreasing. Therefore, by (1) we can see that the subsequence (s_{2n}) is bounded above by every odd partial sum, and the subsequence (s_{2n+1}) is a bounded below by each even partial sum. Then by 2.4.2 (s_{2n}) and (s_{2n+1}) converge to some s and t. Now we have

$$t - s = \lim_{n\to\infty}(s_{2n+1} - 2_{2n}) = \lim_{n\to\infty} a_{2n+1} = 0$$

so $s = t$ and $\lim_n s_n = s$. (**WHY???** Is it because $s = supS$ and $t = \inf S$ where S is the set of subsequential limits.)

To check the last claim, note that $s_{2k} \leq s \leq s_{2k+1}$, so both $s_{2k+1} - s$ and $s - s_{2k}$ are bounded by $s_{2k+1} - s_{2k} = a_{2k+1} \leq a_{2k}$ (**WHY????**). So whether n is even or odd, we have $|s - s_n| \leq a_n$. ∎

Chapter 3

Continuity

3.1 Continuous Functions

In this book/note, we will be concerned with functions f such that $\operatorname{dom}(f) \subseteq \mathbb{R}$ and such that f is a real-valued function. We consider the *natural domain* as "the largest subset of $\mathbb{R}$ on which the function is a well defined real-valued function.

Definition 3.1.1. The function f is *continuous* at x_0 in $\operatorname{dom}(f)$ if, for every sequence (x_n) in $\operatorname{dom}(f)$ converging to x_0, we have $\lim_n f(x_n) = f(x_0)$. If f is continuous at each point of a set $S \subseteq \operatorname{dom}(f)$, then f is said to be *continuous* on S. The function f is said to be *continuous* if it is continuous on $\operatorname{dom}(f)$.

Theorem 3.1.2. *f is continuous at x_0 in $\operatorname{dom}(f)$ if and only if*

$$\forall \epsilon > 0,\ \exists \delta > 0 \quad \textit{such that} \quad (x \in \operatorname{dom}(f)) \wedge (|x - x_0| < \delta) \implies |f(x) - f(x_0)| < \epsilon \quad (1)$$

Proof.

$\implies$: Suppose f is continuous at x_0 but (1) does not hold. In other words, there exists $\epsilon > 0$ so that

$$(x \in \operatorname{dom}(f)) \wedge (|x - x_0| < \delta) \implies |f(x) - f(x_0)| < \epsilon$$

fails for each $\delta > 0$. In particular the implication

$$(x \in \operatorname{dom}(f)) \wedge (|x - x_0| < \frac{1}{n}) \implies |f(x) - f(x_0)| < \epsilon$$

fails for each $n \in \mathbb{N}$. Thus for each $n \in \mathbb{N}$ there exists x_n in $\operatorname{dom}(f)$ such that $|x_n - x_0| < \frac{1}{n}$ and yet $|f(x_0) - f(x_n)| \geq \epsilon$. Hence we have $|f(x_0) - f(x_n)| \geq \epsilon \implies \lim f(x_n) \neq f(x_0)$. This contradicts to the definition of continuity 3.1.1.

$\impliedby$: Suppose (1) holds and consider a (x_n) in $\operatorname{dom}(f)$ such that $\lim x_n = x_0$. Let $\epsilon > 0$. By (1) there exists $\delta > 0$ such that

$$(x \in \operatorname{dom}(f)) \wedge (|x - x_0| < \delta) \implies |f(x) - f(x_0)| < \epsilon$$

Since $\lim x_n = x_0$ we have

$$n > N \implies |x_n - x_0| < \delta \implies |f(x_n) - f(x_0)| < \epsilon$$

Thus $\lim f(x_n) = f(x_0)$

■

The condition $(x \in \operatorname{dom}(f)) \wedge (|x - x_0| < \delta) \implies |f(x) - f(x_0)| < \epsilon$ in the book is a little bit confusing. In other words, it means

$$\forall x \in \operatorname{dom}(f),\ |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$$

Corollary 3.1.2.1 (Discontinuity). *To use ϵ-δ property to prove the discontinuity, we need to show that*

$$\exists \epsilon > 0 \quad \textit{such that} \quad \forall \delta > 0,\ \exists x \in \textit{dom}\,(f) \quad \textit{such that} \quad |x - x_0| < \delta \quad \textit{but} \quad |f(x) - f(x_0)| \geq \epsilon$$

Theorem 3.1.3. *Let f be a real-valued function with $\textit{dom}\,(f) \subseteq \mathbb{R}$. f is continuous at x_0 if and only if for every monotonic sequence (x_n) in $\textit{dom}\,(f)$ converging to x_0, we have $\lim f(x_n) = f(x_0)$.*

Theorem 3.1.4. *If f is continuous at x_0 in $\textit{dom}\,(f)$, then $|f|$ and kf, for $k \in \mathbb{R}$, are continuous at x_0.*

Proof. Since f is continuous at x_0, we have $\lim f(x_n) = f(x_0)$. Since $\lim kf(x_n) = k \lim f(x_n) = kf(x_0)$, this proves kf is continuous at x_0. Since $\lim f(x_n) = f(x_0)$, we have

$$n > N \implies |f(x_n) - f(x_0)| < \epsilon$$

Since $||f(x_n)| - |f(x_0)|| \leq |f(x_n) - f(x_0)|$, we have

$$n > N \implies \big||f(x_n)| - |f(x_0)|\big| \leq |f(x_n) - f(x_0)| < \epsilon$$

so $\lim |f(x_n)| = |f(x_0)|$ ■

Theorem 3.1.5. *Let f and g be real-valued fucntions that are continuous at x_0 in $\mathbb{R}$. Then*

(i) $f + g$ is continuous at x_0;

(ii) fg is continuous at x_0;

(iii) f/g is continuous ar x_0 if $g(x_0) \neq 0$.

Proof. We use the basic definition of continuity 3.1.1 and the basic theorems of limit. ■

Theorem 3.1.6. *If f is continuous at x_0 and g is continuous at $f(x_0)$, then the composite function $g \circ f$ is continuous at x_0.*

Proof. Given that $x_0 \in \operatorname{dom}(f)$ and $f(x_0) \in \operatorname{dom}(g)$, let (x_n) be a sequence in $\{x \in \operatorname{dom}(f) : f(x) \in \operatorname{dom}(g)\}$ converging to x_0. Since f is continuous at x_0, we have $\lim f(x_n) = f(x_0)$. Since the sequence $(f(x_n))$ converges to $f(x_0)$ and g is continuous at $f(x_0)$, we also have $\lim g(f(x_n)) = g(f(x_0))$ which is $\lim g \circ f(x_n) = g \circ f(x_0)$. ■

Conclusion

Suppose f, g are real-valued functions. If f and g are continuous at x_0, then the following functions are also continuous at x_0 (as long as x_0 is also in the domain of them):

- $(f+g)(x_0) = f(x_0) + g(x_0)$
- $(fg)(x_0) = f(x_0)g(x_0)$
- $k \in \mathbb{R}$ $(kf)(x_0) = k(f(x_0))$
- $(f/g)(x_0) = f(x_0)/g(x_0)$
- $(|f|)(x_0) = |f(x_0)|$
- $(\max\{f, g\})(x_0) = \max\{f(x_0), g(x_0)\}$
- $(\min\{f, g\})(x_0) = \min\{f(x_0), g(x_0)\}$

3.2 Properties of Continuous Functions

A real-valued function f is said to be *bounded* if $\{f(x) : x \in \mathrm{dom}(f)\}$ is a bounded set. i.e. if there exists a real number M such that $|f(x)| \leq M$ for all $x \in \mathrm{dom}(f)$.

Theorem 3.2.1. *Let f be a continuous real-valued function on a* closed *interval $[a,b]$. Then f is a bounded function. Moreover, f assume its maximum and minimum values on $[a,b]$; that is there exist x_0, y_0 in $[a,b]$ such that $f(x_0) \leq f(x) \leq f(y_0)$ for all $x \in [a,b]$.*

Proof. First assume f is not bounded on $[a,b]$. Then for each $n \in \mathbb{N}$ there corresponds an $x_n \in [a,b]$ such that $|f(x_n)| > n \implies \lim_{k\to\infty} |f(x_{n_k})| = +\infty$. By 2.5.6, since (x_n) is bounded by $[a,b]$ it has a subsequence (x_{n_k}) that converges to some real number $x_0 \in [a,b]$. Since f is continuous, we have $\lim f(x_{n_k}) = f(x_0) < \infty$, which is a contradiction. Thus, f is bounded.

Not since f is bounded, $M = \sup\{f(x) : x \in [a,b]\}$ is finite. For each $n \in \mathbb{N}$ there exists $y_n \in [a,b]$ such that $M - \frac{1}{n} < f(y_n) \leq M$. Hence we have $\lim f(y_n) = M$ by Squezze formula. By 2.5.6 there is a subsequence (y_{n_k}) of (y_n) converging to some limit $y_0 \in [a,b]$. Since y is continuous at y_0, we have $\lim_{k\to\infty} f(y_{n_k}) = f(y_0)$. Since $(f(y_{n_k}))$ is also a subsequence of $(f(y_n))$, by 2.5.4 $\lim_{k\to\infty} f(y_{n_k}) = \lim_{n\to\infty} f(y_n) = M$. Thus $f(y_0) = M$ meaning that f achieves its maximum at y_0.

Apply the same method to $-f$, and we get $-f$ achieves its maximum at some $x_0 \in [a,b]$. In other words, f achieves its minimum at x_0. ∎

Theorem 3.2.2 (Intermediate Value Theorem)**.** *If f is a continuous real-valued function on an interval I, then f has the intermediate value property on I: Whenever $a, b \in I$, if $a < b$ and y lies between $f(a)$ and $f(b)$ [i.e. $f(a) < y < f(b)$ or $f(b) < y < f(a)$], then there exists at least one x in (a,b) such that $f(x) = y$.*

Proof. Let's focus on the case that $f(a) < y < f(b)$ since the other case is similar. Let $S = \{x \in [a,b] : f(x) < y\}$. Since $a \in S$ the set S is nonempty, and $x_0 = \sup S$ represents a number in $[a,b]$. For each $n \in \mathbb{N}$, $x_0 - \frac{1}{n}$ is not an upper bound for S, so there exists $s_n \in S$ such that $x_0 - \frac{1}{n} < s_n \leq x_0$. Thus $\lim s_n = x_0$ and since $f(s_n) < y$ for all n, we have

$$f(x_0) = \lim f(s_n) \leq y$$

because f is continuous at x_0. Let $t_n = \min\{b, x_0 + \frac{1}{n}\}$. Since $x_0 < t_n \leq x_0 + \frac{1}{n}$ we have $\lim t_n = x_0$. Each t_n belongs to $[a,b]$ but not to S, so $f(t_n) \geq y$ for all n. Therefore,

$$f(x_0) = \lim f(t_n) \geq y$$

because f is continuous at x_0. Thus $f(x_0) = y$. ∎

Corollary 3.2.2.1 (Fixed Point Theorem)**.** *Let f be a function $f : [0,1] \to [0,1]$. If f is continuous, then f has a fixed point, i.e., there exists $x \in [0,1]$ such that $f(x) = x$.*

Proof. Define $g(x) = f(x) - x$. Note that g is continuous.

Case 1: If $f(0) = 0$ or $f(1) = 1$, th en the statement is true trivially.

Case 2: If $f(0) \neq 0$ and $f(1) \neq 1$, then $g(0) = f(0) - 0 > 0$ and $g(1) = f(1) - 1 < 0$. Since $g(1) < 0 < g(0)$, by 3.2.2 there exists $x \in (0, 1)$ such that $g(x) = 0 \implies f(x) = x$.

■

Corollary 3.2.2.2. *Let f and g be continuous functions on $[a, b]$ such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove $f(x_0) = g(x_0)$ for at least one x_0 in $[a, b]$.*

Proof. See Ex. 18.5 ■

Corollary 3.2.2.3. *If f is a continuous real-valued function on an interval I, then the set $f(I) = \{f(x) : x \in I\}$ is also an interval or a single point.*

Proof. By **??** the set $J = f(I)$ has the property:

$$(y_0, y_1 \in J \quad \text{and} \quad y_0 < y < y_1) \implies y \in J$$

If $\inf J < \sup J$, then such a set J will be an interval. Consider $\inf J < y < \sup J$. Then there exist $y_0, y_1 \in J$ so that $y_0 < y < y_1$, so $y \in J$ by the above property. We showed that

$$\inf J < y < \sup J \implies y \in J$$

so J is an interval with endpoints $\inf J$ and $\sup J$ ■

Theorem 3.2.3. *Let g be a strictly increasing function on an interval J such that $g(J)$ is an interval I. Then g is continuous on J.*

Proof. Consider an non-endpoint x_0 of J. Since g is strictly increasing, $g(x_0)$ is also not an endpoint of I, so $\exists \epsilon_0 > 0$ such that $(g(x_0) - \epsilon_0, g(x_0) + \epsilon_0) \subseteq I$.

Let $\epsilon > 0$ and we can assume $\epsilon < \epsilon_0$ (**WHY???**). Then $\exists x_1, x_1 \in J$ such that $g(x_1) = g(x_0) - \epsilon$ and $g(x_2) = g(x_0) + \epsilon$. This means $x_1 < x_0 < x_2$ because g is increasing. Also if $x_1 < x < x_2$, then $g(x_1) < g(x) < g(x_2)$, hence $g(x_0) - \epsilon < g(x) < g(x_0) + \epsilon$, and hence $|g(x) - g(0)| < \epsilon$. Now set $\delta = \min\{x_2 - x_0, x_0 - x_1\}$, then

$$|x - x_0| < \delta \implies x_1 < x < x_2 \implies |g(x) - g(x_0)| < \epsilon$$

Thus g is continuous on J. ■

Theorem 3.2.4. *Let f be a continuous strictly increasing function on some interval I. Then $f(I)$ is an interval J by 3.2.2.3 and f^{-1} represents a function with domain J. The function f^{-1} is a continuous strictly increasing function on J.*

Proof. Obviously f^{-1} is still strictly increasing. Since f^{-1} maps J onto I, by 3.2.3 f^{-1} is continuous. ∎

Theorem 3.2.5. *Let f be a* one-to-one *continuous function on an interval I. Then f is strictly increasing or strictly decreasing.*

Proof. Firstly we want to show that

$$\text{if } a < b < c \text{ in } I, \text{ then } f(b) \text{ lies between } f(a) \text{ and } f(c) \tag{1}$$

Assume it's false so $f(b) > \max\{f(a), f(c)\}$. Select y so that $f(b) > y > \max\{f(a), f(c)\}$. By **??** applied to $[a, b]$ and $[b, c]$, $\exists x_1 \in (a, b)$ and $x_2 \in (b, c)$ such that $f(x_1) = f(x_2) = y$. This contradicts to the one-to-one property of f.

Now select any $a_0 < b_0$ in I and suppose, say, that $f(a_0) < f(b_0)$. We need to show f is strictly increasing on I. By (1) we have

$$\begin{aligned} f(x) < f(a_0) &\quad \text{for} \quad x < a_0 \\ f(a_0) < f(x) < f(b_0) &\quad \text{for} \quad a_0 < x < b_0 \\ f(b_0) < f(x) &\quad \text{for} \quad x > b_0 \end{aligned}$$

In particular,

$$f(x) < f(a_0) \quad \text{for all} \quad x < a_0 \tag{2}$$

$$f(a_0) < f(x) \quad \text{for all} \quad x > a_0 \tag{3}$$

Now consider any $x_1 < x_2$ in I.

$$\begin{aligned} x_1 \le a_0 \le x_2 &\implies f(x_1) < f(x_2) \quad \text{by (2) and (3)} \\ x_1 < x_2 < a_0 &\implies f(x_1) < f(a_0) \quad \text{by (2)} \implies f(x_1) < f(x_2) \quad \text{by (1)} \\ a_0 < x_1 < x_2 &\implies f(a_0) < f(x_2) \quad \text{by (2)} \implies f(x_1) < f(x_2) \quad \text{by (1)} \end{aligned}$$

∎

3.3 Uniform Continuity

Sometimes we want to know when the δ in 3.1.2 can be chosen to depend only on $\epsilon > 0$ and S, so that δ does not depend on the particular point x_0.

Definition 3.3.1. Let f be a real-valued function defined on a set $S \subseteq \mathbb{R}$. Then f is uniformly ocntinuous on S if

$$\text{for each } \epsilon > 0 \text{ there exists } \delta > 0 \text{ such that}$$
$$\forall x, y \in S,\ |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

We will say f is uniformly continuous if f is uniformly continuous on $\operatorname{dom} f$.

Theorem 3.3.2. *If a real-valued function f is uniformly continuous on an open interval (a, b), then f is* bounded *on (a, b).*

Theorem 3.3.3. *If f is continuous on a closed interval $[a, b]$, then f is uniformly continuous on $[a, b]$.*

Proof. Assume f is not uniformly continuous on $[a, b]$, then there exists $\epsilon > 0$ such that for each $\delta > 0$, there exists $x, y \in [a, b]$ such that $|x - y| < \delta$ but $|f(x) - f(y)| \geq \epsilon$. Thus for each $n \in \mathbb{N}$, since $\frac{1}{n} > 0$, there exist $x_n, y_n \in [a, b]$ such that $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) \geq f(y_n)| \geq \epsilon$. By 2.5.6, since (x_n) is bounded in $[a, b]$, it has a subsequence (x_{n_k}) converging to $x_0 \in [a, b]$. Clearly we can also have a subsequence (y_{n_k}) converging to x_0. Because f is continuous at x_0, we have

$$f(x_0) = \lim_{x \to \infty} f(x_{n_k}) = \lim_{k \to \infty} f(y_{n_k})$$

so

$$\lim_{k \to \infty} [f(x_{n_k}) - f(y_{n_k})] = 0$$

However, since $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon > 0$ for all k, we have a contradiction. ■

Theorem 3.3.4. *If f is uniformly continuous on a set S and (s_n) is a Cauchy sequence in S, then $(f(s_n))$ is a Cauchy sequence.*

Proof. Let (s_n) be a Cauchy sequence in S and let $\epsilon > 0$. Since f is uniformly continuous on S, there exists $\delta > 0$ so that

$$\forall x, y \in S,\ |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Since (s_n) is a Cauchy sequence, there exists N so that

$$m, n > N \implies |s_n - s_m| < \delta \implies |f(s_n) - f(s_m)| < \epsilon.$$

Thus $(f(s_n))$ is indeed a Cauchy sequence. ■

Corollary 3.3.4.1. *Let (s_n) be a Cauchy sequence in S and f be a function on S. If $(f(s_n))$ is not a Cauchy sequence, then f is not* uniformly continuous.

Proof. The contrapositive of 3.3.4. ■

Corollary 3.3.4.2. *If f is uniformly continuous on (a, b), then f is a bounded function on (a, b). i.e. If f is an unbounded function on (a, b), then f is not uniformly continuous on (a, b).*

Theorem 3.3.5 (Continuous Extension Theorem)**.** *A real-valued function f on (a, b) is uniformly continuous on (a, b) if and only if it can be extended to a continuous function $\tilde{f}$ on $[a, b]$.*

Proof.

$\implies$: Suppose (s_n) is a sequence in (a, b) converging to a, then (s_n) is a Cauchy sequence by 2.4.11, so $(f(s_n))$ is also a Cauchy sequence by **??** and converging again by 2.4.11. Hence we have the following claim:

$$\text{if } (s_n) \text{ is a sequence in } (a, b) \text{ converging to } a\text{, then } (f(s_n)) \text{ converges} \tag{1}$$

Create a sequence $(u_n) = (s_1, t_1, s_2, t_2, \dots)$ where (t_n) is also a sequence converging to a. Clearly $\lim u_n = a$ and $\lim f(u_n)$ exists due to (1). Therefore, $(f(s_n))$ and $(f(t_n))$ are subsequences of $(f(u_n))$ both converge to $\lim f(u_n)$ by 2.5.4, so $\lim f(s_n) = \lim f(t_n)$. Hence we have the following claim:

$$\text{if } (s_n) \text{ and } (t_n) \text{ are sequences in } (a, b) \text{ converging to } a\text{, then } \lim f(s_n) = \lim f(t_n) \tag{2}$$

Now we define

$$\tilde{f}(a) = \lim f(s_n) \text{ for any sequence } (s_n) \text{ in } (a, b) \text{ converging to } a \tag{3}$$

(1) guarantees $\lim f(s_n)$ exists, and (2) guarantees $\tilde{f}(a)$ is not ambiguous. Thus, $\tilde{f}$ is continuous at a. Similar method for $\tilde{f}(b)$.

$\impliedby$: Since $\tilde{f}$ is continuous on $[a, b]$, $\tilde{f}$ is also uniformly continuous on $[a, b]$ by 3.3.3, so clearly f is uniformly continuous on (a, b).

■

Theorem 3.3.6. *Let f be a continuous function on an interval I [I may be bounded or unbounded]. Let I° be the interval obtained by removing from I any endpoints that happen to be in I. If f is differentiable on I° and if f' is bounded on I°, then f is uniformly continuous on I.*

3.4 Continuity in Metric Space

Definition 3.4.1 (Image and Preimage)**.** Let X and Y be two sets. Let a function $f : X \to Y$. Let $E \subseteq X$ and $U \subseteq Y$. We define the *image of E under f* as

$$f(E) = \{f(x) : x \in E\};$$

define the *preimage of A under f* as

$$f^{-1}(A) = \{x \in X : f(x) \in A\}.$$

Theorem 3.4.2. *Let X and Y be two sets, and let $f : X \to Y$, let $E \subseteq X$, and let $A, B \subseteq Y$. Thenthe following assertions are true:*

(a) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(b) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(c) $f^{-1}(A^{\mathsf{C}}) = (f^{-1}(A))^{\mathsf{C}}$.

(d) $f^{-1}(A) \subseteq f^{-1}B$ *if* $A \subseteq B$.

(e) $E \subseteq f^{-1}(f(E))$.

Proof. See worksheet 12. ■

Definition 3.4.3 (Continuity)**.** Let (X, d_X) and (Y, d_Y) be two metric spaces, and let $f : X \to Y$. The following are three *equivalent* definitions of continuity at a point $x_0 \in X$.

1. (ϵ-δ definition) For any $\epsilon > 0$, there exists $\delta > 0$ such that $x \in X$ and $d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \epsilon$.
2. (sequential definition) For any sequence (x_n) in X converging to x_0, the sequence $f(x_n)$ converges to $f(x_0)$.
3. (topological definition) For any open set U in Y such that $f(x_0) \in U$, there exists an open set V in X such that $x_0 \in V \subseteq f^{-1}(U)$.

Remark. See worksheet 12 for the proof of equivalence.

‘

Theorem 3.4.4. *f is continuous (on its domain) if and only if $f^{-1}(U)$ is open in X for every open set U in Y. i.e. a function is continuous if and only if the preimage of every open set is open.*

Proof. See worksheet 12. ■

Definition 3.4.5 (Uniform Continuity)**.** Let (X, d_X) and (Y, d_Y) be two metric spaces. A function $f : X \to Y$ is *uniformly continuous* on $E \subseteq X$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$x_1, x_2 \in E \text{ and } d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \epsilon.$$

Theorem 3.4.6. *Let (X, d_X) and (Y, d_Y) be two metric spaces. Let a function $f : X \to Y$ be continuous. Suppose a subset $E \subseteq X$ is compact.*

(i) $f(E)$ is compact. i.e., the image of a compact set under continuous function is still compact.

(ii) f is uniformly continuous on E. i.e., a continuous function on a compact set is also uniformly continuous.

Proof.

(i) Let $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ be an open cover of $f(E)$. Then $\{f^{-1}(U_\alpha)\}_{\alpha\in\mathcal{A}}$ is an open cover of E since f is continuous and by 3.4.4. Since E is compact, there exists a finite subcover $\{f^{-1}(U_{\alpha_i})\}_i^n$ of E. Thus

$$E \subseteq \bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i}) = f^{-1}\left(\bigcup_{i=1}^{n} U_{\alpha_i}\right)$$

implies

$$f(E) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$$

which is a finite subcover of $f(E)$.

(ii) TODO

■

Corollary 3.4.6.1. *Let f be a continuous function $f : X \to \mathbb{R}$. Suppose E is a compact subset of X. Then*

(i) $f(E)$ is closed and bounded;

(ii) There exists $u, v \in E$ such that $f(u) = \inf_{x\in E} f(x) = \inf f(E)$ and $f(v) = \sup_{x\in E} f(x) = \sup f(E)$. i.e. f attains its maximum *and* minimum *on E.*

Proof. (i) Follows from the definition of compact set $f(E)$.

(ii) For any compact set, its supremum and infimum are in the set.

■

Theorem 3.4.7. *Let S be a subset of $\mathbb{R}$. Suppose a function $f : S \to \mathbb{R}$. If f is continuous on an interval $I \subseteq S$ like $[a, b], (a, b), (a, b]$ where $a, b \in \mathbb{R} \cup \{\pm\infty\}$, then $f(I)$ is a singleton or an interval.*

Proof. TODO ∎

Remark. Singleton is a set with exactly one element.

Theorem 3.4.8 (Continuous Extension Theorem)**.** *Let (X, d) be a metric space. Let $E \subseteq X$ and f be a function $f : E \to R$*

(i) If f is uniformly continuous function on E, then f can be extended to a (uniformly) continuous function on $\overline{E} = E \cup E'$.

(ii) If f can be extended to a uniformly continuous function on $\overline{E} = E \cup E'$, then f is uniformly continuous function on E.

Proof. TODO ∎

Remark. Be careful that if f is *only continuous* (not uniformly) on $\overline{E}$, then (ii) fails.

3.5 Limits of Functions

Definition 3.5.1. Let $S \subset \mathbb{R}$ and $a \in \mathbb{R}$ or a symbol ∞ or $-\infty$ that is the limit of some sequence in S, and let L be a real number or symbol $+\infty$ or $-\infty$. We write $\lim_{x\to a^S} f(x) = L$ if

$$f \text{ is a function defined on } S,$$

and

$$\text{for every sequence } (x_n) \text{ in } S \text{ with limit } a, \text{ we have } \lim_{n\to\infty} f(x_n) = L.$$

Recall the definition of continuity, now we can say that a function f is continuous at a in $\operatorname{dom}(f) = S \iff \lim_{x\to a^S} f(x) = f(a)$. Also notice that when limits exist, they are unique. In other words, there is only one L equals to $\lim_{x\to a^S} f(x)$.

Now let's define the various standard limit concepts for functions.

Definition 3.5.2.

(a) For $a \in \mathbb{R}$ and a function f we write $\lim_{x\to a} f(x) = L$ provided $\lim_{x\to a^S} f(x) = L$ for some set $S = J\backslash\{a\}$ where J is an open interval containing a. Such a limit $\lim_{x\to a^S}$ is called the *[two-sided]* limit of f at a. Note that neither $f(a)$ needs to be defined or $\lim_{x\to a} f(x)$ needs to be equal $f(a)$, unless we want to say f is continuous at a.

(b) For $a \in \mathbb{R}$ and a function f we write $\lim_{x\to a^+} f(x) = L$ provided $\lim_{x\to a^S} f(x) = L$ for some open interval $S = (a, b)$. This is called the *[right-hand]* limit. Again f need not be defined at a.

(c) For $a \in \mathbb{R}$ and a function f we write $\lim_{x\to a^-} f(x) = L$ provided $\lim_{x\to a^S} f(x) = L$ for some open interval $S = (c, a)$. This is called the *[left-hand]* limit.

(d) For a function f we write $\lim_{x\to\infty} f(x) = L$ provided $\lim_{x\to\infty^S} f(x) = L$ for some interval $S = (c, \infty)$. Likewise, For a function f we write $\lim_{x\to-\infty} f(x) = L$ provided $\lim_{x\to-\infty^S} f(x) = L$ for some interval $S = (-\infty, b)$

Theorem 3.5.3. *Let f_1 and f_2 be functions for which the limits $L_1 = \lim_{x\to a^S} f_1(x)$ and $L_2 = \lim_{x\to a^S} f_2(x)$ exist and are finite. Then*

(i) $\lim_{x\to a^S}(f_1 + f_2)(x)$ *exists and equals* $L_1 + L_2$*;*

(ii) $\lim_{x\to a^S}(f_1 f_2)(x)$ *exists and equals* $L_1 L_2$*;*

(iii) $\lim_{x\to a^S}(f_1/f_2)(x)$ *exists and equals* L_1/L_2 *provided* $L_2 \neq 0$ *and* $f_2(x) \neq 0$ *for* $x \in S$*.*

Proof. By the assumption since both f_1 and f_2 are defined on S and a is the limit of some sequence in S, clearly the functions $f_1 + f_2$ and $f_1 f_2$ are defined on$S$ and so is f_1/f_2 if

$f_2(x) \neq 0$ for $x \in S$.

By the assumption, for every sequence (x_n) in S with limit a, we have $L_1 = \lim_{n\to\infty} f_1(x_n)$ and $L_2 = \lim_{n\to\infty} f_2(x_n)$. Now by the basic theorems of the limits, we have

$$\lim_{n\to\infty}(f_1+f_2)(x_n) = \lim_{n\to\infty} f_1(x_n) + \lim_{n\to\infty} f_2(x_n) = L_1 + L_2$$

and

$$\lim_{n\to\infty}(f_1 f_2)(x_n) = \left[\lim_{n\to\infty} f_1(x_n)\right] \cdot \left[\lim_{n\to\infty} f_2(x_n)\right] = L_1 L_2$$

Similar proof for (iii). ■

Theorem 3.5.4. *Let f be a function for which the limit $L = \lim_{x\to a^S} f(x)$ exists and is finite. If g is a function defined on $\{f(x) : x \in S\} \cup \{L\}$ that is continuous at L, then $\lim_{x\to a^S} g \circ f(x)$ exists and equals $g(L)$.*

Proof. First note that $g \circ f$ is defined on S by our assumptions. Consider a sequence (x_n) in S with limit a. Then we have $L = \lim_{n\to\infty} f(x_n)$. Since g is continuous at L, it follows that

$$g(L) = \lim_{n\to\infty} g(f(x_n)) = \lim_{n\to\infty} g \circ f(x_n)$$

Hence $\lim_{n\to a^S} g \circ f(x_n) = g(L)$. ■

Be careful that for this theorem to work, g needs to be **continuous** at L.

Theorem 3.5.5. *Let f be a function defined on a subset S of $\mathbb{R}$, let a be a real number that is the limit of some sequence in S, and let L be a real number, then $\lim_{x\to a^S} f(x) = L$ if and only if*

for each $\epsilon > 0$ there exists $\delta > 0$ such that $x \in S$ and $|x-a| < \delta$ imply $|f(x) - L| < \epsilon$ (1)

Proof.

$\Longrightarrow$: Suppose $\lim_{n\to a^S} f(x) = L$ but (1) does not hold. This means there exists $\epsilon > 0$ such that for each $\delta > 0$ and $n \in \mathbb{N}$**WHY???**, there exists $x_n \in S$ such that $|x_n - a| < \delta$ but $|f(x) - L| \geq \epsilon$. Hence x_n is a sequence in S with limit a but $\lim_{n\to\infty} f(x_n) = L$ fails. This is a contradiction.

$\Longleftarrow$: Consider an arbitrary sequence (x_n) in S such that $\lim_{n\to\infty} x_n = a$. Thus, choose $\epsilon = \delta$ and there exists N such that

$$n > N \implies |x_n - a| < \delta \implies |f(x_n) - L| < \epsilon.$$

The last implication comes from the assumption, so $\lim_{n\to a^S} f(x) = L$.

■

Corollary 3.5.5.1. *Let f be a function defined on $J\backslash\{a\}$ for some open interval J containing a, and let L be a real number. Then $\lim_{x\to a^s} f(x) = L$ if and only if*

$$\text{for each } \epsilon > 0 \text{ there exists } \delta > 0 \text{ such that } 0 < |x-a| < \delta \implies |f(x) - L| < \epsilon \quad (1)$$

Corollary 3.5.5.2. *Let f be a function defined on some interval (a,b), and let L be a real number. Then $\lim_{x\to a^+} f(x) = L$ if and only if*

$$\text{for each } \epsilon > 0 \text{ there exists } \delta > 0 \text{ such that } a < x < a+\delta \implies |f(x) - L| < \epsilon \quad (1)$$

Now let's give some general conditions for the limit of function in different situations: $\lim_{x\to s} f(x) = L \iff$

- L is finite:
  - $s = a$: for each $\epsilon > 0$ there exists $\delta > 0$ such that $0 < |x-a| < \delta$ implies $|f(x) - L| < \epsilon$.
  - $s = a^+$: for each $\epsilon > 0$ there exists $\delta > 0$ such that $a < x < a + \delta$ implies $|f(x) - L| < \epsilon$.
  - $s = a^-$: for each $\epsilon > 0$ there exists $\delta > 0$ such that $a - \delta < x < a$ implies $|f(x) - L| < \epsilon$.
  - $s = \infty$: for each $\epsilon > 0$ there exists $\alpha < \infty$ such that $x > \alpha$ implies $|f(x) - L| < \epsilon$.
  - $s = -\infty$: for each $\epsilon > 0$ there exists $\alpha > -\infty$ such that $x < \alpha$ implies $|f(x)-L| < \epsilon$.
- $L = +\infty$:
  - $s = a$: for each $M > 0$ there exists $\delta > 0$ such that $0 < |x-a| < \delta$ implies $f(x) > M$.
  - $s = a^+$: for each $M > 0$ there exists $\delta > 0$ such that $a < x < a + \delta$ implies $f(x) > M$.
  - $s = a^-$: for each $M > 0$ there exists $\delta > 0$ such that $a - \delta < x < a$ implies $f(x) > M$.
  - $s = \infty$: for each $M > 0$ there exists $\alpha < \infty$ such that $x > \alpha$ implies $f(x) > M$.
  - $s = -\infty$: for each $M > 0$ there exists $\alpha > -\infty$ such that $x < \alpha$ implies $f(x) > M$.
- $L = -\infty$:
  - $s = a$: for each $M < 0$ there exists $\delta > 0$ such that $0 < |x-a| < \delta$ implies $f(x) < M$.
  - $s = a^+$: for each $M < 0$ there exists $\delta > 0$ such that $a < x < a + \delta$ implies $f(x) < M$.

- $s = a^-$: for each $M < 0$ there exists $\delta > 0$ such that $a - \delta < x < a$ implies $f(x) < M$.
- $s = \infty$: for each $N < 0$ there exists $\alpha < \infty$ such that $x > \alpha$ implies $f(x) < N$.
- $s = -\infty$: for each $N < 0$ there exists $\alpha > -\infty$ such that $x < \alpha$ implies $f(x) < N$.

Theorem 3.5.6. *Let f be a function defined on $J \backslash \{a\}$ for some open interval J containing a. Then $\lim_{x\to a} f(x)$ exists $\iff$ the limits $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ both exist and are equal to each other, thereby all three limits are equal.*

Proof.

$\implies$: Suppose L is finite. Since $\lim_{x\to a} f(x) = L$, (1) in 3.5.5.1 holds, and then (1) in 3.5.5.2 also holds. Thus we have $\lim_{x\to a^+} f(x) = L$; similarly for $\lim_{x\to a^-} f(x) = L$.

If L is infinite, say $+\infty$, then consider an arbitrary $M > 0$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \implies f(x) > M \tag{2}$$

Then clearly

$$a < x < a + \delta \implies f(x) > M \tag{3}$$

and

$$a - \delta < x < a \implies f(x) > M \tag{4}$$

so $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = +\infty$.

$\impliedby$: Suppose L is finite and $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$. Consider $\epsilon > 0$, then we apply 3.5.5.2 and its analogue for a^- to get $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$a < x < a + \delta_1 \implies |f(x) - L| < \epsilon$$

and

$$a - \delta_2 < x < a \implies |f(x) - L| < \epsilon.$$

If $\delta = \min\{\delta_1, \delta_2\}$, then

$$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon,$$

so $\lim_{x\to a} f(x) = L$ by 3.5.5.1.

If L is infinite, say $+\infty$, so $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = +\infty$ then for each $M > 0$ there exists $\delta_1 > 0$ such that (2) holds, and there exists $\delta_2 > 0$ so that (3) holds. Then (1) holds with $\delta = \min\{\delta_1, \delta_2\}$. We conclude $\lim_{x\to a} f(x) = +\infty$.

∎

Chapter 4

Sequences and Series of Functions

4.1 Power Series

Definition 4.1.1. Given a sequence $(a_n)_{n=0}^{\infty}$ of real numbers and $x_0 \in \mathbb{R}$, the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is called a power series, which is a function of x provided it converges for some or all x. Note that any power series always converges at $x = x_0$ (with convention $0^0 = 1$). One of the following holds for a power series with coefficients (a_n):

(a) The power series converge for all $x \in \mathbb{R}$;

(b) The power series converges only for $x = x_0$;

(c) The power series converges for all x in some bounded interval centered at x_0; the interval may be open, half-open, or closed.

Theorem 4.1.2. *For the power series $\sum a_n(x-x_0)^n$, let*

$$\beta = \limsup |a_n|^{1/n}$$

and

$$R := \begin{cases} \frac{1}{\beta} & \text{if } 0 < \beta < \infty, \\ \infty & \text{if } \beta = 0, \\ 0 & \text{if } \beta = \infty. \end{cases}$$

Then

(i) The power series converges for $|x-x_0| < R$;

(ii) The power series diverges for $|x-x_0| > R$.

We call R the radius of convergence *for the power series. Note that we need to check $|x-x_0| = R$ cases individually.*

Proof. The proof follows easily from the Root Test 2.8.4. Define $\alpha_x = \limsup |a_n(x-x_0)^n|^{1/n}$, then we have

$$\alpha_x = \limsup |a_n(x-x_0)^n|^{1/n} = \limsup |x-x_0||a_n|^{1/n} = |x| \cdot \limsup |a_n|^{1/n} = \beta|x-x_0|$$

Now we need to consider three different cases:

1. Suppose $0 < R < +\infty$. Then $\alpha_x = \frac{|x-x_0|}{R}$. If $|x-x_0| < R$, then $\alpha_x < 1$, so the series converge by the root test. Likewise, if $|x-x_0| > R$, then $\alpha_x > 1$ and the series diverges.

2. Suppose $R = +\infty$. Then $\beta = 0$ and $\alpha_x = 0$ no matter what $|x-x_0|$ is. Hence the power series converges for all x.

3. Suppose $R = 0$. Then $\beta = +\infty$ and $\alpha_x = +\infty$ for $x \neq x_0$. Thus the series diverges for $|x-x_0| > 0 = R$ by the root test.

■

Corollary 4.1.2.1. *If* $\lim \left|\frac{a_n}{a_n+1}\right|$ *exists, then it is equal to the radius of convergence of the power series.*

Proof. If $\lim \left|\frac{a_n}{a_{n+1}}\right|$ exists, so does $\lim \left|\frac{a_{n+1}}{a_n}\right| = \alpha$. By 2.6.2, $\limsup |a_n|^{\frac{1}{n}} = \alpha$. Then

$$\lim \left|\frac{a_n}{a_{n+1}}\right| \begin{cases} \frac{1}{\alpha} & \text{if } 0 < \alpha < \infty, \\ \infty & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha = \infty, \end{cases}$$

which is equal to R, the radius of convergence. ■

Definition 4.1.3 (Interval of Convergence)**.** The interval of convergence of the power series $\sum a_n(x - x_0)^n$ is the set $\{x \in \mathbb{R}$: the series of real numbers $\sum a_n(x - x_0)^n$ converges$\}$.

4.2 Uniform Convergence

Definition 4.2.1 (Pointwise Convergence)**.** Let (X, d) be a metric space. Let (f_n) be a sequence of real-valued functions defined on a set $E \subseteq X$. The sequence (f_n) *converges pointwise* on E to a function f defined on E if

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for each} \quad x \in S.$$

We often write $\lim f_n = f$ *pointwise [on E]* or $f_n \to f$ *pointwise [on E]*

Observe that saying $f_n \to f$ *pointwise [on E]* is equivalent to the following:

for each $\epsilon > 0$ and x in E there exists N such that $|f_n(x) - f(x)| < \epsilon$ for $n > N$.

Definition 4.2.2 (Uniform Convergence (i))**.** Let (X, d) be a metric space. Let (f_n) be a sequence of real-valued functions defined on a set $E \subseteq X$. The sequence (f_n) *converges uniformly* on E to a function f defined on E if

for each $\epsilon > 0$ there exists a number N such that
$|f_n(x) - f(x)| < \epsilon$ for all $x \in E$ and all $n \geq N$.

We write $\lim f_n = f$ *uniformly [on E]* or $f_n \to f$ *uniformly [on E]*

Remark.
- Comparing to pointwise convergence, here for each $\epsilon > 0$, N works for all the $x \in E$.
- Note that if $f_n \to f$ uniformly on E and if $\epsilon > 0$, then there exists N such that $f(x) - \epsilon < f_n(x) < f(x) + \epsilon$ for *all* $x \in E$ and $n \geq N$. i.e. for $n \geq N$ the graph of f_n lies in the strip between the graphs of $f - \epsilon$ and $f + \epsilon$.

Definition 4.2.3 (Uniform Convergence (ii))**.** Let (X, d) be a metric space. Let (f_n) be a sequence of real-valued functions defined on a set $E \subseteq X$. The sequence (f_n) *converges uniformly* on E to a function f defined on E if

$$\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in E\} = 0.$$

Remark. This is an alternative definition to uniform convergence. See details in worksheet 14. We can decide whether a sequence (f_n) converges uniformly to f by calculating $\sup\{|f_n(x) - f(x)| : x \in X\}$ for each n. If $f_n - f$ is differentiable, we may use calculus to find these suprema.

Example. $f_n(x) = x^n$ does not converge uniformly to $f(x) = \begin{cases} 0 & x \in [0, 1) \\ 1 & x = 1. \end{cases}$

Proof. Let $\epsilon = \frac{1}{2}$. Then for each $N \in \mathbb{N}$, since $f_N(x) = x^N$ is continuous at $x_0 = 1$, there exists $\delta > 0$ such that $|x - x_0| < \delta \implies |x^N - 1| < \frac{1}{2}$. Then $f_N\left(1 - \frac{\delta}{2}\right) = \left(1 - \frac{\delta}{2}\right)^N > \frac{1}{2}$. Thus at $x_0 = 1$, no $N \in \mathbb{N}$ can work for the definition of uniform convergence. ■

Theorem 4.2.4 (Uniform Limit Theorem)**.** *The uniform limit of a continuous function is continuous. More precisely, let (f_n) be a sequence of real-valued functions defined on $E \subseteq X$. Suppose $f_n \to f$ uniformly on E, and suppose $E = dom(f)$. If f_n is continuous at x_0 in E for each $n \in \mathbb{N}$, then f is continuous at x_0. [so if each f_n is continuous on S, then f is continuous on S.]*

Proof. The critical inequality for this proof is

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \tag{1}$$

Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that

$$n > N \implies \forall x \in E,\ |f_n(x) - f(x)| < \frac{\epsilon}{3}$$

since $f_n \to f$ uniformly on E. In particular,

$$\forall x \in E,\ |f_{N+1}(x) - f(x)| < \frac{\epsilon}{3} \tag{2}$$

Since f_{N+1} is continuous at x_0 there is a $\delta > 0$ such that

$$\forall x \in E,\ |x - x_0| < \delta \implies |f_{N+1}(x) - f_{N+1}(x_0)| < \frac{\epsilon}{3}; \tag{3}$$

Now apply (1) with $n = N + 1$, (2) twice and (3) once to conclude

$$\forall x \in E,\ |x - x_0| < \delta \implies |f(x) - f(x_0)| < 3 \cdot \frac{\epsilon}{3} = \epsilon;$$

Thus f is continuous at x_0. ■

Remark. The contrapositive of this theorem is useful to show f_n does not uniformly converge to f on E: If f is not continuous at $x_0 \in E$ but f_n is continuous at x_0, then the statement that "$f_n \to f$ uniformly on E" is **incorrect**.

4.3 More on Uniform Convergence

Definition 4.3.1 (Uniformly Cauchy). Let (X, d) be a metric space, and let $E \subseteq X$. A sequence (f_n) of functions defined on a set E is *uniformly Cauchy on S* if

$$\text{for each } \epsilon > 0 \text{ there exists a number } N \text{ such that}$$
$$|f_m(x) - f_n(x)| < \epsilon \text{ for all } x \in E \text{ and all } m, n \geq N.$$

Definition 4.3.2 (Uniform Convergence for Series of Functions). Let (X, d) be a metric space, and let $E \subseteq X$. A series of functions $\sum_{n=1}^{\infty} g_n$ on E is *uniformly convergent to the function G on E* if

$$\forall \epsilon > 0\ \exists N \in \mathbb{N}\ n \geq N \implies \left| \sum_{k=1}^{n} g_k(x) - G(x) \right| < \epsilon \text{ for all } x \in E.$$

Definition 4.3.3 (Uniform Cauchy Criterion). Let (X, d) be a metric space, and let $E \subseteq X$. A series of functions $\sum_{n=1}^{\infty} g_n$ on E satisfy the *uniform Cauchy criterion* if

$$\forall \epsilon > 0\ \exists N \in \mathbb{N}\ n \geq m \geq N \implies \left| \sum_{k=m}^{n} g_k(x) \right| < \epsilon \text{ for all } x \in E.$$

Remark. There is an analogue between the Cauchy criterion for a normal series $\sum a_k$ and the one for a series of functions $\sum g_k$: The sequence of partial sums of a series $\sum_{k=0}^{\infty} g_k$ of functions is uniformly Cauchy on a set $E \iff$ the series satisfies the *uniform Cauchy criterion* on E.

Theorem 4.3.4. *Let (X, d) be a metric space, and let $E \subseteq X$. A sequence of functions (f_n) is* uniformly Cauchy *if and only if (f_n) converges uniformly.*

Proof. $\implies$: Use $\frac{\epsilon}{2}$ and triangular inequality $|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_n(x)|$.

$\impliedby$: Since (f_n) is uniformly Cauchy, $(f_n(x_0))$ is a Cauchy sequence for each $x_0 \in E$. Thus $(f_n(x_0))$ is in $\mathbb{R}$ and thereby convergent. Now for each $x \in E$, define $f(x) = \lim_{n\to\infty} f_n(x)$. This defines a function f on E such that $f_n \to f$ pointwise on E.

Let $\epsilon > 0$. Since (f_n) is uniformly Cauchy, there is a number N such that

$$|f_m(x) - f_n(x)| < \frac{\epsilon}{2} \quad \text{for all} \quad x \in E \quad \text{and all} \quad m, n \geq N \tag{1}$$

Consider $n \geq N$ and $x \in E$. (1) tells us that $f_n(x)$ lies in $(f_N(x) - \frac{\epsilon}{2}, f_N(x) + \frac{\epsilon}{2})$ for all $n \geq N$. Therefore, $f(x) = \lim_{n\to\infty} f_n(x)$ lies in $[f_N(x) - \frac{\epsilon}{2}, f_N(x) + \frac{\epsilon}{2}]$. In other words,

$$|f_n(x) - f(x)| \leq \frac{\epsilon}{2} \quad \text{for all} \quad x \in E \quad \text{and} \quad n \geq N$$

Then of course

$$|f_n(x) - f(x)| \le \epsilon \quad \text{for all} \quad x \in E \quad \text{and} \quad n \ge N$$

Thus $f_n \to f$ uniformly on S, as desired.

■

Corollary 4.3.4.1. *$\sum_{k=0}^{\infty} g_k$ satisfies the uniform Cauchy criterion if and only if it converges.*

Theorem 4.3.5. *Consider a series $\sum_{k=0}^{\infty} g_k$ of functions on a set $S \subseteq \mathbb{R}$. Suppose each g_k is continuous on S and the series converges uniformly on S. Then the series $\sum_{k=0}^{\infty} g_k$ represents a continuous function on S.*

Proof. Each partial sum $f_n = \sum_{k=1}^{n} g_k$ is continuous and the sequence (f_n) converges uniformly on S. Hence the limit function is continuous by 4.2.4. ■

Theorem 4.3.6. *If a series $\sum_{k=0}^{\infty} g_k$ of functions satisfies the Cauchy criterion uniformly on a set S, then the series converges uniformly on S.*

Proof. Let $f_n = \sum_{k=0}^{n} g_k$. The sequence (f_n) of partial sums is uniformly Cauchy on S, so (f_n) converges uniformly on S by 4.3.4. ■

Theorem 4.3.7 (Weierstrass M-test)**.** *Let (M_k) be a sequence of nonnegative real numbers where $\sum M_k < \infty$, i.e., $\sum M_k$ converges. If $|g_k(x)| \le M_k$ for all x in a set E and $k \in \mathbb{N}$, then $\sum g_k$ converges uniformly on E.*

Proof. We want to verify the Cauchy criterion of such $\sum g_k$ on S. Let $\epsilon > 0$. Since the series $\sum M_k$ converges, it satisfies the Cauchy criterion in 2.8.1. So there exists a number N such that

$$n \ge m > N \implies \sum_{k=m}^{n} M_k < \epsilon.$$

Hence if $n \ge m > N$ and x is in S, then

$$\left|\sum_{k=m}^{n} g_k(x)\right| \le \sum_{k=m}^{n} |g_k(x)| \le \sum_{k=m}^{n} M_k < \epsilon.$$

Thus the series $\sum g_k$ satisfies the Cauchy criterion uniformly on S, and 4.3.6 shows that it converges uniformly on S. ■

Theorem 4.3.8. *If the series $\sum g_n$ converges uniformly on a set S, then*

$$\lim_{n\to\infty} \sup\{|g_n(x)| : x \in S\} = 0.$$

Theorem 4.3.9. *Let (f_n) be a sequnece of bounded functions on a set S. If $f_n \to f$ uniformly on E, then f is a bounded function on E.*

Proof. See Ex 25.5 ∎

Theorem 4.3.10. *Let $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ be a power series with radius of convergence $R > 0$ [possibly $R = +\infty$]. If $0 < R_0 < R$, then the power series converges uniformly on $[x_0 - R_0, x_0 + R_0]$ to a continuous function.*

Proof. Consider $0 < R_0 < R$. First observe that the series $\sum a_n(x-x_0)^n$ and $\sum |a_n|(x-x_0)^n$ have the same radius of convergence $R > 0$. Since $x_0 + R_0 \in (x_0 - R, x_0 + R)$, we have $\sum |a_n|(x_0+R_0-x_0)^n < \infty$, i.e. $\sum M_n = \sum |a_n|R_0^n$ converges. Clearly we have $|a_n(x-x_0)^n| \leq |a_n|R_0^n = M_n$ for all $x \in [x_0 - R_0, x_0 + R_0]$, so the series $\sum a_n(x-x_0)^n$ converges uniformly on $[x_0 - R_0, x_0 + R_0]$ by 4.3.7. Then by 4.3.5 since each $a_n(x-x_0)^n$ is continuous, the limit of the series of the functions is also continuous. ∎

Corollary 4.3.10.1. *The power series $\sum a_n(x-x_0)^n$ with radius of convergence $R > 0$ converges to a continuous function on the open interval $(x_0 - R, x_0 + R)$.*

Proof. If $x \in (x_0 - R, x_0 + R)$, then exists $R_0 < R$ such that $x \in (x_0 - R_0, x_0 + R_0)$. By the previous theorem 4.3.10, the limit of the series $\sum a_n(x-x_0)^n$ is continuous on $[x_0-R_0, x_0+R_0]$ and so is at x. ∎

Theorem 4.3.11 (Dini's Theorem)**.** *If (f_n) is a sequence of continous functions on $[a, b]$ such that $(f_n(x))$ is nondecreasing for each $x \in [a, b]$ and $f_n \to f$ pointwise for some continuous function f, then $f_n \to f$ uniformly on $[a, b]$.*

Proof. See worksheet 15 ∎

Lemma 4.3.12. *If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence 1 and the series converges at $x = 1$, then f is continuous on $[0, 1]$.*

Proof. See worksheet 15. ∎

Theorem 4.3.13 (Abel's Theorem)**.** *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with finite positive radius of convergence R. If the series converges at $x = R$, then f is continuous at $x = R$; if the series converges at $x = -R$, then f is continuous at $x = -R$.*

Proof.

Case 1. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence 1 and the series converges at $x = 1$. We will prove f is continuous on $[0, 1]$. We need to show that f is continuous on $[0,1]$. By subtracting a constant from f, we may assume $f(1) = \sum_{n=0}^{\infty} a_n = 0$. Let $f_n(x) = \sum_{k=0}^{n} a_k x^k$ and $s_n = \sum_{k=0}^{n} a_k = f_n(1)$ for $n = 0, 1, 2, \ldots$.. Since $f_n(x) \to f(x)$ pointwise on $[0,1]$ and each f_n is continuous, by 4.2.4 it suffices to show $f_n \to f$ uniformly on $[0,1]$. By 4.3.4 it suffices to show the convergence is uniformly Cauchy.

For $m < n$, we have

$$\begin{aligned} f_n(x) - f_m(x) &= \sum_{k=m+1}^{n} a_k x^k \\ &= \sum_{k=m+1}^{n} (s_k - s_{k-1}) x^k \\ &= \sum_{k=m+1}^{n} s_k x^k - x \sum_{k=m+1}^{n} s_{k-1} x^{k-1} \\ &= \sum_{k=m+1}^{n} s_k x^k - x \sum_{k=m}^{n-1} s_k x^k \end{aligned}$$

and therefore

$$f_n(x) - f_m(x) = s_n x^n - s_m x^{n+1} + (1-x) \sum_{k=m+1}^{n-1} s_k x^k \tag{1}$$

By the definition of s_n, we have $\lim s_n = \sum_{k=0}^{\infty} a_k = f(1) = 0$. Given $\epsilon > 0$, there is an integer N so that $|s_n| < \frac{\epsilon}{3}$ for all $n \geq N$. Then for $n > m \geq N$ and $x \in [0,1)$, we have

$$\begin{aligned} \left| (1-x) \sum_{k=m+1}^{n-1} s_k x^k \right| &\leq \frac{\epsilon}{3} (1-x) \sum_{k=m+1}^{n-1} x^k \\ &= \frac{\epsilon}{3} (1-x) x^{m+1} \frac{1 - x^{n-m-1}}{1-x} \\ &< \frac{\epsilon}{3} \end{aligned} \tag{2}$$

Since $\left|(1-x)\sum_{k=m+1}^{n-1} s_k x^k\right| < \frac{\epsilon}{3}$ for $x = 1$, combining (1) and (2), for $n > m \geq N$ and $x \in [0,1]$,

$$|f_n(x) - f_m(x)| \leq |s_n| x^n + |s_m| x^{m+1} + \frac{\epsilon}{3} < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Thus the sequence (f_n) is uniformly Cauchy on $[0,1]$, and its limit f is continuous.

Case 2. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, $0 < R < \infty$, and the series converges at $x = R$. Let $g(x) = f(Rx)$ and note that

$$g(x) = \sum_{n=0}^{\infty} a_n R^n x^n \quad \text{for} \quad |x| < 1.$$

This series has radius of convergence 1, and it converges at $x = 1$. By Case 1, g is continuous at $x = 1$. Since $f(x) = g(\frac{x}{R})$, it follows that f is continuous at $x = R$.

Case 3. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, $0 < R < \infty$, and the series converges at $x = -R$. Let $h(x) = f(-x)$ and note that

$$h(x) = \sum_{n=0}^{\infty} (-1)^n a_n x^n \quad \text{for} \quad |x| < R.$$

This series for h converges at $x = R$, so h is continuous at $x = R$ by Case 2. It follows that $f(x) = h(-x)$ is continuous at $x = -R$.

■

4.4 Differentiation and Integration of Power Series

Chapter 5

Differentiation

5.1 Basic Properties of the Derivative

Limits of functions

Definition 5.1.1 (Limit of function (i)). We denote

$$\lim_{x\to c} f(x) = L$$

as that for every sequence $(x_n) \subseteq \operatorname{dom}(f)\setminus\{c\}$ such that $x_n \to c$ where $c \in \mathbb{R}\cup\{\pm\infty\}$, we have $f(x_n) \to L$.

Remark. Observe that followed by the definition above, f is continuous at c if and only if $\lim_{x\to c} f(x) = f(c)$.

Definition 5.1.2 (Limit of function (ii)). Alternatively, we can also claim the ϵ-δ definition of the limit of function as $\lim_{x\to c} f(x) = L$ if

$$\forall \epsilon > 0\ \exists \delta > 0\ 0 < |x - c| < \delta \implies |f(x) - L| < \epsilon$$

Definition 5.1.3 (Left-hand Limit). We write $\lim_{x\to c^-} f(x) = L$ if there exists $a < c$ such that $(a, c) \subseteq \operatorname{dom}(f)$ and for any sequence $(x_n) \subseteq (a, c)$ such that $x_n \to c$, we have $f(x_n) \to L$.

Theorem 5.1.4. $\lim_{x\to c} f(x)$ *exists if and only if* $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$, *in which case, all equal.*

Derivative

Definition 5.1.5. Let f be a real-valued function defined on an open interval containing a point x. Define *difference quotient* on $\operatorname{dom}(f)\backslash\{x\}$ as

$$\varphi_x(y) = \frac{f(y)-f(x)}{y-x}.$$

We say f is *differentiable* at x, or f has a derivative at a, if the limit

$$\lim_{y\to x}\varphi_x(y) = \lim_{y\to x}\frac{f(y)-f(x)}{y-x}$$

exists and is finite. We will write $f'(x)$ for the derivative of f at x:

$$f'(x) = \lim_{y\to x}\varphi_x(y) = \lim_{y\to x}\frac{f(y)-f(x)}{y-x}$$

whenever this limit exists and is finite.

Remark.
- f is *differentiable on a set E* if f is differentiable at every point $x \in E$.
- f is *differentiable* if f is differentiable at every point $x \in \operatorname{dom}(f)$.
- We can consider f' as a function. The domain of f' is the set of points at which f is differentiable; thus $\operatorname{dom}(f') \subseteq \operatorname{dom}(f)$.

Theorem 5.1.6. *If f is differentiable at a point x, then f is continuous at x.*

Proof. We are given $f'(x) = \lim_{y\to x}\frac{f(y)-f(x)}{y-x}$, and we need to prove $\lim_{y\to x} f(y) = f(x)$. We have

$$f(y) = (y-x)\frac{f(y)-f(x)}{y-x} + f(x)$$

for $y \in \operatorname{dom}(f), y \neq x$. Since $\lim_{y\to x}(y-x) = 0$ and $\lim_{y\to x}\frac{f(y)-f(x)}{y-x}$ exists and is finite, by 3.5.3(ii), $\lim(y-x)\frac{f(y)-f(x)}{y-x} = 0$. Thus $\lim_{y\to x} f(y) = f(x)$. ■

Theorem 5.1.7. *Let f and g be functions that are differentiable at the point a. Each of the functions cf, $f+g$, fg, and f/g is also differentiable at a, except f/g is $g(a) = 0$ since f/g is not defined at a in this case. The formulas are:*

(i) $(cf)'(a) = c \cdot f'(a)$;

(ii) $(f+g)'(a) = f\prime(a) + g'(a)$;

(iii) *Product rule:* $(fg)'(a) = f(a)g'(a) + f'(a)g(a)$;

(iv) *Quotient rule:* $(f/g)'(a) = \frac{[g(a)f'(a)-f(a)g'(a)]}{g^2(a)}$ *if* $g(a) \neq 0$.

Proof.

(i)
$$(cf)'(a) = \lim_{x\to a} \frac{(cf)(x) - (cf)(a)}{x-a} = \lim_{x\to a} c \cdot \frac{f(x)-f(a)}{x-a} = c \cdot f'(a)$$

(ii)
$$\frac{(f+g)(x) - (f+g)(a)}{x-a} = \frac{f(x)-f(a)}{x-a} + \frac{g(x)-g(a)}{x-a}$$

(iii) See the textbook.

(iv) See the textbook.

■

Theorem 5.1.8 (Chain Rule)**.** *If f is differentiable at a and g is differentiable at $f(a)$, then the composite function $g \circ f$ is differentiable at a and we have $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$.*

If f is differentiable on an interval I and if g is differentiable on $\{f(x) : x \in I\}$, then $(g \circ f)'$ is exactly $(g' \circ f) \cdot f'$ on I.

5.2 The Mean Value Theorem

Lemma 5.2.1. *Let f be defined on an open interval I containing x. If f attains its maximum (or minimum) at x and f is differentiable at x, then $f'(x) = 0$.*

Proof. We'll only prove the maximum case since the minimum case is similar. Suppose f attains its maximum at x. Argue by contradiction and suppose $f'(x) > 0$, i.e. $\lim_{y \to x} \varphi_x(y) > 0$. By the ϵ-δ definition of derivative, there exists $\delta > 0$ such that $0 < |y - x| < \delta$ implies $\varphi_x(y) > 0$, i.e. $\frac{f(y)-f(x)}{y-x} > 0$. It follows that there exists y such that $0 < |y - x| < \delta$ and $y > x$, so $f(y) > f(x)$ which is a contradiction. The proof of the case $f'(x) < 0$ will be similar. ■

Remark. If $f'(x) \neq 0$, then f does not attains its maximum nor minimum at x.

Theorem 5.2.2 (Rolle's Theorem)**.** *Suppose f is continuous on $[a, b]$ and differentiable on (a, b), and that $f(a) = f(b)$. Then there exists $x \in (a, b)$ such that $f'(x) = 0$.*

Proof. Since f is continuous on the compact set $[a, b]$, it attains its maximum and minimum on $[a, b]$. If a or b achieves the maximum, then so is b or a. Thus there must be some $x \in (a, b)$ achieves the minimum. Then use the above lemma. If neither a or b achieves the maximum, then Thus there must be some $x \in (a, b)$ achieves the maximum. Then use the above lemma. ■

Theorem 5.2.3 (Mean Value Theorem)**.** *Let f be a continuous function on $[a, b]$ which is differentiable on (a, b). Then there exists $x \in (a, b)$ such that $f'(x) = \frac{f(b)-f(a)}{b-a}$.*

Proof. Consider $g(x) = \big(f(b) - f(a)\big)x - (b - a)f(x)$ and apply Rolle's theorem. ■

Corollary 5.2.3.1. *If f is differentiable on (a, b), and $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b).*

Proof. Show the contrapositive. ■

Corollary 5.2.3.2. *If f, g are differentiable on (a, b) and $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f = g + C$ for some $c \in \mathbb{R}$.*

Proof. We have $(f - g)'(x) = 0$ for all x. By the previous corollary, $f - g$ is a constant function, so $f = g + C$. ■

Corollary 5.2.3.3. *Let f be a differentiable function on (a, b).*

(i) If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing;

(ii) If $f'(x) < 0$ for all $x \in (a, b)$, then f is strictly decreasing;

Proof.

(i) Suppose $x_1, x_2 \in (a, b)$ and $x_1 < x_2$. Then f is differentiable on (x_1, x_2) and by **??**, there exist $c \in (x_1, x_2) \subset (a, b)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0$$

which implies $f(x_2) > f(x_1)$.

(ii) Similar to (i).

■

Theorem 5.2.4 (Generalized Mean Value Theorem)**.** *Suppose f and g are continuous on $[a, b]$ and differentiable on (a, b). Prove that there exists $x \in (a, b)$ such that*

$$f'(x)(g(b) - g(a)) = g'(x)(f(b) - f(a)).$$

Proof. Consider $h(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x)$. Show that $h(a) = h(b)$ and then apply Rolle's Theorem 5.2.2. ■

Theorem 5.2.5. *If f is a differentiable function on (a, b) with bounded derivative, then f is uniformly continuous on (a, b).*

Proof. Let $\epsilon > 0$. Since f' is bouned, there exists $M > 0$ such that $|f'(x)| \leq M$ for every $x \in (a, b)$. Take $\delta = \epsilon/M$. Suppose $x, y \in (a, b)$ such that $x < y$ and $|x - y| < \delta = \epsilon/M$. By MVT 5.2.3, $\exists c \in (x, y)$ $\left|\frac{f(x)-f(y)}{x-y}\right| = |f'(c)| \leq M$. Thus $|f(x) - f(y)| \leq M|x - y| < M \cdot \epsilon/M = \epsilon$, and hence f is uniformly continuous on (a, b). ■

Remark. The converse does not hold generally. For example, consider $f(x) = \sqrt{x}$ on $(0, 1)$.

Theorem 5.2.6 (Intermediate Value Theorem for Derivatives)**.** *Let f be a differentiable function on (a, b). If $a < x < y < b$ and c is between $f'(x)$ and $f'(y)$, then there exists $z \in (x, y)$ such that $f'(z) = c$.*

5.3 L'Hospital's Rule

Theorem 5.3.1. *Suppose* f, g *are differentiable on* (a, b) *and* $g'(x) \neq 0$ *for where* $-\infty \leq a < b \leq \infty$. *Let* $s \in \{a, b\}$. *If* $\lim_{x\to s} \frac{f'(x)}{g'(x)} = L$ *(*$-\infty \leq L \leq \infty$*) and either*

(i) $\lim_{x\to s} f(x) = \lim_{x\to s} g(x) = 0$*; or*

(ii) $\lim_{x\to s} g(x) = \pm\infty$,

then $\lim_{x\to s} \frac{f(x)}{g(x)} = L$.

Chapter 6

Useful Tricks

1. Here is one of the most important techniques in real analysis.
   (a) If $a \le b + \epsilon$ for any $\epsilon > 0$, then $a \le b$.
   (b) If $a \ge b - \epsilon$ for any $\epsilon > 0$, then $a \ge b$.
   (c) If $|a - b| < \epsilon$ for any $\epsilon > 0$, then $|a - b| = 0$.
2. Let S be a bounded nonempty subset of $\mathbb{R}$ and suppose $\sup S \notin S$. Then there is a (strictly) increasing sequence (s_n) of points in S such that $\lim s_n = \sup S$.
3. A point x is a limit point of a set $E \subseteq X$ if and only if $x = \lim x_n$ for some sequence x_n of points in $E \backslash \{x\}$.
4. Let (s_n) be a convergent sequence.
   - If $s_n \ge a$ for all but finitely many n, then $\lim s_n \ge a$.
   - If $s_n \le b$ for all but finitely many n, then $\lim s_n \le b$.
5. (Squeeze Theorem) If $a_n \le s_n \le b_n$ for all n and $\lim a_n = \lim b_n = s$, then $\lim s_n = s$.
6. Assume all $s_n \neq 0$ and that the limit $L = \lim \left|\frac{s_{n+1}}{s_n}\right|$ exists.
   (a) If $L < 1$, then $\lim s_n = 0$.
   (b) If $L > 1$, then $\lim |s_n| = +\infty$.
7. The set $\mathbb{Q}$ of rational number can be listed as a sequence (r_n). Given any real number a there exists a subsequence (r_{n_k}) of (r_n) converging to a.
8. Given two **convergent** sequences (s_n) and t_n. If there exists $N \in \mathbb{N}$ such that $s_n \le t_n$ for all $n \ge N_0$, then $\lim s_n \le \lim t_n$.
9. In general, if $A \subseteq B$, then $\inf A \ge \inf B$ and $\sup A \le \sup B$.