Math 104 Homework 6

UC Berkeley, Summer 2021

Due by Friday, July 30, 11:59pm PDT

---

1. (Ross 18.10) Suppose f is continuous on $[0,2]$ and $f(0) = f(2)$. Prove that there exist $x, y \in [0,2]$ such that $|y - x| = 1$ and $f(x) = f(y)$. (Hint: Consider $g(x) = f(x+1) - f(x)$ on $[0,1]$.)

2. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on an open interval (a,b), then f is *bounded* on (a,b), i.e. there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in (a,b)$.

3. (a) Let f and g be two continuous real-valued functions on $\mathbb{R}$. Prove that if $f(q) = g(q)$ for every $q \in \mathbb{Q}$, then $f(x) = g(x)$ for all $x \in \mathbb{R}$.
(b) Let (X, d_X) and (Y, d_Y) be two metric spaces, and let f and g be two continuous functions from X to Y. Formulate and prove a generalization of part (a).

4. For any rational number $q \in \mathbb{Q}$, let $\varphi(q) := \min\{n \in \mathbb{N} : \exists m \in \mathbb{Z} \text{ such that } q = \frac{m}{n}\}$. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \\ \frac{1}{\varphi(x)} & \text{if } x \in \mathbb{Q}. \end{cases}$$

Prove that f is discontinuous at every $x \in \mathbb{Q}$ and continuous at every $x \in \mathbb{R} \setminus \mathbb{Q}$.

5. (a) Let (X, d) be a metric space. Consider the metric space $(X \times X, d^*)$ where $d^*((x,y),(u,v)) = \max\{d(x,u), d(y,v)\}$ (see Homework 5 Problem 1.) Show that the original metric $d : X \times X \to \mathbb{R}$ is a uniformly continuous real-valued function on the metric space $X \times X$.
(b) Let E be a nonempty compact subset of X, and let $\delta = \sup\{d(x,y) : x, y \in E\}$. Use part (a) and Homework 5 Problem 1(d) to prove that there exist $x, y \in E$ such that $d(x,y) = \delta$ (cf. Homework 4 Problem 3.)

6. (a) Let (X, d) be a metric space, and let A be any nonempty subset of X. Define $f : X \to \mathbb{R}$ by $f(x) := d(x, A) = \inf\{d(x,y) : y \in A\}$ (see Homework 4 Problem 4.) Show that f is uniformly continuous on X. (Hint: Carefully argue the following skeleton of implications: $d(x,A) \leq d(x,a) \leq d(x,y) + d(y,a) \Rightarrow d(y,a) \geq d(x,A) - d(x,y) \Rightarrow d(y,A) \geq d(x,A) - d(x,y) \Rightarrow d(x,A) - d(y,A) \leq d(x,y)$.)
(b) Let E be a nonempty compact subset of X. Use part (a) to show that there exists $x_0 \in E$ such that $f(x_0) = \inf\{d(x,A) : x \in E\}$. In particular, if $A = \{a\}$ is a *singleton* (a set with only one element), then E has a closest element to a (cf. Homework 4 Problem 4.)
(c) Prove that if E is a nonempty compact subset of X and A is a closed subset of X and $E \cap A = \emptyset$, then $\inf\{d(x,a) : x \in E, a \in A\} > 0$ (there is a "gap" between E and A.)
(d) Find a counterexample to show that the conclusion in part (c) can fail if E is assumed to be closed but not compact.

7. Let (X, d_X) be a discrete metric space, and let (Y, d_Y) be any metric space. Prove that any function $f : X \to Y$ is continuous.

8. Let (X, d) be a metric space. A *contraction* is a continuous function $f : X \to X$ with the property that there exists $c < 1$ that $d(f(x), f(y)) \leq c \cdot d(x, y)$ for all $x, y \in X$. Prove that if X is complete, then every contraction on X has a unique *fixed point*. (A fixed point of f is an element $x \in X$ such that $f(x) = x$.) (*Hint*: Construct a sequence beginning with some $x_0 \in X$ by repeatedly applying f; then argue that the sequence is Cauchy and hence convergent by completeness of X and verify that the limit is in fact a fixed point. Don't forget to show uniqueness.)

9. Let (X, d) be a metric space, and let $f : X \to \mathbb{R}$ be a continuous function. Define $Z(f) := \{x \in X : f(x) = 0\}$. Prove that $Z(f)$ is a closed subset of X.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(U)$ is open for every open set $U \subseteq \mathbb{R}$. Prove that f is monotonic.