Q1

Let $f(0) = f(2) = c$. Then we have

$$g(0) = f(0+1) - f(0) = f(1) - c,$$
$$g(1) = f(1+1) - f(1) = c - f(1).$$

If $f(1) = c$, then $g(0) = g(1) = 0 \implies f(1) = f(0)$ and $f(2) = f(1)$, as desired.
If $f(1) > c$, then $g(1) < 0 < g(0) \implies \exists x_0 \in [0,1]\ f(x_0+1) - f(x_0) = g(x_0) = 0$ by intermediate value theorem. Thus we have $|(x_0+1) - x_0| = 1$ and $f(x_0+1) = f(x_0)$ as desired.
If $f(1) < c$, then $g(0) < 0 < g(1) \implies \exists x_0 \in [0,1]\ f(x_0+1) - f(x_0) = g(x_0) = 0$ by intermediate value theorem. Thus we have $|(x_0+1) - x_0| = 1$ and $f(x_0+1) = f(x_0)$ as desired, completing the proof.

Q2

(Contrapositive) Suppose f is unbounded on (a,b), i.e., $\forall M < 0\ \exists x \in (a,b)\ |f(x)| > M$. Thus we can construct a sequence $(x_n) \in (a,b)$ such that $\forall n \in \mathbb{N}\ |f(x_n)| > n$. Since (x_n) is bounded in (a,b), it has a convergent subsequence $(x_{n_k}) \in (a,b)$, which is also Cauchy sequence. Clearly $\forall k \in \mathbb{N}\ |f(x_{n_k})| > n_k$ which implies $f(x_{n_k})$ is not convergent and hence not Cauchy. Thus f is not uniformly continuous on (a,b).

Q3

(a) Since f and g are continuous on $\mathbb{R}$, $f-g$ is also continuous on R. Let $r \in \mathbb{R}\backslash\mathbb{Q}$. Suppose $f(r) \neq g(r)$, i.e. $(f-g)(r) = c_r \neq 0$. Let $\epsilon = |c_r|$, then for each $\delta > 0$ $\exists q \in \mathbb{Q}$ such that

$$\begin{aligned} r < q < r + \delta \text{ and } |(f-g)(q) - (f-g)(r)| &= |0 - |c_r|| \\ &= |c_r| \\ &\geq \epsilon \end{aligned}$$

implying that $f-g$ is not continuous at r which is a contradiction. Thus $f(r) = g(r)$ for each $r \in \mathbb{R}\backslash\mathbb{Q}$. Since $f(q) = g(q)$ for every $q \in \mathbb{Q}$, we have $f(x) = f(x)$ for every $x \in \mathbb{R}$.

(b)

Q4

Q5

Q6

Q7

Q8

Q9

Q10