# Math 104 Homework 5

UC Berkeley, Summer 2021

Due by Friday, July 23, 11:59pm PDT

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**1.** Let $(X, d_X)$ and $(Y, d_Y)$ be two metric spaces. Let $X \times Y := \{(x, y) : x \in X, y \in Y\}$ and define the function $d : (X \times Y) \times (X \times Y) \to \mathbb{R}$ by

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.$$

(a) Show that $d$ defines a metric on $X \times Y$.
(b) Show that $E$ is a compact set in $X$ and $F$ is a compact set in $Y$, then $E \times F$ is compact in $X \times Y$.

**2.** Prove that if $\sum a_n$ is a convergent series of nonzero terms then $\sum \frac{1}{a_n}$ diverges.

**3.** (Ross 14.8) Show that if $\sum a_n$ and $\sum b_n$ are two convergent series of nonnegative real numbers, then $\sum \sqrt{a_n b_n}$ converges. (Hint: Show that $\sqrt{a_n b_n} \leq a_n + b_n$ for all $n$.)

**4.** (Ross 14.14) Let $(a_n)$ be a sequence of real numbers such that $\liminf |a_n| = 0$. Prove that there exists a subsequence $(a_{n_k})$ of $(a_n)$ such that $\sum_{k=1}^{\infty} a_{n_k}$ converges.

**5.** Give an example of a convergent series $\sum a_n$ for which $\sum a_n^2$ diverges.

**6.** (Ross 15.7) (a) Prove that if $(a_n)$ is a nonincreasing sequence of real numbers and if $\sum a_n$ converges, then $\lim n a_n = 0$. (Hint: Consider $|a_N + a_{N+1} + \ldots + a_n|$ for suitable $N$.) Note that this gives an alternative proof that $\sum \frac{1}{n}$ diverges.

**7.** Determine whether each of the following series converges or diverges and prove it.

$$\text{(a)} \sum_{n=1}^{\infty} \frac{n}{(n+1)!} \quad \text{(b)} \sum_{n=1}^{\infty} \frac{a^n}{n!} \quad (a \in \mathbb{R}) \quad \text{(c)} \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}} \quad \text{(d)} \sum_{n=2}^{\infty} \frac{1}{n \log n}$$

**8.** (Ross 17.5) (a) Prove that for any $n \in \mathbb{N}$ the function $f(x) = x^n$ is continuous.
(b) Prove that every polynomial function $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ is continuous.

**9.** (a) Prove that the function

$$f(x) = \begin{cases} 1 \text{ if } x \geq 0, \\ 0 \text{ if } x < 0 \end{cases}$$

is discontinuous at 0.

(b) Prove that the function

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is discontinuous at 0.

**10.** Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q}, \\ 0 \text{ if } x \notin \mathbb{Q}. \end{cases}$$

Show that $f$ is discontinuous at every $r \in \mathbb{R}$.