Thursday, June 24

- No class on Monday, June 28.

Sequences

Def: A <u>sequence</u> of real numbers is function $\mathbb{N} \longrightarrow \mathbb{R}$ (our convention).

domain codomain

can represent sequences as such

$$\begin{aligned} s(1) &= s_1 \\ s(2) &= s_2 \\ s(3) &= s_3 \\ &\vdots \end{aligned} \longleftrightarrow (s_1, s_2, s_3, \dots) = (s_n)_{n \in \mathbb{N}} = (s_n).$$

Examples

- $s_n = \frac{1}{n}$: $\left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$
- $s_n = n$: $(1, 2, 3, \ldots)$
- $s_n = 0$ (for all n) : $(0, 0, \ldots)$

  $s_n \equiv 0$

  identically
- $s_n = (-1)^n$: $(-1, 1, -1, 1, \ldots)$
- $s_n = \left(1 + \frac{1}{n}\right)^n$: $\left(2, \frac{9}{4}, \frac{64}{27}, \ldots\right)$

  (converges to e)
- $(s_n) = (3, 3.1, 3.14, 3.141, 3.1415, \ldots)$

  (converges to π)

Want to formalize the idea of terms getting "closer and closer" to some number.

<u>Def</u>: A sequence (s_n) of real numbers <u>converges</u> if there exists $s \in \mathbb{R}$ such that

$$\boxed{\text{for any } \varepsilon > 0, \text{ there exists } N \in \mathbb{N} \text{ such that } n \geq N \text{ implies } |s_n - s| < \varepsilon.}$$

textbook def: N can be a non-natural #. $n > N$ instead.

Intuitively: No matter how small you pick $\varepsilon > 0$, <u>eventually</u> the sequence will stay within ε of s.

at some point, and forever after.

threshold N

In this case, we say that (s_n) <u>converges to s</u>. s is called the <u>limit</u> of (s_n).

$\boxed{\lim_{n \to \infty} s_n = s}$ or $\boxed{s_n \to s}$.

Def: If a sequence does not converge, then we say it **diverges**.

Examples:

a) $s_n = \frac{1}{n}$. Prove that $s_n \to 0$. Archimedean property

Proof: Let $\varepsilon > 0$. By A.P., there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Then if $n \geq N$, then $|s_n - 0| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$.

b) $s_n \equiv 0$. Prove that $s_n \to 0$.

Proof: Let $\varepsilon > 0$. Set $N = 1$. Then if $n \geq N$, then $|s_n - 0| = |0 - 0| = 0 < \varepsilon$.

c) $s_n = (-1)^n$. Prove that (s_n) diverges.

Proof: (Contradiction) Suppose $s_n \to s$ for some $s \in \mathbb{R}$.

Consider $\varepsilon = 1$. Then $2 = 1-(-1) = 1 - s + s - (-1) \leq \underbrace{|1-s|}_{<1} + \overbrace{\underbrace{|s-(-1)|}_{<1}}^{=|s+1|=|-s-1|} < 2$. Contradiction.

There exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |s_n - s| < 1 \Rightarrow |1-s| < 1$ and $|-1-s| < 1$.

+1 or -1

Definition: Divergence to $+\infty$ or $-\infty$.

what should it mean for a sequence to be "getting closer and closer to ∞"?

A sequence (s_n) <u>diverges to ∞</u> if for any $M \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $s_n > M$.

"no matter how large"

Likewise for (s_n) diverging to $-\infty$.

In this case, we write $\lim_{n\to\infty} s_n = \infty$.

Technicality:

converge — limit exists, equals a real number

diverge:
- diverge to ∞ or $-\infty$ — limit exists, equals ∞ or $-\infty$
- not div. to ∞ or $-\infty$ — limit does not exist.

Limit theorems for sequences

Def: A sequence (s_n) is bounded if the set $\{s_n : n \in \mathbb{N}\}$ is bounded, i.e. there exists $M > 0$ such that $|s_n| \le M$ for all n.

$\in \mathbb{R}$

Theorem: Convergent sequences are bounded.

Proof: Suppose (s_n) converges to s. Let $\varepsilon = 1$. Then there exists $N \in \mathbb{N}$ such that $n \ge N$ implies

$|s_n - s| < 1 \implies |s_n| < |s| + 1$. Set $M = \max\{|s_1|, |s_2|, \ldots, |s_{N-1}|, |s|+1\}$

$$|s_n| = |s_n - s + s| \le |s_n - s| + |s|$$
$$\Rightarrow |s_n| - |s| \le |s_n - s| < 1$$
$$\Rightarrow |s_n| < |s| + 1.$$

$|s_1| \le M$
$|s_2| \le M$
$\vdots$
$|s_{N-1}| \le M$
$n \ge N$: $|s_n| < |s| + 1 \le M$.

Then $|s_n| \le M$ for all n.

Math 104 Worksheet 3

UC Berkeley, Summer 2021

Thursday, June 24

Prove the following basic limit theorems using the rigorous definition of a limit.

1. If $r \in \mathbb{R}$ and (s_n) converges to s, then (rs_n) converges to rs, i.e. $\lim(rs_n) = r\lim(s_n)$.

Proof. (Completed) Assume $r \neq 0$, since otherwise the result is trivial. Let $\varepsilon > 0$. (The goal is to find $N \in \mathbb{N}$ such that $|rs_n - rs| < \varepsilon$ for all $n \geq N$.) Since $s_n \to s$, there exists $N \in \mathbb{N}$ such that $|s_n - s| < \varepsilon/|r|$ for all $n \geq N$. Then $|rs_n - rs| < \varepsilon$ for all $n \geq N$, as desired. □

2. If (s_n) converges to s and (t_n) converges to t, then $(s_n + t_n)$ converges to $s + t$, i.e. $\lim(s_n + t_n) = \lim s_n + \lim t_n$.

Proof. Let $\varepsilon > 0$. Goal: … Show there exists $N \in \mathbb{N}$ s.t. $n \geq N$ implies $|s_n + t_n - (s+t)| < \varepsilon$.

(*Hint:* $|s_n + t_n - (s+t)| = |(s_n - s) + (t_n - t)| \leq |s_n - s| + |t_n - t|$.)

Since $s_n \to s$, there exists $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies $|s_n - s| < \varepsilon/2$.
Since $t_n \to t$, there exists $N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies $|t_n - t| < \varepsilon/2$. Set $N = \max(N_1, N_2)$. Then if $n \geq N$,

$$|s_n + t_n - (s+t)| \leq \underbrace{|s_n - s|}_{<\varepsilon/2} + \underbrace{|t_n - t|}_{<\varepsilon/2} < \varepsilon$$

□

3. If (s_n) converges to s and (t_n) converges to t, then $(s_n t_n)$ converges to st, i.e. $\lim(s_n t_n) = (\lim s_n) \cdot (\lim t_n)$.

Proof. Since (s_n) converges, it is a bounded sequence, so there exists $M \in \mathbb{R}$ such that $|s_n| \leq M$. Let $\varepsilon > 0$. Goal: … Show there exists N s.t. $n \geq N \Rightarrow |s_n t_n - st| < \varepsilon$.

(*Hint:* $|s_n t_n - st| = |s_n t_n - s_n t + s_n t - st| \leq |s_n t_n - s_n t| + |s_n t - st| = |s_n| \cdot |t_n - t| + |t| \cdot |s_n - s|$.)

Since $t_n \to t$, $\exists N_1 : n \geq N_1 \Rightarrow |t_n - t| < \dfrac{\varepsilon}{2M}$.

Since $s_n \to s$, $\exists N_2 : n \geq N_2 \Rightarrow |s_n - s| < \dfrac{\varepsilon}{2(|t|+1)}$.

Set $N = \max(N_1, N_2)$. Then $n \geq N \Rightarrow$

$$|s_n t_n - st| \leq \underbrace{|s_n|}_{<M} \underbrace{|t_n - t|}_{<\frac{\varepsilon}{2M}} + \underbrace{|t|}_{<|t|+1} \underbrace{|s_n - s|}_{<\frac{\varepsilon}{2(|t|+1)}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□