Math 104 Homework 2 Solutions

UC Berkeley, Summer 2021

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1. (a) Prove that if $0 \neq q \in \mathbb{Q}$ and $r \in \mathbb{R} \setminus \mathbb{Q}$, then $qr \in \mathbb{R} \setminus \mathbb{Q}$.
(Note: $\mathbb{R} \setminus \mathbb{Q} = \{x \in \mathbb{R} : x \notin \mathbb{Q}\}$ is the set of irrational numbers.)
(b) Prove that the set of irrational numbers is dense in $\mathbb{R}$, i.e. for any $a, b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < r < b$.

Solution. (a) If $qr = p \in \mathbb{Q}$, then $r = \frac{1}{q} \cdot p \in \mathbb{Q}$ (since $\mathbb{Q}$ is closed under multiplication); this contradicts the premise that $r \in \mathbb{R} \setminus \mathbb{Q}$.
(b) Let $a, b \in \mathbb{R}$ with $a < b$. Then $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$, so by the density of $\mathbb{Q}$ in $\mathbb{R}$ there exists $q \in \mathbb{Q}$ such that $\frac{a}{\sqrt{2}} < q < \frac{b}{\sqrt{2}}$. Then $a < q\sqrt{2} < b$, and $q\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$ by part (a).

2. (Ross 4.16) Prove that $\sup\{r \in \mathbb{Q} : r < a\} = a$ for any $a \in \mathbb{R}$.

Solution. Let $a \in \mathbb{R}$. a is clearly an upper bound for the set $S = \{r \in \mathbb{Q} : r < a\}$. Now let $b < a$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $q \in \mathbb{Q}$ such that $b < q < a$. But then $q \in S$, so b is not an upper bound for S. Therefore a is the least upper bound for S and hence $\sup S = a$.

3. Let A and B be two nonempty bounded subsets of **positive** real numbers. Define the set

$$AB := \{ab : a \in A, b \in B\}.$$

(a) Prove that $\sup(AB) = \sup(A) \cdot \sup(B)$.
(b) Find a counterexample to show that the identity in part (a) does not hold in general if the positivity restriction on the elements of A and B is removed.

Solution. Let $\alpha = \sup A$ and $\beta = \sup B$. For any $a \in A$ and $b \in B$, we have $ab \leq \alpha\beta$, so $\alpha\beta$ is an upper bound for AB and hence $\sup(AB) \leq \alpha\beta$. To show that $\sup(AB) \geq \alpha\beta$, it suffices to show that $\sup(AB) \geq \alpha\beta - \varepsilon$ for any $\varepsilon > 0$. Let $\varepsilon > 0$. Set $\delta = \min\{\alpha, \beta, \frac{\varepsilon}{\alpha+\beta}\}$. Then $\alpha - \delta > 0$ and $\beta - \delta > 0$, and there exist $a \in A$ and $b \in B$ such that $a > \alpha - \delta$ and $b > \beta - \delta$; then

$$\sup(AB) \geq ab > (\alpha-\delta)(\beta-\delta) = \alpha\beta - (\alpha+\beta)\delta + \delta^2 > \alpha\beta - (\alpha+\beta)\delta \geq \alpha\beta - (\alpha+\beta)\frac{\varepsilon}{\alpha+\beta} = \alpha\beta - \varepsilon.$$

4. Squeeze theorem. (a) Suppose that (a_n), (b_n), and (s_n) are three sequences such that $a_n \leq s_n \leq b_n$ for all n, and $\lim a_n = \lim b_n = s$. Prove that $\lim s_n = s$.
(b) Suppose that (s_n) and (t_n) are two sequences such that $|s_n| \leq t_n$ for all n and $\lim t_n = 0$. Prove that $\lim s_n = 0$.
(c) Let (s_n) be the sequence given by $s_n = \frac{1}{n} \sin n$. Prove that $\lim s_n = 0$.

Solution. (a) Let $\varepsilon > 0$. There exist $N_1, N_2 \in \mathbb{N}$ such that $|a_n - s| < \varepsilon$ for all $n \geq N_1$ and $|b_n - s| < \varepsilon$ for all $n \geq N_2$. Set $N = \max(N_1, N_2)$. Then for all $n \geq N$, we have $s - \varepsilon < a_n \leq s_n \leq b_n < s + \varepsilon$, so $|s_n - s| < \varepsilon$.
(b) The condition $|s_n| \leq t_n$ is equivalent to $-t_n \leq s_n \leq t_n$. Then $\lim t_n = 0$ and $\lim(-t_n) = -\lim t_n = 0$, so by part (a) it follows that $\lim s_n = 0$.
(c) Since $|s_n| = |\frac{1}{n} \sin n| \leq \frac{1}{n}$ for all n and $\lim \frac{1}{n} = 0$, it follows that $\lim s_n = 0$ by part (b).
5. (Ross 8.9) Show that if (s_n) converges and $s_n \geq a$ for all but finitely many n, then $\lim s_n \geq a$.

Solution. (Contrapositive) Suppose $\lim s_n = s < a$. Let $\varepsilon = a - s$. There exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|s_n - s| < \varepsilon$, so $s_n < s + \varepsilon = a$. This shows $s_n < a$ for infinitely many n.

6. (Ross 8.10(a)) Let (s_n) be a convergence sequence, and suppose that $\lim s_n > a$. Prove that there exists $N \in \mathbb{N}$ such that $s_n > a$ for all $n \geq N$.

Solution. Suppose that $\lim s_n = s > a$. Let $\varepsilon = s - a$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|s_n - s| < \varepsilon$, so $s_n > s - \varepsilon = a$.

7. Show that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

Solution. Since $\frac{n!}{n^n} = \frac{1}{n} \cdot \frac{2}{n} \cdots \frac{n}{n} \leq \frac{1}{n}$ and $\lim \frac{1}{n} = 0$, it follows that $\lim \frac{n!}{n^n} = 0$.

8. (Ross 9.12) Let (s_n) be a sequence such that $s_n \neq 0$ for all n and $L = \lim|s_{n+1}/s_n|$ exists.
(a) Show that if $L < 1$, then $\lim s_n = 0$. (*Hint*: Find $\varepsilon > 0$ such that $L + \varepsilon < 1$, and obtain $N \in \mathbb{N}$ such that $|s_{n+1}| < (L+\varepsilon)|s_n|$ for $n \geq N$; then show $|s_n| < (L+\varepsilon)^{n-N}|s_N|$ for $n > N$. You may use that fact that for any real number a with $|a| < 1$, $\lim a^n = 0$; see example 9.7(b) in the textbook for a proof.)
(b) Show that if $L > 1$, then $\lim|s_n| = \infty$. (*Hint*: Apply (a) to the sequence $1/|s_n|$.)
(c) Show that $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ for all $a \in \mathbb{R}$.

Solution. (a) Let ε be such that $L + \varepsilon < 1$. There exists $N \in \mathbb{N}$ such that for all $n \geq N$, $L - \varepsilon < |\frac{s_{n+1}}{s_n}| < L + \varepsilon$. This implies that $|s_{n+1}| < (L + \varepsilon)|s_n|$ for all $n \geq N$. From this we deduce that $|s_n| < (L + \varepsilon)^{n-N}|s_N|$ for each $n > N$; since $L + \varepsilon < 1$, it follows that $\lim|s_n| = 0$ and hence $\lim s_n = 0$.
(b) If $L > 1$, then if (t_n) is the sequence given by $t_n = 1/s_n$, we have $\lim|\frac{t_{n+1}}{t_n}| = \lim|\frac{s_n}{s_{n+1}}| = \frac{1}{L} < 1$, so $\lim|t_n| = 0$. Thus $\lim|\frac{1}{t_n}| = \lim|s_n| = \infty$.
(c) Let $s_n = \frac{a^n}{n!}$. Then $\frac{s_{n+1}}{s_n} = \frac{a^{n+1}/(n+1)!}{a^n/n!} = \frac{a}{n+1}$, so $\lim|\frac{s_{n+1}}{s_n}| = 0$. Hence $\lim s_n = 0$ by part (a).

9. (Ross 10.6) Let (s_n) be a sequence such that $|s_{n+1} - s_n| < 2^{-n}$ for all n. Prove that (s_n) is a Cauchy sequence and hence converges. (*Hint*: $\sum_{n=N}^{\infty} 2^{-n} = 2^{-N+1}$.)

Solution. Let $\varepsilon > 0$. Let $N \in \mathbb{N}$ be such that $2^{-N+1} < \varepsilon$ (such an N exists because the sequence $s_n = 2^{-n}$ converges to 0.) Then for $m, n \geq N$ with $m \geq n$, we have

$$\begin{aligned} |s_m - s_n| &= |s_m - s_{m-1} + s_{m-1} - s_{m-2} + s_{m-2} - \ldots - s_{n+1} + s_{n+1} - s_n| \\ &\leq |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + \ldots + |s_{n+1} - s_n| \\ &\leq \sum_{n=N}^{\infty} |s_{n+1} - s_n| < \sum_{n=N}^{\infty} 2^{-n} = 2^{-N+1} < \varepsilon. \end{aligned}$$

10. Let S be a bounded nonempty subset of $\mathbb{R}$ and suppose that $\sup S \notin S$. Prove that there is a (strictly) increasing sequence (s_n) of points in S such that $\lim s_n = \sup S$.

Solution. Let $\alpha = \sup S \notin S$. Since $\alpha - 1$ cannot be an upper bound of S, there exists $s_1 \in S$ such that $s_1 > \alpha - 1$, and since $\alpha \notin S$, $s_1 < \alpha$, so $\alpha - 1 < s_1 < \alpha$. Now having already found $s_1, \ldots, s_n$ with $s_n < \alpha$, since $\max(s_n, \alpha - \frac{1}{n+1}) < \alpha$ we can find $s_{n+1} \in S$ such that $s_{n+1} > \max(s_n, \alpha - \frac{1}{n+1})$, and since $\alpha \notin S$, $s_{n+1} < \alpha$. Then by our inductive construction, (s_n) is strictly increasing, and $|s_n - \alpha| < \frac{1}{n}$ and hence $\lim s_n = \alpha$.