Math 104 Worksheet 1

UC Berkeley, Summer 2021

Monday, June 21

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1. Prove the following consequences of the field properties. Continue from the first step(s) that have been provided for some of the problems.

(a) If $a + c = b + c$, then $a = b$.

Proof. Suppose $a + c = b + c$. Then

$$a = a + 0 = a + (c + (-c)) = \dots (a+c)+(-c) = (b+c)+(-c) = b+(c+(-c))$$
$$= b+0 = b.$$

$\square$

(b) $a \cdot 0 = 0$ for all a.

Proof. $0 + a \cdot 0 = a \cdot 0 = a \cdot (0 + 0) = \dots a \cdot 0 + a \cdot 0$. By Part (a), $0 = a \cdot 0$.

(The result should follow by applying assertion (a).)

$\square$

(c) $(-a)b = -ab$ for all a, b.

Proof. $ab + (-a)b = \dots (a+(-a))b = 0 \cdot b = 0 = ab + (-ab)$. By Part (a), $(-a)b = -ab$.

(The result should follow by applying assertion (a).)

$\square$

(d) $-(-a) = a$ for all a.

Proof. $-(-a) = -(-a) + 0 = -(-a) + a + (-a) = -(-a) + (-a) + a$
$= 0 + a = a$.

$\square$

(e) $(-a)(-b) = ab$ for all a, b.

Proof. (Use assertions (c) and (d).)

$(-a)(-b) = -(a(-b)) = -(-ab) = ab$

$\square$

(f) If $ab = 0$, then $a = 0$ or $b = 0$ for all a, b.

Proof. ~~Suppose that a and b are both nonzero. (Show that $ab \neq 0$ by using assertion (b).)~~

Suppose $ab = 0$.

Case 1. $a = 0$. Done.

Case 2. $a \neq 0$.

$a^{-1}ab = a^{-1}0 = 0$

$\overset{\shortparallel}{b}$.

$\square$

$A \Rightarrow B$

Contrapositive:

not $B \Rightarrow$ not A.

2. Prove the following consequences of the properties of an ordered field. Continue from the first step(s) that have been provided for some of the problems.

(a) If $a \le b$, then $-b \le -a$.

Proof. Suppose that $a \le b$. Then

$$-b = (a + (-a)) + (-b) = a + ((-a) + (-b)) \le b + ((-a) + (-b)) = \ldots = -a.$$

□

(b) If $0 \le a$ and $0 \le b$, then $0 \le ab$.

Proof. $0 = 0 \cdot b \le \ldots ab$.

□

(c) $0 \le a^2$ for all a.

Proof. (Consider the two cases $0 \le a$ and $a \le 0$.)

Case 1. $0 \le a$. Follows from (b).

Case 2. $a \le 0$. Then $0 \le -a$. $\Rightarrow 0 \le (-a)(-a) = a^2$.

□

(d) $0 < 1$.

Proof. (First justify $0 \le 1$, then justify $0 \ne 1$.)

$0 \le 1^2 = 1$, $0 \ne 1$.

If $0 = 1$,

$x = x \cdot 1 = x \cdot 0 = 0$

(all elements $= 0$, i.e there is only one element.)

□

(e) If $0 < a$, then $0 < a^{-1}$.

Proof. Let $0 < a$, and suppose that $a^{-1} \le 0$. Then $0 \le -a^{-1}$ by assertion (a). (Show that $0 \le -1$, a contradiction.)

$0 \le -a^{-1} \Rightarrow 0 = 0 \cdot a \le -a^{-1} a = -1$.

□

(f) If $0 < a < b$, then $0 < b^{-1} < a^{-1}$.

Proof. By Part (e), $0 < b^{-1}$ and $0 < a^{-1} \Rightarrow a^{-1} b^{-1} > 0$.
by Part (b).

$a < b \Rightarrow a a^{-1} b^{-1} < b a^{-1} b^{-1}$

$b^{-1} < a^{-1}$.

□

Recall Yesterday – discussed why we might want to study $\mathbb{R}$.
($\mathbb{Q}$ not good enough).

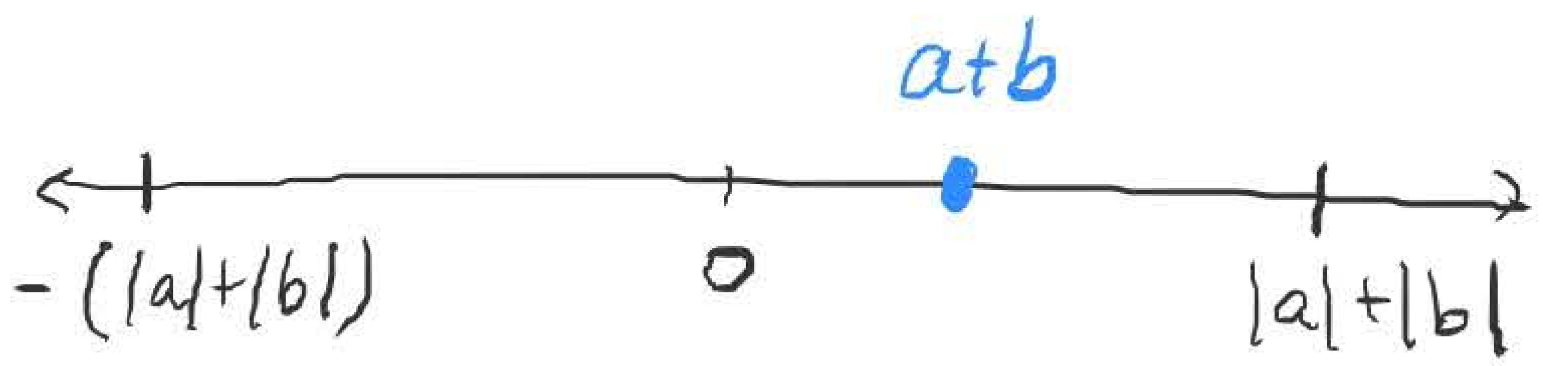
- gaps ($\sqrt{2} \notin \mathbb{Q}$)
- no LUBP

Before we proceed further:
There is a notion of distance on $\mathbb{R}$.

- **Def**: abs. value $|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0. \end{cases}$

- **Def**: The distance between $a, b \in \mathbb{R}$ is defined as $d(a,b) = |a-b|$.

distance function



Proposition

(i) $|a| \geq 0$ for all $a \in \mathbb{R}$.

(ii) $|ab| = |a| \cdot |b|$ for all $a, b \in \mathbb{R}$.

(iii) $|a+b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$.

Proof: (i) By definition.

(ii) Easy – check all 4 cases.

(iii) $-|a| \leq a \leq |a|$

$-|b| \leq b \leq |b|$.

Add: $-(|a|+|b|) \leq a+b \leq |a|+|b|$.

$\Rightarrow |a+b| \leq |a|+|b|$. $\square$

Corollary (Triangle inequality)

$|a-c| \leq |a-b| + |b-c|$ for all $a, b, c \in \mathbb{R}$.

Proof:

$|a-c| = |a-b+b-c|$

$\leq |a-b| + |b-c|$ by (iii).

Fact: $|a| \le b \iff -b \le a \le b$.

Exercise: Prove that if $a, b \in \mathbb{R}$, $a < b$, then $a < \frac{a+b}{2} < b$.

Proof: $a = \frac{a+a}{2} < \frac{a+b}{2} < \frac{b+b}{2} = b$.

Corollary: If $a, b \in \mathbb{R}$, $a < b$, then there exists $x \in \mathbb{R}$ such that $a < x < b$.

Corollary: If $a, b \in \mathbb{R}$, $a < b$, there ~~there~~ exist infinitely many $x \in \mathbb{R}$ such that $a < x < b$.

Foreshadowing – denseness of $\mathbb{Q}$ in $\mathbb{R}$.

Notion of infinity

Introduce the symbols ∞ and $-\infty$.

- for any $a \in \mathbb{R}$, $a < \infty$ and $-\infty < a$.

Important: ∞ and $-\infty$ are not real numbers!

Just symbols which we understand intuitively with the purpose of notational convenience.

e.g. $(-\infty, a) = \{x \in \mathbb{R} : x < a\}$.

$[a, \infty)$

$(-\infty, -\infty) = \mathbb{R}$.

Extended real numbers: $\mathbb{R} \cup \{\infty, -\infty\}$.

- still ordered, but not a field.

not a field.

$a + c = b + c \Rightarrow a = b$.

$1 + \infty = 2 + \infty \not\Rightarrow 1 = 2$.

$\boxed{\text{Let } S \text{ be a nonempty subset of } \mathbb{R}}$

Def: The maximum of S, or $\max S$, is an element $x \in S$ such that $s \leq x$ for all $s \in S$.

Ex: $\max\{2,3,4\} = 4$.

maximum of $(1,2) = \{x \in \mathbb{R} : 1 < x < 2\}$ does not exist.

Def: The minimum ...

Def: An upper bound of S is a real number u such that $s \leq u$ for all $s \in S$.

(note: $\max S$, if it exists, is always an upper bound of S).

10 is an upper bound for $\{2,3,4\}$.

Def: A lower bound ...

<u>Def</u>: The <u>supremum</u> of S, or <u>$\sup S$</u> is the least upper bound of S. If S is bounded above,

(i) $s \leq \sup S$ for all $s \in S$ (upper bound)

(ii) (How can we make the notion of "least" upper bound rigorous?)
Idea: anything less than $\sup S$ cannot be an upper bound.

- If $m < \sup S$, then there exists $s \in S$ such that $s > m$.

<u>**OR**</u> $\boxed{\text{For any } \varepsilon > 0}$, there exists $s \in S$ such that $s > \sup S - \varepsilon$.

Welcome to real analysis!

If S is not bounded above, then define $\sup S = \infty$.

Exercise: Define the <u>infimum</u>, or greatest lower bound.