Math 104 Worksheet 11

UC Berkeley, Summer 2021

Wednesday, July 21

Exercise 1. (a) Prove that the function $f(x) = |x|$ is continuous on $\mathbb{R}$ using the $\varepsilon - \delta$ definition of continuity. (Hint: Recall the reverse triangle inequality, $||a| - |b|| \le |a - b|$ for any $a, b \in \mathbb{R}$.)

Let $x_0 \in \mathbb{R}$. Let $\varepsilon > 0$. Choose $\delta = \varepsilon$.

Then if $|x - x_0| < \delta = \varepsilon$, then $||x| - |x_0|| \le |x - x_0| < \varepsilon$.

$f \circ g(x_0)$.

(b) Prove that if g is continuous at x_0, then the function $|g|$ is continuous at x_0.

Let $\varepsilon > 0$. There exists $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |g(x) - g(x_0)| < \varepsilon$. Then for $|x - x_0| < \delta$, have $||g(x)| - |g(x_0)|| \le |g(x) - g(x_0)| < \varepsilon$.

Exercise 2. (a) Prove that $\max(a, b) = \frac{1}{2}(a + b) + \frac{1}{2}|a - b|$ for any $a, b \in \mathbb{R}$.

$a \ge b$: $\frac{1}{2}(a+b) + \frac{1}{2}\underbrace{|a-b|}_{=a-b} = \frac{1}{2}(a+b) + \frac{1}{2}(a-b) = a$.

$a < b$: $\frac{1}{2}(a+b) + \frac{1}{2}\underbrace{|a-b|}_{=b-a} = \frac{1}{2}(a+b) + \frac{1}{2}(b-a) = b$.

(b) Prove that $\min(a, b) = -\max(-a, -b)$ for any $a, b \in \mathbb{R}$.

$a \ge b$: $-\max(-a, -b) = -(-b) = b$.

$a < b$: $-\max(-a, -b) = -(-a) = a$.

(c) Prove that if f and g are continuous at x_0, then the functions $\max(f, g)$ and $\min(f, g)$ are continuous at x_0.

$\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|$ is continuous.

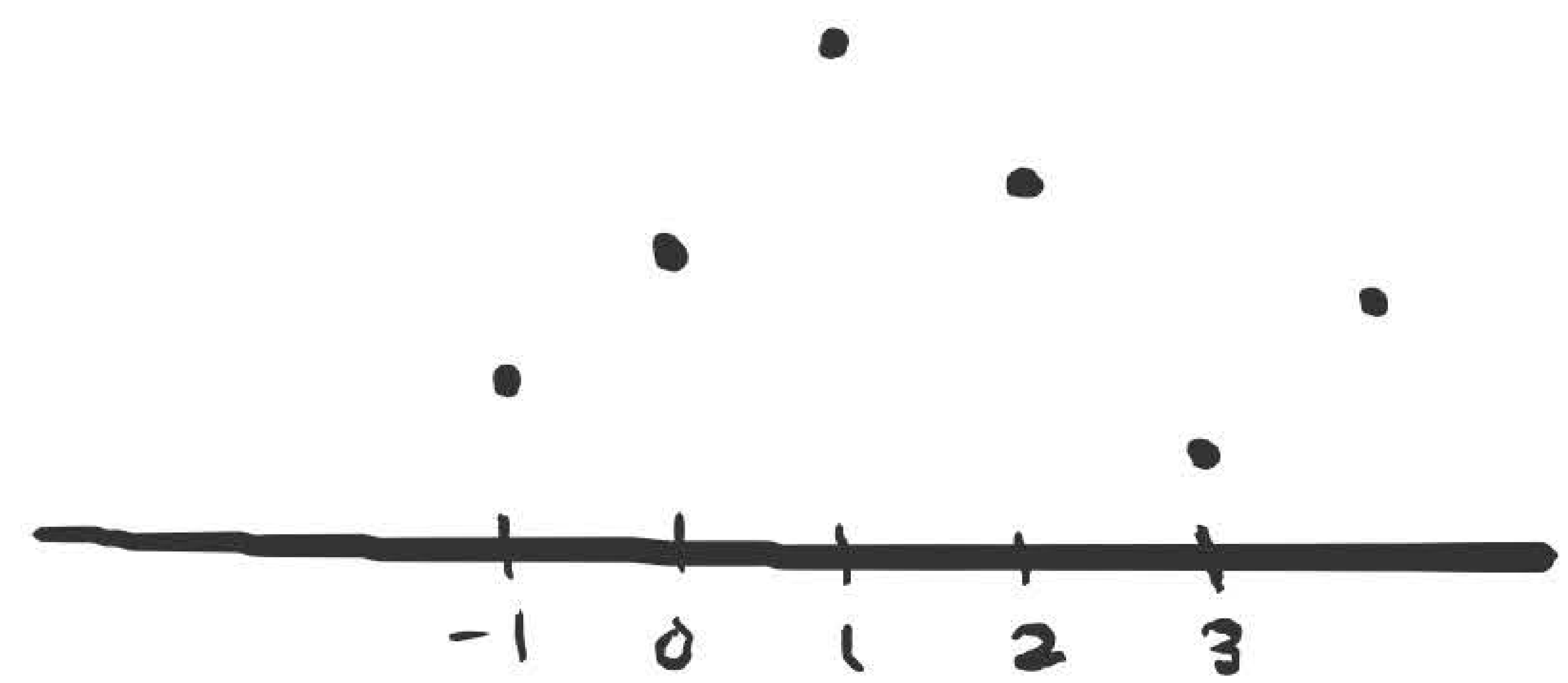
$\min(f, g) = -\max(-f, -g)$ is continuous.

Recall: Claim that any $f: \mathbb{Z} \to \mathbb{R}$ is continuous.

Let $x_0 \in \mathbb{Z}$. Let $\varepsilon > 0$. Let $\delta = \frac{1}{2}$.

Then if $x \in \mathrm{dom}(f) = \mathbb{Z}$ and $|x - x_0| < \delta = \frac{1}{2}$

then $x = x_0$, so $|f(x) - f(x_0)| = 0 < \varepsilon$.



Alt. def of continuity (sequential definition)

Theorem: f is continuous at $x_0 \in \mathrm{dom}(f)$ if and only if for every sequence $(x_n) \subseteq \mathrm{dom}(f)$ such that $x_n \to x_0$, we have that $f(x_n) \to f(x_0)$.

(f cont at $x_0 \Leftrightarrow f$ preserve limits toward x_0).

$$\lim f(x_n) = f(x_0) = f(\lim x_n).$$

Notes on processes with cyclically exchangeable increments

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1 Introduction

These notes collect together a number of results regarding processes with cyclically exchangeable increments. The main focus is on exact results with a discrete time parameter, and their combinatorial interpretations in terms of lattice walks. But it is known that such discrete time results are often the keys to analogous results for processes with cyclically exchangeable increments in continuous time, especially Lévy processes and their bridges. See for instance [**?**]

2 Known results

General notation. Let

$$S = (S_0 = 0, S_1, \dots, S_n)$$

denote a real-valued process constructed as the partial sums $S_n = X_1 + \cdots + X_n$ of a sequence of n random increments $(X_1, \dots, X_n)$. Set

$$\underline{S}_k := \min_{0 \le i \le k} S_i; \qquad \overline{S}_k := \max_{0 \le i \le k} S_i; \tag{2.1}$$

and define random argmin and argmax sets by

$$\underline{\mathcal{A}}_k := \{i : 0 \le i \le k, S_i = \underline{S}_k\}; \qquad \overline{\mathcal{A}}_k := \{i : 0 \le i \le k, S_i = \overline{S}_k\}; \tag{2.2}$$

Let $\#A$ be the number of elements in a set A. The sets $\mathcal{L}_{\le}$ and $\mathcal{L}_{<}$ of of *weak* and *strict descending ladder indices* of S are

$$\mathcal{L}_{\le} := \{k : 1 \le k \le n, S_k = \underline{S}_{k-1}\} = \{k : 0 \le k \le n, k \in \underline{\mathcal{A}}_k\} \tag{2.3}$$
$$\mathcal{L}_{<} := \{k : 1 \le k \le n, S_k < \underline{S}_{k-1}\} \subseteq \mathcal{L}_{\le} \tag{2.4}$$

with sets of *weak* and *strict ascending ladder indices* $\mathcal{L}_{\geq}$ and $\mathcal{L}_{>}$ defined similarly.
Dwass [**?**]
For $x \geq 0$ and $n \geq 1$ let

$$L_{n,x} := \#\{k \in \mathcal{L}_{\geq} : \overline{S}_k \geq \overline{S}_n - x\}$$

which is the number of weak ascending ladder indices at which the current value of the path is within distance x of the ultimate maximum of the path up to time n. Note that for each $x \geq 0$, there is the identity of events

$$(L_{n,x} = 0) = \cap_{i=1}^{n}(S_i < 0).$$

Following a similar result of Alili, Chaumont and Doney [**?**, Corollary 1], Chaumont [**?**] gave the following result.

Proposition 2.1 () [Chaumont [**?**, Lemma 3]]*For every sequence S with cyclically exchangeable increments and $\mathbb{P}(\cap_{i=1}^{n}(S_i < 0)) = 0$, and every Borel set B*

$$\mathbb{E}\frac{1(S_n = \overline{S} \in A)}{L_{n,S_n}} = \frac{\mathbb{P}(S_n \in A)}{n} \tag{2.5}$$

This is a strange formulation. It appears that on the event $(S_n = \overline{S})$ which restricts the expectation on the left side, there is the equality of random variables $L_{n,S_n} = \#\mathcal{L}_{\geq} \cap [n]$, so the formula is equivalent to

$$\mathbb{E}(1/(\#\mathcal{L}_{\geq} \cap [n]) \,|\, S_n, n \in \mathcal{L}_{\geq}) = 1/n \tag{2.6}$$

which appears to be a consequence or variant of Feller's Lemma for cyclically stationary sequences. Note also that condition $\mathbb{P}(\cap_{i=1}^{n}(S_i < 0)) = 0$ holds iff $\mathbb{P}(S_n \geq 0) = 1$. Conditioning on S_n does not effect cyclic stationarity, so the above reduces immediately to

$$\mathbb{E}(1/(\#\mathcal{L}_{\geq} \cap [n]) \,|\, n \in \mathcal{L}_{\geq}) = 1/n \tag{2.7}$$

for every cyclically stationary process.
xxx should recall and compare results of Dwass, Takacs, Feller.
The following is a clean discrete form of Vervaat's transformation which is [8, Exercise 6.1.1] This concerns an integer valued walk

$$S = (S_0 = 0, S_1, \ldots, S_n = -1)$$

from $(0, 0)$ to $(n, -1)$ with increments $X_1, \ldots, X_n$ which are are cyclically exchangeable, assuming the walk is *left continuous*, meaning that $\mathbb{P}(X_i \geq -1) = 1$. Let $\underline{S}_k := \min\{S_0, \ldots, S_k\}$ and $M_n := \min\{k \leq n : S_k = \underline{S}_n\}$ the first time at the minimum, and let $S^{\#}$ be the walk with increments shifted to start at time M_n, that is $(X_{M_n+i}, 1 \leq i \leq n)$ with $M_n + i$ taken modulo n. Then

- M_n is uniform on $\{1, \dots, n\}$
- $S^{\#} \stackrel{d}{=} (S \mid M_n = n)$
- M_n and $S^{\#}$ are independent.

The event $(M_n = n)$ is also the event that S is a first passage bridge from initial level 0 to target level -1, which first reaches the target level at time n. The exercise refers to [10] for this result. But the only discussion of such transformations in that article is at the bottom of page 14 which does something slightly different. Still, I think the above is correct, and a consequence of the central result in the book of Takács [14] that

- if $k_1, \dots, k_n$ are n nonnegative integers with sum $k \leq n$, then among the n cyclic permutations of the sequence $(k_1, \dots, k_n)$ there are exactly $n - k$ for which $k_1 + \cdots + k_r \leq r$ for all $r \in [n]$.

Earlier forms of this *cycle lemma* appear earlier in work of Dvoretzky and Motzkin [7] and Raney [11]. The article of Dershowitz and Zaks [5] reviews these results and their applications.

3 Possibly new results

I am currently interested in the following variant of Vervaat's transformation, which is implicit in the description of concave majorants of random walks in [1, §7] but which does not seem to be fully formulated or proved there. I have known this result (or at least believed it) for a few years, since that work on concave majorants.

Consider a bridge $S = (S_0, \cdots, S_n)$ with n cyclically stationary increments $X_1, \cdots, X_n$ and $S_0 = S_n = 0$. So S is a $(0,0)$ to $(n,0)$ bridge. Let

$$A := A(S) := \{i : 0 \leq i < n, S_i = \underline{S}_{n-1}\} \subseteq \{0, 1, \dots, n-1\}$$

be the random argmin set of $S_0, ...S_{n-1}$. Note that $\underline{S}_{n-1} = \underline{S}_n$ because $S_0 = S_n = 0$, but that the final index n with $S_n = 0$ is specifically excluded in the definition of A. In particular, if $S \geq 0$, meaning $\underline{S}_n \geq 0$, or $S_i \geq 0$ for all $0 \leq i \leq n$, then $|A|$ is the number of excursions of S away from 0, that is the number of returns of S to 0. In particular,

$$S \text{ is a positive excursion with no intermediate returns to } 0 \iff A = \{0\} \iff |A| = 1.$$

It is a key observation that

- if the path of the walk S and hence $|A(S)|$ are regarded as functions of the sequence of increments $(X_1, \ldots, X_n)$ subject to $\sum_{i=1}^n X_i = 0$, then the number of times $|A(S)|$ that the walk attains its minimal value on $[0, n)$ is a function that is invariant under cyclic permutations of $(X_1, \ldots, X_n)$.

This observation leads to:

Theorem 3.1 *Let S be a bridge from $(0,0)$ to $(n,0)$ with real-valued cyclically exchangeable increments $(X_1, \ldots, X_n)$. Let T be a random index which conditionally given S is uniformly distributed on the random argmin set $A = S(S)$ restricted to $\{0, 1, \ldots, n-1\}$:*

$$P(T = a \,|\, S) = \frac{1(a \in A)}{|A|} \qquad (0 \leq a < n) \tag{3.1}$$

and let S^* be the walk with increments $(X_{T+1}, \ldots. X_{T+n})$ cyclically shifted to start at time T, with $+$ modulo n. So by construction, S^* is a non-negative bridge with $|A(S^*)| = |A(S)|$ excursions. Then

- *T has uniform distribution on $\{0, \ldots, n-1\}$*
- *the distribution of S^* is that of S given $S \geq 0$, biased by the invers of its number of returns to 0, meaning that for every non-negative measurable function g of sequences $s = (s_0, \ldots, s_n)$*

$$\mathbb{E} g(S^*) = \frac{\mathbb{E}[g(S)|A|^{-1} \;\, | \, S \geq 0]}{\mathbb{E}(|A|^{-1} \;\, | \, S \geq 0)} = \frac{\mathbb{E} g(S)|A|^{-1} 1(S \geq 0)}{\mathbb{E}\, |A|^{-1} 1(S \geq 0)} \tag{3.2}$$

- *the random index T is independent of the nonnegative bridge S^*.*

Corollary 3.2 *Suppose that $(P(|A| = m) = 1$ for some $m \in [n]$, which assuming $(P(|A| = m) > 0$ may be achieved by conditioning S on the event $(|A| = m)$, without disturbing the cyclic invariance of increments of S. $S^* \stackrel{d}{=} (S \,|\, S \geq 0)$.*

For the simplest random walk bridge S with ± 1 increments , and $m = 1$, this corollary is due to Vervaat (1979). The assertion that the unique argmin this case has uniform distribution is an Exercise in Feller Vol 1.
It is easy to fall into the trap of thinking that because Corollary 3.2 holds for S with $P(|A| = m) = 1$ every fixed m, the conclusion that $S^* \stackrel{d}{=} (S \,|\, S \geq 0)$ must also be valid for S with random $|A|$. But the theorem shows this is false!

Corollary 3.3 *For a bridge S from $(0,0)$ to $(n,0)$ with cyclically exchangeable increments,*

- *the law of $|A|$ given $(S \geq 0)$ is the law of $|A|$ biased by $|A|$;*
- *the law of $|A|$ is the law of $|A|$ given $(S \geq 0)$ biased by $1/|A|$.*

Consequently, the law of $|A|$ is identical to the law of $|A|$ given $(S \geq 0)$ iff $|A|$ is constant, meaning $\mathbb{P}(|A| = m) = 1$ for some $m \in [n]$.

Put another way, the laws of S^* and of $(S \,|\, S \geq 0)$ are identical iff one or other of these laws assigns probability one to paths with some fixed number m of excursions above their minimal value. Otherwise, these laws are not identitical. They share a common family of conditional distributions given $|A|$. But their distributions of $|A|$ are necessarily different if $|A|$ has a non-degenerate distribution under either of these laws.

4 Application to simple random walk

For the simple random walk bridge S of length $2n$ with increments of ± 1, with all $\binom{2n}{n}$ bridges of semilength n equally likely, the law of S given $S \geq 0$ places equal probability $1/C_n$ on each of the C_n *Dyck paths* from $(0,0)$ to $(2n,0)$. Here C_n is the nth *Catalan number*:

$$C_n = \#[\pm 1 \text{ bridges from } (0,0) \text{to } (2n,0)] = \frac{1}{(n+1)}\binom{2n}{n}. \tag{4.1}$$

The number of Dyck paths with k returns to 0 is known [6, (6.22)] to be

$$\#[\pm 1 \text{ bridges from } (0,0) \text{to } (2n,0) \text{ with } |A| = k] = \frac{k}{2n-k}\binom{2n-k}{n} \qquad (1 \leq k \leq n). \tag{4.2}$$

For S a uniformly distributed random ± 1 bridge from $(0,0)$ to $(2n,0)$, with argmin set A, this translates into

$$\mathbb{P}(|A| = k \,|\, S \geq 0) = \frac{k}{2n-k}\binom{2n-k}{n}\frac{1}{C_n} \qquad (1 \leq k \leq n). \tag{4.3}$$

This probabilistic equivalent of (4.2) is one of many consequences of a general formula from the fluctuation theory of random walks [**?**]: for every random walk $(S_0 = 0, S_1, S_2, \ldots)$ with i.i.d. increments,

$$\sum_{n=0}^{\infty} \mathbb{P}(\underline{S}_n \geq 0, S_n = 0)x^n = \exp\left(\sum_{n=1}^{\infty} \mathbb{P}(S_n = n)\frac{x^n}{n}\right) \qquad (|x| < 1). \tag{4.4}$$

By application of the above corollary, for the simple random walk bridge S of length $2n$, the unconditional distribution of size $|A|$ of the random argmin set $S(A)$ derived from the distribution (4.2) by inverse-size biasing. That is

$$\mathbb{P}(|A| = k) = \frac{1}{2n-k}\binom{2n-k}{n}\frac{1}{B_n} \qquad (1 \le k \le n) \tag{4.5}$$

where B_n is a constant of normalization, determined by the requirement that these probabilities sum to 1:

$$B_n = \sum_{k=1}^{n} \frac{1}{2n-k}\binom{2n-k}{n} = \frac{1}{2n}\binom{2n}{n}. \tag{4.6}$$

Granted (4.5) the second equality in (4.6) is found without calculation. For $k = n$ there are exactly 2 bridges s from $(0,0)$ to $(2n,0)$ with ± 1 increments and $|A(s)| = n$, namely the bridge s with alternating increments $x_i = \pm(-1)^i$ for $0 \le i \le 2n$, and $-s$. So for $k = n$ the left side of (4.5) is the ratio 2 over $\binom{2n}{n}$. After multiplication by $\binom{2n}{n}$, the enumerative equivalent of (4.5) is that the number of ± 1 walk bridges from $(0,0)$ to $(2n,0)$ which attain their minimal value k times in $[0,2n)$ is

$$\#[\pm 1 \text{ bridges from } (0,0) \text{to } (2n,0) \text{ with } |A| = k] = \frac{2n}{2n-k}\binom{2n-k}{n} \qquad (1 \le k \le n). \tag{4.7}$$

The following display compares the well known Catalan triangle of numbers of non-negative $\pm$ bridges of length $2n$ with $|A| = k$ to the less well known but closely related triangle of numbers of $\pm$ bridges of length $2n$ with $|A| = k$, for A the set of times before time $2n$ that the bridge attains its overall minimal value. In each triangle, the range of the row inxex n is $1 \le n \le 8$ and the rangle of column index k is $1 \le k \le n$. Each entry $B_{n,k}$ in the second triangle classifing unrestricted bridges of length $2n$ by their number of excursions above their minimal value is greater than the corresponding entry $C_{n,k}$ in the first triangle classifing non-negative bridges in the same way by the factor of $2n/k$:

$$B_{n,k} = \frac{2n}{2n-k}\binom{2n-k}{n} = \frac{2n}{k}\frac{k}{2n-k}\binom{2n-k}{n} = \frac{2n}{k}B_{n,k}. \tag{4.8}$$

Some patterns implied by this relation and apparent in the displays of $C_{n,k}$ on the left and $B_{n,k}$ on the right are the simple evaluations down the main diagonal

$$B_{n,n} = \frac{2n}{n}C_{n,n} = 2 \times 1 = 2 \qquad (n \ge 1) \tag{4.9}$$

and down the subdiagonal diagonal

$$B_{n,n-1} = \frac{2n}{n-1}C_{n,n-1} = \frac{2n}{n-1} \times (n-1) = 2n \qquad (n \ge 2). \tag{4.10}$$

$$\begin{pmatrix} 1 &&&&&&& \\ 1 & 1 &&&&&& \\ 2 & 2 & 1 &&&&& \\ 5 & 5 & 3 & 1 &&&& \\ 14 & 14 & 9 & 4 & 1 &&& \\ 42 & 42 & 28 & 14 & 5 & 1 && \\ 132 & 132 & 90 & 48 & 20 & 6 & 1 & \\ 429 & 429 & 297 & 165 & 75 & 27 & 7 & 1 \end{pmatrix} \quad \begin{pmatrix} 2 &&&&&&& \\ 4 & 2 &&&&&& \\ 12 & 6 & 2 &&&&& \\ 40 & 20 & 8 & 2 &&&& \\ 140 & 70 & 30 & 10 & 2 &&& \\ 504 & 252 & 112 & 42 & 12 & 2 && \\ 1848 & 924 & 420 & 168 & 56 & 14 & 2 & \\ 6864 & 3432 & 1584 & 660 & 240 & 72 & 16 & 2 \end{pmatrix} \tag{4.11}$$

The first triangle, with nth row sum the Catalan number C_n equal to the first entry of the next row, is a transposed version of Catalan's triangle, with many enumerative interpretations indicated in the OEIS A033184. The second triangle has nth row sum which is the central binomial coefficient $\binom{2n}{n}$, that is OEIS A000984 with dozens of interpretations. Surpringly this triangle which provides an enumerative expansion of the central binomial coefficients does not seem to have its own entry in the OEIS, though large fractions of it appear in A217234, and there are doubtless many known bijections b from ± 1 bridges to other sets H_n enumerated by $\binom{2n}{n}$ through which this triangle acquires further interpretations as enumerations of H_n according to the value of $|A| \circ b^{-1}$. How interesting this might be depends on how natural the functional $|A| \circ b^{-1}$ is as an attribute of elements of H_n, and how difficult it may be to enumerate elements of H_n directly by this function.

Unlike the Catalan numbers C_n, with the direct enumerative interpretation (4.1), the sequence of rational numbers B_n defined by (4.6), so B_n is the sum over over all C_n Dyck paths s of length $2n$ of the inverse of the number of returns of s to 0, seems harder to interpret enumeratively. The B_n for $1 \leq n \leq 10$ are

$$\left\{1, \frac{3}{2}, \frac{10}{3}, \frac{35}{4}, \frac{126}{5}, 77, \frac{1716}{7}, \frac{6435}{8}, \frac{24310}{9}, \frac{46189}{5}\right\}. \tag{4.12}$$

The sequences of numerators and denominators of these rational numbers appear as A201058 and A201059, with the fraction defined as the central binomial coefficient $\binom{2n}{n}$ normalized by $2n$, without mention of the summation identity in (4.6), or any probabilistic or combinatorial interpretation of these fractions comparable to the above probabilistic interpretation of B_n/C_n as the mean value over all Dyck paths of the inverse of the number of returns to 0. The only reference to the literature in these OEIS entries is an accidental appearance of the fraction B_n due to what is evidently a typographical error, as the fraction is claimed to be a positive integer for all n, which it certainly is not. The list of all $n \leq 200$ such that B_n is an integer is

$$1, 6, 15, 28, 42, 45, 66, 77, 91, 110, 126, 140, 153, 156, 170, 187, 190$$

This is OEIS A058008. The structure of this sequence does not seem very obvious. However, a great deal is known about the prime factorization of binomial coefficients, so it would not be surprising if this sequence was subpject of known results in number theory.

4.1 Relation to known results

Révész [12, Theorem 12.25] attributes to Endre Csáki a formula which determines the distribution of $|\mathrm{argmax}\{S_0, \dots, S_n\}|$ for a simple random walk $S_n = X_1 + \cdots + X_n$ started at $S_0 = 0$, with independent increments with Bernoulli (1/2) distribution on $\{\pm 1\}$. Replacing S_n by $-S_n$ gives the same formula for the distribution of $|A_n|$ where

$$A_n := \mathrm{argmin}\{S_0, S_1, \dots, S_n\} \tag{4.13}$$

is the random set of times at which the walk attains its minimal value over times k with $0 \le k \le n$. Note that in contrast to the earlier discusison, the argmin set now includes the final index n. So if $(S_i, 0 \le i \le n)$ happens to be a positive excursion of length n, then $|A_n| = 2$. Csáki's formula is

$$\mathbb{P}(|A_n| > k) = 2^{-k}\mathbb{P}(\overline{S}_{n-k} \ge k) \qquad (0 \le k \le \lfloor n/2 \rfloor) \tag{4.14}$$

where $\overline{S}_n := \max_{0 \le i \le n} S_i$. For each $n \ge 0$ and $0 \le m \le n$ the reflection principle for simple random walk determines the tail probabilities of $\overline{S}_n$ by

$$\mathbb{P}(\overline{S}_n \ge m) = \mathbb{P}(S_n = m) + 2\mathbb{P}(S_n > m) = \mathbb{P}(S_n = m) + 2\sum_{k > m} \mathbb{P}(S_n = k) \qquad (n, m \ge 0) \tag{4.15}$$

where the point probabilities in the distribution of S_n are well known to be

$$\mathbb{P}(S_n = k) = \binom{n}{(n-k)/2} 2^{-n} \qquad (0 \le k \le n) \tag{4.16}$$

with the binomial coefficient defined to be 0 if $(n - k)/2$ is not an integer. The point probabilities in the distribution of $|A_n|$ on $\{0, 1, \dots, ??\}$ are given by

$$\mathbb{P}(|A_n| = k) = \mathbb{P}(|A_n| > k - 1) - \mathbb{P}(|A_n| > k) \qquad (0 \le k \le \lfloor n/2 \rfloor) \tag{4.17}$$

where the strict tail probabilities can be read from (4.14) and $\mathbb{P}(|A_n| > k-1) = 1$ if $k = 0$. It is well known that the binomial tail sum in (4.15) is a Gaussian hypergeometric ${}_2F_1$ evaluation which typically does not simplify. It appears that some simplification occurs in (4.17) due to the differencing. But a careful analysis would be required to obtain a credible general simplification for this point probability function. Numerical evaluations show the sequence of distributions of $|A_n|$ determined by Csáki's formula has some remarkable properties. Here is this sequence of distributions of $|A_n|$ displayed as a row-stochastic matrix with zero entries blank:

$$[P(|A_n| = k)]_{0 \le n \le 9, 1 \le k \le 5} = \begin{pmatrix} 1 & & & & \\ 1 & & & & \\ \frac{3}{4} & \frac{1}{4} & & & \\ \frac{3}{4} & \frac{1}{4} & & & \\ \frac{11}{16} & \frac{1}{4} & \frac{1}{16} & & \\ \frac{11}{16} & \frac{1}{4} & \frac{1}{16} & & \\ \frac{21}{32} & \frac{1}{4} & \frac{5}{64} & \frac{1}{64} & \\ \frac{21}{32} & \frac{1}{4} & \frac{5}{64} & \frac{1}{64} & \\ \frac{163}{256} & \frac{1}{4} & \frac{11}{128} & \frac{3}{128} & \frac{1}{256} \\ \frac{163}{256} & \frac{1}{4} & \frac{11}{128} & \frac{3}{128} & \frac{1}{256} \end{pmatrix} \tag{4.18}$$

Two immediately surprising features of this display are

- the *even-odd symmetry*, that each even numbered row is identical to the following odd numbered row; that is the identity in distribution

$$|A_{2n}| \stackrel{d}{=} |A_{2n+1}| \qquad (n \ge 0) \tag{4.19}$$

  exhibited above for $0 \le n \le 4$, and easily confirmed to be valid for $1 \le n \le 100$ using *Mathematica*. (So what is the probability that (4.19) holds for all positive integers n?! Should we ask Laplace about the sun rising?)

- the constancy of the second column from row $n = 2$ on:

$$\mathbb{P}(|A_n| = 2) = 1/4 \text{ for } n \ge 2. \tag{4.20}$$

Neither of these properties of the sequence of distributions of $|A_n|$ derived from the simple ± 1 walk seems obvious without a careful analysis.

Problem 4.1 *Establish these properties using Csáki's formula* (4.14) *or otherwise.*

Surprisingly, the even-odd symmetry (4.19) does not seem obvious by a bijective argument. But it is hard to believe there is not some relatively simple bijection to explain this symmetry. A deeper analysis of the even-odd symmetry may involve a fuller description of the stochastic mechanism whereby the random argmin set A_n and its size $|A_n|$ evolve as stochastic processes indexed by $n = 0, 1, 2, \ldots$, starting from $A_0 = \{0\}$ with $|A_0| = 1$. Note that if A_n is shifted to $A_n^- := \{a - \min A_n, a \in A_n\}$ so that always $0 \in A_n^-$, it follows from known results in the fluctuation theory of random walks that A_n converges in distribution to the set of times of renewals in a transient renewal process in which the total number of renewals after time 0 has a geometric(1/2) distribution, and the spacing between renewals, given that it is finite, is distributed according to the distribution of the return time to 0 of the unconditioned random walk. See e.g. Vervaat (1979) and Pitman-Tang (in preparation). This is related to the decomposition of (S_n) at its first minimum time in terms of the Feller chains, treated by Bertoin and others.

An immediate consequence of this general fluctuation theory is that $|A_n|$ converges in distribution to $1 + G_{1/2}$, where G_p is geometric (p) with $\mathbb{P}(G_p \geq n) = (1-p)^n$.

While this asymptotic analysis of the limit distribution of $|A_n|$ is basically known, the features of the exact distribution of A_n observed above do not seem to have been treated in the literature. Neither does there seem to be any discussion of A_n^- or of $|A_n| = |A_n^-|$ as processes. This is not entirely straightforward, as it appears that neither of these processes is Markovian without supplementary variables. Another encoding which may be more amenable is to consider the set valued process $(n - A_n, n \geq 0)$ which does appear to be Markov with stationary transition probabilities. One way to think about the evolution of $(n - A_n, n \geq 0)$ is to suppose that the increment X_{n+1} is added to the beginning rather than the end of the sequence up to time n. The analysis of this process $(n - A_n, n \geq 0)$ appears to be relatively straightforward in terms of the standard renewal theory of Markov chains with a countable state space. However, this chain $(n - A_n, n \geq 0)$ is easily seen to be null-recurrent, while its functional $|A_n| = |n - A_n|$ exhibits convergence in distribution more commonly associated with positive recurrent chains. So some care is required in this analysis.

A first step towards understanding these processes better, and possibly generalizing the discussion to other random walks besides the simple ± 1 walk, would be to provide a direct probabilistic or combinatorial proof of the even-odd symmetry. That done, the discussion reduces to an analysis of the evolution of these processes viewed just at even times. The discussion may involve the *reflecting walk*

$$R_n := S_n - \underline{S}_n \qquad (n \geq 0) \tag{4.21}$$

which is well known to be a time-homogeneous Markov chain for any walk S_n with i.i.d. increments. For the simple random walk, this chain (R_n) is has fair ± 1 increments starting from any positive value, and transition probabilities $0 \to 0$ and $0 \to 1$ with probability 1/2 each.

Consider for some fixed n the step from A_{2n} to A_{2n+1}. Observe that

- with probability 1/2 the next increment $X_{2n+1} = +1$, in which case the argmin set is unchanged: $A_{2n+1} = A_{2n}$.
- with probability 1/2 the next increment $X_{2n+1} = -1$, in which case the argmin set might or might not change:

– if $R_{2n} = 0$ then the increment of -1 creates a new minimum at a level never before attained, hence $A^{-}_{2n+1} = (2n+1) - A_{2n+1} = \{0\}$ and $|A_{2n+1}| = 1$, which might or might not be a change, according to whether $|A_{2n}| > 1$ or $|A_{2n}| = 1$.

– if $R_{2n} = 1$ then the increment of -1 creates a new minimum at the same level as the previous values which attained $\underline{S}_{2n}$. Consequently, the argmin set A_{2n} increments to $A_{2n+1} = A_{2n} \cup \{2n+1\}$, and $|A_{2n+1}| = |A_{2n}| + 1$.

– if $R_{2n} > 1$ then the increment of -1 decrements both S_{2n} and R_{2n} by 1, but that leaves $R_{2n+1} \geq 1$, so cannot create a new minimum; in this case there is no change to the argmin set: $A_{2n+1} = A_{2n}$.

Overall, considering these possibilities for the change between $|A_{2n}|$ and $|A_{2n+1}|$, the joint distribution of these two variables has an $(n+1) \times (n+1)$ transition probability matrix say

$$Q_{2n}(i,j) = \mathbb{P}(|A_{2n+1}| = j \mid |A_{2n}| = i) \qquad (i,j \geq 1) \tag{4.22}$$

with $Q_{2n}(i,j) > 0$ iff either $j = i$ or $j = i+1$ or $j = 1$. The even-odd symmetry holds for a particular n iff this transition matrix Q_{2n} leaves the distribution of A_{2n} invariant. So the problem of confirming the even-odd symmetry is reduced to checking a system of stationary equations of the form $\pi Q_{2n} = \pi$ for a specified stationary distribution π depending on n, which is obviously unique because Q_{2n} has all diagonal entries strictly positive. An analysis through this transition matrix Q_{2n} should provide an alternate descriptions of the distribution of $|A_n|$. Surprisingly, this even-odd symmetry is not immediately obvious from the above analysis. To provide an explicit formula for entries of the transition matris $Q_{2n}(i,j)$ it seems necessary to consider the joint distribution of $|A_{2n}|$ and R_{2n}, or at least the value in $\{0,1,2\}$ of $R_{2n} \vee 2$ which given R_{2n} determines what the possible transitions of $|A_{2n}|$ are. Considering that the distributions of both A_{2n} and A_{2n+1} are determined by enumerations of subsets of the set of 2^{2n+1} paths in $\{\pm 1\}^{2n+1}$, it is clear in principle that this entire probabilistic discussion can be reframed in purely combinatorial terms, which in the end should reduce matters to suitable bijective transformations on path space.

The formula $\mathbb{P}(|A_n| = 2) = 1/4$ for $n \geq 2$ suggests there may be simplifications of Csáki's formula (4.14) to give nice expressions for the sequence of probabilities $\mathbb{P}(|A_n| = k)$ or the sequence of path counts $2^n P(|A_n| = k)$ at least for small k. Following are some indications of what these sequences appear to be, cutting some corners by "experimental mathematics" using the OEIS to identify the sequences, and other settings in which the same sequences arise. Trusting that the even-odd symmetry holds for all n, it is only of interest to consider even values of n. The following matrix displays the counts

$$\#(A_{2n} = k) := 2^{2n}\mathbb{P}(|A_{2n} = k))$$

with rows indexed by $0 \le n \le 8$ and columns indexed by $1 \le k \le 9$, so the nth row sum is 2^{2n}:

$$\begin{pmatrix} 1 & & & & & & & & \\ 3 & 1 & & & & & & & \\ 11 & 4 & 1 & & & & & & \\ 42 & 16 & 5 & 1 & & & & & \\ 163 & 64 & 22 & 6 & 1 & & & & \\ 638 & 256 & 93 & 29 & 7 & 1 & & & \\ 2510 & 1024 & 386 & 130 & 37 & 8 & 1 & & \\ 9908 & 4096 & 1586 & 562 & 176 & 46 & 9 & 1 & \\ 39203 & 16384 & 6476 & 2380 & 794 & 232 & 56 & 10 & 1 \end{pmatrix} \tag{4.23}$$

Remarkably, the rows of this number triangle do not appear in the OEIS, though many of its columns can be found there. This *Csáki triangle* therefore appears to provide a unified approach to many previous enumerations, and to be worthy of further analysis. Looking at the columns one by one, the first column is

$$\#(A_{2n} = 1) = \sum_{i=0}^{n} \binom{2n}{i} = \frac{1}{2}\left(2^{2n} + \binom{2n}{n}\right) \tag{4.24}$$

which follows easily from Csáki's formula. This is OEIS A032443 which gives several combinatorial interpretations and references to

- D. Phulara and L. W. Shapiro, Descendants in ordered trees with a marked vertex, Congressus Numerantium, 205 (2011), 121-128.
- A. Bernini, F. Disanto, R. Pinzani and S. Rinaldi, Permutations defining convex permutominoes, J. Int. Seq. 10 (2007) # 07.9.7.
- M. Klazar, Twelve countings with rooted plane trees, European Journal of Combinatorics 18 (1997), 195-210; Addendum, 18 (1997), 739-740.
- Mircea Merca, A Note on Cosine Power Sums J. Integer Sequences, Vol. 15 (2012), Article 12.5.3.

but does not mention Csáki's interpretation of this sequence as the number of $\pm$ walks of length $2n$ with exactly one minimal value. The next column appears to be

$$\#(A_{2n} = 2) = 2^{2n-2} 1(n > 0) \tag{4.25}$$

which should be easily checkable from Csáki's formula. This is OEIS A000302 up to a shift of index, which has several combinatorial interpretations. Some checking should determine whether any of these can be easily put into bijective correspondence with the number of $\pm$ walks of length $2n$ with exactly two minimal values. Why this particular column of Csáki's triangle should magically simplify to 4^{n-1} for $n > 0$, while other columns have more complicated expressions, does not seem obvious, and invites a bijective explanation.

The next column

$$0, 0, 1, 5, 22, 93, 386, 1586, 6476, 26333, 106762$$

(where this and all other columns listed below are indexed by $0, 1, 2, \ldots$) appears to be

$$\#(A_{2n} = 3) = 2^{2n-3} - \binom{2n-3}{n-1} \qquad (n > 1). \tag{4.26}$$

This is OEIS A000346 up to a shift of index, again with several combinatorial interpretations, but not obviously including the present interpretation as the number of $\pm$ walks of length $2n$ with exactly three minimal values. References include the Phulara-Shapiro paper cited above, and

- T. Myers and L. Shapiro, Some applications of the sequence 1, 5, 22, 93, 386, ... to Dyck paths and ordered trees, Congressus Numerant., 204 (2010), 93-104.
- R. Bacher, On generating series of complementary plane trees arXiv:math/0409050 [math.CO], 2004.
- E. A. Bender, E. R. Canfield and R. W. Robinson, The enumeration of maps on the torus and the projective plane, Canad. Math. Bull., 31 (1988), 257-271; see p. 270.
- D. E. Davenport, L. K. Pudwell, L. W. Shapiro, L. C. Woodson, The Boundary of Ordered Trees, 2014.
- P. Flajolet and R. Sedgewick, Analytic Combinatorics, 2009; see page 185
- Mircea Merca, A Special Case of the Generalized Girard-Waring Formula J. Integer Sequences, Vol. 15 (2012), Article 12.5.7. - From N. J. A. Sloane, Nov 25 2012
- D. Merlini, R. Sprugnoli and M. C. Verri, Waiting patterns for a printer, FUN with algorithm'01, Isola d'Elba, 2001.
- D. Merlini, R. Sprugnoli and M. C. Verri, The tennis ball problem, J. Combin. Theory, A 99 (2002), 307-344 ($A_n fors = 2$).
- W. T. Tutte, On the enumeration of planar maps. Bull. Amer. Math. Soc. 74 1968 64-74.

- T. R. S. Walsh and A. B. Lehman, Counting rooted maps by genus, J. Comb. Thy B13 (1972), 122-141 and 192-218 (eq. 5, b=1).

The next column

$$0, 0, 0, 1, 6, 29, 130, 562, 2380, 9949, 41226$$

appears to be

$$\#(A_{2n} = 4) = 2^{2n-4} - \binom{2n-3}{n-2} \qquad (n > 2) \tag{4.27}$$

which is OEIS A008549$[n-2]$, again with several combinatorial interpretations, many of them involving lattice path enumerations, but again not obviously including the present interpretation as the number of $\pm$ walks of length $2n$ with exactly 4 minimal values. References include

- José Agapito, Ângela Mestre, Maria M. Torres, and Pasquale Petrullo, On One-Parameter Catalan Arrays, Journal of Integer Sequences, Vol. 18 (2015), Article 15.5.1.

- Jean-Christophe Aval, A Boussicault, P Laborde-Zubieta, M Pétréolle, Generating series of Periodic Parallelogram polyominoes, arXiv preprint arXiv:1612.03759, 2016

- R. Bacher, On generating series of complementary plane trees arXiv:math/0409050 [math.CO], 2004.

- C. Banderier, Analytic combinatorics of random walks and planar maps, PhD Thesis, 2001.

- Adrien Boussicault, P. Laborde-Zubieta, Periodic Parallelogram Polyominoes, arXiv preprint arXiv:1611.03766 [math.CO], 2016.

- A. Burstein and S. Elizalde, Total occurrence statistics on restricted permutations, arXiv preprint arXiv:1305.3177 [math.CO], 2013.

- R. Chapman, Moments of Dyck paths, Discrete Math., 204 (1999), 113-117.

- Guo-Niu Han, Enumeration of Standard Puzzles

- Milan Janjić, Pascal Matrices and Restricted Words, J. Int. Seq., Vol. 21 (2018), Article 18.5.2.

- N. G. Johansson, Efficient Simulation of the Deutsch-Jozsa Algorithm, Master's Project, Department of Electrical Engineering & Department of Physics, Chemistry and Biology, Linkoping University, April, 2015.

- M. Jones, S. Kitaev, J. Remmel, Frame patterns in n-cycles, arXiv preprint arXiv:1311.3332 [math.CO], 2013.
- Henri Mühle, Symmetric Chain Decompositions and the Strong Sperner Property for Noncrossing Partition Lattices, arXiv preprint arXiv:1509.06942v1 [math.CO], 2015.
- Ran Pan, Jeffrey B. Remmel, Paired patterns in lattice paths, arXiv:1601.07988 [math.CO], 2016.
- E. Pergola, Two bijections for the area of Dyck paths, Discrete Math., 241 (2001), 435-447.
- W.-J. Woan, Area of Catalan Paths, Discrete Math., 226 (2001), 439-444.

The next column

$$0, 0, 0, 0, 1, 7, 37, 176, 794, 3473, 14893$$

appears to be

$$\#(A_{2n} = 5) = 2^{2n-5} + \binom{2n-5}{n-3} - \binom{2n-3}{n-2} \tag{4.28}$$

which is OEIS A0064199$[n-3]$, again with several combinatorial interpretations, many of them involving lattice path enumerations, but again not obviously including the present interpretation as the number of $\pm$ walks of length $2n$ with exactly 5 minimal values. References for this one include Phularaq and Shapiro cited above, and

- Jason Bandlow and Kendra Killpatrick, An area-to-inv bijection between Dyck paths and 312-avoiding permutations,Electron. J. Combin. 8 (2001), no. 1, Research Paper 40, 16 pp.
- Miklós Bóna, Surprising Symmetries in Objects Counted by Catalan Numbers, Electronic J. Combin., 19 (2012), #P62, eq. (5).
- Pudwell, Lara; Scholten, Connor; Schrock, Tyler; Serrato, Alexa Noncontiguous pattern containment in binary trees, ISRN Comb. 2014, Article ID 316535, 8 p. (2014), Table 1.
- R. P. Stanley, F. Zanello, The Catalan case of Armstrong's conjecture on core partitions, arXiv preprint arXiv:1312.4352 [math.CO], 2013.
- T. R. S. Walsh, A. B. Lehman, Counting rooted maps by genus. III: Nonseparable maps, J. Combinatorial Theory Ser. B 18 (1975), 222-259.

This OEIS entry gives the generating function $C^3(x)x/(1-4x)$ where $C(x)$ is the Catalan GF, which may be an instance of a more general GF for other k.
The next column

$$0, 0, 0, 0, 0, 1, 8, 46, 232, 1093, 4944, 21778, 94184, 401930, 1698160, 7119516$$

appears to be

$$\#(A_{2n} = 6) = 2^{2n-6} - \frac{1}{n-1}\binom{2n-4}{n-2} \qquad (n > 2) \tag{4.29}$$

which is OEIS A0064199$[n-2]$, with the remarkable interpretation of $a_n =$ A0064199 $[n]$, as

$$a_n = 2^{2n-2} - nC_n$$

with C_n the Catalan number of non-negative walk bridges of length $2n$, and a_n equal to the sum of valley heights over all Dyck paths. Why this should be, up to a shift, the number of $\pm$ walks of length $2n$ with exactly 6 minimal values is very far from clear!. It is quite unexpected that the formula for $\#(A_{2n} = 6)$ should be simpler than for $\#(A_{2n} = 5)$. What is going on here?
The next column $\#(A_{2n} = 7)$ is

$$0, 0, 0, 0, 0, 0, 1, 9, 56, 299, 1471, 6885, 31180, 137980, 600370, 2579130$$

which does not seem to appear in OEIS, and neither does $\#(A_{2n} = 8)$

$$0, 0, 0, 0, 0, 0, 0, 1, 10, 67, 378, 1941, 9402, 43796, 198440, 880970$$

Still, there are several remarkable aspects to this. One is how simple the explicit and varied the formulas are for $\#(A_{2n} = k)$ for $1 \le k \le 7$. Presumably some reasonably simple general formula for $\#(A_{2n} = k)$ can be derived either from Csáki's formula or otherwise. Another remarkable aspect is how varied the combinatorial interpretations of these sequences are, without direct reference to their present interpretation.

5 Further references

[2] [4] [3]

6 Further notes on the literature

From the review of [5] in MR by Ira Gessel:

A sequence $x_1, x_2, \ldots, x_\ell$ of 0's and 1's is called k-dominating, for some $k = 1, 2, \ldots$, iff for every positive integer i with $1 \leq i \leq \ell$, the number of 0's in $x_1, \ldots, x_i$ is more than k times the number of 1's. The result of Dvoretzky and Motzkin is that for each sequence $x_1, \ldots, x_{m+n}$ of m 0's and n 1's, with $m \geq kn$, there exactly $m - kn$ of the $m + n$ cyclic permutations $x_j, x_{j+1}, \ldots, x_{m+n} \ldots, p_{j-1}$ for $j \in [m+n]$ that that are k-dominating. The authors give two proofs of the cycle lemma and two applications to counting trees. The number of ordered forests of s trees, each a t-ary tree, with n internal nodes and $tn + s - n$ leaves is

$$\frac{s}{tn+s}\binom{tn+s}{n}$$

The article has many references.

See also the more recent survey [9] of enumerations of trees and forests related to branching processes and random walks.

In a Galton-Watson branching process with offspring distribution $(p_0, p_1, \ldots)$ started with k individuals, the distribution of the total progeny is identical to the distribution of the first passage time to $-k$ for a random walk started at 0 which takes steps of size j with probability p_{j+1} for $j \geq -1$. The formula for this distribution, due to Otter and Good in the setting of branching processes, and to Kemperman in the setting of random walks, is a probabilistic expression of the Lagrange inversion formula for the coefficients in the power series expansion of $h^k(y)$ in terms of those of $p(x)$ for $h(y)$ defined implicitly by $h(y) = yp(h(y))$ The Lagrange inversion formula is the analytic counterpart of various enumerations of trees and forests which generalize Cayley's formula kn^{n-k-1} for the number of rooted forests labeled by $[n]$ whose set of roots is $[k]$, These known results are derived by elementary combinatorial methods without appeal to the Lagrange formula, which is then obtained as a byproduct. This approach unifies and extends a number of known identities involving the distributions of various kinds of random trees and random forests.

See also Stanley [13, Chapter 5] and [8, Chapter 6] for further discussion.

References

[1] Josh Abramson and Jim Pitman. Concave majorants of random walks and related Poisson processes. *Combin. Probab. Comput.*, 20(5):651–682, 2011.

[2] Sami Assaf, Noah Forman, and Jim Pitman. The quantile transform of simple walks and Brownian motion. *Electron. J. Probab.*, 20:no. 90, 39, 2015.

[3] L. Chaumont, D. G. Hobson, and M. Yor. Some consequences of the cyclic exchangeability property for exponential functionals of Lévy processes. In *Séminaire de Probabilités, XXXV*, volume 1755 of *Lecture Notes in Math.*, pages 334–347. Springer, Berlin, 2001.

[4] Loïc Chaumont and Gerónimo Uribe Bravo. Shifting processes with cyclically exchangeable increments at random. In *XI Symposium on Probability and Stochastic Processes*, volume 69 of *Progr. Probab.*, pages 101–117. Birkhäuser/Springer, Cham, 2015.

[5] Nachum Dershowitz and Shmuel Zaks. The cycle lemma and some applications. *European J. Combin.*, 11(1):35–40, 1990.

[6] Emeric Deutsch. Dyck path enumeration. *Discrete Math.*, 204(1-3):167–202, 1999.

[7] A. Dvoretzky and Th. Motzkin. A problem of arrangements. *Duke Math. J.*, 14:305–313, 1947.

[8] J. Pitman. *Combinatorial stochastic processes*, volume 1875 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2006. Lectures from the 32nd Summer School on Probability Theory held in Saint-Flour, July 7–24, 2002.

[9] Jim Pitman. Enumerations of trees and forests related to branching processes and random walks. In *Microsurveys in discrete probability (Princeton, NJ, 1997)*, volume 41 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 163–180. Amer. Math. Soc., Providence, RI, 1998.

[10] Jim Pitman. Brownian motion, bridge, excursion, and meander characterized by sampling at independent uniform times. *Electron. J. Probab.*, 4:no. 11, 33, 1999.

[11] George N. Raney. Functional composition patterns and power series reversion. *Trans. Amer. Math. Soc.*, 94:441–451, 1960.

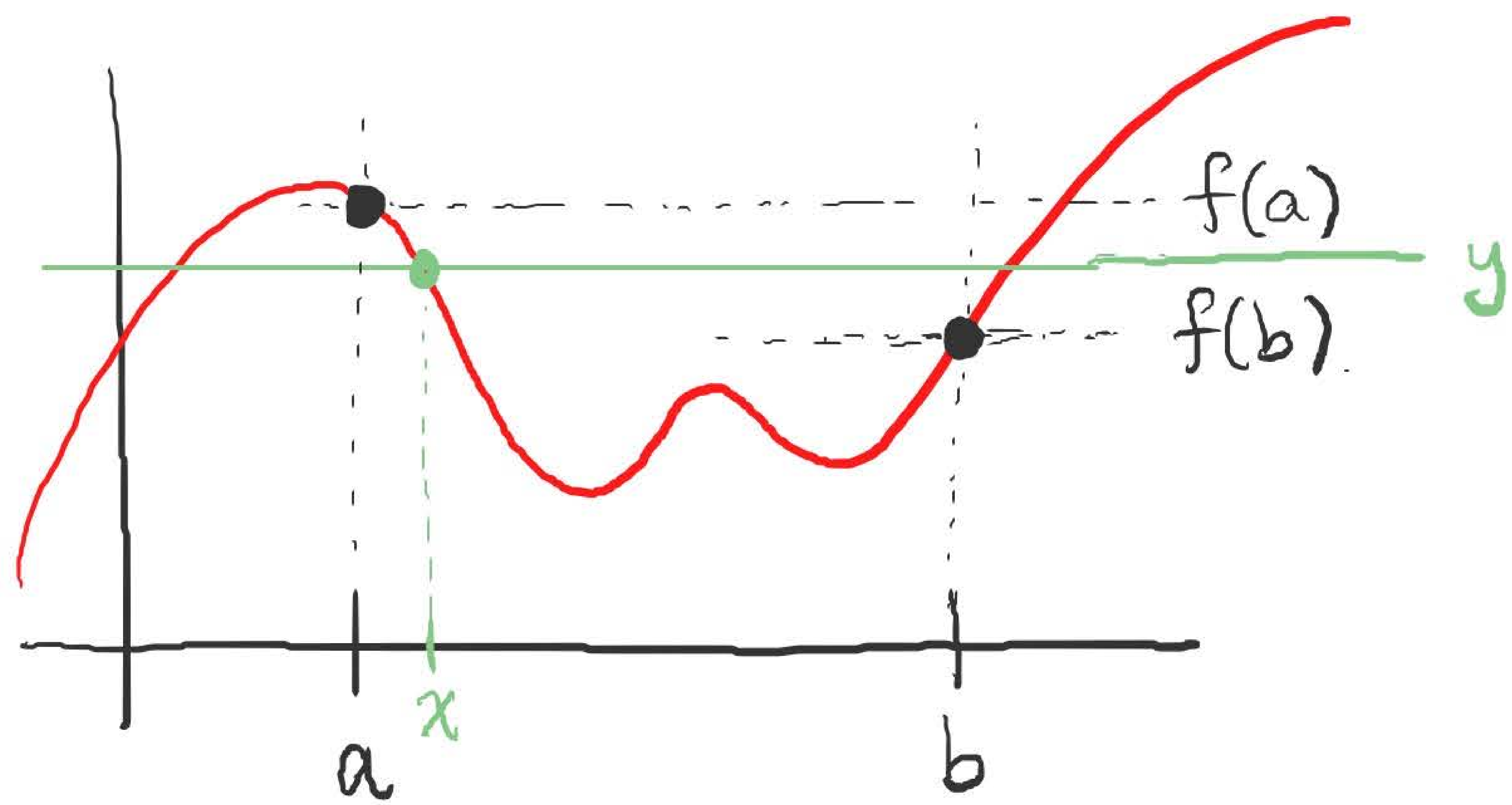
[12] Pál Révész. *Random walk in random and non-random environments*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, third edition, 2013.

[13] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[14] Lajos Takács. *Combinatorial methods in the theory of stochastic processes*. John Wiley & Sons, Inc., New York-London-Sydney, 1967.

Intermediate value theorem

If f is a real-valued function which is continuous on a closed interval $I \subseteq \mathrm{dom}(f)$, then f has the intermediate value property on I: whenever $a, b \in I$ with $a < b$, and y is (strictly) between $f(a)$ and $f(b)$, then there exists $x \in (a, b)$ such that $f(x) = y$.



Proof: $\Rightarrow]$ Suppose f is continuous at x_0.

Let $(x_n) \subseteq \mathrm{dom}(f)$ such that $x_n \to x_0$.

(Goal: Show that $f(x_n) \to f(x_0)$.)

Let $\varepsilon > 0$. There exists $\delta > 0$ such that

$x \in \mathrm{dom}(f)$, $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

Since $x_n \to x_0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|x_n - x_0| < \delta$.

Then if $n \geq N$, since $|x_n - x_0| < \delta$, then $|f(x_n) - f(x_0)| < \varepsilon$.

$\Leftarrow]$ (Contrapositive) Suppose f is NOT continuous at x_0.

There exist $\varepsilon > 0$ such that $\boxed{\text{for all } \delta > 0,}$ there exists $x \in \mathrm{dom}(f)$, $|x - x_0| < \delta$, but $|f(x) - f(x_0)| \geq \varepsilon$.

For each $n \in \mathbb{N}$, there exists $x_n \in \mathrm{dom}(f)$, $|x_n - x_0| < \frac{1}{n}$, but $|f(x_n) - f(x_0)| \geq \varepsilon$. Then $x_n \to x_0$, but $f(x_n) \not\to f(x_0)$.

Wednesday, July 21

Warm-Up: Let $C \in \mathbb{R}$. Prove that the function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = Cx$ is continuous.
(ε-δ definition)

Recall: f is continuous at $x_0 \in \text{dom}(f)$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x \in \text{dom}(f),\ |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon.$$

Def: f is continuous on $S \subseteq \text{dom}(f)$ if f is continuous at every $x_0 \in S$.

Def: f is continuous if f is continuous at every $x_0 \in \text{dom}(f)$.

Proof: Let $x_0 \in \text{dom}(f) = \mathbb{R}$. Let $\varepsilon > 0$.

Case 1: $C = 0$.
Let $\delta = 1$. Then if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| = |0 - 0| < \varepsilon$.

Case 2: $C \neq 0$.
Let $\delta = \frac{\varepsilon}{|C|}$. Then if $|x - x_0| < \delta = \frac{\varepsilon}{|C|}$, then

$$|f(x) - f(x_0)| = |Cx - Cx_0| = |C| \cdot \underbrace{|x - x_0|}_{< \frac{\varepsilon}{|C|}} < \varepsilon.$$

Used: f, g continuous at x_0 $\Longrightarrow$ $f+g$ continuous at x_0.

Proof: Suppose $x_n \longrightarrow x_0$.

$$f(x_n) \longrightarrow f(x_0), \quad g(x_n) \longrightarrow g(x_0).$$

$$\therefore \underbrace{(f+g)(x_n)}_{f(x_n)+g(x_n)} \longrightarrow \underbrace{(f+g)(x_0)}_{f(x_0)+g(x_0)}.$$

Let $\varepsilon > 0$. There exists

$\delta_1 > 0$: $|x - x_0| < \delta_1 \Rightarrow |f(x) - f(x_0)| < \frac{\varepsilon}{2}$.

$\delta_2 > 0$: $|x - x_0| < \delta_2 \Rightarrow |g(x) - g(x_0)| < \frac{\varepsilon}{2}$.

$x \in dom(f)$

Set $\delta = \min(\delta_1, \delta_2)$. $|x - x_0| < \delta \Rightarrow \left| f(x) + g(x) - (f(x_0) + g(x_0)) \right|$

$$\leq \underbrace{|f(x) - f(x_0)|}_{< \varepsilon/2} + \underbrace{|g(x) - g(x_0)|}_{< \varepsilon/2}$$

$$< \varepsilon.$$

Defining new functions from existing ones.

Suppose f, g real-valued functions, $\mathrm{dom}(f) = \mathrm{dom}(g)$.

codomain $= \mathbb{R}$

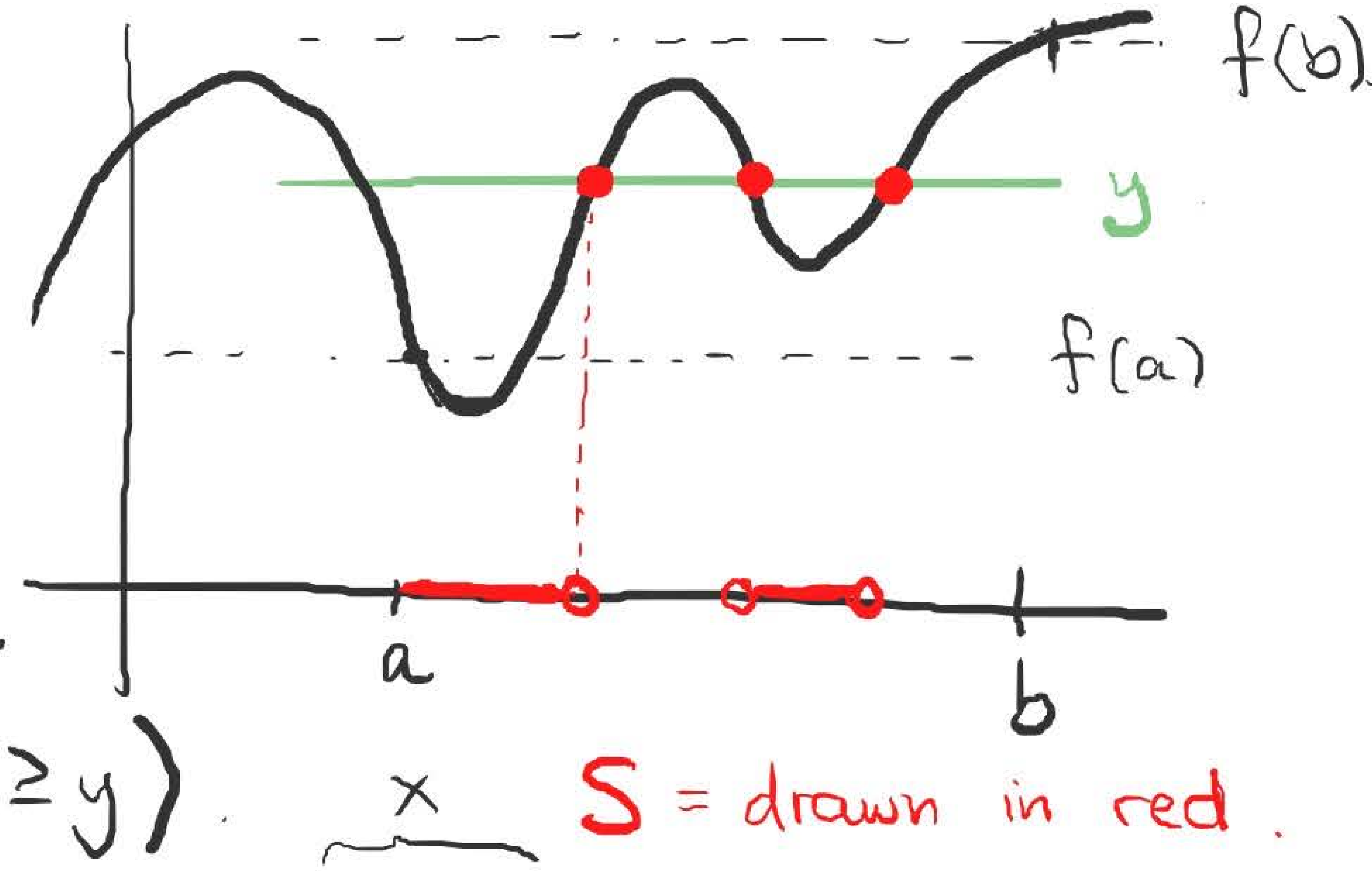
- $f+g$ $\qquad (f+g)(x) = f(x) + g(x)$
- fg
- kf, $\quad k \in \mathbb{R}$.
- f/g
- $|f|$ $\qquad |f|(x) = |f(x)|$.
- $\max(f,g)$, $\min(f,g)$.

Proposition: If f and g are continuous at x_0, then so are the above functions. $\longrightarrow$ Worksheet.

Proof: Assume (WLOG) $f(a) < y < f(b)$.

Define $S = \{ x \in [a,b] : f(x) < y \}$

(Goal: Show that $f(\underbrace{\sup S}_{x}) = y$)



S is bounded, $\sup S$ exists, $\sup S \leq b$.

Let $x = \sup S$. (Show $f(x) \leq y$ and $f(x) \geq y$).

For each $n \in \mathbb{N}$, there exists $x_n \in S$: $x_n > \sup S - \frac{1}{n}$.

Then $x_n \to x$, since f is continuous, $\underbrace{f(x_n)}_{< y} \to f(x)$.

$\Rightarrow \boxed{f(x) \leq y}$

Consider the sequence $t_n = \min\left(b, x + \frac{1}{n}\right)$. $t_n \geq x$, $t_n \to x$.

Since f is continuous, $\underbrace{f(t_n)}_{\geq y} \to f(x) \Rightarrow \boxed{f(x) \geq y}$

$\therefore f(x) = y$.