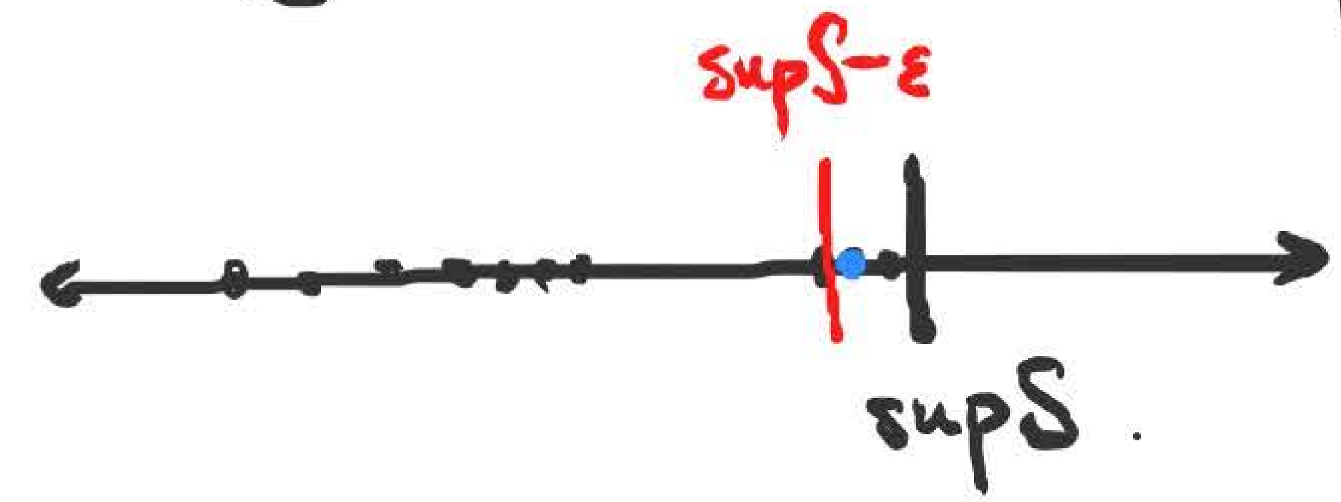
Wednesday, June 23

Official department tutors
  - see Piazza

Recall: Defined **supremum**: least upper bound
infimum: greatest lower bound.

if $S \subseteq \mathbb{R}$ is nonempty and bounded above:

- $s \leq \sup S$ for all $s \in S$.
- For any $\varepsilon > 0$, there exists $s \in S$ such that $s > \sup S - \varepsilon$.

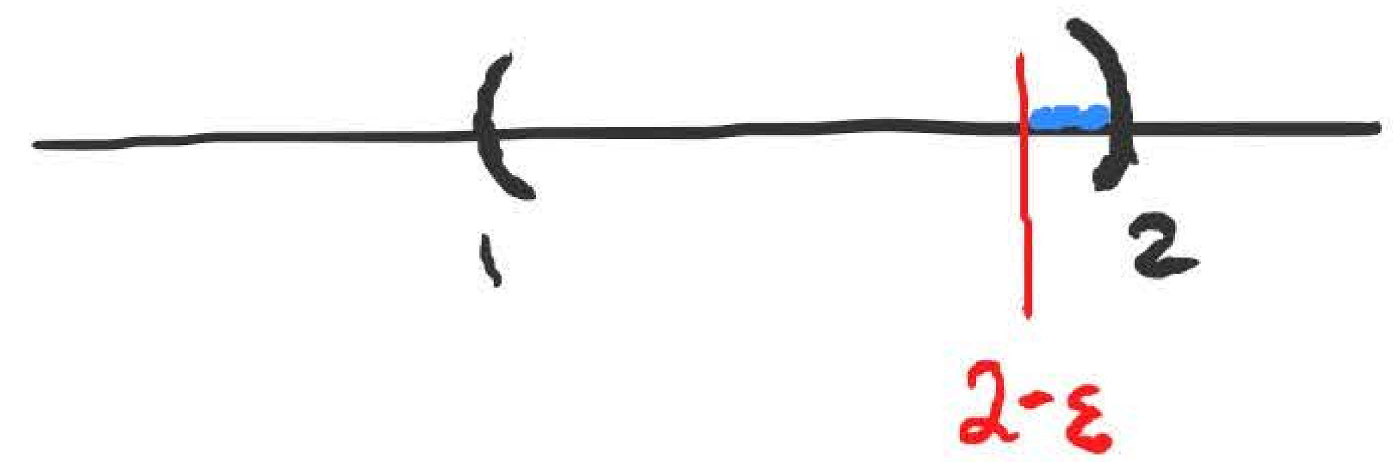


if S is not bounded above,
set $\sup S = \infty$ (convention).

Ex. $S = (1,2) = \{ x \in \mathbb{R} : 1 < x < 2 \}$.

$\max S$ does not exist.

$\sup S = 2$.



Easy facts:

- If a set S has finitely many elements, then $\max S$ exists.
- If $\max S$ exists, then $\sup S = \max S$.
- For any $S \neq \emptyset$, $\inf S \leq \sup S$. (Proof: For any $s \in S$: $\inf S \leq s \leq \sup S$).

Convention: $\inf \emptyset = \infty$

$\sup \emptyset = -\infty$

if A, then B

$\equiv$ if NOT B, then NOT A.

Math 104 Worksheet 2

UC Berkeley, Summer 2021

Tuesday, June 22

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1. The following theorem is a fundamental idea in real analysis, and it is one of the most important techniques in the subject.

Theorem.

(a) $\boxed{\text{If } a \leq b + \varepsilon \text{ for any } \varepsilon > 0}$, then $a \leq b$.

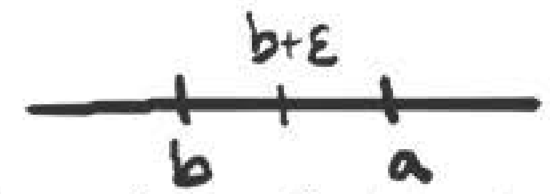
(b) If $a \geq b - \varepsilon$ for any $\varepsilon > 0$, then $a \geq b$.

negation of $a \leq b$.

$\neg(\forall \varepsilon > 0,\ a \leq b + \varepsilon)$

$\exists \varepsilon > 0\ \ \neg(a \leq b + \varepsilon)$

$\exists \varepsilon > 0 :\ a > b + \varepsilon$



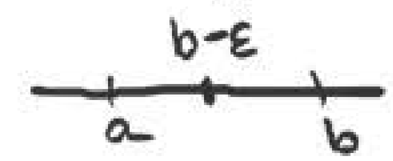
Proof. (a) Suppose that $a > b$. Let $\varepsilon = (a - b)/2 > 0$. Then $a > b + \varepsilon$, so the statement that $a \leq b + \varepsilon$ for any $\varepsilon > 0$ is not true.

negation of $a \leq b + \varepsilon$.

(b) (Your turn)

Suppose $a < b$. Let $\varepsilon = (b-a)/2 > 0$.
Then $a < b - \varepsilon$.

□



2. Given two nonempty bounded subsets A and B of $\mathbb{R}$, define the set

$$A + B = \{a + b : a \in A, b \in B\}.$$

Theorem.

(a) $\sup(A + B) = \sup(A) + \sup(B)$.

(b) $\inf(A + B) = \inf(A) + \inf(B)$.

Proof. Given two quantities x and y, if you are asked to show that $x = y$, a common technique is to show that $x \leq y$ **and** $x \geq y$, since if both are true then $x = y$.

(a) *Strategy*: We will show that both inequalities (i) $\sup(A + B) \leq \sup(A) + \sup(B)$ **and** (ii) $\sup(A + B) \geq \sup(A) + \sup(B)$ are true.

(i) For any pair of elements $a \in A$ and $b \in B$, since the supremum of a set is **an** upper bound for the set, we have that $a \leq \sup(A)$ and $b \leq \sup(B)$. Therefore, $a + b \leq \sup(A) + \sup(B)$. Since this is true for any $a \in A$ and $b \in B$, it follows that $c \leq \sup(A) + \sup(B)$ for all $c \in A + B$. That means that $\sup(A) + \sup(B)$ is an upper bound for $A + B$. (**Complete the proof by explaining why** $\sup(A + B)$ **must be less than** $\sup(A) + \sup(B)$.)

by def of $\sup(A+B)$, it is the LEAST upper bd, and $\sup A + \sup B$ is just AN upper bound

(ii) To show that $\sup(A + B) \geq \sup(A) + \sup(B)$, we will use the technique from Problem 1. Let $\varepsilon > 0$. The goal is to show that $\boxed{\sup(A + B) \geq \sup(A) + \sup(B) - \varepsilon}$. If we can find $a \in A$ and $b \in B$ such that $a + b \geq \sup(A) + \sup(B) - \varepsilon$, the boxed inequality would follow because... (why?) $\sup(A+B) \geq a + b \geq \sup A + \sup B - \varepsilon$.

Now explain why it is possible to find such a and b. (*Hint:* $\sup(A) - \frac{\varepsilon}{2}$ is **not** an upper bound for A.)

$\exists a \in A :\ a > \sup A - \frac{\varepsilon}{2}$

$\exists b \in B :\ b > \sup B - \frac{\varepsilon}{2}$

(b) (Your turn)

□

$a + b > \sup A + \sup B - \varepsilon$ ..

Recall: LUBP.

X has the LUBP if for any subset $S \subseteq X$ which is bounded above, $\sup S$ exists (in X).

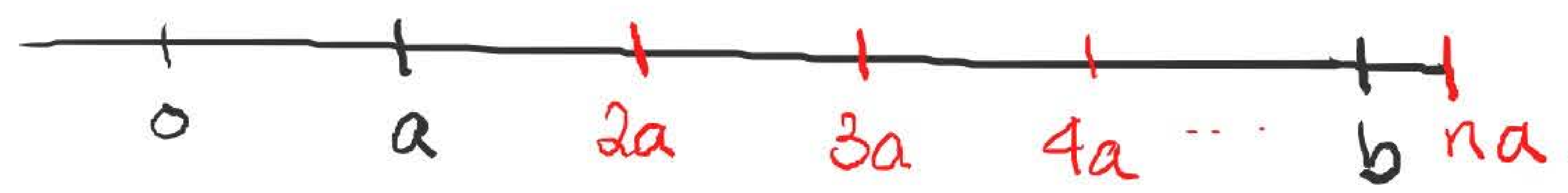
Ex. $X = \mathbb{Q}$, $S = \{ q \in \mathbb{Q} : 0 \leq q \leq \sqrt{2} \}$.

Completeness Axiom: $\mathbb{R}$ has the LUBP.

Corollary: Every subset of $\mathbb{R}$ which is bounded below has a greatest lower bound.

Archimedean property of $\mathbb{R}$

If $a>0$ and $b>0$, then there exists $n \in \mathbb{N}$ such that $na > b$.



Proof: (Contradiction)

Suppose there exist $a>0$ and $b>0$ such that $na \le b$ for all $n \in \mathbb{N}$.

Let $S = \{na : n \in \mathbb{N}\}$. So b is an upper bound for S.

By Completeness Axiom, $\sup S$ exists.

Then $\sup S - a$ is not an upper bound for S,

so there exists $m \in \mathbb{N}$ such that $\underbrace{ma}_{\text{an element of } S} > \sup S - a$.

or $\underbrace{(m+1)a}_{\in S} > \sup S$. Contradiction.

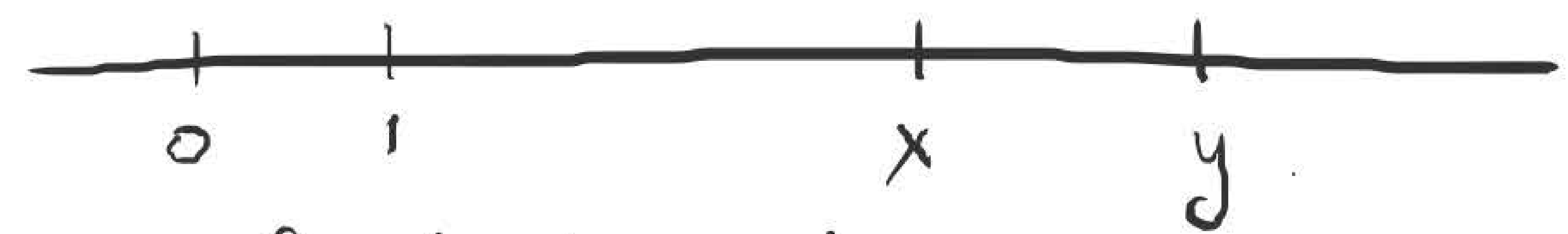
<u>Corollary</u> (Set $a=1$). For any $b>0$, there exists $n\in\mathbb{N}$ such that $n>b$.

<u>Corollary</u> (Set $b=1$) For any $a>0$, there exists $n\in\mathbb{N}$ such that $\boxed{\frac{1}{n}<a}$ or $na>1$.

more useful in practice.

Denseness of $\mathbb{Q}$ in $\mathbb{R}$.

<u>Lemma</u>: If $x,y\in\mathbb{R}$ such that $\overbrace{y-x>1}^{y>x+1}$, then there exists $m\in\mathbb{Z}$ such that $x<m<y$.



<u>Proof</u>: Case 1: $x\geq 0$.

Let $S=\{n\in\mathbb{Z}_{\geq 0}: n\leq x\}$. By Corollary of Archimedean property, S has finitely many elements, so $k=\max S$ exists.

$$x \underset{k+1\notin S}{<} \underbrace{k+1}_{\in\mathbb{Z}} \underset{k\leq x}{\leq} x+1<y.$$

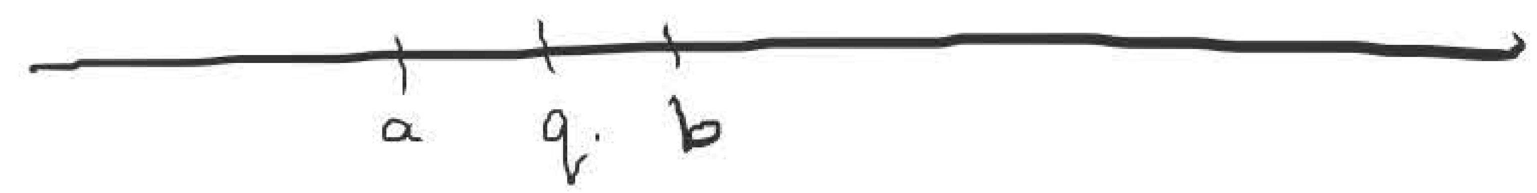
$x<k+1<y$.

Case 2: $x<0$.

Then $-x>0$. By Corollary of A.P., there exists $N\in\mathbb{N}$ such that $N>-x$. Consider $x+N>0$ and $(y+N)-(x+N)>1$. By Case 1, there exists $m\in\mathbb{Z}$ such that $x+N<m<y+N$. Then $x<m-N<y$.

Theorem: For any $a, b \in \mathbb{R}$ with $a < b$, there exists $q \in \mathbb{Q}$ such that $a < q < b$.

Proof: Want to show that there exist $m, n \in \mathbb{Z}$ such that



positive

$$a < \frac{m}{n} < b$$

or equivalently, $na < m < nb$.

By A.P., there exists $n \in \mathbb{N}$ such that $n\overbrace{(b-a)}^{>0} > 1$, i.e. $nb - na > 1$. By Lemma, there exists an integer $m \in \mathbb{Z}$ such that

$$na < m < nb.$$

Hence $a < \frac{m}{n} < b$.