University of California, Berkeley
Math 104: Introduction to Analysis

Instructor: Theodore Zhu

Midterm Exam

July 13, 2017

10:10 AM – 11:55 AM

Name: ______________________________

Student ID: _________________________

Instructions. This is a closed-book, closed-notes, closed-electronics exam. Please write carefully and clearly in the spaces provided. If you run out of space for a problem, you may continue on the reverse side of the page, or on the extra pages at the end. Cross out any work that you do not want to be graded. Unless otherwise specified, show all work and justify any nontrivial claims. **You may use any results from lecture and homework problems, but you must clearly state the result that you are using.**

Question	Points	Score
1	10	
2	5	
3	5	
4	5	
5	5	
6	5	
Total:	35	

1. **Short Answer.** No justification required.

   (a) (2 points) In the metric space $\mathbb{R}$ with standard Euclidean metric, give an example of an infinite set S of rational numbers such that S is a closed and bounded subset of $\mathbb{R}$.

   $S = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$

   (b) (2 points) Give an example of a sequence (s_n) of real numbers such that $\limsup s_n = \infty$, $\liminf s_n = -\infty$, and the sequence $(\bar{s}_n)$ defined by $\bar{s}_n := \frac{s_1 + \ldots + s_n}{n}$ converges.

   $(1, -1, \sqrt{2}, -\sqrt{2}, \sqrt{3}, -\sqrt{3}, \sqrt{4}, -\sqrt{4}, \ldots)$

   (c) (2 points) In the metric space $\mathbb{R}^2$ with standard Euclidean metric, let

   $$E := [0,1] \times (0,1) = \{(x,y) : 0 \leq x \leq 1,\ 0 < y < 1\} \subseteq \mathbb{R}^2.$$

   Give an example of an open cover of E which has no finite subcover.

   $\big\{(-1,2) \times (\frac{1}{n}, 1)\big\}_{n \in \mathbb{N}}$

(d) (2 points) Give an example of a metric space (X, d) and a nonempty set $E \subsetneq X$ such that E is both open and closed in X. (The notation $E \subsetneq X$ means $E \subseteq X$ and $E \neq X$.)

Let X be any set with at least 2 elements. Let d be the discrete metric on X. Let $E = \{x\}$ for any $x \in X$.

(e) (2 points) Give an example of a metric space (X, d) and a set $E \subseteq X$ such that E is closed and bounded, but not compact.

$X = \{\text{bounded sequences of real numbers}\}$, $d\big((s_n), (t_n)\big) = \sup\{|s_n - t_n| : n \in \mathbb{N}\}$, $E = \big\{(s_n) \in X : |s_n| \leq 1 \text{ for all } n\big\}$.

2. (5 points) Let (s_n) and (t_n) be two sequences of real numbers such that (s_n) is nondecreasing, (t_n) is nonincreasing, $s_n \leq 104 \leq t_n$ for all $n \in \mathbb{N}$, and $|t_n - s_n| < \frac{1}{n}$ for all n. Prove that (s_n) and (t_n) both converge and $\lim s_n = \lim t_n = 104$.

Solution. (s_n) is nondecreasing and bounded above by 104, and (t_n) is nonincreasing and bounded below by 104, so by the monotone convergence theorem, (s_n) converges to some $s \in \mathbb{R}$ and (t_n) converges to some $t \in \mathbb{R}$. Note that $|t_n - s_n| < \frac{1}{n}$ for all n implies that $\lim(t_n - s_n) = 0$. Then

$$t - s = \lim t_n - \lim s_n = \lim(t_n - s_n) = 0,$$

so $t = s$. Since $s_n \leq 104 \leq t_n$ for all n, it follows that $s \leq 104 \leq t = s$, and therefore $s = t = 104$.

3. (5 points) Let (X, d) be a discrete metric space, so for any $x, y \in X$,

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Prove that a set $E \subseteq X$ is compact if and only if E is a finite set.

Solution. Suppose E is finite, say $E = \{x_1, \ldots, x_n\}$. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of E. Since $E \subseteq \bigcup_{\alpha \in A} U_\alpha$, for each $1 \leq k \leq n$ there exists U_{α_k} such that $x \in U_{\alpha_k}$. Then $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ is a finite subcover of $\{U_\alpha\}_{\alpha \in A}$.
For the converse, suppose E is infinite. Observe that $\{B_{1/2}(x)\}_{x \in E} = \{x\}_{x \in E}$ is an open cover of E. But the union of any finite subcollection of this open cover only contains finitely many points and therefore cannot cover E.

4. (5 points) Let (X, d) be a metric space and let $E \subseteq X$. Let E' denote the set of all limit points of E. Prove that E' is closed.

Solution. Let $x \in (E')^c$, so x is not a limit point of E. Then there exists $r > 0$ such that $(B_r(x) \setminus \{x\}) \cap E = \emptyset$, so $B_r(x) \setminus \{x\} \subseteq E^c$. Let $y \in B_r(x)$ and let $s = \min\{d(x,y), r - d(x,y)\} > 0$. Then $B_s(y) \subseteq B_r(x) \setminus \{x\} \subseteq E^c$, so $B_s(y) \cap E = \emptyset$ and hence y is not a limit point of E so $y \in (E')^c$. Therefore $B_r(x) \subseteq (E')^c$, so $(E')^c$ is open and thus E' is closed.

5. (5 points) Let (s_n) be a bounded sequence of real numbers. Suppose $\alpha \in \mathbb{R}$ has the property that for any $\beta > \alpha$, there exists $N \in \mathbb{N}$ such that $s_n < \beta$ for all $n \geq N$. Prove that $\limsup s_n \leq \alpha$.

Solution. Let $\varepsilon > 0$. By the property of α in the problem, there exists $N \in \mathbb{N}$ such that $s_n < \alpha + \varepsilon$ for all $n \geq N$. Then $v_N := \sup\{s_n : n \geq N\} \leq \alpha + \varepsilon$ and hence $\limsup s_n \leq \alpha + \varepsilon$. Since this is true for any $\varepsilon > 0$, it follows that $\limsup s_n \leq \alpha$.

6. (5 points) Let (X, d) be a metric space. Prove that if (x_n) and (y_n) are two Cauchy sequences in X, then the sequence $\big(d(x_n, y_n)\big)_{n\in\mathbb{N}}$ of real numbers converges. (Hint: $\mathbb{R}$ is complete. Note that $d(x_n, y_n) \le d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n)$.)

Solution. Let $\varepsilon > 0$. Since (x_n) and (y_n) are Cauchy, there exist N_1 and N_2 such that $d(x_m, x_n) < \varepsilon/2$ for all $m, n \ge N_1$ and $d(y_m, y_n) < \varepsilon/2$ for all $m, n \ge N_2$. Set $N := \max\{N_1, N_2\}$. Let $m, n \ge N$. By the triangle inequality,

$$d(x_n, y_n) \le d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n)$$

and

$$d(x_m, y_m) \le d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m),$$

so

$$|d(x_m, y_m) - d(x_n, y_n)| \le d(x_m, x_n) + d(y_m, y_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore the sequence $\big(d(x_n, y_n)\big)_{n\in\mathbb{N}}$ is Cauchy and therefore convergent by completeness of $\mathbb{R}$.

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