University of California, Berkeley
Math 104: Introduction to Analysis

Instructor: Theodore Zhu

Final Exam

August 10, 2017

10:10 AM – 11:55 AM

Name: ______________________________

Student ID: _________________________

Instructions. This is a closed-book, closed-notes, closed-electronics exam. Please write carefully and clearly in the spaces provided. If you run out of space for a problem, you may continue on the reverse side of the page, or on the extra pages at the end. Cross out any work that you do not want to be graded. Unless otherwise specified, show all work and justify any nontrivial claims. **You may use any results from lecture and homework problems, but you must clearly state the result that you are using.**

Question	Points	Score
1	12	
2	8	
3	5	
4	10	
5	10	
Total:	45	

1. **Short Answer.** Unless otherwise stated, no justification is required.

   (a) (2 points) Give an example of a sequence (s_n) of nonzero real numbers such that

   $$\limsup |s_n|^{1/n} < \limsup \left|\frac{s_{n+1}}{s_n}\right|.$$

   $(1, 2, 1, 2, 1, 2, \ldots)$

   (b) (2 points) Let (X, d_X) and (Y, d_Y) be two metric spaces.
   TRUE or FALSE: A function $f : X \to Y$ is continuous if and only if $f^{-1}(E)$ is a closed set in X for every closed set E in Y. **Justify your answer**.

   TRUE. $f^{-1}(E)^c = f^{-1}(E^c)$ is open for any closed set E in Y if and only if f is continuous, so $f^{-1}(E)$ is closed for any closed set E in Y if and only if f is continuous.

(c) (2 points) Let f be a bounded function on $[0,1]$, and suppose that f^2 is integrable on $[0,1]$. Does it follow that f^3 is integrable on $[0,1]$? **Justify your answer**.

NO. Consider

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ -1 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

(d) (2 points) Compute the Taylor series expansion of

$$f(x) = \frac{1}{(1-x)^2}$$

about $x_0 = 0$. What is the radius of convergence?

$T^{f,0}(x) = \sum_{n=0}^{\infty}(n+1)x^n$, $R = 1$

(e) (2 points) Give an example of a sequence of differentiable functions (f_n) on some open interval I such that (f_n) converges uniformly on I but (f_n') does not converge uniformly on I.

$$f_n(x) = \frac{x^{n+1}}{n+1} \qquad \text{on } (0,1)$$

(f) (2 points) State precisely any **ONE** of the following theorems: Taylor's theorem, fundamental theorem of calculus (either one of the two parts), Abel's theorem, dominated convergence theorem. (If you state more than one, the least correct one will be graded.)

2. Let $C([0,1])$ denote the set of all continuous real-valued functions on $[0,1]$. For $f, g \in C([0,1])$, define $D(f,g) := \sup\{|f(x) - g(x)| : x \in [0,1]\}$.

(a) (5 points) Prove that D defines a metric on $C([0,1])$.

(i) Continuous functions on $[0,1]$ must be bounded, so for $f, g \in C([0,1])$,

$D(f,g) = \sup\{|f(x)-g(x)| : x \in [0,1]\} \leq \sup\{|f(x)| : x \in [0,1]\} + \sup\{|g(x)| : x \in [0,1]\} < \infty$.

(ii) $D(f,f) = \sup\{|f(x) - f(x)| : x \in [0,1]\} = \sup\{0\} = 0$;
If $f, g \in C([0,1])$ and $f \neq g$, then there exists $x_0 \in [0,1]$ such that $f(x_0) \neq g(x_0)$ or $|f(x_0) - g(x_0)| > 0$, so $D(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\} \geq |f(x_0) - g(x_0)| > 0$.

(iii) Since $|f(x) - g(x)| = |g(x) - f(x)|$ for every $x \in [0,1]$, $D(f,g) = D(g,f)$.

(iv)

$$
\begin{aligned}
D(f,h) &= \sup\{|f(x) - h(x)| : x \in [0,1]\} \\
&\leq \sup\{|f(x) - g(x)| + |g(x) - h(x)| : x \in [0,1]\} \\
&\leq \sup\{|f(x) - g(x) : x \in [0,1]\} + \sup\{|g(x) - h(x)| : x \in [0,1]\} \\
&= D(f,g) + D(g,h).
\end{aligned}
$$

(b) (3 points) Is $(C([0,1]), D)$ a complete metric space? Justify your answer.

YES. A Cauchy sequence of functions in this metric space is precisely a uniformly Cauchy sequence, which converges uniformly. By the uniform limit theorem, the limit function is continuous.

Extra page

3. (5 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, nondecreasing function, and let S be a nonempty bounded set of real numbers. Prove that $\sup f(S) = f(\sup S)$.

Proof. Since f is nondecreasing and $s \leq \sup S$ for every $s \in S$, $f(s) \leq \sup S$ for every $s \in S$. Hence $\sup S$ is an upper bound for $f(S)$, so $\sup f(S) \leq f(\sup S)$. For the other direction, let $\varepsilon > 0$. Since f is continuous, there exists $\delta > 0$ such that $|f(\sup S) - f(s)| < \varepsilon$ whenever $|\sup S - s| < \delta$. There exists $s_0 \in S$ such that $\sup S \geq s_0 > \sup S - \delta$. Then $\sup f(S) \geq f(s_0) > f(\sup S) - \varepsilon$. Thus $\sup f(S) \geq f(\sup S)$.

$\square$

4. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 1$ such that

(i) $a_n \geq 0$ for all n;
(ii) $\sum_{n=0}^{\infty} a_n = 1$;
(iii) $\sum_{n=1}^{\infty} n a_n > 1$;
(iv) $0 < a_0 < 1$;
(v) $a_n > 0$ for some $n \geq 2$.

(a) (5 points) Prove that there exists $b < 1$ such that $f(x) < x$ for all $x \in (b, 1)$. (Hint: Let $g(x) = f(x) - x$. Consider $g'(1)$ and continuity of g'.)

Proof. Let $g(x) = f(x) - x$. Then $g'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} - 1$ so $g'(1) = \sum_{n=1}^{\infty} n a_n - 1 > 0$ by (iii). Since $R > 1$, g' is continuous at 1 so there exists $\delta > 0$ such that $|g'(x) - g'(1)| < g'(1)$ (which implies $g'(x) > 0$) whenever $|x - 1| < \delta$. Let $b = 1 - \delta$. Note that $g(1) = f(1) - 1 = \sum_{n=0}^{\infty} a_n - 1 = 0$. Then for $x \in (b, 1)$, by the mean value theorem there exists $c \in (x, 1)$ such that

$$\frac{g(1) - g(x)}{1 - x} = \frac{-g(x)}{1 - x} = g'(c) > 0,$$

so $g(x) < 0$, i.e. $f(x) < x$.

$\square$

(b) (5 points) Prove that f has a unique fixed point in $(0, 1)$. In other words, prove that there exists exactly one point $x_0 \in (0, 1)$ such that $f(x_0) = x_0$.
(You may use the conclusion in part (a) even if you are unable to prove it.)
(Hint: Prove existence and uniqueness separately. For uniqueness, consider f'' and $f(1)$.)

Proof. Since $0 < a_n < 1$, $g(0) = f(0) - 0 = a_0 > 0$. For any $x \in (b, 1)$ (from part (a)), $g(x) < 0$. Since g is continuous on $[-1, 1] \subseteq (-R, R)$, by the intermediate value theorem there exists $x_0 \in (0, x) \subseteq (0, 1)$ such that $g(x_0) = 0$, or $f(x_0) = x_0$.
To prove uniqueness, suppose (for contradiction) that f has two distinct fixed points $x_0, y_0 \in (0, 1)$ with $x_0 < y_0$. By the mean value theorem there exists $c_1 \in (x_0, y_0)$ such that $f'(c_1) = 1$, and there exists $c_2 \in (y_0, 1)$ such that $f'(c_2) = 1$. But $f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} > 0$ for all $x \in (0, 1)$ since $a_n > 0$ for some $n \geq 2$, which implies that f' is strictly increasing on $(0, 1)$ – contradiction.

$\square$

Extra page

5. Let g be an bounded function on $[a, b]$. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function, i.e. there exists $C > 0$ such that $|f(x) - f(y)| \le C|x - y|$ for all $x, y \in \mathbb{R}$.

   (a) (5 points) Prove that for any subset $S \subseteq [a, b]$,

$$M(f \circ g, S) - m(f \circ g, S) \le C\big(M(g, S) - m(g, S)\big).$$

(Hint: Introduce $\varepsilon > 0$.)

Proof. Let $\varepsilon > 0$. There exist $x, y \in S$ such that

$$\begin{aligned} f \circ g(x) &> M(f \circ g, S) - \frac{\varepsilon}{2}, \\ f \circ g(y) &< m(f \circ g, S) + \frac{\varepsilon}{2}. \end{aligned}$$

Then

$$\begin{aligned} M(f \circ g, S) - m(f \circ g, S) - \varepsilon &< f \circ g(x) - f \circ g(y) \\ &\le |f \circ g(x) - f \circ g(y)| \\ &\le C|g(x) - g(y)| \\ &\le C(M(g, S) - m(g, S)). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$M(f \circ g, S) - m(f \circ g, S) \le C(M(g, S) - m(g, S)).$$

$\square$

(b) (5 points) Prove that if g is integrable on $[a, b]$, then $f \circ g$ is integrable on $[a, b]$. (Note: You may use the conclusion in part (a) even if you are unable to prove it.)

Proof. Suppose g is integrable on $[a, b]$. Let $\varepsilon > 0$. There exists a partition $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ of $[a, b]$ such that $U(g, P) - L(g, P) < \varepsilon / C$. Then

$$\begin{aligned}
U(f \circ g, P) - L(f \circ g, P) &= \sum_{k=1}^{n} [M(f \circ g, [t_{k-1}, t_k]) - m(f \circ g, [t_{k-1}, t_k])] \cdot (t_k - t_{k-1}) \\
&\leq C \sum_{k=1}^{n} [M(g, [t_{k-1}, k]) - m(g, [t_{k-1}, t_k])] \cdot (t_k - t_{k-1}) \\
&= C(U(g, P) - L(g, P)) \\
&< \varepsilon.
\end{aligned}$$

$\square$

Extra page

Extra page