# Math 104 Homework 8

UC Berkeley, Summer 2021
Due: N/A

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**1.** Let $f(x) = |x|^3$ on $\mathbb{R}$. Compute $f'(x)$ and $f''(x)$ for $x \in \mathbb{R}$, and show that $f'''(0)$ does not exist.

**2.** Let $f$ be a function defined on an open interval $I$ containing $x_0$ which is twice differentiable on $I$ such that $f''$ is continuous at $x_0$. Prove that for any $\varepsilon > 0$, there exists $h > 0$ such that

$$\left| \frac{f(x_0 + h) - f(x_0 - h)}{2h} - f'(x_0) \right| < h \cdot \varepsilon.$$

(Hint: Use Taylor's theorem about $x_0$.)

**3.** (Ross 32.8) Let $f$ be an integrable function on $[a, b]$. Prove that $f$ is integrable on any closed subinterval $[c, d] \subseteq [a, b]$.

**4.** (a) Let $S = \{x_1, \ldots, x_n\}$ be a finite subset of $[a, b]$. Let $f$ be the function

$$f(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases}$$

Prove that $f$ is integrable and $\int_a^b f = 0$.
(b) Let $g$ be an integrable function on $[a, b]$, and suppose $h$ is a function on $[a, b]$ such that $h(x) = g(x)$ for all but finitely many points $x$ in $[a, b]$. Prove that $h$ is integrable and $\int_a^b h = \int_a^b g$.

**5.** Show that the function

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is integrable on $[0, 1]$.

**6.** Let $f$ be an integrable function on $[a, b]$.
(a) Let $C > 0$ be such that $|f(x)| \leq C$ for all $x \in [a, b]$. Show that for any $P \in \Pi_{[a,b]}$,

$$U(f^2, P) - L(f^2, P) \leq 2C(U(f, P) - L(f, P)).$$

(Hint: $c^2 - d^2 = (c + d)(c - d)$.)
(b) Prove that $f^2$ is integrable on $[a, b]$.
(c) Prove that if $g$ and $h$ are integrable functions on $[a, b]$, then $gh$ is integrable on $[a, b]$. (Hint: $cd = \frac{1}{4}[(c + d)^2 - (c - d)^2]$.)

**7.** (Ross 33.13) Suppose $f$ and $g$ are continuous functions on $[a,b]$ such that $\int_a^b f = \int_a^b g$. Prove that there exists $x \in (a,b)$ such that $f(x) = g(x)$.

**8.** (Ross 33.9) Let $(f_n)$ be a sequence of integrable functions on $[a,b]$ such that $f_n \to f$ uniformly on $[a,b]$. Prove that $f$ is integrable and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

**9.** A sequence $(f_n)$ of integrable functions on $[a,b]$ is said to **converge in** $L^1([a,b])$ to an integrable function $f$ if $\lim_{n\to\infty} \int_a^b |f_n - f| = 0$. Find an example of a sequence of integrable functions $(f_n)$ on $[0,1]$ and an integrable function $f$ on $[0,1]$ such that $f_n \to f$ pointwise and $\lim_{n\to 0} \int_0^1 f_n = \int_0^1 f$, but $f_n$ does not converge to $f$ in $L^1([0,1])$.

**10.** Please complete the **Course Evaluations**. Evaluations are completely anonymous, and they are not released to the instructor and department until three weeks after the end of the course. All feedback is greatly appreciated!