Theorem (Root test)

Let $\sum a_n$ be a series and let $\alpha = \limsup |a_n|^{\frac{1}{n}}$.

The series $\sum a_n$

(i) converges absolutely if $\alpha < 1$.

(ii) diverges if $\alpha > 1$

(Inconclusive if $\alpha = 1$).

$\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$.

(need to use fact that $n^{\frac{1}{n}} \to 1$ as $n \to \infty$).

$\limsup |a_n|^{\frac{1}{n}} = \lim v_n$

$\sup\{|a_m|^{\frac{1}{m}} : m \geq n\}$

Proof: (i) Suppose $\alpha < 1$. Let $\beta \in (\alpha, 1)$, so $\alpha < \beta < 1$.

There exists $N \in \mathbb{N}$ s.t. $n \geq N$ implies $|a_n|^{\frac{1}{n}} < \beta$.

Hence $|a_n| < \beta^n$ for $n \geq N$. Since $\sum_{n=1}^{\infty} \beta^n$ converges,

$(\beta < 1)$

by comparison test, $\sum |a_n|$ converges.

(ii) Suppose $\alpha > 1$. $\underbrace{\limsup |a_n|^{\frac{1}{n}}}_{\lim v_n} > 1$. For every n, $v_n = \sup\{\underline{|a_m|^{\frac{1}{m}}} : m \geq n\} > 1$.

Hence $|a_n|^{\frac{1}{n}} > 1$ infinitely often. Then $|a_n| > 1$ infinitely often.

Then $a_n \not\to 0$, so $\sum a_n$ diverges.

Recall: $\sum a_n$ converges $\Rightarrow a_n \to 0$.

Ratio test

Let $\sum a_n$ be a series of nonzero real numbers.

(i) If $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum a_n$ converges absolutely.

(ii) If $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum a_n$ diverges.

(otherwise no information).

Note: weaker version: if $\lim \left|\frac{a_{n+1}}{a_n}\right| = L \begin{matrix} <1 \cdots \\ >1 \cdots \end{matrix}$

Proof: Let $\alpha = \limsup |a_n|^{\frac{1}{n}}$.

By Lemma:

(i) If $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$,

$\Rightarrow \alpha < 1 \Rightarrow$ conv. absolutely by Root test.

Lemma:

$\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq \limsup |a_n|^{\frac{1}{n}} \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$.

(ii) If $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$,

$\Rightarrow \alpha > 1 \Rightarrow$ diverge by Root test.

Monday, July 19

If you feel that your performance was hindered by the typo, schedule an appointment with me between now and class tomorrow — respond to upcoming Piazza post.

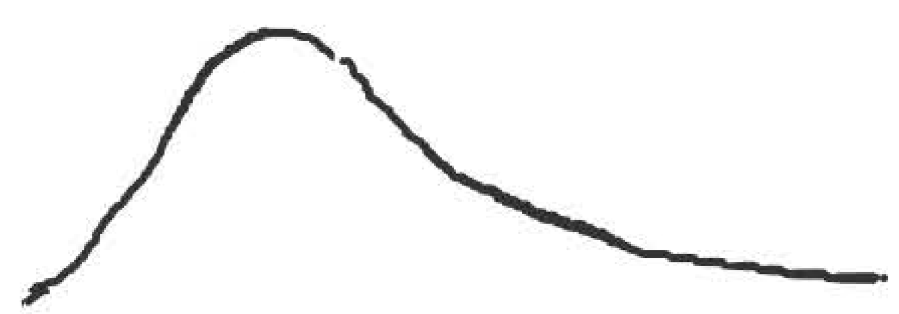
Exam Stats (out of 30).

Max: 25

Mean: 14.75

Median: 13

SD: 5.37



<u>Recall</u> :

- comparison test.
  - proved that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ diverges for } p<1. \qquad (p\text{-series}).$$

Soon : $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p>1$.

Math 104 Worksheet 10

UC Berkeley, Summer 2021

Monday, July 19

Lemma. Let (s_n) be a sequence of nonzero real numbers. Then

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| \le \liminf |s_n|^{1/n} \le \boxed{\limsup |s_n|^{1/n} \le \limsup \left|\frac{s_{n+1}}{s_n}\right|.}$$

Proof. (The proof here uses the fact that $\lim n^{1/n} = 1$ and $\lim a^{1/n} = 1$ for any constant $a > 0$; see p.49 in the textbook for the proofs.) The second inequality is trivial. For the third inequality:

Let $L := \limsup \left|\frac{s_{n+1}}{s_n}\right|$. If $L = \infty$ then the inequality is trivial, so assume $L \in \mathbb{R}$. Let $\varepsilon > 0$. (Goal: Show that $\limsup |s_n|^{1/n} \le L + \varepsilon$.) There exists $N \in \mathbb{N}$ such that $\left|\frac{s_{n+1}}{s_n}\right| < L + \varepsilon$ for $n \ge N$. Then for $n > N$,

$$|s_n| = \underbrace{\left|\frac{s_n}{s_{n-1}}\right|}_{<L+\varepsilon} \cdot \underbrace{\left|\frac{s_{n-1}}{s_{n-2}}\right|}_{<L+\varepsilon} \cdots \underbrace{\left|\frac{s_{N+1}}{s_N}\right|}_{<L+\varepsilon} \cdot |s_N| < (L+\varepsilon)^{n-N}|s_N| = (L+\varepsilon)^n \cdot \underbrace{\frac{|s_N|}{(L+\varepsilon)^N}}_{\text{constant.}} = C(L+\varepsilon)^n$$

where $C := \frac{|s_N|}{(L+\varepsilon)^N}$. Hence $|s_n|^{1/n} < C^{1/n}(L+\varepsilon)$ for $n > N$, so

$$\limsup |s_n|^{1/n} \le \limsup \underbrace{C^{1/n}}_{\to 1}(L+\varepsilon) = L + \varepsilon.$$

Exercise 1. Using the same strategy as above, prove the first inequality.

Let $L = \liminf \left|\frac{s_{n+1}}{s_n}\right|$. If $L = 0$, trivial.

Assume $L \in \mathbb{R}$, $L > 0$.

Let $\varepsilon > 0$. (Show that $\liminf |s_n|^{\frac{1}{n}} \ge L - \varepsilon$).

$\varepsilon < L$.

There exists $N \in \mathbb{N}$ such that $n \ge N$ implies $\left|\frac{s_{n+1}}{s_n}\right| > L - \varepsilon$. Then for $n > N$,

$$|s_n| = \underbrace{\left|\frac{s_n}{s_{n-1}}\right|}_{>L-\varepsilon} \cdots \underbrace{\left|\frac{s_{N+1}}{s_N}\right|}_{<L-\varepsilon} |s_N| > (L-\varepsilon)^{n-N} |s_N| = C(L-\varepsilon)^n$$

$$|s_n|^{\frac{1}{n}} > C^{\frac{1}{n}}(L-\varepsilon).$$

$$\liminf |s_n|^{\frac{1}{n}} \ge L - \varepsilon.$$

$\Leftrightarrow \lim \left|\frac{s_{n+1}}{s_n}\right| = \infty.$

Assume $\liminf \left|\frac{s_{n+1}}{s_n}\right| = \infty$.

Show $\lim |s_n|^{\frac{1}{n}} = \infty$.

Let $M > 0$. Since $\lim \left|\frac{s_{n+1}}{s_n}\right| = \infty$, there exists $N \in \mathbb{N}$ s.t. $n \ge N \Rightarrow \left|\frac{s_{n+1}}{s_n}\right| > M+1$

For $n > N$.

$$|s_n| = \underbrace{\left|\cdot\right|}_{>M+1} \cdots \underbrace{\left|\cdot\right|}_{>M+1} \cdot |s_N| > C(M+1)^n$$

$$|s_n|^{\frac{1}{n}} > C^{\frac{1}{n}}(M+1) > M$$

Since RHS $\to M+1$ ...