Thursday, July 1

- No class on Monday, July 5.

$f: \mathbb{R} \to \mathbb{R}$

Ex $f(x) = x^2$.

Range$(f) = [0,\infty)$.

Recall Notion of a subsequence.

$(s_1, s_2, s_3, \ldots)$

- $(s_1, s_2, s_3, s_5, s_8, s_{13}, s_{21}, \ldots)$ is a subsequence
- $(s_5, s_4, s_7, s_6, \ldots)$ is not a subsequence.

Ex: $\mathbb{Q}$ is countable: there is a bijection from $\mathbb{N}$ to $\mathbb{Q}$.

Therefore we have a sequence $(q_n) = (q_1, q_2, q_3, \ldots)$ such that $\{q_n : n \in \mathbb{N}\} = \mathbb{Q}$.

"enumeration of $\mathbb{Q}$"

<u>Proposition</u>: Let (q_n) be an enumeration of $\mathbb{Q}$.
Then for any $a \in \mathbb{R}$, there exists a subsequence (q_{n_k}) of (q_n) such that $q_{n_k} \longrightarrow a$.

<u>Proof</u>: ("Inductive construction" — See HW2 #10)
Construct (q_{n_k}) inductively.
First there exists $r_1 \in \mathbb{Q}$ such that $a-1 < r_1 < a+1$ ($\mathbb{Q}$ dense in $\mathbb{R}$).
There exists $n_1 \in \mathbb{N}$ such that $q_{n_1} = r_1$.
Having already found $n_1, \dots, n_k$:
find $n_{k+1} > n_k$ such that $a - \frac{1}{k+1} < q_{n_{k+1}} < a + \frac{1}{k+1}$.
(why is this always possible?)
There are infinitely many rationals between $a - \frac{1}{k+1}$ and $a + \frac{1}{k+1}$.
Not all of these rationals come before the n_k^{th} term in (q_n).
By construction, $q_{n_k} \longrightarrow a$.

<u>Theorem</u>: Let (s_n) be a sequence of real numbers. If $s_n \to s$, then $s_{n_k} \to s$ for every subsequence (s_{n_k}) of (s_n).

<u>Proof</u>: Suppose $s_n \to s \in \mathbb{R}$. Let (s_{n_k}) be a subsequence of (s_n).

$(s_{n_k})_{k\in\mathbb{N}} = (s_{n_1}, s_{n_2}, s_{n_3}, \ldots)$.

Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|s_n - s| < \varepsilon$.

There exists $K \in \mathbb{N}$ such that $n_K \geq N$.

Then for $k \geq K$, $|s_{n_k} - s| < \varepsilon$ since $n_k \geq N$.

Also can prove similar for $s \in \{\infty, -\infty\}$.

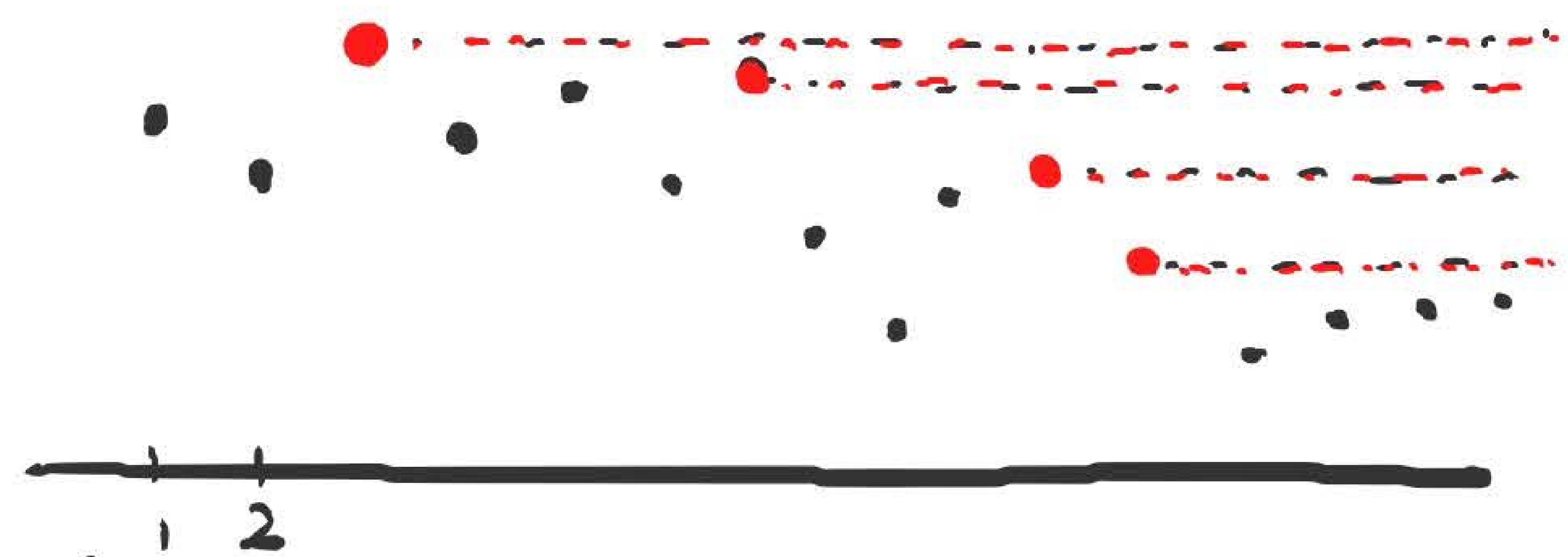
Theorem: Every sequence of real numbers has a monotonic subsequence.

Proof: Let $S = \{ n \in \mathbb{N} : s_n > s_m \text{ for all } m > n \}$

terms which are (strictly) larger than everything afterward.

Case 1: S has infinitely many elements

$\longrightarrow$ decreasing subsequence.



Case 2: S has finitely many elements.

Let $m = \max S$.

Let $n_1 = m+1$. Having already found

$n_1, n_2, \dots, n_k$: since $n_k > m$, $n_k \notin S$, so there exists

$n_{k+1} > n_k$ such that $s_{n_{k+1}} \geq s_{n_k}$.

(s_{n_k}) is nondecreasing.

<u>Corollary</u>: (Bolzano-Weierstrass Theorem)

Every bounded sequence has a convergent subsequence.

<u>Proof</u>: Apply monotone convergence theorem with previous theorem.

<u>Def</u>: Let (s_n) be a sequence of real numbers.

A <u>subsequential limit</u> of (s_n) is any $a \in \mathbb{R} \cup \{\infty, -\infty\}$ such that there exists a subsequence (s_{n_k}) of (s_n) for which $s_{n_k} \longrightarrow a$.

Math 104 Worksheet 5

UC Berkeley, Summer 2021

Wednesday, June 30

The aim of this worksheet is to prove the equivalence of two definitions of lim sup. (The analogous definitions for lim inf will also be equivalent, with nearly an identical proof.)

Definition 1. Given a sequence (s_n) of real numbers, we define

$$\limsup s_n := \begin{cases} \lim_{n\to\infty}\Big(\sup\{s_m : m \geq n\}\Big) & \text{if } (s_n) \text{ is bounded from above,} \\ +\infty & \text{if } (s_n) \text{ is not bounded from above.} \end{cases}$$

Definition 2. Given a sequence (s_n) of real numbers, let $L \subseteq \mathbb{R} \cup \{\pm\infty\}$ denote the set of *subsequential limits* of (s_n). Define $\limsup s_n := \sup L$.

Theorem. The two definitions of lim sup above are equivalent.

Proof. Let (s_n) be a sequence of real numbers.

Case 1. (s_n) is NOT bounded from above. Then according to Definition 1, $\limsup s_n = \infty$. On the other hand, since (s_n) is not bounded from above, it should be possible to construct a subsequence (s_{n_k}) of (s_n) such that $\lim_{k\to\infty} s_{n_k} = \infty$.

• Exercise 1. Inductively construct a subsequence (s_{n_k}) of (s_n) such that $\lim_{k\to\infty} s_{n_k} = \infty$.

1 is not an upper bound for (S_n), so there exists $n_1 \in \mathbb{N}$: $S_{n_1} > 1$.
Having already found $n_1, n_2, \dots, n_k$: can find $n_{k+1} > n_k$ s.t. $S_{n_{k+1}} > k+1$.
$S_{n_k} > k \Rightarrow S_{n_k} \to \infty$.

Therefore, $\infty \in L$ and hence $\limsup s_n = \sup L = \infty$.

Case 2. (s_n) is bounded from above. Then the goal is to show that

$$\sup L = \lim_{n\to\infty}\Big(\sup\{s_m : m \geq n\}\Big) \qquad \Big(= \lim_{n\to\infty} v_n\Big).$$

To do this, we will prove inequality in both directions.

• Exercise 2. To show that $\sup L \boxed{\leq} \lim v_n$, it suffices to show that for every subsequence (s_{n_k}) of (s_n) such that $\lim_{k\to\infty} s_{n_k}$ exists, we have the inequality $\lim_{k\to\infty} s_{n_k} \leq \lim v_n$ (why?). Now prove the assertion.

lim V_n is an upper bound for L
sup L is the L.U.B.

$S_{n_k} \leq V_{n_k} = \sup\{S_m : m \geq n_k\}$. Since (v_n) converges V, (V_{n_k}) also converges to V
$\lim S_{n_k} \leq \lim V_{n_k} = \lim V_n$.

• Exercise 3. To show that $\sup L \boxed{\geq} \lim v_n$, it suffices to show that there exists a subsequence (s_{n_k}) of (s_n) such that $\lim_{k\to\infty} s_{n_k} = \lim v_n$ (why?). Now inductively construct such a subsequence.

lim $V_n \in L$; sup L is an upper bd of L.

Find n_1: $S_{n_1} > V_1 - 1$
$\sup\{S_m : m \geq 1\}$
Having already found $n_1, \dots, n_k$: find $n_{k+1} > n_k$ such that $S_{n_{k+1}} > V_{n_k+1} - \frac{1}{k+1}$
$\sup\{S_m : m \geq n_k + 1\}$

$\underbrace{V_{n_{k-1}+1} - \frac{1}{k}}_{\xrightarrow{k\to\infty} \lim V_n} < S_{n_k} \leq V_{n_k} \xrightarrow{k\to\infty} \lim V_n$

Use squeeze theorem.
$\therefore \lim S_{n_k} = \lim V_n$

Theorem: Let (s_n) be a sequence in $\mathbb{R}$, let L denote the set of subsequential limits of (s_n). Then

(i) $L \neq \emptyset$.

(ii) $\lim s_n$ exists if and only if $\underbrace{|L|}_{\#\text{ of elements in } L} = 1$, in which case $L = \{\lim s_n\}$

(iii) if (s_n) is not bounded above, then $\infty \in L$.

" below " $-\infty \in L$.

(iv) $\limsup s_n \in L$ and $\liminf s_n \in L$.

(v) $\sup L = \limsup s_n$ and $\inf L = \liminf s_n$.

We've developed quite a bit of theory about the real numbers!
What are some of the important properties of $\mathbb{R}$ that we've used?

- $\mathbb{Q}$ dense $\mathbb{R}$
- Archimedean
- completeness
- Order field
- Distance

Def: A metric space (X, d) is a set X equipped with a "distance function" $d: X \times X \longrightarrow [0, \infty)$ satisfy:

- $\forall x,y$ $d(x,y) \geq 0$ and $d(x,y) = 0 \Leftrightarrow x = y$. (positive definiteness)
- $\forall x,y$ $d(x,y) = d(y,x)$ (symmetry)
- $\forall x,y,z$ $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality).

Ex: $\mathbb{R}$ with $d(x,y) = |x-y|$.
$\mathbb{R}^2$ with $d((x_1,y_1),(x_2,y_2)) = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$.

We can generalize some of the concepts that we developed in $\mathbb{R}$.

<u>Def</u>: A sequence (x_n) in a metric space (X, d) <u>converges</u> to x if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$.

<u>Def</u>: A sequence (x_n) is <u>Cauchy</u> if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $m, n \geq N$ implies $d(x_m, x_n) < \varepsilon$.

<u>Def</u>: A metric space (X, d) is <u>complete</u> if every Cauchy sequence converges.

metric space.

<u>Ex</u>: $\mathbb{Q}$ is not complete.

<u>Exercise</u>: If (x_n) converges, then it is Cauchy.