image: $f(E)=\{f(x): x\in E\}$

preimage: $f^{-1}(A)=\{x\in X: f(x)\in A\}$

Math 104 Worksheet 12

UC Berkeley, Summer 2021

Thursday, July 22

Let X and Y be two sets, and let $f: X \to Y$, let $E \subseteq X$, and let $A, B \subseteq Y$.

Exercise 1. Prove the following assertions.

(a) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

Proof. $x \in f^{-1}(A \cap B) \iff f(x) \in A \cap B \iff f(x) \in A$ and $f(x) \in B \iff$ $x \in f^{-1}(A)$ and $x \in f^{-1}(B) \iff x \in f^{-1}(A) \cap f^{-1}(B)$.

(b) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

$x\in f^{-1}(A\cup B) \Leftrightarrow f(x)\in A\cup B \Leftrightarrow f(x)\in A$ or $f(x)\in B$

$\Leftrightarrow x\in f^{-1}(A)$ or $x\in f^{-1}(B) \Leftrightarrow x\in f^{-1}(A)\cup f^{-1}(B)$.

(c) $f^{-1}(A^c) = (f^{-1}(A))^c$.

$x\in f^{-1}(A^c) \Leftrightarrow f(x)\in A^c \Leftrightarrow f(x)\notin A$

$\Leftrightarrow x\notin f^{-1}(A) \Leftrightarrow x\in (f^{-1}(A))^c$.

(d) $f^{-1}(A) \subseteq f^{-1}(B)$ if $A \subseteq B$.

$x\in f^{-1}(A) \Leftrightarrow f(x)\in A \Rightarrow f(x)\in B \Leftrightarrow x\in f^{-1}(B)$.

$f(x)\in A \Leftrightarrow x\in f^{-1}(A)$.

(e) $E \subseteq f^{-1}(f(E))$

$x\in E \Rightarrow f(x)\in f(E) \Leftrightarrow x\in f^{-1}(f(E))$.

(f) Find a counterexample to show that the statement $E = f^{-1}(f(E))$ is not always true.

$f(x)=x^2$. $E=(0,\infty)$. $f(E)=(0,\infty)$

$f^{-1}(f(E)) = \mathbb{R}\setminus\{0\}$.

Let (X, d_X) and (Y, d_Y) be two metric spaces, and let $f : X \to Y$. The following are three definitions of continuity at a point $x_0 \in X$.

$f(x) \in B_\varepsilon(f(x_0))$

1. (ε-δ definition) For any $\varepsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \varepsilon$ whenever $x \in X$ and $d_X(x, x_0) < \delta$. $x \in B_\delta(x_0)$.
2. (sequential definition) For any sequence (x_n) in X converging to x_0, the sequence $f(x_n)$ converges to $f(x_0)$.
3. (topological definition) For any open set U in Y such that $f(x_0) \in U$, there exists an open set V in X such that $x_0 \in V \subseteq f^{-1}(U)$.

Theorem. The three definitions above are equivalent.

Exercise 2. Prove the preceding theorem.
(a) Prove (2) $\Rightarrow$ (1).

Skip.

(b) Prove (1) $\Rightarrow$ (3). Let U be an open set in Y, $f(x_0) \in U$. Since U is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(f(x_0)) \subseteq U$. There exists $\delta > 0$ such that $x \in \boxed{B_\delta(x_0)}$ implies $f(x) \in B_\varepsilon(f(x_0)) \subseteq U$. Then $x_0 \in \underbrace{B_\delta(x_0)}_{V} \subseteq f^{-1}(U)$.

(c) Prove (3) $\Rightarrow$ (2).
Let $(x_n) \subseteq X$, $x_n \to x_0$.
Let $\varepsilon > 0$. Show there exists N: $n \geq N$ implies $f(x_n) \in \underbrace{B_\varepsilon(f(x_0))}_{U}$.
There exists an open set $V \subseteq X$ such that $x_0 \in V \subseteq f^{-1}(U)$.
There exists $\delta > 0$ such that $B_\delta(x_0) \subseteq V$. Since $x_n \to x_0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies that $x_n \in B_\delta(x_0) \subseteq V \subseteq f^{-1}(U)$.
Then for $n \geq N$, $f(x_n) \in U = B_\varepsilon(f(x_0))$.

Exercise 3. Using the topological definition of continuity at a point, prove that f is continuous (on its domain) if and only if $f^{-1}(U)$ is open in X for every open set U in Y.
(A function is continuous if and only if the preimage of every open set is open.)

$\Rightarrow$| Suppose topological def. holds at every $x \in X$.
Let $U \subseteq Y$ be open. Let $x \in f^{-1}(U)$.
(Show x is an interior point of $f^{-1}(U)$). $f(x) \in U$.
There exists open $V \subseteq X$ such that $x \in V \subseteq f^{-1}(U)$.
Since V is open, there exists $r > 0$ such that $B_r(x) \subseteq V \subseteq f^{-1}(U)$.

$\Leftarrow$| Suppose $f^{-1}(U)$ open (in X) whenever U open (in Y). Let $x \in X$. Let $U \subseteq Y$ open, $f(x) \in U$. $f^{-1}(U)$ is open. $x \in f^{-1}(U)$.
Take this to be V.

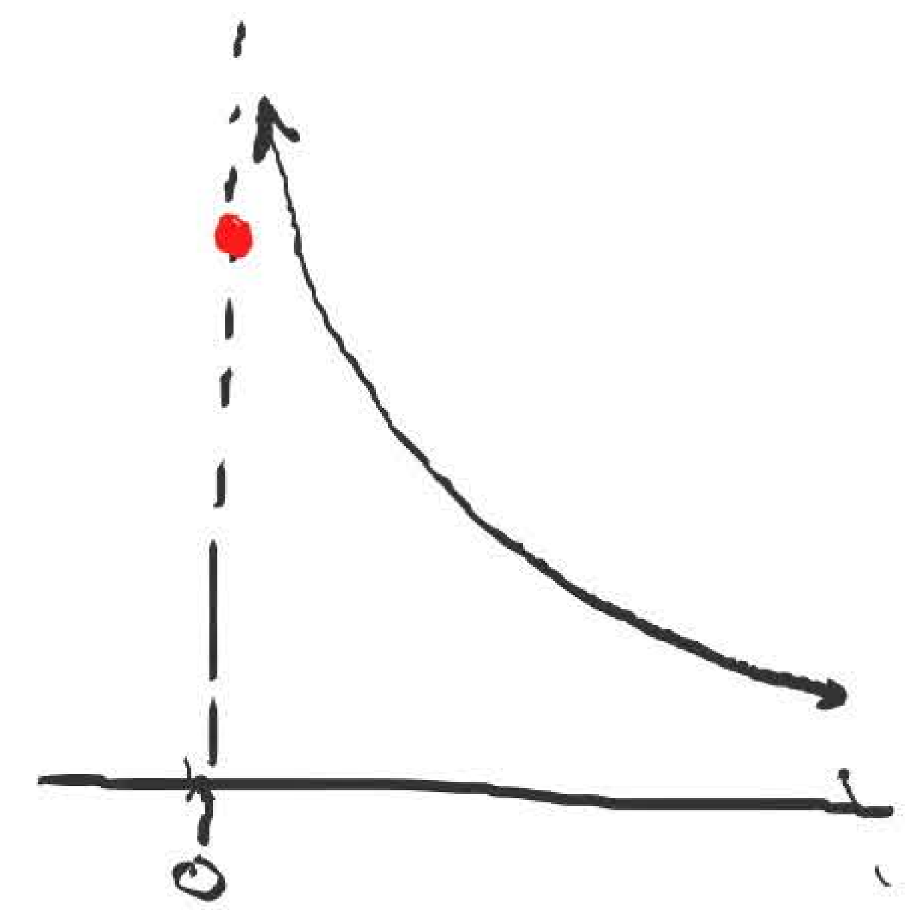
Monday, July 26

Recall:

- uniform continuity
  - on closed and bounded intervals, uniform continuity = reg. continuity
  - continuous extension theorem

    uniformly cont. function of (a,b)

    $\Leftrightarrow$ can be extended to a continuous function on $[a,b]$.



$(X, d_X), (Y, d_Y)$ metric spaces

<u>Def</u>: $f: X \to Y$ is <u>uniformly continuous</u> on $E \subseteq X$ if

for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$x_1, x_2 \in E$ and $d_X(x_1, x_2) < \delta$ implies $d_Y(f(x_1), f(x_2)) < \varepsilon$.

<u>Theorem</u>: Let $f: X \to Y$ be continuous. Let $E \subseteq X$ be compact.

(i) $f(E)$ is compact. (image of a compact set under cont. function is compact)

(ii) f is uniformly continuous on E. (recall: cont. function on closed, bdd intervals)

<u>Proof</u>:

(i) Let $\boxed{\{U_\alpha\}_{\alpha \in A} \text{ be an open cover of } f(E)}$. Then $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ is an open cover of E.

Since E is compact, there exists a finite subcover $f^{-1}(U_{\alpha_1}), \dots, f^{-1}(U_{\alpha_n})$, i.e.

$E \subseteq \bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i}) = f^{-1}\left(\bigcup_{i=1}^{n} U_{\alpha_i}\right) \Rightarrow f(E) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$ ← finite subcover of $\{U_\alpha\}_{\alpha \in A}$.

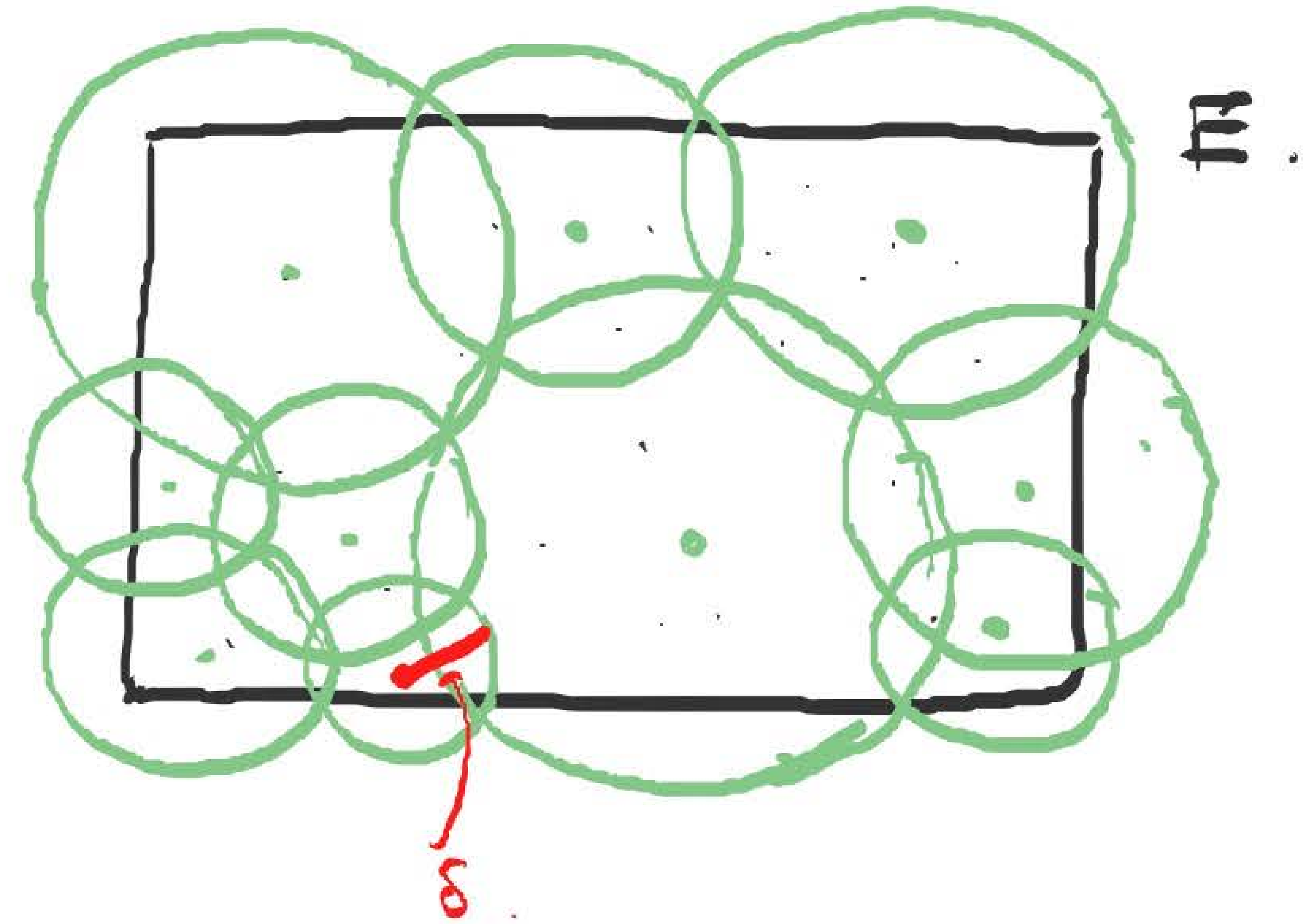
(ii) Let $\varepsilon > 0$.

For each $x \in E$, there exists $\delta_x > 0$ such that $z \in E, d_X(x,z) < \delta_x \Rightarrow d_Y(f(x), f(z)) < \frac{\varepsilon}{2}$.

For each $x \in E$, let $U_x = B_{\frac{1}{2}\delta_x}(x)$. $\{U_x\}_{x \in E}$ is an open cover of E. Since E is compact, there exists a finite subcover, $U_{x_1}, \ldots, U_{x_n}$. Let $\delta = \frac{1}{2}\min\{\delta_{x_1}, \ldots, \delta_{x_n}\}$

$= \min\{\frac{1}{2}\delta_{x_1}, \ldots, \frac{1}{2}\delta_{x_n}\}$

Show this δ satisfies unif. cont (with our ε).

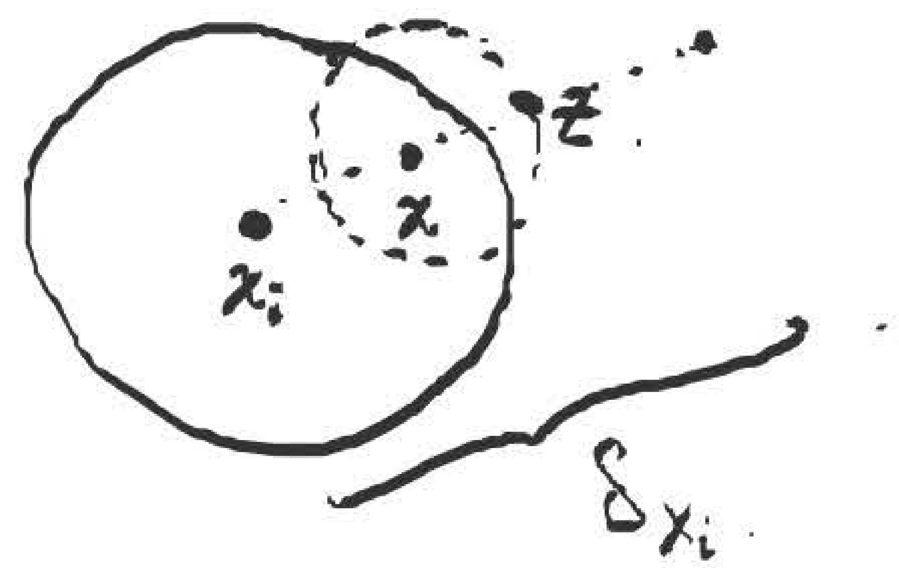


Let $x, z \in E$, $d_X(x,z) < \delta$

$x \in U_{x_i} = B_{\frac{1}{2}\delta_{x_i}}(x_i)$ for some i.

Since $d_X(x,z) < \delta \leq \frac{1}{2}\delta_{x_i}$, $d_X(z, x_i) < \delta_{x_i}$.

(and $d_X(x, x_i) < \delta_{x_i}$)



Then $d_Y(f(x), f(x_i)) < \frac{\varepsilon}{2}$.

$d_Y(f(z), f(x_i)) < \frac{\varepsilon}{2}$.

Hence $d_Y(f(x), f(z)) \leq \underbrace{d_Y(f(x), f(x_i))}_{< \varepsilon/2} + \underbrace{d_Y(f(x_i), f(z))}_{< \varepsilon/2} < \varepsilon$.