Q1

(a) We need to show both directions:

$\Longrightarrow$:

Case 1. If $b \geq 0$, then $|b| \leq a \implies b \leq a$ by the definition of absolute value.

Case 2. If $b < 0$, then

$$\begin{aligned} |b| \leq a &\implies -b \leq a \quad \text{by the defintion of } |b| \\ &\implies -a \leq -(-b) \quad \text{by theorem 3.2(i)} \\ &\implies -a \leq b \end{aligned}$$

Thus combining both cases, we have $-a \leq b \leq a$.

$\Longleftarrow$:

Case 1. If $b \geq 0$, then $b = |b|$ by the definition of absolute value and $a \geq b \geq 0$. From the assumption $b \leq a$, we have $|b| \leq a$.

Case 2. If $b < 0$, then $-b = |b|$ by the definition of absolute value. From the assumption $-a \leq b$, we have $-b \leq -(-a) = a$ by 3.2(i). Thus $|b| \leq a$. Thus combining both cases, we have $|b| \leq a$.

(b) From part (a), we only need to show that $-|a-b| \leq |a| - |b| \leq |a-b|$. Observe that $|a| = |a-b+b| \leq |a-b| + |b| \implies |a| - |b| \leq |a-b| + |b| - |b| = |a-b|$. The second inequality comes from Triangle Inequality 3.7. The implication comes from properties A4&O4.

Also observe that $|b| = |b-a+a| \leq |b-a| + |a|$ by Triangle Inequality. This implies

$$\begin{aligned} |b| - |a| \leq |b-a| &\implies -|b-a| \leq -(|b| - |a|) \\ &\implies -|-(a-b)| \leq -|b| + |a| \\ &\implies -|a-b| \leq |a| - |b| \end{aligned}$$

Thus we have $-|a-b| \leq |a| - |b| \leq |a-b|$ which implies $\big||a| - |b|\big| \leq |a-b|$.

Q2

(a) We have

$$\begin{aligned}
2(\sqrt{n+1}-\sqrt{n}) &= 2(\sqrt{n+1}-\sqrt{n})\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \\
&= \frac{2}{\sqrt{n+1}+\sqrt{n}} \\
&< \frac{2}{\sqrt{n}+\sqrt{n}} \quad \text{by } \sqrt{n+1}>\sqrt{n} \\
&= \frac{1}{\sqrt{n}}
\end{aligned}$$

Thus we prove the first inequality. Again we have

$$\begin{aligned}
2(\sqrt{n}-\sqrt{n-1}) &= 2(\sqrt{n}-\sqrt{n-1})\cdot\frac{\sqrt{n}+\sqrt{n-1}}{\sqrt{n}+\sqrt{n-1}} \\
&= \frac{2}{\sqrt{n}+\sqrt{n-1}} \\
&> \frac{2}{\sqrt{n}+\sqrt{n}} \quad \text{by } \sqrt{n-1}<\sqrt{n} \\
&= \frac{1}{\sqrt{n}}
\end{aligned}$$

Thus we prove the second inequality, completing the proof.

(b) We have

$$\begin{aligned}
\sum_{k=1}^{n}\frac{1}{\sqrt{k}} &> \sum_{k=1}^{n} 2(\sqrt{k+1}-\sqrt{k}) \quad \text{by assertion (a)} \\
&= 2(\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n+1}-1-\sqrt{2}-\sqrt{3}-\cdots-\sqrt{n}) \\
&= 2(\sqrt{n+1}-1) \\
&> 2(\sqrt{n}-1) \\
&= 2\sqrt{n}-2
\end{aligned}$$

Thus $\sum_{k=1}^{n}\frac{1}{\sqrt{k}} > 2\sqrt{n}-2$, completing the proof.

(c)
- Induction Hypothesis: For all integer $n \geq 2$, $\sum_{k=1}^{n}\frac{1}{\sqrt{k}} < 2\sqrt{n}-1$.
- Base Case $n=2$: $\sum_{k=1}^{2}\frac{1}{\sqrt{k}} = 1+\frac{1}{\sqrt{2}} < 2\sqrt{2}-1$.

- Induction Step $n+1$:

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} &= \sum_{k=1}^{n} \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{n+1}} \\
&< 2\sqrt{n} - 1 + \frac{1}{\sqrt{n+1}} \quad \text{by the hypothesis} \\
&< 2\sqrt{n} - 1 + 2(\sqrt{n+1} - \sqrt{n}) \quad \text{by (a)} \\
&= 2\sqrt{n+1} - 1
\end{aligned}$$

Thus we have $\sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} < 2\sqrt{n+1} - 1$, completing the induction step and the proof.

Q3

The question can be reformulated to: show that for each $\epsilon > 0$, let $b_1 = b + \epsilon > b$ with $a, b \in \mathbb{R}$, if $a \leq b_1 = b + \epsilon$, then $a \leq b$. Suppose the implication does not hold which is $\exists a \in \mathbb{R},\ (a \leq b_1 = b + \epsilon) \wedge (a > b)$. Then let $\epsilon = \frac{a-b}{2}$, and we will have

$$\begin{aligned} a \leq b + \epsilon &\implies a \leq b + \frac{a-b}{2} \\ &\implies \frac{a}{2} \leq \frac{b}{2} \\ &\implies a \leq b \end{aligned}$$

The last implication contradicts to the assumption that $a > b$. Thus the implication does hold.

Q4

We will use proof by contradiction. Suppose $\sup S > \inf T$. Then by the definition of $\sup S$, $\exists s \in S$ such that $s > \inf T$. Otherwise, $\inf T$ is an upper bound of S smaller than $\sup S$. This implies that there exists $s \in S$ such that $\forall t \in T,\ s > \inf T \geq t \implies s > t$. This contradicts to the condition that $s \leq t$ for all $s \in S$ and $t \in T$. Thus $\sup S \leq \inf T$, completing the proof.

Q5

(a) A does not have a minimum or maximum; $\inf(A) = 0$ and $\sup(A) = \infty$.

(b) B does not have a minimum, but $\max(B) = 2$; $\inf(B) = 0$ and $\sup(B) = 2$.

(c) $\min(C) = -1$, but C does not have a maximum; $\inf(C) = -1$ and $\sup(C) = \infty$.

(d) $\min(D) = 0$ and $\max(D) = 3$; $\inf(D) = 0$ and $\sup(D) = 3$.

(e) $\min(E) = 2$ but E does not have a maximum; $\inf(E) = 2$ and $\sup(E) = \infty$.

(f) Essentially $F = \{1\}$, so the minimum, maximum, infimum, and supremum of F are all 1.