Tuesday, July 27

Recall: $(X, d_X), (Y, d_Y)$ metric spaces.

$f: X \to Y$, E compact subset of X.

(i) $f(E)$ is compact

(ii) f uniformly continuous on E.

Corollary: $f: X \to \mathbb{R}$ continuous, E compact subset of X.

(i) $f(E)$ is closed and bounded (f is bounded on E)

Def: $f(E)$ is bounded.

(ii) There exists $u, v \in E$ such that

$$f(u) = \inf_{x \in E} f(x) \quad \text{and} \quad f(v) = \underbrace{\sup_{x \in E} f(x)}_{\sup f(E).}$$

i.e. f attains its minimum and maximum on E.

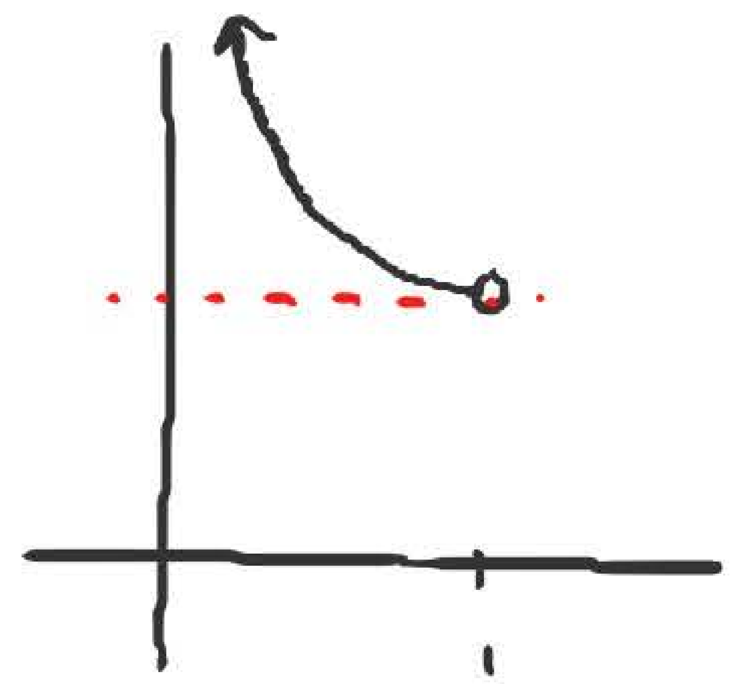
Proof: (i) Trivial

(ii) Recall from HW: if $F \subseteq \mathbb{R}$ is compact, then $\sup F \in F$ and $\inf F \in F$.

$\inf f(E) \in f(E)$, i.e. there exists $u \in E$ such that $f(u) = \inf f(E) \ldots$

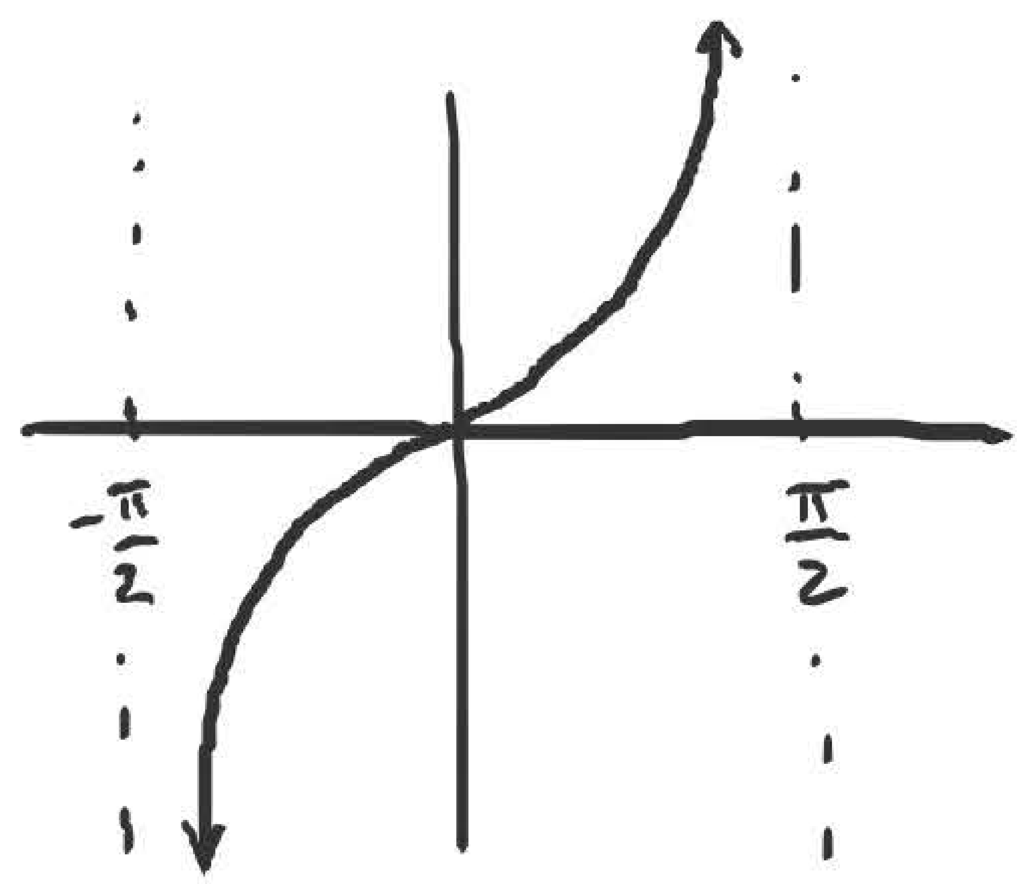
$\underline{\text{Ex}}$ $f(x) = \frac{1}{x}$ on $\underbrace{(0,1)}_{\text{not compact}}$.

$$f((0,1)) = (1, \infty)$$



$f(x) = \tan x$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$$f\left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right) = (-\infty, \infty)$$

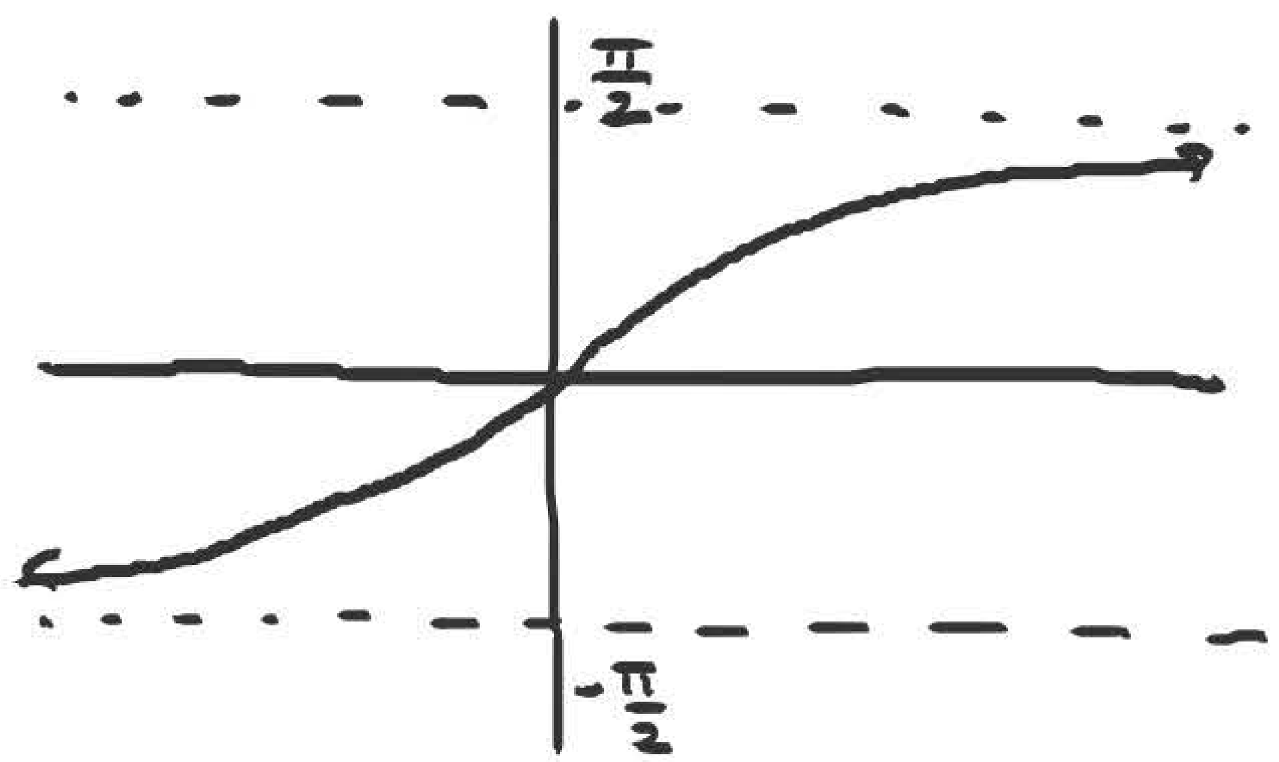


$f(x) = \arctan x$ on $\mathbb{R}$.

$$f(\mathbb{R}) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

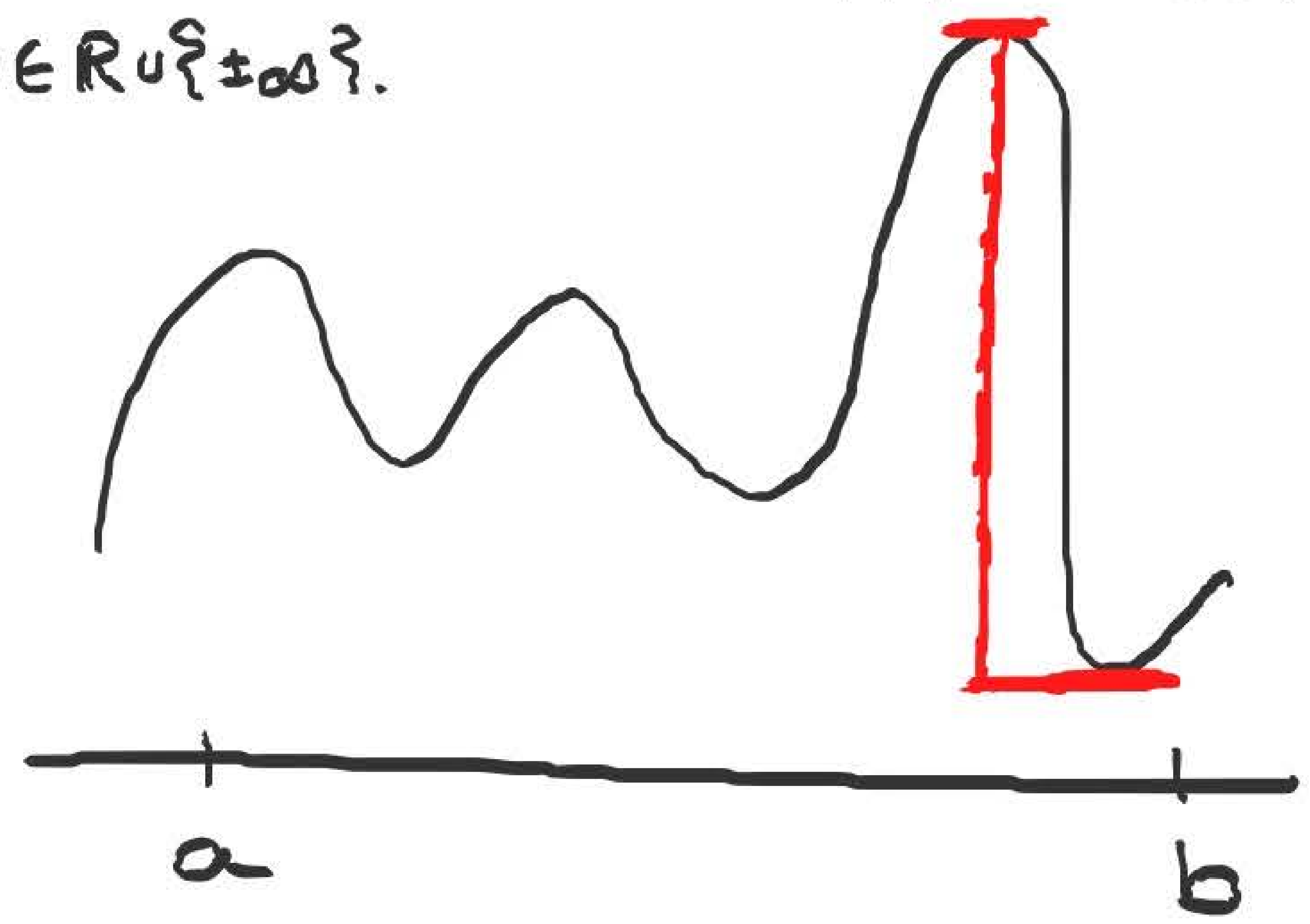
$\inf f(\mathbb{R}) = -\frac{\pi}{2} \notin f(\mathbb{R})$.

$\sup f(\mathbb{R}) = \frac{\pi}{2} \notin f(\mathbb{R})$



Theorem: $S \subseteq \mathbb{R}$. Let $f: S \to \mathbb{R}$.
If f is continuous on an interval $I \subseteq S$, then $f(I)$ is a singleton or an interval.

anything of the form $(a,b), [a,b)$, $(a,b], [a,b]$, $a,b \in \mathbb{R} \cup \{\pm\infty\}$.



Proof: Consider $\inf f(I)$ and $\sup f(I)$.
Know that $\inf f(I) \le \sup f(I)$.

Case 1: $\inf f(I) = \sup f(I)$.
Since $f(I) \neq \emptyset$, $\sup f(I) \neq -\infty$
and $\inf f(I) \neq +\infty$. $\Rightarrow \inf f(I) = \sup f(I) \in \mathbb{R}$.
Then $f(I) = \{\inf f(I)\}$.

Case 2: $\inf f(I) < \sup f(I)$. Goal: Show that $(\inf f(I), \sup f(I)) \subseteq f(I)$.
Let $y \in (\inf f(I), \sup f(I))$. y is not a lower bound for $f(I)$ and y is not an upper bound for $f(I)$, so there exists $y_*, y^* \in f(I)$ such that $\inf f(I) \le y_* < y < y^* \le \sup f(I)$.

$y_* = f(x_*)$, $y^* = f(x^*)$ for some $x_*, x^* \in I$.

By IVT, there exists $x \in I$ (between x_* and x^*) such that $f(x) = y$, so $y \in f(I)$.

Recall: continuous extension theorem. $f:(a,b)\to\mathbb{R}$.

f uniformly continuous on (a,b) $\iff$ f can be extended to a continuous function on $[a,b]$.

Let's generalize.

Continuous extension theorem (general version)

Let (X,d) be a metric space. Let $E\subseteq X$.

If $f:E\to\mathbb{R}$ is uniformly continuous (on E), then f can be extended to a continuous function on $\bar{E}=E\cup E'$.

limit points of E.

(Alt statement: $f:E\to\mathbb{R}$ uniformly cont. $\iff$ can extend to unif. cont. $\tilde{f}$ on $\bar{E}$).

not necessarily compact.

Proof: For each $x \in \bar{E} \setminus E$, let (x_n) be a sequence in E which converges to x. Since (x_n) is convergent, it is Cauchy, hence $\underbrace{(f(x_n))}_{\subseteq \mathbb{R}}$ is Cauchy, therefore $(f(x_n))$ converges. Define $\tilde{f}(x) = \lim f(x_n)$.

Need to show that $\tilde{f}(x)$ is well-defined.

Let $(y_n) \subseteq E$, $y_n \to x$. Consider $(x_1, y_1, x_2, y_2, \dots) = (z_n)$.

$\dots \quad \lim f(z_n) = \lim f(x_n) = \lim f(y_n)$. Good.

Now we need to prove continuity of $\tilde{f}$.

Let $\varepsilon > 0$ (Goal: Show that there exists $\delta > 0$ such that $x, y \in \bar{E}$, $d(x,y) < \delta \Rightarrow |\tilde{f}(x) - \tilde{f}(y)| < \varepsilon$)

- there exists $\delta' > 0$ such $s, t \in E$ (not $\bar{E}$), $d(s,t) < \delta' \Rightarrow |f(s) - f(t)| < \frac{\varepsilon}{3}$.
- set $\delta = \frac{\delta'}{3}$.
- Let $x, y \in \bar{E}$ with $d(x,y) < \delta$ (Show $|\tilde{f}(x) - \tilde{f}(y)| < \varepsilon$).
- Let $(x_n), (y_n) \subseteq E$ such that $x_n \to x$ and $y_n \to y$.
- There exist $N_1, N_2 \in \mathbb{N}$ such that
  - $n \geq N_1$ implies $d(x_n, x) < \delta$.
  - $n \geq N_2$ implies $d(y_n, y) < \delta$.

  Let $N = \max(N_1, N_2)$.
- For $n \geq N$,

$$d(x_n, y_n) \leq \underbrace{d(x_n, x)}_{<\delta} + \underbrace{d(x, y)}_{<\delta} + \underbrace{d(y, y_n)}_{<\delta} < 3\delta = \delta'.$$

- For $n \geq N$, $\quad |f(x_n) - f(y_n)| < \frac{\varepsilon}{3}$.
- Since $f(x_n) \longrightarrow \tilde{f}(x)$ and $f(y_n) \longrightarrow \tilde{f}(y)$, there exist M_1, M_2:

  $n \geq M_1$ implies $|f(x_n) - \tilde{f}(x)| < \frac{\varepsilon}{3}$

  $n \geq M_2$ implies $|f(y_n) - \tilde{f}(y)| < \frac{\varepsilon}{3}$.

  Let $M = \max(M_1, M_2, N)$. Then

$$\underbrace{|\tilde{f}(x) - \tilde{f}(y)|}_{< \varepsilon} \leq \underbrace{|\tilde{f}(x) - f(x_M)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f(x_M) - f(y_M)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f(y_M) - \tilde{f}(y)|}_{< \frac{\varepsilon}{3}}$$

Question: Can we generalize further — e.g. does the codomain of f need to be $\mathbb{R}$?

Yes; replace codomain $\mathbb{R}$ with (Y, d_Y) a complete metric space.

Power series

Given $(a_0, a_1, a_2, \ldots)$ of real numbers and $x_0 \in \mathbb{R}$,

$$\underbrace{\sum_{n=0}^{\infty} a_n (x-x_0)^n}_{\text{an object}}$$ is called a <u>power series</u>.

If we plug in a real number for x, then we get a series of real numbers — may or may not converge.

Can think of the power series as a function $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ on the set of x-values for which the series converge.

- <u>Ex</u>: $f(x) = \sum_{n=0}^{\infty} x^n$ on $(-1,1)$. $f(x) = \frac{1}{1-x}$

- any power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ always converges at $x = x_0$. $a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots$

(Convention: $0^0 = 1$) $\lim_{x \to 0^+} x^0 = 1$, $\lim_{x \to 0^+} 0^x = 0$, $\lim_{x \to 0^+} x^x = 1$

Combinatorial: # of functions from a set of m elements to a set of n elements is n^m. $|\{f: \emptyset \to \emptyset\}| = 1$.