Math 104 Homework 3 Solutions

UC Berkeley, Summer 2021

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1. Let (s_n) be a sequence of real numbers and let $s \in \mathbb{R}$. Prove that every subsequence of (s_n) has a subsequence that converges to s if and only if (s_n) converges to s.

Solution. If (s_n) converges to s, then every subsequence of (s_n) converges to s, and consequently every subsequence of every subsequence of (s_n) converges to s. For the converse, we prove the contrapositive: suppose that (s_n) does not converges to s. Then for some $\varepsilon > 0$, the set $\{n : |s_n - s| \geq \varepsilon\}$ has infinitely many elements, so we can construct a subsequence (s_{n_k}) of (s_n) such that $|s_{n_k} - s| \geq \varepsilon$ for all k; hence no subsequence of (s_{n_k}) converges to s.

2. Let (s_n) and (t_n) be two bounded sequences of real numbers. Prove that

$$\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n.$$

(*Hint*: First show that $\sup\{s_m + t_m : m \geq n\} \leq \sup\{s_m : m \geq n\} + \sup\{t_m : m \geq n\}$.)

Solution. For any $n \in \mathbb{N}$ and $M \geq n$, $s_M \leq \sup\{s_m : m \geq n\}$ and $t_M \leq \sup\{t_m : m \geq n\}$, so $s_M + t_M \leq \sup\{s_m : m \geq n\} + \sup\{t_m : m \geq n\}$. Therefore $\sup\{s_m : m \geq n\} + \sup\{t_m : m \geq n\}$ is an upper bound for the set $\{s_m + t_m : m \geq n\}$, so

$$\sup\{s_m + t_m : m \geq n\} \leq \sup\{s_m : m \geq n\} + \sup\{t_m : m \geq n\}.$$

Since the limits on both sides exist as $n \to \infty$, it follows that

$$\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n.$$

3. Let (s_n) be a bounded sequence of real numbers, and let $k \in \mathbb{R}$.
(a) Let S be a bounded set of real numbers, and define the set

$$-S := \{-s : s \in S\}.$$

Prove that $\sup(-S) = -\inf S$.
(b) Prove that if $k \geq 0$, then $\limsup(ks_n) = k \cdot \limsup(s_n)$.
(c) Prove that if $k < 0$, then $\limsup(ks_n) = k \cdot \liminf(s_n)$.

Solution. (a) Let $\varepsilon > 0$. There exists $s \in S$ such that $s < \inf S + \varepsilon$. Then $-s \in -S$, so $\sup(-S) \geq -s > -\inf S - \varepsilon$. On the other hand, there exists $s \in S$ such that $-s > \sup(-S) - \varepsilon$. Since $s \geq \inf S$, it follows that $\inf S \leq s < -\sup(-S) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $\sup(-S) \geq -\inf S$ and $\inf S \leq -\sup(-S)$ (or rearranging, $\sup(-S) \leq -\inf S$), and hence $\sup(-S) = -\inf S$.
(b) The case $k = 0$ is trivial. For $k > 0$, it suffices to show that for any $n \in \mathbb{N}$, $\sup\{ks_m : m \geq n\} = k \cdot \sup\{s_m : m \geq n\}$. Since $ks_m \leq k \cdot \sup\{s_m : m \geq n\}$ for all $m \geq n$, it follows that $\sup\{ks_m : m \geq n\} \leq k \cdot \sup\{s_m : m \geq n\}$. For the opposite inequality, let $\varepsilon > 0$. There exists $M \geq n$ such that $s_M > \sup\{s_m : m \geq n\} - \varepsilon/k$. Then

$\sup\{ks_m : m \geq n\} \geq ks_M > k(\sup\{s_m : m \geq n\} - \varepsilon/k) = k \cdot \sup\{s_m : m \geq n\} - \varepsilon$, and since this holds for any $\varepsilon > 0$ it follows that $\sup\{ks_m : m \geq n\} \geq k \cdot \sup\{s_m : m \geq n\}$. Therefore $\sup\{ks_m : m \geq n\} = k \cdot \sup\{s_m : m \geq n\}$, and taking limits yields $\limsup(ks_n) = k \cdot \limsup(s_n)$.
(c) If $k < 0$ then $-\frac{1}{k} > 0$, so by parts (a) and (b) we have

$$\liminf(s_n) = \lim_{n\to\infty}\left(\inf\left\{\frac{1}{k} \cdot ks_m : m \geq n\right\}\right) = -\lim_{n\to\infty}\left(\sup\left\{-\frac{1}{k} \cdot ks_m : m \geq n\right\}\right)$$
$$= -\limsup\left(-\frac{1}{k} \cdot ks_n\right) = \frac{1}{k}\limsup(ks_n).$$

4. (a) (Ross 12.8) Let (s_n) and (t_n) be two bounded sequences of nonnegative real numbers. Prove that

$$\limsup s_n t_n \leq (\limsup s_n)(\limsup t_n).$$

(b) Show that the inequality in part (a) fails if the nonnegativity restriction is removed.

Solution. (a) For any $n \in \mathbb{N}$ and $M \geq n$, $s_M t_M \leq \sup\{s_m : m \geq n\} \cdot \sup\{t_m : m \geq n\}$, and hence $\sup\{s_m t_m : m \geq n\} \leq \sup\{s_m : m \geq n\} \cdot \sup\{t_m : m \geq n\}$. Since the limits exist on both sides as $n \to \infty$, it follows that $\limsup s_n t_n \leq (\limsup s_n)(\limsup t_n)$.
(b) Let $s_n = t_n = (-1)^n - 1$. Then $\limsup s_n t_n = 4$, but $\limsup s_n = \limsup t_n = 0$.

5. Let (s_n) be a sequence of real numbers, and for each n define

$$\bar{s}_n := \frac{s_1 + s_2 + \ldots + s_n}{n}.$$

(a) Prove that $\limsup \bar{s}_n \leq \limsup s_n$ and $\liminf \bar{s}_n \geq \liminf s_n$.
(*Hint*: For the first inequality, the case $\limsup s_n = \infty$ is trivial. The case $\limsup s_n = -\infty$ needs to be considered separately from the case $\limsup s_n = \alpha \in \mathbb{R}$. For the latter case, let $\varepsilon > 0$ and show that $\limsup \bar{s}_n \leq \alpha + \varepsilon$. To show this, observe that if $N \in \mathbb{N}$ and $v_N = \sup\{s_n : n \geq N\}$, then for $n \geq N$,

$$\bar{s}_n \leq \frac{s_1 + \ldots + s_{N-1}}{n} + \frac{(n - N + 1)v_N}{n} = \frac{s_1 + \ldots + s_{N-1} - (N-1)v_N}{n} + v_N.$$

The proof of the second inequality mirrors the proof of the first.)
(b) Prove that if $\lim s_n$ exists, then $\lim \bar{s}_n$ exists and $\lim \bar{s}_n = \lim s_n$.
(c) Give an example of a sequence (s_n) such that $\lim s_n$ does not exist but $\lim \bar{s}_n = 0$.
(d) Describe an example of a bounded sequence (s_n) such that $(\bar{s}_n)$ diverges.

Solution. (a) The case $\limsup s_n = \infty$ is trivial. Consider the case $\limsup s_n = -\infty$; this happens if and only if $\lim s_n = -\infty$. Observe that if $N \in \mathbb{N}$ and $v_N = \sup\{s_n : n \geq N\}$, then for $n \geq N$,

$$\bar{s}_n \leq \frac{s_1 + \ldots + s_{N-1}}{n} + \frac{(n - N + 1)v_N}{n} = \frac{s_1 + \ldots + s_{N-1} - (N-1)v_N}{n} + v_N.$$

Case 1: $\limsup s_n = -\infty$. Let $M \in \mathbb{R}$. There exists $N_1 \in \mathbb{N}$ such that $v_{N_1} < M - 1$. Then

$$\frac{s_1 + \ldots + s_{N_1-1} - (N_1 - 1)v_{N_1}}{n} =: \varphi_{N_1}(n) \longrightarrow 0$$

as $n \to \infty$, so there exists $N_2 \geq N_1$ such that $\varphi_{N_1}(n) < 1$ for all $n \geq N_2$. Then for all $n \geq N_2$,

$$\bar{s}_n \leq \varphi_{N_1}(n) + v_{N_1} < 1 + M - 1 = M.$$

Therefore, $\limsup \bar{s}_n = \lim \bar{s}_n = -\infty$.
Case 2: $\limsup s_n = \alpha \in \mathbb{R}$. Let $\varepsilon > 0$. There exists $N_1 \in \mathbb{N}$ such that $|v_{N_1} - \alpha| < \varepsilon/2$, so $v_{N_1} < \alpha + \varepsilon/2$. Since $\varphi_{N_1}(n) \to 0$ as $n \to \infty$, there exists $N_2 \geq N_1$ such that for all $n \geq N_2$, $|\varphi_{N_1}(n)| < \varepsilon/2$, so $\varphi_{N_1}(n) < \varepsilon/2$ for $n \geq N_2$. Then for $n \geq N_2$,

$$\bar{s}_n \leq \varphi_{N_1}(n) + v_{N_1} < \varepsilon/2 + \alpha + \varepsilon/2 = \alpha + \varepsilon.$$

Therefore, $\limsup \bar{s}_n \leq \alpha = \limsup s_n$.

The proof for the corresponding result for $\liminf$ mirrors the proof for $\limsup$.
(b) $s_n = (-1)^n$.
(c) Define s_n as follows: $s_1 = 1$; $s_n = 0$ for $2 \leq n \leq 10$; $s_n = 1$ for $11 \leq n \leq 100$; $s_n = 0$ for $101 \leq n \leq 1000$; $s_n = 1$ for $1001 \leq n \leq 10000$; $s_n = 0$ for $10001 \leq n \leq 100000$; and so on. Then we see that $\bar{s}_1 = 1$, $\bar{s}_{10} = 0.1$, $\bar{s}_{100} \geq 0.9$, $\bar{s}_{1000} \leq 0.1$, $\bar{s}_{10000} \geq 0.9$, $\bar{s}_{100000} \leq 0.1$, and so on. The sequence $(\bar{s}_n)$ exceeds 0.9 infinitely often and dips below 0.1 infinitely often, and therefore diverges. More compactly, we can write down a formula for the sequence as

$$s_n = \begin{cases} 1 & \text{if } \lceil \log_{10} n \rceil \text{ is even;} \\ 0 & \text{if } \lceil \log_{10} n \rceil \text{ is odd.} \end{cases}$$

6. Define $d : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ by

$$d(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{k} |y_j - x_j|.$$

(a) Show that d is a metric on $\mathbb{R}^k$.
(b) Show that $(\mathbb{R}^k, d)$ is a complete metric space.

Solution. (a) (i) If $\mathbf{x} = \mathbf{y}$, then $x_j = y_j$ for all j and hence $d(\mathbf{x}, \mathbf{y}) = 0$. If $\mathbf{x} \neq \mathbf{y}$, then there exists j such that $x_j \neq y_j$, so $|y_j - x_j| > 0$ and hence $d(\mathbf{x}, \mathbf{y}) > 0$. (ii) Symmetry is trivial. (iii) For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^k$,

$$d(\mathbf{x}, \mathbf{z}) = \sum_{j=1}^{k} |z_j - x_j| \leq \sum_{j=1}^{k} \big(|x_j - y_j| + |y_j - z_j|\big) = d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}).$$

(b) Let $(\mathbf{x}^{(n)})$ be a Cauchy sequence in $\mathbb{R}^k$. Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $d(\mathbf{x}^{(m)}, \mathbf{x}^{(n)}) < \varepsilon$. Then for each $1 \leq j \leq k$, $|x_j^{(m)} - x_j^{(n)}| \leq d(\mathbf{x}^{(m)}, \mathbf{x}^{(n)}) < \varepsilon$, so each of the sequences $(x_k^{(n)})$ in $\mathbb{R}$ is Cauchy, and hence $(x_j^{(n)})$ converges to some x_j for each $1 \leq j \leq k$. Let $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$. Let $\varepsilon' > 0$. For each $1 \leq j \leq k$, there exists $N_j \in \mathbb{N}$ such that $|x_j^{(n)} - x_j| < \varepsilon'/k$ for all $n \geq N_j$. Let $N = \max\{N_1, \ldots, N_k\}$. Then for $n \geq N$, $d(\mathbf{x}^{(n)}, \mathbf{x}) = \sum_{j=1}^k |x_j^{(n)} - x_j| < \sum_{j=1}^k \frac{\varepsilon'}{k} = \varepsilon'$. Hence $(\mathbf{x}^{(n)})$ converges, so $(\mathbb{R}^k, d)$ with d as defined is complete.

7. Let (X, d) be a metric space and let $E \subseteq X$. Prove that x is a limit point of E if and only if there exists a sequence (x_n) of points in $E \setminus \{x\}$ such that (x_n) converges to x.

Solution. Suppose that x is a limit point of E. There exists $x_1 \in B_1(x) \setminus \{x\}$. Having already found $x_1, \ldots, x_{k-1}$, we can find x_k such that $x_k \in B_{1/k}(x) \setminus \{x\}$. By construction, the sequence (x_n) converges to x. For the converse, if (x_n) is a sequence of points in $E \setminus \{x\}$ which converges to x, then for any $r > 0$, there exists $N \in \mathbb{N}$ such that $x_n \in B_r(x)$ for all $n \geq N$, and since $x_n \in E \setminus \{x\}$ for all n, it follows that x is a limit point of E.

8. Let (X, d) be a metric space and suppose $E \subseteq F \subseteq X$. Prove that $E' \subseteq F'$.
(Notation: E' denotes the set of limit points of E.)

Solution. Let $x \in E'$. Then for any $r > 0$, there exists $y \in E \subseteq F$ such that $y \in B_r(x)$, so $x \in F'$.

9. Let (X, d) be a metric space and let $E \subseteq X$. Let $\overline{E} = E \cup E'$.
(a) Prove that $\overline{E}$ is closed.
(b) Prove that $E = \overline{E}$ if and only if E is closed.
(c) Prove that if F is a closed set such that $E \subseteq F$, then $\overline{E} \subseteq F$.

Solution. (a) We will show that $(\overline{E})^c$ is open. Let $x \in (\overline{E})^c$. Then $x \notin E$ and $x \notin E'$, so there exists $r > 0$ such that $B_r(x) \cap E = \emptyset$. I claim that also $B_r(x) \cap E' = \emptyset$. If not, then there exists $y \in B_r(x) \cap E'$, so for $s := r - d(x, y) > 0$, there exists $z \in B_s(y) \cap E$; but then $d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + s = r$, so $z \in B_r(x) \cap E$ which contradicts $B_r(x) \cap E = \emptyset$. Hence $B_r(x) \cap \overline{E} = \emptyset$, so $B_r(x) \subseteq (\overline{E})^c$. Therefore $(\overline{E})^c$ is open.
(b) If E is closed, then $E' \subseteq E$ so $E = \overline{E}$. The other direction follows from part (a).
(c) Suppose F is a closed set and $E \subseteq F$. By Problem 8, $E' \subseteq F' \subseteq F$, so $\overline{E} = E \cup E' \subseteq F$.

10. Let (X, d) be a metric space and let $E \subseteq X$. Let E° denote the set of all interior points of E.
(a) Prove that E° is open.
(b) Prove that $E = E^\circ$ if and only if E is open.
(c) Prove that if F is an open set such that $F \subseteq E$, then $F \subseteq E^\circ$.

Solution. (a) Let $x \in E^\circ$. There exists $r > 0$ such that $B_{2r}(x) \subseteq E$. Let $y \in B_r(x)$. Then for any $z \in B_r(y)$, $d(z,x) \leq d(z,y) + d(y,x) < 2r$, so $z \in B_{2r}(x) \subseteq E$ and hence $B_r(y) \subseteq E$. Therefore, $B_r(x) \subseteq E^\circ$.
(b) If E is open, then $E = E^\circ$ by definition. The other direction follows from part (a).
(c) Suppose F is an open set and $F \subseteq E$. Then for any $x \in F$, there exists $r > 0$ such that $B_r(x) \subseteq F \subseteq E$, so $x \in E^\circ$.