Math 104 Midterm Exam A (Printout Version)

UC Berkeley, Summer 2021

Thursday, July 15, 4:10pm - 6:00pm PDT

Problem 1. Short answers. No justification required for examples. (2 points each)

(a) Please copy verbatim the following text, followed by your signature. This MUST be handwritten UNLESS you are writing your entire exam electronically.

"As a member of the UC Berkeley community, I will act with honesty, integrity, and respect for others during this exam. The work that I will upload is my own work. I will not collaborate with or contact anyone during the exam, search online for problem solutions, or otherwise violate the instructions for this examination."

(b) Give an example of a sequence (s_n) of real numbers such that $\limsup s_n = \infty$, $\liminf s_n = 104$, and (s_n) has no nonincreasing subsequences.

$$S_n = 104 + (-1)^n n^{(-1)^n}$$

(c) What are the possible sizes of the set of interior points of a set in a metric space? (For example, is it possible for a set to have 0 interior points? 1? 2? Infinite?)

$\mathbb{N} \cup \{0, \infty\}$

(d) Let (X, d) be a metric space, and let $E \subseteq X$.
Assertion A: There exists $r > 0$ such that for every $x \in E$ we have $B_r(x) \subseteq E$.
Assertion B: E is open.
In general, is it true that (i) A imples B? (ii) B imply A?

(i) True

(ii) False

(e) Consider the set $S = \{(x, y) : 0 < \sqrt{x^2 + y^2} \leq 1\} \subseteq \mathbb{R}^2$. Find an open cover of S which has no finite subcover.

$$\left\{ \left\{ (x,y) : \frac{1}{n} < \sqrt{x^2+y^2} < 2 \right\} \right\}_{n \in \mathbb{N}}$$

Problem 2. (5 points) Prove that

$$\inf\underbrace{\{x - y + z : x > y > z > 0\}}_{S} = 0.$$

Proof: Since $x > y$ and $z > 0$, $\underbrace{x-y}_{>0} + \underbrace{z}_{>0} > 0$, so 0 is a lower bound for S.

Let $\varepsilon > 0$. Observe that $\underbrace{\varepsilon - \frac{\varepsilon}{2} + \frac{\varepsilon}{3}}_{\in S} = \frac{5}{6}\varepsilon < \varepsilon$ so ε is not a lower bound for S.

Therefore $\inf S = 0$.

Problem 3. (5 points) Let (s_n) be a sequence of real numbers. Suppose that $\lim|s_n| = s \in \mathbb{R}$. Prove that there exists a convergent subsequence (s_{n_k}) such that $|\lim s_{n_k}| = s$.

Proof: Since the sequence $(|s_n|)$ converges, it is bounded, hence so is the sequence (s_n). By the Bolzano-Weierstrass theorem, there exists a convergent subsequence (s_{n_k}). Let $x = \lim s_{n_k}$. Then

$$s = \lim|s_n| = \lim|s_{n_k}| = |x| = |\lim s_{n_k}|.$$

every subsequence of a convergent sequence converges to the same limit as the original sequence

$s_{n_k} \to x$
$\Rightarrow |s_{n_k}| \to |x|$

Problem 4. (5 points) Let (s_n) be a bounded sequence of real numbers. Let

$$v_n = \sup\{s_m : m \geq n\}$$

for each $n \in \mathbb{N}$. Prove that $\liminf(v_n - s_n) = 0$.

<u>Proof</u>: Let $\sigma_n = \inf\{\overbrace{v_m - s_m}^{\geq 0} : m \geq n\}$. Note that $\sigma_n \geq 0$.

Let $\varepsilon > 0$. There exists $m \geq n$ such that $s_m > v_n - \varepsilon$.

Then

$$v_m - s_m \underset{\uparrow}{\leq} v_n - s_m < \varepsilon$$

(v_n) is nonincreasing

hence $\sigma_n < \varepsilon$.

Therefore, $\sigma_n = 0$ for all n. By definition,

$$\liminf(v_n - s_n) = \lim \sigma_n = 0.$$

constant sequence of all 0's

Problem 5. (5 points) Let (X, d) be a metric space, and let $E \subseteq X$. Suppose that (x_n) is a Cauchy sequence in E which does not converge to any point in E. Construct an open cover of E which has no finite subcover, and prove that this is the case.
(Hint: For the solution that I have in mind, the open cover is constructed by taking an open ball around each $x \in E$ with carefully chosen radius.)

Proof: For each $x \in E$, since $x_n \not\to x$, there exists $r_x > 0$ such that for any $N \in \mathbb{N}$, there exists $n \geq N$ such that $d(x_n, x) \geq r_x$. Let $s_x = \frac{r_x}{2}$.

Consider the open cover $\{B_{s_x}(x)\}_{x \in E}$ of E.

Let $B_{s_{y_1}}(y_1), \ldots, B_{s_{y_k}}(y_k)$ be any finite subcollection of the open cover. Let $\varepsilon = \min(s_{y_1}, \ldots, s_{y_k})$.

Since (x_n) is Cauchy, there exists $N \in \mathbb{N}$ such that $m, n \geq N$ implies that $d(x_m, x_n) < \varepsilon$.

For each $i = 1, \ldots, k$, there exists $n_i \geq N$ such that $d(x_{n_i}, y_i) \geq r_{y_i}$. Then

$$d(x_N, y_i) \geq \underbrace{d(x_{n_i}, y_i)}_{\geq r_{y_i} = 2s_{y_i}} - \underbrace{d(x_{n_i}, x_N)}_{< \varepsilon \leq s_{y_i}} > s_{y_i}$$

so $x_N \notin B_{s_{y_i}}(y_i)$.

So $x_N \in E$ but $x_N \notin \bigcup_{i=1}^{k} B_{s_{y_i}}(y_i)$.