Math 104 Worksheet 13

UC Berkeley, Summer 2021

Tuesday, July 27

Theorem. Let $x_0 \in \mathbb{R}$. For the power series $\sum a_n(x-x_0)^n$, let $\beta := \limsup |a_n|^{1/n}$ and

$$R := \begin{cases} \frac{1}{\beta} & \text{if } 0 < \beta < \infty, \\ \infty & \text{if } \beta = 0, \\ 0 & \text{if } \beta = \infty. \end{cases}$$

(i) The power series converges for $|x - x_0| < R$.
(ii) The power series diverges for $|x - x_0| > R$.
R as defined above is called the **radius of convergence** of the power series.

Proof. Exercise 1. Restate the **root test** for a series of real numbers $\sum b_n$.

Exercise 2. Treat x as a fixed value, so $\sum a_n(x-x_0)^n = \sum b_n$ where $b_n = a_n(x-x_0)^n$. Compute the quantity of interest in the root test for this series; express your answer in terms of x and β.

Exercise 3. Consider the three cases $0 < \beta < \infty$, $\beta = 0$, and $\beta = \infty$ separately to justify the conclusion of the theorem.

Corollary. If $\lim \left|\frac{a_n}{a_{n+1}}\right|$ exists, then it is equal to the radius of convergence of the power series.

Exercise 4. Prove the preceding corollary.

Definition. The **interval of convergence** of the power series $\sum a_n(x-x_0)^n$ is the set $\{x \in \mathbb{R} : \text{ the series of real numbers } \sum a_n(x-x_0)^n \text{ converges}\}$. Note that the theorem guarantees that this set is an interval (which can be open, closed, or half-open-half-closed.)

Exercise 5. For each of the following power series, find the interval of convergence.

(a) $\sum \frac{1}{n!}x^n$

(b) $\sum x^n$

(c) $\sum \frac{1}{n}x^n$

(d) $\sum \frac{1}{n^2}x^n$

(e) $\sum n^{104}x^n$

(f) $\sum n!x^n$