Math 104 Worksheet 3

UC Berkeley, Summer 2021
Thursday, June 24

Prove the following basic limit theorems using the rigorous definition of a limit.

1. If $r \in \mathbb{R}$ and (s_n) converges to s, then (rs_n) converges to rs, i.e. $\lim(rs_n) = r\lim(s_n)$.

Proof. (Completed) Assume $r \neq 0$, since otherwise the result is trivial. Let $\varepsilon > 0$. (The goal is to find $N \in \mathbb{N}$ such that $|rs_n - rs| < \varepsilon$ for all $n \geq N$.) Since $s_n \to s$, there exists $N \in \mathbb{N}$ such that $|s_n - s| < \varepsilon/|r|$ for all $n \geq N$. Then $|rs_n - rs| < \varepsilon$ for all $n \geq N$, as desired. $\square$

2. If (s_n) converges to s and (t_n) converges to t, then $(s_n + t_n)$ converges to $s + t$, i.e. $\lim(s_n + t_n) = \lim s_n + \lim t_n$.

Proof. Let $\varepsilon > 0$. Goal: ...

(*Hint*: $|s_n + t_n - (s + t)| = |(s_n - s) + (t_n - t)| \leq |s_n - s| + |t_n - t|$.)

$\square$

3. If (s_n) converges to s and (t_n) converges to t, then $(s_n t_n)$ converges to st, i.e. $\lim(s_n t_n) = (\lim s_n) \cdot (\lim t_n)$.

Proof. Since (s_n) converges, it is a bounded sequence, so there exists $M \in \mathbb{R}$ such that $|s_n| \leq M$. Let $\varepsilon > 0$. Goal: ...

(*Hint*: $|s_n t_n - st| = |s_n t_n - s_n t + s_n t - st| \leq |s_n t_n - s_n t| + |s_n t - st| = |s_n| \cdot |t_n - t| + |t| \cdot |s_n - s|$.)

$\square$

4. Let (s_n) be a sequence of nonzero real numbers, and suppose (s_n) converges to $s \neq 0$. Then
(a) $\inf\{|s_n| : n \in \mathbb{N}\} > 0$;
(b) The sequence $(1/s_n)$ converges to $1/s$.

Proof.
(a) *Hint*: The proof is similar to the proof that convergent sequences are bounded.

(b) Let $\varepsilon > 0$. Goal: ...

Let $m = \inf\{|s_n| : n \in \mathbb{N}\}$. By part (a), $m > 0$. Let $N \in \mathbb{N}$ be such that $|s - s_n| <$ ______ for all $n \geq \mathbb{N}$. Then

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s - s_n}{s_n s}\right| = \frac{|s - s_n|}{|s_n| \cdot |s|} \leq \frac{|s - s_n|}{m|s|} < \underline{\qquad\qquad\qquad}$$

$\square$

5. Suppose that (s_n) converges to s and (t_n) converges to t. If $s \neq 0$ and $s_n \neq 0$ for all n, then (t_n/s_n) converges to t/s.

Proof. Hint: Use two of the previous problems on this worksheet.

$\square$