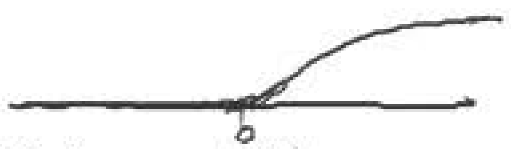
Math 104 Worksheet 17

UC Berkeley, Summer 2021

Wednesday, August 4

Consider the function

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$$



Exercise 1. Show that $f'(0) = 0$. (*Hint:* Consider the left and right limits separately.)

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{0}{x} = 0. \qquad \Rightarrow \lim_{x\to 0} \varphi_0(x) = 0 = f'(0)$$

$y = \frac{1}{x}$

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{x} = \boxed{\lim_{y\to\infty} \frac{y}{e^y}} \underset{\text{L'H.}}{=} \lim_{y\to\infty} \frac{1}{e^y} = 0$$

Exercise 2. Show by induction that for $x > 0$, $f^{(n)}(x)$ has the form

$$q_n\left(\frac{1}{x}\right)e^{-1/x}$$

where $q_n(t)$ is a polynomial in t.

Base case: $n=0$. $f(x) = q_0\left(\frac{1}{x}\right)e^{-\frac{1}{x}}$ where $q_0(x) = 1$.

Induction: Suppose true for some n: $f^{(n)}(x) = q_n\left(\frac{1}{x}\right)e^{-\frac{1}{x}}$.

Then
$$f^{(n+1)}(x) = q_n'\left(\frac{1}{x}\right)\cdot\frac{-1}{x^2}e^{-\frac{1}{x}} + q_n\left(\frac{1}{x}\right)e^{-\frac{1}{x}}\cdot\frac{1}{x^2}.$$
$$= \underbrace{\left[-q_n'\left(\frac{1}{x}\right)\cdot\frac{1}{x^2} + \frac{1}{x^2}q_n\left(\frac{1}{x}\right)\right]}_{\text{polynomial}} e^{-\frac{1}{x}}.$$

polynomial $q_{n+1}(x) = x^2\left(q_n(x) - q_n'(x)\right)$ evaluated at $\frac{1}{x}$.

Recall: $\lim_{y\to\infty} \frac{p(y)}{e^y} = 0$ (p: any polynomial)

Exercise 3. Show by induction that $f^{(n)}(0) = 0$ for all n. (Therefore, $T^{f,0}(x) \equiv 0$, so $f(x) \neq T^{f,0}(x)$ for all $x > 0$.)

Base case. $f^{(0)}(0) = f(0) = 0$.

Induction: Suppose $f^{(n)}(0) = 0$ for some n.

$$\lim_{x\to 0^+} \frac{f^{(n)}(x) - f^{(n)}(0)}{x-0} = \lim_{x\to 0^+} \frac{q_n\left(\frac{1}{x}\right)e^{-\frac{1}{x}}}{x} = \lim_{x\to 0^+} \underbrace{\frac{1}{x}q_n\left(\frac{1}{x}\right)}_{\text{polynomial}} e^{-\frac{1}{x}} \overset{y=\frac{1}{x}}{=} \lim_{y\to\infty} \frac{p(y)}{e^y} = 0.$$

$p(x) = x\,q_n(x)$ eval. at $\frac{1}{x}$.

$$T^{f,0}(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = 0.$$

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Recall. Let f be a bounded function on $[a, b]$. For a **partition** $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ we define

$$U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1}) \quad \text{and} \quad L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1})$$

where $M(f, S) = \sup\{f(x) : x \in S\}$ and $m(f, S) = \inf\{f(x) : x \in S\}$. Then we define

$$U(f) = \inf\{U(f, P) : P \in \Pi_{[a,b]}\} \quad \text{and} \quad L(f) = \sup\{L(f, P) : P \in \Pi_{[a,b]}\}$$

where $\Pi_{[a,b]}$ is the set of all partitions of $[a, b]$.

Definition. If $P, P^* \in \Pi_{[a,b]}$ and $P \subseteq P^*$, P^* is called a **refinement** of P.

Exercise 1. Prove that if P^* is a refinement of P, then

$$L(f, P) \leq L(f, P^*) \leq U(f, P^*) \leq U(f, P).$$

Proof. Let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$. For each subinterval $I_k = [a_{k-1}, a_k]$, P^* induces a partition $P_k^* = \{s \in P^* : a_{k-1} \leq s \leq a_k\} = \{a_{k-1} = s_0 < \ldots < s_m = a_k\}$ of I_k. (Complete the proof.)

Exercise 2. Prove that if $P, Q \in \Pi_{[a,b]}$, then $L(f, P) \leq U(f, Q)$. (Hint: Use Exercise 1.)

Let $g(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(t-x_0)^k + \frac{M(t-x_0)^n}{n!} - f(t).$ $(t \in I)$.

- $g(x_0) = f(x_0) - f(x_0) = 0$.

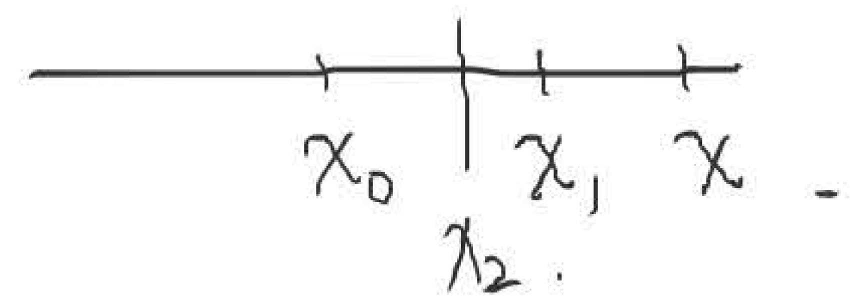
$$g'(t) = \sum_{k=1}^{n-1} \frac{f^{(k)}(x_0)}{(k-1)!}(t-x_0)^{k-1} + \frac{M(t-x_0)^{n-1}}{(n-1)!} - f'(t).$$

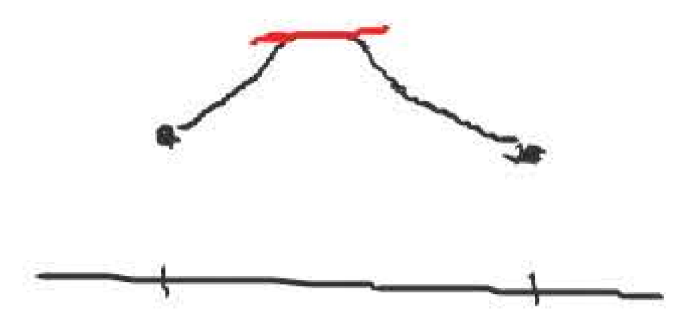
$$g'(x_0) = f'(x_0) - f'(x_0) = 0.$$

$$g''(x_0) = 0$$

$$\vdots$$

$$g^{(n-1)}(x_0) = 0.$$





- $g(x) = 0$, $g(x_0) = 0$.

Rolle's theorem $\Rightarrow$ there exists x_1 between x and x_0 such that

$$g'(x_1) = 0.$$

$g'(x_0) = 0$, $g'(x_1) = 0$ $\Rightarrow$ there exists x_2 : $g^{(2)}(x_2) = 0$.

...

there exists x_n : $g^{(n)}(x_n) = 0$.

$$g^{(n)}(t) = M - f^{(n)}(t)$$

$$0 = g^{(n)}(x_n) = M - f^{(n)}(x_n) \Rightarrow \boxed{f^{(n)}(x_n) = M}$$

Riemann integral

Let f be a bounded function on a closed interval $[a,b]$.
For $S \subseteq [a,b]$, define

$$M(f,S) = \sup\{f(x) : x \in S\}, \qquad m(f,S) = \inf\{f(x) : x \in S\}.$$

least upper bd of f on S greatest lower bd of f on S.

<u>Def</u>: A <u>partition</u> of $[a,b]$ is a finite ordered subset of the form $P = \{a = t_0 < t_1 < t_2 < \ldots < t_n = b\}$.

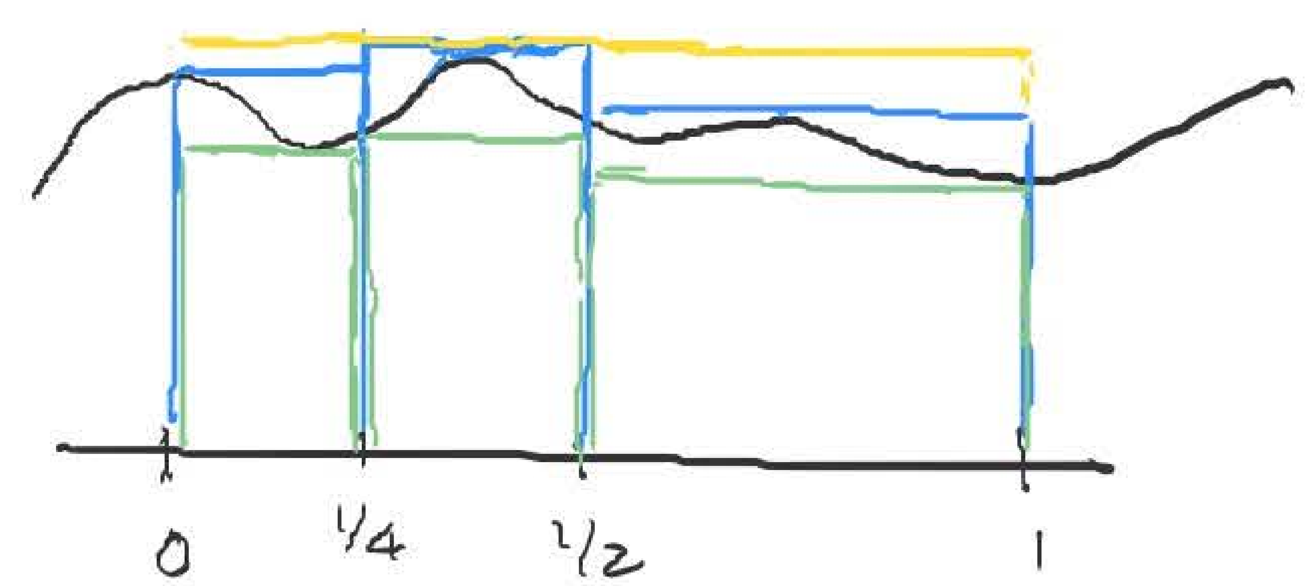
<u>Ex</u>. $P = \{0, \frac{1}{4}, \frac{1}{2}, 1\}$ is a partition of $[0,1]$.

<u>Def</u>: The <u>upper Darboux sum</u> $U(f,P)$ of f w.r.t. $P = \{t_0, \ldots, t_n\}$.

$$U(f,P) = \sum_{k=1}^{n} M(f, I_k) \cdot \ell(I_k) \quad \text{where } I_k = [t_{k-1}, t_k],\ \ell(I_k) = t_k - t_{k-1}.$$

The <u>lower Darboux sum</u> $L(f,P)$ ...

$$L(f,P) = \sum_{k=1}^{n} m(f, I_k)\,\ell(I_k).$$



Observe:

$$-\infty < m(f,[a,b])(b-a) \le L(f,P) \le U(f,P) \le M(f,[a,b])(b-a) < \infty.$$

Def: The <u>upper Darboux integral</u> $U(f)$ of f over $[a,b]$ is

$$U(f) = \inf \{ U(f,P) : P \in \Pi_{[a,b]} \}.$$

set of all partitions of $[a,b]$.

Def: The <u>lower Darboux integral</u>

$$L(f) = \sup \{ L(f,P) : P \in \Pi_{[a,b]} \}.$$

By previous observation, $U(f), L(f) \in \mathbb{R}$.

Taylor's theorem: Let f be defined on an open interval I containing x_0 such that the n^{th} derivative of f exists at every point in I.

$\Rightarrow$ first n derivatives of f exist.

Then for each $x \in I \setminus \{x_0\}$, there exists α_x between x and x_0 such that

$$R_n^{f, x_0}(x) = \frac{f^{(n)}(\alpha_x)}{n!}(x-x_0)^n, \quad \text{i.e.}$$

$$f(x) = \underbrace{\sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}_{\text{Taylor polynomial}} + \boxed{\underbrace{\frac{f^{(n)}(\alpha_x)}{n!}(x-x_0)^n}_{R_n^{f, x_0}(x)}}.$$

"error term"

Ideally, would like

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

Don't have this.

Do have:

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n)}(\alpha_x)}{k!}(x-x_0)^k.$$

Proof:

Scratch work: Rearrange the above:

$$f^{(n)}(\alpha_x) = \underbrace{\frac{n!}{(x-x_0)^n}\left(f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \right)}_{=: M}$$

Want to show that there exists α_x between x and x_0 such that $\boxed{f^{(n)}(\alpha_x) = M}$.

Corollary: If f is infinitely differentiable on an interval I containing x_0 and there exists $M > 0$ such that

$$|f^{(n)}(x)| \le M \quad \text{for all } n \ge 0,\ x \in I$$

then $f(x) = T^{f,x_0}(x)$ for all $x \in I$.

Proof: By Taylor's theorem, for each $n \in \mathbb{N}$ and $x \in I$, there exists $\alpha_x^{(n)}$ between x and x_0 such that

$$\left| R_n^{f,x_0}(x) \right| = \left| \frac{f^{(n)}(\alpha_x^{(n)})}{n!} (x - x_0)^n \right| \le \frac{M}{n!} |x - x_0|^n \xrightarrow[n \to \infty]{} 0.$$

Ex $f(x) = e^x$ infinitely differentiable on $(-\infty, \infty)$.

$$T^{f,0}(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$

Let $x^* \in \mathbb{R}$. $x^* \in \underbrace{\left(-|x^*|-1,\ |x^*|+1\right)}_{I}$. $|f^{(n)}(x)| \le e^x \le e^{|x^*|+1}$ for all n, all $x \in I$.

By corollary, $T^{f,0}(x^*) = f(x^*)$, i.e.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{for all } x \in \mathbb{R}.$$

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Taylor series

Def: $f^{(n)}$ denotes the n^{th} derivative of f.

Def: f is infinitely differentiable at x_0 if $f^{(n)}(x_0)$ exists for all $n \in \mathbb{N}$.

existence implies $f^{(n-1)}$ exists on an open interval containing x_0.

Def: Let f be a function defined on an open interval I containing x_0.
If f is infinitely differentiable at x_0, define the Taylor series for f about x_0 as the power series

$$\sum_{k=0}^{\infty} \underbrace{\frac{f^{(k)}(x_0)}{k!}}_{a_k} (x-x_0)^k =: T^{f,x_0}(x)$$

on interval of converge of the Taylor series.

Def: The n^{th} remainder (of the above) is $R_n^{f,x_0}(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$.

Observation: for any $x \in I$, $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \Leftrightarrow R_n^{f,x_0}(x) \to 0$ as $n \to \infty$.