University of California, Berkeley
Math 104: Introduction to Analysis

Instructor: Theodore Zhu

Midterm Exam

July 13, 2017

10:10 AM – 11:55 AM

Name: ______________________________

Student ID: _________________________

Instructions. This is a closed-book, closed-notes, closed-electronics exam. Please write carefully and clearly in the spaces provided. If you run out of space for a problem, you may continue on the reverse side of the page, or on the extra pages at the end. Cross out any work that you do not want to be graded. Unless otherwise specified, show all work and justify any nontrivial claims. **You may use any results from lecture and homework problems, but you must clearly state the result that you are using.**

Question	Points	Score
1	10	
2	5	
3	5	
4	5	
5	5	
6	5	
Total:	35	

1. **Short Answer.** No justification required.

   (a) (2 points) In the metric space $\mathbb{R}$ with standard Euclidean metric, give an example of an infinite set S of rational numbers such that S is a closed and bounded subset of $\mathbb{R}$.

   (b) (2 points) Give an example of a sequence (s_n) of real numbers such that $\limsup s_n = \infty$, $\liminf s_n = -\infty$, and the sequence $(\bar{s}_n)$ defined by $\bar{s}_n := \frac{s_1+\ldots+s_n}{n}$ converges.

   (c) (2 points) In the metric space $\mathbb{R}^2$ with standard Euclidean metric, let

$$E := [0,1] \times (0,1) = \{(x,y) : 0 \leq x \leq 1,\ 0 < y < 1\} \subseteq \mathbb{R}^2.$$

   Give an example of an open cover of E which has no finite subcover.

(d) (2 points) Give an example of a metric space (X, d) and a nonempty set $E \subsetneq X$ such that E is both open and closed in X. (The notation $E \subsetneq X$ means $E \subseteq X$ and $E \neq X$.)

(e) (2 points) Give an example of a metric space (X, d) and a set $E \subseteq X$ such that E is closed and bounded, but not compact.

2. (5 points) Let (s_n) and (t_n) be two sequences of real numbers such that (s_n) is nondecreasing, (t_n) is nonincreasing, $s_n \leq 104 \leq t_n$ for all $n \in \mathbb{N}$, and $|t_n - s_n| < \frac{1}{n}$ for all n. Prove that (s_n) and (t_n) both converge and $\lim s_n = \lim t_n = 104$.

3. (5 points) Let (X, d) be a discrete metric space, so for any $x, y \in X$,

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Prove that a set $E \subseteq X$ is compact if and only if E is a finite set.

4. (5 points) Let (X, d) be a metric space and let $E \subseteq X$. Let E' denote the set of all limit points of E. Prove that E' is closed.

5. (5 points) Let (s_n) be a bounded sequence of real numbers. Suppose $\alpha \in \mathbb{R}$ has the property that for any $\beta > \alpha$, there exists $N \in \mathbb{N}$ such that $s_n < \beta$ for all $n \geq N$. Prove that $\limsup s_n \leq \alpha$.

6. (5 points) Let (X, d) be a metric space. Prove that if (x_n) and (y_n) are two Cauchy sequences in X, then the sequence $\big(d(x_n, y_n)\big)_{n\in\mathbb{N}}$ of real numbers converges.
(Hint: $\mathbb{R}$ is complete. Note that $d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n)$.)

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