Q1

Let $f(0) = f(2) = c$. Then we have

$$g(0) = f(0+1) - f(0) = f(1) - c,$$
$$g(1) = f(1+1) - f(1) = c - f(1).$$

If $f(1) = c$, then $g(0) = g(1) = 0 \implies f(1) = f(0)$ and $f(2) = f(1)$, as desired.

If $f(1) > c$, then $g(1) < 0 < g(0) \implies \exists x_0 \in (0,1)\ f(x_0+1) - f(x_0) = g(x_0) = 0$ by intermediate value theorem. Thus we have $|(x_0+1) - x_0| = 1$ and $f(x_0+1) = f(x_0)$ as desired.

If $f(1) < c$, then $g(0) < 0 < g(1) \implies \exists x_0 \in (0,1)\ f(x_0+1) - f(x_0) = g(x_0) = 0$ by intermediate value theorem. Thus we have $|(x_0+1) - x_0| = 1$ and $f(x_0+1) = f(x_0)$ as desired, completing the proof.

Q2

(Contrapositive) Suppose f is unbounded on (a, b), i.e., $\forall M > 0\ \exists x \in (a, b)\ |f(x)| > M$. Thus we can inductively construct a sequence $(x_n) \in (a, b)$ such that $\forall n \in \mathbb{N}\ |f(x_n)| > n$. Since (x_n) is bounded in (a, b), it has a convergent subsequence $(x_{n_k}) \in (a, b)$ which is also Cauchy. Clearly $\forall k \in \mathbb{N}\ |f(x_{n_k})| > n_k$ which implies $f(x_{n_k})$ is not convergent and hence not Cauchy. Since (x_n) is Cauchy but $f(x_n)$ is not Cauchy, f is not uniformly continuous on (a, b).

Q3

(a) Since f and g are continuous on $\mathbb{R}$, $f-g$ is also continuous on R. Let $r \in \mathbb{R}\backslash\mathbb{Q}$. Suppose $f(r) \neq g(r)$, i.e. $(f-g)(r) = c_r \neq 0$. Let $\epsilon = |c_r|$, then for each $\delta > 0$ $\exists q \in \mathbb{Q}$ such that

$$\begin{aligned} r < q < r + \delta \text{ and } |(f-g)(q) - (f-g)(r)| &= |0 - |c_r|| \\ &= |c_r| \\ &= \epsilon \end{aligned}$$

implying that $f-g$ is not continuous at r which is a contradiction. Thus $f(r) = g(r)$ for each $r \in \mathbb{R}\backslash\mathbb{Q}$. Since $f(q) = g(q)$ for every $q \in \mathbb{Q}$, we have $f(x) = f(x)$ for every $x \in \mathbb{R}$.

(b) We generalize part (a) to "For any dense subset E of X, if $f(x) = g(x)$ for every $x \in E$, then $f(x) = g(x)$ for all $x \in X$". The proof is the following:

Note that $f-g$ is continuous on X and hence on $X\backslash E$. For any $x \in E$, the statement is trivially true by the statement itself. Let $x \in E\backslash X$. Suppose $f(x) \neq g(x)$, i.e. $f(x) - g(x) = c_x \neq 0$. Let $\epsilon = |c_x|$. Since E is dense in X, for each $\delta > 0$ $\exists y \in B_\delta(x) \cap E$ such that

$$|(f-g)(y) - (f-g)(x)| = |0 - (f-g)(x)| = |(f-g)(x)| = |c_x| = \epsilon.$$

Thus $f-g$ is not continuous at x by definition, which is a contradiction. Thus $f(x) = g(x)$ for each $x \in X\backslash E$, completing the proof.

Q4

Let $x_0 \in \mathbb{Q}$. Then $f(x_0) = \frac{1}{\varphi(x_0)} \neq 0$. Consider a sequence $(x_n) \subseteq \mathbb{R}$ where $\forall n \in \mathbb{N}$ $x_n = x_0 + \frac{\sqrt{2}}{n}$. Clearly (x_n) converges to x_0 but each term of it is irrational, i.e. $\forall n \in \mathbb{N}$ $f(x_n) = 0$. Thus $f(x_n)$ converges to 0 which is different from the limit of (x_n). Hence f is discontinuous at x_0.

Let $x_0 \in \mathbb{R} \backslash \mathbb{Q}$. Define $\delta_n = \min\{\left|x_0 - \frac{k}{n}\right| : k \in \mathbb{Z}\}$ for each $n \in \mathbb{N}$. Note that $\delta_n > 0$ because x_0 is irrational. Consider an arbitrary $\epsilon > 0$. By Archimedean Property there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Take $\delta = \min\{\delta_k\}_{k=1}^{N}$. Then for each $x \in \mathbb{R}$ such that $|x - x_0| < \delta$, if $x \in \mathbb{R}\backslash\mathbb{Q}$, then $|f(x) - f(x_0)| = |0 - 0| = 0 < \epsilon$ trivially; if $x \in \mathbb{Q}$, since x is not the multiple of any $\frac{1}{n}$ where $n \leq N$, $|f(x) - f(x_0)| = \left|\frac{1}{\varphi(x)}\right| < \frac{1}{N} < \epsilon$. Thus f is continuous at x_0, completing the proof.

Q5

(a) Let $\epsilon > 0$ and $\delta = \frac{\epsilon}{2}$. Suppose two points $(x, y), (u, v) \in (X \times X)$ such that $d^*((x, y), (u, v)) < \delta$, i.e., $\max\{d(x, u), d(y, v)\} < \delta \implies d(x, u) < \frac{\epsilon}{2}$ and $d(y, v) < \frac{\epsilon}{2}$. Then we have

$$d(u, v) \leq d(u, x) + d(x, y) + d(y, v) < \frac{\epsilon}{2} + d(x, y) + \frac{\epsilon}{2} = d(x, y) + \epsilon \tag{1}$$

and

$$d(x, y) \leq d(x, u) + d(u, v) + d(v, y) < \frac{\epsilon}{2} + d(x, y) + \frac{\epsilon}{2} = d(u, v) + \epsilon. \tag{2}$$

Both (1) and (2) implies

$$d(u, v) - d(x, y) < \epsilon \quad \text{and} \quad d(x, y) - d(u, v) < \epsilon.$$

i.e.

$$|d(x, y) - d(u, v)| < \epsilon.$$

Thus d is uniformly continuous by definition.

(b) Since E is a compact set, by HW5 1(b) $E \times E$ is a compact set as well. Since d is uniformly continuous on $X \times X$, d is also uniformly continuous on $E \times E$ which is a subset of $X \times X$. When $E \times E$ is compact, d attains its maximum on $E \times E$, i.e. $\exists x, y \in E$ $d(x, y) = \sup d(E \times E) = \delta$.

Q6

(a) First let's prove the hint. Let $x, y \in X$. Thus $\forall a \in A$

$$\begin{aligned} d(x, A) &= \inf\{d(x,a) : a \in A\} \\ &\leq d(x,a) \\ &\leq d(x,y) + d(y,a), \end{aligned}$$

where the last inequality comes from the triangular inequality of d. Thus we have $\forall a \in A\ d(y,a) \geq d(x,A) - d(x,y)$. Since $d(x,A) - d(x,y)$ is a lower bound of $\{d(x,a) : a \in A\}$, $d(y,A) \geq d(x,A) - d(x,y)$, i.e., $d(x,A) - d(y,A) \leq d(x,y)$. Now let $\epsilon > 0$ and let $\delta = \epsilon$, then $d(x,y) < \delta \implies |f(x) - f(y)| = |d(x,A) - d(y,A)| \leq d(x,y) < \epsilon$ which implies f is uniformly continuous on X.

(b) From (a), f is continuous on X, and hence f is continuous on $E \subseteq X$. Since E is compact, f achieves its minimum on E. i.e. $\exists x_0 \in E\ f(x_0) = \min\{f(x) : x \in E\} = \inf\{d(x,A) : x \in E\}$, as desired.

If $A = \{a\}$, then $\exists x_0 \in E\ f(x_0) = \inf\{d(x,A) : x \in E\} = \inf\{\inf\{d(x,y) : y \in A\} : x \in E\} = \inf\{\inf\{d(x,a)\} : x \in E\} = \inf\{d(x,a) : x \in E\}$, i.e. x_0 is the closet element in E to a.

(c) First notice that since $d(x,a) \geq 0$ for each $x \in E$ and $a \in A$, $\inf\{d(x,a) : x \in E, a \in A\} \geq 0$. By part (b), since E is a nonempty compact subset of X, there exists $x_0 \in E$ such that $f(x_0) = \inf\{d(x,A) : x \in E\} = \inf\{d(x,a) : x \in E, a \in A\}$. Note that $f(x_0) \geq 0$. Suppose $f(x_0) = 0$, then $\forall \epsilon > 0\ \exists a_\epsilon \in A\ d(x_0, a_\epsilon) < \epsilon$. Since $E \cap A = \emptyset$, $a_\epsilon \neq x_0$. Thus we have $\forall \epsilon > 0\ (B_\epsilon(x_0) \backslash \{x_0\}) \cap A \neq \emptyset$, i.e. x_0 is a limit point of A. Since A is closed, x_0 is in A, and hence $x_0 \in E \cap A \implies E \cap A \neq \emptyset$ which is a contradiction. Thus $f(x_0) \neq 0$, i.e. $\inf\{d(x,a) : x \in E, a \in A\} = f(x_0) > 0$.

(d) Consider ($\mathbb{R}^2$, Euclidean Metric). Let $E = \{(x,y) : x > 0, y \geq \frac{1}{x}\}$ and $A = \{(x,y) : x > 0, y \leq -\frac{1}{x}\}$. Clearly E and A are both closed but not bounded, and $\inf\{d(x,a) : x \in E, a \in A\} = 0$ because as $x \to \infty$ the boundary lines of two sets approaches to x-axis infinitely but never actually reach to x-axis.

Q7

Consider an arbitrary open set U in Y. By the well-known property of discrete metric space, any subset of X is open. Let f be an arbitrary function $f : X \to Y$. Since $f^{-1}(U)$ is in X, $f^{-1}(U)$ is open and hence f is continuous.

Q8

Let x_0 be an arbitrary fixed point in X. Define a sequence $(x_n) = (f(x_0), f(f(x_0)), \dots)$ such that for each $n \in \mathbb{N}$ $x_n = f^n(x_0)$. If $d(f(x_0), x_0) = 0$, then $f(x_0) = x_0$, and hence x_0 is a fixed point. Otherwise, since $c < 1$, $\sum c^n$ converges, and hence for each $\epsilon > 0$ $\exists N \in \mathbb{N}$ $m \geq n \geq N \implies \left|\sum_{k=n}^{m-1} c^k\right| < \frac{\epsilon}{d(f(x_0),x_0)}$ by Cauchy criterion. Observe that

$$\begin{aligned}
d(x_m, x_n) = d(f^m(x_0), f^n(x_0)) &\leq d(f^m(x_0), f^{m-1}(x_0)) + \cdots + d(f^{n+1}(x_0), f^n(x_0)) \\
&\leq c^{m-1} d(f(x_0), x_0) + \cdots + c^n d(f(x_0), x_0) \\
&= \left(\sum_{k=n}^{m-1} c^k\right) d(f(x_0), x_0) \\
&< \frac{\epsilon}{d(f(x_0), x_0)} \cdot d(f(x_0), x_0) \\
&= \epsilon.
\end{aligned}$$

Thus (x_n) is Cauchy, and by completeness of X, (x_n) converges to $x' \in X$. Since f is continuous on X, $f(x_n)$ converges to $f(x')$. Note that $f(x_n)$ is actually equivalent to (x_{n+1}), so (x_{n+1}) converges to the same limit of (x_n). i.e. $f(x_n) \to x'$. By the uniqueness of limit, $f(x') = x'$, i.e. x' is a fixed point of f.

To prove the uniqueness of fixed point, suppose there are two distinct fixed points of f which are $f(x_0) = x_0$ and $f(y_0) = y_0$. Notice that

$$d(x_0, y_0) = d(f(x_0), f(y_0)) \leq c \cdot d(x_0, y_0) < d(x_0, y_0)$$

when $c < 1$, which is a contradiction. Thus, f has a unique fixed point.

Q9

By the definition of $Z(f)$, $Z(f)$ is a subset of X trivially.

Let $x_0 \in (Z(f))'$. Since x_0 is a limit point of $Z(f)$, there exists a sequence (x_n) of points in $Z(f)\backslash\{x_0\}$ such that (x_n) converges to x_0. Since f is continuous on X, $f(x_n)$ converges to $f(x_0)$. Because each term of (x_n) is in $Z(f)$, $f(x_n) = 0$ for each $n \in \mathbb{N}$, and hence $f(x_n)$ converges to 0. Since the limit of a sequence is unique, $f(x_0) = 0 \implies x_0 \in Z(f) \implies (Z(f))' \subseteq Z(f) \implies Z(f)$ is closed, as desired.

Q10

(Contradiction) Suppose f is non-monotonic, i.e. WLOG $\exists x, y, z \in \mathbb{R}$ with $x < y < z$ such that $f(x) < f(y)$ and $f(y) > f(z)$. Since $[x, z]$ is closed and bounded in $\mathbb{R}$, $[x, z]$ is compact. We know f is continuous on $[x, z]$, so $\exists a \in [x, z]$ such that $f(a) = \max f([x, z])$. Because $f(x) < f(y)$ and $f(y) > f(z)$, $a \neq x$ and $a \neq z$, and hence $a \in (x, z)$ which is open. By the assumption in the question, $f((x, z))$ is open in $\mathbb{R}$, and $f(a) = \max f((a, z))$. However, an open set $f((x, z))$ cannot have a maximum because if $f(a)$ actually exists, then at $f(a)$, for each $\epsilon > 0$ there exists a number c such that $f(a) < c < f(a) + \epsilon$, which means $f(a)$ is not an interior point of $f((x, z))$. Thus $f((x, z))$ is not open, which is a contradiction, completing the proof.