Math 104 Worksheet 18

UC Berkeley, Summer 2021
Thursday, August 5

Recall. Let f be a bounded function on $[a,b]$. For a **partition** $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ we define

$$U(f,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k]) \cdot (t_k - t_{k-1}) \quad \text{and} \quad L(f,P) = \sum_{k=1}^{n} m(f,[t_{k-1},t_k]) \cdot (t_k - t_{k-1})$$

where $M(f,S) = \sup\{f(x) : x \in S\}$ and $m(f,S) = \inf\{f(x) : x \in S\}$. Then we define

$$U(f) = \inf\{U(f,P) : P \in \Pi_{[a,b]}\} \quad \text{and} \quad L(f) = \sup\{L(f,P) : P \in \Pi_{[a,b]}\}$$

where $\Pi_{[a,b]}$ is the set of all partitions of $[a,b]$.

Definition. If $P, P^* \in \Pi_{[a,b]}$ and $P \subseteq P^*$, P^* is called a **refinement** of P.

Exercise 1. Prove that if P^* is a refinement of P, then

$$L(f,P) \leq L(f,P^*) \leq U(f,P^*) \leq U(f,P).$$

Proof. Let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$. For each subinterval $I_k = [a_{k-1}, a_k]$, P^* induces a partition $P_k^* = \{s \in P^* : a_{k-1} \leq s \leq a_k\} = \{a_{k-1} = s_0 < \ldots < s_m = a_k\}$ of I_k. (Complete the proof.)

Exercise 2. Prove that if $P, Q \in \Pi_{[a,b]}$, then $L(f,P) \leq U(f,Q)$.
(Hint: Use Exercise 1.)

Exercise 3. Prove that $L(f) \leq U(f)$.

Definition. f is **integrable/Darboux integrable/Riemann integrable** if $L(f) = U(f)$.

Lemma. Let f and g be two bounded functions on $[a, b]$. Then

(i) $\inf\{U(f, P)+U(g, P) : P \in \Pi_{[a,b]}\} = \inf\{U(f, P) : P \in \Pi_{[a,b]}\}+\inf\{U(g, P) : P \in \Pi_{[a,b]}\}$;

(ii) $\sup\{L(f, P)+L(g, P) : P \in \Pi_{[a,b]}\} = \sup\{L(f, P) : P \in \Pi_{[a,b]}\}+\sup\{L(g, P) : P \in \Pi_{[a,b]}\}$.

Exercise 4. Prove part (i) of the preceding lemma.