Math 104 Final Exam

Wenhao Pan

TOTAL POINTS

31 / 40

QUESTION 1

Problem 1 13 pts

1.1 a **2 / 2**

✓ **+ 2 pts Correct**

+ 0 pts Incorrect

1.2 b **2 / 2**

✓ **+ 2 pts Correct**

+ 0 pts Incorrect

1.3 c **4 / 4**

✓ **+ 2 pts (i) Correct**

+ 0 pts (i) Incorrect

✓ **+ 2 pts (ii) Correct**

+ 0 pts (ii) Incorrect or insufficient justification

1.4 d **4 / 4**

✓ **+ 4 pts (i) and (ii) both correct**

+ 2 pts (i) or (ii) correct

+ 0 pts Incorrect

1.5 e **1 / 1**

✓ **+ 1 pts Correct**

+ 0 pts Incorrect

QUESTION 2

2 Problem 2 **6 / 6**

✓ **+ 6 pts Correct with valid proof**

+ 5 pts Correct with almost valid proof or minor error

+ 1 pts Some progress

+ 0 pts Minimal progress

QUESTION 3

3 Problem 3 **1 / 6**

+ 6 pts Correct

+ 5 pts Almost correct

✓ **+ 1 pts Some progress**

+ 0 pts Minimal progress

QUESTION 4

Problem 4 8 pts

4.1 a **4 / 4**

✓ **+ 4 pts Correct**

+ 1 pts Some progress

+ 0 pts Minimal progress

4.2 b **4 / 4**

✓ **+ 4 pts Correct**

+ 3 pts Almost correct

+ 1 pts Some progress

+ 0 pts Minimal progress

QUESTION 5

Problem 5 7 pts

5.1 a **2 / 2**

✓ **+ 2 pts Correct**

+ 0 pts Incorrect

5.2 b **1 / 5**

+ 5 pts Correct

+ 4 pts Correct construction but lacking rigorous proof

✓ **+ 1 pts Some progress**

+ 0 pts Minimal progress

QUESTION 6

6 Problem 6 **0 / 0**

+ 0 pts Correct

+ **0 pts** Minimal progress

✓ + **0 pts** **N/A**

Math 104 Final Exam A (Printout Version)

UC Berkeley, Summer 2021

Friday, August 12, 4:10pm - 6:00pm PDT

Problem 1. Short answers. No justification required for examples.

(a) (2 points) Please copy verbatim the following text, followed by your signature. This MUST be handwritten UNLESS you are writing your entire exam electronically.

"As a member of the UC Berkeley community, I will act with honesty, integrity, and respect for others during this exam. The work that I will upload is my own work. I will not collaborate with or contact anyone during the exam, search online for problem solutions, or otherwise violate the instructions for this examination."

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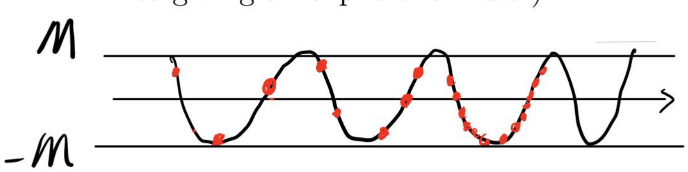
Wenhao Pan

(b) (2 points) Give an example of a bounded divergent sequence (s_n) of real numbers such that $\lim(s_{n+1} - s_n) = 0$. (Suggestion: It may be easier to describe an example as opposed to giving an explicit formula.)



Sn ✗ converge. diff. between each term is closing

$\exists \varepsilon > 0$

An example will be like the graph above. Red points represent each term. Each term is bounded by M (i.e. $|S_n| \le M$), so all the terms are on a wave/sin func. like path. The vertical distances between each points are decreasing as $n \to \infty$. However, as long as all terms are moving along the wave/sin func. like path, Sn will not converge to any value on the path.

1.1 a 2 / 2

✓ **+ 2 pts** Correct

+ 0 pts Incorrect

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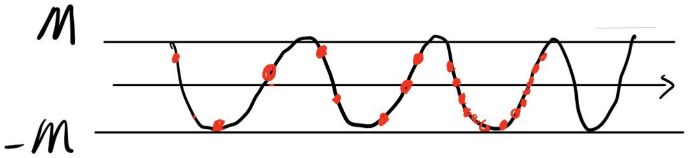

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1.2 b 2 / 2

✓ + 2 pts Correct

+ 0 pts Incorrect

(c) Let f be a function defined on (a,b). Let P be some property that f may or may not satisfy on any given subset of (a,b).

Assertion $[A]$: f satisfies the property P on every closed interval $[s,t] \subseteq (a,b)$.
Assertion $[B]$: f satisfies the property P on (a,b).

(i) (2 points) Give an example of a property P for which $[A]$ implies $[B]$.

f is $\underbrace{\text{continuous}}_{P}$ on every closed interval $[s,t] \subseteq (a,b)$

let $x_0 \in (a,b)$. find $[c,d]$ s.t. $x \in [c,d] \Rightarrow f$ is cont. on $[c,d]$

i.e. f is cont. at x_0

$\therefore f$ is cont. on (a,b)

(ii) (2 points) Is it true for any property P that $[A]$ implies $[B]$? Justify your answer.

No. For example, consider $f(x) = \frac{1}{x}$ on $(0,1)$.
Clearly, $\frac{1}{x}$ is $\underbrace{\text{uniformly continuous}}_{P}$ on any closed interval $\subseteq (0,1)$ since $\frac{1}{x}$ is continuous on $(0,1)$, but clearly $\frac{1}{x}$ is not $\underbrace{\text{uniformly continuous}}_{P}$ on $(0,1)$
i.e. $[A] \not\Rightarrow [B]$

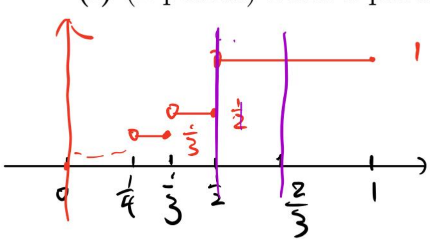
(d) Consider the function f defined on $[0,1]$ given by

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right], n \in \mathbb{N} \\ 0 & \text{if } x = 0 \end{cases}$$

$\frac{4}{6} - \frac{3}{6}$

(i) (2 points) Find a partition P of $[0,1]$ such that $U(f,P) - L(f,P) \leq \frac{1}{3}$.

$a > \frac{1}{2}$



$P = \{0, \frac{1}{2}, \frac{2}{3}, 1\}$

$\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$

$[\frac{1}{2}, a]$

$\frac{1}{2}(a - \frac{1}{2}) = \frac{1}{12}$

$(a - \frac{1}{2}) = \frac{1}{6}$

$a = \frac{1}{6} + \frac{1}{2} = \frac{4}{6} = \frac{2}{3}$

14

50

$8+6+36$

$[0, \frac{1}{2}]$

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$\frac{1}{2}(\frac{1}{2} - 0) = \frac{1}{4}$.

$[a, 1] = 0$

$\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

(ii) (2 points) Compute $U(f,P)$ and $L(f,P)$ for your partition P.

$$U(f,P) = M(f,[0,\tfrac{1}{2}]) \cdot \tfrac{1}{3} + M(f,[\tfrac{1}{2},\tfrac{2}{3}]) + M(f,[\tfrac{2}{3},1])$$
$$= \tfrac{1}{2} \cdot \tfrac{1}{2} + 1 \cdot \tfrac{1}{6} + 1 \cdot \tfrac{2}{3} = \tfrac{1}{4} + \tfrac{1}{6} + \tfrac{2}{3} = \tfrac{3+2+8}{12} = \tfrac{13}{12}$$

$$L(f,P) = m(f,[0,\tfrac{1}{2}]) \cdot \tfrac{1}{3} + m(f,[\tfrac{1}{2},\tfrac{2}{3}]) + m(f,[\tfrac{2}{3},1])$$
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$$\therefore U(f,P) - L(f,P) = \tfrac{13}{12} - \tfrac{9}{12} = \tfrac{4}{12} = \tfrac{1}{3} \leq \tfrac{1}{3}$$

(e) (1 point) Submit your exam on time via Gradescope, and correctly assign pages to every problem you submit.

1.3 C **4 / 4**

- ✓ **+ 2 pts** **(i) Correct**
- **+ 0 pts** (i) Incorrect
- ✓ **+ 2 pts** **(ii) Correct**
- **+ 0 pts** (ii) Incorrect or insufficient justification

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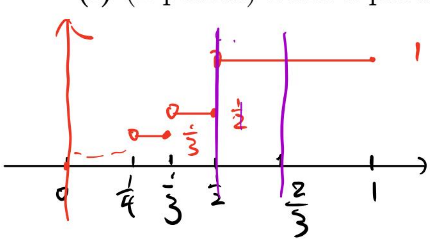

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1.4 d 4 / 4

✓ + **4 pts** (i) and (ii) both correct

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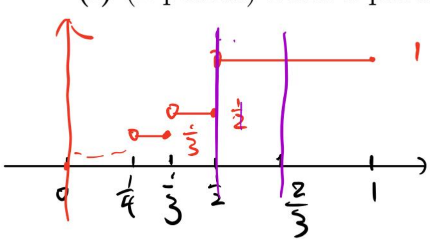

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$[0, \frac{1}{2}]$ $\quad a > \frac{1}{2}$

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1.5 e 1 / 1

✓ + 1 pts Correct

+ 0 pts Incorrect

Problem 2. (6 points) Suppose the series $\sum_{n=0}^{\infty}(-1)^n a_n$ converges, but not absolutely. Let $M > 0$. What is the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{a_n}{M^n} x^n$? Prove your answer. (Note: There is no assumption that the a_n are nonnegative.)

$$\beta = \limsup \left|\frac{a_n}{M^n}\right|^{\frac{1}{n}} = \limsup \frac{|a_n|^{\frac{1}{n}}}{|M^n|^{\frac{1}{n}}} = \limsup \frac{|a_n|^{\frac{1}{n}}}{(M^n)^{\frac{1}{n}}} = \frac{1}{M} \limsup |a_n|^{\frac{1}{n}}$$

$(M>0)$

$\sum(-1)^n a_n$ converges but not absolutely $\Rightarrow \alpha = \limsup |(-1)^n a_n|^{\frac{1}{n}}$

$= \limsup |a_n|^{\frac{1}{n}} = 1$

(if $\alpha<1$, then $\sum(-1)^n a_n$ converges absolutely by root test

if $\alpha>1$, then $\sum(-1)^n a_n$ diverges by root test ✗)

Thus $\beta = \frac{1}{M} \cdot 1 = \frac{1}{M}$ and $R = \frac{1}{\beta}$ ($\beta = \frac{1}{M} > 0$ (since $M>0$) and $\beta < \infty$)

$= M$. i.e. the radius of convergence is M.

2 Problem 2 6 / 6

✓ + **6 pts** **Correct with valid proof**

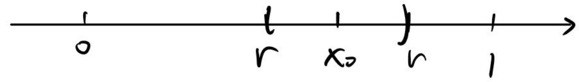
+ **5 pts** Correct with almost valid proof or minor error

+ **1 pts** Some progress

+ **0 pts** Minimal progress

Problem 3. (6 points) Suppose (f_n) is a sequence of functions defined on $[0,1]$, and suppose f is a function defined on $[0,1]$ such that for each $x \in [0,1]$, there exists $r > 0$ such that $f_n \to f$ uniformly on $(x-r, x+r) \cap [0,1]$ (which is $B_r(x)$ in the metric space $[0,1]$.) Prove that $f_n \to f$ uniformly on $[0,1]$.

Suppose $f_n \not\to f$ uniformly on $[0,1]$



$\exists \varepsilon_0 > 0$, $\forall N \in \mathbb{N}$, $\exists N_0 \geq N$ and $\exists x_0 \in [0,1]$ s.t. $|f_{\varepsilon_0}(x_0) - f(x_0)| \geq \varepsilon_0$.

By the assumption, $\exists r > 0$ s.t. $f_n \to f$ uniformly on $(x_0 - r, x_0 + r) \cap [0,1]$

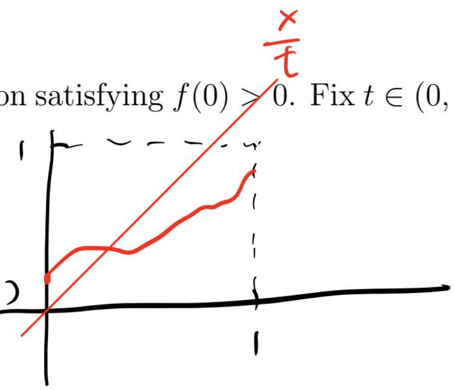
i.e. $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, $n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon$ for all $x \in (x_0 - r, x_0 + r) \cap [0,1]$

Clearly $x_0 \in (x_0 - r, x_0 + r) \cap [0,1]$, so we have a contradiction, completing the proof

3 Problem 3 **1 / 6**

- \+ **6 pts** Correct
- \+ **5 pts** Almost correct
- ✓ + **1 pts** **Some progress**
- \+ **0 pts** Minimal progress

Problem 4. Let $f : [0, 1] \to [0, 1]$ be a continuous function satisfying $f(0) > 0$. Fix $t \in (0, 1]$ and define the set $S_t = \{x \in [0, 1] : \frac{x}{f(x)} = t\}$.



(a) (4 points) Prove that S_t is nonempty.

$g(x) = f(x) - \frac{x}{t}$ (g is continuous on $[0,1]$ since f and $\frac{x}{t}$ are continuous)

$g(0) = f(0) - \frac{0}{t} > 0$

$g(1) = f(1) - \frac{1}{t} \le 1 - \frac{1}{t} \le 1 - 1 = 0$ (since $0 < t \le 1 \Rightarrow \frac{1}{t} \ge 1 \Rightarrow -\frac{1}{t} \le -1$

If $g(1) = 0$, then $f(0) = \frac{0}{t} \Rightarrow t = \frac{0}{f(0)} \Rightarrow 0 \in S_t$

If $g(1) < 0$, then by Intermediate Value Thm, $\exists x_0 \in (0, 1)$ s.t.

$g(x_0) = 0$

i.e. $t = \frac{x_0}{f(x_0)} \Rightarrow x_0 \in S_t$

Thus S_t is nonempty.

(b) (4 points) Suppose also that f is differentiable on $(0, 1)$ and $\underbrace{|f'(x)| < \frac{1}{t} \text{ for all } x \in (0, 1)}_{(1)}$.
Prove that S_t contains exactly 1 element.

Suppose S_t has at least 2 elements, x and y.

Then $x, y \in [0, 1]$ and $\frac{x}{f(x)} = t$ and $\frac{y}{f(y)} = t$ and $x \neq y$

WLOG suppose $x < y$. Then by Mean Value Thm

$\exists c \in (x, y)$ $f'(c) = \frac{f(y) - f(x)}{y - x} = \frac{\frac{y}{t} - \frac{x}{t}}{y - x} = \frac{1}{t}$.

Since $(x, y) \subseteq [0, 1]$, $c \in (0, 1)$ and $|f'(c)| = |\frac{1}{t}| = \frac{1}{t}$.

This is a contradiction to assumption (1), so there are at most 1 element. By part (a), there is at least 1 element, so we have exactly 1 element in S_t

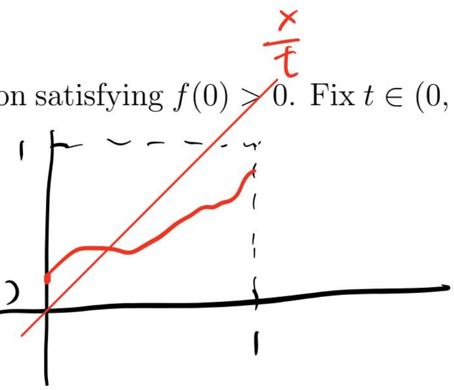
4.1 a 4 / 4

✓ + **4 pts** Correct

+ **1 pts** Some progress

+ **0 pts** Minimal progress

Problem 4. Let $f : [0,1] \to [0,1]$ be a continuous function satisfying $f(0) > 0$. Fix $t \in (0,1]$ and define the set $S_t = \{x \in [0,1] : \frac{x}{f(x)} = t\}$.



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Then $x, y \in [0,1]$ and $\frac{x}{f(x)} = t$ and $\frac{y}{f(y)} = t$ and $x \ne y$

WLOG suppose $x < y$. Then by Mean Value Thm

$\exists c \in (x,y)$ $f'(c) = \frac{f(y) - f(x)}{y - x} = \frac{\frac{y}{t} - \frac{x}{t}}{y - x} = \frac{1}{t}$.

Since $(x,y) \subseteq [0,1]$, $c \in (0,1)$ and $|f'(c)| = |\frac{1}{t}| = \frac{1}{t}$.

This is a contradiction to assumption (1), so there are at most 1 element. By part (a), there is at least 1 element, so we have exactly 1 element in S_t

4.2 b **4 / 4**

- ✓ **+ 4 pts** Correct
- **+ 3 pts** Almost correct
- **+ 1 pts** Some progress
- **+ 0 pts** Minimal progress

Problem 5. Let (s_n) be a sequence of real numbers such that $\lim(s_{n+1} - s_n) = 0$. Suppose there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ such that $s_n = \alpha$ for infinitely many n and $s_n = \beta$ for infinitely many n. Let $s \in (\alpha, \beta)$.

(a) (2 points) Prove that for any $N \in \mathbb{N}$, there exist $N_2 > N_1 \geq N$ such that $s_{N_1} = \alpha$ and $s_{N_2} = \beta$.

By contradiction. suppose $\exists N \in \mathbb{N}$, $\forall N_2 > N_1 \geq N$, $S_{N_1} \neq \alpha$ or $S_{N_2} \neq \beta$.

This means $\forall n \geq N$, $S_n \neq \alpha$, contradicting to that we have infinitely many $S_n = \alpha$.

Thus $\forall N \in \mathbb{N}$, $\exists N_1, N_2 \in \mathbb{N}$ with $N_2 > N_1 \geq N$ s.t. $S_{N_1} = \alpha$ and $S_{N_2} = \beta$

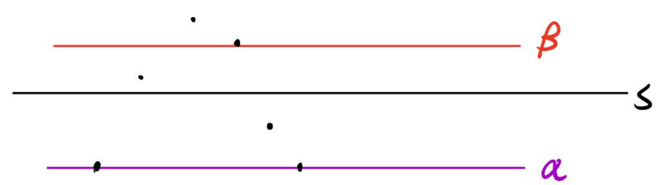
(b) (5 points) Inductively construct a subsequence (s_{n_k}) of (s_n) such that $s_{n_k} \to s$.

$\alpha < s < \beta$.

Let $N = 1$. then by (a)

We have $n_2 > n_1 \geq 1$ s.t.

$S_{n_2} = \beta$ and $S_{n_1} = \alpha$



Having already selected $n_1 < n_2 < \cdots < n_{2k}$ s.t. odd terms are α and even terms are β. Take $N = n_{2k}+1$, by (a) we can select n_{2k+1} and n_{2k+2} s.t. $S_{n_{2k+1}} = \alpha$ and $S_{n_{2k+2}} = \beta$.

Since $\lim S_{n+1} - S_n = 0$, $\lim S_{n_{k+1}} - S_{n_k} = 0$.

It follows that $S_{n_k} \to S$

5.1 a **2 / 2**

✓ **+ 2 pts** Correct

+ 0 pts Incorrect

Problem 5. Let (s_n) be a sequence of real numbers such that $\lim(s_{n+1} - s_n) = 0$. Suppose there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ such that $s_n = \alpha$ for infinitely many n and $s_n = \beta$ for infinitely many n. Let $s \in (\alpha, \beta)$.

(a) (2 points) Prove that for any $N \in \mathbb{N}$, there exist $N_2 > N_1 \geq N$ such that $s_{N_1} = \alpha$ and $s_{N_2} = \beta$.

By contradiction. suppose $\exists N \in \mathbb{N}$, $\forall N_2 > N_1 \geq N$, $S_{N_1} \neq \alpha$ or $S_{N_2} \neq \beta$.

This means $\forall n \geq N$, $S_n \neq \alpha$, contradicting to that we have infinitely many $S_n = \alpha$.

Thus $\forall N \in \mathbb{N}$, $\exists N_1, N_2 \in \mathbb{N}$ with $N_2 > N_1 \geq N$ s.t. $S_{N_1} = \alpha$ and $S_{N_2} = \beta$

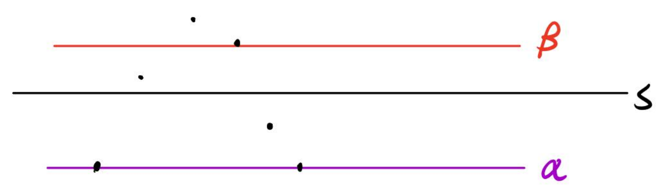
(b) (5 points) Inductively construct a subsequence (s_{n_k}) of (s_n) such that $s_{n_k} \to s$.

$\alpha < s < \beta$.

Let $N=1$. then by (a)

We have $n_2 > n_1 \geq 1$ s.t.

$S_{n_2} = \beta$ and $S_{n_1} = \alpha$



Having already selected $n_1 < n_2 < \cdots < n_{2k}$ s.t. odd terms are α and even terms are β. Take $N = n_{2k}+1$, by (a) we can select n_{2k+1} and n_{2k+2} s.t. $S_{n_{2k+1}} = \alpha$ and $S_{n_{2k+2}} = \beta$.

Since $\lim S_{n+1} - S_n = 0$, $\lim S_{n_{k+1}} - S_{n_k} = 0$.

It follows that $S_{n_k} \to S$

5.2 b **1 / 5**

- \+ **5 pts** Correct
- \+ **4 pts** Correct construction but lacking rigorous proof
- ✓ + **1 pts** **Some progress**
- \+ **0 pts** Minimal progress

6 Problem 6 **0 / 0**

+ **0 pts** Correct

+ **0 pts** Minimal progress

✓ + **0 pts** **N/A**