Math 104 Homework 4 Solutions

UC Berkeley, Summer 2021

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1. Let (X, d) be a metric space. Prove that for any $x \in X$ and $r > 0$, the set

$$C_r(x) := \{y \in X : d(x, y) \leq r\}$$

is closed. ($C_r(x)$ is called the "closed ball of radius r centered at x.")

Solution. Let $y \in C_r(x)^c$. Then $d(x, y) > r$. Let $s = d(x, y) - r$. Then if $z \in B_s(y)$, by the triangle inequality $d(z, x) \geq d(x, y) - d(y, z) > r$, so $z \in C_r(x)^c$. Therefore $B_s(y) \subseteq C_r(x)^c$, so y is an interior point of $C_r(x)^c$. This proves that $C_r(x)^c$ is open, so $C_r(x)$ is closed.

2. (Ross 13.13) Let E be a compact nonempty subset of $\mathbb{R}$. Show that $\sup E$ and $\inf E$ belong to E.

Solution. Since E is a compact subset of $\mathbb{R}$, E must be closed and bounded, so $\sup E < \infty$ and $\inf E > -\infty$. For any $\varepsilon > 0$, $\sup E - \varepsilon$ is not an upper bound for E, so there exists $x \in E$ such that $x > \sup E - \varepsilon$. It follows that either $\sup E \in E$ or $\sup E$ is a limit point of E, but since E is closed, E contains all of its limit points, and therefore $x \in E$. Likewise, $\inf E \in E$.

3. (Ross 13.14) Let E be a compact nonempty subset of $\mathbb{R}^k$, and let δ be the diameter of E, i.e. $\delta = \sup\{d(\mathbf{x}, \mathbf{y}) : \mathbf{x}, \mathbf{y} \in E\}$. Show that there exist $\mathbf{x_0}, \mathbf{y_0} \in E$ such that $d(\mathbf{x_0}, \mathbf{y_0}) = \delta$.

Solution. The theorem actually holds in a general metric space (X, d) (not necessarily $\mathbb{R}^k$). For each $n \in \mathbb{N}$, there exist $x_n \in E$ and $y_n \in E$ such that $d(x_n, y_n) > \delta - \frac{1}{n}$. Since E is compact, (x_n) has a subsequence (x_{n_k}) which converges to some $x \in E$. Furthermore, the corresponding subsequence (y_{n_k}) has a subsequence $(y_{n_{k_j}})$ which converges to some $y \in E$. Then $d(x_{n_{k_j}}, y_{n_{k_j}}) > \delta - \frac{1}{n_{k_j}}$ for each j. Let $\varepsilon > 0$. Then there exists J_1 such that $j \geq J_1$ implies $d(x_{n_{k_j}}, x) < \frac{\varepsilon}{3}$, J_2 such that $j \geq J_2$ implies $d(y, y_{n_{k_j}}) < \frac{\varepsilon}{3}$, and J_3 such that $j \geq J_3$ implies $\frac{1}{n_{k_j}} < \frac{\varepsilon}{3}$. Set $J = \max\{J_1, J_2, J_3\}$. Then for $j \geq J$, by the triangle inequality we have

$$d(x_{n_{k_j}}, y_{n_{k_j}}) \leq d(x_{n_{k_j}}, x) + d(x, y) + d(y, y_{n_{k_j}})$$

so

$$d(x, y) \geq d(x_{n_{k_j}}, y_{n_{k_j}}) - d(x_{n_{k_j}}, x) - d(y, y_{n_{k_j}}) > \delta - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} = \delta - \varepsilon.$$

Since ε is an arbitrary positive number, it follows that $d(x, y) \geq \delta$, and therefore $d(\mathbf{x}, \mathbf{y}) = \delta$.

4. Let (X, d) be a metric space, and let E be a compact set in X. For any $x \in X$, define

$$d(x, E) := \inf\{d(x, y) : y \in E\}.$$

Prove that for any $x \in X$, there exists $y \in E$ such that $d(x, y) = d(x, E)$.

Solution. For each $n \in \mathbb{N}$, there exists $y_n \in E$ such that $d(y_n, x) < d(x, E) + \frac{1}{n}$. Since E is compact, there exists a subsequence (y_{n_k}) of (y_n) which converges to some $y \in E$. Let $\varepsilon > 0$. There exists K_1 such that $k \geq K_1$ implies $\frac{1}{n_k} < \frac{\varepsilon}{2}$, and K_2 such that $k \geq K_2$ implies $d(y_{n_k}, y) < \frac{\varepsilon}{2}$. Set $K = \max\{K_1, K_2\}$. Then for $k \geq K$, by the triangle inequality

$$d(y, x) \leq d(y, y_{n_k}) + d(y_{n_k}, x) < \frac{\varepsilon}{2} + d(x, E) + \frac{\varepsilon}{2} = d(x, E) + \varepsilon.$$

Since ε is an arbitrary positive number, it follows that $d(y, x) \leq d(x, E)$, and therefore $d(y, x) = d(x, E)$.

5. Consider the metric space $(\mathbb{Q}, d)$ with $d(x, y) = |y - x|$. Let $E = \{q \in \mathbb{Q} : \sqrt{2} < q < \sqrt{3}\}$. Show that E is closed and (obviously) bounded, but not compact.

Solution. Let $x \in E^c$. If $x < \sqrt{2}$, then $B_{\sqrt{2}-x}(x) \subseteq E^c$, and if $x > \sqrt{3}$, $B_{x-\sqrt{3}}(x) \subseteq E^c$, so E^c is open and therefore E is closed. E is bounded because $|x| < \sqrt{3}$ for all $x \in E$. But E is not compact: consider the open cover $\{U_n\}_{n \in \mathbb{N}}$ of E where

$$U_n := \{q \in \mathbb{Q} : \sqrt{2} + \frac{1}{n} < q < \sqrt{3}\}.$$

It is easy to check that U_n is an open set for each n, and that $\bigcup_{n=1}^{\infty} U_n = E$. If we take any finite subcollection $\{U_{n_1}, \ldots, U_{n_k}\}$, then $\bigcup_{j=1}^{k} U_{n_j} = U_N$ where $N := \max\{n_1, \ldots, n_k\}$. By denseness of $\mathbb{Q}$ in $\mathbb{R}$, there exists $p \in \mathbb{Q}$ such that $\sqrt{2} < p < \sqrt{2} + \frac{1}{N}$, which shows that $E \not\subseteq U_N$. Thus $\{U_n\}_{n \in \mathbb{N}}$ has no finite subcover, so E is not compact.

6. Let (X, d) be a metric space. We say that a collection closed sets $\{E_\alpha\}_{\alpha \in A}$ in X has the *finite intersection property* if the intersection of any finite subcollection of $\{E_\alpha\}_{\alpha \in A}$ is nonempty. Show that if every collection of closed sets $\{E_\alpha\}_{\alpha \in A}$ with the finite intersection property has nonempty intersection, i.e. $\bigcap_{\alpha \in A} E_\alpha \neq \emptyset$, then X is compact.
(Note: The converse was proven in class.)

Solution. (Contrapositive) Suppose that X is not compact. Then there exists an open cover $\{U_\alpha\}_{\alpha \in A}$ of X which has no finite subcover. Consider the collection of closed sets $\{U_\alpha^c\}_{\alpha \in A}$. If $\{U_{\alpha_1}^c, \ldots, U_{\alpha_n}^c\}$ is any finite subcollection of $\{U_\alpha^c\}_{\alpha \in A}$, then

$$U_{\alpha_1}^c \cap \cdots \cap U_{\alpha_n}^c = (U_{\alpha_1} \cup \cdots \cup U_{\alpha_n})^c \neq \emptyset$$

because $U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} \neq X$ (otherwise it would be a finite subcover of the open cover $\{U_\alpha\}_{\alpha \in A}$ of X), so $\{U_\alpha^c\}_{\alpha \in A}$ satisfies the finite intersection property. However,

$$\bigcap_{\alpha \in A} U_\alpha^c = \Big(\bigcup_{\alpha \in A} U_\alpha\Big)^c = X^c = \emptyset.$$

7. Let X be the set of all bounded sequences of real numbers. Define the function
$d : X \times X \to \mathbb{R}$ by $d\big((s_n), (t_n)\big) = \sup\{|t_n - s_n| : n \in \mathbb{N}\}$. We proved that d is a metric on X on Worksheet 7. Show that $\{(s_n) \in X : |s_n| \leq 1 \text{ for all } n\} \subseteq X$ is closed and bounded, but not compact.

Solution. Observe that $E := \{(s_n) \in X : |s_n| \leq 1 \text{ for all } n\} = C_1\big((0, 0, 0, \ldots)\big)$ in the notation of Problem 1. Therefore it is both closed and bounded. To see that E is not compact, consider the sequence of coordinate sequences $(1, 0, 0, \ldots)$, $(0, 1, 0, 0, \ldots)$, $(0, 0, 1, 0, 0, \ldots), \ldots$ in E. In any subsequence of this sequence of sequences, the distance between successive terms will always be equal to 1, so any subsequence will not be Cauchy and therefore will not converge. This implies that E is not compact, since any sequence of points in a compact set has a convergent subsequence.