Math 104 Worksheet 13

UC Berkeley, Summer 2021

Tuesday, July 27

Theorem. Let $x_0 \in \mathbb{R}$. For the power series $\sum a_n(x-x_0)^n$, let $\beta := \limsup |a_n|^{1/n}$ and

$$R := \begin{cases} \frac{1}{\beta} & \text{if } 0 < \beta < \infty, \\ \infty & \text{if } \beta = 0, \\ 0 & \text{if } \beta = \infty. \end{cases}$$

(i) The power series converges for $|x - x_0| < R$.
(ii) The power series diverges for $|x - x_0| > R$.
R as defined above is called the **radius of convergence** of the power series.

Proof. Exercise 1. Restate the **root test** for a series of real numbers $\sum b_n$.

$\sum b_n$ converges if $\limsup |b_n|^{\frac{1}{n}} < 1$.

$\sum b_n$ diverges if $\limsup |b_n|^{\frac{1}{n}} > 1$.

Exercise 2. Treat x as a fixed value, so $\sum a_n(x-x_0)^n = \sum b_n$ where $b_n = a_n(x-x_0)^n$. Compute the quantity of interest in the root test for this series; express your answer in terms of x and β.

$$\limsup |b_n|^{\frac{1}{n}} = \limsup |a_n(x-x_0)^n|^{\frac{1}{n}} = \limsup |x-x_0| \cdot |a_n|^{\frac{1}{n}}$$
$$= \boxed{|x-x_0| \cdot \underbrace{\limsup |a_n|^{\frac{1}{n}}}_{\beta}} = \beta |x-x_0|.$$

Exercise 3. Consider the three cases $0 < \beta < \infty$, $\beta = 0$, and $\beta = \infty$ separately to justify the conclusion of the theorem.

Case 1: $0 < \beta < \infty$.

$\sum a_n(x-x_0)^{\frac{1}{n}}$ converges if $\beta|x-x_0| < 1 \Leftrightarrow |x-x_0| < \frac{1}{\beta} = R$.

diverges if $\beta|x-x_0| > 1 \Leftrightarrow |x-x_0| > \frac{1}{\beta} = R$.

Case 2: $\beta = 0$. Then $\beta|x-x_0| = 0 < 1$.

$\sum a_n(x-x_0)^{\frac{1}{n}}$ converge for all $x \in \mathbb{R}$, i.e. $|x-x_0| < \infty = R$.

Case 3: $\beta = \infty$. $\limsup |a_n(x-x_0)^n|^{\frac{1}{n}} = \infty > 1$ for all $x \neq x_0$.

$\sum a_n(x-x_0)^{\frac{1}{n}}$ diverges for $|x-x_0| > 0 = R$.

Corollary. If $\lim \left|\frac{a_n}{a_{n+1}}\right|$ exists, then it is equal to the radius of convergence of the power series.

Exercise 4. Prove the preceding corollary.

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \le \limsup |a_n|^{\frac{1}{n}} \le \limsup \left|\frac{a_{n+1}}{a_n}\right|$$

If $\lim \left|\frac{a_n}{a_{n+1}}\right|$ exists, so does $\underbrace{\lim \left|\frac{a_{n+1}}{a_n}\right|}_{\alpha}$, and $\limsup |a_n|^{\frac{1}{n}} = \alpha$.

Then $\lim \left|\frac{a_n}{a_{n+1}}\right| = \begin{cases} \frac{1}{\alpha} & \text{if } 0<\alpha<\infty \\ 0 & \text{if } \alpha=\infty \\ \infty & \text{if } \alpha = 0 \end{cases} = R.$

Definition. The **interval of convergence** of the power series $\sum a_n(x-x_0)^n$ is the set $\{x \in \mathbb{R} :$ the series of real numbers $\sum a_n(x-x_0)^n$ converges$\}$. Note that the theorem guarantees that this set is an interval (which can be open, closed, or half-open-half-closed.)

Exercise 5. For each of the following power series, find the interval of convergence.

(a) $\sum \frac{1}{n!}x^n$ $\quad \lim \left|\frac{a_n}{a_{n+1}}\right| = \lim \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} = \lim (n+1) = \infty.$

$(-\infty, \infty)$

(b) $\sum x^n$ $\quad \limsup |a_n|^{\frac{1}{n}} = 1. \Rightarrow R=1.$

$|a_n| = 1$ for all n.

Convergence on $(-1,1)$.
Does not conv at -1 or 1.
Interval of conv: $\boxed{(-1,1)}$.

(c) $\sum \frac{1}{n}x^n$ $\quad \lim \left|\frac{a_n}{a_{n+1}}\right| = \lim \frac{\frac{1}{n}}{\frac{1}{n+1}} = 1$

$\Rightarrow R=1$. Does not conv at 1.
Conv. at -1.
Int. of conv.: $\boxed{[-1,1)}$

(d) $\sum \frac{1}{n^2}x^n$ $\quad \lim \left|\frac{a_n}{a_{n+1}}\right| = \lim \frac{\frac{1}{n^2}}{\frac{1}{(n+1)^2}} = 1.$
$R=1$.
Int. of conv: $[-1,1]$.

(e) $\sum n^{104}x^n$
$\lim \left|\frac{a_n}{a_{n+1}}\right| = \lim \frac{n^{104}}{(n+1)^{104}} = 1$. $R=1$. Int of conv: $(-1,1)$.

formalate a general result for series of the form $\sum n^p x^n$

(f) $\sum n!x^n$ $\quad \lim \left|\frac{a_n}{a_{n+1}}\right| = \lim \frac{n!}{(n+1)!} = \lim \frac{1}{n+1} = 0 \Rightarrow R=0.$
Converges at only $\{0\}$.

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Natural question:

Is $f(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k$ a continuous function on $(x_0 - R, x_0 + R)$?

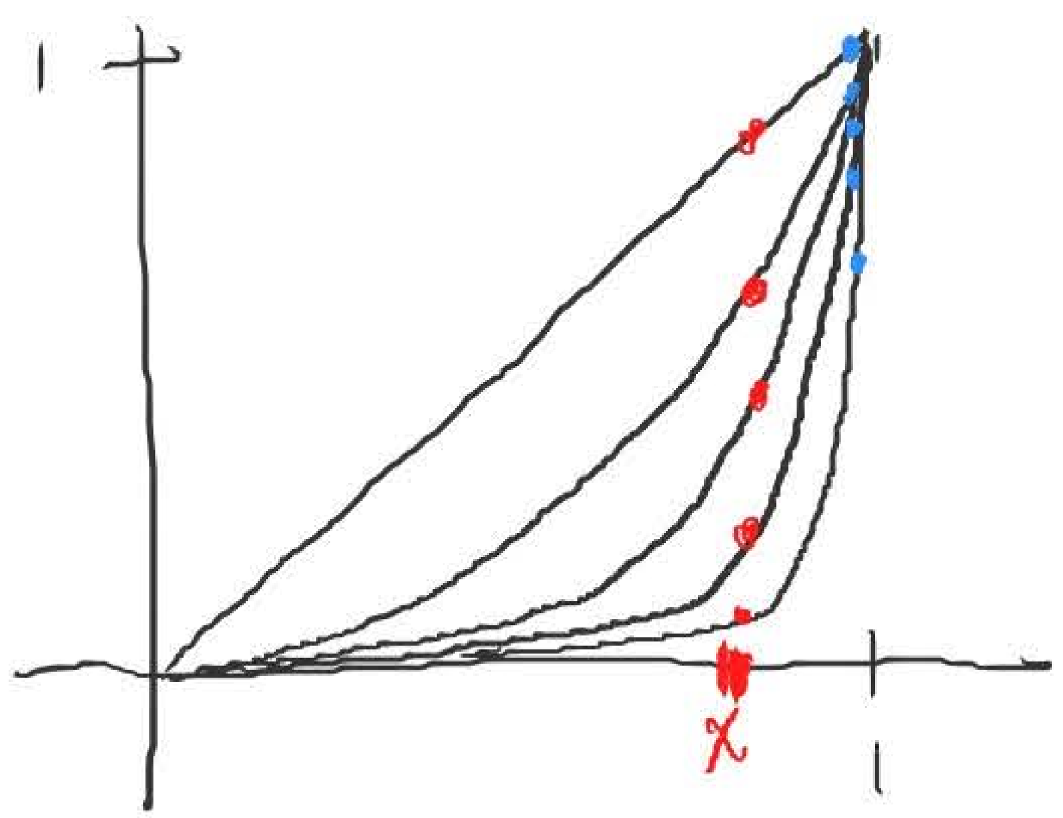
We know that $f_n(x) = \sum_{k=0}^{n} a_k (x-x_0)^k$ is continuous (polynomial).

Think at f as the "limit" of (f_n) (needs to be formalized).

Let (X,d) be a metric space. Let (f_n) be a sequence of real-valued functions defined on $E \subseteq X$.

<u>Def</u>: (f_n) converges <u>pointwise on E</u> to a function f if for each $x \in E$, $\lim_{n\to\infty} f_n(x) = f(x)$. Write "$f_n \to f$ pointwise"

<u>Ex</u>. $f_n(x) = x^n$ on $[0,1]$ converges pointwise to $f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1. \end{cases}$



f_n all continuous, $\to f$ not continuous.

Observe: pointwise convergence in this example is not occurring at the same rate across the domain.

Uniform convergence : (X, d).

(f_n) converges uniformly on E to a function f if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(x) - f(x)| < \varepsilon$ for every $x \in E$.

works for all $x \in E$.

Compare to pointwise convergence :

For any $\varepsilon > 0$ and $x \in E$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(x) - f(x)| < \varepsilon$.

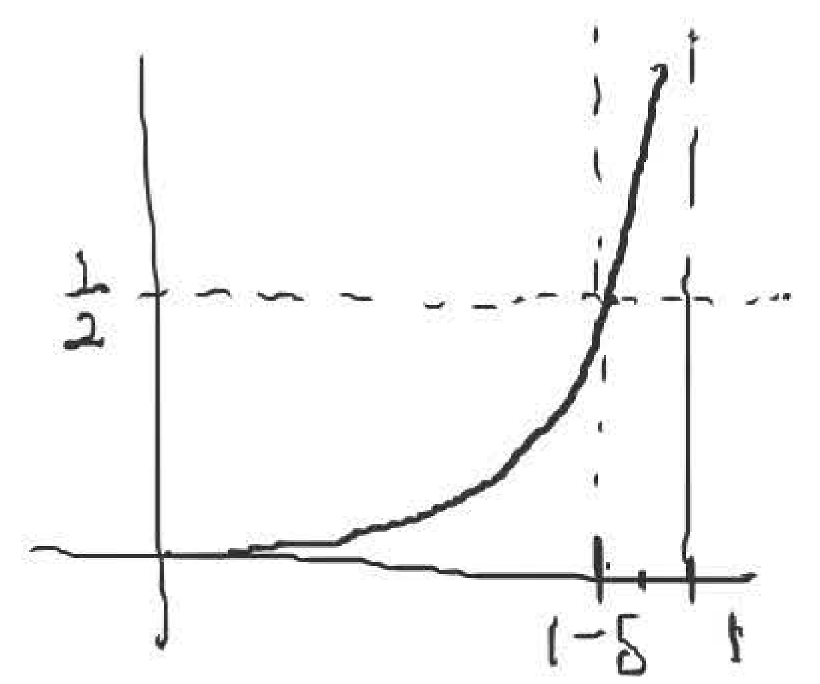
depends on x.

Claim : $f_n(x) = x^n$ does not converge uniformly to $f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$.

Proof: Let $\varepsilon = \frac{1}{2}$. Let $N \in \mathbb{N}$. Since $f_N(x) = x^N$ is continuous at $x = 1$, there exists $\delta > 0$ such that $|x - 1| < \delta \Rightarrow |x^N - 1| < \frac{1}{2}$.

Then $f_N(1 - \frac{\delta}{2}) = (1 - \frac{\delta}{2})^N > \frac{1}{2}$.

So there does not exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(x) - \underbrace{f(x)}_{=0 \text{ for } 0 \leq x < 1}| < \frac{1}{2}$.



Theorem: Uniform limit theorem.

"The uniform limit of continuous functions is continuous."

More precisely, let (X, d) be a metric space.
Let (f_n) be a sequence of real-valued functions defined on $E \subseteq X$.
If $f_n \to f$ uniformly and f_n is continuous at $x_0 \in E$ for all n,
then f is continuous at x_0. (if f_n is continuous on E, then f is continuous on E).

Proof: Key inequality:

$$|f(x) - f(x_0)| \leq \underbrace{|f(x) - f_n(x)|} + \underbrace{|f_n(x) - f_n(x_0)|}_{\text{make small by cont. of } f_n} + \underbrace{|f_n(x_0) - f(x_0)|}.$$

make small by uniform convergence of f_n to f.