Q1

(a) We will use proof by contradiction. Suppose $qr \in \mathbb{Q}$, then let $qr = \frac{a}{b}$ where $a, 0 \neq b \in \mathbb{Z}$. Since $0 \neq q \in \mathbb{Q}$, let $q = \frac{c}{d}$ where $0 \neq c, 0 \neq d \in \mathbb{Z}$. Then we have

$$
\begin{aligned}
qr = \frac{a}{b} &\implies \frac{c}{d} \cdot r = \frac{a}{b} \\
&\implies r = \frac{a}{b} \cdot \frac{d}{c} \\
&\implies r = \frac{ad}{bc} \\
&\implies r \text{ is rational}
\end{aligned}
$$

The last implication comes from that $a, 0 \neq b, 0 \neq d, 0 \neq c \in \mathbb{Z} \implies ad, 0 \neq bc \in \mathbb{Z}$. However by the condition $r \in \mathbb{R} \backslash \mathbb{Q}$, we have a contradiction. Thus $qr \in \mathbb{R} \backslash \mathbb{Q}$, completing the proof.

(b) We all know $\sqrt{2}$ is an irrational number. Since $\frac{1}{\sqrt{2}} > 0$, $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$. Because $\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}} \in \mathbb{R}$ and by the denseness of $\mathbb{Q}$, there exists $0 \neq q \in \mathbb{Q}$ such that $\frac{a}{\sqrt{2}} < q < \frac{b}{\sqrt{2}}$. This implies $a < q\sqrt{2} < b$. By (a), $q\sqrt{2} \in \mathbb{R} \backslash \mathbb{Q}$, completing the proof.

Q2

We need to verify two properties of supremum.

(i) $\forall q \in \{r \in \mathbb{Q} : r < a\},\ q < a \implies q \leq a = \sup\{r \in \mathbb{Q} : r < a\}$.

(ii) Consider $a' < a$, then by the denseness of $\mathbb{Q}$, there exists $q \in \mathbb{Q}$ such that $a' < q < a$. This implies $q \in \{r \in \mathbb{Q} : r < a\}$ and $q > a'$.

We can see that a satisfying both properties of supremum, completing the proof.

Q3

(a) We need to prove both inequalities.

$\geq$: By the definition of supremum, we have $\forall a \in A, b \in B,\ ab \leq \sup(AB)$. This implies

$$\begin{aligned}\frac{\sup(AB)}{a} \geq b &\implies \frac{\sup(AB)}{a} \geq \sup(B)\\ &\implies \frac{\sup(AB)}{\sup(B)} \geq a\\ &\implies \frac{\sup(AB)}{\sup(B)} \geq \sup(A)\\ &\implies \sup(AB) \geq \sup(A) \cdot \sup(B),\end{aligned}$$

completing this part of the proof.

$\leq$: By the definition of supremum, we have $\forall a \in A, b \in B,\ 0 < a \leq \sup(A)$ and $0 < b \leq \sup(B)$. This implies $\forall a \in A, b \in B,\ ab \leq \sup(A) \cdot b \leq \sup(A) \cdot \sup(B)$. Thus $\sup(A) \cdot \sup(B)$ is an upper bound of AB, and $\sup(AB) \leq \sup(A) \cdot \sup(B)$.

(b) Let $A = \{-1, 1\}$ and $B = \{-3, 1\}$, then $AB = \{3, -1, -3, 1\}$. Thus $\sup(AB) = 3 \neq 1 = 1 \cdot 1 = \sup(A) \cdot \sup(B)$.

Q4

(a) If we can show that $a_n \leq s_n$ for all $n \implies \lim a_n \leq \lim s_n$, then similarly we will have $\lim s_n \leq \lim b_n$. Now since $s = \lim a_n \leq \lim s_n \leq \lim b_n = s$, all the inequalities actually achieve the equality, so $\lim s_n = \lim a_n = \lim b_n = s$.

From $a_n \leq \ s_n$, we have $a_n - s_n \leq 0$. Let $s = \lim a_n - s_n$. Suppose $s > 0$, then by the denseness of $\mathbb{Q}$ and the definition of limit, there exists $0 < \epsilon < s$ and $N \in \mathbb{N}$, such that $n \geq N$ implies

$$\begin{aligned} |a_n - s_n - s| < \epsilon < s &\implies |a_n - s_n - s| < s \\ &\implies -s < a_n - s_n - s < s \\ &\implies 0 < a_n - s_n < 2s \end{aligned}$$

The last implication implies $a_n > s_n$ which is a contradiction to the condition that $a_n \leq s_n$ for all $n \in \mathbb{N}$. Thus $\lim(a_n - s_n) = s \leq 0 \implies \lim a_n - \lim s_n \leq 0 \implies \lim a_n \leq \lim s_n$, completing the proof.

(b) First observe that $|s_n| \leq t_n \implies -t_n \leq s_n \leq t_n$ for all n. Since $\lim t_n = 0$, we have $\lim(-t_n) = \lim(-1) \cdot (t_n) = -1 \cdot \lim t_n = -1 \cdot 0 = 0$ from theorem 9.2. Thus by (a) we have $0 = \lim(-t_n) \leq \lim s_n \leq \lim t_n = 0$, implying $\lim s_n = 0$.

(c) Observe that $|s_n| = |\frac{1}{n} \sin n| = |\frac{1}{n}| \cdot |\sin n| \leq |\frac{1}{n}| \cdot 1 = \frac{1}{n}$. Since $\lim \frac{1}{n} = 0$, by (b) we have $\lim s_n = \lim \frac{1}{n} \sin n = 0$.

Remark. The original proof of (a) above has a small issue that to prove the lemma $a_n \leq s_n$ for all $n \implies \lim a_n \leq \lim s_n$, we need to show $\lim s_n$ first exists. However, I did not show that. See more standard solution in *hw2sol.pdf*.

Q5

We will use proof by contradiction. Suppose $\lim s_n < a$, then $\lim s_n - a < 0$. Let $s = \lim s_n - a < 0$. Since $-s > 0$ and the denseness of $\mathbb{Q}$, there exists $0 < \epsilon < -s$. By the definition of limit, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies

$$\begin{aligned}
|s_n - a - s| < \epsilon < -s &\implies |s_n - a - s| < -s \\
&\implies s_n - a - s < -s \\
&\implies s_n - a < 0 \\
&\implies s_n < a
\end{aligned}$$

This is a contradiction to $s_n \geq a$ for all but finitely many n since we can find a N such that for all infinite $n \geq N$, we have $s_n < a$. Thus $\lim s_n \geq a$, completing the proof.

Q6

Let $s = \lim s_n > a$. By the definition of limit, select $\epsilon = s - a > 0$, there exists $N \in \mathbb{N}$ such that

$$
\begin{aligned}
n \geq N &\implies |s_n - s| < \epsilon = s - a \\
&\implies -(s-a) < s_n - s < s - a \\
&\implies -s + a + s < s_n - s + s < s - a + s \\
&\implies s_n > a
\end{aligned}
$$

Thus complete the proof.

Q7

Observe that

$$\begin{aligned}\frac{n!}{n^n} &= \frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdot\dots\cdot\frac{1}{n}\\ &\le 1\cdot 1\cdot 1\cdot\dots\cdot\frac{1}{n}\\ &= \frac{1}{n}\end{aligned}$$

Since $\frac{n!}{n^n} > 0 = \lim 0$ and $\lim\frac{1}{n} = 0$, by Squeeze theorem, we have $0 = \lim 0 \le \lim\frac{n!}{n^n} \le \lim\frac{1}{n} = 0$, implying $\lim\frac{n!}{n^n} = 0$.

Q8

(a) Since $L < 1 \implies 1 - L > 0$, by the denseness of Q, there exists $0 < \epsilon < 1 - L$. This implies $L + \epsilon < 1$. By the definition of limit, there exists $N \in \mathbb{N}$ such that

$$\begin{aligned} n \geq N \implies & \left| \left| \frac{s_{n+1}}{s_n} \right| - L \right| < \epsilon \\ \implies & \left| \frac{s_{n+1}}{s_n} \right| < L + \epsilon \\ \implies & \frac{|s_{n+1}|}{|s_n|} < L + \epsilon \\ \implies & |s_{n+1}| < (L + \epsilon)|s_n| \end{aligned}$$

Since n is an arbitrary integer $\geq N$, the last implication can be also applied to $n+2, n+3, \ldots$. For example, $|s_{n+2}| < (L+\epsilon)|s_{n+1}| < (L+\epsilon)(L+\epsilon)|s_n| = (L+\epsilon)^2|s_n|$. Thus we can conclude that for $n > N$, $|s_n| < (L+\epsilon)^{n-N}|s_N|$.

Observe that $\lim (L+\epsilon)^{n-N}|s_N| = |s_N| \cdot \lim (L+\epsilon)^{n-N} = |s_N| \cdot 0 = 0$ since $L + \epsilon < 1$. Therefore, by Squeeze theorem $0 = -1 \cdot 0 = \lim -(L+\epsilon)^{n-N}|s_N| < \lim s_n < \lim (L+\epsilon)^{n-N}|s_N| = 0$, implying $\lim s_n = 0$.

(b) let $t_n = \frac{1}{|s_n|}$, then we have

$$\begin{aligned} \lim \left| \frac{t_{n+1}}{t_n} \right| &= \lim \left| \frac{s_n}{s_{n+1}} \right| \\ &= \lim \frac{1}{\left| \frac{s_{n+1}}{s_n} \right|} \\ &= \frac{1}{L} \\ &< 1 \end{aligned}$$

Apply (a) to $\lim \left| \frac{t_{n+1}}{t_n} \right|$ and we get $\lim t_n = 0 \implies \lim \frac{1}{t_n} = \lim |s_n| = \infty$, completing the proof.

(c) Let $s_n = \frac{a^n}{n!}$, then

$$\begin{aligned} \lim \left| \frac{s_{n+1}}{s_n} \right| &= \lim \left| \frac{a^{n+1}}{(n+1)!} \cdot \frac{n!}{a^n} \right| \\ &= \lim \left| \frac{a}{n} \right| \\ &= |a| \cdot \lim \frac{1}{n} \\ &= |a| \cdot 0 \\ &= 0 \end{aligned}$$

Since $L = 0 < 1$, we have $\lim s_n = \lim \frac{a^n}{n!} = 0$

Q9

WLOG, consider $m \geq n \geq N$ where $N > 1 - \log_2 \epsilon$, then we have

$$\begin{aligned}
|s_m - s_n| &= |s_m - s_{m-1} + \cdots + s_{n+1} - s_n| \\
&\leq |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + \cdots + |s_{n+2} - s_{n+1}| + |s_{n+1} - s_n| \\
&< \sum_{k=n}^{m-1} 2^{-k} \quad \text{by the assumption} \\
&< \sum_{k=N}^{n-1} 2^{-k} + \sum_{k=n}^{m-1} 2^{-k} + \sum_{k=m}^{\infty} 2^{-k} \quad \text{since all terms are positive} \\
&= \sum_{k=N}^{\infty} 2^{-k} \\
&= 2^{-N+1} \quad \text{by the hint} \\
&< 2^{-(1-\log_2 \epsilon)+1} \quad \text{by } N > 1 - \log_2 \epsilon \\
&= 2^{\log_2 \epsilon} \\
&= \epsilon.
\end{aligned}$$

Thus (s_n) is a Cauchy sequence and hence converges.

Q10

We will use inductive construction.

- Base case: Since $\sup S - 1 < \sup S$, there exists $s \in S$ such that $s > \sup S - 1$. Because $\sup S \notin S$, $s < \sup S$ instead of $s \leq \sup S$. Let $s_1 = s$ and we have $\sup S - 1 < s_1 < \sup S$.

- Induction step: Given $s_1, \ldots, s_k \in S$ such that $s_1 < \cdots < s_k$ and $\sup S - \frac{1}{j} < s_j < \sup S$ for $j = 1, \ldots, k$. Since $s_k < \sup S$, there exists $s \in S$ such that $s_k < s < \sup S$. Also since $\sup S - \frac{1}{k+1} < \sup S$, there exists $t \in S$ such that $\sup S - \frac{1}{k+1} < t < \sup S$. Select $s_{k+1} = \max\{s, t\}$, then we have $s_k < s_{k+1} \in S$ and $\sup S - \frac{1}{k+1} < s_{k+1} < \sup S$.

Thus we inductively construct a strictly increasing sequence such that for each $k \in \mathbb{N}$, $\sup S - \frac{1}{k} < s_k < \sup S$. By Squeeze Lemma, $\limsup(S - \frac{1}{k}) < \lim s_k < \limsup S$, implying $\lim s_k = \sup S$.