Math 104 Homework 4

UC Berkeley, Summer 2021

Due by Saturday, July 17, 11:59pm PDT

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1. Let (X, d) be a metric space. Prove that for any $x \in X$ and $r > 0$, the set

$$C_r(x) := \{y \in X : d(x, y) \leq r\}$$

is closed. ($C_r(x)$ is called the "closed ball of radius r centered at x.")

2. (Ross 13.13) Let E be a compact nonempty subset of $\mathbb{R}$. Show that $\sup E$ and $\inf E$ belong to E.

3. Let (X, d) be a metric space. Let E be a nonempty compact subset of X, and let δ be the diameter of E, i.e. $\delta = \sup\{d(\mathbf{x}, \mathbf{y}) : \mathbf{x}, \mathbf{y} \in E\}$. Show that there exist $\mathbf{x_0}, \mathbf{y_0} \in E$ such that $d(\mathbf{x_0}, \mathbf{y_0}) = \delta$.

4. Let (X, d) be a metric space, and let E be a compact set in X. For any $x \in X$, define

$$d(x, E) := \inf\{d(x, y) : y \in E\}.$$

Prove that for any $x \in X$, there exists $y \in E$ such that $d(x, y) = d(x, E)$.

5. Consider the metric space $(\mathbb{Q}, d)$ with $d(x, y) = |y - x|$. Let $E = \{q \in \mathbb{Q} : \sqrt{2} < q < \sqrt{3}\}$. Show that E is closed and (obviously) bounded, but not compact.

6. Let (X, d) be a metric space. We say that a collection closed sets $\{E_\alpha\}_{\alpha \in A}$ in X has the *finite intersection property* if the intersection of any finite subcollection of $\{E_\alpha\}_{\alpha \in A}$ is nonempty. Show that if every collection of closed sets $\{E_\alpha\}_{\alpha \in A}$ with the finite intersection property has nonempty intersection, i.e. $\bigcap_{\alpha \in A} E_\alpha \neq \emptyset$, then X is compact.

7. Let X be the set of all bounded sequences of real numbers. Define the function $d : X \times X \to \mathbb{R}$ by $d\big((s_n), (t_n)\big) = \sup\{|t_n - s_n| : n \in \mathbb{N}\}$. We proved that d is a metric on X on Worksheet 7. Show that $\{(s_n) \in X : |s_n| \leq 1 \text{ for all } n\} \subseteq X$ is closed and bounded, but not compact.