## Q1

We will use proof by contradiction. Consider $y \in (C_r(x))'$ and suppose $y \in (C_r(x))^{\mathsf{C}}$. Then we have $d(x,y) > r \implies d(x,y) - r > 0$. Take $s$ with $0 < s < d(x,y) - r$. $\forall z \in B_s(y)$ by triangular inequality of $d$ we have

$$\begin{aligned} d(x,z) &\geq d(x,y) - d(y,z) \\ &> d(x,y) - s \\ &> d(x,y) - (d(x,y) - r) \\ &= r. \end{aligned}$$

Thus $z \in (C_r(x))^{\mathsf{C}} \implies B_s(y) \subseteq (C_r(x))^{\mathsf{C}} \implies B_s(y) \cap C_r(x) = \emptyset$. This is a contradiction to our assumption that $y$ is a limit point of $C_r(x)$. Thus $y \in C_r(x)$, and hence $C_r(x)$ is closed by definition.

## Q2

We will use proof by contradiction for both $\sup E$ and $\inf E$. Because $E$ is compact in $\mathbb{R}$, $E$ is closed and bounded. Thus both $\sup E$ and $\inf E$ exist.

Suppose $\sup E \notin E$, then $\forall r > 0\ \exists x \in E\ \ \sup E - r < x < \sup E < \sup E + r$. Thus $x \in (\sup E - r, \sup E + r) = B_r(\sup E) \implies (B_r(\sup E) \backslash \{\sup E\}) \cap E \neq \emptyset$ since $x \neq \sup E$. By the definition $\sup E \in E' \subseteq E$ because $E$ is closed. Then we have $\sup E \in E$ which is a contradiction. Thus $\sup E \in E$.

Suppose $\inf E \notin E$, then $\forall r > 0\ \exists x \in E\ \ \inf E - r < \inf E < x < \inf E + r$. Thus $x \in (\inf E - r, \inf E + r) = B_r(\inf E) \implies (B_r(\inf E) \backslash \{\inf E\}) \cap E \neq \emptyset$ since $x \neq \inf E$. By the definition $\inf E \in E' \subseteq E$ because $E$ is closed. Then we have $\inf E \in E$ which is a contradiction. Thus $\inf E \in E$.

## Q3

Suppose $\forall x, y \in E$ $d(x, y) \neq \delta$. By HW 2.10, we can construct a pair of sequences $(x_n)$ and $(y_n)$ in $E$ such that $\forall n \in \mathbb{N}$ $\max\{\delta - \frac{1}{n}, d(x_{n-1}, y_{n-1})\} < d(x_n, y_n) < \delta$. By the construction, it is clear than $(d(x_n, y_n)) \to \delta$ and $(d(x_n, y_n))$ is increasing. Since $E$ is compact, $(x_n)$ has a convergent subsequence $(x_{n_k}) \to x_0 \in E$. Moreover, $(d(x_{n_k}, y_{n_k})) \to \delta$ because $(d(x_n, y_n)) \to \delta$. Again, by the compactness of $E$, $(y_{n_k})$ has a convergent subsequence $(y_{n_{k_l}}) \to y_0 \in E$, and hence $(x_{n_{k_l}}) \to x_0$ and $(d(x_{n_{k_l}}, y_{n_{k_l}})) \to \delta$.

Now by trapezoid inequality, $\forall l \in \mathbb{N}$

$$d(x_{n_{k_l}}, y_{n_{k_l}}) \leq d(x_{n_{k_l}}, x_0) + d(x_0, y_0) + d(y_0, y_{n_{k_l}}).$$

i.e.

$$d(x_0, y_0) \geq d(x_{n_{k_l}}, y_{n_{k_l}}) - d(x_{n_{k_l}}, x_0) - d(y_0, y_{n_{k_l}}). \tag{1}$$

For each $\epsilon > 0$, there exists $L_1 \in \mathbb{N}$ $l \geq L_1 \implies d(x_{n_{k_l}}, x_0) < \frac{\epsilon}{3}$; there exists $L_2 \in \mathbb{N}$ $l \geq L_2 \implies d(y_0, y_{n_{k_l}}) < \frac{\epsilon}{3}$; there exists $L_3 \in \mathbb{N}$ $d(x_{n_{k_{L_3}}}, y_{n_{k_{L_3}}}) > \delta - \frac{\epsilon}{3}$. By the previous construction of $d((x_n), (y_n))$, $l \geq L_3 \implies d(x_{n_{k_l}}, y_{n_{k_l}}) \geq d(x_{n_{k_{L_3}}}, y_{n_{k_{L_3}}}) > \delta - \frac{\epsilon}{3}$. Take $L = \max\{L_1, L_2, L_3\}$, and hence with (1) $l \geq L$ implies

$$\begin{aligned} d(x_0, y_0) &> \delta - \frac{\epsilon}{3} - d(x_{n_{k_l}}, x_0) - d(y_0, y_{n_{k_l}}) \\ &> \delta - \frac{\epsilon}{3} - \frac{\epsilon}{3} - \frac{\epsilon}{3} \\ &= \delta - \epsilon. \end{aligned}$$

Thus $\delta \leq d(x_0, y_0) \leq \delta \implies d(x_0, y_0) = \delta$. Since $x_0, y_0 \in E$, we have a contradiction, so $\exists x_0, y_0 \in E$ $d(x_0, y_0) = \delta$.

# Q4

If $x \in E$, then it is trivially true.

If $x \in X \backslash E$, then suppose $\forall y \in E\ d(x, y) \neq d(x, E)$.i.e. $\forall y \in E\ d(x, E) < d(x, y)$. By similar argument in HW 2.10 and Q3, we can construct a $(y_n) \in E$ such that $\forall n \in \mathbb{N}\ d(x, E) < d(x, y_n) < \min\{d(x, E) + \frac{1}{n}, d(x, y_{n-1})\}$. Thus $(d(x, y_n)) \to d(x, E)$ and $(d(x, y_n))$ is decreasing. Since $E$ is compact, $(y_n)$ has a convergent subsequence $(y_{n_k}) \to y_0 \in E$, and hence $(d(x, y_{n_k})) \to d(x, E)$.

For each $\epsilon > 0$, there exists $K_1 \in \mathbb{N}\ k \geq K_1 \implies d(y_{n_{K_1}}, y_0) < \frac{\epsilon}{2}$; there exists $K_2 \in \mathbb{N}\ d(x, y_{n_{K_2}}) < d(x, E) + \frac{\epsilon}{2}$. By the previous construction, $k \geq K_2 \implies (d(x, y_{n_k})) \leq d(x, y_{n_{K_2}}) < d(x, E) + \frac{\epsilon}{2}$. Take $K = \max\{K_1, K_2\}$, then by triangular inequality $k \geq K$ implies

$$
\begin{aligned}
d(x, y_0) &\leq d(x, y_{n_k}) + d(y_{n_k}) && (1) \\
&< d(x, E) + \frac{\epsilon}{2} + \frac{\epsilon}{2} && (2) \\
&= d(x, E) + \epsilon. && (3)
\end{aligned}
$$

Thus when $y_0 \in E$, $d(x, E) \leq d(x, y_0) \leq d(x, E) \implies d(x, y_0) = E$, which is a contradiction. Thus $\exists y_0 \in E\ d(x, y_0) = d(x, E)$.

# Q5

- Consider $x \in E'$. Then there exists a sequence $(x_n)$ of points in $E\backslash\{x\}$ such that $x_n \to x$. Since we are dealing with set $\mathbb{Q}$, $x \in \mathbb{Q}$. Suppose $x \leq \sqrt{2}$, or actually $x < \sqrt{2}$ since $\sqrt{2}$ is irrational, then $\exists x < r < \sqrt{2}$. Obviously $(B_{r-x}(x)\backslash\{x\}) \cap E = \emptyset \implies$ x is not a limit point of $E$, which is a contradiction. Thus $x > \sqrt{2}$.

  Similarly, suppose $x \geq \sqrt{3}$, or actually $x > \sqrt{3}$ since $\sqrt{3}$ is irrational, then $\exists \sqrt{3} < r < x$. Obviously $(B_{x-r}(x)\backslash\{x\}) \cap E = \emptyset \implies$ x is not a limit point of $E$, which is a contradiction. Thus $x < \sqrt{3}$.

  Now we have $\sqrt{2} < x < \sqrt{3} \implies x \in E$, so $E$ is closed.

- Let $x = 0$, then $\forall y \in E$ $d(x, y) < \sqrt{3}$, so $E$ is bounded.

- Consider an open cover $\{(\sqrt{2}, r)\}_{r \in (\sqrt{2}, \sqrt{3})}$ of $E$. It does not have a finite subcover since for any finite subcover $\{(\sqrt{2}, r_i)\}_{i=1}^{n}$, there is always a rational number exclusively between $\max\{r_i : j = 1, \ldots, n\}$ and $\sqrt{3}$ that is not covered by $\{(\sqrt{2}, r)\}_{i=1}^{n}$. Thus $E$ is not compact.

## Q6

Given an open cover $\{\mathcal{U}_\alpha\}_{\alpha\in\mathcal{A}}$ of $X$, we have

$$\bigcup_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha = X \implies \bigcap_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha^{\mathsf{C}} = \left(\bigcup_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha\right)^{\mathsf{C}} = X^{\mathsf{C}} = \emptyset.$$

Since each $\mathcal{U}_\alpha$ is open, its complement $\mathcal{U}_\alpha^{\mathsf{C}}$ is closed. Observe that the collection of closed sets $\{\mathcal{U}_\alpha^{\mathsf{C}}\}_{\alpha\in\mathcal{A}}$ does not have the finite intersection property, and hence there exists a finite subfamily $\mathcal{B}$ of $\mathcal{A}$ such that

$$\bigcap_{\alpha\in\mathcal{B}} \mathcal{U}_\alpha^{\mathsf{C}} = \emptyset.$$

i.e.

$$\bigcup_{\alpha\in\mathcal{B}} \mathcal{U}_\alpha = \left(\bigcap_{\alpha\in\mathcal{B}} \mathcal{U}_\alpha^{\mathsf{C}}\right)^{\mathsf{C}} = \emptyset^{\mathsf{C}} = X.$$

Thus $\{\mathcal{U}_\alpha\}_{\alpha\in\mathcal{A}}$ has a finite subcover $\{\mathcal{U}_\alpha\}_{\alpha\in\mathcal{B}}$, and hence $X$ is compact.

## Q7

Denote $E = \{(s_n) \in X : |s_n| \leq 1 \text{ for all } n\}$. Let $(t_n) = 0$. It is in $X$ because it is bounded by 0. Then $\forall (s_n) \in E\ d((s_n),(t_n)) = \sup\{|s_n| : n \in \mathbb{N}\} \leq 1$, so $E$ is bounded.

Consider $(s_n) \in E'$. Then $\forall r > 0\ (B_r((s_n)))\backslash\{(s_n)\}) \cap E \neq \emptyset$. In other words,

$$\forall r > 0\ \exists (s_n) \neq (t_n) \in X\ \forall n \in \mathbb{N}\ |t_n - s_n| < r \text{ and } |t_n| \leq 1.$$

For the purpose of contradiction, suppose $\exists N \in \mathbb{N}\ |s_N| > 1$. If $s_N > 1$, then let $r = \frac{s_N - 1}{2}$. It follows $|t_N - s_N| < \frac{s_N-1}{2} \implies t_N > 1$, which is a contradiction. If $s_N < -1$, then let $r = \frac{-1-s_N}{2}$. It follows $|t_N - s_N| < \frac{-1-s_N}{2} \implies t_N < -1$, which is a contradiction. Thus $\forall n \in \mathbb{N}\ |s_n| \leq 1 \implies (s_n) \in E$, and hence $E$ is closed.

For each $(s_n) \in E$, define an open set $\mathcal{U}_{(s_n)} = \{(x_n) \in X : d((x_n),(s_n)) < 1\}$. Then $\{\mathcal{U}_{(s_n)}\}_{(s_n)\in E}$ is an open cover of $E$ trivially. Now consider sequences in $E$: $(x_n^{(1)}) = (1,-1,-1,\dots)$, $(x_n^{(2)}) = (-1,1,-1,\dots)$, $(x_n^{(3)}) = (-1,-1,1,\dots)$. For each $j = 1,2,\dots$, all terms in $(x_n^{(j)})$ are $-1$ except $x_j^{(j)} = 1$. Observe that for any $i \neq j$, $d((x_n^{(i)}),(x_n^{(j)})) = 2$. By the construction, any distinct sequences $(x_n^{(j)})$ cannot belong to the same open set $\mathcal{U}_{s_n}$ because otherwise, $d((x_n^{(i)}),(x_n^{(j)})) \leq d((x_n^{(i)}),(s_n)) + d((s_n),(x_n^{(j)})) < 1 + 1 = 2$, which is a contradiction. Thus, any finite subcover of $\{\mathcal{U}_{(s_n)}\}_{(s_n)\in E}$ cannot cover all such sequences $(x_n^{(j)})$, so $E$ is not compact.