Math 104 Worksheet 6

UC Berkeley, Summer 2021

Tuesday, July 6

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1. Let X be the set of all **bounded** sequences of real numbers. Define the function $d : X \times X \to \mathbb{R}$ by $d\big((s_n), (t_n)\big) = \sup\{|t_n - s_n| : n \in \mathbb{N}\}$.

Exercise 1. Verify that $d\big((s_n), (t_n)\big) < \infty$ for any bounded sequences (s_n) and (t_n) in $\mathbb{R}$.

Exercise 2. Verify that d satisfies the requirements of a metric, so (X, d) is a metric space.

Exercise 3. (Challenge) Prove that (X, d) is a **complete** metric space.
Hint: Begin with a Cauchy sequence (of sequences) $((s_1^{(n)}, s_2^{(n)}, s_3^{(n)}, \ldots))_{n\in\mathbb{N}}$ in X. Justify that $(s_k^{(n)})$ converges for each k, say to s_k^*. Prove that $((s_1^{(n)}, s_2^{(n)}, s_3^{(n)}, \ldots))_{n\in\mathbb{N}}$ converges to $(s_1^*, s_2^*, s_3^* \ldots)$.

2. Let (X, d) be a metric space. The following are two equivalent definitions of a **closed** set.
Definition 1. A set $E \subseteq X$ is **closed** if every limit point of E is an element of E.
Definition 2. A set $E \subseteq X$ is **closed** if E^c is open.

<u>Exercise 4</u>. Prove that the two definitions of a closed set above are equivalent; that is, show that for $E \subseteq X$, E^c is open if and only if every limit point of E is an element of E.

$\Longleftarrow$ Suppose that every limit point of E is an element of E. Let $x \in E^c$. (Show that x is an interior point of E^c.)

$\Longrightarrow$ Suppose that E^c is open. Let x be a limit point of E. (Show that $x \in E$ by showing that x is not an interior point of E^c and therefore $x \notin E^c$.)