Math 104 Worksheet 18

UC Berkeley, Summer 2021

Thursday, August 5

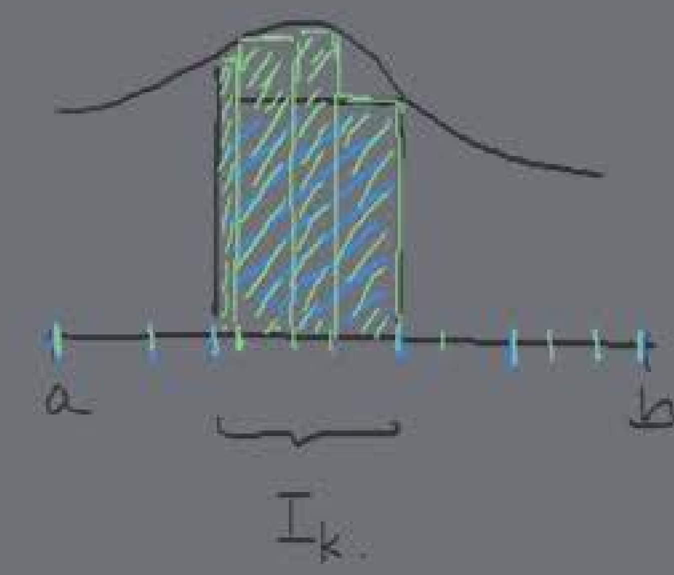
Recall. Let f be a bounded function on $[a, b]$. For a **partition** $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ we define

$$U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1}) \quad \text{and} \quad L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1})$$

where $M(f, S) = \sup\{f(x) : x \in S\}$ and $m(f, S) = \inf\{f(x) : x \in S\}$. Then we define

$$U(f) = \inf\{U(f, P) : P \in \Pi_{[a,b]}\} \quad \text{and} \quad L(f) = \sup\{L(f, P) : P \in \Pi_{[a,b]}\}$$

where $\Pi_{[a,b]}$ is the set of all partitions of $[a, b]$.

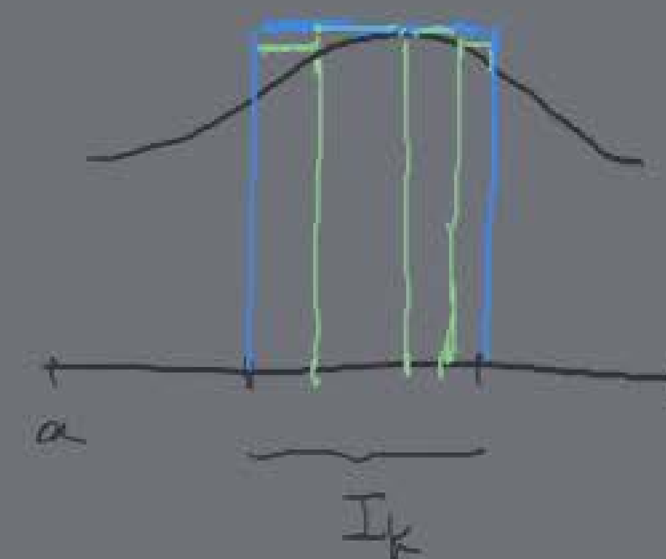


Definition. If $P, P^* \in \Pi_{[a,b]}$ and $P \subseteq P^*$, P^* is called a **refinement** of P.

Exercise 1. Prove that if P^* is a refinement of P, then

$$L(f, P) \leq L(f, P^*) \leq U(f, P^*) \leq U(f, P).$$

Proof. Let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$. For each subinterval $I_k = [a_{k-1}, a_k]$, P^* induces a partition $P_k^* = \{s \in P^* : a_{k-1} \leq s \leq a_k\} = \{a_{k-1} = s_0 < \ldots < s_m = a_k\}$ of I_k. (Complete the proof.)



$$L(f, P) = \sum_{k=1}^{n} \underbrace{m(f, I_k) \cdot \ell(I_k)}$$

for P^*, $m(f, I_k)\ell(I_k) \leq \sum_{i=1}^{m} m(f, J_i) \cdot \ell(J_i)$

$L(f, P) \leq L(f, P^*)$. Likewise, $U(f, P) \geq U(f, P^*)$.

Exercise 2. Prove that if $P, Q \in \Pi_{[a,b]}$, then $L(f, P) \leq U(f, Q)$.
(Hint: Use Exercise 1.)

Consider the refinement $P \cup Q$ (of both P and Q)

$\underline{L(f, P)} \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq \underline{U(f, Q)}$.

Exercise 3. Prove that $L(f) \le U(f)$.

For any $P, Q \in \Pi_{[a,b]}$. $L(f,P) \le U(f,Q)$.

So $U(f,Q)$ is an upper bound for $\{L(f,P) : P \in \Pi_{[a,b]}\}$.

So $L(f) \le U(f,Q)$. True for any $Q \in \Pi_{[a,b]}$,

hence $L(f)$ is a lower bound for $\{U(f,Q) : Q \in \Pi_{[a,b]}\}$. $\therefore$ $L(f) \le U(f)$.

Definition. f is **integrable/Darboux integrable/Riemann integrable** if $L(f) = U(f)$.

Lemma will be used to prove $\int f+g = \int f + \int g$.

Lemma. Let f and g be two bounded functions on $[a,b]$. Then

(i) $\inf\{U(f,P)+U(g,P) : P \in \Pi_{[a,b]}\} = \inf\{U(f,P) : P \in \Pi_{[a,b]}\}+\inf\{U(g,P) : P \in \Pi_{[a,b]}\}$;

(ii) $\sup\{L(f,P)+L(g,P) : P \in \Pi_{[a,b]}\} = \underbrace{\sup\{L(f,P) : P \in \Pi_{[a,b]}\}}_{L(f)}+\underbrace{\sup\{L(g,P) : P \in \Pi_{[a,b]}\}}_{L(g)}$.

$A+B = \{a+b \ a \in A, b \in B\}$

Exercise 4. Prove part (i) of the preceding lemma.

(i) $\ge$ trivial: for any partition Q,

$$U(f,Q) + U(g,Q) \ge \inf\{U(f,P)\} + \inf\{U(g,P)\}.$$

$$\Rightarrow \inf\{U(f,Q)+U(g,Q) : Q \in \Pi_{[a,b]}\} \ge \inf\{U(f,P)\} + \inf\{U(g,P)\}.$$

(ii) $\le$ Let $\varepsilon > 0$.

WTS: $\inf\{U(f,P)+U(g,P)\} \le \underbrace{\inf\{U(f,P)\}+\frac{\varepsilon}{2}}_{\ge U(f,P_1) \text{ for some } P_1} + \underbrace{\inf\{U(g,P)\}+\frac{\varepsilon}{2}}_{\ge U(g,P_2) \text{ for some } P_2}$.

$P^* = P_1 \cup P_2$

$$\inf\{U(f,P)+U(g,P)\} \le U(f,P^*)+U(g,P^*) \le U(f,P_1)+U(g,P_2) \le \inf\{U(f,P)\}+\frac{\varepsilon}{2}+\inf\{U(g,P)\}+\frac{\varepsilon}{2}.$$

$$= \inf\{U(f,P)\}+\inf\{U(g,P)\}+\varepsilon.$$

<u>Better notation</u>: $U_a^b(f)$, $L_a^b(f)$.

If f is <u>integrable</u> on $[a,b]$, we define the <u>integral of f</u> on $[a,b]$ as

(Riemann or Darboux)

$$\int_a^b f(x)\,dx = L_a^b(f) = U_a^b(f).$$

<u>Ex</u>: On $[0,1]$, $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$

For any partition P, $L(f,P) = 0 \Rightarrow L(f) = 0$

$U(f,P) = 1 \Rightarrow U(f) = 1$. $\qquad L(f) \neq U(f)$.

$\therefore$ f is not Riemann / Darboux integrable.

(Not in the course) f <u>is</u> Lebesgue integrable, with integral 0.

<u>Def</u>: The <u>mesh</u> of a partition $P = \{a = t_0 < \ldots < t_n = b\}$ is

$$\text{mesh}(P) = \max\{t_1 - t_0,\ t_2 - t_1,\ \ldots,\ t_n - t_{n-1}\}$$

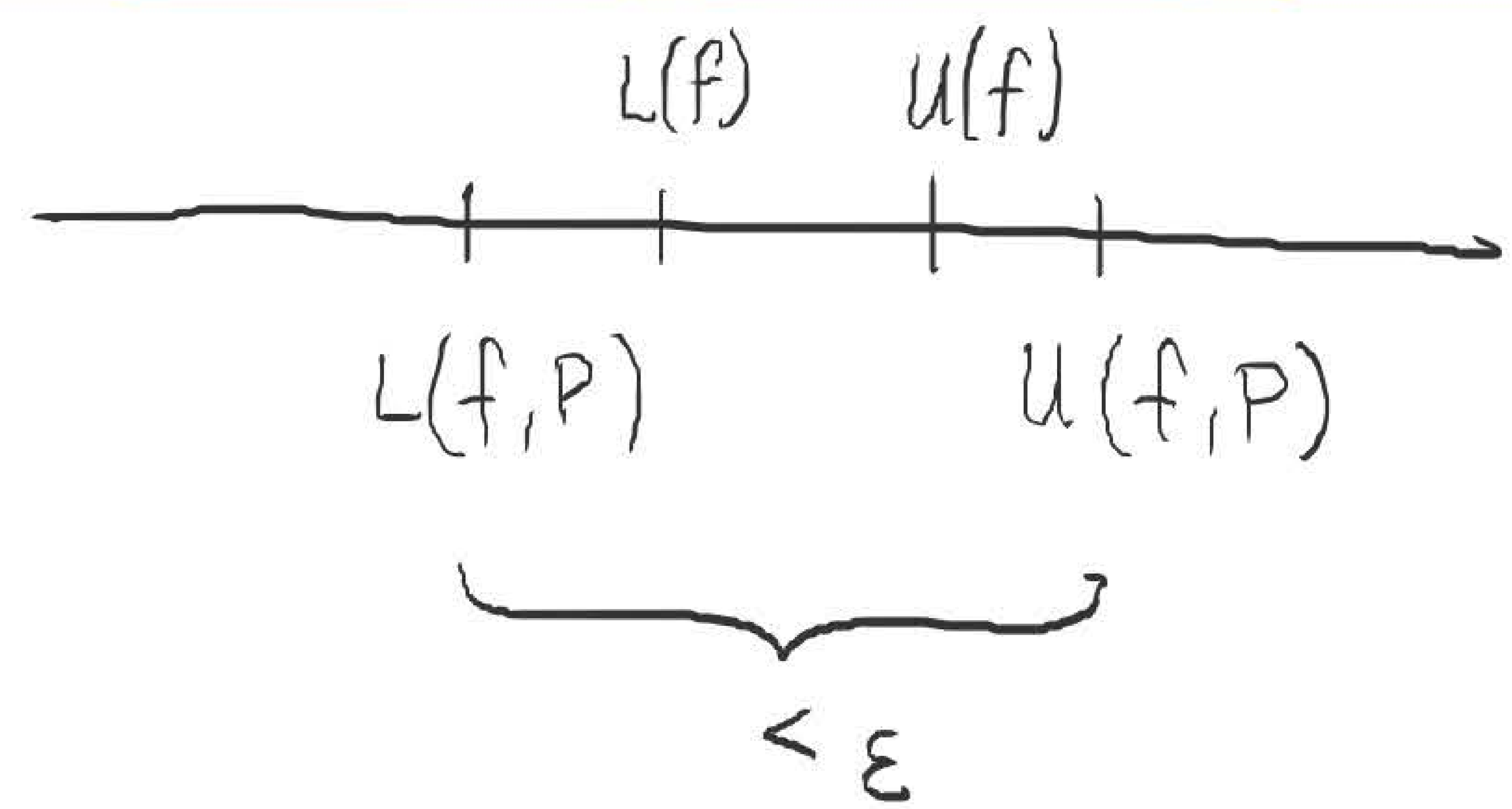
= length of longest subinterval.

Theorem: Every monotonic function f on $[a,b]$ is integrable.

Proof: (nonincreasing case) Assume $f(a) > f(b)$ (else f is constant).

Let $\varepsilon > 0$. $\boxed{\text{Goal: find } P \in \Pi_{[a,b]} \text{ such that } U(f,P) - L(f,P) \le \varepsilon}$

Common method: if true, the $U(f) - L(f) < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, follows that $U(f) = L(f)$.



Let $P \in \Pi_{[a,b]}$ such that $\operatorname{mesh}(P) < \dfrac{\varepsilon}{f(b)-f(a)}$.

$$\underline{U(f,P) - L(f,P)} = \sum_{k=1}^{n} \underbrace{\Big[\underbrace{M(f, I_k)}_{f(t_{k-1})} - \underbrace{m(f, I_k)}_{f(t_k)} \Big]}_{= f(t_{k-1}) - f(t_k) \ge 0} \underbrace{\ell(I_k)}_{< \frac{\varepsilon}{f(b)-f(a)}}$$

$[t_{k-1}, t_k]$

$$\le \sum_{k=1}^{n} \big(f(t_{k-1}) - f(t_k) \big) \cdot \frac{\varepsilon}{f(b)-f(a)}$$

$$= \frac{\varepsilon}{f(b)-f(a)} \Big(\cancel{f(t_1)} - \underbrace{f(t_0)}_{=f(a)} + \cancel{f(t_2)} - \cancel{f(t_1)} + \cancel{\cdots} + \underbrace{f(t_n)}_{=f(b)} - \cancel{f(t_{n-1})} \Big)$$

$$= \underline{\varepsilon}.$$

Theorem: Every continuous function f on $[a,b]$ is integrable.

Proof: Let $\varepsilon > 0$. Since f is continuous on $[a,b]$, it is uniformly continuous, so there exists $\delta > 0$ such that $x, y \in [a,b]$, $|x-y| < \delta$, then $|f(x) - f(y)| < \dfrac{\varepsilon}{b-a}$.

Let $P \in \Pi_{[a,b]}$ such that $\operatorname{mesh}(P) < \delta$.

$$U(f,P) - L(f,P) = \sum_{k=1}^{n} \Big(\underbrace{M(f, I_k) - m(f, I_k)}_{< \frac{\varepsilon}{b-a}} \Big) \cdot \ell(I_k).$$

needs careful argument!

$$< \frac{\varepsilon}{b-a} \underbrace{\sum_{k=1}^{n} \ell(I_k)}_{b-a}$$

$$= \varepsilon.$$

Theorem: Let f, g be integrable functions on $[a,b]$ and let $c \in \mathbb{R}$.

Then

(i) cf is integrable and $\int_a^b (cf)(x)\,dx = c\int_a^b f(x)\,dx$.

(ii) $f+g$ is integrable and $\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

Proof: (i) $c=0$: trivial.

$\{c\cdot s : s \in S\}$.

$c>0$: recall: $\sup(cS) = c\cdot\sup(S)$, $\inf(cS) = c\cdot\inf(S)$.

$\Rightarrow M(cf, S) = c\cdot M(f,S)$, $m(cf,S) = c\cdot m(f,S)$ for any $S \subseteq [a,b]$.

$\Rightarrow U(cf,P) = \sum_{k=1}^{n} \underbrace{M(cf, I_k)}_{= c\cdot M(f,I_k)} \ell(I_k) = c\cdot U(f,P)$ and $L(cf,P) = c\cdot L(f,P)$.

$\therefore \{U(cf,P) : P \in \Pi_{[a,b]}\} = c\cdot\{U(f,P) : P\in\Pi_{[a,b]}\}$

$\Rightarrow U_a^b(cf) = c\,U_a^b(f) = c\,L_a^b(f) = L_a^b(cf)$. Likewise for L instead of U.

$\Rightarrow cf$ is integrable, $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.

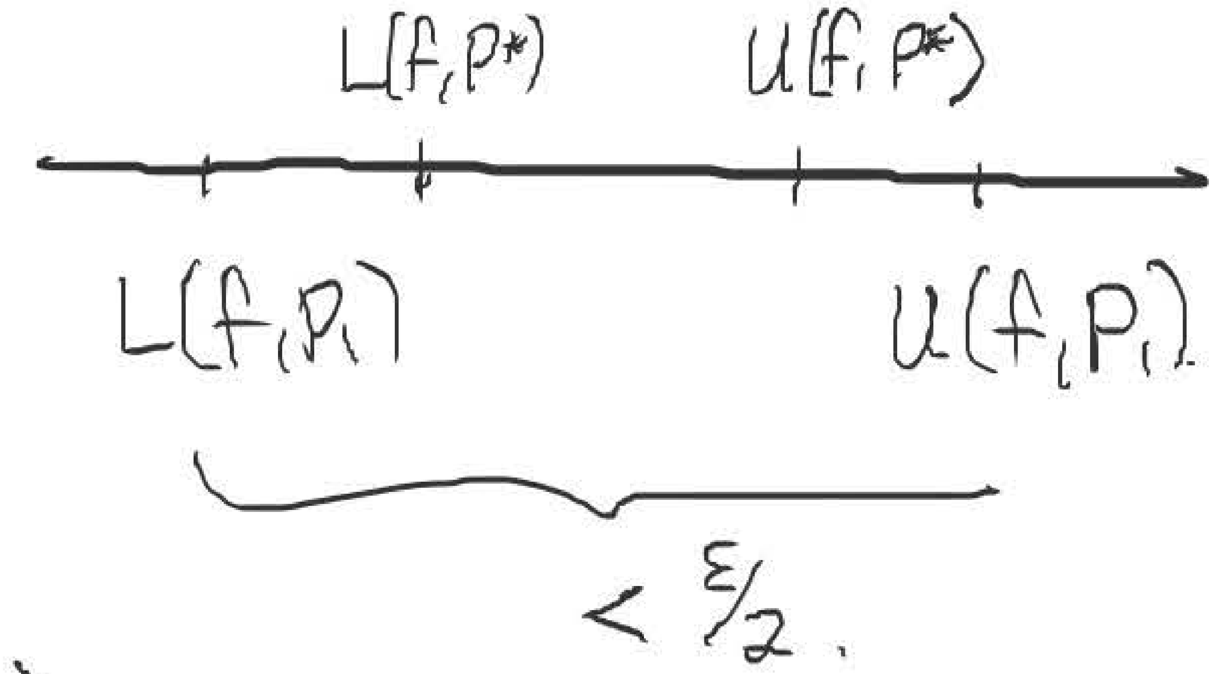
$c<0$: similar; $\sup S = -\inf(-S)$, $\inf(S) = -\sup(-S)$.

(ii) Let $\varepsilon > 0$. Let P_1 : $U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{2}$.

P_2 : $U(g, P_2) - L(g, P_2) < \frac{\varepsilon}{2}$.

Let $P^* = P_1 \cup P_2$. $U(f, P^*) - L(f, P^*) < \frac{\varepsilon}{2}$.

$U(g, P^*) - L(g, P^*) < \frac{\varepsilon}{2}$.



$\Rightarrow$

$$U(f+g, P^*) - L(f+g, P^*) \underset{?}{\le} \left(U(f, P^*) + U(g, P^*)\right) - \left(L(f, P^*) + L(g, P^*)\right) < \varepsilon$$

$\sum \underline{M(f+g, I_k)}\, \ell(I_k)$ $\qquad$ $\sum \underline{[M(f, I_k) + M(g, I_k)]} \cdot \ell(I_k)$.

$M(f+g, I_k) \le M(f, I_k) + M(g, I_k)$.

- $U(f+g, P^*) \le U(f, P^*) + U(g, P^*)$.
- $L(f+g, P^*) \ge L(f, P^*) + L(g, P^*)$.

$\Rightarrow -L(f+g, P^*) \le -\left(L(f, P^*) + L(g, P^*)\right)$.

$U(f+g, P^*) - L(f+g, P^*) < \varepsilon$. $\therefore$ $f+g$ is integrable.

Note that for any $P \in \Pi_{[a,b]}$,

$$L(f,P) + L(g,P) \leq L(f+g,P) \leq U(f+g,P) \leq U(f,P) + U(g,P).$$

Then

$$\begin{aligned}
U_a^b(f+g) = \inf\{U(f+g,P) : P \in \Pi_{[a,b]}\} &\leq \inf\{U(f,P) + U(g,P) : P \in \Pi_{[a,b]}\}. \\
&= \inf\{U(f,P) : P \in \Pi_{[a,b]}\} + \inf\{U(g,P) : P \in \Pi_{[a,b]}\} \\
\text{by worksheet lemma} \quad &= U_a^b(f) + U_a^b(g) \\
&= L_a^b(f) + L_a^b(g) \\
&= \sup\{L(f,P) : P \in \Pi_{[a,b]}\} + \sup\{L(g,P) : P \in \Pi_{[a,b]}\} \\
&\leq \sup\{L(f,P) + L(g,P) : P \in \Pi_{[a,b]}\} \\
&\leq \sup\{L(f+g,P) : P \in \Pi_{[a,b]}\} \\
&= L_a^b(f+g).
\end{aligned}$$

$\therefore U_a^b(f+g) \leq L_a^b(f+g)$,

so $U_a^b(f+g) = L_a^b(f+g)$.