# Math 104 Worksheet 8

UC Berkeley, Summer 2021

Monday, July 12

---

On Worksheet 8, we showed that in a metric space $(X, d)$, if $E$ is a compact set then every sequence in $E$ has a convergent subsequence (whose limit lies in $E$). This worksheet guides a proof of the converse.

**Lemma 1.** Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of $E$. If every sequence in $E$ has a convergent subsequence whose limit is in $E$, then there exists $\varepsilon > 0$ such that for every $x \in E$, there exists $\alpha \in A$ such that $B_\varepsilon(x) \subseteq U_\alpha$.

*Proof.* (Contrapositive) Suppose that for any $\varepsilon > 0$, there exists $x \in E$ such that $B_\varepsilon(x) \nsubseteq U_\alpha$ for all $\alpha \in A$. Then for each $n \in \mathbb{N}$, there exists $x_n$ such that $V_n := B_{1/n}(x_n) \nsubseteq U_\alpha$ for all $\alpha \in A$.

Claim: $(x_n)$ does not have a convergent subsequence.

Exercise 1. Prove the claim by contradiction. (*Hint:* If $(x_n)$ did have a convergent subsequence, then the limit $x$ would be in $U_\alpha$ for some $\alpha$. Since $U_\alpha$ is open, there is an open ball around $x$ that fits inside $U_\alpha$. Show that some $V_n$ fits inside that ball.)

**Lemma 2.** If every sequence in $E$ has a convergent subsequence whose limit is in $E$, then for any $\varepsilon > 0$ there exists a finite collection $x_1, \ldots, x_n$ of points in $E$ such that $E \subseteq \bigcup_{i=1}^n B_\varepsilon(x_i)$.

*Proof.* (Contrapositive) Suppose that for some $\varepsilon > 0$, $E$ cannot be covered by finitely many open balls of radius $\varepsilon$.

Exercise 2. Construct (inductively) a sequence $(x_n)$ in $E$ such that $d(x_m, x_n) \geq \varepsilon$ for any $m \neq n$. Explain why this sequence has no convergent subsequence.

**Theorem.** $E$ is compact if and only if every sequence in $E$ has a convergent subsequence whose limit is in $E$.

*Proof.* (The forward direction has already been proven.) Suppose every sequence in $E$ has a convergent subsequence whose limit is in $E$. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of $E$.

Exercise 3. Use Lemma 1 and Lemma 2 to show that $\{U_\alpha\}_{\alpha \in A}$ has a finite subcover.