# Q1

(a)

- Positive Definiteness: $\forall (x_1, y_1), (x_2, y_2) \in (X \times Y)$ $d_X(x_1, x_2) \geq 0$ and $d_Y(y_1, y_2) \geq 0 \implies d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} \geq 0$ since $d_X$ and $d_Y$ are proper metrics. Also because

$$\begin{aligned}\max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} = 0 &\iff 0 \leq d_X(x_1, x_2) \leq 0 \text{ and } 0 \leq d_Y(y_1, y_2) \leq 0 \\ &\iff d_X(x_1, x_2) = 0 \text{ and } d_Y(y_1, y_2) = 0 \\ &\iff x_1 = x_2 \text{ and } y_1 = y_2 \\ &\iff (x_1, y_1) = (x_2, y_2)\end{aligned}$$

  $d$ is positive definite.

- Symmetry: Since $d_X$ and $d_Y$ are proper metrics, it is clear that

$$\max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} = \max\{d_X(x_2, x_1), d_Y(y_2, y_1)\}.$$

  Thus $d$ is symmetric.

- Triangular Inequality: Since $d_X$ and $d_Y$ are proper metrics, it is clear that for each $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in (X \times Y)$ we have

$$\begin{aligned}d_X(x_1, x_2) &\leq d_X(x_1, x_3) + d_X(x_3, x_2), \\ d_Y(y_1, y_2) &\leq d_Y(y_1, y_3) + d_Y(y_3, y_2).\end{aligned}$$

  If $d_X(x_1, x_2) \leq d_Y(y_1, y_2)$, then

$$\begin{aligned}\max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} &= d_Y(y_1, y_2) \\ &\leq d_Y(y_1, y_3) + d_Y(y_3, y_2) \\ &\leq \max\{d_X(x_1, x_3), d_Y(y_1, y_3)\} + \max\{d_X(x_3, x_2), d_Y(y_3, y_2)\}.\end{aligned}$$

  If $d_X(x_1, x_2) > d_Y(y_1, y_2)$, then

$$\begin{aligned}\max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} &= d_X(x_1, x_2) \\ &\leq d_X(x_1, x_3) + d_X(x_3, x_2) \\ &\leq \max\{d_X(x_1, x_3), d_Y(y_1, y_3)\} + \max\{d_X(x_3, x_2), d_Y(y_3, y_2)\}.\end{aligned}$$

  Thus combining two cases, we have

$$d((x_1, y_1), (x_2, y_2)) \leq d((x_1, y_1), (x_3, y_3)) + d((x_3, y_3), (x_2, y_2)),$$

  completing the proof.

(b) Consider an arbitrary sequence $(x_n, y_n)$ in $E \times F$. Note that $(x_n) \in E$ and $(y_n) \in F$. Since $E$ is compact, $(x_n)$ has a subsequence $(x_{n_k})$ converging to $x_0$ in $E$. Moreover, since $F$ is compact, $(y_{n_k})$ has a subsequence $(y_{n_{k_l}})$ converging to $y_0$ in $F$. Now $(x_n, y_n)$ has a subsequence $(x_{n_{k_l}}, y_{n_{k_l}})$ converging to $(x_0, y_0)$ in $E \times F$, so $E \times F$ is compact.

## Q2

Since $\sum a_n$ is convergent, $\lim a_n = 0$. i.e. Let $\epsilon = 1$ $\exists N_1 \in \mathbb{N}$ $n \geq N_1 \implies |a_n| < 1 \implies \left|\frac{1}{a_n}\right| = \frac{1}{|a_n|} > 1$. Thus $\lim \frac{1}{n} \neq 0$, implying that $\sum \frac{1}{a_n}$ does not converge, and hence diverges.

## Q3

Since $\sum(a_n + b_n) = \sum a_n + \sum b_n$, it is clear that $\sum(a_n + b_n)$ converges when both $\sum a_n$ and $\sum b_n$ converge. Observe that when for each $n \in N$ $a_n$ and $b_n$ are nonnegative, we have

$$\left|\sqrt{a_n b_n}\right| = \sqrt{a_n b_n} \leq \sqrt{2a_n b_n} \leq \sqrt{a_n^2 + b_n^2 + 2a_n b_n} = \sqrt{(a_n + b_n)^2} = a_n + b_n.$$

By comparison test, $\sum \sqrt{a_n b_n}$ converges.

## Q4

Since $\liminf |a_n| = 0$, there exists a subsequence $|a_{n_k}|$ such that $\lim |a_{n_k}| = 0$, i.e. $\forall \epsilon > 0 \; \exists K \in \mathbb{N} \; k \geq K \implies |a_{n_k}| < \epsilon$. Thus select $\epsilon = 1$ and according to the definition of limit, we can select $n_{k_1}$ such that $\left|a_{n_{k_1}}\right| < 1$. Having already found $n_{k_1} < n_{k_2} < \cdots < n_{k_l}$ such that $\left|a_{n_{k_l}}\right| < \frac{1}{l^2}$, we can choose $n_{k_{l+1}} > n_{k_1}$ such that $\left|a_{n_{k_{l+1}}}\right| < \frac{1}{(l+1)^2}$ since we have infinitely many terms smaller than $\frac{1}{(l+1)^2}$.

Now we have a series $\sum a_{n_{k_{l+1}}}$ such that $\left|a_{n_{k_l}}\right| < \frac{1}{l^2}$. Then by comparison test, since $\sum \frac{1}{l^2}$ converges, $\sum a_{n_{k_l}}$ also converges. Note $(a_{n_{k_l}})$ is a subsequence of $(a_n)$.

## Q5

Let $\sum a_n = \sum \frac{(-1)^n}{\sqrt{n}}$. It is an alternating series because

$$1 > \frac{1}{\sqrt{2}} > \frac{1}{\sqrt{3}} > \cdots > 0 \text{ and } \lim \frac{1}{\sqrt{n}} = 0.$$

Thus $\sum a_n$ converges. Then $\sum a_n^2 = \sum (\frac{(-1)^n}{\sqrt{n}})^2 = \sum \frac{1}{n}$is the harmonic series and hence $\sum a_n^2$ diverges.

## Q6

Since $\sum a_n$ converges, it satisfies the Cauchy Criterion, i.e.

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} \ n \geq m \geq N \implies \left| \sum_{k=m}^{n} a_n \right| < \frac{\epsilon}{2}.$$

Let $m = N$, then we have

$$\begin{aligned} \frac{\epsilon}{2} &> |a_N + a_{N+1} + \cdots + a_n| \\ &\geq (n - N + 1)|a_n| \quad \text{since } (a_n) \text{ is nonincreasing.} \end{aligned}$$

If $n \geq 2N$, then $n \leq 2(n-N) < 2(n-N+1)$. Thus

$$|na_n| = n|a_n| < 2(n-N+1)|a_n| < 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

i.e. $\lim na_n = 0$.

## Q7

(a) First observe that all terms are nonzero. Using ratio test, we have

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n+1}{(n+2)!} \cdot \frac{(n+1)!}{n}\right| = \lim \left|\frac{n+1}{n^2-2n}\right| = 0.$$

Thus $\limsup \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$, and hence the series converges (absolutely).

(b) If $a = 0$, then the series converges trivially. If $a \neq 0$, first observe all terms are nonzero, and then we have

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{a^{n+1}}{(n+1)!} \cdot \frac{n!}{a^n}\right| = \lim \left|\frac{a}{n+1}\right| = 0.$$

Thus $\limsup \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$, and hence the series converges (absolutely) by the ratio test.

(c) Let $(a_n) = \frac{1}{\sqrt{n}}$. Note that we have

$$\frac{1}{\sqrt{1}} > \frac{1}{\sqrt{2}} > \frac{1}{\sqrt{3}} > \cdots > 0 \text{ and } \lim \frac{1}{\sqrt{n}} = 0.$$

Thus by alternating series test, the series converges.

(d) Observe that

$$\sum_{n=2}^{\infty} \frac{1}{n \log n} > \int_2^{\infty} \frac{1}{n \log n}\, dn = \infty.$$

Thus by the integral test, the series diverges.

## Q8

## Q9

## Q10