Math 104 Homework 1

UC Berkeley, Summer 2021

Due by Friday, June 25, 11:59pm PDT

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1. Reverse triangle inequality (Ross 3.5)
(a) Show that $|b| \leq a$ if and only if $-a \leq b \leq a$.
(b) Prove that $||a| - |b|| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

2. Prove that

$$2\sqrt{n} - 2 < \sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n} - 1$$

for any integer $n \geq 2$, by following the steps below.
(a) Prove that for any $n \in \mathbb{N}$,

$$2(\sqrt{n+1} - \sqrt{n}) < \frac{1}{\sqrt{n}} < 2(\sqrt{n} - \sqrt{n-1}).$$

(b) Prove that for any integer $n \geq 2$,

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > 2\sqrt{n} - 2.$$

(c) Use induction to prove that for all integers $n \geq 2$,

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n} - 1.$$

3. (Ross 3.8) Let $a, b \in \mathbb{R}$. Show that if $a \leq b_1$ for every $b_1 > b$, then $a \leq b$.

4. (Ross 4.8) Let S and T be nonempty subsets of $\mathbb{R}$ such that $s \leq t$ for all $s \in S$ and $t \in T$. Prove that $\sup S \leq \inf T$.

5. Consider the following sets:

$$A = (0, \infty) \qquad B = \{\tfrac{1}{m} + \tfrac{1}{n} : m, n \in \mathbb{N}\}, \qquad C = \{x^2 - 1 : x \in \mathbb{R}\},$$

$$D = [0,1] \cup [2,3] \qquad E = \bigcup_{n=1}^{\infty} [2n, 2n+1], \qquad F = \bigcap_{n=1}^{\infty} (1 - \tfrac{1}{n}, 1 + \tfrac{1}{n}).$$

For each set, determine its minimum and maximum if they exist. In addition, determine each set's infimum and supremum (if the set is unbounded, answer in terms of ∞.)