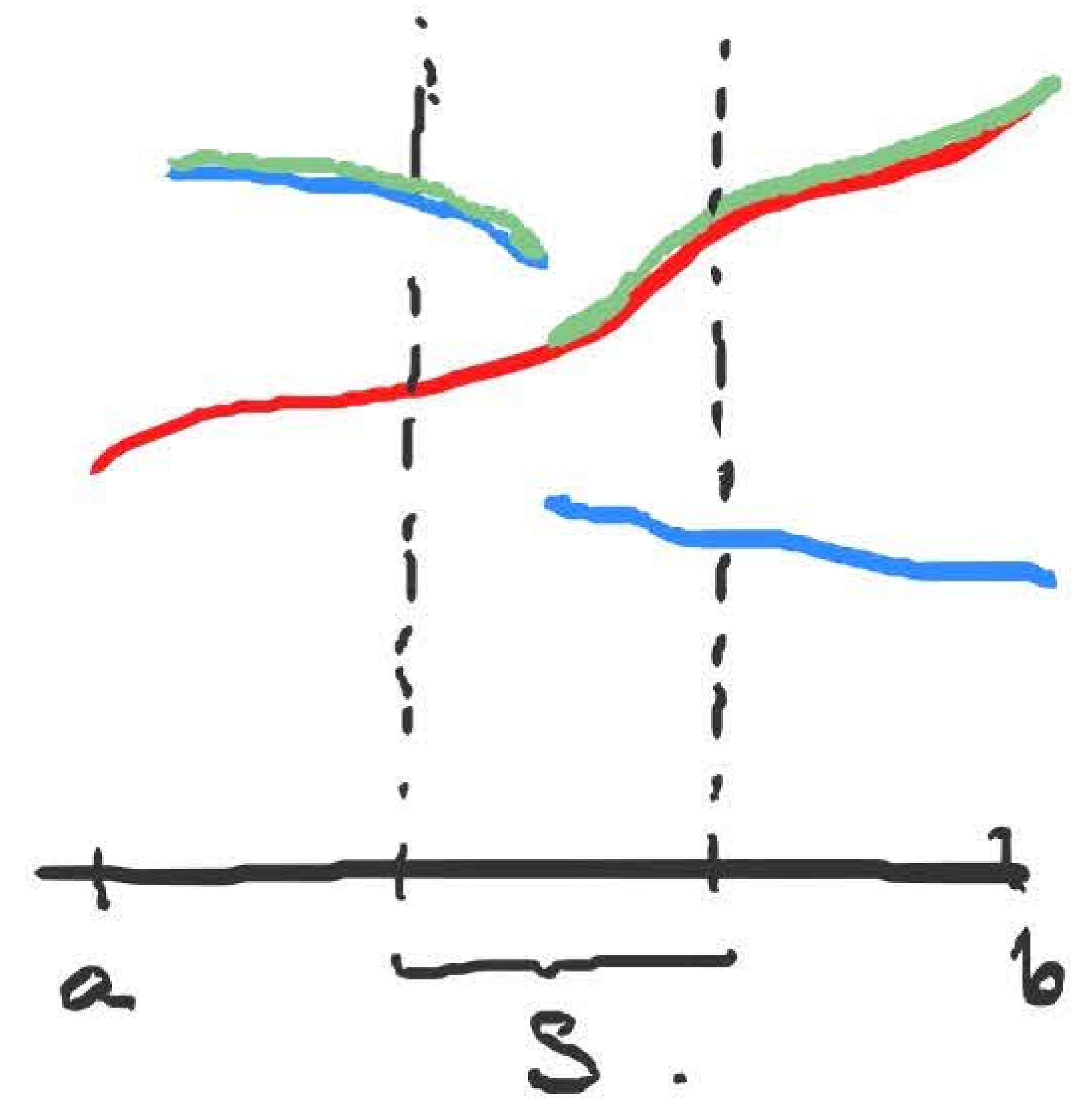
Tuesday, August 10



Warm-up: Show that if f and g are integrable on $[a,b]$, then $\max(f,g)$ is integrable on $[a,b]$.

- $$M(\max(f,g),S) - \underbrace{m(\max(f,g),S)}_{\geq m(f,S) \text{ and } \geq m(g,S)} =: A.$$

$$\leq \underbrace{M(\max(f,g),S)}_{= M(f,S) \text{ or } = M(g,S)} - m(f,S) \quad \text{AND} \quad \leq \underbrace{M(\max(f,g),S)}_{= M(f,S) \text{ or } = M(g,S)} - m(g,S).$$

$$\leq M(f,S) - m(f,S) \quad \text{OR} \quad \leq M(g,S) - m(g,S)$$

either $\boxed{\leq M(f,S) - m(f,S)}$ OR $\leq M(g,S) - m(f,S)$ AND $\leq M(f,S) - m(g,S)$ OR $\boxed{\leq M(g,S) - m(g,S)}$

Case 1: $M(f,S) \geq M(g,S)$.

$\boxed{A \leq M(f,S) - m(f,S)}$

OR

Case 2: $M(f,S) < M(g,S)$

$\boxed{A \leq M(g,S) - m(g,S)}$

$$M(\max(f,g),S) - m(\max(f,g),S)$$

$$\Rightarrow \quad \leq \big(M(f,S) - m(f,S)\big) + \big(M(g,S) - m(g,S)\big)$$

Let $\varepsilon > 0$. Let $P_1, P_2 \in \Pi_{[a,b]}$ such that

$$U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2}.$$

$$U(g,P_2) - L(g,P_2) < \frac{\varepsilon}{2}.$$

Let $P^* = P_1 \cup P_2$.

$$U(\max(f,g),P^*) - L(\max(f,g),P^*) = \sum_{k=1}^{n} \Big(\underbrace{M(\max(f,g),I_k) - m(\max(f,g),I_k)}_{\leq (M(f,I_k)-m(f,I_k)) + (M(g,I_k)-m(g,I_k))}\Big)\, \ell(I_k).$$

$$= \underbrace{\sum_{k=1}^{n}\big(M(f,I_k)-m(f,I_k)\big)\,\ell(I_k)}_{U(f,P^*)-L(f,P^*) < \frac{\varepsilon}{2}} + \underbrace{\sum_{k=1}^{n}\big(M(g,I_k)-m(g,I_k)\big)\,\ell(I_k)}_{U(g,P^*)-L(g,P^*) < \frac{\varepsilon}{2}}.$$

$$< \varepsilon.$$

Recall

- monotonic functions are integrable
- continuous functions are integrable
- f, g integrable, $c \in \mathbb{R}$ $\Rightarrow$ cf, $f+g$ integrable and

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx, \quad \int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

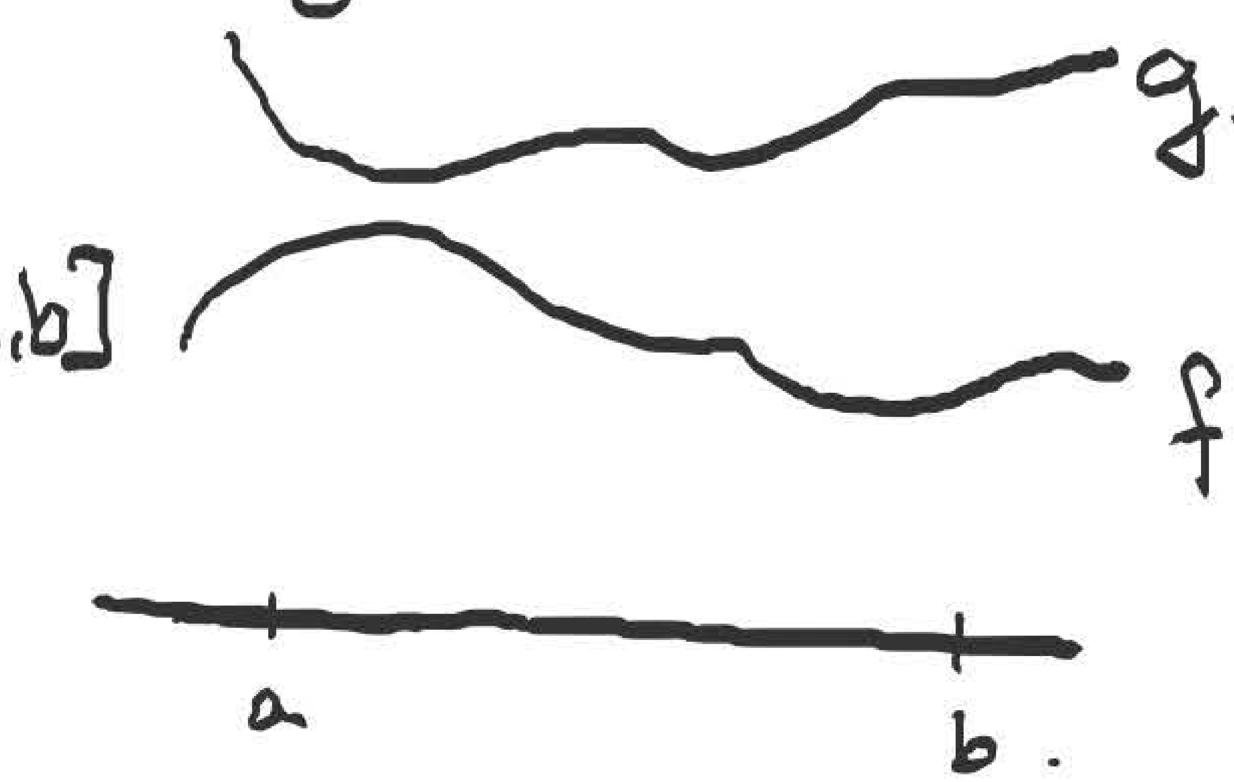
Today:

- Prove a few more basic integration laws
- Prove FTC.

Tomorrow:

- Discuss integral convergence theorems and integration/differentiation of power series.

<u>Theorem</u>: (i) If f and g are integrable on $[a,b]$ and $f(x) \le g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.



(ii) If g is continuous on $[a,b]$ and $g(x) \ge 0$ for all $x \in [a,b]$ and $\int_a^b g(x)\,dx = 0$, then $g \equiv 0$ on $[a,b]$.

<u>Proof</u>: (i) Let $h = g - f$. $h \ge 0 \Rightarrow L_a^b(h) \ge 0 \Rightarrow \int_a^b h(x)\,dx \ge 0$

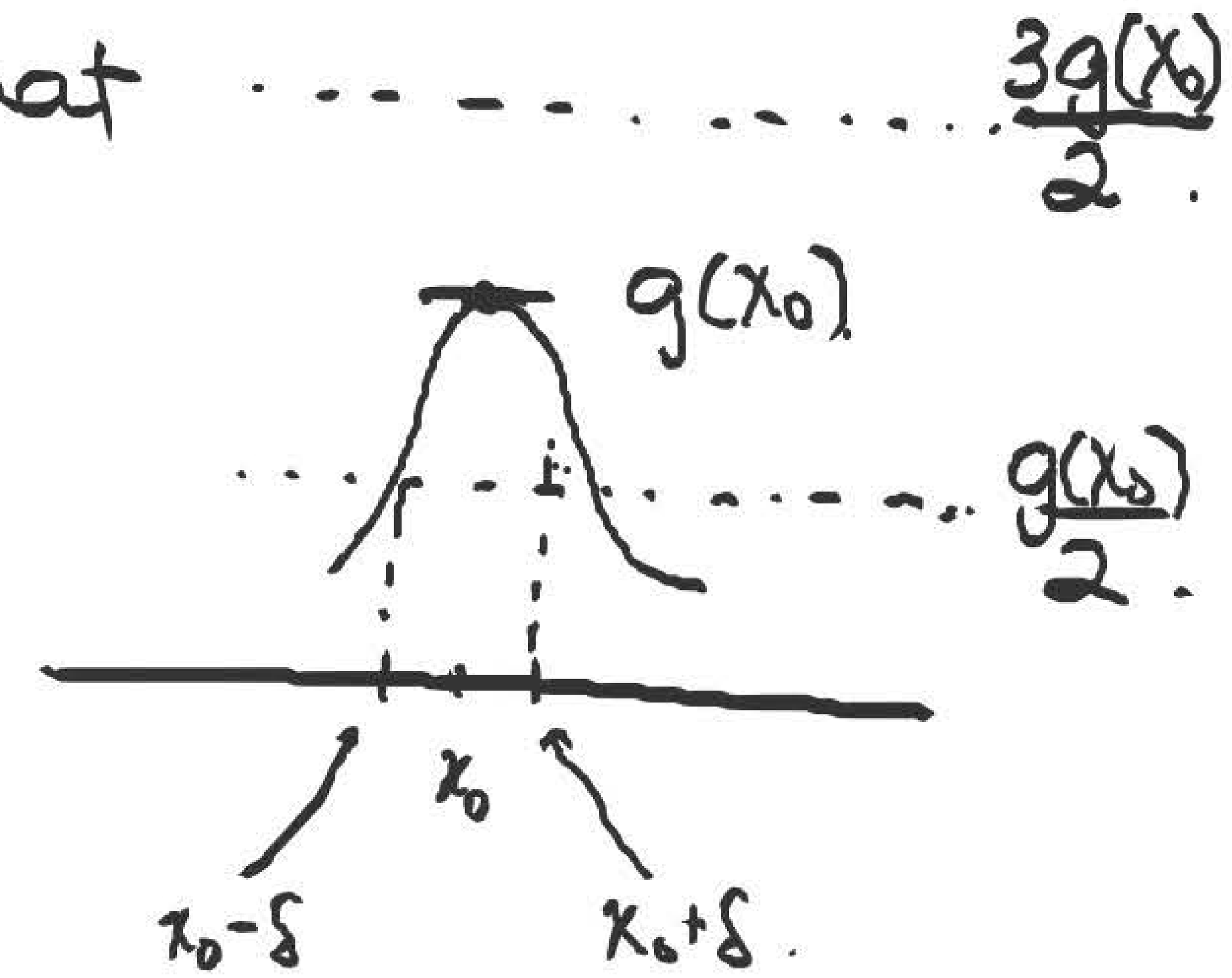
$$\int_a^b g(x)\,dx = \underbrace{\int_a^b h(x)\,dx}_{\ge 0} + \int_a^b f(x)\,dx \ge \int_a^b f(x)\,dx.$$

(Contrapositive).

(ii) Suppose g is cont. on $[a,b]$ and $g(x) \ge 0$ for $x \in [a,b]$.

Suppose $g \not\equiv 0$ on $[a,b]$, i.e. there exists $x_0 \in [a,b]$ such that $g(x_0) > 0$. Let $\varepsilon = \frac{g(x_0)}{2}$. There exists $\delta > 0$ such that

$|x - x_0| < \delta \Rightarrow |g(x) - g(x_0)| < \varepsilon \Rightarrow g(x) > \frac{g(x_0)}{2}$.



Consider partition $P = \{a = t_0 < x_0 - \frac{\delta}{2} < x_0 + \frac{\delta}{2} < t_3 = b\}$.

$$L(g,P) = \underbrace{m(g,[a,x_0-\tfrac{\delta}{2}])(x_0-\tfrac{\delta}{2}-a)}_{\ge 0} + \underbrace{\underbrace{m(g,[x_0-\tfrac{\delta}{2},x_0+\tfrac{\delta}{2}])}_{> \frac{g(x_0)}{2} > 0}\,\delta}_{>0} + \underbrace{\qquad\qquad}_{\ge 0}.$$

> 0.

$\therefore L_a^b(g) > 0 \Rightarrow \int_a^b g(x)\,dx > 0$.

Theorem: If f is integrable on $[a,b]$, then $|f|$ is integrable on $[a,b]$ and $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$.

Proof: $|f| = \max(f,0) + \max(-f,0) \Rightarrow |f|$ is integrable by Warm-Up.

Note that

$$-|f| \leq f \leq |f|.$$

By previous theorem,

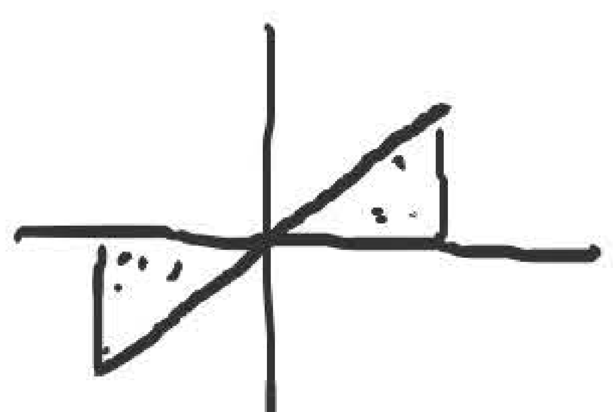
$$-\int_a^b |f(x)|dx \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|dx$$

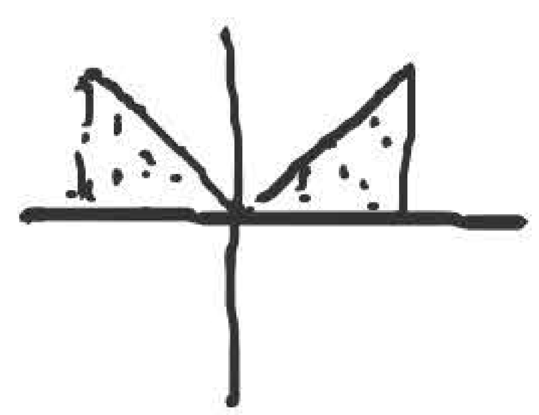
$$\Leftrightarrow \left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx.$$

$$-B \leq A \leq B$$
$$\Leftrightarrow |A| \leq B.$$

Ex. $f(x) = x$ on $[-1,1]$.

$$\int_{-1}^1 x\,dx = 0.$$

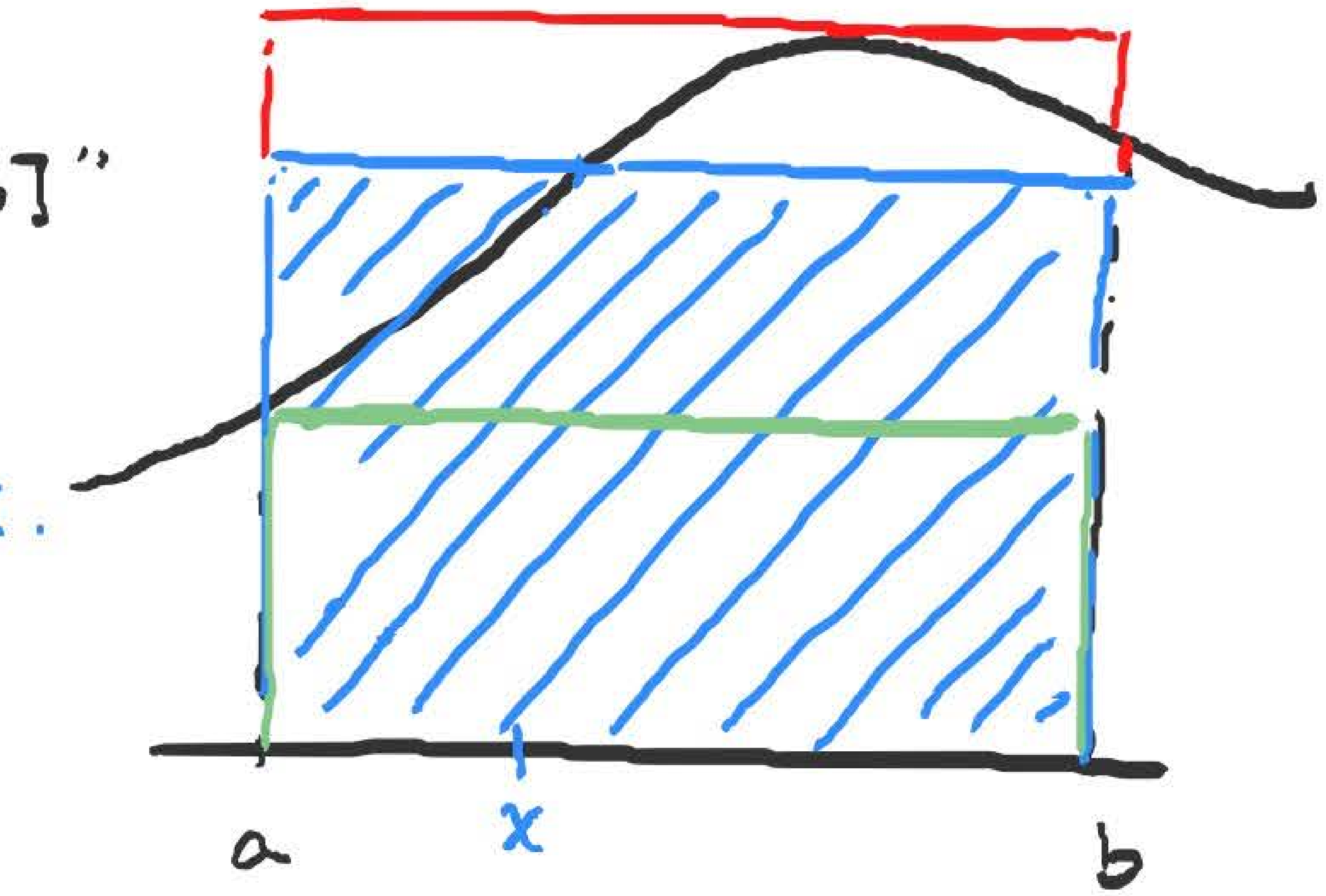




Theorem: (intermediate value theorem for integrals).

If f is continuous on $[a,b]$, then there exists $x^* \in (a,b)$ such that $f(x^*) = \underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{\text{"average of } f \text{ over } [a,b]\text{"}}$.

$(b-a)f(x^*) = \int_a^b f(x)\,dx$.



Proof: Case 1: f is constant.

Trivial.

Case 2: f not constant.

Let $M = \max\{f(x) : x \in [a,b]\}$

$m = \min\{f(x) : x \in [a,b]\}$.

$M > m$.

There exist x_0, z_0 : $f(x_0) = M$, $f(z_0) = m$.

$M - f \geq 0$, $f - m \geq 0$ nonnegative, continuous, not identically 0.

constant function

$\Rightarrow \int_a^b (M - f(x))\,dx > 0$, $\int_a^b (f(x) - m)\,dx > 0$.

$\Rightarrow \underbrace{\int_a^b m\,dx}_{m(b-a)} < \int_a^b f(x)\,dx < \underbrace{\int_a^b M\,dx}_{M(b-a)}$.

$\Rightarrow m < \frac{1}{b-a}\int_a^b f(x)\,dx < M$

Since f is continuous, $f(x_0) = M$, $f(z_0) = m$, by IVT, there exists x^* between x_0 and z_0 such that $f(x^*) = \frac{1}{b-a}\int_a^b f(x)\,dx$.

Fundamental Theorem of Calculus : relating integration to differentiation.

Def: A bounded function f on (a,b) is integrable on $[a,b]$ if any extension of f to $[a,b]$ is integrable.

FTC 1 : If f is integrable on $[a,b]$ and F is continuous on $[a,b]$ and differentiable on (a,b) and $F'(x)=f(x)$ for all $x\in(a,b)$, then

$$\int_a^b f(x)\,dx = F(b)-F(a).$$

F is an antiderivative of f on (a,b).

Proof: (Key ingredient : mean value theorem).

Let $\varepsilon>0$. Since f is integrable on $[a,b]$ there exists $P=\{a=t_0<\dots<t_n=b\}$ such that $U(f,P)-L(f,P)<\varepsilon$.

For $1\le k\le n$: $f(x_k)=F'(x_k)=\dfrac{F(t_k)-F(t_{k-1})}{t_k-t_{k-1}}$ for some $x_k\in(t_{k-1},t_k)$ by MVT.

$$f(x_k)\cdot(t_k-t_{k-1}) = F(t_k)-F(t_{k-1}).$$

Then

$$F(b)-F(a)=\sum_{k=1}^{n}\left(F(t_k)-F(t_{k-1})\right)=\sum_{k=1}^{n}\underbrace{f(x_k)}_{m(f,I_k)\le f(x_k)\le M(f,I_k)}\cdot(t_k-t_{k-1}).$$

$$\Rightarrow\quad L(f,P)\le \begin{matrix}F(b)-F(a)\\ \int_a^b f(x)\,dx\end{matrix}\le U(f,P) \Longrightarrow \left|F(b)-F(a)-\int_a^b f(x)\,dx\right|<\varepsilon.$$

Ex. $\int_a^b x^2\,dx$.

$F(x)=\frac{1}{3}x^3$. $F'(x)=f(x)$.

$\therefore \int_a^b x^2\,dx=\frac{1}{3}b^3-\frac{1}{3}a^3$.

FTC 2: Let f be integrable on $[a,b]$. For x in $[a,b]$, let $F(x) = \int_a^x f(t)\,dt$.

(i) Then F is continuous on $[a,b]$. (ii) If f is continuous at $x_0 \in (a,b)$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

(f is cont on $[a,b] \Rightarrow F$ is an antiderivative of f on (a,b).).

Proof:

(i) (Show uniform continuity of F).

Let $\varepsilon > 0$. Let C be such that $|f(x)| \le C$ on $[a,b]$.

Set $\delta = \frac{\varepsilon}{C}$. Then for $y > x$, $|y-x| < \delta = \frac{\varepsilon}{C}$,

$$|F(y)-F(x)| = \left|\int_a^y f(t)dt - \int_a^x f(t)dt\right| \le \left|\int_x^y f(t)dt\right| \le \int_x^y \underbrace{|f(t)|}_{\le C}dt \le \int_x^y C\,dt = C\underbrace{(y-x)}_{<\varepsilon/C} < \varepsilon.$$

(ii) Suppose f is cont. at $x_0 \in (a,b)$. For $x \ne x_0$:

(div by $x-x_0$) $$\frac{F(x)-F(x_0)}{x-x_0} = \frac{\int_{x_0}^x f(t)\,dt}{x-x_0}.$$

Convention: if $a<b$, $\int_b^a f(x)dx = -\int_a^b f(x)dx$.

(subtract $f(x_0)$) $$\underbrace{\boxed{\frac{F(x)-F(x_0)}{x-x_0}} - f(x_0)}_{\text{Show } \lim_{x\to x_0}(\_\_\_\_) = 0.} = \int_{x_0}^x \frac{(f(t)-f(x_0))}{x-x_0}\,dt.$$

$$\left(f(x_0) = \int_{x_0}^x \underbrace{\frac{f(x_0)}{x-x_0}}_{\text{constant w.r.t. } t} dt \right)$$

Let $\varepsilon > 0$. Since f is cont. at x_0, there exists $\delta > 0$ such that $|y - x_0| < \delta \Rightarrow |f(y) - f(x_0)| < \varepsilon$. Then for x such that $|x - x_0| < \delta$,

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| = \left| \int_{x_0}^{x} \frac{f(t) - f(x_0)}{x - x_0} \, dt \right|$$

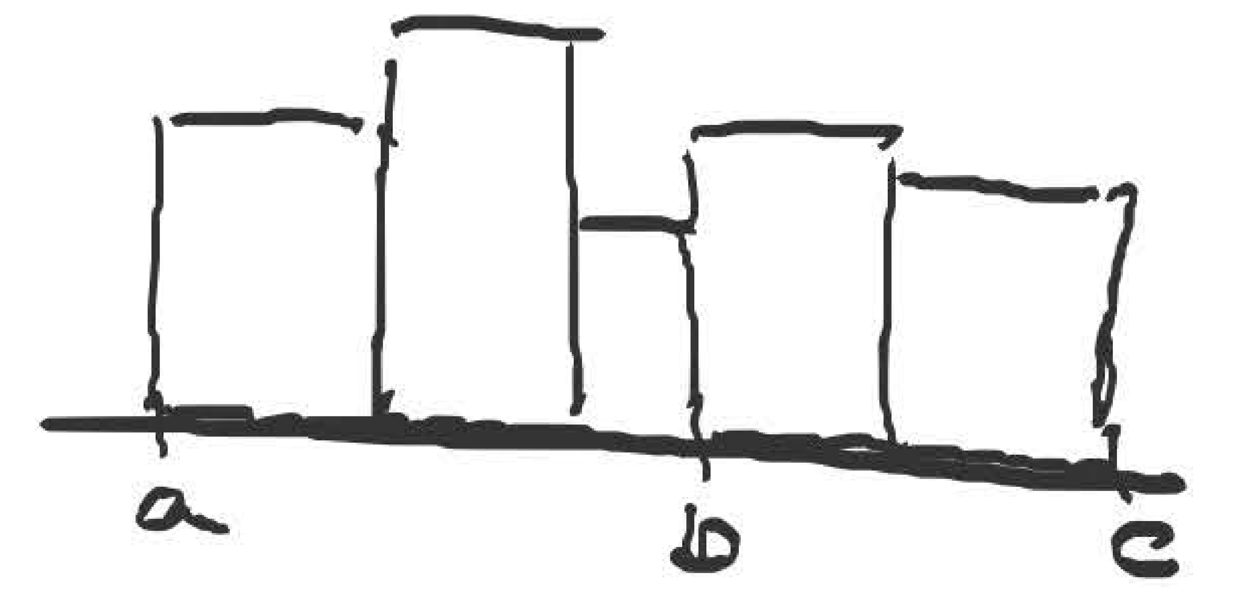
For case $x > x_0$.

$$\leq \frac{1}{x - x_0} \underbrace{\int_{x_0}^{x} \underbrace{|f(t) - f(x_0)|}_{< \varepsilon} \, dt}_{< \varepsilon (x - x_0).}$$

$$< \varepsilon.$$

$$\Rightarrow \lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0).$$

Theorem: If f is integrable on $[a,b]$ and f is integrable on $[b,c]$, then f is integrable on $[a,c]$ and $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$.



Proof: Let $\varepsilon > 0$.

Let $P_1 \in \Pi_{[a,b]}$: $U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2}$.

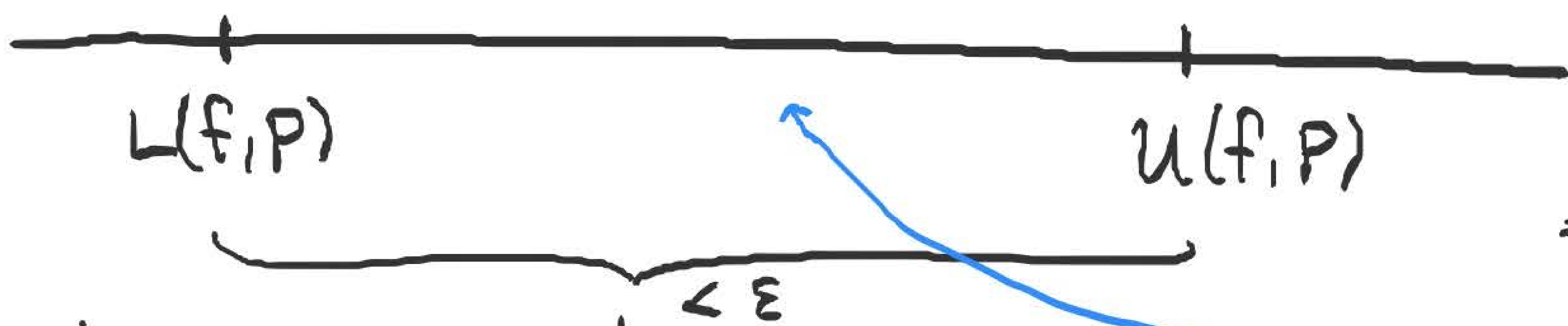
Let $P_2 \in \Pi_{[b,c]}$: $U(f,P_2) - L(f,P_2) < \frac{\varepsilon}{2}$.

Let $P = P_1 \cup P_2$ (note: not a refinement). $P \in \Pi_{[a,c]}$.

$$
\begin{aligned}
U(f,P) - L(f,P) &= U(f,P_1) + U(f,P_2) - \big(L(f,P_1) + L(f,P_2)\big) \\
&= \underbrace{U(f,P_1) - L(f,P_1)}_{<\varepsilon/2} + \underbrace{U(f,P_2) - L(f,P_2)}_{<\varepsilon/2} \\
&< \varepsilon .
\end{aligned}
$$

- $L(f,P) \le \boxed{\int_a^c f(x)dx} \le U(f,P)$.
- $L(f,P_1) \le \int_a^b f(x)dx \le U(f,P_1)$
  $L(f,P_2) \le \int_b^c f(x)dx \le U(f,P_2)$.

$\Rightarrow L(f,P) \le \boxed{\int_a^b f(x)dx + \int_b^c f(x)dx} \le U(f,P)$.



$\Rightarrow \left| \int_a^c f(x)dx - \left(\int_a^b f(x)dx + \int_b^c f(x)dx\right)\right| < \varepsilon \Rightarrow \int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$.