Math 104 Homework 1 Solutions

UC Berkeley, Summer 2021

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1. Reverse triangle inequality (Ross 3.5)
(a) Show that $|b| \le a$ if and only if $-a \le b \le a$.
(b) Prove that $||a| - |b|| \le |a - b|$ for all $a, b \in \mathbb{R}$.

Solution. (a) $|b| \le a \iff b \le a$ and $-b \le a \iff -a \le b \le a$.
(b) By part (a), it suffices to show that $-|a-b| \le |a| - |b| \le |a-b|$. The first inequality holds because $|b| = |b - a + a| \le |b - a| + |a|$, and the second inequality holds because $|a| = |a - b + b| \le |a - b| + |b|$.

$\square$

2. Prove that

$$2\sqrt{n} - 2 < \sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n} - 1$$

for any integer $n \ge 2$, by following the steps below.
(a) Prove that for any $n \in \mathbb{N}$,

$$2(\sqrt{n+1} - \sqrt{n}) < \frac{1}{\sqrt{n}} < 2(\sqrt{n} - \sqrt{n-1}).$$

(b) Prove that for any integer $n \ge 2$,

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > 2\sqrt{n} - 2.$$

(c) Use induction to prove that for all integers $n \ge 2$,

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n} - 1.$$

Solution. (a) We have

$$2(\sqrt{n+1} - \sqrt{n}) = 2(\sqrt{n+1} - \sqrt{n}) \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{2}{\sqrt{n+1} + \sqrt{n}} < \frac{2}{\sqrt{n} + \sqrt{n}} = \frac{1}{\sqrt{n}}$$

and

$$2(\sqrt{n} - \sqrt{n-1}) = 2(\sqrt{n} - \sqrt{n-1}) \cdot \frac{\sqrt{n} + \sqrt{n-1}}{\sqrt{n} + \sqrt{n-1}} = \frac{2}{\sqrt{n} + \sqrt{n-1}} > \frac{2}{\sqrt{n} + \sqrt{n}} = \frac{1}{\sqrt{n}}.$$

(b) By part (a),

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sum_{k=1}^{n} 2(\sqrt{k+1} - \sqrt{k}) = 2\sqrt{n+1} - 2 > 2\sqrt{n} - 2.$$

(c) For the base case $n = 2$, we have $\sum_{k=1}^{2} \frac{1}{\sqrt{k}} = 1 + \frac{1}{\sqrt{2}} < 1 + 2(\sqrt{2} - \sqrt{1}) = 2\sqrt{2} - 1$, where the inequality comes from applying the second inequality in part (a). Now suppose that the inequality holds for some $n \geq 2$, so

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n} - 1.$$

Then

$$\sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} = \underbrace{\frac{1}{\sqrt{n+1}}}_{<2(\sqrt{n+1}-\sqrt{n})} + \underbrace{\sum_{k=1}^{n} \frac{1}{\sqrt{k}}}_{<2\sqrt{n}-1} < 2\sqrt{n+1} - 1.$$

□

3. (Ross 3.8) Let $a, b \in \mathbb{R}$. Show that if $a \leq b_1$ for every $b_1 > b$, then $a \leq b$.

Solution. (Contrapositive) Suppose $a > b$. Let $b_1 \in \mathbb{R}$ such that $b < b_1 < a$. Then $b_1 > b$ but it is not true that $a \leq b_1$.

□

4. (Ross 4.8) Let S and T be nonempty subsets of $\mathbb{R}$ such that $s \leq t$ for all $s \in S$ and $t \in T$. Prove that $\sup S \leq \inf T$.

Solution. (Contradiction) Suppose that $\sup S > \inf T$. Then $\inf T$ is not an upper bound for S, so there exists $s \in S$ such that $s > \inf T$. In the same vein, s is not a lower bound for T, so there exists $t \in T$ such that $t < s$. This contradicts the hypothesis that $s \leq t$ for all $s \in S$ and $t \in T$.

□

5. Consider the following sets:

$$A = (0, \infty) \qquad B = \{\tfrac{1}{m} + \tfrac{1}{n} : m, n \in \mathbb{N}\}, \qquad C = \{x^2 - 1 : x \in \mathbb{R}\},$$

$$D = [0,1] \cup [2,3] \qquad E = \bigcup_{n=1}^{\infty} [2n, 2n+1], \qquad F = \bigcap_{n=1}^{\infty} (1 - \tfrac{1}{n}, 1 + \tfrac{1}{n}).$$

For each set, determine its minimum and maximum if they exist. In addition, determine each set's infimum and supremum (if the set is unbounded, answer in terms of ∞.)

Solution. (a) $\min A$ does not exist, $\max A$ does not exist, $\inf A = 0$, $\sup A = \infty$
(b) $\min B$ does not exist, $\max B = 2$, $\inf B = 0$, $\sup B = 2$
(c) $\min C = -1$, $\max C$ does not exist, $\inf C = -1$, $\sup C = \infty$
(d) $\min D = 0$, $\max D = 3$, $\inf D = 0$, $\sup D = 3$
(e) $\min E = 2$, $\max E$ does not exist, $\inf E = 2$, $\sup E = \infty$
(f) Note that $F = \{1\}$. $\min F = \max F = \inf F = \sup F = 1$.

□