Math 104 Worksheet 7

UC Berkeley, Summer 2021

Thursday, July 8

Let (X, d) be a metric space.

Theorem. Closed subsets of compact sets are compact.

Proof. Let E be a compact set, and let $F \subseteq E$ be a closed set. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of F. Goal: ____.
(*Hint*: Expand $\{U_\alpha\}_{\alpha \in A}$ to an open cover of E. Note that F is closed.)
<u>Exercise 1.</u> Complete the proof.

Theorem. Every sequence in a compact set has a convergent subsequence.

Proof. Let E be a compact set, and let (x_n) be a sequence of points in E. Consider the set $S = \{x_n : n \in \mathbb{N}\}$.
<u>Exercise 2.</u> Explain why if S is finite, then (x_n) has a convergent subsequence.

If S is infinite, then it suffices to show that S has a limit point in E. Why?

Suppose (for contradiction) that no point in E is a limit point of S.
<u>Exercise 3.</u> Construct an open cover of E which has no finite subcover. This would imply that E is not compact, which is a contradiction.