Math 104 Worksheet 10

UC Berkeley, Summer 2021
Monday, July 19

Lemma. Let (s_n) be a sequence of nonzero real numbers. Then

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| \leq \liminf |s_n|^{1/n} \leq \limsup |s_n|^{1/n} \leq \limsup \left|\frac{s_{n+1}}{s_n}\right|.$$

Proof. (The proof here uses the fact that $\lim n^{1/n} = 1$ and $\lim a^{1/n} = 1$ for any constant $a > 0$; see p.49 in the textbook for the proofs.) The second inequality is trivial. For the third inequality:

Let $L := \limsup \left|\frac{s_{n+1}}{s_n}\right|$. If $L = \infty$ then the inequality is trivial, so assume $L \in \mathbb{R}$. Let $\varepsilon > 0$. (Goal: Show that $\limsup |s_n|^{1/n} \leq L+\varepsilon$.) There exists $N \in \mathbb{N}$ such that $\left|\frac{s_{n+1}}{s_n}\right| < L+\varepsilon$ for $n \geq N$. Then for $n > N$,

$$|s_n| = \left|\frac{s_n}{s_{n-1}}\right| \cdot \left|\frac{s_{n-1}}{s_{n-2}}\right| \cdots \left|\frac{s_{N+1}}{s_N}\right| \cdot |s_N| < (L+\varepsilon)^{n-N}|s_N| = (L+\varepsilon)^n \cdot \frac{|s_N|}{(L+\varepsilon)^N} = C(L+\varepsilon)^n$$

where $C := \frac{|s_N|}{(L+\varepsilon)^N}$. Hence $|s_n|^{1/n} < C^{1/n}(L+\varepsilon)$ for $n > N$, so

$$\limsup |s_n|^{1/n} \leq \limsup C^{1/n}(L+\varepsilon) = L+\varepsilon.$$

<u>Exercise 1</u>. Using the same strategy as above, prove the first inequality.

Exercise 2. Determine whether or not the following series converges:

$$\sum_{n=1}^{\infty} \left(\frac{2}{(-1)^n - 3} \right)^n .$$

Exercise 3. Consider the series $\sum a_n$ with $a_n = 2^{(-1)^n - n}$, so $\sum a_n = 2 + \frac{1}{4} + \frac{1}{2} + \frac{1}{16} + \frac{1}{8} + \frac{1}{64} + \dots$
(a) Use the comparison test to show that $\sum a_n$ converges.

(b) Show that the ratio test gives no information.

(c) Use the root test to show that $\sum a_n$ converges.

(d) Can you find a series for which the ratio test proves convergence or divergence, but the root test gives no information?

Exercise 4. The series in Exercise 3 is an example of a series for which the root test proves convergence but the ratio test gives no information. Find an example of a series for which the root test proves **divergence** but the ratio test gives no information.