Math 104 Worksheet 4

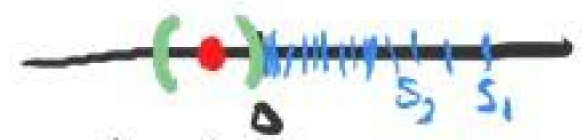
UC Berkeley, Summer 2021

Tuesday, June 29

---

1. Let (s_n) be a sequence of nonnegative real numbers which converges to s.

(a) Show that $s \geq 0$. (*Hint*: Argue by contradiction.)



Suppose $s < 0$.

$(\varepsilon = |s|)$ There exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|s_n - s| < |s| \Rightarrow s_n < 0$.

Contradiction.

(b) Show that $\sqrt{s_n} \to \sqrt{s}$. (*Hint*: Consider the cases $s = 0$ and $s > 0$ separately. For the case $s > 0$, observe that $|\sqrt{s_n} - \sqrt{s}| = \frac{|s_n - s|}{\sqrt{s_n} + \sqrt{s}} \leq \frac{|s_n - s|}{\sqrt{s}}$.)

$s = 0$: Let $\varepsilon > 0$.

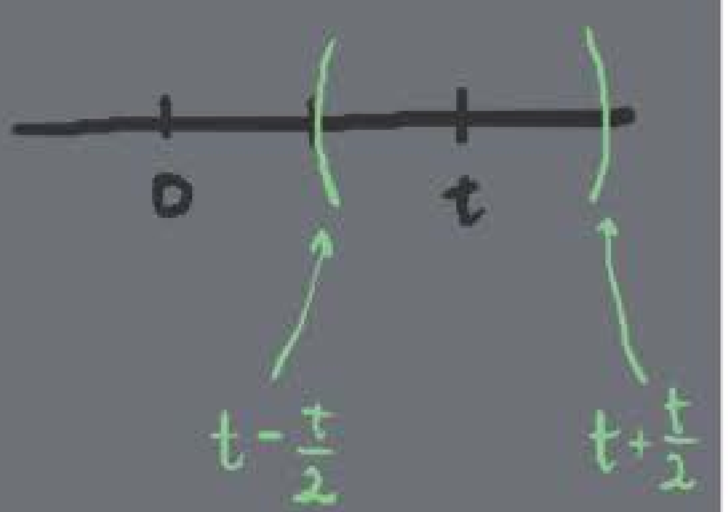
Since $s_n \to s$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|s_n - s| = \boxed{s_n < \varepsilon^2}$

Then $n \geq N$ implies $|\sqrt{s_n} - 0| = \sqrt{s_n} < \varepsilon$.

$s > 0$: Let $\varepsilon > 0$. Since $s_n \to s$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|s_n - s| < \varepsilon\sqrt{s}$. $\cdots$ $|\sqrt{s_n} - \sqrt{s}| \leq \frac{|s_n - s|}{\sqrt{s}} < \frac{\varepsilon\sqrt{s}}{\sqrt{s}} = \varepsilon$.

2. (Theorem 9.9) Let (s_n) and (t_n) be sequences such that $\lim s_n = \infty$ and (t_n) converges to $t > 0$. Then $\lim s_n t_n = \infty$.

Proof. Let $M > 0$. Goal: Show that there exists $N \in \mathbb{N}$ such that ... $n \geq N$ implies $s_n t_n > M$.



First, since $t_n \to t > 0$, we can find $N_1 \in \mathbb{N}$ such that $|t_n - t| < \frac{t}{2}$ for all $n \geq N_1$. Then $t_n \geq \frac{t}{2}$ for all $n \geq N_1$. Now since $s_n \to \infty$, there exists $N_2 \in \mathbb{N}$ such that

$$s_n > \underline{2M/t}$$

for all $n \geq N_2$. Set $N = \max(N_1, N_2)$. Then for $n \geq N$,

$$s_n t_n > \underline{\frac{2M}{t} \cdot \frac{t}{2}} = M.$$

$\square$

3. Give an example of . . .

1. a sequence (s_n) of rational numbers which converges to an irrational number.

$$s_n = \left(1+\frac{1}{n}\right)^n \longrightarrow e$$

2. a sequence (s_n) of irrational numbers which converges to a rational number.

$$s_n = \sqrt[n]{n} \; (n \geq 2), \quad t_n = \frac{\pi}{n}$$

3. a divergent sequence (s_n) such that $(|s_n|)$ converges.

$$s_n = (-1)^n$$

4. a sequence (s_n) of nonzero real numbers which converges to 0 such that the sequence $(1/s_n)$ does not have a limit.

$$s_n = \frac{(-1)^n}{n} \qquad \left(-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots\right)$$

$$\left(\frac{1}{s_n}\right) = (-1, 2, -3, 4, \ldots)$$

5. two divergent sequences (s_n) and (t_n) such that the sequence $(s_n + t_n)$ converges.

$$s_n = (-1)^n \qquad (-1, 1, -1, 1, \ldots)$$
$$t_n = (-1)^{n+1} \qquad (1, -1, 1, -1, \ldots) \qquad (0, 0, \ldots).$$

6. a sequence (s_n) of nonzero real numbers and a divergent sequence (t_n) such that the sequence $(s_n t_n)$ converges.

same as #5

7. two convergent sequences (s_n) and (t_n) such that $s_n < t_n$ for all n and $\lim s_n = \lim t_n$.

$$s_n = \frac{1}{n+1}, \quad t_n = \frac{1}{n}$$

8. a divergent sequence (s_n) of positive real numbers such that $\lim|s_{n+1}/s_n| = 1$. (cf. Homework 2 Problem 8)

$$s_n = n$$

9. a bounded divergent sequence (s_n) such that $|s_n|$ is strictly increasing.

$$s_n = (-1)^n \cdot \frac{n}{n+1} \qquad \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$$

10. a divergent sequence (s_n) such that $|s_{n+1} - s_n| < \frac{1}{n}$ for all n.

$$s_n = \sum_{k=1}^{n} \frac{1}{n} \qquad s_{n+1} - s_n = \frac{1}{n+1} < \frac{1}{n}.$$

Recall For a bounded sequence (s_n),

defined $u_n = \inf\{s_m : m \geq n\}$ nondecreasing

$v_n = \sup\{s_m : m \geq n\}$. nonincreasing.

Define $\liminf s_n = \lim u_n$ (def for when (s_n) is bounded).

$\limsup s_n = \lim v_n$.

Ex: $\left(1, -\frac{2}{1}, 2, -\frac{3}{2}, 3, -\frac{4}{3}, 4, -\frac{5}{4}, \ldots\right) = (s_n)$

$u_1 = -2$, $u_2 = -2$, $u_3 = -\frac{3}{2}$, $u_4 = -\frac{3}{2}$, $u_5 = u_6 = -\frac{4}{3}$, $u_7 = u_8 = -\frac{5}{4}$, ...

$u_n \longrightarrow -1$. $\Rightarrow \liminf s_n = -1$.

(s_n) not bounded above $\Rightarrow \limsup s_n = \infty$ (by definition/convention).

Exercise: Prove that for any (s_n),

$\liminf s_n \leq \limsup s_n$.

$u_n \leq v_n$ for all n.

Theorem: Let (s_n) be a sequence of real numbers. Then $\lim s_n$ exists (in $\mathbb{R}$ or $\pm\infty$) if and only if $\liminf s_n = \limsup s_n$, in which case $\lim s_n = \limsup s_n = \liminf s_n$.

Proof: (finite case)

$\Rightarrow$] Suppose $\lim s_n = s \in \mathbb{R}$. Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|s_n - s| < \varepsilon$, i.e.

$$s - \varepsilon < s_n < s + \varepsilon \quad \text{for all } n \geq N.$$

$\Rightarrow s - \varepsilon \leq u_N$ $\qquad$ $\Rightarrow v_N \leq s + \varepsilon$

$\liminf s_n$

(u_n) is nondecreasing $\Rightarrow \lim u_n \geq s - \varepsilon$. $\Longrightarrow \liminf s_n \geq s$.

(v_n) is nonincreasing $\Rightarrow \lim v_n \leq s + \varepsilon$. $\Longrightarrow \limsup s_n \leq s$

$\limsup s_n$

$\Leftarrow$] Suppose $\underbrace{\liminf s_n}_{\lim u_n} = \underbrace{\limsup s_n}_{\lim v_n} = s \in \mathbb{R}$. $\Rightarrow s \leq \liminf s_n \leq \limsup s_n \leq s$.

$\Rightarrow \liminf s_n = \limsup s_n = s$.

Let $\varepsilon > 0$. There exists N_1: $n \geq N_1$ implies $|u_n - s| < \varepsilon \Longrightarrow \overset{\inf\{s_m : m \geq n\}}{u_n} > s - \varepsilon \Longrightarrow s_n > s - \varepsilon$.

There exists N_2: $n \geq N_2$ implies $|v_n - s| < \varepsilon \Longrightarrow v_n < s + \varepsilon \Longrightarrow s_n < s + \varepsilon$.

Let $N = \max(N_1, N_2)$. $n \geq N$ implies $s - \varepsilon < s_n < s + \varepsilon$, i.e. $|s_n - s| < \varepsilon$.

Cauchy sequences

Def: A sequence (s_n) of real numbers is Cauchy if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $m, n \geq N$ implies $|s_m - s_n| < \varepsilon$.

(intuitively: u_N and v_N are close to each other)

u_N: $\inf\{s_n : n \geq N\}$ $\quad$ v_N: $\sup\{s_n : n \geq N\}$

Theorem: A sequence (s_n) of real numbers converges if and only if it is Cauchy.

Proof: $\Rightarrow$) Suppose $s_n \to s \in \mathbb{R}$. Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|s_n - s| < \frac{\varepsilon}{2}$.

Then for $m, n \geq N$, $|s_m - s_n| = |s_m - s + s - s_n| \leq \underbrace{|s_m - s|}_{<\varepsilon/2} + \underbrace{|s - s_n|}_{<\varepsilon/2} < \varepsilon$.

$\Leftarrow$) Suppose (s_n) is Cauchy. Let $\varepsilon > 0$.

(Goal: Show $\liminf s_n = \limsup s_n$ by showing that $\limsup s_n - \liminf s_n \leq \varepsilon$)

There exists $N \in \mathbb{N}$ such that $m, n \geq N$ implies $|s_m - s_n| < \frac{\varepsilon}{2}$.

In particular, $|s_n - s_N| < \frac{\varepsilon}{2}$ (fixed $m = N$), i.e. $s_N - \frac{\varepsilon}{2} < s_n < s_N + \frac{\varepsilon}{2}$ for $n \geq N$.

$\Rightarrow$ $u_N \geq s_N - \frac{\varepsilon}{2}$, $v_N \leq s_N + \frac{\varepsilon}{2}$.

$$s_N - \frac{\varepsilon}{2} \leq u_N \leq \liminf s_n \leq \limsup s_n \leq v_N \leq s_N + \frac{\varepsilon}{2}.$$

"Completeness of $\mathbb{R}$" $\quad \Rightarrow \limsup s_n - \liminf s_n \leq \varepsilon.$

Subsequences

Def: Let (s_n) be a sequence of real numbers.

A subsequence of (s_n) is a sequence $(s_{n_k})_{k \in \mathbb{N}}$ where $n_k \in \mathbb{N}$ and $n_1 < n_2 < n_3 < \ldots$

(original sequence but throwing away some terms).

might be a lot.

Ex $(1, 2, 3, 4, 5, 6, \ldots) = (s_n)$.

$(2, 4, 6, 8, \ldots)$ is a subsequence.

$n_1 = 2, \; n_2 = 4, \; n_3 = 6, \ldots$

2^{nd} term of (s_n)

4^{th} term.