Q1

(a)
- Positive Definiteness: $\forall (x_1, y_1), (x_2, y_2) \in (X \times Y)$ $d_X(x_1, x_2) \geq 0$ and $d_Y(y_1, y_2) \geq 0 \implies d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} \geq 0$ since d_X and d_Y are proper metrics. Also because

$$\begin{aligned}
\max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} = 0 &\iff 0 \leq d_X(x_1, x_2) \leq 0 \text{ and } 0 \leq d_Y(y_1, y_2) \leq 0 \\
&\iff d_X(x_1, x_2) = 0 \text{ and } d_Y(y_1, y_2) = 0 \\
&\iff x_1 = x_2 \text{ and } y_1 = y_2 \\
&\iff (x_1, y_1) = (x_2, y_2)
\end{aligned}$$

  d is positive definite.

- Symmetry: Since d_X and d_Y are proper metrics, it is clear that

$$\max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} = \max\{d_X(x_2, x_1), d_Y(y_2, y_1)\}.$$

  Thus d is symmetric.

- Triangular Inequality: Since d_X and d_Y are proper metrics, it is clear that for each $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in (X \times Y)$ we have

$$\begin{aligned}
d_X(x_1, x_2) &\leq d_X(x_1, x_3) + d_X(x_3, x_2), \\
d_Y(y_1, y_2) &\leq d_Y(y_1, y_3) + d_Y(y_3, y_2).
\end{aligned}$$

  If $d_X(x_1, x_2) \leq d_Y(y_1, y_2)$, then

$$\begin{aligned}
\max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} &= d_Y(y_1, y_2) \\
&\leq d_Y(y_1, y_3) + d_Y(y_3, y_2) \\
&\leq \max\{d_X(x_1, x_3), d_Y(y_1, y_3)\} + \max\{d_X(x_3, x_2), d_Y(y_3, y_2)\}.
\end{aligned}$$

  If $d_X(x_1, x_2) > d_Y(y_1, y_2)$, then

$$\begin{aligned}
\max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} &= d_X(x_1, x_2) \\
&\leq d_X(x_1, x_3) + d_X(x_3, x_2) \\
&\leq \max\{d_X(x_1, x_3), d_Y(y_1, y_3)\} + \max\{d_X(x_3, x_2), d_Y(y_3, y_2)\}.
\end{aligned}$$

  Thus combining two cases, we have

$$d((x_1, y_1), (x_2, y_2)) \leq d((x_1, y_1), (x_3, y_3)) + d((x_3, y_3), (x_2, y_2)),$$

  completing the proof.

(b) Consider an arbitrary sequence (x_n, y_n) in $E \times F$. Note that $(x_n) \in E$ and $(y_n) \in F$. Since E is compact, (x_n) has a subsequence (x_{n_k}) converging to x_0 in E. Moreover, since F is compact, (y_{n_k}) has a subsequence $(y_{n_{k_l}})$ converging to y_0 in F. Now (x_n, y_n) has a subsequence $(x_{n_{k_l}}, y_{n_{k_l}})$ converging to (x_0, y_0) in $E \times F$, so $E \times F$ is compact.

Q2

Since $\sum a_n$ is convergent, $\lim a_n = 0$. i.e. Let $\epsilon = 1$ $\exists N_1 \in \mathbb{N}$ $n \geq N_1 \implies |a_n| < 1 \implies \left|\frac{1}{a_n}\right| = \frac{1}{|a_n|} > 1$. Thus if $\lim \frac{1}{a_n}$ exists, $\lim \frac{1}{a_n} \neq 0$, implying that $\sum \frac{1}{a_n}$ does not converge, and hence diverges.

Q3

Since $\sum(a_n+b_n)=\sum a_n+\sum b_n$, it is clear that $\sum(a_n+b_n)$ converges when both $\sum a_n$ and $\sum b_n$ converge. Observe that when for each $n \in N$ a_n and b_n are nonnegative, we have

$$\left|\sqrt{a_n b_n}\right| = \sqrt{a_n b_n} \leq \sqrt{2a_n b_n} \leq \sqrt{a_n^2+b_n^2+2a_n b_n} = \sqrt{(a_n+b_n)^2} = a_n+b_n.$$

By comparison test, $\sum \sqrt{a_n b_n}$ converges.

Q4

Since $\liminf |a_n| = 0$, there exists a subsequence $|a_{n_k}|$ such that $\lim |a_{n_k}| = 0$, i.e. $\forall \epsilon > 0\ \exists K \in \mathbb{N}\ k \geq K \implies |a_{n_k}| < \epsilon$. Thus select $\epsilon = 1$ and according to the definition of limit, we can select n_{k_1} such that $\left|a_{n_{k_1}}\right| < 1$. Having already found $n_{k_1} < n_{k_2} < \cdots < n_{k_l}$ such that $\left|a_{n_{k_l}}\right| < \frac{1}{l^2}$, we can choose $n_{k_{l+1}} > n_{k_l}$ such that $\left|a_{n_{k_{l+1}}}\right| < \frac{1}{(l+1)^2}$ since we have infinitely many terms smaller than $\frac{1}{(l+1)^2}$.

Now we have a series $\sum a_{n_{k_l}}$ such that $\left|a_{n_{k_l}}\right| < \frac{1}{l^2}$ for each $l \in \mathbb{N}$. Then by comparison test, since $\sum \frac{1}{l^2}$ converges, $\sum a_{n_{k_l}}$ also converges. Note $(a_{n_{k_l}})$ is a subsequence of (a_n).

Q5

Let $\sum a_n = \sum \frac{(-1)^n}{\sqrt{n}}$. It is an alternating series because

$$1 > \frac{1}{\sqrt{2}} > \frac{1}{\sqrt{3}} > \cdots > 0 \text{ and } \lim \frac{1}{\sqrt{n}} = 0.$$

Thus $\sum a_n$ converges. Then $\sum a_n^2 = \sum (\frac{(-1)^n}{\sqrt{n}})^2 = \sum \frac{1}{n}$ is the harmonic series and hence $\sum a_n^2$ diverges.

Q6

Since $\sum a_n$ converges, it satisfies the Cauchy Criterion, i.e.

$$\forall \epsilon > 0\ \exists N \in \mathbb{N}\ n \geq m \geq N \implies \left|\sum_{k=m}^{n} a_n\right| < \frac{\epsilon}{2}.$$

Let $m = N$, then we have $\forall \epsilon > 0$,

$$\begin{aligned} \frac{\epsilon}{2} &> |a_N + a_{N+1} + \cdots + a_n| \\ &\geq (n - N + 1)|a_n| \quad \text{since } (a_n) \text{ is nonincreasing.} \end{aligned}$$

If $n \geq 2N$, then $n \leq 2(n - N) < 2(n - N + 1)$. Thus

$$|na_n| = n|a_n| < 2(n - N + 1)|a_n| < 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

i.e. $\lim na_n = 0$.

Q7

(a) First observe that all terms are nonzero. Using ratio test, we have

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n+1}{(n+2)!} \cdot \frac{(n+1)!}{n}\right| = \lim \left|\frac{n+1}{n^2+2n}\right| = 0.$$

Thus $\limsup \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$, and hence the series converges (absolutely).

(b) If $a = 0$, then the series converges trivially. If $a \neq 0$, first observe all terms are nonzero, and then we have

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{a^{n+1}}{(n+1)!} \cdot \frac{n!}{a^n}\right| = \lim \left|\frac{a}{n+1}\right| = 0.$$

Thus $\limsup \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$, and hence the series converges (absolutely) by the ratio test.

(c) Let $(a_n) = \frac{1}{\sqrt{n}}$. Note that we have

$$\frac{1}{\sqrt{2}} > \frac{1}{\sqrt{3}} > \cdots > 0 \text{ and } \lim \frac{1}{\sqrt{n}} = 0.$$

Thus by alternating series test, the series converges.

(d) Observe that

$$\sum_{n=2}^{\infty} \frac{1}{n \log n} > \int_2^{\infty} \frac{1}{n \log n}\, dn = \infty.$$

Thus by the integral test, the series diverges.

Q8

(a) If $x = 0$, then $f(x) = 0^n = 0$ is continuous trivially. Now show that $g(x) = x$ is continuous. For each $x_0 \in \mathbb{R}$ and each $\epsilon > 0$, let $\delta = \epsilon$, then it is clear that $x \in \text{dom}(f)$ and $|x - x_0| < \delta = \epsilon \implies |f(x) - f(x_0)| = |x - x_0| < \epsilon$. Thus $g(x) = x$ is continuous. Then for each $x_0 \in \mathbb{R}$ and each $n > 2$ in $\mathbb{N}$, suppose $f'(x) = x^{n-1}$ is continuous at x_0, since $f(x) = x^n = x^{n-1} \cdot x = f'(x) \cdot g(x)$, $f(x)$ is continuous at x_0 by theorem 17.4(ii). Moreover, we've shown $f(x) = x^n$ is continuous for each $n \in \mathbb{N}$, completing the proof.

(b) By part (a), $1, x, x^2, \ldots, x^n$ are all continuous for each $n \in N$. Then by theorem 17.3, $a_0, a_1 x, \ldots, a_n x^n$ are continuous for each $a_0, a_1, \ldots, a_n \in \mathbb{R}$. Thus using theorem 17.4(i) inductively, $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is continuous.

Q9

(a) Consider $(x_n) = -\frac{1}{n}$. Obviously, $(x_n) \to 0$. Let $x_0 = 0$. For each $n \in N$, $x_n < 0 \implies f(x_n) = 0$, and hence $f(x_n) \to 0$. However, since $f(x_0) = 1$, $f(x_n) \nrightarrow f(x_0)$, $f(x)$ is discontinuous at 0.

(b) Consider $(x_n) = \frac{2}{(4n-3)\pi}$. Obviously, $(x_n) \to 0$. Let $x_0 = 0$. For each $n \in N$, $f(x_n) = \sin\left(\frac{(4n-3)\pi}{2}\right) = 1$, and hence $f(x_n) \to 1$. However, since $f(x_0) = 0$, $f(x_n) \nrightarrow f(x_0)$, $f(x)$ is discontinuous at 0.

Q10

- If $r \in \mathbb{Q}$, then consider the sequence $(x_n) = r + \frac{\sqrt{2}}{n}$. Obviously, $(x_n) \to r \in \mathbb{Q} \implies f(r) = 1$. However, $\forall n \in \mathbb{N}\ x_n \in \mathbb{R}\backslash\mathbb{Q} \implies f(x_n) = 0$. Thus $f(x_n) \to 0 \neq f(r)$, and hence $f(x)$ is discontinuous at r.

- If $r \in \mathbb{R}\backslash\mathbb{Q}$, then there exists a sequence (q_n) of rational numbers such that $(q_n) \to r$. Then $\forall n \in \mathbb{N}\ f(q_n) = 1$, but $f(r) = 0$. Thus $f(q_n) \nrightarrow f(r)$, and hence f is discontinuous at r.

  Combining two cases above, we've shown that f is discontinuous at every $r \in \mathbb{R}$.