Thursday, July 29

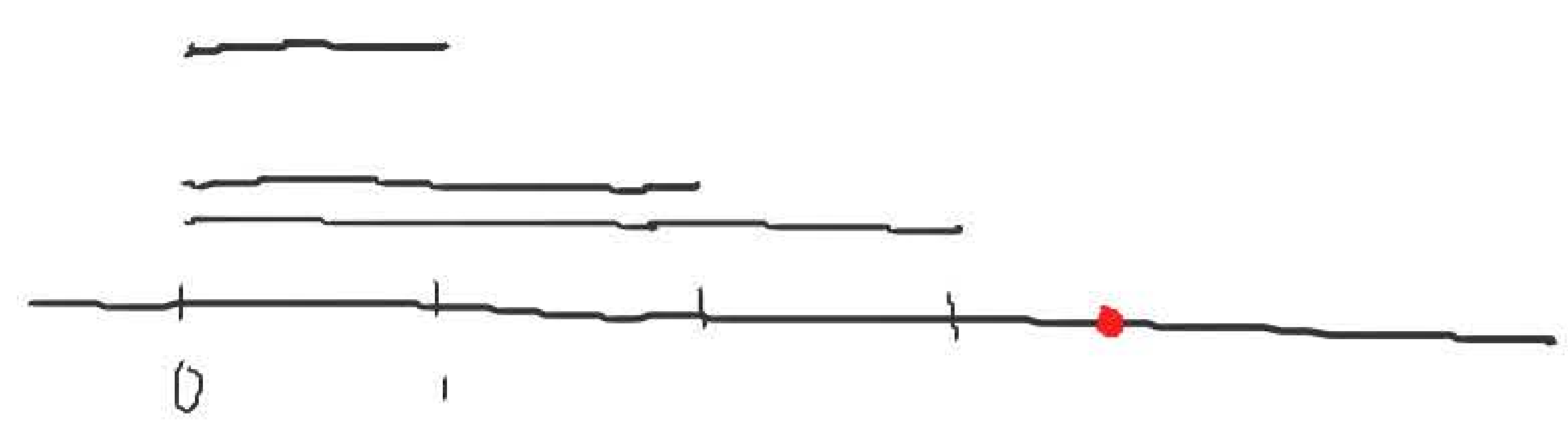
Recall: sequence of function (f_n) on E.

- pointwise convergence of (f_n)
- uniform convergence of (f_n).
- Uniform limit theorem:
  uniform limit of continuous functions is continuous.

$\chi_A : \mathbb{R} \to \mathbb{R}$, $\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$.

zero function

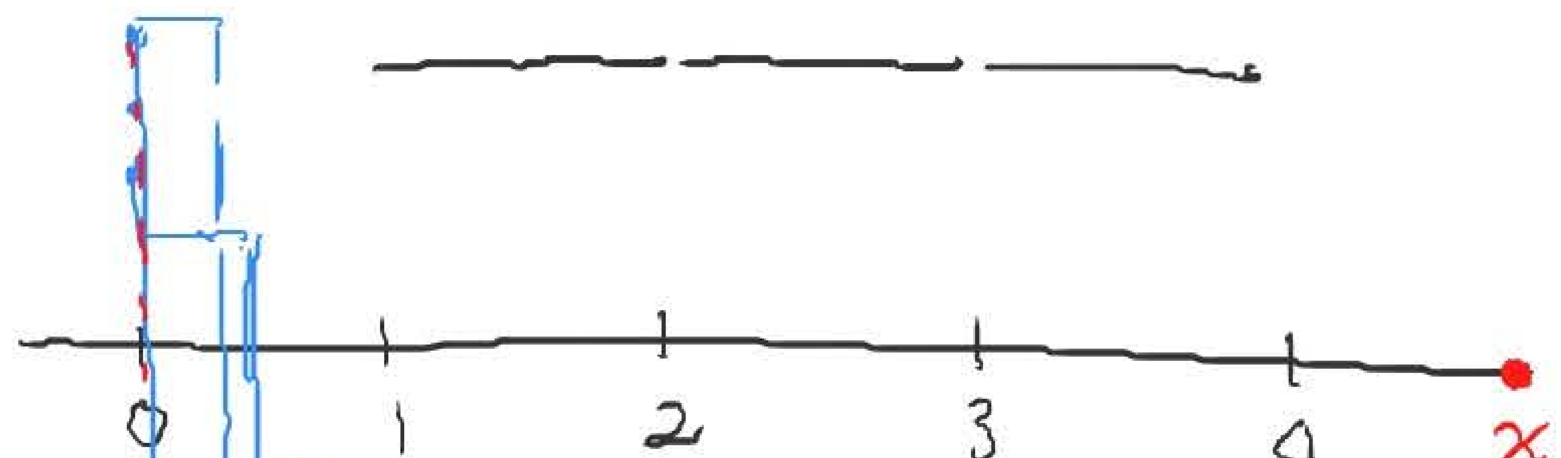
- $f_n(x) = \frac{1}{n} \chi_{[0,n]}$



$f_n \to 0$ pointwise

$f_n \to 0$ uniformly

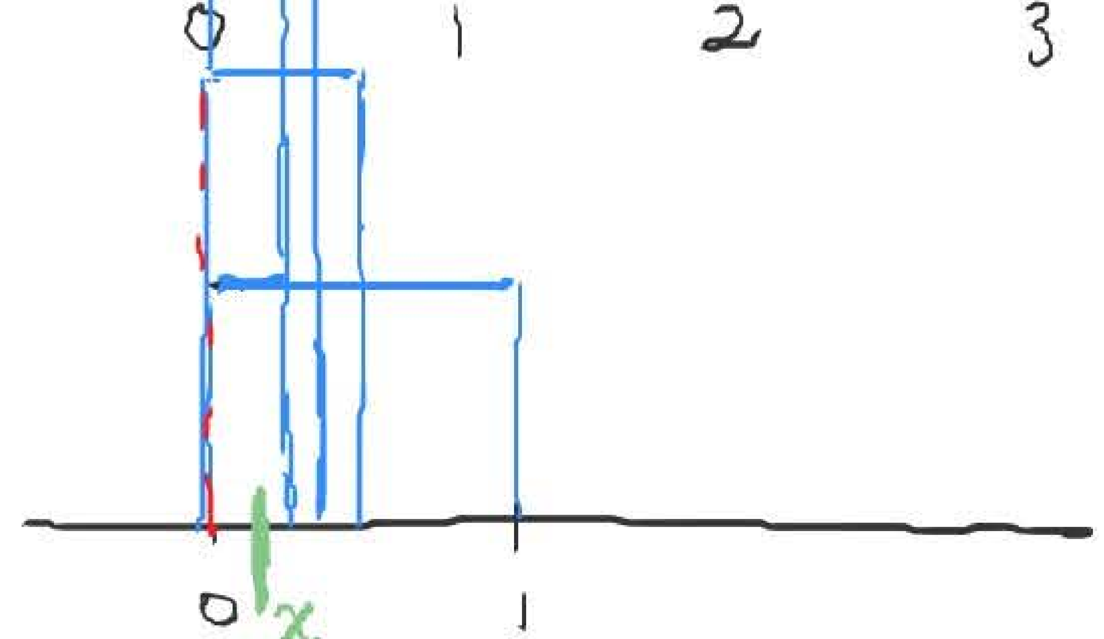
- $f_n(x) = \chi_{[n,n+1]}$



$f_n \to 0$ pointwise.

$f_n \not\to 0$ uniformly.

- $f_n(x) = n\chi_{(0,\frac{1}{n}]}$.



$f_n \to 0$ pointwise

$f_n \not\to 0$ uniformly.

Math 104 Worksheet 14

UC Berkeley, Summer 2021

Wednesday, July 28

Let (X, d) be a metric space, and let $E \subseteq X$.

Exercise 1. Show that $D(f, g) := \sup\{|f(x) - g(x)| : x \in E\}$ defines a metric on the space of bounded real-valued functions on E, $B(X) := \{f : E \to \mathbb{R} : f \text{ is bounded}\}$.

See previous worksheet.

$f_n \to f$ w.r.t. metric: $\forall \varepsilon > 0$, $\exists N$: $n \geq N \Rightarrow \underbrace{D(f_n, f)}_{\sup\{|f_n(x) - f(x)| : x \in E\}} < \varepsilon$.

$\Rightarrow |f_n(x) - f(x)| < \varepsilon$ for all $x \in E$.

i.e. uniform convergence.

$|f_n(x) - f(x)| < \varepsilon/2$. $\forall x \in E$.

$\Rightarrow \sup\{|f_n(x) - f(x)| : x \in E\} \leq \frac{\varepsilon}{2} < \varepsilon$.

Exercise 2. What does it mean for a sequence of functions in $B(X)$ to converge?

(f_n) converging means (f_n) converge uniformly.

Alternate definition of unif conv. $\lim_{n \to \infty} \underbrace{\sup\{|f_n(x) - f_n(x)| : x \in E\}}_{D(f_n, f)} = 0$.

Exercise 3. Formulate a definition for a sequence of functions (f_n) on E to be **uniformly Cauchy**.

$\forall \varepsilon > 0$ $\exists N$: $m, n \geq N \Rightarrow D(f_m, f_n) < \varepsilon$.

$\Rightarrow |f_m(x) - f_n(x)| < \varepsilon$ for all $x \in E$.

Exercise 4. Formulate a definition for a series of functions $\sum_{n=1}^{\infty} g_n$ on E to be **uniformly convergent on E**.

$(g_1, g_1 + g_2, g_1 + g_2 + g_3, \dots)$ conv. uniformly to some G

$\forall \varepsilon > 0$ $\exists N$: $n \geq N \Rightarrow \left| \sum_{k=1}^{n} g_k(x) - G(x) \right| < \varepsilon$ for all $x \in E$.

$\left| \sum_{k=n+1}^{\infty} g_k(x) \right| < \varepsilon$ for all $x \in E$.

$\sum_{n=0}^{\infty} x^n$ converges uniformly on $[-\frac{1}{2}, \frac{1}{2}]$ to $\frac{1}{1-x}$

Exercise 5. Formulate a definition for a series $\sum_{n=1}^{\infty} g_n$ to satisfy the **uniform Cauchy criterion**.

$\forall \varepsilon > 0$ $\exists N$: $n \geq m \geq N$: $\left| \sum_{k=m}^{n} g_k(x) \right| < \varepsilon$. for all $x \in E$.

Theorem: (f_n) is uniformly Cauchy if and only if (f_n) converges uniformly.

Proof: $\Leftarrow$ Let $\varepsilon > 0$. There exists f and $N \in \mathbb{N}$: $n \geq N$ implies $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ for all x.

Then for $m, n \geq N$, $|f_m(x) - f_n(x)| \leq \underbrace{|f_m(x) - f(x)|}_{< \varepsilon/2} + \underbrace{|f(x) - f_n(x)|}_{< \varepsilon/2} < \varepsilon$ for all x.

$\Rightarrow$ Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$: $m, n \geq N$ implies $|f_m(x) - f_n(x)| < \varepsilon/2$ for all x.

$(f_n(x))$ is Cauchy for any x.

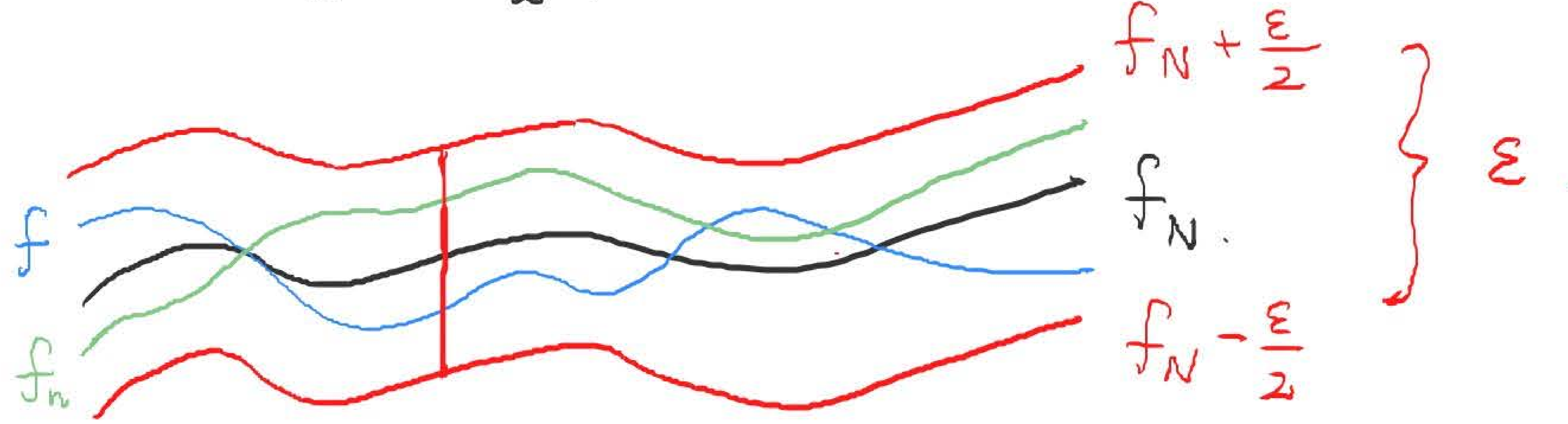
seq. of real numbers

$\Rightarrow (f_n(x))$ converges to something ; define $f(x) = \lim_{n\to\infty} f_n(x)$.

$n \geq N$: $f_N(x) - \frac{\varepsilon}{2} < f_n(x) < f_N(x) + \frac{\varepsilon}{2}$ for all x.

$\Rightarrow f_N(x) - \frac{\varepsilon}{2} \leq f(x) \leq f_N(x) + \frac{\varepsilon}{2}$.

$\Rightarrow |f_n(x) - f(x)| < \varepsilon$.



Corollary: $\sum_{k=0}^{\infty} g_k$ satisfies the uniform Cauchy criterion if and only if it converges.

Weierstrass M-test: Let (g_k) be a sequence of functions on $E \subseteq X$. Let (M_k) be a sequence of nonnegative real numbers such that $\sum M_k$ converges. If $|g_k(x)| \leq M_k$ for all $x \in E$, then $\sum g_k$ converges uniformly on E.

Proof: (Just need to check uniform Cauchy criterion)

Let $\varepsilon > 0$. There exists N: $n \geq m \geq N$ implies $\sum_{k=m}^{n} M_k < \varepsilon$.

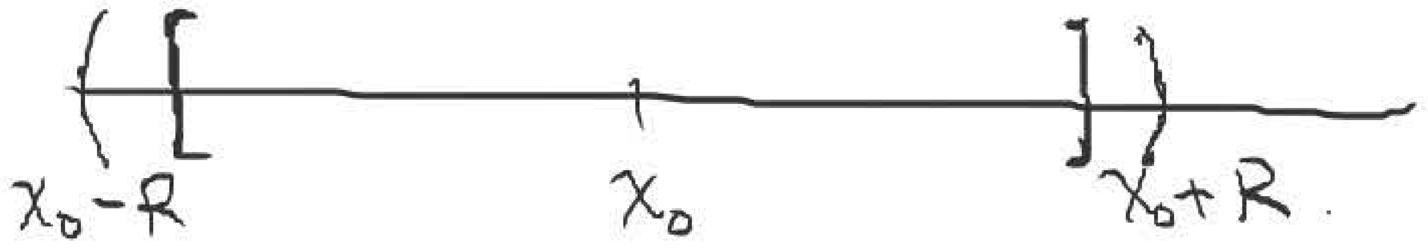
Then

$$\left| \sum_{k=m}^{n} g_k(x) \right| \leq \sum_{k=m}^{n} |g_k(x)| \leq \sum_{k=m}^{n} M_k < \varepsilon \quad \text{for all } x \in E.$$

Therefore $\sum g_k$ satisfies the uniform Cauchy criterion, so it converges uniformly.

Theorem: Let $\overbrace{\sum_{n=0}^{\infty} a_n(x-x_0)^n}^{\text{series of functions}}$ be a power series with radius of convergence $R>0$. If $0<R_0<R$, then the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges uniformly on $[x_0-R_0, x_0+R_0]$.

(to a continuous function on $[x_0-R_0, x_0+R_0]$.)



Proof: Observe that $\sum a_n(x-x_0)^n$ and $\sum |a_n|(x-x_0)^n$ have the same radius of convergence $R>0$.

$0<R_0<R$: $\sum |a_n| \underbrace{(x_0+R_0-x_0)^n}_{R_0^n} < \infty$ $\qquad x_0+R_0 \in (x_0-R, x_0+R)$

$$\sum \underbrace{|a_n| R_0^n}_{M_n} \text{ converges.}$$

For all $x \in [x_0-R_0, x_0+R_0]$, $|a_n(x-x_0)^n| \leq |a_n| R_0^n = M_n$.

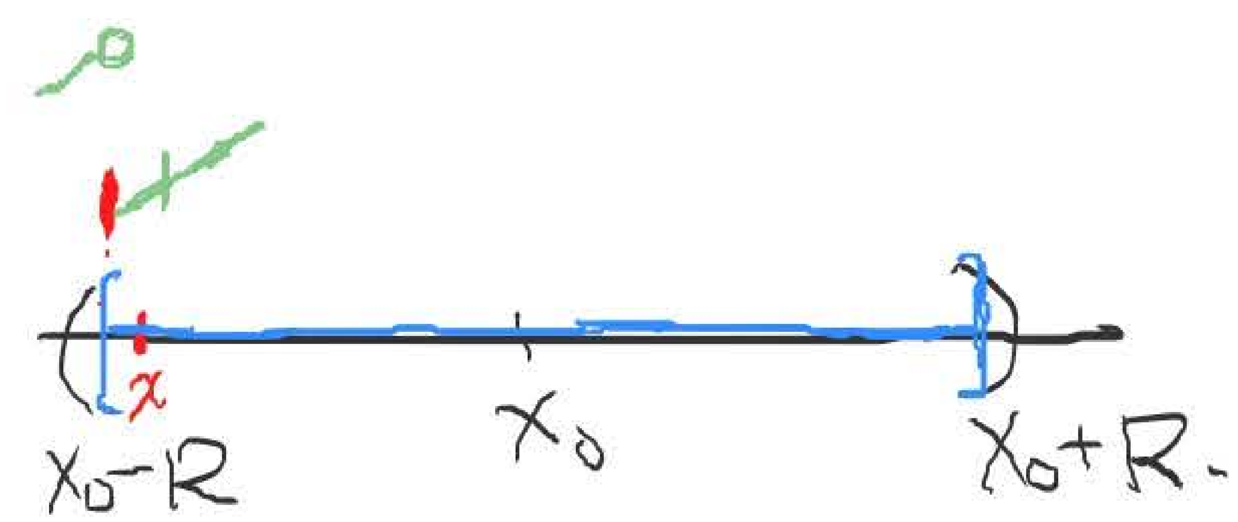
Weierstrass M-test $\Rightarrow$ $\sum a_n x^n$ converges uniformly on $[x_0-R_0, x_0+R_0]$

to a continuous function on $[x_0-R_0, x_0+R_0]$

by uniform limit theorem.

Corollary: Let $\sum a_n(x-x_0)^n$ be a power series with radius of convergence $R>0$.

Then $\sum a_n(x-x_0)^n$ converges to a continuous function on (x_0-R, x_0+R).



Proof: Let $x \in (x_0-R,\ x_0+R)$.

There exists $0<R_0<R$ such that

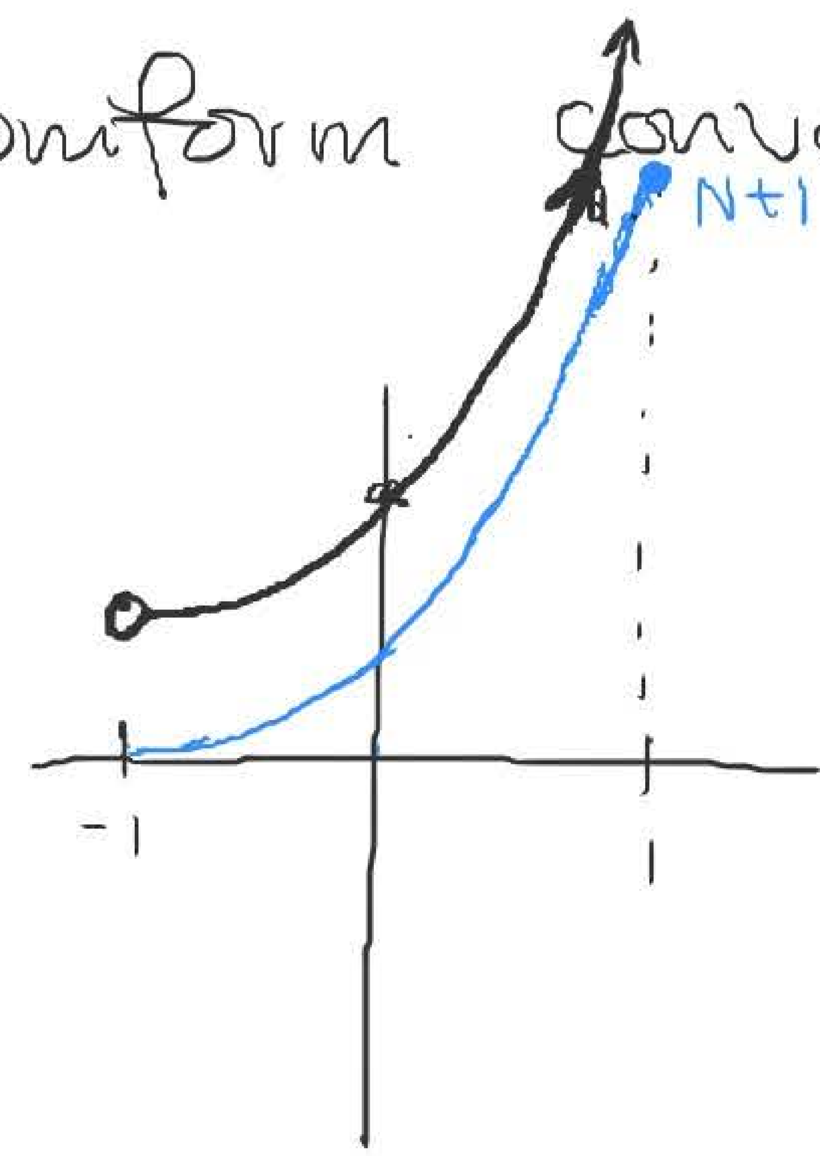
$$x \in (x_0-R_0,\ x_0+R_0)$$

By previous theorem, $\sum a_n(x-x_0)^n$ is continuous on $[x_0-R_0,\ x_0+R_0]$, so it is continuous at x.

In general, don't have uniform convergence on $(x_0-R,\ x_0+R)$.

Ex. $\sum x^n \longrightarrow \dfrac{1}{1-x}$

$$\sum_{n=0}^{N} x^n = \begin{cases} \dfrac{1-x^{N+1}}{1-x} & x \neq 1 \\ N+1 & x=1 \end{cases}$$



Question: What if the power series converges at one or both endpoints of the interval of convergence?

Answer: Yes (see Worksheet 15).