Wednesday, July 14

Midterm tomorrow

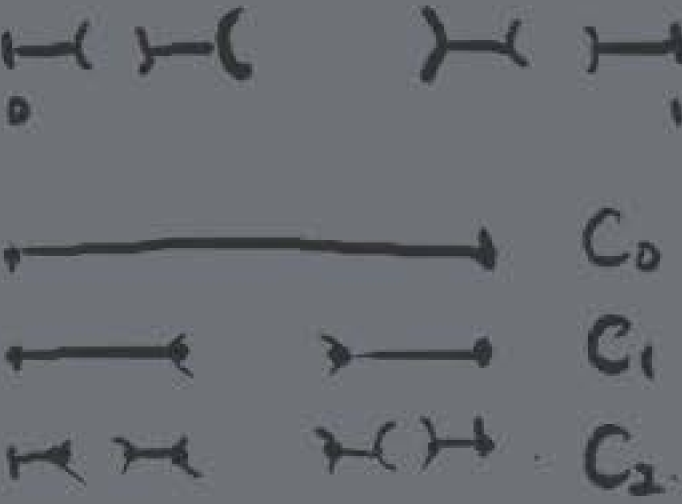
- check confirmation email
  from Tuesday.

Math 104 Worksheet 9

UC Berkeley, Summer 2021

Tuesday, July 13

Exercise 1. Justify the following facts about the Cantor set $\mathcal{C}$. $\subseteq \mathbb{R}$,



(a) $\mathcal{C}$ is compact. intersection of closed sets is closed. bound.

$C = \bigcap_{i=0}^{\infty} C_i$

(b) $\mathcal{C}$ does not contain any intervals.

Suppose $(a,b) \subseteq C$. There exists $n \in \mathbb{N}$: $(\frac{1}{3})^n < b-a$.

$(a,b) \not\subseteq C_n$ is a union of a bunch of disjoint intervals of length $(\frac{1}{3})^n$.

$C \subseteq C_n$

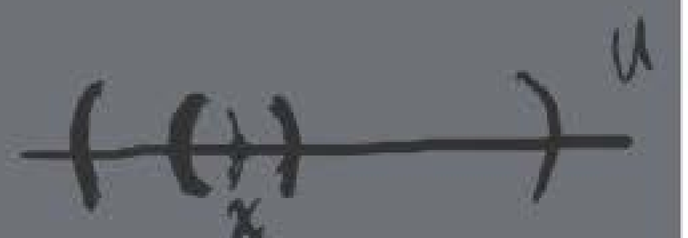
(c) $\mathcal{C}$ does not have any interior points.

Let $x \in C$. For any $r>0$, $B_r(x) = (x-r, x+r) \not\subseteq C$ by (b).

(d) Every point in $\mathcal{C}$ is a limit point of $\mathcal{C}$.

Let $x \in C$. Let $r>0$. There exists $n \in \mathbb{N}$: $(\frac{1}{3})^n < r$. $x \in C_n$. x is an element of some closed interval of length $(\frac{1}{3})^n$ contained in C_n. The two endpoints of this interval will lie in $B_r(x) \cap C$. ∴ x is a limit point.

(e) Every point in $\mathcal{C}$ is a limit point of $\mathcal{C}^c$.



Let $x \in C$. Let $r>0$. $B_r(x) = (x-r, x+r) \not\subseteq C$ $\Rightarrow$ there exists $y \in B_r(x)$ s.t. $y \notin C$ hence $y \in C^c$.

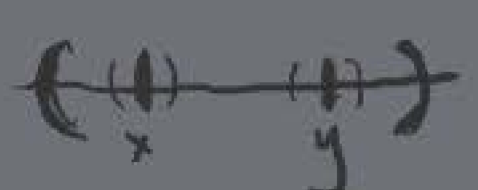
Exercise 2. Prove that any open set in $\mathbb{R}$ is an at most countable disjoint union of open intervals. (*Hint:* If $U \subseteq \mathbb{R}$ is open, for any $x \in U$, any interval containing x can be expanded to the largest interval in U containing x.)

U open. For each $x \in U$, there exists $r>0$: $B_r(x) \subseteq U$. $(x-r, x+r) \subseteq U$.

Let $s_x = \inf\{a < x : (a,x) \subseteq U\}$

$t_x = \sup\{b > x : (x,b) \subseteq U\}$

Let $I_x = (s_x, t_x)$.



$I_x = I_y$.

Check: $I_x \subseteq U$. ✓

Claim: $U = \bigcup_{x \in U} I_x$.

Check (obvious).

Check: If $x, y \in U$, then either $I_x = I_y$, or $I_x \cap I_y = \emptyset$.

Argue: only at countably many distinct intervals.

for each interval I, pick $q_I \in \mathbb{Q} \cap I$. by denseness of $\mathbb{Q}$ the map $I \mapsto q_I$ is injective.

Recall $\displaystyle\sum_{k=m}^{\infty} a_k \overset{\text{def}}{=\!=} \lim_{n\to\infty} S_n$

partial sums of $\displaystyle\sum_{k=m}^{\infty} a_k$.

$(S_n) = (a_m, a_m + a_{m+1}, \ldots)$.

Inherit notions of convergence/divergence, Cauchy criterion.

Theorem: A series converges if and only if it satisfies Cauchy criterion.

Corollary: If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Question: Is the converse true?

$a_n = \frac{1}{n}$.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$$

Harmonic series.

Theorem (Comparison test).

Let $\sum a_n$ be a series such that $a_n \geq 0$ for all n.

(i) If $\sum a_n$ converges and $|b_n| \leq a_n$ for all n, then $\sum b_n$ converges.

(ii) If $\sum a_n = \infty$, and $b_n \geq a_n$ for all n, then $\sum b_n = \infty$.

diverges to ∞

Proof: (i) (Show that $\sum b_n$ satisfies the Cauchy criterion).

Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that $n \geq m \geq N$ implies $\sum_{k=m}^{n} a_k < \varepsilon$.

Then for $n \geq m \geq N$, $\left| \sum_{k=m}^{n} b_k \right| \leq \sum_{k=m}^{n} |b_k| \leq \sum_{k=m}^{n} a_k < \varepsilon$.

(ii) Let (s_n) be the sequence of partial sums of $\sum a_n$.

Let (t_n) be the sequence of partial sums of $\sum b_n$.

$s_n \leq t_n$ for all n. $s_n \rightarrow \infty \Rightarrow t_n \rightarrow \infty$.

$a_1 + a_2 + a_3 + \cdots$

$b_1 + b_2 + b_3 + \cdots$

Corollary: Absolutely convergent series converge.

$\sum a_n$ abs. conv mean that $\sum |a_n|$ converges.

Proof:

$$a_n \le |a_n|.$$

If $\sum a_n$ converges absolutely, then $\sum \underbrace{|a_n|}_{b_n}$ converges.

By comparison test (i), $|a_n| \le b_n$ for all n $\Rightarrow \sum a_n$ converges.

Ex. Harmonic series.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots + \frac{1}{16} + \frac{1}{17} + \cdots + \frac{1}{32} + \cdots$$

smaller $$1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{\frac{1}{2}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{\frac{1}{2}} + \underbrace{\frac{1}{16} + \cdots + \frac{1}{16}}_{\frac{1}{2}} + \underbrace{\frac{1}{32} + \cdots + \frac{1}{32}}_{\frac{1}{2}}.$$

diverges

$\therefore$ Harmonic series diverge by comparison test.