Tuesday, July 6

Recall: metric space

(X, d)

set — distance function

- $d(x,y) \geq 0$ and $d(x,y) = 0$ iff $x = y$.
- $d(x,y) = d(y,x)$
- $d(x,z) \leq d(x,y) + d(y,z)$.

Examples $\mathbb{R}$, $\mathbb{R}^k$ with $d(\vec{x},\vec{y}) = \sqrt{\sum_{i=1}^{k} (y_i - x_i)^2}$

ordered k-tuples

- See HW 3 for an example of another metric on $\mathbb{R}^k$. (#6)

- Discrete metric space.

X any set

discrete metric

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y. \end{cases}$$

- Property 1 ✓
- Property 2 ✓
- $d(x,z) \leq d(x,y) + d(y,z)$?

Two cases:

- $x = z$. LHS $= 0$ ✓
- $x \neq z$. LHS $= 1$.
  On RHS, $y \neq x$ or $y \neq z$, so
  $d(x,y) = 1$ or $d(y,z) = 1$ (or both).

Note: Discrete metric spaces give lots of nice examples/counterexamples to various assertions.

Let (X, d) be a metric space.

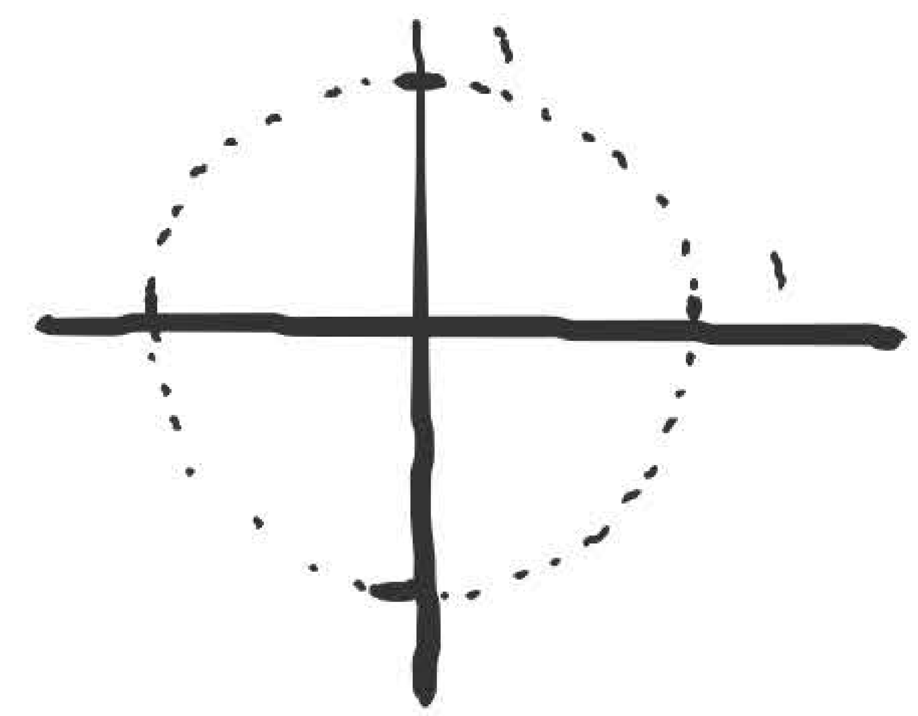
Def: For $x \in X$ and $r > 0$, the <u>open ball of radius r centered at x</u> is the set

$$B_r(x) = \{ y \in X : d(y,x) < r \}.$$

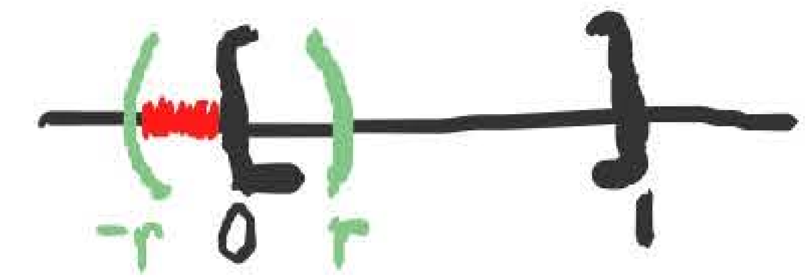
Note: For $\mathbb{R}^k$, if the distance function is not specified, always assume the Euclidean metric.

<u>Examples</u>

- In $\mathbb{R}$, $B_1(2) = (1,3)$. open interval from 1 to 3.
- In $\mathbb{R}^2$, $B_1((0,0))$ is the open unit disk.



- In $X \neq \emptyset$ with discrete metric,
  - $B_{1/2}(x) = \{x\}$
  - $B_2(x) = X$.



Def: For $E \subseteq X$, $x \in E$ is an <u>interior point</u> of E if there exists $r > 0$ such that $B_r(x) \subseteq E$.

Lesson: dependence on X.

<u>Examples</u>:

- In $\mathbb{R}$, $E = (0,1)$. $\frac{1}{2}$ is an interior point of E since $B_{1/4}(\frac{1}{2}) = (\frac{1}{4}, \frac{3}{4}) \subseteq E$.
  $\frac{1}{100}$ works too since $B_{\frac{1}{100}}(\frac{1}{100}) = (0, \frac{2}{100}) \subseteq E$.
- In $\mathbb{R}$, 0 is <u>not</u> an interior point of $[0,1]$. For any $r>0$, $B_r(0) = (-r, r) \not\subseteq E$.
- In the metric space $[0,1]$ with usual distance function, 0 is an interior point of $[0,1]$.

Def: A set $E \subseteq X$ is <u>open</u> if every point $x \in E$ is an interior point of E.

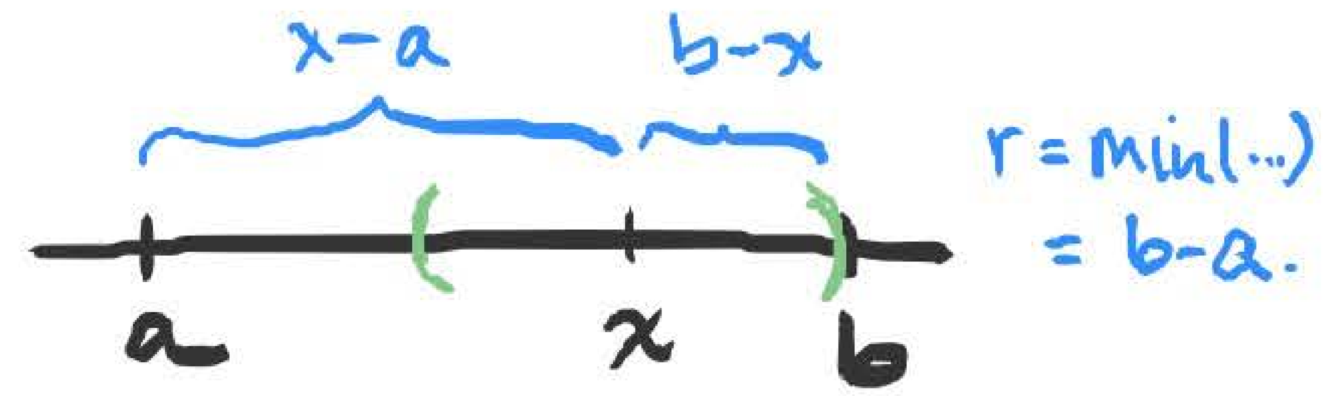
if $x \in E$, then x is an interior point of E.

A	B	A⇒B
T	T	T
T	F	F
F	T	T
F	F	T

Examples

- In $\mathbb{R}$, (a,b), (a,∞), $(-\infty,a)$ are open sets.

  Proof: Let $x \in (a,b)$. Let $r = \min(x-a,\, b-x) > 0$. $B_r(x) \subseteq (a,b)$.



- In $\mathbb{R}$, $[a,b]$, $[a,b)$, $(a,b]$ are <u>not</u> open.

  a is not an interior point

  b is not an interior point

- In $\mathbb{R}$, $\mathbb{Q}$ is <u>not</u> open. $q \in \mathbb{Q}$: For any $r > 0$, $B_r(q) \not\subseteq \mathbb{Q}$.

  contains lots of irrational numbers.

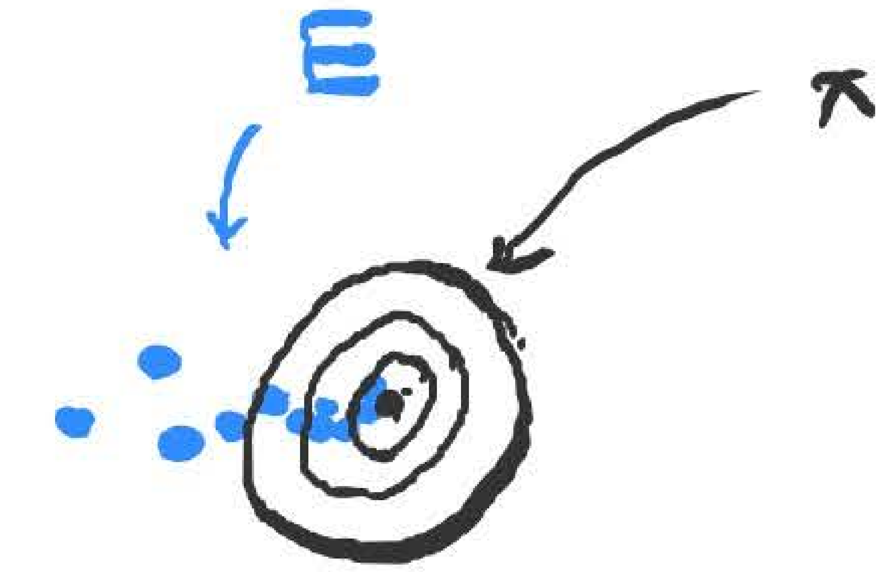
- In any metric space, X and $\emptyset$ are open.

  by definition, $B_r(x) = \{y \in X : d(y,x) < r\} \subseteq X$.

  "vacuously true" that $\emptyset$ is open.

Def: For a set $E \subseteq X$, the __complement__ of E is the set $E^c = X \setminus E = \{x \in X : x \notin E\}$.

"set-minus"



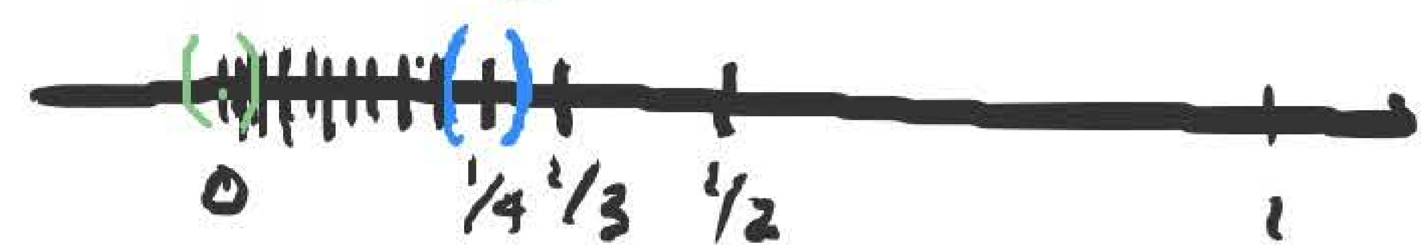
Def: For a set $E \subseteq X$, a point $x \in X$ is a __limit point__ of E if for any $r > 0$, we have that $\Big(\underbrace{B_r(x) \setminus \{x\}}_{\text{elements in } B_r(x) \text{ excluding } x.}\Big) \cap E \neq \emptyset$,

i.e. for any radius r, no matter how small, there is some element of E which sits in $B_r(x)$, other than x itself.

__Examples__:

- In $\mathbb{R}$, the set of limit points of $(0,1)$ is $[0,1]$.
- In $\mathbb{R}$, the only limit point of $\underbrace{\{\frac{1}{n} : n \in \mathbb{N}\}}_{E}$ is 0.
- In $\mathbb{R}$, the set of limit points of $\mathbb{Q}$ is $\mathbb{R}$. (by denseness of $\mathbb{Q}$ in $\mathbb{R}$).

$\frac{1}{4}$ is not an interior point E.



$\underbrace{B_r(0)}_{(-r,r)} \cap E \neq \emptyset$.

By A.P., there exists $N \in \mathbb{N}$: $\frac{1}{N} < r \Rightarrow \frac{1}{N} \in B_r(0)$

Def: For a set $E \subseteq X$, $\boxed{x \in E}$ is called an **isolated point** is x is not a limit point of E.

Examples: In $\mathbb{R}$, every integer is an isolated point of $\mathbb{Z}$.

In $\mathbb{R}$, the set $\mathbb{Q}$ has no isolated points.

In $\mathbb{R}$, every element of $\{\frac{1}{n} : n \in \mathbb{N}\}$ is an isolated point.

Notation: $E' :=$ set of all limit points of E.

Also: $E^\circ :=$ set of all interior points of E.

Def: A set $E \subseteq X$ is **closed** if $E' \subseteq E$, i.e.

E contains all of its limit points.

Examples:

- In $\mathbb{R}$, $[0,1]$ is closed. $[a,\infty)$, $(-\infty, a]$ are closed.
- In $\mathbb{R}$, the set $\{\frac{1}{n} : n \in \mathbb{N}\}$ is not closed. But $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is closed.
- In any metric space, X and $\emptyset$ are closed.

Definition: A set $E \subseteq X$ is **bounded** if for some $x \in X$ are $M > 0$ such that $d(x,y) \leq M$ for all $y \in E$.

- In $\mathbb{R}$, E is bounded if there exists M s.t. $\underbrace{|x|}_{d(x,0)} \leq M$ for all $x \in E$.

Def: The **closure** of E in X is $\bar{E} = E \cup E'$.

Def: A set $E \subseteq X$ is **dense** in X if $\bar{E} = X$.

- $\mathbb{Q}$ is dense in $\mathbb{R}$ (as is $\mathbb{R} \setminus \mathbb{Q}$).
- In any metric space (X,d), X is dense in X.