University of California, Berkeley
Math 104: Introduction to Analysis

Instructor: Theodore Zhu

Final Exam

August 9, 2018

10:10 AM – 11:55 AM

Name: ______________________________

Student ID: _________________________

Instructions. This is a closed-book, closed-notes, closed-electronics exam. Please write carefully and clearly in the spaces provided. If you run out of space for a problem, you may continue on the reverse side of the page, or on the extra pages at the end. Cross out any work that you do not want to be graded. Unless otherwise specified, show all work and justify any nontrivial claims. **You may use any results from lecture and homework problems, but you must clearly state the result that you are using.**

Question	Points	Score
1	14	
2	7	
3	6	
4	6	
5	6	
6	6	
Total:	45	

1. **Short Answer.** Unless otherwise stated, no justification is required.

   (a) (2 points) Give an example of a continuous function f and a Cauchy sequence (s_n) such that $f(s_n)$ is NOT Cauchy.

   (b) (2 points) Give an example of a power series with a closed and bounded interval of convergence.

   (c) (2 points) Give two examples of power series: one with radius of convergence 0, and the other with radius of convergence ∞.

(d) (2 points) Give an example of a sequence of functions (f_n) which converges pointwise but not uniformly.

(e) (2 points) Give an example of a function that is discontinuous at every point in its domain.

(f) (2 points) Give an example of a metric space (X, d) which is NOT compact, but can be expressed as a countable union of compact sets.

(g) (2 points) Give an example of a continuous function f and an open set $U \subseteq \mathrm{dom}(f)$ such that $f(U)$ is NOT open.

2. **Definitions and theorems.** Complete sentences are not required.

   (a) (1 point) State either one of the two parts of the **fundamental theorem of calculus**.

   (b) (1 point) State the **intermediate value theorem** for continuous functions.

   (c) (1 point) Given a partition $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ of $[a, b]$, what is the **upper Darboux sum** $U(f, P)$? (Give a formula.)

(d) (1 point) Given metric spaces (X, d_X) and (Y, d_Y), what does it mean for a function $f : X \to Y$ to be **continuous** at a point $x_0 \in X$? (Provide any one of the three equivalent definitions.)

(e) (1 point) What does it mean for a series of real numbers $\sum a_n$ to satisfy the **Cauchy criterion**?

(f) (1 point) State the **continuous extension theorem** (in any level of generality).

(g) (1 point) State the **uniform limit theorem**.

3. (6 points) Prove that every compact metric space is complete.

4. (6 points) Recall that a real-valued function f is **Lipschitz continuous** on $E \subseteq \text{dom}(f) \subseteq \mathbb{R}$ if there exists $C > 0$ such that $|f(y) - f(x)| \leq C|y - x|$ for all $x, y \in E$.

   Prove that if f is differentiable on (a, b) and f' is bounded on (a, b), then f is Lipschitz continuous on (a, b).

5. (6 points) Let (X, d) be a metric space. Let $f_n : X \to \mathbb{R}$ be a sequence of uniformly continous functions such that (f_n) converges to some function f uniformly on X. Prove that f is uniformly continuous.

6. (6 points) Let f be an integrable function on $[a, b]$. Let $x_0 \in [a, b]$, and suppose g is a function on $[a, b]$ such that $g(x_0) \neq f(x_0)$ and $g(x) = f(x)$ for all $x \neq x_0$. Prove that g is integrable and $\int_a^b g(x)dx = \int_a^b f(x)dx$.

   Note: For this problem, you may **NOT** cite results from homework without proof.

Extra page

Extra page