MATH 104 Exercise Solutions

Wenhao Pan

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Introduction

This note is a collection of the solutions to the recommended exercises of *Elementary Analysis* by Kenneth A. Ross.

Chapter 8

Q6a

Chapter 9

Q4

(a) $s_1 = 1, s_2 = \sqrt{2}, s_3 = \sqrt{\sqrt{2}+1}, s_4 = \sqrt{\sqrt{\sqrt{2}+1}+1}$.

(b) Since (s_n) converges, $\lim(s_n) = \lim(s_{n+1}) = s$ which implies

$$\begin{aligned} \lim s_n = \lim \sqrt{s_n + 1} &= s \\ \lim s_n + 1 &= s^2 \\ \lim s_n &= s^2 - 1 \\ s &= s^2 - 1 \end{aligned}$$

Thus solve the last equation for s to get $s = \frac{1+\sqrt{5}}{2}$ since $s_n > 0$ for all n.

Q9

(c): Let $s = \lim(s_n - t_n)$. Suppose $s > 0$, then $\exists N_1$ $n > N_1 \implies |s_n - t_n - s| < s \implies s_n > t_n$. This contradicts to the condition that there exists N_0 such that $n > N_0 \implies s_n \leq t_n$.

Q10

(a) Since $\lim s_n = +\infty$ and $k < 0$, for each $\frac{M}{k} > 0$, there exists N such that $n > N \implies s_n > \frac{M}{k}$. Thus for each $M > 0$, $n > N \implies ks_n > k \cdot \frac{M}{k} = M$, so $\lim ks_n = +\infty$.

Q11

(a) Suppose $\inf\{t_n : n \in \mathbf{N}\} = m$. For each $M > 0$, consider two cases of $M - m$:

Case 1: If $M - m \leq 0$ then $M > M - m$. Thus we have there exists N_1 such that $n > N_1 \implies s_n > M - m \implies s_n + m > M \implies s_n + t_n \geq s_n + m > M$, so $\lim(s_n + t_n) = +\infty$.

Case 2: If $M - m > 0$, then there exists N_1 such that $n > N_1 \implies s_n > M - m \implies s_n + m > M \implies s_n + t_n \geq s_n + m > M$, so $\lim(s_n + t_n) = +\infty$.

(b) We want to show that $\lim t_n > -\infty \implies \inf\{t_n : n \in \mathbf{N}\} > -\infty$. Since $\lim t_n \neq -\infty$, there exists M with $-\infty < M < 0$ such that $\forall N \in \mathbf{N}\ \exists n > N\ t_n > M$. This implies $\inf\{t_n : n \in \mathbf{N}\} \geq M > -\infty$. Then we can apply (a).

(c) Since (t_n) is bounded, $\exists M \in \mathbf{R}$ such that $\forall n \in \mathbf{N}\ |t_n| \leq M$. This implies for all n, $t_n \geq -M \implies -M \leq \inf\{t_n : n \in \mathbf{N}\} \implies \inf t_n : n \in \mathbf{N} > -\infty$. Then we can apply (a).

Q18

(a) Let $S = 1 + a + a^2 + \cdots + a^n$, then $a \cdot S = a + a^2 + a^3 + \cdots + a^{n+1}$. Then subtract aS from S to get $S - aS = 1 - a^{n+1} \implies S = \frac{1-a^{n+1}}{1-a}$.

(b) $\lim_n(1 + a + a^2 + \cdots + a^n) = \lim_n \frac{1-a^{n+1}}{1-a} = \frac{1}{1-a} \lim(1 - a^{n+1}) = \frac{1}{1-a}(1 - \lim a^n) = \frac{1}{1-a}(1 - 0) = \frac{1}{1-a}$ when $|a| < 1$.

(c) $\frac{1}{1-1/3} = \frac{3}{2}$.

(d) If $a \geq 1$, then $\lim_n(1 + a + a^2 + \cdots + a^n) \geq \lim(1 + 1 + 1 + \cdots + 1) = \lim n = +\infty$. Thus $\lim_n(1 + a + a^2 + \cdots + a^n) = \infty$.

Chapter 10

Q9

(a) $s_2 = (\frac{1}{2}) \cdot 1^2 = \frac{1}{2}$; $s_3 = (\frac{2}{3}) \cdot (\frac{1}{2})^2 = \frac{1}{2 \cdot 3}$; $s_4 = \frac{3}{4} \cdot (\frac{1}{2 \cdot 3})^2 = \frac{1}{2^2 \cdot 3 \cdot 4}$

(b) Observe that s_n is nonincreasing(monotone) and bounded by 1, so s_n converges and hence $\lim s_n$ exists.

(c) Since $\lim s_n$ exists, assume $\lim s_n = s$. Then $s = \lim s_{n+1} = \lim(\frac{n}{n+1})s_n^2 = \lim(\frac{n}{n+1})s^2 = s^2$. Then solve the equation for s to get $s = 1$ or $s = 0$. Since $s_2 < 1$ and s_n is strictly decreasing, $s = 0$.

Chapter 11

Q8

First we want to show that $\inf\{s_n : n > N\} = -\sup\{-s_n : n > N\}$:

$\leq$: Let $\inf\{s_n : n > N\} = m$, then we have

$$\begin{aligned} \forall n > N \; s_n \geq m &\implies \forall n > N \; -s_n \geq -m \\ &\implies \sup\{-s_n : n > N\} \leq -m \\ &\implies m \leq -\sup\{-s_n : n > N\}. \end{aligned}$$

Thus $\inf\{s_n : n > N\} \leq -\sup\{-s_n : n > N\}$.

$\geq$: Let $-\sup\{-s_n : n > N\} = M$, then we have

$$\begin{aligned} \sup\{-s_n : n > N\} = -M &\implies \forall n > N \; -s_n \leq -M \\ &\implies \forall n > N \; M \leq s_n \\ &\implies M \leq \inf\{s_n : n > N\}. \end{aligned}$$

Thus $\inf\{s_n : n > N\} \geq -\sup\{-s_n : n > N\}$.

Thus $\inf\{s_n : n > N\} = -\sup\{-s_n : n > N\}$. Then $\lim_N \inf\{s_n : n > N\} = \lim_N(-\sup\{-s_n : n > N\}) = -\lim_N \sup\{-s_n : n > N\} = -\lim_N \sup(-s_n)$.

Chapter 12

Q2

$\implies$: Since $0 \geq \liminf |s_n| \leq \limsup |s_n| = 0$, we have $lim \inf |s_n| = \limsup |s_n| = 0 \implies \lim |s_n| = 0$. Since $\forall n \in \mathbf{N}\ -|s_n| \leq s_n \leq |s_n|$, by Squeeze Formula $\lim s_n = 0$.

$\impliedby$: From $\lim s_n = 0$, we know $\forall \epsilon > 0\ \exists N \in \mathbf{N}$

$$\begin{aligned} n \geq N &\implies |s_n - 0| < \epsilon \\ &\implies \big||s_n| - |0|\big| \leq |s_n - 0| < \epsilon \\ &\implies \big||s_n| - 0\big| < \epsilon \\ &\implies \lim |s_n| = 0. \end{aligned}$$

Q7

Q9b

Since $\liminf t_n > 0$, $\exists N_1\ m = \inf\{t_n : n \geq N_1\} > 0$. From $\limsup s_n = +\infty$, we know $\forall \frac{M}{m} > 0\ \exists N_2\ \sup\{s_n : n \geq N_1\} > \frac{M}{m}$. Now take $N = \max\{N_1, N_2\}$, we have $\forall n \geq N$

$$\sup\{s_n t_n : n \geq N\} \geq s_n t_n \geq s_n \cdot m.$$

This implies

$$\begin{aligned} \sup\{s_n t_n : n \geq N\} &\geq \sup\{s_n \cdot m : n \geq N\} \\ &= m \cdot \sup\{s_n : n \geq N\} \quad \text{since } m > 0 \\ &> m \cdot \frac{M}{m} \\ &= M, \end{aligned}$$

completing the proof.

Q13

Let's prove $\liminf s_n = \sup A$. Consider $N \in \mathbf{N}$ then $\forall n \geq N\ u_N = \inf\{s_n : n \geq N\} \leq s_n$. i.e. $\{n \in \mathbf{N} : s_n < u_n\} \subseteq \{1, \ldots, N-1\}$. Thus $u_N \in A \implies u_N \leq \sup A$, and hence $\lim_N u_N = \liminf s_n \leq \sup A$.

Now consider arbitrary $a \in A$. Let $N_0 = \max\{n \in \mathbf{N} : s_n < a\} < \infty$ since all but finitely many $s_n < a$. Then $s_n \geq a$ for $n > N_0$. Thus for $N > N_0$ we have $u_N = \inf\{s_n : n \geq N\} \geq a$. It follows that $\lim_N u_N = \liminf s_n \geq a$. Since a is arbitrary, $\liminf s_n$ is an upper bound of A, and hence $\liminf s_n \geq \sup A$.

Chapter 13

Q3b

No, since $d^*((1,1,1,\dots),(0,0,0,\dots)) = \sum_{j=1}^{\infty} |1-0| = \sum_{j=1}^{\infty} 1 = \infty$, and we don't allow distance function to be valued as ∞.

Q5a

We want to show both directions.

$\subseteq$: Consider $u \in \bigcap_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha^{\mathsf{C}}$, then we have

$$\begin{aligned}
\forall\alpha\in\mathcal{A}\ u \in \mathcal{U}_\alpha^{\mathsf{C}} &\implies \forall\alpha\in\mathcal{A}\ u \notin \mathcal{U}_\alpha && (1)\\
&\implies u \notin \bigcup_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha && (2)\\
&\implies u \in \left(\bigcup_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha\right)^{\mathsf{C}}. && (3)
\end{aligned}$$

(1) $\implies$ (2) because

$$(\neg\,(u\in\mathcal{U}_1)) \wedge (\neg\,(u\in\mathcal{U}_2)) \wedge \cdots = \neg\,((u\in\mathcal{U}_1 \vee (u\in\mathcal{U}_2) \vee \cdots) = \neg\left(u \in \bigcup \mathcal{U}_i\right)$$

Thus $\bigcap_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha^{\mathsf{C}} \subseteq \left(\bigcup_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha\right)^{\mathsf{C}}$.

$\supseteq$: Consider $u \in \left(\bigcup_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha\right)^{\mathsf{C}}$, then we have

$$\begin{aligned}
u \notin \bigcup_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha &\implies \forall\alpha\in\mathcal{A}\ u \notin \mathcal{U}_\alpha\\
&\implies \forall\alpha\in\mathcal{A}\ u \in \mathcal{U}_\alpha^{\mathsf{C}}\\
&\implies u \in \bigcap_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha^{\mathsf{C}}
\end{aligned}$$

Thus $\bigcap_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha^{\mathsf{C}} \supseteq \left(\bigcup_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha\right)^{\mathsf{C}}$, and hence $\bigcap_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha^{\mathsf{C}} = \left(\bigcup_{\alpha\in\mathcal{A}} \mathcal{U}_\alpha\right)^{\mathsf{C}}$.