Math 104 Final Exam (Compact Version)

UC Berkeley, Summer 2020

Thursday, August 13, 12:00pm - 2:00pm PDT

Problem 1. Short answers. No justification required for examples. (3 points each)
(a) Please copy verbatim the following text, followed by your signature. This MUST be handwritten UNLESS you are writing your entire exam electronically.

"As a member of the UC Berkeley community, I will act with honesty, integrity, and respect for others during this exam. The work that I will upload is my own work. I will not collaborate with or contact anyone during the exam, search online for problem solutions, or otherwise violate the instructions for this examination."

(b) Given an example of a compact subset of $\mathbb{R}$ with exactly 3 limit points.

(c) Find all subsequential limits of the sequence $s_n = (-1)^n n^{(-1)^n} + (-1)^n$.

(d) Consider the function f defined on $[0, 2]$ given by

$$f(x) = \begin{cases} 0 \text{ for } x \in [0, 1) \\ 1 \text{ for } x \in [1, 2]. \end{cases}$$

Find a partition P of $[0, 2]$ such that $U(f, P) - L(f, P) \leq \frac{1}{3}$.

(e) Let (f_n) be a sequence of continuous functions defined on an open interval (a, b) which converges pointwise to a function f on (a, b). Suppose that for every closed interval $[c, d]$ contained within (a, b), (f_n) converges uniformly on $[c, d]$. Prove or explain carefully **in no more than four sentences** why f must be continuous on (a, b).

Problem 2. (10 points) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series such that for some constant $C > 0$, $|a_n| \geq C$ for infinitely many $n \in \mathbb{N}$. Show that the power series has radius of convergence $R \leq 1$.

Problem 3. Let (X, d) be a metric space.
(a) (5 points) Show that for any $x_1, x_2, x_3, x_4 \in X$,

$$|d(x_1, x_2) - d(x_3, x_4)| \leq d(x_1, x_3) + d(x_2, x_4).$$

(b) (5 points) Suppose $f : X \to X$ satisfies $d(f(x), f(y)) \leq d(x, y)$ for any $x, y \in X$. Let $g : X \to \mathbb{R}$ be the function given by $g(x) = d(x, f(x))$. Show that g is uniformly continuous (on X).

Problem 4. Let $f : [0, 1] \to [0, 1]$ be a continuous function satisfying $f(0) > 0$. Fix $t \in (0, 1]$ and define the set $S_t = \{x \in [0, 1] : \frac{x}{f(x)} = t\}$.
(a) (5 points) Prove that $\inf S_t > 0$.
(b) (5 points) Prove that $\inf S_t \in S_t$.

Problem 5. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function satisfying $f(0) > 0$, $f(1) = 1$, and $f'(1) > 1$.
(a) (5 points) Show that there exists $x_0 \in (0, 1)$ such that $f(x_0) < x_0$.
(b) (5 points) Show that there exists $y_0 \in (0, 1)$ such that $f(y_0) = y_0$.