Thursday, July 22

- Midterm solutions posted
- Regrade requests open until class on Monday.
  - Two types:

    Type 1: "What is wrong with my solution?"

    unlimited

    Type 2: "I think I deserve more partial credit."

    - Limit 1 per student (doesn't count if you get points back).
    - Unlikely to work — "No changes were made."

<u>Theorem</u>: If f is continuous on a closed interval $[a,b]$, then f is uniformly continuous on $[a,b]$.

(cont. and unif. cont are equivalent notions on $\underbrace{\text{closed, bounded}}$ intervals)

<u>Proof</u>: (Contradiction) Suppose f is continuous on $[a,b]$, compact? but not uniformly continuous on $[a,b]$. Then there exists $\varepsilon>0$, such that for any $\delta>0$, there exist $x,y\in[a,b]$ such that $|x-y|<\delta$ but $|f(x)-f(y)|\geq\varepsilon$.

For each $n\in\mathbb{N}$, there exist $x_n,y_n\in[a,b]$: $\boxed{|x_n-y_n|<\frac{1}{n}}$, $\boxed{|f(x_n)-f(y_n)|\geq\varepsilon}$

(x_n) has a convergent subsequence $x_{n_k}\to\alpha\in[a,b]$.

$$|y_{n_k}-\alpha|\leq\underbrace{|y_{n_k}-x_{n_k}|}_{\to 0}+\underbrace{|x_{n_k}-\alpha|}_{\to 0}\Rightarrow y_{n_k}\to\alpha.$$

Then

$$\lim_{k\to\infty}\overbrace{\Big|\underbrace{f(x_{n_k})}_{\to f(\alpha)}-\underbrace{f(y_{n_k})}_{\to f(\alpha)}\Big|}^{\geq\varepsilon}=0. \quad \text{Contradiction.}$$

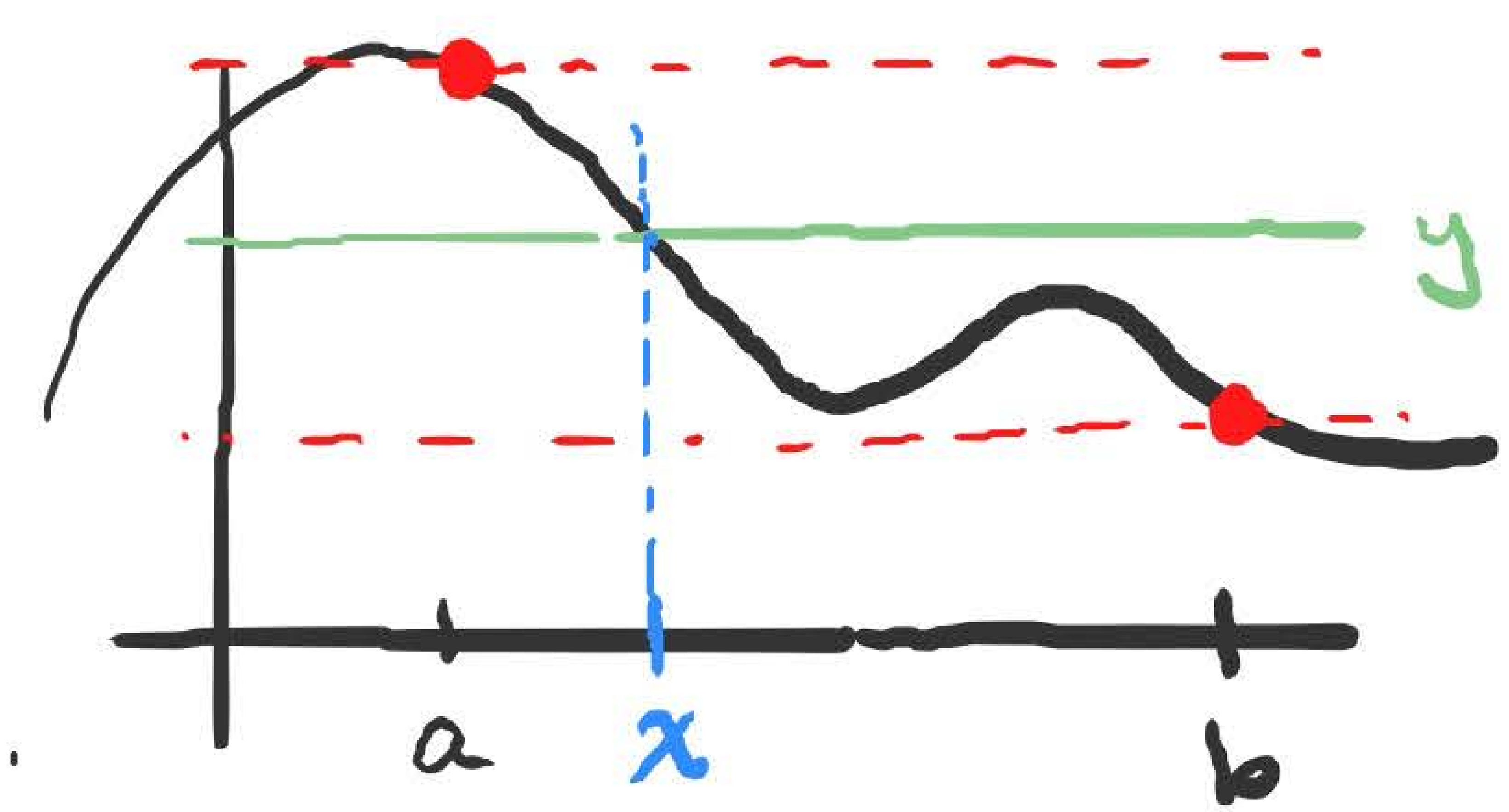
$f'(x)=\frac{1}{2\sqrt{x}}$.

$\boxed{\text{Consider } f(x)=\sqrt{x} \text{ on } [0,1]}$

Recall: intermediate value theorem:

f cont on $[a,b] \Rightarrow$ for any y (strictly) between $f(a)$ and $f(b)$, there exists $x \in (a,b)$: $f(x) = y$.

Corollary (fixed point theorem)



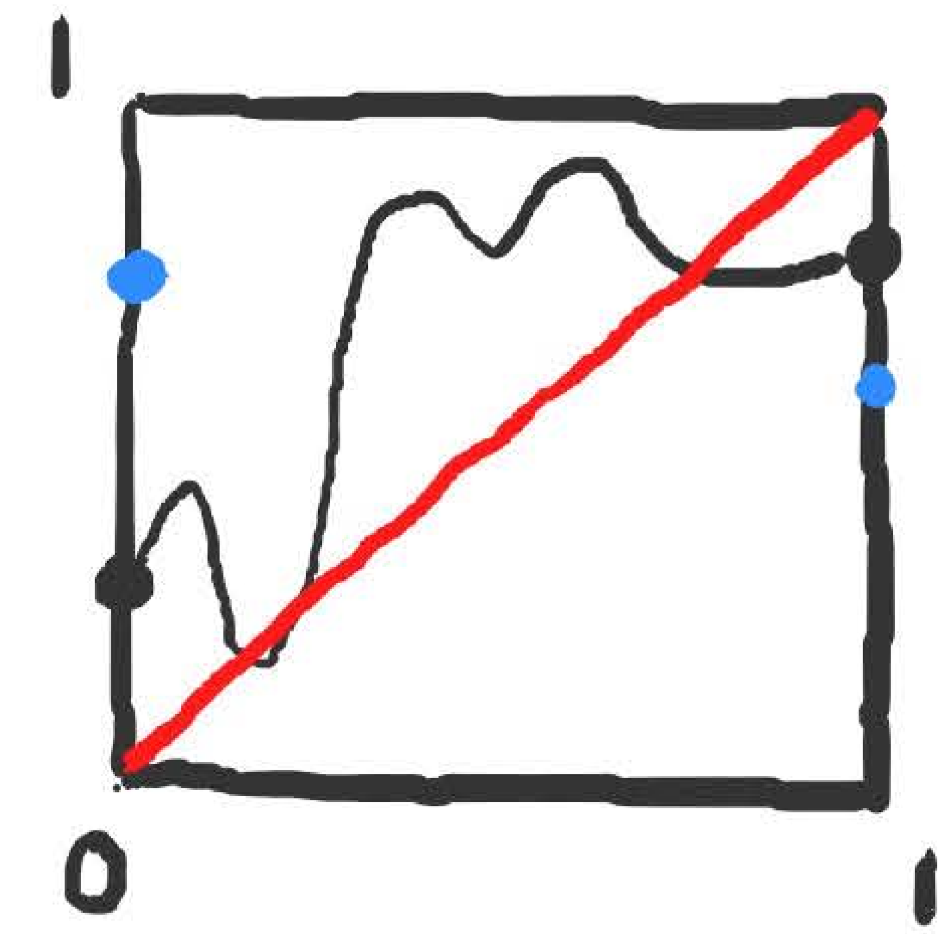
If $f: [0,1] \to [0,1]$ is continuous, then f has a fixed point, i.e. there exists $x \in [0,1]$ such that $f(x) = x$.

Proof: Define $g(x) = f(x) - x$. g is continuous.

Case 1: $f(0) = 0$ or $f(1) = 1$. Done.

Case 2: $f(0) \neq 0$, $f(1) \neq 1$. $g(0) > 0$ and $g(1) < 0$.

0 is between $g(0)$ and $g(1)$. By IVT, there exists $x \in (0,1)$ such that $g(x) = 0 \Rightarrow f(x) = x$.



<u>Lemma</u>: If f is uniformly continuous on $S \subseteq \mathbb{R}$ and (s_n) is a Cauchy sequence in S, then $(f(s_n))$ is Cauchy.

(unif. cont. functions preserve Cauchy property of sequences)

<u>Proof</u>: Let $\varepsilon > 0$. There exists $\delta > 0$ such that

$$x, y \in S,\ |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon. \quad (*)$$

Since (s_n) is Cauchy, there exists $N \in \mathbb{N}$ such that

$$m, n \geq N \text{ implies } |s_m - s_n| < \delta \underset{(*)}{\Rightarrow} |f(s_m) - f(s_n)| < \varepsilon.$$

<u>Def</u>: Given a function f, we say that $\tilde{f}$ is an <u>extension</u> of f if $\mathrm{dom}(f) \subseteq \mathrm{dom}(\tilde{f})$ and $\tilde{f}(x) = f(x)$ for all $x \in \mathrm{dom}(f)$.

i.e. $\tilde{f}\big|_{\mathrm{dom}(f)} = f$.

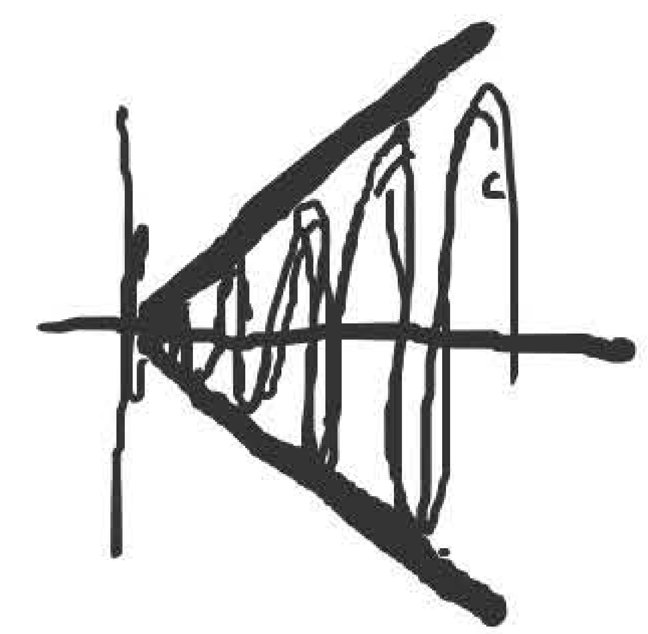
restriction.

<u>Ex</u>. $f(x) = x \sin\left(\frac{1}{x}\right)$ on $(0,1]$.

Consider
$$\tilde{f}(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \in (0,1] \\ 0 & x = 0. \end{cases}$$

$-|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|$.

$$\lim_{x \to 0^+} x \sin\left(\tfrac{1}{x}\right) = 0.$$

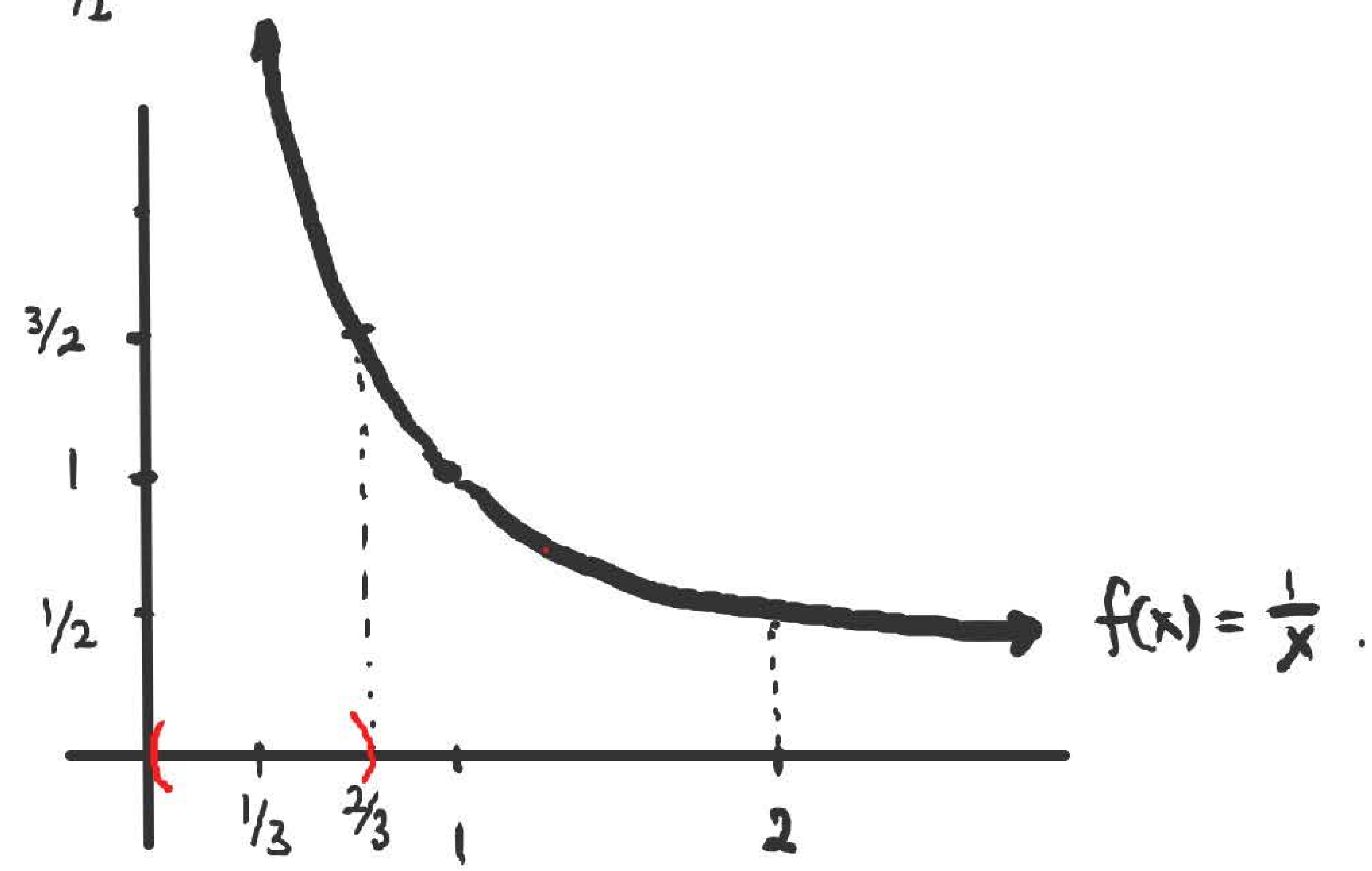


$\tilde{f}$ is a <u>continuous</u> extension.

$\tilde{f}$ is uniformly continuous.

$\Rightarrow f$ is uniformly continuous.

Ex Look at $f(x) = \frac{1}{x}$ on $(0, \infty)$



Let $\varepsilon = \frac{1}{2}$.

Let $x = 1$. $\delta = \frac{1}{3}$ works.

But for $x = \frac{1}{3}$? $\delta = \frac{1}{3}$ doesn't work!

No δ works for every x value!

Notion of <u>uniform continuity</u>.

regular continuity: for every $x \in \mathrm{dom}(f)$, for every $\varepsilon > 0$, there exists $\delta > 0$ (depends on x) such that $y \in \mathrm{dom}(f)$, $|y - x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon$.

<u>uniform continuity</u>: for every $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in \mathrm{dom}(f)$, $|y - x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon$.

Remark: uniform continuity implies regular continuity.

Claim: $f(x) = \frac{1}{x}$ is NOT uniformly continuous on $(0,\infty)$.

Proof: Let $\varepsilon = 1$. Let $\delta > 0$. Set $x = \delta$. $f(x) = \frac{1}{\delta}$.

We can find $y \in (0, 2\delta)$ very close to 0 so that $f(y)$ will be very large. To be precise, consider $0 < y = \frac{\delta}{1+\delta} < x$. Then $|y-x| < \delta$, but $|f(y) - f(x)| = 1$ (not $< \varepsilon$).

Claim: $f(x) = \frac{1}{x}$ is uniformly continuous on $[a,\infty)$ for any $a > 0$.

Let $\varepsilon > 0$. Set $\delta = \varepsilon \cdot a^2$.

$$|x-y| < \delta \implies |f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y-x}{xy}\right| \leq \frac{\overbrace{|y-x|}^{< \varepsilon a^2}}{a^2} < \varepsilon .$$

Exercise: a) Show that $f(x) = x^2$ is NOT uniformly continuous on $\mathbb{R}$.

b) Show that $f(x) = x^2$ is uniformly continuous on any closed interval $[a,b]$.

Preimages of functions.

$f : A \to B$.

$S \subseteq B$.

$$f^{-1}(S) = \{ x \in A : f(x) \in S \}.$$

preimage of S

Ex. $f(x) = x^2$ on $\mathbb{R}$.

$$f^{-1}([1,\infty)) = (-\infty,-1] \cup [1,\infty)$$

$$f^{-1}(\{0\}) = \{0\}.$$

$$f^{-1}((-\infty,0)) = \emptyset.$$

Math 104 Worksheet 12

UC Berkeley, Summer 2021
Thursday, July 22

Let X and Y be two sets, and let $f : X \to Y$, let $E \subseteq X$, and let $A, B \subseteq Y$.

Exercise 1. Prove the following assertions.

(a) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

Proof. $x \in f^{-1}(A \cap B) \iff f(x) \in A \cap B \iff f(x) \in A$ and $f(x) \in B \iff$ $x \in f^{-1}(A)$ and $x \in f^{-1}(B) \iff x \in f^{-1}(A) \cap f^{-1}(B)$.

subset of codomain

$$f^{-1}(A) = \{x \in X : f(x) \in A\}.$$

preimage of A

(b) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

subset of domain

$$f(E) = \{f(x) : x \in E\}.$$

image of E

(c) $f^{-1}(A^c) = \left(f^{-1}(A)\right)^c$.

(d) $f^{-1}(A) \subseteq f^{-1}(B)$ if $A \subseteq B$.

(e) $E \subseteq f^{-1}\big(f(E)\big)$

(f) Find a counterexample to show that the statement $E = f^{-1}\big(f(E)\big)$ is not always true.

Theorem (continuous extension theorem)

A real-valued function f on (a,b) is uniformly continuous on (a,b) if and only if it can be extended to a continuous function on $[a,b]$.

Proof: $\Leftarrow$] $\tilde{f}$ cont on $[a,b] \Rightarrow \tilde{f}$ unif cont on $[a,b] \Rightarrow \tilde{f}$ unif cont on (a,b) $\Rightarrow f$ unif cont on (a,b).

$\Rightarrow$] Suppose f is uniformly continuous on (a,b).

We want to define values for $\tilde{f}(a)$ and $\tilde{f}(b)$.

Let $(s_n) \subseteq (a,b)$, $s_n \to a \Rightarrow (s_n)$ is Cauchy $\Rightarrow (f(s_n))$ is Cauchy.

$\Rightarrow (f(s_n))$ converges, $f(s_n) \longrightarrow \alpha$.

Set $\tilde{f}(a) = \alpha$. Need to prove that $\tilde{f}$ is continuous at a.

It suffices to show that for any $(t_n) \subseteq (a,b)$ such that $t_n \to a$, we have $f(t_n) \to \alpha$. Consider

$$(u_n) = (s_1, t_1, s_2, t_2, s_3, t_3, \ldots) \longrightarrow a.$$

(u_n) is Cauchy $\Rightarrow (f(u_n))$ converges $\Rightarrow f(u_n) \to \alpha \Rightarrow f(t_n) \to \alpha$.

$(f(s_n))$ is a subseq. of $(f(u_n))$ which conv. to α.

Do the same for $\tilde{f}(b)$.