Math 104 Worksheet 5

UC Berkeley, Summer 2021

Wednesday, June 30

The aim of this worksheet is to prove the equivalence of two definitions of $\limsup$. (The analogous definitions for $\liminf$ will also be equivalent, with nearly an identical proof.)

Definition 1. Given a sequence (s_n) of real numbers, we define

$$\limsup s_n := \begin{cases} \lim_{n\to\infty}\Big(\sup\{s_m : m \geq n\}\Big) & \text{if } (s_n) \text{ is bounded from above,} \\ +\infty & \text{if } (s_n) \text{ is not bounded from above.} \end{cases}$$

Definition 2. Given a sequence (s_n) of real numbers, let $L \subseteq \mathbb{R} \cup \{\pm\infty\}$ denote the set of *subsequential limits* of (s_n). Define $\limsup s_n := \sup L$.

Theorem. The two definitions of $\limsup$ above are equivalent.

Proof. Let (s_n) be a sequence of real numbers.

Case 1. (s_n) is NOT bounded from above. Then according to Definition 1, $\limsup s_n = \infty$. On the other hand, since (s_n) is not bounded from above, it should be possible to construct a subsequence (s_{n_k}) of (s_n) such that $\lim_{k\to\infty} s_{n_k} = \infty$.

• <u>Exercise 1</u>. Inductively construct a subsequence (s_{n_k}) of (s_n) such that $\lim_{k\to\infty} s_{n_k} = \infty$.

Therefore, $\infty \in L$ and hence $\limsup s_n = \sup L = \infty$.

Case 2. (s_n) is bounded from above. Then the goal is to show that

$$\sup L = \lim_{n\to\infty}\Big(\sup\{s_m : m \geq n\}\Big) \qquad \Big(= \lim_{n\to\infty} v_n\Big).$$

To do this, we will prove inequality in both directions.

• <u>Exercise 2</u>. To show that $\sup L \boxed{\leq} \lim v_n$, it suffices to show that for every subsequence (s_{n_k}) of (s_n) such that $\lim_{k\to\infty} s_{n_k}$ exists, we have the inequality $\lim_{k\to\infty} s_{n_k} \leq \lim v_n$ (why?). Now prove the assertion.

• <u>Exercise 3</u>. To show that $\sup L \boxed{\geq} \lim v_n$, it suffices to show that there exists a subsequence (s_{n_k}) of (s_n) such that $\lim_{k\to\infty} s_{n_k} = \lim v_n$ (why?). Now inductively construct such a subsequence.

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