Math 104 Worksheet 2

UC Berkeley, Summer 2021
Tuesday, June 22

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1. The following theorem is a fundamental idea in real analysis, and it is one of the most important techniques in the subject.
Theorem.
(a) If $a \leq b + \varepsilon$ for any $\varepsilon > 0$, then $a \leq b$.
(b) If $a \geq b - \varepsilon$ for any $\varepsilon > 0$, then $a \geq b$.

Proof. (a) Suppose that $a > b$. Let $\varepsilon = (a - b)/2 > 0$. Then $a > b + \varepsilon$, so the statement that $a \leq b + \varepsilon$ for any $\varepsilon > 0$ is not true.
(b) (Your turn)

$\square$

2. Given two nonempty bounded subsets A and B of $\mathbb{R}$, define the set

$$A + B = \{a + b : a \in A, b \in B\}.$$

Theorem.
(a) $\sup(A + B) = \sup(A) + \sup(B)$.
(b) $\inf(A + B) = \inf(A) + \inf(B)$.

Proof. Given two quantities x and y, if you are asked to show that $x = y$, a common technique is to show that $x \leq y$ **and** $x \geq y$, since if both are true then $x = y$.

(a) *Strategy*: We will show that both inequalities (i) $\sup(A + B) \leq \sup(A) + \sup(B)$ **and** (ii) $\sup(A + B) \geq \sup(A) + \sup(B)$ are true.

(i) For any pair of elements $a \in A$ and $b \in B$, since the supremum of a set is **an** upper bound for the set, we have that $a \leq \sup(A)$ and $b \leq \sup(B)$. Therefore, $a + b \leq \sup(A) + \sup(B)$. Since this is true for any $a \in A$ and $b \in B$, it follows that $c \leq \sup(A) + \sup(B)$ for all $c \in A + B$. That means that $\sup(A) + \sup(B)$ is an upper bound for $A + B$. (**Complete the proof by explaining why** $\sup(A + B)$ **must be less than** $\sup(A) + \sup(B)$.)

(ii) To show that $\sup(A + B) \geq \sup(A) + \sup(B)$, we will use the technique from Problem 1. Let $\varepsilon > 0$. The goal is to show that $\boxed{\sup(A + B) \geq \sup(A) + \sup(B) - \varepsilon}$. If we can find $a \in A$ and $b \in B$ such that $a + b \geq \sup(A) + \sup(B) - \varepsilon$, the boxed inequality would follow because… (why?)
Now explain why it is possible to find such a and b. (*Hint*: $\sup(A) - \frac{\varepsilon}{2}$ is **not** an upper bound for A.)

(b) (Your turn) $\square$