Monday, July 12

- Midterm this Thursday
  - regular exam: 4:10 – 6:00 PM (until 6:10 PM to submit)
  - alternate exam: 12:10 – 2:00 AM (until 2:10 AM to submit)

    Must post reply to Piazza thread to sign up.
  - mock email to be sent Tuesday.

Recall E is compact if every open cover of E has a finite subcover.

Math 104 Worksheet 7

UC Berkeley, Summer 2021

Thursday, July 8

Let (X, d) be a metric space.

Theorem. Closed subsets of compact sets are compact.

Proof. Let E be a compact set, and let $F \subseteq E$ be a closed set. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of F. Goal: Show that there exists a finite subcover of $\{U_\alpha\}_{\alpha\in A}$ of F.
(*Hint*: Expand $\{U_\alpha\}_{\alpha \in A}$ to an open cover of E. Note that F is closed.) F^c is open.
Exercise 1. Complete the proof.

Consider $\{U_\alpha\}_{\alpha\in A} \cup \{F^c\}$; this is an open cover of E. Since E is compact, this open cover has a finite subcover $U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}$, maybe F^c. Then $U_{\alpha_1}, \ldots, U_{\alpha_n}$ is a finite subcover of F.

$F \subseteq \bigcup_{\alpha\in A} U_\alpha$

$F^c \subseteq F^c$

$\underbrace{F \cup F^c}_{=X} \subseteq \underbrace{\bigcup_{\alpha\in A} U_\alpha \cup F^c}_{=X}$

$=$

Theorem. Every sequence in a compact set has a convergent subsequence.

Proof. Let E be a compact set, and let (x_n) be a sequence of points in E. Consider the set $S = \{x_n : n \in \mathbb{N}\}$.
Exercise 2. Explain why if S is finite, then (x_n) has a convergent subsequence.

Then there exists some value which is repeated infinitely many times $\longrightarrow$ take constant subsequence equal to that value.

If S is infinite, then it suffices to show that S has a limit point in E. Why?

E

Can inductively construct a subsequence of (x_n) which converges to this limit.

Suppose (for contradiction) that no point in E is a limit point of S.
Exercise 3. Construct an open cover of E which has no finite subcover. This would imply that E is not compact, which is a contradiction.

For every $x \in E$, since x is not a limit point of S, there exists $r_x > 0$ such that $(B_{r_x}(x) \setminus \{x\}) \cap S = \emptyset$.
$\{B_{r_x}(x)\}_{x\in E}$ is an open cover of E. $|B_{r_x}(x) \cap S| \leq 1$.
Any finite subcollection $B_{r_1}(x_1), \ldots, B_{r_n}(x_n)$ cover at most n elements of S.

$\{B_{r_x}(x)\}_{x\in E}$ has no finite subcover. Contradiction.

Corollary: If $\{K_\alpha\}_{\alpha \in A}$ is a collection of compact sets, then $\bigcap_{\alpha \in A} K_\alpha$ is compact.

Recall: Compact sets are closed.

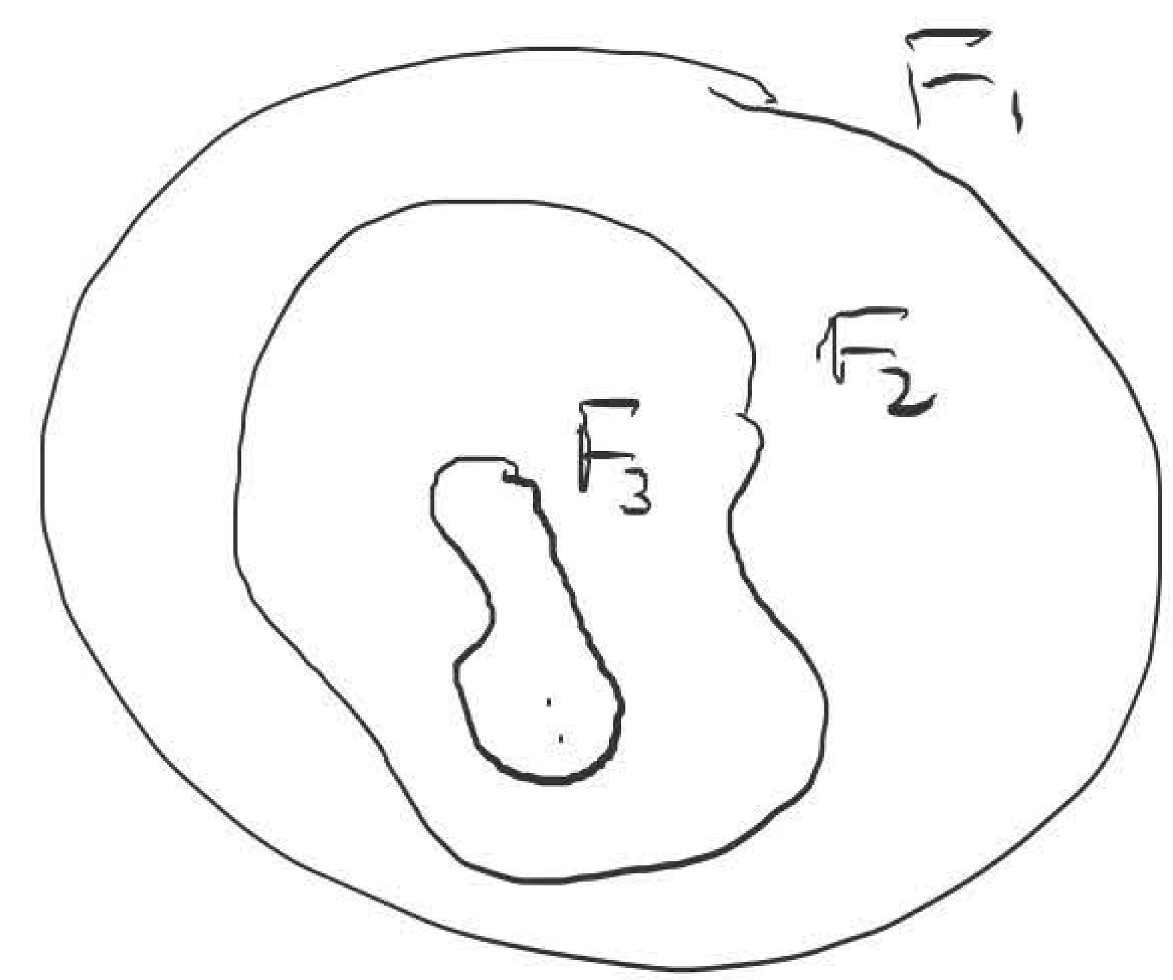
Ex. Infinite discrete metric space.

$\{\{x\}\}_{x \in X}$ is an infinite collection of compact sets. Their union is X. (why is X not compact?)

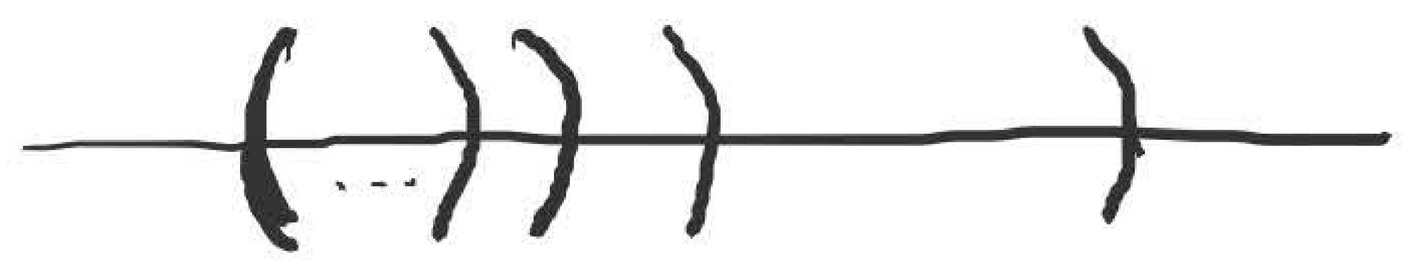
is an open cover of X with no finite subcover.

Theorem: Let (F_n) be a sequence of closed, bounded, nonempty sets in $\mathbb{R}^k$ such that $F_1 \supseteq F_2 \supseteq \dots$.

Then $F = \bigcap_{n=1}^{\infty} F_n \neq \emptyset$.



Example in $\mathbb{R}$: $\bigcap_{n=1}^{\infty} \underbrace{\left(0, \frac{1}{n}\right)}_{\text{not compact}} = \emptyset$.



Proof: For each n, let $x_n \in F_n$. Since $(x_n) \subseteq \overset{\text{bounded}}{F_1}$, it has a convergent subsequence $(x_{n_k}) \longrightarrow x$.

Claim: $x \in \bigcap_{n=1}^{\infty} F_n$. (Strategy: Show that $x \in F_N$ for each N).

Let $N \in \mathbb{N}$. Let $K \in \mathbb{N}$ such that $n_K \geq N$.

Then for $k \geq K$, $x_{n_k} \in \underbrace{F_{n_k} \subseteq F_N}_{\text{since } n_k \geq N}$.

$(x_{n_1}, \dots, \underbrace{x_{n_K}, x_{n_{K+1}}, \dots}_{\in F_N \leftarrow \text{closed}}) \Rightarrow x \in F_N$.

<u>Theorem</u>: Let (X,d) be a metric space.

Suppose $\{E_\alpha\}_{\alpha\in A}$ is a collection of compact sets such that $\bigcap_{\alpha\in B} E_\alpha \neq \emptyset$ for any finite $B \subseteq A$.

Then $\bigcap_{\alpha\in A} E_\alpha \neq \emptyset$.

<u>Proof</u>: Fix $E^* \subseteq \{E_\alpha\}_{\alpha\in A}$. (proof by contradiction).

Suppose no point in E^* is in every E_α.

Let $U_\alpha = E_\alpha^c$ ← open sets.

$\{U_\alpha\}_{\alpha\in A}$ is an open cover of E^*.

Since E^* is compact, there is a finite subcover $U_{\alpha_1}, \dots, U_{\alpha_n}$ of E^*.

for every point $x \in E^*$, there exists $\alpha\in A$ such that $x \notin E_\alpha$.

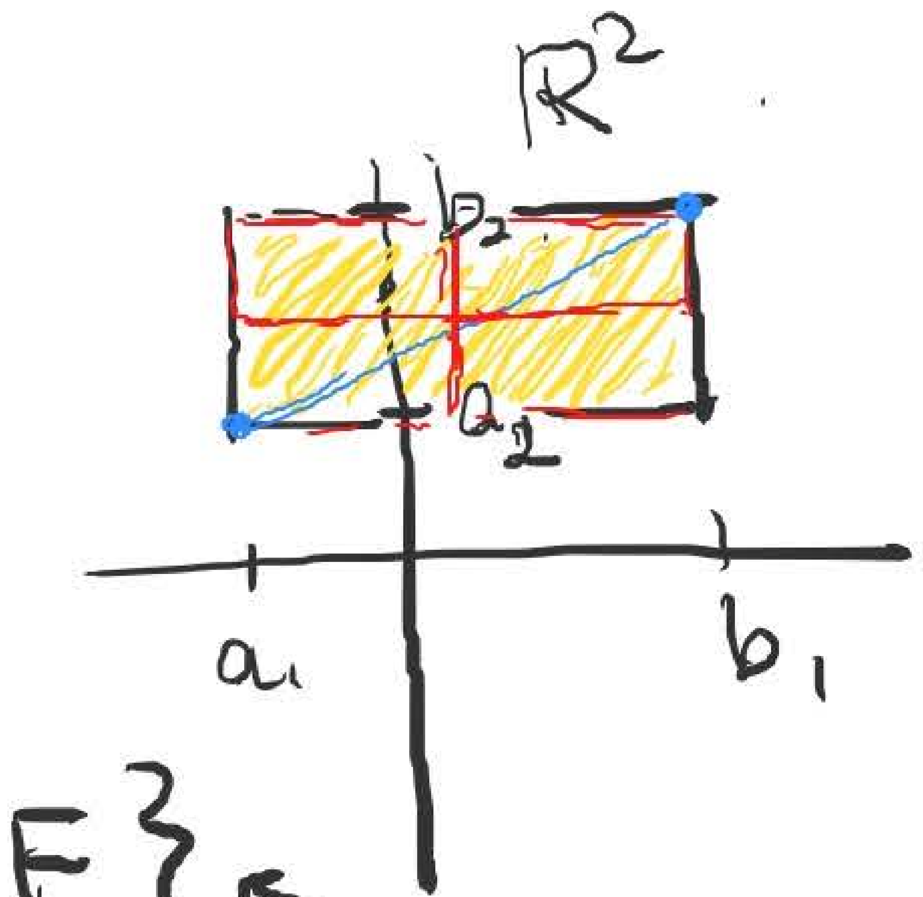
so $x \in E_\alpha^c = U_\alpha$.

$$E^* \subseteq \bigcup_{i=1}^n U_{\alpha_i} = \bigcup_{i=1}^n E_{\alpha_i}^c = \left(\bigcap_{i=1}^n E_{\alpha_i}\right)^c \Rightarrow \overbrace{E^* \cap \bigcap_{i=1}^n E_{\alpha_i}}^{E^*\cap E_{\alpha_1}\cap \dots \cap E_{\alpha_n}} = \emptyset.$$

Contradiction.

Def: A **k-cell** is a subset of $\mathbb{R}^k$ of the form

$$[a_1, b_1] \times [a_2, b_2] \times \dots \times [a_k, b_k]$$



Theorem: Every k-cell in $\mathbb{R}^k$ is compact.

Proof: Let F be a k-cell.

"diameter"

$F = [a_1, b_1] \times \dots \times [a_k, b_k]$. Let $\delta = \sup\{d(x,y) : x, y \in F\}$ $=: \mathrm{diam}(F)$.

Suppose F is not compact. $\qquad = \sqrt{\sum_{i=1}^{k} (b_i - a_i)^2}$.

So there exists an open cover of F $\{U_\alpha\}_{\alpha \in A}$ which has no finite subcover.

$\Rightarrow$ There exists $F_1 \subseteq F$, $\mathrm{diam}(F_1) = \frac{1}{2}\delta$, such that no finite subcollection of $\{U_\alpha\}_{\alpha \in A}$ covers F_1.

(see picture — if for each sub-region, there is a finite subcollection of $\{U_\alpha\}_{\alpha \in A}$ which cover it, then putting them together gives a finite subcover of F).

Having found $F_1, \dots, F_{k-1}$, find $F_k \subseteq F_{k-1}$ such that ... $\mathrm{diam}(F_k) = \frac{1}{2^k}\delta$.

$F_1 \supseteq F_2 \supseteq \dots$ closed, nonempty $\Rightarrow \bigcap_{n=1}^{\infty} F_n \neq \emptyset$. Let $x \in \bigcap_{n=1}^{\infty} F_n \subseteq F$.

$\{U_\alpha\}_{\alpha\in A}$ is an open cover of F,

there exists U_β such that $x \in U_\beta$.

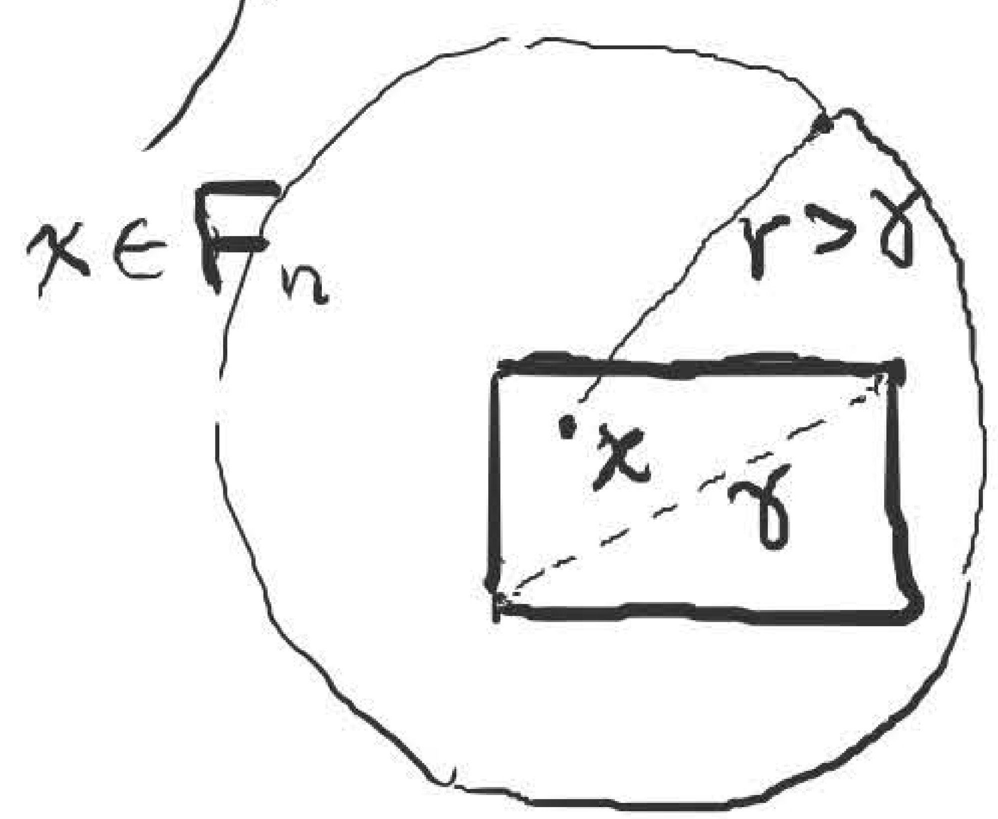
Since U_β is open, x is an interior point of U_β,

so there exists $r>0$ such that $B_r(x) \subseteq U_\beta$.

But since there exists $n \in \mathbb{N}$ such

that $\frac{1}{2^n}\delta < r$,

$$F_n \subseteq B_r(x) \subseteq U_\beta. \quad \text{Contradiction.}$$



<u>Theorem</u>: In $\mathbb{R}^k$, a set is compact if and only if it is closed and bounded.