MATH 104 Notes

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Contents

Chapter 1

Introduction

1.1 The Set $\mathbb{N}$ of Natural Numbers

We denote the set $\{1, 2, 3, \dots\}$ of all *positive integers* by $\mathbb{N}$. Each positive integer n has a successor, namely $n + 1$. The following is 5 properties of $\mathbb{N}$:

N1. 1 belongs to $\mathbb{N}$.

N2. If $n \in \mathbb{N}$, then its successor $n + 1 \in \mathbb{N}$.

N3. 1 is not the successor of any element in $\mathbb{N}$.

N4. If n and m in $\mathbb{N}$ have the same successor, then $n = m$.

N5. A subset of $\mathbb{N}$ which contains 1, and which contains $n+1$ whenever it contains n, must equal $\mathbb{N}$.

Axiom **N5** is the basis of mathematical induction, which asserts all the statements $P_1, P_2, P_3, \dots$ are true provided

(I_1) P_1 is true,

(I_2) P_{n+1} is true whenever P_n is true.

1.2 The Set $\mathbb{Q}$ of Rational Numbers

Definition 1.2.1. A number is called an *algebraic number* if it satisfies a polynomial equation

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0$$

where the coefficients $c_0, c_1, \ldots, c_n$ are integers, $c_n \neq 0$ and $n \geq 1$.

Rational numbers are always algebraic numbers. If $r = \frac{m}{n}$ is a rational number [$m, n \in \mathbb{Z}$ and $n \neq 0$], then it satisfies the equation $nx - m = 0$.

Theorem 1.2.2 (Rational Zeros Theorem)**.** *Suppose $c_0, c_1, \ldots, c_n$ are integers and r is a rational number satisfying the polynomial equation*

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0 \tag{1}$$

where $n \geq 1$, $c_n \neq 0$ and $c_0 \neq 0$. Let $r = \frac{c}{d}$ where c, d are integers having no common factors and $d \neq 0$. Then $c \,|\, c_0$ and $d \,|\, c_n$.

In other words, the only rational candidates for solutions of (1) have the form $\frac{c}{d}$ where c divides c_0 and d divides c_n.

Proof. We are given

$$c_n \left(\frac{c}{d}\right)^n + c_{n-1} \left(\frac{c}{d}\right)^{n-1} + \cdots + c_1 \left(\frac{c}{d}\right) + c_0 = 0$$

Multiply both sides by d^n and obtain

$$c_n c^n + c_{n-1} c^{n-1} d + c_{n-2} c^{n-2} d^2 + \cdots + c_2 c^2 d^{n-2} + c_1 c d^{n-1} + c_0 d^n = 0$$

Solve for $c_0 d^n$ and obtain

$$c_0 d^n = -c[c_n c^{n-1} + c_{n-1} c^{n-2} d + \cdots + c_2 c d^{n-2} + c_1 d^{n-1}]$$

Since c and d^n have no common factors, c divides c_0. Do the same thing and solve for $c_n c^n$ and we will see d divides c_n. ■

Corollary 1.2.2.1. *Consider the polynomial equation*

$$x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0$$

where the coefficients $c_0, c_1, \ldots, c_{n-1}$ are integers and $c_0 \neq 0$. Any rational solution of this equation must be an integer that divides c_0.

Proof. By the Rational Zeros Theorem 1.2.2, the denominator of r must divide the coefficient of x^n, which is 1. Thus r is an integer dividing c_0. ■

1.3 The Set $\mathbb{R}$ of Real Numbers

The set $\mathbb{Q}$ of Rational numbers also have the following properties for addition and multiplication:

A1. $a + (b + c) = (a + b) + c$ for all a, b, c.

A2. $a + b = b + a$ for all a, b.

A3. $a + 0 = a$ for all a.

A4. For each a, there is an element $-a$ such that $a + (-a) = 0$.

M1. $a(bc) = (ab)c$ for all a, b, c.

M2. $ab = ba$ for all a, b.

M3. $a \cdot 1 = a$ for all a.

M4. For each $a \neq 0$, there is an element a^{-1} such that $aa^{-1} = 1$.

DL $a(b + c) = ab + ac$ for all a, b, c.

The set $\mathbb{Q}$ also has an order structure $\leq$ satisfying

O1. Given a and b, either $a \leq b$ or $b \leq a$.

O2. If $a \leq b$ and $b \leq a$, then $a = b$.

O3. If $a \leq b$ and $b \leq c$, then $a \leq c$.

O4. If $a \leq b$, then $a + c \leq b + c$.

O5. If $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

Theorem 1.3.1. *The following are consequences of the field properties:*

(i) $a + c = b + c \implies a = b$;

(ii) $a \cdot 0 = 0$ *for all* a;

(iii) $(-a)b = -ab$ *for all* a, b;

(iv) $(-a)(-b) = ab$ *for all* a, b;

(v) $(ac = bc) \wedge (c \neq 0) \implies a = b$;

(vi) $ab = 0 \implies (a = 0) \vee (b = 0)$ *for* $a, b, c \in \mathbb{R}$.

for $a, c, c \in \mathbb{R}$.

Theorem 1.3.2. *The following are consequences of the properties of an ordered field:*

(i) $a \leq b \implies -b \leq -a$*;*

(ii) $(a \leq b) \wedge (c \leq 0) \implies bc \leq ac$*;*

(iii) $(0 \leq a) \wedge (0 \leq b) \implies 0 \leq ab$*;*

(iv) $0 \leq a^2$ *for all* a*;*

(v) $0 < 1$*;*

(vi) $0 < a \implies 0 < a^{-1}$*;*

(vii) $0 < a < b \implies 0 < b^{-1} < a^{-1}$*;*

for $a, c, c \in \mathbb{R}$*.*

Note that $a < b$ can be represented as $(a \leq b) \wedge (a < b)$.

Definition 1.3.3. We define

$$|a| = a \text{ if } a \geq 0 \text{ and } |a| = -a \text{ if } a \leq 0$$

An useful fact: $|a| \leq b \iff -b \leq a \leq b$.

Definition 1.3.4. For numbers a and b we define $\mathrm{dist}(a, b) = |a - b|$; $\mathrm{dist}(a, b)$ represents the *distance between* a *and* b.

Theorem 1.3.5.

(i) $|a| \geq 0$ *for all* $a \in \mathbb{R}$*.*

(ii) $|ab| = |a| \cdot |b|$ *for all* $a, b \in \mathbb{R}$*.*

(iii) $|a + b| \leq |a| + |b|$ *for all* $a, b \in \mathbb{R}$*.*

Corollary 1.3.5.1. $\mathit{dist}(a, c) \leq \mathit{dist}(a, b) + \mathit{dist}(b, c)$ *for all* $a, b, c \in \mathbb{R}$*. This is equivalent to* $|a - c| \leq |b - c| + |b - c|$*.*

Theorem 1.3.6 (Triangle Inequality)**.** $|a + b| \leq |a| + |b|$ *for all* a, b*.*

Corollary 1.3.6.1 (Reverse Triangular Inequality). $\big||a|-|b|\big| \leq |a-b|$ *for all* $a, b \in \mathbb{R}$.

Here is one of the most important techniques in real analysis.

(a) If $a \leq b + \epsilon$ for any $\epsilon > 0$, then $a \leq b$.

(b) If $a \geq b - \epsilon$ for any $\epsilon > 0$, then $a \geq b$.

(c) If $|a - b| < \epsilon$ for any $\epsilon > 0$, then $|a - b| = 0$.

Proof. The proof for two cases is similar, so I will only show (a) here. Suppose that $a > b$. Let $\epsilon = (a - b)/2 > 0$. Then $a > b + \epsilon$, so the statement that $a \leq b + \epsilon$ for any $\epsilon > 0$ is not true. ■

1.4 The Completeness Axiom

The completeness axiom for $\mathbb{R}$ ensure us $\mathbb{R}$ has no "gaps".

Definition 1.4.1. Let S be a nonempty subset of $\mathbb{R}$.

(a) If S contains a largest element s_0 [that is, $s_0 \in S$ and $\forall s \in S,\ s \leq s_0$], then we call s_0 the *maximum* of S and write $s_0 = \max S$.

(b) If S contains a smallest element s_0 [that is, $s_0 \in S$ and $\forall s \in S,\ s \geq s_0$], then we call s_0 the *minimum* of S and write $s_0 = \min S$.

Open intervals like $(a, b) = \{x \in \mathbb{R} : a < x \leq b\}$ have no minimum or maximum since the endpoints a and b is not in the interval.

Definition 1.4.2. Let S be a nonempty subset of $\mathbb{R}$.

(a) If a real number M satisfies $s \leq M$ for all $s \in S$, then M is called an *upper bound* of S and the set S is said to be *bounded above*.

(b) If a real number m satisfies $m \leq s$ for all $s \in S$, then m is called an *lower bound* of S and the set S is said to be *bounded below*.

(c) The set S is said to be *bounded* if it is bounded above and bounded below. Thus S is bounded if there exist real numbers m and M such that $S \subseteq [m, M]$.

The maximum of a set is always an upper bound for the set. Likewise, the minimum of a set is always a lower bound for the set.

Definition 1.4.3. Least Upper Bound Property (LUBP)
An ordered set S has the LUBP if every nonempty subset $\mathcal{A} \subset S$ that has an upper bound has a least upper bound in S.

Note that the set $\mathbb{Q}$ of rational number does not satisfy the LUBP but $\mathbb{R}$ does. e.g. $(A) = \{q \in \mathbb{Q} : q^2 < 2\}$.

Definition 1.4.4. Let S be a nonempty subset of $\mathbb{R}$.

(a) If S is bounded above and S has a least upper bound, then we will call it the *supremum* of S and denote it by $\sup S$.

(b) If S is bounded below and S has a greatest lower bound, then we will call it the *infimum* of S and denote it by $\inf S$.

If S is bounded above, then $M = \sup S$ if and only if (i) $s \leq M$ for all $s \in S$, and (ii) whenever $M_1 < M$, there exists $s_1 \in S$ such that $s_1 > M_1$. Or for each $\epsilon > 0$, there exists

$s \in S$ such that $s > \sup S - \epsilon$.

Note that for a positive set $S = \{s : s > 0\}$, its infimum is not always positive. Example: $\{\frac{1}{n} : n \in \mathbb{N}\}$. Each element is positive but the infimum is 0.

Here are some basic facts:

- If a set S has finitely many elements, then $\max S$ exists.
- If $\max S$ exists, then $\sup S = \max S$.
- For any set $S \neq \varnothing$, $\inf S \leq \sup S$

Theorem 1.4.5 (Completeness Axiom)**.** *Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. In other words, $\sup S$ exists and is a real number.*

Note that the completeness axiom does not hold for $\mathbb{Q}$.

Corollary 1.4.5.1. *Every nonempty subset S of $\mathbb{R}$ that is bounded below has a greatest lower bound. In other words, $\inf S$ exists and is a real number.*

Theorem 1.4.6 (Archimedean Property)**.** *If $a > 0$ and $b > 0$, then for some positive integer n, we have $na > b$.*

Corollary 1.4.6.1. *(Set $a = 1$). For any $b > 0$, there exists $n \in \mathbb{N}$ such that $n > b$*

Corollary 1.4.6.2. *(Set $b = 1$). For any $a > 0$, there exists $n \in \mathbb{N}$ such that $na > 1 \implies \frac{1}{n} < a$.*

Lemma 1.4.7. *If $x, y \in \mathbb{R}$ such that $y - x > 1$, then there exists $m \in \mathbb{Z}$ such that $x < m < y$.*

Proof.

Case 1: $x \geq 0$. Let $S = \{n \in \mathbb{Z}_+ \, n \leq x\}$. By the corollary of Archimedean property 1.4.6 (set $a = 1$), S has finitely many elements, so $k = \max S$ exists. Then we have

$$x < k + 1 \leq x + 1 < y$$

where $k + 1$ is an integer.

Case 2: $x < 0$. Then $-x > 0$. By the corollary of Archimedean property 1.4.6 (set $a = 1$), there exists $N \in \mathbb{N}$ such that $N > -x$. Consider $x + N > 0$ and $(y + N) - (x + N) > 1$. By Case 1, there exists $m \in \mathbb{Z}$ such that $x + N < m < y + N$. Then $x < m - N < y$ where $m - N$ is an integer.

■

Theorem 1.4.8 (Denseness of $\mathbb{Q}$)**.** *If* $a, b \in \mathbb{R}$ *and* $a < b$*, then there is a rational* $r \in \mathbb{Q}$ *such that* $a < r < b$*.*

Proof. By Archimean property 1.4.6, there exists $n \in \mathbb{N}$ such that $n(b-a) > 1$. i.e. $nb - na > 1$. By 1.4.7, there exists an integer $m \in \mathbb{Z}$ between na and nb. Thus $na < m < nb \implies a < \frac{m}{n} < b$. ■

1.5 The Symbols $+\infty$ and $-\infty$

The symbols $+\infty$ and $-\infty$ are extremely useful even though they are **not** real numbers. So for each real number a, $-\infty < a < \infty$. If a set S is not bounded above, we define $\sup S = +\infty$. Likewise, if S is not bounded below, then we define $\inf S = -\infty$.

We can extend real numbers to $\mathbb{R} \cup \{-\infty, \infty\}$. Notice that this is not a **field**, so it does not satisfy all field properties.

For emphasis, we recapitulate:

Let S be any nonempty subset of $\mathbb{R}$. The *symbols* $\sup S$ and $\inf S$ always make sense. If S is not bounded above, then $\sup S$ is a *real* number; otherwise $\sup S = +\infty$. If S is bounded below, then $\inf S$ is a *real* number; otherwise $\inf S = -\infty$. Moreover, we have $\inf S \leq \sup S$.

Chapter 2

Sequences

2.1 Limits of Sequences

A *sequence* is a function whose domain is $\{n \in \mathbf{Z} : n \geq m,\ m \text{ is usually } 1 \text{ or } 0\}$. We usually denote a sequence by s and its value at n by s_n. $(s_n)_{n=m}^{\infty} = (s_m, s_{m+1}, \ldots)$. $(s_n)_{n\in N}$ represents the sequence with $m = 1$.

Example 2.1.1.

- $(s_n)_{n\in N}$ where $s_n = \frac{1}{n^2}$ is the sequence $(1, \frac{1}{4}, \frac{1}{9}, \ldots)$
- $(a_n)_{n=0}^{\infty}$ where $a_n = (-1)^n$ is the sequence $(1, -1, 1, -1, 1, \ldots)$

The "limits" of a sequence is a real number that the values s_n are close to for large values of n.

Definition 2.1.2. A sequence (s_n) of real numbers is said to **converge** to the real number s provided that

$$\forall \epsilon > 0,\ \exists N,\ n > N \Rightarrow |s_n - s| < \epsilon.$$

If (s_n) converges to s, we write $\lim_{n\to\infty} s_n = s$ or $s_n \to s$. s is the *limit* of the sequence (s_n). A sequence that does not converge (i.e. it has no *limit*) is said to *diverge*. Notice that in the definition, instead of simple ϵ, we can also use some other complicated forms with some extra constants like $M\epsilon$, $\frac{\epsilon}{c}$, $a^2\epsilon$ and so on.

Intuitively, the definition means that no matter how small you pick $\epsilon > 0$, **eventually** the sequence will stay within ϵ of s at some point (the threshold N) and forever after.

Theorem 2.1.3. *The limit of a sequence (s_n) is unique. i.e.* $(\lim s_n = s) \wedge (lim s_n = t) \Rightarrow s = t$.

Proof. By the definition of limit, we have

$$n > N_1 \Rightarrow |s_n - s| < \frac{\epsilon}{2}$$
$$n > N_2 \Rightarrow |s_n - t| < \frac{\epsilon}{2}$$

For $n > \max\{N_1, N_2\}$, by Triangular Inequality **??**,

$$|s - t| = |(s - s_n) + (s_n - t)| \leq |s - s_n| + |s_n - t| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

This shows $\forall \epsilon > 0,\ |s - t| < \epsilon \Rightarrow |s - t| = 0 \Rightarrow s = t$ ∎

Theorem 2.1.4.

- *If $s_n \geq a$ for all but finitely many n, then $\lim s_n \geq a$.*
- *If $s_n \leq b$ for all but finitely many n, then $\lim s_n \leq b$.*

Theorem 2.1.5 (Squeeze Lemma)**.** *If $a_n \leq s_n \leq b_n$ for all n and $\lim a_n = \lim b_n = s$, then $\lim s_n = s$.*

2.2 A Discussion about Proofs

This section gives several examples of proofs with some discussion using the definition of the limit of a sequence.

Example 2.2.1. Prove $\lim \frac{1}{n^2} = 0$.

Discussion. According to the definition of the limit, we need to consider an $\epsilon > 0$ such that $|\frac{1}{n^2} - 0| < \epsilon$ for $n >$ some$N$. $|\frac{1}{n^2} - 0| < \epsilon$ implies that $\frac{1}{\epsilon} < n^2$or$\frac{1}{\sqrt{\epsilon}} < n$. Thus we can suppose $N = \frac{1}{\sqrt{\epsilon}}$ and check if we reverse our reasoning into proof, it still makes sense.

Proof. Let $\epsilon > 0$ and $N = \frac{1}{\sqrt{\epsilon}}$, then

$$\begin{aligned} n > N &\Rightarrow \epsilon > \frac{1}{n^2} \\ &\Rightarrow \frac{1}{n^2} - 0 < \epsilon - 0 \\ &\Rightarrow \left|\frac{1}{n^2} - 0\right| < \epsilon \end{aligned}$$

This proofs $\lim \frac{1}{n^2} = 0$ according to the definition of the limit 2.1.2. ■

Example 2.2.2. Prove $\lim \frac{3n+1}{7n-4} = \frac{3}{7}$

Discussion. Just like the last example, we can start from the definition 2.1.2 to get a suitable N.

Proof. Let $\epsilon > 0$ and $N = \frac{19}{49\epsilon} + \frac{4}{7}$, then

$$\begin{aligned} n > N &\Rightarrow 7n > \frac{19}{7\epsilon} + 4 \\ &\Rightarrow \frac{19}{7(7n-4)} < \epsilon \\ &\Rightarrow \frac{3n+1}{7n-4} - \frac{3}{7} < \epsilon \\ &\Rightarrow \left| \frac{3n+1}{7n-4} - \frac{3}{7} \right| < \epsilon \qquad \text{since } n > 0 \end{aligned}$$

This proofs $\lim \frac{3n+1}{7n-4} = \frac{3}{7}$ according to the definition of the limit 2.1.2. ∎

Example 2.2.3. Prove $\lim \frac{4n^3+3n}{n^3-6} = 4$

Discussion. Since $\frac{4n^3+3n}{n^3-6} - 4 = \frac{3n+24}{n^3-6}$, when $n > 1$, we can find an upper bound for $\frac{3n+24}{n^3-6}$ so that the bound $< \epsilon \Rightarrow \left|\frac{3n+24}{n^3-6}\right| < \epsilon$. Finding an upper bound for a fraction is equivalent to finding a upper bound for its numerator and a lower bound for its denominator. We know $3n + 24 \leq 27n$ for $n > 1$. Also we note $n^3 - 6 \geq \frac{n^3}{2} \Rightarrow n > 2$. Thus we can have $\frac{3n+24}{n^3-6} < \frac{27n}{n^3/2} < \epsilon \Rightarrow n > \sqrt{\frac{54}{\epsilon}}$, provided $n > 2$.

Proof. Let $\epsilon > 0$ and $N = \max\{2, \sqrt{\frac{54}{\epsilon}}\}$, then

$$\begin{aligned} n > N &\Rightarrow (n > \sqrt{\frac{54}{\epsilon}}) \wedge (n > 2) \\ &\Rightarrow (\frac{27n}{n^3/2} < \epsilon) \wedge (\frac{n^3}{2} \leq n^3 - 6) \wedge (27n \geq 3n + 24) \\ &\Rightarrow \frac{3n+24}{n^3-6} < \frac{27n}{n^3/2} < \epsilon \\ &\Rightarrow \left| \frac{4n^3+3n}{n^3-6} - 4 \right| < \epsilon \end{aligned}$$

This proofs $\lim \frac{4n^3+3n}{n^3-6} = 4$ according to the definition of the limit 2.1.2. ∎

Example 2.2.4. Show that $a_n = (-1)^n$ does not converge.

Discussion. Assume $\lim(-1)^n = a$, and we can see that no matter what a is, either 1 or -1 is at least 1 from a, so it means $|(-1)^n - a| < 1$ will not hold for all large n.

Proof. Suppose $\lim(-1)^n = a$ and $\epsilon = 1$. By 2.1.2, $|(-1)^n - a| < 1 \Rightarrow (|1 - a| < 1) \wedge (|-1 - a| < 1)$. Now by **??**, $2 = |1 - a + a - (-1)| \leq |1 - a| + |a - (-1)| < 1 + 1 = 2$, which is a contradiction. ∎

Example 2.2.5. Let (s_n) be a sequence of nonnegative real numbers and suppose $s = \lim s_n$. Note $s \geq 0$. Prove $\lim \sqrt{s_n} = \sqrt{s}$

Proof. There are two cases.

1. $s > 0$: Let $\epsilon > 0$. $\lim s_n = s \Rightarrow (\exists N,\ n > N \Rightarrow |s_n - s| < \sqrt{s}\epsilon)$. $n > N$ also implies

$$|\sqrt{s_n} - \sqrt{s}| = \frac{(\sqrt{s_n} - \sqrt{s})(\sqrt{s_n} + \sqrt{s})}{\sqrt{s_n} + \sqrt{s}} = \frac{|s_n - s|}{\sqrt{s_n} + \sqrt{s}} \leq \frac{|s_n - s|}{\sqrt{s}} < \frac{\sqrt{s}\epsilon}{\sqrt{s}} = \epsilon$$

2. $s = 0$: EXERCISE 8.3

■

Example 2.2.6. Let (s_n) be a convergent sequence of real numbers such that $s_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim s_n = s \neq 0$. Prove $\inf\{|s_n| : n \in \mathbb{N}\} > 0$

Proof. Let $\epsilon = \frac{|s|}{2}$. Since $\lim s_n = s$,

$$n > N \Rightarrow |s_n - s| < \frac{|s|}{2} \Rightarrow |s_n| \geq \frac{|s|}{2}$$

The last implication is because otherwise

$$|s| = |s - s_n + s_n| \leq |s - s_n| + |s_n| < \frac{|s|}{2} + \frac{|s|}{2} = |s|$$

which is a contradiction. Now if we set $m = \min\{\frac{|s|}{2}, |s_1|, |s_2|, \ldots, |s_N|\}$, then clearly we have $m > 0$ since and $|s_n| \geq m$ for all $n \in \mathbb{N}$. Thus $\inf\{|s_n| : n \in \mathbb{N}\} \geq m > 0$**WHY???** ■

2.3 Limit Theorems for Sequences

Definition 2.3.1. A sequence (s_n) is said to be *bounded* if $\exists M,\ \forall n,$ such that $|s_n| \leq M$

Theorem 2.3.2. *Convergent sequences are bounded.*

Proof. Let (s_n) be a convergent sequence and $\lim s_n = s$, then select $\epsilon = 1$ and we have

$$n > N \Rightarrow |s_n - s| < 1$$

From the reverse triangular inequality 1.3.6.1, $|s_n| - |s| \leq |s_n - s| < 1 \Rightarrow |s_n| < |s| + 1$ when $n > N$. Thus define $M = \max\{|s|+1, |s_1|, |s_2|, \dots, |s_N|\}$, then $|s_n| < M$ for all n. ■

Theorem 2.3.3. *If the sequence (s_n) converges to s and $k \in \mathbb{R}$, then (ks_n) converges to ks. i.e.* $\lim(ks_n) = k \cdot \lim s_n$.

Proof. Assume $k \neq 0$ and let $\frac{\epsilon}{|k|}$, then there exists N such that

$$n > N \Rightarrow |s_n - s| < \frac{\epsilon}{|k|} \Rightarrow |ks_n - ks| < \epsilon$$

■

Theorem 2.3.4. *If (s_n) and (t_n) converge to s and t, then $(s_n + t_n)$ converges to $s + t$. That is,*

$$\lim(s_n + t_n) = \lim s_n + \lim t_n.$$

Proof. From 2.1.2, we know

$$n > N_1 \Rightarrow |s_n - s| < \frac{\epsilon}{2}$$
$$n > N_2 \Rightarrow |t_n - t| < \frac{\epsilon}{2}$$

Thus, let $N = \max\{N_1, N_2\}$,

$$n > N \Rightarrow |s_n + t_n - (s + t)| \leq |s_n - s| + |t_n - t| < \epsilon$$

■

Theorem 2.3.5. *If (s_n) and (t_n) converge to s and t, then $(s_n t_n)$ converges to st. That is,*

$$\lim(s_n t_n) = (\lim s_n)(\lim t_n)$$

Proof. Let $\epsilon > 0$. By 2.3.2, $|s_n| \leq M$ for some $M > 0$. From 2.1.2, we have

$$\begin{aligned} n > N_1 &\Rightarrow |s_n - s| < \frac{\epsilon}{2(|t|+1)} \\ n > N_2 &\Rightarrow |t_n - t| < \frac{\epsilon}{2M} \end{aligned}$$

Thus, let $N = \max\{N_1, N_2\}$,

$$\begin{aligned} n > N \Rightarrow |s_n t_n - st| &= |s_n t_n - s_n t + s_n t - st| \\ &\leq |s_n t_n - s_n t| + |s_n t - st| \\ &= |s_n| \cdot |t_n - t| + |t| \cdot |s_n - s| \\ &\leq M \cdot \frac{\epsilon}{2M} + |t| \cdot \frac{\epsilon}{2(|t|+1)} \\ &= \epsilon \end{aligned}$$

■

Lemma 2.3.6. *If $(s_n) \to s \neq 0$ and $s_n \neq 0$ and for all n, then* $\inf |s_n| : n \in \mathbb{N} > 0$.

Proof. Since $(s_n) \to s$, select $\epsilon = \frac{|s|}{2}$ and we have $n \geq N \implies |s_n - s| < \frac{|s|}{2}$, which implies $|s_n| > \frac{|s|}{2}$. Thus select $m = \min\{s_1, \ldots, s_N, \frac{|s|}{2}\}$, and then $|s_n| \geq m$ for all n. Since $m > 0$ and m is a lower bound of $(|s_n|)$, $\inf\{|s_n| : n \in \mathbb{N}\} \geq m > 0$. ■

Lemma 2.3.7. *If (s_n) converges to s, $s_n \neq 0$ for all n, and $s \neq 0$, then $(1/s_n)$ converges to $1/s$.*

Proof. Let $\epsilon > 0$. Since there exists $m > 0$ such that $|s_n| \geq m$ for all n. By 2.1.2, we have

$$\begin{aligned} n > N &\Rightarrow |s - s_n| < \epsilon \cdot m|s| \\ &\Rightarrow \left| \frac{1}{s_n} - \frac{1}{s} \right| = \frac{|s - s_n|}{|s_n s|} \leq \frac{|s - s_n|}{m|s|} < \epsilon. \end{aligned}$$

■

Theorem 2.3.8. *Suppose (s_n) and (t_n) converge to s and t. If $s \neq 0$ and $s_n \neq 0$ for all n, then (t_n/s_n) converges to t/s.*

Proof. By 2.3.7, $(1/s_n)$ converges to $1/s$, so

$$\lim \frac{t_n}{s_n} = \lim \frac{1}{s_n} \cdot \lim t_n = \frac{1}{s} \cdot t = \frac{t}{s}$$

by 2.3.5. ■

Theorem 2.3.9.

(a) $\lim_{n\to\infty}(\frac{1}{n^p}) = 0$ *for* $p > 0$.

(b) $\lim_{n\to\infty} a^n = 0$ *if* $|a| < 1$.

(c) $\lim(n^{1/n}) = 1$.

(d) $\lim_{n\to\infty} a^{1/n} = 1$ *for* $a > 0$.

Proof.

(a) Let $N = (\frac{1}{\epsilon})^{1/p}$ and the rest is easy.

(b) If $a = 0$ then it's obvious. Otherwise, since $|a| < 1$ we can write $|a| = \frac{1}{1+b}$ where $b > 0$. Since $(1+b)^n \geq 1 + nb > nb$,

$$|a^n - 0| = |a^n| = \frac{1}{(1+b)^n} < \frac{1}{nb}$$

. Then let $N = \frac{1}{\epsilon b}$ and finish the proof.

(c) Let $s_n = (n^{1/n}) - 1$ and note $s_n \geq 0$ for all n. By 2.3.4, we only need to show $\lim s_n = 0$. $1 + s_n = (n^{1/n}) \Rightarrow n = (1+s_n)^n$. For $n \geq 2$, the binomial expansion tells

$$n = (1+s_n)^n \geq 1 + ns_n + \frac{1}{2}n(n-1)s_n^2 > \frac{1}{2}n(n-1)s_n^2.$$

This implies $s_n < \sqrt{\frac{2}{n-1}}$ for $n \geq 2$. Now we can suppose $N = \frac{\epsilon}{\epsilon - 2}$ to finish the proof.

(d) If $a \geq 1$, then for $n \geq a$ we have $1 \leq a^{1/n} \leq n^{1/n}$. Since $\lim n^{1/n} = 1$, by Squeeze Theorem we have $\lim a^{1/n} = 1$. Now if $0 < a < 1$, then $\frac{1}{a} > 1$, so $\lim(\frac{1}{a})^{1/n} = 1$ from above. By 2.3.7, $\lim a^{1/n} = 1$.

∎

Definition 2.3.10. For a (s_n), we write $\lim s_n = +\infty$ provided for each $M > 0$ there is a number N wuch that $n > N \Rightarrow s_n > M$. Similarly, we write $\lim s_n = -\infty$ provided for each $M < 0$ there is a number N wuch that $n > N \Rightarrow s_n < M$.

This implies that if $\lim s_n > -\infty$, $\exists T,\ \forall n, s_n > T$. $\lim s_n < \infty$, $\exists T,\ \forall n, s_n < T$. Be careful that we say $\lim s_n = +\infty$ as (s_n) **diverges** to ∞, **not converge** to ∞.

Example 2.3.11. Prove that $\lim(\sqrt{n} + 7) = +\infty$.

Proof. Let $M > 0$ and let $N = (M-7)^2$. Then $n > N \Rightarrow \sqrt{n} + 7 > M$. ∎

Example 2.3.12. Prove $\lim \frac{n^2+3}{n+1} = +\infty$

Discussion. We want to find a simpler lower bound for $\frac{n^2+3}{n+1} = +\infty$.

Proof. Let N=2M. Then
$$\frac{n^2+3}{n+1} > \frac{n^2}{2n} = \frac{1}{2}n > M.$$
∎

Theorem 2.3.13. *Let* $\lim s_n = +\infty$ *and* $\lim t_n > 0$. *Then* $\lim s_n t_n = +\infty$.

Proof. Let $M > 0$ and select an m so that $0 < m < \lim t_n$. It is clear that there exists N_1 so that
$$n > N_1 \Rightarrow t_n > m$$
Since $\lim s_n = +\infty$, there exists N_2 so that
$$n > N_2 \Rightarrow s_n > \frac{M}{m}$$
Thus $n > \max\{N_1, N_2\} \Rightarrow s_n t_n > \frac{M}{m} \cdot m = M$. ∎

Theorem 2.3.14. *For a* (s_n) *of* positive *real numbers, we have* $\lim s_n = +\infty$ *if and only if* $\lim(\frac{1}{s_n}) = 0$.

Proof. We need to show it in both directions.

$\Rightarrow$: Let $\epsilon > 0$ and $M = \frac{1}{\epsilon}$. Since $\lim s_n = +\infty$, $n > N \Rightarrow s_n > M = \frac{1}{\epsilon}$. Therefore, $n > N \Rightarrow \left|\frac{1}{s_n} - 0\right| < \epsilon$.

$\Leftarrow$: Let $M > 0$ and $\epsilon = \frac{1}{M}$, then $n > N \Rightarrow \left|\frac{1}{s_n} - 0\right| < \epsilon = \frac{1}{M}$. Since $s_n > 0$, we have
$$n > N \Rightarrow 0 < \frac{1}{s_n} < \frac{1}{M} \Rightarrow s_n > M.$$
∎

Theorem 2.3.15. *Assume all* $s_n \neq 0$ *and that the limit* $L = \lim \left|\frac{s_{n+1}}{s_n}\right|$ *exists.*

(a) If $L < 1$, *then* $\lim s_n = 0$.

(b) If $L > 1$, *then* $\lim |s_n| = +\infty$.

Proof. See exercise 9.12 and HW2 Q8. ∎

2.4 Monotone Sequences and Cauchy Sequence

Definition 2.4.1. (s_n) is called an *increasing sequence (or nondecreasing)* if $\forall n,\ s_n \leq s_{n+1}$ and $s_n \leq s_m$ whenever $n < m$. Similarly, (s_n) is called an *decreasing sequence (or nonincreasing)* if $\forall n,\ s_n \geq s_{n+1}$. An increasing or decreasing sequence is called *monotone* or *monotonic* sequence.

Theorem 2.4.2. *All bounded monotone sequences converge.*

Proof. Let (s_n) be a bounded increasing sequence, $S = \{s_n : n \in \mathbb{N}\}$. We can say $u = \sup S$ since (s_n) is bounded by 1.4.5. Since $u - \epsilon < u$, there exists N such that $s_N > u - \epsilon \Rightarrow \forall n > N,\ s_n > u - \epsilon$. Since u is the supremum, $u - \epsilon < s_n \leq u \Rightarrow |s_n - u| < \epsilon$. The proof for decreasing sequence is in exercise 10.2. ■

From the proof procedure above, we can see that bounded monotone sequences **converge to its infimum or supremum**

Discussion of Decimals

Notice that real numbers are simply decimal expansions. For a decimal expansion like $K.d_1d_2d_3d_4\cdots$, we can define a sequence by

$$s_n = K + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}$$

where K is an nonnegative integer and each $d_j \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. When trying to $\lim s_n$, the formula of geometric series could help:

$$\lim_{n\to\infty} a(1 + r + r^2 + \cdots + r^n) = \frac{a}{1-r} \quad \text{for} \quad |r| < 1;$$

There are two important reversible facts:

1. Different decimal expansions can represent the same real number.

2. Every nonnegative real number has at least one decimal expansion

Theorem 2.4.3.

(i) If (s_n) is an unbounded increasing sequence, then $\lim s_n = +\infty$.

(ii) If (s_n) is an unbounded decreasing sequence, then $\lim s_n = -\infty$.

Proof.

(i) Let $M > 0$. Since $\{s_n : n \in \mathbb{N}\}$ is unbounded and bounded by s_1, it must be unbounded above. Thus there must be some $N \in \mathbb{N}$ so that $s_N > M$. Since s_n is increasing, we have $n > N \Rightarrow s_n \geq s_N > M$, so $\lim s_n = +\infty$.

(ii) Exercise 10.5

∎

Theorem 2.4.4.

- *If (s_n) is a bounded and nonincreasing sequence, then $\lim s_n = \inf\{s_n : n \in \mathbb{N}\}$.*
- *If (s_n) is a bounded and nondecreasing sequence, then $\lim s_n = \sup\{s_n : n \in \mathbb{N}\}$.*

Corollary 2.4.4.1. *If (s_n) is monotone, then $\lim s_n$ is always meaningful. i.e. $\lim s_n = s, \ +\infty, \ or \ -\infty$.*

Suppose (s_n) is bounded. Define $u_n = \inf\{s_m : m \geq n\}$ and $v_n = \sup s_m : m \geq n$. Then observe that (u_n) is nondecreasing and (v_n) is nonincreasing since as n increases, the set has fewer elements. i.e. we have fewer choices for infimum and supremum. In general, if $A \subseteq B$, then $\inf A \geq \inf B$ and $\sup A \leq \sup B$.

Definition 2.4.5. Let (s_n) be a sequence in $\mathbb{R}$, define

- $\limsup s_n = \lim\limits_{N\to\infty} \sup\{s_n : n > N\}$
- $\liminf s_n = \lim\limits_{N\to\infty} \inf\{s_n : n > N\}$

If (s_n) is not bounded above. $\sup\{s_n : n > N\} = +\infty$ for all N and we decree $\limsup s_n = +\infty$. Likewise, if (s_n) is not bounded below. $\inf\{s_n : n > N\} = -\infty$ for all N and we decree $\liminf s_n = -\infty$.

Notice that $\limsup s_n$ need not equal to $\sup\{s_n : n > N\}$, but $\limsup s_n \leq \sup\{s_n : n > N\}$

Theorem 2.4.6. *Let (s_n) be a sequence in $\mathbb{R}$.*

(i) If $\lim s_n$ is defined, then $\liminf s_n = \lim s_n = \limsup s_n$.

(ii) If $\liminf s_n = \limsup s_n$, then $\lim s_n$ is defined and $\lim s_n = \liminf s_n = \limsup s_n$.

Proof. We use the notation $u_N = \inf\{s_n : n > N\}$, $v_N = \sup\{s_n : n > N\}$, $u = \lim u_N = \liminf s_n$ and $v = \lim v_N = \limsup s_n$.

(i) • $\lim s_n = +\infty$: Let $M > 0$ then there is a N so that $n > N \Rightarrow s_n > M$. Then

$u_N = \inf\{s_n : n > N\} \geq M$. This means $m > N \Rightarrow u_m \geq M \Rightarrow \lim u_N = \liminf s_n = +\infty$.

- $\lim s_n = -\infty$: It is similar to the previous proof.
- $\lim s_n = s$: Let $\epsilon > 0$ then $n > N \Rightarrow |s_n - s| < \epsilon$, so

$$v_N = \sup\{s_n : n > N\} \leq s + \epsilon.$$

Also $m > N \Rightarrow v_m \leq s + \epsilon$ since v_n is nonincreasing, so

$$\limsup s_n = \lim v_m \leq s + \epsilon \Rightarrow \limsup s_n \leq s = \lim s_n.$$

A similar argument shows $\lim s_n \leq \liminf s_n$. Since $\liminf s_n \leq \limsup s_n$ we get

$$\liminf s_n = \lim s_n = \limsup s_n$$

(ii)
- If $\liminf s_n = \limsup s_n = s$, then we have

$$|s - \sup s_n : n > N_0| < \epsilon$$

which implies $\sup\{s_n : n > N_0\} < s + \epsilon \Rightarrow \forall n > N_0,\ s_n < s + \epsilon$. Similarly, we have

$$|s - \inf s_n : n > N_1| < \epsilon$$

which implies $\inf\{s_n : n > N_1\} > s - \epsilon \Rightarrow \forall n > N_1,\ s_n > s - \epsilon$. Therefore,

$$\forall n > \max\{N_0, N_1\},\ s - \epsilon < s_n < s + \epsilon \Rightarrow |s_n - s| < \epsilon$$

- If $\liminf s_n = \limsup s_n = +\infty$, then

$$\liminf s_n = +\infty \Rightarrow \forall M > 0,\ \inf\{s_n : n > N_0\} > M \Rightarrow n > N_0,\ s_n > M.$$

- If $\liminf s_n = \limsup s_n = -\infty$, then

$$\limsup s_n = -\infty \Rightarrow \forall M < 0,\ \sup\{s_n : n > N_0\} < M \Rightarrow n > N_0,\ s_n < M.$$

■

Definition 2.4.7. A (s_n) is called a *Cauchy sequence* if

$$\forall \epsilon > 0,\ \exists N \text{ such that } m, n > N \Rightarrow |s_n - s_m| < \epsilon$$

Lemma 2.4.8. *Convergent sequences are Cauchy sequences.*

Proof. Suppose $\lim s_n = s$. Let $\epsilon > 0$ then

$$\begin{aligned} n, m > N &\Rightarrow |s_n - s| < \frac{\epsilon}{2} \text{ and } |s_m - s| < \frac{\epsilon}{2} \\ &\Rightarrow |s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s - s_m| < \epsilon. \end{aligned}$$

■

Lemma 2.4.9. *Cauchy sequences are bounded.*

Proof. By 2.4.7 and set $\epsilon = 1$ we have

$$m, n > N \Rightarrow |s_n - s_m| < 1$$

In particular $n > N \Rightarrow |s_n| - |s_{N+1}| \leq |s_n - s_{N+1}| < 1$. Let $M = \max\{|s_{N+1}| + 1, |s_1|, |s_2|, \ldots, |s_N|\}$ then $|s_n| \leq M$. ■

Theorem 2.4.10. *A sequence is a convergent sequence if and only if it is a Cauchy sequence.*

Proof. We've already proved the right direction so we only need to proved the left direction by showing $\liminf s_n = \limsup s_n$ from 2.4.6. Let $\epsilon > 0$ and since (s_n) is a Cauchy sequence, there exists N such that

$$m, n < N \Rightarrow |s_n - s_m| < \epsilon \Rightarrow \forall m > N,\ v_N = \sup\{s_n : n > N\} \leq s_m + \epsilon$$

Now $v_N - \epsilon$ becomes a lower bound for $\{s_m : m > N\}$ so $v_N - \epsilon \leq \inf\{s_m : m > N\} = u_N$. Thus

$$\limsup s_n \leq v_N \leq u_N + \epsilon \leq \liminf s_n + \epsilon$$

Since this is true for all $\epsilon > 0$, $\limsup s_n$ cannot be greater than $\liminf s_n$ (imagine ϵ is extremely small, then $\limsup s_n > \liminf s_n + \epsilon$). Thus we have $\liminf s_n \geq \limsup s_n$. $\liminf s_n \leq \limsup s_n$ is obviously true, so we have the equality. ■

2.5 Subsequences

Definition 2.5.1. Suppose $(s_n)_{n\in\mathbb{N}}$ is a sequence. A *subsequence* of this sequence is $(t_k)_{k\in\mathbb{N}}$ where for each k there is a positive integer n_k such that

$$n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$$

and

$$t_k = s_{n_k}.$$

Thus (t_k) is just a selection of some [possibly all] of the s_n's taken in order.

For the subset $\{n_1, n_2, \dots\}$ there is a natural function σ given by $\sigma(k) = n_k$ for $k \in \mathbb{N}$. The function σ "selects" an infinite subset of $\mathbb{N}$ in order. Then the subsequence of s corresponding to σ is simply the composite function $t = s \circ \sigma$. That is

$$t_k = t(k) = s \circ \sigma(k) = s(\sigma(k)) = s(n_k) = s_{n_k} \quad \text{for} \quad k \in N.$$

Notice that σ needs to be an *increasing* function.

Recall that the set $\mathbb{Q}$ of rational numbers is *countable*: there is a bijection from $\mathbb{N}$ to $\mathbb{Q}$. Therefore we have a sequence $(q_n) = (q_1, q_2, q_3, \dots)$ such that $\{q_n : n \in \mathbb{N}\} = \mathbb{Q}$. Then we have the following proposition:

Theorem 2.5.2. *Let (q_n) be an enumeration of $\mathbb{Q}$. Then for any $a \in \mathbb{R}$, there exists a subsequence (q_{n_k}) of (q_n) such that $q_{n_k} \to a$.*

Proof. First there exists $r_1 \in \mathbb{Q}$ such that $a - 1 < r_1 < a + 1$ by the denseness of $\mathbb{Q}$. Since (q_n) is an enumeration of $\mathbb{Q}$, there exists $n_1 \in \mathbb{N}$ such that $q_{n_1} = r_1$.

Given that we've already constructed $n_1, \dots, n_k$ such that $a - \frac{1}{j} < q_{n_j} < a + \frac{1}{j}$ for $j = 1, \dots, k$. Since there are infinitely many rational numbers between $a - \frac{1}{k+1}$ and $a + \frac{1}{k+1}$, and only finite many of them have been selected as $q_{n_1}, \dots, q_{n_k}$, we are able to find $n_{k+1} > n_k$ such that $a - \frac{1}{k+1} < q_{n_{k+1}} < a + \frac{1}{k+1}$.

Now we have (q_{n_k}) such that $a - \frac{1}{k} < q_{n_k} < a + \frac{1}{k}$ for each $k \in \mathbb{N}$. Thus by Squeeze Lemma, $\lim_k q_{n_k} = a$. ∎

Theorem 2.5.3. *Let (s_n) be a sequence.*

(i) If t is in $\mathbb{R}$ then there is a subsequence of (s_n) converging to t if and only if the set $\{n \in \mathbb{N} : |s_n - t| < \epsilon\}$ is infinite *for all $\epsilon > 0$.*

(ii) If (s_n) is unbounded above, it has a subsequence with limit $+\infty$.

(iii) If (s_n) is unbounded below, it has a subsequence with limit $-\infty$.

In each case, the subsequence can be taken to be monotonic.

Proof. The forward implications are easy to check. Let's check these backward implications:

(i) First suppose $\{n \in \mathbb{N} : s_n = t\}$ is infinite. Then we can simply create a subsequence $(s_{n_k})_{k\in\mathbb{N}}$ such that $s_{n_k} = t$ for all k.

Otherwise, suppose $\{n \in \mathbb{N} : s_n = t\}$ is finite. Then

$$\{n \in \mathbb{N} : 0 < |s_n - t| < \epsilon\} \quad \text{is infinite for all} \quad \epsilon > 0.$$

Since these sets are equal to

$$\{n \in \mathbb{N} : t - \epsilon < s_n < t\} \cup \{n \in \mathbb{N} : t < s_n < t + \epsilon\}$$

and these sets get smaller as $\epsilon \to 0$ we have

$$\{n \in \mathbb{N} : t - \epsilon < s_n < t\} \quad \text{is infinite for all} \quad \epsilon > 0 \tag{1}$$

or

$$\{n \in \mathbb{N} : t < s_n < t + \epsilon\} \quad \text{is infinite for all} \quad \epsilon > 0 \tag{2}$$

otherwise for sufficiently small $\epsilon > 0$ the sets in both (1) and (2) would be finite.

Assume (1) holds, now we want to construct a $(s_{n_k})_{k\in\mathbb{N}}$ satisfying

$$t - 1 < s_{n_1} < t \quad \text{and} \quad \max\{s_{n_{k-1}}, t - \frac{1}{k}\} \leq s_{n_k} < t \quad \text{for} \quad k \geq 2 \tag{3}$$

Assume $n_1, \ldots, n_{k-1}$ have been selected satisfying (3) and show how to select n_k. This is called "inductive definition" or "definition by induction". A subsequence satisfying (3) is a monotone increasing sequence and by Squeeze Formula $\lim_k s_{n_k} = t$. Here is the construction: By (1) we can select n_1 such that $t - 1 < s_{n_1} < t$. Suppose we've selected $n_1, \ldots, n_{k-1}$ so that $n_1 < n_2, \cdots < n_{k-1}$ and

$$\max\{s_{n_{j-1}}, t - \frac{1}{j}\} \leq s_{n_j} < t \quad \text{for} \quad j = 2, \ldots, k-1 \tag{4}$$

By using (1) with $\epsilon = max\{s_{n_{k-1}}, t - \frac{1}{k}\}$, we can select $n_k > n_{k-1}$ satisfying (4) for $j = k$, so (3) also holds for k.

(ii) Given $n_1 = 1$ and $n_1 < \cdots < n_{k-1}$, select $n_k > n_{k-1}$ so that $s_{n_k} > \max\{s_{n_{k-1}}, k\}$. This is possible since (s_n) is unbounded above. Then the subsequence will be monotonically unbounded above thereby have limit $+\infty$.

■

Theorem 2.5.4. *If (s_n) converges, then every subsequence converges to the same limit.*

Proof. Let (s_{n_k}) denote a subsequence of (s_n). Note that $n_k \geq k$ for all k. Let $s = \lim s_n$ and $\epsilon > 0$. There exists N so that $n > N \Rightarrow |s_n - s| < \epsilon$. Since $n_k \geq k > N$, $|s_{n_k} - s| < \epsilon$. Thus

$$\lim_{k\to\infty} s_{n_k} = s.$$

■

In the other way, if there are two subsequences of (s_n) with different limits, (s_n) does not converge.

Theorem 2.5.5. *Every sequence* (s_n) *has a monotonic subsequence.*

Proof. Define n-th term is *dominant* if it is greater than every term which follows it

$$s_m < s_n \quad \text{for all} \quad m > n \tag{1}$$

- Case 1: Suppose there are infinitely many dominant terms, then we can easily construct a monotone decreasing subsequence.
- Case 2: Suppose there are only finitely many dominant terms. Select n_1 so that s_{n_1} is beyond all the dominant terms of the sequence. Then

  $$\text{given } N \geq n_1 \text{ there exists } m > N \text{ such that} \quad s_m \geq s_N. \tag{2}$$

  Suppose $n_1, \dots, n_{k-1}$ have been selected so that

  $$n_1 < n_2 < \cdots < n_{k-1} \tag{3}$$

  and

  $$s_{n_1} \leq \cdots \leq s_{n_{l-1}} \tag{4}$$

  Apply (2) with $N = n_{k-1}$ we can select $n_k > n_{k-1}$ such that $s_{n_k} \geq s_{n_{k-1}}$. Then the procedure continues by induction and we obtain an increasing subsequence.

■

Theorem 2.5.6 (Bolzano-Weierstrass Theorem)**.** *Every bounded sequence has a convergent subsequence.*

Proof. If (s_n) is a bounded sequence, it has a bounded monotonic subsequence which converges. ■

Definition 2.5.7. Let (s_n) be a sequence in $\mathbb{R}$. A *subsequential limit* is any real number or symbol $+\infty$ or $-\infty$ that is the limit of some subsequence of (s_n).

Theorem 2.5.8. *Let* (s_n) *be any sequence. There exists a monotonic subsequence whose limit is* $\limsup s_n$*, and there exists a monotonic subsequence whose limit is* $\liminf s_n$.

Proof. If (s_n) is not bounded above, then by 2.5.3(ii) there is a monotonic subsequence with limit $+\infty = \limsup s_n$. The proof for not bounded below is similar.

Now if (s_n) is bounded above, then let $t = \limsup s_n$, and consider $\epsilon > 0$. There exists N_0 so that

$$\sup\{s_n : n > N\} < t + \epsilon \quad \text{for} \quad N \geq N_0.$$

In particular, $s_n < t + \epsilon$ for all $n > N_0$. We now claim

$$\{n \in \mathbb{N} : t - \epsilon < s_n < t + \epsilon\} \quad \text{is infinite.} \tag{1}$$

Otherwise, there exists $N_1 > N_0$ so that $s_n \leq t - \epsilon$ for $n > N_1$(WHY???). Then $\sup\{s_n : n > N\} \leq t - \epsilon$ for $N \geq N_1$, so that $\limsup s_n < t$, a contradiction. Since (1) holds true for all $\epsilon > 0$, 2.5.3(i) shows that there is a monotonic subsequence verges to $t = \limsup s_n$. ■

Theorem 2.5.9. *Let (s_n) be any sequence in $\mathbb{R}$, and let S denote the set of subsequential limits of (s_n).*

(i) S is nonempty.

(ii) $\sup S = \limsup s_n$ and $\inf S = \liminf s_n$.

(iii) $\lim s_n$ exists if and only if S has exactly one element, namely $\lim s_n$.

(iv) $\limsup s_n \in S$ and $\liminf s_n \in S$.

Proof.

(i) By the last theorem.

(ii) Consider any limit t os a subsequence (s_{n_k}) of (s_n). By 2.4.6 $t = \liminf_k s_{n_k} = \limsup_k s_{n_k}$. Since $n_k > k$ for all k, we have $\{s_{n_k} : k > N\} \subseteq \{s_n : n > N\}$ for each $N \in \mathbb{N}$. Therefore

$$\liminf_n s_n \leq \liminf_k s_{n_k} = t = \limsup_k s_{n_k} \leq \limsup_n s_n$$

The inequality below holds true for all t in S, so

$$\liminf s_n \leq \inf S \leq \sup S \leq \limsup s_n$$

By the last theorem we know both $\liminf s_n$ and $\limsup s_n$ is in S, so (ii) holds.

(iii) This is simply a reformulation of 2.4.6.

(iv) This is from 2.5.8

■

Theorem 2.5.10. *Let S denote the set of subsequential limits of a sequence (s_n). Suppose $(t_n)$is a sequence in $S \cap \mathbb{R}$ and that $t = \lim t_n$. Then t belongs to S.*

Proof. Suppose t is finite, then some t_N is in $(t-\epsilon, t+\epsilon)$. Let $\delta = \min\{t+\epsilon-t_N, t_N-t+\epsilon\}$, so that

$$(t_N - \delta, t_N + \delta) \subseteq (t - \epsilon, t + \epsilon)$$

Since t_N is a subsequential limit, the set $\{n \in \mathbb{N} : s_n \in (t_N - \delta, t_N + \delta)\}$ is infinite, so the set $\{n \in \mathbb{N} : s_n \in (t - \epsilon, t + \epsilon)\}$ is also infinite. Thus by 2.4.6 t itself is a subsequential limit of (s_n).

If $t = +\infty$, then clearly the sequence (s_n) is unbounded above, so a subsequence of (s_n) has limit $+\infty$ by 2.4.6. Thus $+\infty$ is also in S. A similar argument applies if $t = -\infty$. ∎

2.6 lim sup's and lim inf's

Theorem 2.6.1. *If (s_n) converges to a positive real number s and (t_n) is any sequence, then*

$$\limsup s_n t_n = s \cdot \limsup t_n.$$

Here we allow the conventions $s \cdot (+\infty) = +\infty$ and $s \cdot (-\infty) = -\infty$ for $s > 0$.

Proof. We first want to show

$$\limsup s_n t_n \geq s \cdot \limsup t_n. \tag{1}$$

We have three cases. Let $\beta = \limsup t_n$.

1. Suppose β is finite. By **??**, there exists a subsequence (t_{n_k}) of (t_n) such that $\lim_{k\to\infty} t_{n_k} = \beta$. We also have $\lim_{k\to\infty} s_{n_k} = s$ by 2.5.4, so $\lim_{k\to\infty} s_{n_k} t_{n_k} = s\beta$ thus $(s_{n_k} t_{n_k})$ is a subsequence of $(s_n t_n)$ converging to $s\beta$, and therefore $s\beta \leq \limsup s_n t_n$ by 2.5.9. Thus (1) holds.

2. Suppose $\beta = +\infty$. Then there exists a subsequence (t_{n_k}) of (t_n) converging to $+\infty$. Since $\lim_{k\to\infty} s_{n_k} = s > 0$, $\lim_{k\to\infty} s_{n_k} t_{n_k} = +\infty$. Hence $\limsup s_n t_n = +\infty$. Thus (1) holds.

3. Suppose $\beta = -\infty$. Then the right-hand side of (1) is equal to $-\infty$. Hence (1) is obviously true.

To show $\limsup s_n t_n \leq s \cdot \limsup t_n$, we may ignore the first few terms of (s_n) and assume all $s_n \neq 0$. Then we can write $\lim \frac{1}{s_n} = \frac{1}{s}$. Now we apply (1) with s_n replaced by $\frac{1}{s_n}$ and t_n replaced by $s_n t_n$:

$$\limsup t_n = \limsup (\frac{1}{s_n})(s_n t_n) \geq (\frac{1}{s}) \limsup s_n t_n,$$

which is

$$\limsup s_n t_n \leq s \cdot \limsup t_n$$

Therefore we have $\limsup s_n t_n = s \cdot \limsup t_n$. ∎

Theorem 2.6.2. *Let (s_n) be any sequence of nonzero real numbers. Then we have*

$$\liminf \left| \frac{s_{n+1}}{s_n} \right| \leq \liminf |s_n|^{1/n} \leq \limsup |s_n|^{1/n} \leq \limsup \left| \frac{s_{n+1}}{s_n} \right|$$

Proof. The middle inequality is obvious. The first and third inequalities have similar proofs. We will prove the third inequality as below:

Let $\alpha = \limsup |s_n|^{1/n}$ and $L = \limsup \left|\frac{s_{n+1}}{s_n}\right|$. Assume $L < +\infty$. To prove $\alpha \leq L$ it suffices to show

$$a \leq L_1 \quad \text{for any} \quad L_1 > L \tag{1}$$

because if $\exists L_1 > L,\ \alpha > L_1$, then $\alpha > L_1 > L \Rightarrow \alpha > L$ Since

$$L = \limsup \left|\frac{s_{n+1}}{s_n}\right| = \lim_{N\to\infty} \sup\left\{\left|\frac{s_{n+1}}{s_n}\right| : n > N\right\} < L_1$$

there exists a positive integer N such that

$$\sup\left\{\left|\frac{s_{n+1}}{s_n}\right| : n \geq N\right\} < N_1$$

Thus

$$\left|\frac{s_{n+1}}{s_n}\right| < L_1 \quad \text{for} \quad n \geq N \tag{2}$$

Now for $n > N$ we can write

$$|s_n| = \left|\frac{s_n}{s_{n-1}}\right| \cdot \frac{s_{n-1}}{s_{n-2}} \cdots \left|\frac{s_{N+1}}{s_N}\right| \cdot |s_N|.$$

Apply (2) we see that

$$\begin{aligned} |s_n| &< L_1^{n-N}|s_N| \quad \text{for} \quad n > N \\ |s_n| &< L_1^n a \quad \text{for} \quad n > N. \qquad \text{for } a = L_1^{-N}|s_N| \\ |s_n|^{1/n} &< L_1 a^{1/n} \quad \text{for} \quad n > N \end{aligned}$$

Since $\lim_{n\to\infty} a^{1/n} = 1$ we conclude $\alpha = \limsup |s_n|^{1/n} \leq L_1$ ∎

Corollary 2.6.2.1. *If* $\lim \left|\frac{s_{n+1}}{s_n}\right|$ *exists [and equals L], then* $\lim |s_n|^{1/n}$ *exists [and equals L].*

Proof. If $\lim \left|\frac{s_{n+1}}{s_n}\right| = L$, then all four values in the last theorem are equal to L. Hence $\lim |s_n|^{1/n} = L$ by 2.4.6. ∎

2.7 Some Topological Concepts in Metric Spaces

Definition 2.7.1. Let S be a set, and suppose d is a function $d : X \times X \to [0, \infty]$ defined for all pairs (x, y) of elements from S satisfying

1. $d(x, x) = 0$ for all $x \in S$ and $d(x, y) > 0$ for distinct $x, y \in S$. (Positive Definiteness)
2. $d(x, y) = d(y, x)$ for all $x, y \in S$. (Symmetry)
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in S$. (Triangle Inequality)

Such a function d is called a *distance function* or a *metric* on S.

A *metric space* S is a set S together with a metric on it.

Remark. The positive definiteness can be also expressed as $\forall x, y \in S\ d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$. The distance function cannot be $+\infty$.

Definition 2.7.2.

- A sequence (s_n) in a metric space (S, d) converges to s in S if $\lim_{n\to\infty} d(s_n, s) = 0$.
- A sequence (s_n) in S is a *Cauchy sequence* if for each $\epsilon > 0$ there exists an N such that
$$m, n > N \Rightarrow d(s_m, s_n) < \epsilon.$$
- (S, d) is said to be *complete* if every Cauchy sequence in S converges to some element in S.

Definition 2.7.3 (Open Ball)**.** Let (X, d) be a metrc space. For $x \in X$ and $r > 0$, the open ball of radius r centered at x is the set

$$B_r(x) = \{y \in X : d(y, x) < r\}$$

Definition 2.7.4 (Interior Point)**.** Let (X, d) be a metric space. Let E be a subset of X. An element $x \in E$ is *interior* to E if for some $r > 0$ we have

$$B_r(x) \subseteq E$$

We write E° for the set of points in E that are interior to E.

Remark. The relationship between E and X may affect whether a point in E is interior to E. For example, for $E = [0, 1] \subset [-1, 2] = X$, 0 is not interior to $[0, 1]$. However if $E = [0, 1] \subset [0, 1] = X$, then 0 is interior to 0 since there is not point on the left of 0.

Definition 2.7.5 (Open Set)**.** A set $E \subseteq X$ is *open* if every point $x \in E$ is an interior point of E.

The set E is *open* in X if every point in E is interior to E, i.e., if $E = E^\circ$.

We can show that

1. S is open in S.
2. The empty set $\varnothing$ is open in S.
3. The union of *any* collection of open sets is open.
4. The intersection of *finitely many* open sets is again an open set.

Definition 2.7.6 (Complement)**.** For a set $E \subseteq X$, the *complement* of E is the set $E^C = X \backslash E = \{x \in X : x \notin E\}$.

Definition 2.7.7 (Limit Point)**.** For a set $E \subseteq X$, a point $x \in X$ is a *limit point* of E if for any $r > 0$, we have that $(B_r(x) \backslash \{x\}) \cap E \neq \varnothing$.
$E' :=$ set of all limit points of E.

Remark. In other words, x is a limit point of E if for any (small) radius r, there is some element of E which sits in $B_r(x)$, other than x itself.

Definition 2.7.8 (Isolated Point)**.** For a set $E \subseteq X$, $x \in E$ is called an *isolated point* if x is not a limit point of E

Definition 2.7.9 (Closed Set)**.** A set is *closed* if $E' \subseteq E$.

Example 2.7.10.

- In $\mathbb{R}$, $[0,1]$ is closed. $[a, \infty), (-\infty, a]$ are closed.
- In $\mathbb{R}$, the set $\{\frac{1}{n} : n \in \mathbb{N}\}$ is not closed, but $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is closed.
- In any metric space, X and $\varnothing$ are closed.

Definition 2.7.11 (Bounded Set)**.** A set $E \subseteq X$ is bounded if for some $x \in X$ and $M > 0$ such that $d(x, y) \leq M$ for all $y \in E$.

Example 2.7.12. In $\mathbb{R}$, E is bounded if there exists M such that $|x| \leq M$ for all $x \in E$.

Definition 2.7.13 (Closure)**.** The *closure* of E in X is $\bar{E} = E \cup E'$.

Definition 2.7.14 (Dense Set)**.** A set $E \subseteq X$ is *dense* in X if $\bar{E} = X$.

Definition 2.7.15 (Closed Set)**.** Let (S, d) be a metric space. A subset E of S is *closed* if its complement $S \backslash E$ is an open set.

The intersection of any collection of closed sets is closed. The *closure* E^{-} of a set E is the intersection of all closed sets containing E.

The *boundary* of E is the set $E^{-} \backslash E^{\circ}$; points in this set are called *boundary points*.

Remark. Equivalence of two definitions: TODO

Lemma 2.7.16.

- *A sequence $(\boldsymbol{x}^{(n)})$ in $\mathbb{R}^k$ converges if and only if for each $j = 1, 2 \ldots, k$, the sequence $(x_j^{(n)})$ converges in $\mathbb{R}$.*
- *A sequence $(\boldsymbol{x}^{(n)})$ in $\mathbb{R}^k$ is a Cauchy sequence if and only if each sequence $(x_j^{(n)})$ is a Cauchy sequence in $\mathbb{R}$.*

Proof. First assertion: TODO. For the second assertion, we first observe for $\mathbf{x}$,$\mathbf{y} \in \mathbb{R}^k$ and $j = 1, \ldots, k$,

$$|x_j - y_j| \leq d(\mathbf{x}, \mathbf{y}) \leq \sqrt{k} \max\{|x_j - y_j| : j = 1, \ldots, k\} \tag{1}$$

$\Rightarrow$: Suppose $(\mathbf{x}^{(n)})$ is a Cauchy sequence, from the definition we know

$$m, n > N \Rightarrow d(\mathbf{x}^{(m)}, \mathbf{y}^{(n)}) < \epsilon$$

From (1) we see

$$m, n > N \Rightarrow |x_j^{(m)} - x_j^{(n)}| < \epsilon$$

so $(x_j^{(n)})$ is a Cauchy sequence.

$\Leftarrow$: Suppose $(x_j^{(n)})$ is a Cauchy sequence, then for $j = 1, \ldots, k$

$$m, n > N_j \Rightarrow |x_j^{(m)} - x_j^{(n)}| < \frac{\epsilon}{\sqrt{k}}.$$

If $N = \max\{N_1, N_2, \ldots, N_k\}$, then by (1)

$$m, n > N \Rightarrow d(\mathbf{x}^{(m)}, \mathbf{y}^{(n)}) < \epsilon$$

i.e. $(\mathbf{x}^{(n)})$ is a Cauchy sequence.

■

Theorem 2.7.17. *Euclidean k-space $\mathbb{R}^k$ is complete.*

Proof. Consider a Cauchy sequence $(\mathbf{x}^{(n)})$ in $\mathbb{R}^k$. By 2.7.16, each $(x_j^{(n)})$ is a Cauchy sequence. By 2.4.10 each $(x_j^{(n)})$ converges. Thus by 2.7.16 $(\mathbf{x}^{(n)})$ converges. ■

Theorem 2.7.18. *Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.*

Proof. Since $(\mathbf{x}^{(n)})$ is bounded, then each $(x_j^{(n)})$ is bounded in $\mathbb{R}$. By 2.5.6, we could replace $(\mathbf{x}^{(n)})$ by one of its subsequence, say $(\bar{\mathbf{x}}^{(n)})$, whose $(x_1^{(n)})$ converges. By 2.5.6 again, we may replace $(\mathbf{x}^{(n)})$ by a subsequence of $(\mathbf{x}^{(n)})$ such that both $(x_1^{(n)})$ and $(x_2^{(n)})$ converge. $(x_1^{(n)})$ still converges because 2.5.4. Repeating this argument by k times, we obtain a new sequence $(\mathbf{x}^{(n)})$ where each $(x_j^{(n)})$ converges, $j = 1, \ldots, k$, which is a subsequence of the original sequence, and it converges by 2.7.16. ■

Theorem 2.7.19. *Let E be a subset of a metric space (S, d).*

1. *E is closed $\iff$ $E = E^-$.*
2. *E is closed $\iff$ E contains the limit of every convergent sequence of points in E.*
3. *An element is in E^- $\iff$ it is the limit of some sequence of points in E.*
4. *A point is in the boundary of E $\iff$ it belongs to the closure of both E and its complement.*

Theorem 2.7.20. *Let (F_n) be a decreasing sequence [i.e., $F_1 \supseteq F_2 \supseteq \cdots$] of closed bounded nonempty sets in $\mathbb{R}^k$. Then $F = \cap_{n=1}^{\infty} F_n$ is also closed, bounded and nonempty.*

Proof. TODO ■

Definition 2.7.21. Let (S, d) be a metric space. A family $\mathcal{U}$ of open sets is said to be an *open cover* for a set E if each point of E belongs to at least one set in $\mathcal{U}$, i.e.,

$$E \subseteq \bigcup\{U : U \in \mathcal{U}\}.$$

A *subcover* of $\mathcal{U}$ is any subfamily of $\mathcal{U}$ that also covers E. A cover or subcover if *finite* if it contains only finitely many sets; the sets themselves may be infinite.
A set E is *compact* if every oper cover of E has a finite subcover of E.

Theorem 2.7.22. *A subset E of $\mathbb{R}^k$ is compact if and only if it is closed and bounded.*

Proof. TODO ■

WHAT IS A *K-CELL*

Theorem 2.7.23. *Every k-cell F in $\mathbb{R}^k$ is compact.*

Proof. TODO ■

2.8 Series

For an infinite series $\sum_{n=m}^{\infty} a_n$, we say it *converge* provided the sequence (s_n) of partial sums

$$s_n = a_m + a_{m+1} + \cdots + a_n = \sum_{k=m}^{n} a_k$$

also converges to a real number S. i.e.

$$\sum_{n=m}^{\infty} a_n = S \quad \text{means} \quad \lim s_n = S \quad \text{or} \quad \lim_{n\to\infty} \left(\sum_{k=m}^{n} a_k \right) = S$$

A series that does not converge is said to *diverge*, so $\sum_{n=m}^{\infty} a_n$ *diverge* to $+\infty$, $\sum_{n=m}^{\infty} a_n = +\infty$, provided $\lim s_n = +\infty$. Similar for diverging to $-\infty$.

If the terms in $\sum a_n$ are all nonnegative, then the corresponding partial sums (s_n) form an increasing sequence, so $\sum a_n$ either converges or diverges to $+\infty$ by 2.4.2 and 2.4.3. In particular, $\sum |a_n|$ is meaningful for any (s_n) whatever. The series $\sum a_n$ is said to *converge absolutely* or to be *absolutely convergent* if $\sum |a_n|$ converges.

We use $\sum a_n$ to represent $\sum_{n=m}^{\infty} a_n$

Example 2.8.1 (Geometric Series)**.** A series of the form $\sum_{n=0}^{\infty} ar^n$ for constants a and r is called a geometric series. For $r \neq 1$, the partial sums s_n are given by

$$\sum_{k=0}^{n} ar^k = a\frac{1 - r^{n+1}}{1 - r}.$$

Furthermore, if $|r| < 1$, then $\lim_{n\to\infty} r^{n+1} = 0$ and

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}$$

If $a \neq 0$ and $|r| \geq 1$, then (ar^n) does not converge to 0, so $\sum ar^n$ diverges.

Example 2.8.2.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges if and only if} \quad p > 1$$

If $p \leq 1$, $\sum 1/n^p = +\infty$

Definition 2.8.3. We say a series $\sum a_n$ satisfies the *Cauchy criterion* if its sequence (s_n) of partial sums is a Cauchy sequence which is:

$$\forall \epsilon > 0,\ \exists N,\ m, n > N \Rightarrow |s_n - s_m| < \epsilon \tag{1}$$

which is equivalent to

$$\forall \epsilon > 0,\ \exists N,\ n \geq m > N \Rightarrow |s_n - s_{m-1}| < \epsilon. \tag{2}$$

Since $s_n - s_{m-1} = \sum_{k=m}^{n} a_k$, we can write (2) as

$$\forall \epsilon > 0,\ \exists N,\ n \geq m > N \Rightarrow \left|\sum_{k=m}^{n} a_k\right| < \epsilon \tag{3}$$

Theorem 2.8.4. *A series converges $\iff$ it satisfies the Cauchy criterion.*

Proof. By 2.4.10, we know its partial sum converges, so the series also converges. ∎

Corollary 2.8.4.1. *If a series $\sum a_n$ converges, then* $\lim a_n = 0$

Proof. By setting $n = m$ in the condition of 2.8.3, we get

$$(\forall \epsilon > 0,\ \exists N,\ n > N \Rightarrow |a_n| < \epsilon) \Rightarrow \lim a_n = 0$$

∎

A useful contrapositive of this corollary is "If $\lim a_n \neq 0$, then $\sum a_n$ does not converge."

Theorem 2.8.5 (Comparison Test)**.** *Let $\sum a_n$ be a series where $a_n \geq 0$ for all n.*

(i) If $\sum a_n$ converges and $|b_n| \leq a_n$ for all n, then $\sum b_n$ converges.

(ii) If $\sum a_n = +\infty$ and $b_n \geq a_n$ for all n, then $\sum b_n = +\infty$

Proof.

(i) For $n \geq m$ we have

$$\left|\sum_{k=m}^{n} b_k\right| \leq \sum_{k=m}^{n} |b_k| \leq \sum_{k=m}^{n} a_k \tag{1}$$

Since $\sum a_n$ converges, it satisfies 2.8.3(1). Then from (1) we can see $\sum b_n$ also satisfies the Cauchy criterion in 2.8.3(3), and hence $\sum b_n$ converges.

(ii) Since $b_n \geq a_n$ for all n, obviously we have $\sum_{k=m}^{n} b_k \geq \sum_{k=m}^{n} a_k$. Since $\lim \sum_{k=m}^{n} b_k = +\infty$, $\lim \sum_{k=m}^{n} a_k = +\infty$.

∎

Corollary 2.8.5.1. *Absolutely convergent series are convergent.*

Proof. Suppose $\sum b_n$ is absolutely convergent. This means $\sum a_n$ converges where $a_n = |b_n|$ for all n. Then $|b_n| \leq a_n$ and $\sum b_n$ converges trivially from 2.8.5. ∎

Theorem 2.8.6 (Root Test)**.** *Let $\sum a_n$ be a series and let $\alpha = \limsup |a_n|^{1/n}$. The series $\sum a_n$*

(i) converges absolutely if $\alpha < 1$

(ii) diverges if $\alpha > 1$

(iii) Otherwise the test does not provide any useful information.

Proof. (i) Suppose $\alpha < 1$, and select $\epsilon > 0$ so that $\alpha + \epsilon < 1$. Then

$$\alpha - \epsilon < \sup\{|a_n|^{1/n} : n > N\} < \alpha + \epsilon$$

so

$$|a_n| < (a + \epsilon)^n \quad \text{for} \quad n > N.$$

Since $0 < \alpha + \epsilon < 1$, $\sum_{n=N+1}^{\infty}(\alpha + \epsilon)^n$ converges and 2.8.5(i) tells $\sum_{n=N+1}^{\infty} a_n$ converges. Then clearly $\sum a_n$ converges.

(ii) If $\alpha > 1$, then there is a subsequence of $|a_n|^{1/n}$ has limit $\alpha > 1$ by **??**. This means $|a_n| > 1$ for infinitely many choices of n. In particular, (a_n) cannot possibly converge to 0, so $\sum a_n$ cannot converge by the contrapositive of 2.8.4.1.

(iii) Example: $\sum \frac{1}{n}$ diverges but $\sum \frac{1}{n^2}$ converges.

∎

Theorem 2.8.7 (Ratio Test)**.** *A series $\sum a_n$ of nonzero terms*

(i) converges absolutely if $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$,

(ii) diverges if $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$.

(iii) Otherwise $\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$ *and the test gives no information.*

Proof. let $\alpha = \limsup |a_n|^{1/n}$. By **??** we have

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq \alpha \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|.$$

(i) If $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\alpha < 1$ and the series converges by 2.8.6.

(ii) If $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\alpha > 1$ and the series diverges by 2.8.6.

(iii) If $\alpha = 1$, then same reasoning as the proof in 2.8.6(iii).

■

If the terms a^n are nonzero and if $\lim \left|\frac{a_{n+1}}{a_n}\right| = 1$, then $\alpha = \limsup |a_n|^{1/n} = 1$ by 2.6.2.1, so neither the Ratio Test nor the Root Test gives information about the convergence of $\sum a_n$.

2.9 Alternating Series and Integral Tests

Sometimes we can try to check convergence or divergence of series by comparing the partial sums with familiar integrals. By drawing the function a^n and the of rectangles corresponding to the series on a same picture and comparing the areas under the function and the sum of areas of these rectangles, we may get the information about the convergence of the series. For example, if all rectangles are below the function and the integral of the function is finite, then the series converge.

Theorem 2.9.1. $\sum \frac{1}{n^p}$ *converges* $\iff$ $p > 1$.

Proof. By drawing the function $\frac{1}{n^p}$ and the of rectangles corresponding to the series on a same picture, we can get

$$\sum_{k=1}^{n} \frac{1}{k^p} \leq 1 + \int_1^n \frac{1}{x^p} dx = 1 + \frac{1}{p-1}\left(1 - \frac{1}{n^{p-1}}\right) < 1 + \frac{1}{p-1} = \frac{p}{p-1}$$

Thus $\sum_{n=1}^{\infty} \frac{1}{n^p} \leq \frac{p}{p-1} < +\infty$

Suppose $0 < p \leq 1$. Then $\frac{1}{n^p} \geq \frac{1}{n}$ for all n, so $\sum \frac{1}{n^p}$ diverges when $\sum \frac{1}{n}$ diverges by 2.8.5. ∎

Theorem 2.9.2. *Here are the conditions under which an integral test is advisable:*

(a) All comparison, root, and ratio tests do not apply.

(b) The terms a_n of the series are nonnegative.

(c) There is a nice decreasing function f on $[1, \infty)$ such that $f(n) = a_n$ for all n.

(d) The integral of f is easy to calculate or estimate.

If $\lim_{n\to\infty} \int_1^n f(x)dx = +\infty$, then the series diverges. If $\lim_{n\to\infty} \int_1^n f(x)dx < +\infty$, then the series will converge.

Theorem 2.9.3 (Alternating Series Theorem)**.** *If $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots \geq 0$ and $\lim a_n = 0$, then the alternating series $\sum (-1)^{n+1} a_n$ converges. Moreover, the partial sums $s_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$ satisfy $|s - s_n| \leq a_n$ for all n.*

Proof. To prove the series converge we need to show the partial sum (s_n) also converges. Note that the subsequence (s_{2n}) is increasing (accumulative sum of positive a_n) and the subsequence (s_{2n-1}) is decreasing (accumulative sum of negative a_n). We claim

$$s_{2m} \leq s_{2n+1} \quad \text{for all} \quad m, n \in \mathbb{N} \tag{2}$$

Since $s_{2n+1} - s_{2n} = a_{2n+1} \geq 0$, we have $s_{2n} \leq s_{2n+1}$ for all n. Thus if $m \leq n$ in (1) then (1) holds because $s_{2m} \leq s_{2n} \leq s_{2n+1}$, when (s_{2n}) is increasing. If $m \geq n$ in (1), then (1) also holds because $s_{2n+1} \geq s_{2m+1} \geq s_{2m}$ when (s_{2n+1}) is decreasing. Therefore, by (1) we can see that the subsequence (s_{2n}) is bounded above by every odd partial sum, and the subsequence (s_{2n+1}) is a bounded below by each even partial sum. Then by 2.4.2 (s_{2n}) and (s_{2n+1}) converge to some s and t. Now we have

$$t - s = \lim_{n\to\infty}(s_{2n+1} - 2_{2n}) = \lim_{n\to\infty} a_{2n+1} = 0$$

so $s = t$ and $\lim_n s_n = s$. (**WHY???** Is it because $s = supS$ and $t = \inf S$ where S is the set of subsequential limits.)

To check the last claim, note that $s_{2k} \leq s \leq s_{2k+1}$, so both $s_{2k+1} - s$ and $s - s_{2k}$ are bounded by $s_{2k+1} - s_{2k} = a_{2k+1} \leq a_{2k}$ (**WHY????**). So whether n is even or odd, we have $|s - s_n| \leq a_n$. ■

Chapter 3

Continuity

3.1 Continuous Functions

In this book/note, we will be concerned with functions f such that $\mathrm{dom}(f) \subseteq \mathbb{R}$ and such that f is a real-valued function. We consider the *natural domain* as "the largest subset of $\mathbb{R}$ on which the function is a well defined real-valued function.

Definition 3.1.1. The function f is *continuous* at x_0 in $\mathrm{dom}(f)$ if, for every sequence (x_n) in $\mathrm{dom}(f)$ converging to x_0, we have $\lim_n f(x_n) = f(x_0)$. If f is continuous at each point of a set $S \subseteq \mathrm{dom}(f)$, then f is said to be *continuous* on S. The function f is said to be *continuous* if it is continuous on $\mathrm{dom}(f)$.

Theorem 3.1.2. *f is continuous at x_0 in $\mathrm{dom}(f)$ if and only if*

$$\forall \epsilon > 0,\ \exists \delta > 0 \quad \textit{such that} \quad (x \in \mathrm{dom}(f)) \wedge (|x - x_0| < \delta) \implies |f(x) - f(x_0)| < \epsilon \quad (1)$$

Proof.

$\implies$: Suppose f is continuous at x_0 but (1) does not hold. In other words, there exists $\epsilon > 0$ so that

$$(x \in \mathrm{dom}(f)) \wedge (|x - x_0| < \delta) \implies |f(x) - f(x_0)| < \epsilon$$

fails for each $\delta > 0$. In particular the implication

$$(x \in \mathrm{dom}(f)) \wedge (|x - x_0| < \frac{1}{n}) \implies |f(x) - f(x_0)| < \epsilon$$

fails for each $n \in \mathbb{N}$. Thus for each $n \in \mathbb{N}$ there exists x_n in $\mathrm{dom}(f)$ such that $|x_n - x_0| < \frac{1}{n}$ and yet $|f(x_0) - f(x_n)| \geq \epsilon$. Hence we have $|f(x_0) - f(x_n)| \geq \epsilon \implies \lim f(x_n) \neq f(x_0)$. This contradicts to the definition of continuity 3.1.1.

$\impliedby$: Suppose (1) holds and consider a (x_n) in $\mathrm{dom}(f)$ such that $\lim x_n = x_0$. Let $\epsilon > 0$. By (1) there exists $\delta > 0$ such that

$$(x \in \mathrm{dom}(f)) \wedge (|x - x_0| < \delta) \implies |f(x) - f(x_0)| < \epsilon$$

Since $\lim x_n = x_0$ we have

$$n > N \implies |x_n - x_0| < \delta \implies |f(x_n) - f(x_0)| < \epsilon$$

Thus $\lim f(x_n) = f(x_0)$

∎

The condition $(x \in \mathrm{dom}(f)) \wedge (|x - x_0| < \delta) \implies |f(x) - f(x_0)| < \epsilon$ in the book is a little bit confusing. In other words, it means

$$\forall x \in \mathrm{dom}(f),\ |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$$

To use ϵ-δ property to prove the discontinuity, we need to show that

$$\exists\epsilon > 0 \quad \text{such that} \quad \forall\delta > 0, \exists x \in \mathrm{dom}\,(f) \quad \text{such that} \quad |x - x_0| < \delta \quad \text{but} \quad |f(x) - f(x_0)| \geq \epsilon$$

Theorem 3.1.3. *If f is continuous at x_0 in dom(f), then $|f|$ and kf, for $k \in \mathbb{R}$, are continuous at x_0.*

Proof. Since f is continuous at x_0, we have $\lim f(x_n) = f(x_0)$. Since $\lim kf(x_n) = k \lim f(x_n) = kf(x_0)$, this proves kf is continuous at x_0. Since $\lim f(x_n) = f(x_0)$, we have

$$n > N \implies |f(x_n) - f(x_0)| < \epsilon$$

Since $||f(x_n)| - |f(x_0)|| \leq |f(x_n) - f(x_0)|$, we have

$$n > N \implies \big||f(x_n)| - |f(x_0)|\big| \leq |f(x_n) - f(x_0)| < \epsilon$$

so $\lim |f(x_n)| = |f(x_0)|$ ■

Theorem 3.1.4. *Let f and g be real-valued fucntions that are continuous at x_0 in $\mathbb{R}$. Then*

(i) $f + g$ is continuous at x_0;

(ii) fg is continuous at x_0;

(iii) f/g is continuous ar x_0 if $g(x_0) \neq 0$.

Proof. We use the basic definition of continuity 3.1.1 and the basic theorems of limit. ■

Theorem 3.1.5. *If f is continuous at x_0 and g is continuous at $f(x_0)$, then the composite function $g \circ f$ is continuous at x_0.*

Proof. Given that $x_0 \in \mathrm{dom}\,(f)$ and $f(x_0) \in \mathrm{dom}\,(g)$, let (x_n) be a sequence in $\{x \in \mathrm{dom}\,(f) : f(x) \in \mathrm{dom}\,(g)\}$ converging to x_0. Since f is continuous at x_0, we have $\lim f(x_n) = f(x_0)$. Since the sequence $(f(x_n))$ converges to $f(x_0)$ and g is continuous at $f(x_0)$, we also have $\lim g(f(x_n)) = g(f(x_0))$ which is $\lim g \circ f(x_n) = g \circ f(x_0)$. ■

3.2 Properties of Continuous Functions

A real-valued function f is said to be *bounded* if $\{f(x) : x \in \mathrm{dom}(f)\}$ is a bounded set. i.e. if there exists a real number M such that $|f(x)| \le M$ for all $x \in \mathrm{dom}(f)$.

Theorem 3.2.1. *Let f be a continuous real-valued function on a* closed *interval $[a, b]$. Then f is a bounded function. Moreover, f assume its maximum and minimum values on $[a, b]$; that is there exist x_0, y_0 in $[a, b]$ such that $f(x_0) \le f(x) \le f(y_0)$ for all $x \in [a, b]$.*

Proof. First assume f is not bounded on $[a, b]$. Then for each $n \in \mathbb{N}$ there corresponds an $x_n \in [a, b]$ such that $|f(x_n)| > n \implies \lim_{k\to\infty} |f(x_{n_k})| = +\infty$. By 2.5.6, since (x_n) is bounded by $[a, b]$ it has a subsequence (x_{n_k}) that converges to some real number $x_0 \in [a, b]$. Since f is continuous, we have $\lim f(x_{n_k}) = f(x_0) < \infty$, which is a contradiction. Thus, f is bounded.

Not since f is bounded, $M = \sup\{f(x) : x \in [a, b]\}$ is finite. For each $n \in \mathbb{N}$ there exists $y_n \in [a, b]$ such that $M - \frac{1}{n} < f(y_n) \le M$. Hence we have $\lim f(y_n) = M$ by Squezze formula. By 2.5.6 there is a subsequence (y_{n_k}) of (y_n) converging to some limit $y_0 \in [a, b]$. Since y is continuous at y_0, we have $\lim_{k\to\infty} f(y_{n_k}) = f(y_0)$. Since $(f(y_{n_k}))$ is also a subsequence of $(f(y_n))$, by 2.5.4 $\lim_{k\to\infty} f(y_{n_k}) = \lim_{n\to\infty} f(y_n) = M$. Thus $f(y_0) = M$ meaning that f achieves its maximum at y_0.

Apply the same method to $-f$, and we get $-f$ achieves its maximum at some $x_0 \in [a, b]$. In other words, f achieves its minimum at x_0. ■

Theorem 3.2.2 (Intermediate Value Theorem)**.** *If f is a continuous real-valued function on an interval I, then f has the intermediate value property on I: Whenever $a, b \in I$, if $a < b$ and y lies between $f(a)$ and $f(b)$ [i.e. $f(a) < y < f(b)$ or $f(b) < y < f(a)$], then there exists at least one x in (a, b) such that $f(x) = y$.*

Proof. Let's focus on the case that $f(a) < y < f(b)$ since the other case is similar. Let $S = \{x \in [a, b] : f(x) < y\}$. Since $a \in S$ the set S is nonempty, and $x_0 = \sup S$ represents a number in $[a, b]$. For each $n \in \mathbb{N}$, $x_0 - \frac{1}{n}$ is not an upper bound for S, so there exists $s_n \in S$ such that $x_0 - \frac{1}{n} < s_n \le x_0$. Thus $\lim s_n = x_0$ and since $f(s_n) < y$ for all n, we have

$$f(x_0) = \lim f(s_n) \le y$$

because f is continuous at x_0. Let $t_n = \min\{b, x_0 + \frac{1}{n}\}$. Since $x_0 < t_n \le x_0 + \frac{1}{n}$ we have $\lim t_n = x_0$. Each t_n belongs to $[a, b]$ but not to S, so $f(t_n) \ge y$ for all n. Therefore,

$$f(x_0) = \lim f(t_n) \ge y$$

because f is continuous at x_0. Thus $f(x_0) = y$. ■

Corollary 3.2.2.1. *If f is a continuous real-valued function on an interval I, then the set $f(I) = \{f(x) : x \in I\}$ is also an interval or a single point.*

Proof. By 3.2.2 the set $J = f(I)$ has the property:

$$(y_0, y_1 \in J \quad \text{and} \quad y_0 < y < y_1) \implies y \in J$$

If $\inf J < \sup J$, then such a set J will be an interval. Consider $\inf J < y < \sup J$. Then there exist $y_0, y_1 \in J$ so that $y_0 < y < y_1$, so $y \in J$ by the above property. We showed that

$$\inf J < y < \sup J \implies y \in J$$

so J is an interval with endpoints $\inf J$ and $\sup J$ ∎

Theorem 3.2.3. *Let g be a strictly increasing function on an interval J such that $g(J)$ is an interval I. Then g is continuous on J.*

Proof. Consider an non-endpoint x_0 of J. Since g is strictly increasing, $g(x_0)$ is also not an endpoint of I, so $\exists \epsilon_0 > 0$ such that $(g(x_0) - \epsilon_0, g(x_0) + \epsilon_0) \subseteq I$.

Let $\epsilon > 0$ and we can assume $\epsilon < \epsilon_0$ (**WHY???**). Then $\exists x_1, x_1 \in J$ such that $g(x_1) = g(x_0) - \epsilon$ and $g(x_2) = g(x_0) + \epsilon$. This means $x_1 < x_0 < x_2$ because g is increasing. Also if $x_1 < x < x_2$, then $g(x_1) < g(x) < g(x_2)$, hence $g(x_0) - \epsilon < g(x) < g(x_0) + \epsilon$, and hence $|g(x) - g(0)| < \epsilon$. Now set $\delta = \min\{x_2 - x_0, x_0 - x_1\}$, then

$$|x - x_0| < \delta \implies x_1 < x < x_2 \implies |g(x) - g(x_0)| < \epsilon$$

Thus g is continuous on J. ∎

Theorem 3.2.4. *Let f be a continuous strictly increasing function on some interval I. Then $f(I)$ is an interval J by 3.2.2.1 and f^{-1} represents a function with domain J. The function f^{-1} is a continuous strictly increasing function on J.*

Proof. Obviously f^{-1} is still strictly increasing. Since f^{-1} maps J onto I, by 3.2.3 f^{-1} is continuous. ∎

Theorem 3.2.5. *Let f be a* one-to-one *continuous function on an interval I. Then f is strictly increasing or strictly decreasing.*

Proof. Firstly we want to show that

$$\text{if } a < b < c \text{ in } I, \text{ then } f(b) \text{ lies between } f(a) \text{ and } f(c) \tag{1}$$

Assume it's false so $f(b) > \max\{f(a), f(c)\}$. Select y so that $f(b) > y > \max\{f(a), f(c)\}$. By 3.2.2 applied to $[a, b]$ and $[b, c]$, $\exists x_1 \in (a, b)$ and $x_2 \in (b, c)$ such that $f(x_1) = f(x_2) = y$. This contradicts to the one-to-one property of f.

Now select any $a_0 < b_0$ in I and suppose, say, that $f(a_0) < f(b_0)$. We need to show f is strictly increasing on I. By (1) we have

$$\begin{aligned} f(x) < f(a_0) &\quad \text{for} \quad x < a_0 \\ f(a_0) < f(x) < f(b_0) &\quad \text{for} \quad a_0 < x < b_0 \\ f(b_0) < f(x) &\quad \text{for} \quad x > b_0 \end{aligned}$$

In particular,

$$f(x) < f(a_0) \quad \text{for all} \quad x < a_0 \tag{2}$$
$$f(a_0) < f(x) \quad \text{for all} \quad x > a_0 \tag{3}$$

Now consider any $x_1 < x_2$ in I.

$$\begin{aligned}
x_1 \le a_0 \le x_2 &\implies f(x_1) < f(x_2) \quad \text{by (2) and (3)} \\
x_1 < x_2 < a_0 &\implies f(x_1) < f(a_0) \quad \text{by (2)} \implies f(x_1) < f(x_2) \quad \text{by (1)} \\
a_0 < x_1 < x_2 &\implies f(a_0) < f(x_2) \quad \text{by (2)} \implies f(x_1) < f(x_2) \quad \text{by (1)}
\end{aligned}$$

∎

3.3 Uniform Continuity

Sometimes we want to know when the δ in 3.1.2 can be chosen to depend only on $\epsilon > 0$ and S, so that δ does not depend on the particular point x_0.

Definition 3.3.1. Let f be a real-valued function defined on a set $S \subseteq \mathbb{R}$. Then f is uniformly ocntinuous on S if

for each $\epsilon > 0$ there exists $\delta > 0$ such that $\forall x, y \in S,\ |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$.

We will say f is uniformly continuous if f is uniformly continuous on $\operatorname{dom} f$.

Theorem 3.3.2. *If f is continuous on a closed interval $[a, b]$, then f is uniformly continuous on $[a, b]$.*

Proof. Assume f is not uniformly continuous on $[a, b]$, then there exists $\epsilon > 0$ such that for each $\delta > 0$, there exists $x, y \in [a, b]$ such that $|x - y| < \delta$ but $|f(x) - f(y)| \geq \epsilon$. Thus for each $n \in \mathbb{N}$, since $\frac{1}{n} > 0$, there exist $x_n, y_n \in [a, b]$ such that $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) \geq f(y_n)| \geq \epsilon$. By 2.5.6, since (x_n) is bounded in $[a, b]$, it has a subsequence (x_{n_k}) converging to $x_0 \in [a, b]$. Clearly we can also have a subsequence (y_{n_k}) converging to x_0. Because f is continuous at x_0, we have

$$f(x_0) = \lim_{x \to \infty} f(x_{n_k}) = \lim_{k \to \infty} f(y_{n_k})$$

so

$$\lim_{k \to \infty} [f(x_{n_k}) - f(y_{n_k})] = 0$$

However, since $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon > 0$ for all k, we have a contradiction. ■

Theorem 3.3.3. *If f is uniformly continuous on a set S and (s_n) is a Cauchy sequence in S, then $(f(s_n))$ is a Cauchy sequence.*

Proof. Let (s_n) be a Cauchy sequence in S and let $\epsilon > 0$. Since f is uniformly continuous on S, there exists $\delta > 0$ so that

$$\forall x, y \in S,\ |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Since (s_n) is a Cauchy sequence, there exists N so that

$$m, n > N \implies |s_n - s_m| < \delta \implies |f(s_n) - f(s_m)| < \epsilon.$$

Thus $(f(s_n))$ is indeed a Cauchy sequence. ■

The contrapositive of the above them is that "If (s_n) is a Cauchy sequence in S, but $(f(s_n))$ is not a Cauchy sequence, then f is not uniformly continuous on a set S.

Theorem 3.3.4. *A real-valued function f on (a, b) is uniformly continuous on (a, b) if and only if it can be extended to a continuous function $\tilde{f}$ on $[a, b]$.*

Proof.

$\Longrightarrow$: Suppose (s_n) is a sequence in (a,b) converging to a, then (s_n) is a Cauchy sequence by 2.4.10, so $(f(s_n))$ is also a Cauchy sequence by 3.3.3 and converging again by 2.4.10. Hence we have the following claim:

$$\text{if } (s_n) \text{ is a sequence in } (a,b) \text{ converging to } a, \text{ then } (f(s_n)) \text{ converges} \tag{1}$$

Create a sequence $(u_n) = (s_1, t_1, s_2, t_2, \dots)$ where (t_n) is also a sequence converging to a. Clearly $\lim u_n = a$ and $\lim f(u_n)$ exists due to (1). Therefore, $(f(s_n))$ and $(f(t_n))$ are subsequences of $(f(u_n))$ both converge to $\lim f(u_n)$ by 2.5.4, so $\lim f(s_n) = \lim f(t_n)$. Hence we have the following claim:

$$\text{if } (s_n) \text{ and } (t_n) \text{ are sequences in } (a,b) \text{ converging to } a, \text{ then } \lim f(s_n) = \lim f(t_n) \tag{2}$$

Now we define

$$\tilde{f}(a) = \lim f(s_n) \text{ for any sequence } (s_n) \text{ in } (a,b) \text{ converging to } a \tag{3}$$

(1) guarantees $\lim f(s_n)$ exists, and (2) guarantees $\tilde{f}(a)$ is not ambiguous. Thus, $\tilde{f}$ is continuous at a. Similar method for $\tilde{f}(b)$.

$\Longleftarrow$: Since $\tilde{f}$ is continuous on $[a,b]$, $\tilde{f}$ is also uniformly continuous on $[a,b]$ by 3.3.2, so clearly f is uniformly continuous on (a,b).

■

Theorem 3.3.5. *Let f be a continuous function on an interval I [I may be bounded or unbounded]. Let I° be the interval obtained by removing from I any endpoints that happen to be in I. If f is differentiable on I° and if f' is bounded on I°, then f is uniformly continuous on I.*

3.4 Limits of Functions

Definition 3.4.1. Let $S \subset \mathbb{R}$ and $a \in \mathbb{R}$ or a symbol ∞ or $-\infty$ that is the limit of some sequence in S, and let L be a real number or symbol $+\infty$ or $-\infty$. We write $\lim_{x\to a^S} f(x) = L$ if

$$f \text{ is a function defined on } S,$$

and

$$\text{for every sequence } (x_n) \text{ in } S \text{ with limit } a, \text{ we have } \lim_{n\to\infty} f(x_n) = L.$$

Recall the definition of continuity, now we can say that a function f is continuous at a in $\operatorname{dom}(f) = S \iff \lim_{x\to a^S} f(x) = f(a)$. Also notice that when limits exist, they are unique. In other words, there is only one L equals to $\lim_{x\to a^S} f(x)$.

Now let's define the various standard limit concepts for functions.

Definition 3.4.2.

(a) For $a \in \mathbb{R}$ and a function f we write $\lim_{x\to a} f(x) = L$ provided $\lim_{x\to a^S} f(x) = L$ for some set $S = J\backslash\{a\}$ where J is an open interval containing a. Such a limit $\lim_{x\to a^S}$ is called the *[two-sided]* limit of f at a. Note that neither $f(a)$ needs to be defined or $\lim_{x\to a} f(x)$ needs to be equal $f(a)$, unless we want to say f is continuous at a.

(b) For $a \in \mathbb{R}$ and a function f we write $\lim_{x\to a^+} f(x) = L$ provided $\lim_{x\to a^S} f(x) = L$ for some open interval $S = (a, b)$. This is called the *[right-hand]* limit. Again f need not be defined at a.

(c) For $a \in \mathbb{R}$ and a function f we write $\lim_{x\to a^-} f(x) = L$ provided $\lim_{x\to a^S} f(x) = L$ for some open interval $S = (c, a)$. This is called the *[left-hand]* limit.

(d) For a function f we write $\lim_{x\to\infty} f(x) = L$ provided $\lim_{x\to\infty^S} f(x) = L$ for some interval $S = (c, \infty)$. Likewise, For a function f we write $\lim_{x\to-\infty} f(x) = L$ provided $\lim_{x\to-\infty^S} f(x) = L$ for some interval $S = (-\infty, b)$

Theorem 3.4.3. *Let f_1 and f_2 be functions for which the limits $L_1 = \lim_{x\to a^S} f_1(x)$ and $L_2 = \lim_{x\to a^S} f_2(x)$ exist and are finite. Then*

(i) $\lim_{x\to a^S}(f_1 + f_2)(x)$ *exists and equals* $L_1 + L_2$*;*

(ii) $\lim_{x\to a^S}(f_1 f_2)(x)$ *exists and equals* $L_1 L_2$*;*

(iii) $\lim_{x\to a^S}(f_1/f_2)(x)$ *exists and equals* L_1/L_2 *provided* $L_2 \neq 0$ *and* $f_2(x) \neq 0$ *for* $x \in S$.

Proof. By the assumption since both f_1 and f_2 are defined on S and a is the limit of some sequence in S, clearly the functions $f_1 + f_2$ and $f_1 f_2$ are defined on$S$ and so is f_1/f_2 if

$f_2(x) \neq 0$ for $x \in S$.

By the assumption, for every sequence (x_n) in S with limit a, we have $L_1 = \lim_{n\to\infty} f_1(x_n)$ and $L_2 = \lim_{n\to\infty} f_2(x_n)$. Now by the basic theorems of the limits, we have

$$\lim_{n\to\infty}(f_1+f_2)(x_n) = \lim_{n\to\infty} f_1(x_n) + \lim_{n\to\infty} f_2(x_n) = L_1 + L_2$$

and

$$\lim_{n\to\infty}(f_1 f_2)(x_n) = \left[\lim_{n\to\infty} f_1(x_n)\right] \cdot \left[\lim_{n\to\infty} f_2(x_n)\right] = L_1 L_2$$

Similar proof for (iii). ■

Theorem 3.4.4. *Let f be a function for which the limit $L = \lim_{x\to a^S} f(x)$ exists and is finite. If g is a function defined on $\{f(x) : x \in S\} \cup \{L\}$ that is continuous at L, then $\lim_{x\to a^S} g\circ f(x)$ exists and equals $g(L)$.*

Proof. First note that $g \circ f$ is defined on S by our assumptions. Consider a sequence (x_n) in S with limit a. Then we have $L = \lim_{n\to\infty} f(x_n)$. Since g is continuous at L, it follows that

$$g(L) = \lim_{n\to\infty} g(f(x_n)) = \lim_{n\to\infty} g\circ f(x_n)$$

Hence $\lim_{n\to a^S} g\circ f(x_n) = g(L)$. ■

Be careful that for this theorem to work, g needs to be **continuous** at L.

Theorem 3.4.5. *Let f be a function defined on a subset S of $\mathbb{R}$, let a be a real number that is the limit of some sequence in S, and let L be a real number, then $\lim_{x\to a^S} f(x) = L$ if and only if*

$$\text{for each } \epsilon > 0 \text{ there exists } \delta > 0 \text{ such that } x \in S \text{ and } |x-a| < \delta \text{ imply } |f(x) - L| < \epsilon \tag{1}$$

Proof.

$\Longrightarrow$: Suppose $\lim_{n\to a^S} f(x) = L$ but (1) does not hold. This means there exists $\epsilon > 0$ such that for each $\delta > 0$ and $n \in \mathbb{N}$**WHY???**, there exists $x_n \in S$ such that $|x_n - a| < \delta$ but $|f(x) - L| \geq \epsilon$. Hence x_n is a sequence in S with limit a but $\lim_{n\to\infty} f(x_n) = L$ fails. This is a contradiction.

$\Longleftarrow$: Consider an arbitrary sequence (x_n) in S such that $\lim_{n\to\infty} x_n = a$. Thus, choose $\epsilon = \delta$ and there exists N such that

$$n > N \implies |x_n - a| < \delta \implies |f(x_n) - L| < \epsilon.$$

The last implication comes from the assumption, so $\lim_{n\to a^S} f(x) = L$.

■

Corollary 3.4.5.1. *Let f be a function defined on $J\backslash\{a\}$ for some open interval J containing a, and let L be a real number. Then $\lim_{x\to a^s} f(x) = L$ if and only if*

$$\text{for each } \epsilon > 0 \text{ there exists } \delta > 0 \text{ such that } 0 < |x-a| < \delta \implies |f(x) - L| < \epsilon \quad (1)$$

Corollary 3.4.5.2. *Let f be a function defined on some interval (a, b), and let L be a real number. Then $\lim_{x\to a^+} f(x) = L$ if and only if*

$$\text{for each } \epsilon > 0 \text{ there exists } \delta > 0 \text{ such that } a < x < a + \delta \implies |f(x) - L| < \epsilon \quad (1)$$

Now let's give some general conditions for the limit of function in different situations: $\lim_{x\to s} f(x) = L \iff$

- L is finite:
  - $s = a$: for each $\epsilon > 0$ there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$.
  - $s = a^+$: for each $\epsilon > 0$ there exists $\delta > 0$ such that $a < x < a + \delta$ implies $|f(x) - L| < \epsilon$.
  - $s = a^-$: for each $\epsilon > 0$ there exists $\delta > 0$ such that $a - \delta < x < a$ implies $|f(x) - L| < \epsilon$.
  - $s = \infty$: for each $\epsilon > 0$ there exists $\alpha < \infty$ such that $x > \alpha$ implies $|f(x) - L| < \epsilon$.
  - $s = -\infty$: for each $\epsilon > 0$ there exists $\alpha > -\infty$ such that $x < \alpha$ implies $|f(x) - L| < \epsilon$.
- $L = +\infty$:
  - $s = a$: for each $M > 0$ there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $f(x) > M$.
  - $s = a^+$: for each $M > 0$ there exists $\delta > 0$ such that $a < x < a + \delta$ implies $f(x) > M$.
  - $s = a^-$: for each $M > 0$ there exists $\delta > 0$ such that $a - \delta < x < a$ implies $f(x) > M$.
  - $s = \infty$: for each $M > 0$ there exists $\alpha < \infty$ such that $x > \alpha$ implies $f(x) > M$.
  - $s = -\infty$: for each $M > 0$ there exists $\alpha > -\infty$ such that $x < \alpha$ implies $f(x) > M$.
- $L = -\infty$:
  - $s = a$: for each $M < 0$ there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $f(x) < M$.
  - $s = a^+$: for each $M < 0$ there exists $\delta > 0$ such that $a < x < a + \delta$ implies $f(x) < M$.

– $s = a^-$: for each $M < 0$ there exists $\delta > 0$ such that $a - \delta < x < a$ implies $f(x) < M$.
– $s = \infty$: for each $N < 0$ there exists $\alpha < \infty$ such that $x > \alpha$ implies $f(x) < N$.
– $s = -\infty$: for each $N < 0$ there exists $\alpha > -\infty$ such that $x < \alpha$ implies $f(x) < N$.

Theorem 3.4.6. *Let f be a function defined on $J \backslash \{a\}$ for some open interval J containing a. Then $\lim_{x \to a} f(x)$ exists $\iff$ the limits $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ both exist and are equal to each other, thereby all three limits are equal.*

Proof.

$\Longrightarrow$: Suppose L is finite. Since $\lim_{x \to a} f(x) = L$, (1) in 3.4.5.1 holds, and then (1) in 3.4.5.2 also holds. Thus we have $\lim_{x \to a^+} f(x) = L$; similarly for $\lim_{x \to a^-} f(x) = L$.

If L is infinite, say $+\infty$, then consider an arbitrary $M > 0$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \implies f(x) > M \tag{2}$$

Then clearly

$$a < x < a + \delta \implies f(x) > M \tag{3}$$

and

$$a - \delta < x < a \implies f(x) > M \tag{4}$$

so $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = +\infty$.

$\Longleftarrow$: Suppose L is finite and $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$. Consider $\epsilon > 0$, then we apply 3.4.5.2 and its analogue for a^- to get $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$a < x < a + \delta_1 \implies |f(x) - L| < \epsilon$$

and

$$a - \delta_2 < x < a \implies |f(x) - L| < \epsilon.$$

If $\delta = \min\{\delta_1, \delta_2\}$, then

$$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon,$$

so $\lim_{x \to a} f(x) = L$ by 3.4.5.1.

If L is infinite, say $+\infty$, so $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = +\infty$ then for each $M > 0$ there exists $\delta_1 > 0$ such that (2) holds, and there exists $\delta_2 > 0$ so that (3) holds. Then (1) holds with $\delta = \min\{\delta_1, \delta_2\}$. We conclude $\lim_{x \to a} f(x) = +\infty$.

■

Chapter 4

Sequences and Series of Functions

4.1 Power Series

Definition 4.1.1. Given a sequence $(a_n)_{n=0}^{\infty}$ of real numbers, the series $\sum_{n=0}^{\infty} a_n x^n$ is called a power series, which is a function of x provided it converges for some or all x. One of the following holds for a power series with coefficients (a_n):

(a) The power series converge for all $x \in \mathbb{R}$;

(b) The power series converges only for $x = 0$;

(c) The power series converges for all x in some bounded interval centered at 0; the interval may be open, half-open, or closed.

Theorem 4.1.2. *For the power series $\sum a_n x^n$, let*

$$\beta = \limsup |a_n|^{1/n} \quad \textit{and} \quad R = \frac{1}{\beta}$$

[If $\beta = 0$ we set $R = +\infty$, and if $\beta = +\infty$ we set $R = 0$.] Then

(i) The power series converges for $|x| < R$;

(ii) The power series diverges for $|x| > R$.

We call R the radius of convergence *for the power series. For $|x| = R$ we need some extra care.*

Proof. The proof follows easily from the Root Test 2.8.6. Define $\alpha_x = \limsup |a_n x^n|^{1/n}$, then we have

$$\alpha_x = \limsup |a_n x^n|^{1/n} = \limsup |x||a_n|^{1/n} = |x| \cdot \limsup |a_n|^{1/n} = \beta|x|$$

Now we need to consider three different cases:

1. Suppose $0 < R < +\infty$. Then $\alpha_x = \frac{|x|}{R}$. If $|x| < R$, then $\alpha_x < 1$, so the series converge by the root test. Likewise, if $|x| > R$, then $\alpha_x > 1$ and the series diverges.

2. Suppose $R = +\infty$. Then $\beta = 0$ and $\alpha_x = 0$ no matter what x is. Hence the power series converges for all x.

3. Suppose $R = 0$. Then $\beta = +\infty$ and $\alpha_x = +\infty$ for $x \neq 0$. Thus the series diverges for $x \neq 0$ by the root test.

$\blacksquare$

Note that if $\lim \left|\frac{a_{n+1}}{a_n}\right|$ exists, then it equals β by 2.6.2.1. This limit is often easier to calculate than $\limsup |a_n|^{1/n}$.

4.2 Uniform Convergence

Definition 4.2.1. Let (f_n) be a sequence of real-valued functions defined on a set $S \subseteq \mathbb{R}$. The sequence (f_n) *converges pointwise* to a function f defined on S if

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for all} \quad x \in S.$$

We often write $\lim f_n = f$ *pointwise [on S]* or $f_n \to f$ *pointwise [on S]*

Observe that saying $f_n \to f$ *pointwise [on S]* is equivalent to the following:

for each $\epsilon > 0$ and x in S there exists N such that $|f_n(x) - f(x)| < \epsilon$ for $n > N$.

Definition 4.2.2. Let (f_n) be a sequence of real-valued functions defined on a set $S \subseteq \mathbb{R}$. The sequence (f_n) *converges uniformly* on S to a function f defined on S if

for each $\epsilon > 0$ there exists a number N such that
$|f_n(x) - f(x)| < \epsilon$ for all $x \in S$ and all $n > N$.

We write $\lim f_n = f$ *uniformly [on S]* or $f_n \to f$ *uniformly [on S]*

Theorem 4.2.3. *The uniform limit of continuous functions is continuous. More precisely, let (f_n) be a sequence of functions on a set $S \subseteq \mathbb{R}$, suppose $f_n \to f$ uniformly on S, and suppose $S = dom(f)$. If each f_n is continuous at x_0 in S, then f is continuous at x_0. [so if each f_n is continuous on S, then f is continuous on S.]*

Proof. The critical inequality for this proof is

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \tag{1}$$

Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that

$$n > N \implies \forall x \in S,\ |f_n(x) - f(x)| < \frac{\epsilon}{3}$$

since $f_n \to f$ uniformly on S. In particular,

$$\forall x \in S,\ |f_{N+1}(x) - f(x)| < \frac{\epsilon}{3} \tag{2}$$

Since f_{N+1} is continuous at x_0 there is a $\delta > 0$ such that

$$\forall x \in S,\ |x - x_0| < \delta \implies |f_{N+1}(x) - f_{N+1}(x_0)| < \frac{\epsilon}{3}; \tag{3}$$

Now apply (1) with $n = N + 1$, (2) twice and (3) once to conclude

$$\forall x \in S,\ |x - x_0| < \delta \implies |f(x) - f(x_0)| < 3 \cdot \frac{\epsilon}{3} = \epsilon;$$

Thus f is continuous at x_0. ∎

The contrapositive of this theorem is useful to show f_n does not uniformly converge to f on S: If f is not continuous at $x_0 \in S$ but f_n is continuous at x_0, then the statement that "$f_n \to f$ uniformly on S" is **incorrect**.

It is worthy to mention that the uniform convergence can be reformulated as follows: *A sequence* (f_n) *of functions on a set* $\subseteq \mathbb{R}$ *converges uniformly to a function* f *on* $S \iff$

$$\lim_{n\to\infty} \sup\{|f(x) - f_n(x)| : x \in S\} = 0$$

Thus we can decide whether a sequence (f_n) converges uniformly to f by calculating $\sup\{|f(x) - f_n(x)| : x \in S\}$ for each n. If $f - f_n$ is differentiable, we may use calculus to find these suprema.

4.3 More on Uniform Convergence

Here are two important facts about integration

1. If g and h are integrable on $[a,b]$ and if $g(x) \leq h(x)$ for all $x \in [a,b]$, then $\int_a^b g(x)dx \leq \int_a^b h(x)dx$.

2. If g is integrable on $[a,b]$, then

$$\left|\int_a^b g(x)dx\right| \leq \int_a^b |g(x)|dx.$$

Also continuous functions on closed intervals are integrable.

Theorem 4.3.1. *Let (f_n) be a sequence of continuous functions on $[a,b]$, and suppose $f_n \to f$ uniformly on $[a,b]$. Then*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Proof. By **??** $f$is continuous, so $f_n - f$ are all integrable on $[a,b]$. Let $\epsilon > 0$. Since $f_n \to f$ uniformly on $[a,b]$, there exists N such that $|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$ for all $x \in [a,b]$ and all $n > N$. Consequently $n > N$ implies

$$\begin{aligned}\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| &= \left|\int_a^b [f_n(x) - f(x)]dx\right| \\ &\leq \int_a^b |f_n(x) - f(x)|dx \\ &\leq \int_a^b \frac{\epsilon}{b-a}dx \\ &= \epsilon\end{aligned}$$

Thus $\lim\limits_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$. ∎

Definition 4.3.2. A sequence (f_n) of functions defined on a set $S \subseteq \mathbb{R}$ is *uniformly Cauchy on S* if

for each $\epsilon > 0$ there exists a number N such that
$|f_n(x) - f_m(x)| < \epsilon$ for all $x \in S$ and all $m, n > N$.

Notice that uniformly convergent sequences of functions are uniformly Cauchy.

Theorem 4.3.3. *Let (f_n) be a sequence of functions defined and uniformly Cauchy on a set $S \subseteq \mathbb{R}$. Then there exists a function f on S such that $f_n \to f$ uniformly on S.*

Proof. Since (f_n) is uniformly Cauchy, $(f_n(x_0))$ is a Cauchy sequence for each $x_0 \in S$. Now for each $x \in S$, define $f(x) = \lim_{n\to\infty} f_n(x)$. This defines a function f on S such that $f_n \to f$ pointwise on S.

Let $\epsilon > 0$. Since (f_n) is uniformly Cauchy, there is a number N such that

$$|f_n(x) - f_m(x)| < \frac{\epsilon}{2} \quad \text{for all} \quad x \in S \quad \text{and all} \quad m, n < N \tag{1}$$

Consider $m > N$ and $x \in S$. (1) tells us that $f_n(x)$ lies in $(f_m(x) - \frac{\epsilon}{2}, f_m(x) + \frac{\epsilon}{2})$ for all $n > N$. Therefore, $f(x) = \lim_{n\to\infty} f_n(x)$ lies in $[f_m(x) - \frac{\epsilon}{2}, f_m(x) + \frac{\epsilon}{2}]$. In other words,

$$|f(x) - f_m(x)| \leq \frac{\epsilon}{2} \quad \text{for all} \quad x \in S \quad \text{and} \quad m > N$$

Then of course

$$|f(x) - f_m(x)| \leq \epsilon \quad \text{for all} \quad x \in S \quad \text{and} \quad m > N$$

Thus $f_m \to f$ uniformly on S, as desired. ∎

A *series of functions* is an expression $\sum_{k=0}^{\infty} g_k$ or $\sum_{k=0}^{\infty} g_k(x)$ which makes sense provided the sequence of partial sums $\sum_{k=0}^{n} g_k$ converges, or diverges to $+\infty$ or $-\infty$ pointwise. If the sequence of partial sums converges uniformly on a set S to $\sum_{k=0}^{\infty}$, then we say the *series is uniformly convergent* on S.

Theorem 4.3.4. *Consider a series $\sum_{k=0}^{\infty} g_k$ of functions on a set $S \subseteq \mathbb{R}$. Suppose each g_k is continuous on S and the series converges uniformly on S. Then the series $\sum_{k=0}^{\infty} g_k$ represents a continuous function on S.*

Proof. Each partial sum $f_n = \sum_{k=1}^{n} g_k$ is continuous and the sequence (f_n) converges uniformly on S. Hence the limit function is continuous by **??**. ∎

There is an analogue between the Cauchy criterion for a normal series $\sum a_k$ and the one for a series of functions $\sum g_k$: *The sequence of partial sums of a series $\sum_{k=0}^{\infty} g_k$ of functions is uniformly Cauchy on a set S $\iff$ the series satisfies the* Cauchy criterion*[uniformly on S]*:

$$\text{for each } \epsilon > 0 \text{ there exists a number } N \text{ such that}$$
$$n \geq m > N \text{ implies } \left|\sum_{k=m}^{n} g_k(x)\right| < \epsilon \text{ for all } x \in S.$$

Theorem 4.3.5. *If a series $\sum_{k=0}^{\infty} g_k$ of functions satisfies the Cauchy criterion uniformly on a set S, then the series converges uniformly on S.*

Proof. Let $f_n = \sum_{k=0}^{n} g_k$. The sequence (f_n) of partial sums is uniformly Cauchy on S, so (f_n) converges uniformly on S by **??**. ■

Theorem 4.3.6 (Weierstrass M-test)**.** *Let (M_k) be a sequence of nonnegative real numbers where $\sum M_k < \infty$. If $|g_k(x)| \leq M_k$ for all x in a set S, then $\sum g_k$ converges uniformly on S.*

Proof. We want to verify the Cauchy criterion of such $\sum g_k$ on S. Let $\epsilon > 0$. Since the series $\sum M_k$ converges, it satisfies the Cauchy criterion in 2.8.3. So there exists a number N such that

$$n \geq m > N \implies \sum_{k=m}^{n} M_k < \epsilon.$$

Hence if $n \geq m > N$ and x is in S, then

$$\left| \sum_{k=m}^{n} g_k(x) \right| \leq \sum_{k=m}^{n} |g_k(x)| \leq \sum_{k=m}^{n} M_k < \epsilon.$$

Thus the series $\sum g_k$ satisfies the Cauchy criterion uniformly on S, and **??** shows that it converges uniformly on S. ■

If the series $\sum g_n$ converges uniformly on a set S, then

$$\lim_{n\to\infty} \sup\{|g_n(x)| : x \in S\} = 0.$$

4.4 Differentiation and Integration of Power Series

Theorem 4.4.1. *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$ [possibly $R = +\infty$]. If $0 < R_1 < R$, then the power series converges uniformly on $[-R_1, R_1]$ to a continuous function.*

Proof. Consider $0 < R_1 < R$. First observe that the series $\sum a_n x^n$ and $\sum |a_n| x^n$ have the same radius of convergence. Since $|R_1| < R$, we have $\sum a_n R_1^n < \infty$. Clearly we have $|a_n x^n| \leq |a_n| R_1^n$ for all $x \in [-R_1, R_1]$, so the series $\sum a_n x^n$ converges uniformly on $[-R_1, R_1]$ by 4.3.6. Then by **??** since each $a_n x^n$ is continuous, the limit of the series of the functions is also continuous. ∎

Corollary 4.4.1.1. *The power series $\sum a_n x^n$ converges to a continuous function on the open interval $(-R, R)$.*

Proof. If $x_0 \in (-R, R)$, then $x_0 \in (-R_1, R_1)$ for some $R_1 < R$. The theorem shows the limit of the series is continuous at x_0. ∎

Lemma 4.4.2. *If the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, then the power series*

$$\sum_{n=1}^{\infty} n a_n x^{n-1} \quad \textit{and} \quad \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

also have radius of convergence R.

Proof. First observe the series $\sum n a_n x^{n-1}$ and $\sum n a_n x^n$ have the same radius of convergence. Same for $\sum \frac{a_n}{n+1} x^{n+1}$ and $\sum \frac{a_n}{n+1} x^n$. It is because one is x multiple of the other.

For the series $\sum n a_n x^n$, because $\limsup (n|a_n|)^{1/n} = \limsup n^{1/n} |a_n|^{1/n}$ and $\lim n^{1/n} = 1$, so $\limsup (n|a_n|)^{1/n} = \limsup |a_n|^{1/n} = \beta$. Hence $\sum n a_n x^n$ has radius of convergence R.

Similarly, for the series $\sum \frac{a_n}{n+1} x^n$, because $\limsup (\frac{|a_n|}{n+1})^{1/n} = \limsup (\frac{1}{n+1})^{1/n} \cdots |a_n|^{1/n} = 1 \cdot \limsup |a_n|^{1/n} = \beta$ by 2.6.1, so the series $\sum \frac{a_n}{n+1} x^n$ has radius of convergence R. ∎

Theorem 4.4.3. *Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Then*

$$\int_0^x f(t) dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \quad \textit{for} \quad |x| < R$$

Proof. Fix $x < 0$. The case $x > 0$ is similar. On the interval $[x, 0]$, the sequence of partial sums $\sum_{k=0}^{n} a_k t^k$ converges uniformly to $f(t)$ by 4.4.1. Consequently, by 4.3.1 we have

$$\begin{aligned}
\int_x^0 f(t)dt &= \lim_{n\to\infty} \int_x^0 \left(\sum_{k=0}^{n} a_k t^k\right) dt \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} a_k \int_x^0 t^k dt \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} a_k \left[\frac{0^{k+1} - x^{k+1}}{k+1}\right] \\
&= -\left(\sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}\right) \\
&= -\left(\int_0^x f(t)dt\right)
\end{aligned}$$

■

The last theorem shows that a power series can be integrated term-by-term inside its interval of convergence. The next theorem shows that term-by-term differentiation is also legal

Theorem 4.4.4. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ have radius of convergence $R > 0$. Then f is differentiable on $(-R, R)$ and*

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \textit{for} \quad |x| < R$$

Proof. We begin with the series $g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and observe this series converges for $|x| < R$ by 4.4.2. By 4.4.3, we can integrate $g(x)$ term-by-term:

$$\int_0^x g(t)dt = \sum_{n=1}^{\infty} a_n x^n = f(x) - a_0 \quad \text{for} \quad |x| < R.$$

Thus if $0 < R_1 < R$, then

$$f(x) = \int_{-R_1}^{x} g(t)dt + k \quad \text{for} \quad |x| \le R_1$$

where $k = a_0 - \int_{-R_1}^{0} g(t)dt$. Since g is continuous, the Fundamental Theorem of Calculus shows f is differentiable and $f'(x) = g(x)$. Thus

$$f'(x) = g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{for} \quad |x| \le R_1$$

■

Theorem 4.4.5 (Abel's Theorem)**.** *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with finite positive radius of convergence R. If the series converges at $x = R$, then f is continuous at $x = R$; if the series converges at $x = -R$, then f is continuous at $x = -R$.*

Proof.

Case 1. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence 1 and the series converges at $x = 1$. We will prove f is continuous on $[0,1]$. We need to show that f is continuous on $[0,1]$. By subtracting a constant from f, we may assume $f(1) = \sum_{n=0}^{\infty} a_n = 0$. Let $f_n(x) = \sum_{k=0}^{n} a_k x^k$ and $s_n = \sum_{k=0}^{n} a_k = f_n(1)$ for $n = 0, 1, 2, \ldots$. Since $f_n(x) \to f(x)$ pointwise on $[0,1]$ and each f_n is continuous, by 4.2.3 it suffices to show $f_n \to f$ uniformly on $[0,1]$. By 4.3.3 it suffices to show the convergence is uniformly Cauchy.

For $m < n$, we have

$$\begin{aligned} f_n(x) - f_m(x) &= \sum_{k=m+1}^{n} a_k x^k \\ &= \sum_{k=m+1}^{n} (s_k - s_{k-1}) x^k \\ &= \sum_{k=m+1}^{n} s_k x^k - x \sum_{k=m+1}^{n} s_{k-1} x^{k-1} \\ &= \sum_{k=m+1}^{n} s_k x^k - x \sum_{k=m}^{n-1} s_k x^k \end{aligned}$$

and therefore

$$f_n(x) - f_m(x) = s_n x^n - s_m x^{n+1} + (1-x) \sum_{k=m+1}^{n-1} s_k x^k \tag{1}$$

By the definition of s_n, we have $\lim s_n = \sum_{k=0}^{\infty} a_k = f(1) = 0$. Given $\epsilon > 0$, there is an integer N so that $|s_n| < \frac{\epsilon}{3}$ for all $n \geq N$. Then for $n > m \geq N$ and $x \in [0,1)$, we have

$$\begin{aligned} \left| (1-x) \sum_{k=m+1}^{n-1} s_k x^k \right| &\leq \frac{\epsilon}{3} (1-x) \sum_{k=m+1}^{n-1} x^k \\ &= \frac{\epsilon}{3} (1-x) x^{m+1} \frac{1 - x^{n-m-1}}{1-x} \\ &< \frac{\epsilon}{3} \end{aligned} \tag{2}$$

Since $\left|(1-x) \sum_{k=m+1}^{n-1} s_k x^k \right| < \frac{\epsilon}{3}$ for $x = 1$, combining (1) and (2), for $n > m \geq N$ and $x \in [0,1]$,

$$|f_n(x) - f_m(x)| \leq |s_n| x^n + |s_m| x^{m+1} + \frac{\epsilon}{3} < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Thus the sequence (f_n) is uniformly Cauchy on $[0,1]$, and its limit f is continuous.

Case 2. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, $0 < R < \infty$, and the series converges at $x = R$. Let $g(x) = f(Rx)$ and note that

$$g(x) = \sum_{n=0}^{\infty} a_n R^n x^n \quad \text{for} \quad |x| < 1.$$

This series has radius of convergence 1, and it converges at $x = 1$. By Case 1, g is continuous at $x = 1$. Since $f(x) = g(\frac{x}{R})$, it follows that f is continuous at $x = R$.

Case 3. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, $0 < R < \infty$, and the series converges at $x = -R$. Let $h(x) = f(-x)$ and note that

$$h(x) = \sum_{n=0}^{\infty} (-1)^n a_n x^n \quad \text{for} \quad |x| < R.$$

This series for h converges at $x = R$, so h is continuous at $x = R$ by Case 2. It follows that $f(x) = h(-x)$ is continuous at $x = -R$.

■

Chapter 5

Differentiation

5.1 Basic Properties of the Derivative

Definition 5.1.1. Let f be a real-valued function defined on an open interval containing a point a. We say f is *differentiable* at a, or f has a derivative at a, if the limit

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

exists and is finite. We will write $f'(a)$ for the derivative of f at a:

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

whenever this limit exists and is finite.

The domain of f' is the set of points at which f is differentiable; thus $\operatorname{dom}(f')\subseteq\operatorname{dom}(f)$.

Theorem 5.1.2. *If f is differentiable at a point a, then f is continuous at a.*

Proof. We are given $f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$, and we need to prove $\lim_{x\to a}f(x)=f(x)$. We have

$$f(x)=(x-a)\frac{f(x)-f(a)}{x-a}+f(a)$$

for $x\in\operatorname{dom}(f), x\neq a$. Since $\lim_{x\to a}(x-a)=0$ and $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists and is finite, by 3.4.3(ii), $\lim(x-a)\frac{f(x)-f(a)}{x-a}=0$. Thus $\lim_{x\to a}f(x)=f(a)$. ∎

Theorem 5.1.3. *Let f and g be functions that are differentiable at the point a. Each of the functions cf, $f+g$, fg, and f/g is also differentiable at a, except f/g is $g(a)=0$ since f/g is not defined at a in this case. The formulas are:*

(i) $(cf)'(a)=c\cdot f'(a)$;

(ii) $(f+g)'(a)=f\prime(a)+g'(a)$;

(iii) Product rule: $(fg)'(a)=f(a)g'(a)+f'(a)g(a)$;

(iv) Quotient rule: $(f/g)'(a)=\frac{[g(a)f'(a)-f(a)g'(a)]}{g^2(a)}$ *if* $g(a)\neq 0$.

Proof.

(i)

$$(cf)'(a)=\lim_{x\to a}\frac{(cf)(x)-(cf)(a)}{x-a}=\lim_{x\to a}c\cdot\frac{f(x)-f(a)}{x-a}=c\cdot f'(a)$$

(ii)

$$\frac{(f+g)(x)-(f+g)(a)}{x-a}=\frac{f(x)-f(a)}{x-a}+\frac{g(x)-g(a)}{x-a}$$

(iii) See the textbook.

(iv) See the textbook.

■

Theorem 5.1.4 (Chain Rule)**.** *If f is differentiable at a and g is differentiable at $f(a)$, then the composite function $g \circ f$ is differentiable at a and we have $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$.*

If f is differentiable on an interval I and if g is differentiable on $\{f(x) : x \in I\}$, then $(g \circ f)'$ is exactly $(g' \circ f) \cdot f'$ on I.

Chapter 6

Useful Tricks

1. 8.5
2. 8.9
3. 9.9
4. 9.12
5. The set $\mathbb{Q}$ of rational number can be listed as a sequence (r_n). Given any real number a there exists a subsequence (r_{n_k}) of (r_n) converging to a.