Math 104 Homework 6 Solutions

UC Berkeley, Summer 2021

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1. (Ross 18.10) Suppose f is continuous on $[0, 2]$ and $f(0) = f(2)$. Prove that there exist $x, y \in [0, 2]$ such that $|y - x| = 1$ and $f(x) = f(y)$. (Hint: Consider $g(x) = f(x + 1) - f(x)$ on $[0, 1]$.)

Solution. The function $g : [0, 1] \to \mathbb{R}$ defined by $g(x) = f(x + 1) - f(x)$ is continuous. Note that $g(0) = f(1) - f(0)$ and $g(1) = f(2) - f(1) = f(0) - f(1) = -g(0)$. If $g(0) = 0$ then $f(1) = f(0)$. Otherwise, by the intermediate value theorem there exists $x_0 \in (0, 1)$ such that $g(x_0) = 0$; then $f(x_0 + 1) = f(x_0)$.

2. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on an open interval (a, b), then f is *bounded* on (a, b), i.e. there exists $M > 0$ such that $f(x) \leq M$ for all $x \in (a, b)$.

Solution. Suppose f is not bounded on (a, b). Then for each $n \in \mathbb{N}$ there exists $x_n \in (a, b)$ such that $|f(x_n)| > n$. By Bolzano-Weierstrass, (x_n) has a subsequence (x_{n_k}) which converges to some $x \in [a, b]$. Since $|f(x_{n_k})| \to \infty$, if $x \in (a, b)$ then continuity is violated, and if $x \in \{a, b\}$ then the continuous extension theorem is violated. Hence f is not uniformly continuous on (a, b).

3. (a) Let f and g be two continuous real-valued functions on $\mathbb{R}$. Prove that if $f(q) = g(q)$ for every $q \in \mathbb{Q}$, then $f(x) = g(x)$ for all $x \in \mathbb{R}$.
(b) Let (X, d_X) and (Y, d_Y) be two metric spaces, and let f and g be two continuous functions from X to Y. Formulate and prove a generalization of part (a).

Solution. (a) Let $h := f - g$ is continuous on $\mathbb{R}$ and $h(x) = 0$ for all $x \in \mathbb{Q}$. Since every irrational number is a limit point of $\mathbb{Q}$, it follows that $h(x) = 0$ for all $x \in \mathbb{R}$, hence $f(x) = g(x)$ for all $x \in \mathbb{R}$.
(b) Claim: If E is a dense subset of X and $f(x) = g(x)$ for every $x \in E$, then $f(x) = g(x)$ for all $x \in X$. Proof: Let $x \in X \setminus E$. Since E is dense in X, there exists a sequence (x_n) of points in E such that $x_n \to x$. Since f and g are continuous functions on X, $\lim f(x_n)$ and $\lim g(x_n)$ exist and equal $f(x)$ and $g(x)$, respectively. Hence $f(x) = \lim f(x_n) = \lim g(x_n) = g(x)$.a

4. For any rational number $q \in \mathbb{Q}$, let $\varphi(q) := \min\{n \in \mathbb{N} : \exists m \in \mathbb{Z} \text{ such that } q = \frac{m}{n}\}$. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \\ \frac{1}{\varphi(x)} & \text{if } x \in \mathbb{Q}. \end{cases}$$

Prove that f is discontinuous at every $x \in \mathbb{Q}$ and continuous at every $x \in \mathbb{R} \setminus \mathbb{Q}$.

Solution. (i) Let $x \in \mathbb{Q}$, so $f(x) \neq 0$. Since $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$, there exists a sequence (x_n) in $\mathbb{R} \setminus \mathbb{Q}$ such that $x_n \to x$. Then $\lim f(x_n) = 0 \neq f(x)$, so f is not continuous at x. (ii) Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Let $\varepsilon > 0$. Let $M \in \mathbb{N}$ be such that $\frac{1}{M} < \varepsilon$. Let $N := (M-1)!$. There exists $k \in \mathbb{Z}$ such that $\frac{k}{N} < x < \frac{k+1}{N}$. Let $q \in \mathbb{Q}$ such that $\frac{k}{N} < q < \frac{k+1}{N}$. Suppose $\varphi(q) = \frac{j}{n}$ for some $n < M$ and $j \in \mathbb{Z}$; then n divides $N = (M-1)!$ so $q = \frac{j \cdot \frac{N}{n}}{N}$, contradicting $\frac{k}{N} < q < \frac{k+1}{N}$. Hence $\varphi(q) \geq M$, so $f(q) \leq \frac{1}{M} < \varepsilon$. Therefore for $\delta = \min(x - \frac{k}{N}, \frac{k+1}{N} - x)$, if $|y - x| < \delta$ then $|f(y) - f(x)| = |f(y)| < \varepsilon$, so f is continuous at x.

5. (a) Let (X, d) be a metric space. Consider the metric space $(X \times X, d^*)$ where $d^*((x,y),(u,v)) = \max\{d(x,u), d(y,v)\}$ (see Homework 5 Problem 1.) Show that the original metric $d : X \times X \to \mathbb{R}$ is a uniformly continuous real-valued function on the metric space $X \times X$. (Hint: A triangle inequality similar to the one on Midterm Problem 6 might be useful.)
(b) Let E be a nonempty compact subset of X, and let $\delta = \sup\{d(x,y) : x, y \in E\}$. Use part (a) to prove that there exist $x, y \in E$ such that $d(x,y) = \delta$ (cf. Homework 4 Problem 3.)

Solution. (a) Let $\varepsilon > 0$. Set $\delta = \varepsilon/2$. Let $(x,y), (u,v) \in X \times X$ such that $d^*((x,y),(u,v)) < \delta$ (so $d(x,u) < \delta$ and $d(y,v) < \delta$.) By the triangle inequality,

$$d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$$

so

$$d(x,y) - d(x,u) \leq d(u,v) + d(v,y).$$

Similarly,

$$d(u,v) \leq d(u,x) + d(x,y) + d(y,v)$$

so

$$d(u,v) - d(x,y) \leq d(u,x) + d(y,v),$$

and hence

$$|d(u,v) - d(x,y)| \leq d(u,v) + d(y,v) < 2\delta = \varepsilon.$$

(b) By Homework 5 Problem 1(d), $E \times E$ is a compact subset of $X \times X$, so $d(E \times E) = \{d(x,y) : x, y \in E\}$ is a compact subset of $\mathbb{R}$, hence closed and bounded, and therefore $\delta = \sup\{d(x,y) : x, y \in E\} \in d(E \times E)$, so there exists $x, y \in E$ such that $d(x,y) = \delta$.

6. (a) Let (X, d) be a metric space, and let A be any nonempty subset of X. Define $f : X \to \mathbb{R}$ by $f(x) := d(x, A) = \inf\{d(x,y) : y \in A\}$ (see Homework 4 Problem 4.) Show that f is uniformly continuous on X. (Hint: Carefully argue the following skeleton of implications: $d(x,A) \leq d(x,a) \leq d(x,y) + d(y,a) \Rightarrow d(y,a) \geq d(x,A) - d(x,y) \Rightarrow d(y,A) \geq d(x,A) - d(x,y) \Rightarrow d(x,A) - d(y,A) \leq d(x,y)$.)
(b) Let E be a nonempty compact subset of X. Use part (a) to show that there exists $x_0 \in E$ such that $f(x_0) = \inf\{d(x,A) : x \in E\}$. In particular, if $A = \{a\}$ is a *singleton* (a set with only one element), then E has a closest element to a (cf. Homework 4 Problem 4.)
(c) Prove that if E is a nonempty compact subset of X and A is a closed subset of X and $E \cap A = \emptyset$, then $\inf\{d(x,a) : x \in E, a \in A\} > 0$ (there is a "gap" between E and A.)
(d) Find a counterexample to show that the conclusion in part (c) can fail if E is assumed to be closed but not compact.

Solution. (a) Let $\varepsilon > 0$. Let $\delta = \varepsilon$. Let $x, y \in X$ such that $d(x, y) < \delta$. Observe that for any $a \in A$, $d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a)$, so $d(y, a) \geq d(x, A) - d(x, y)$. Since this holds for every $a \in A$, it follows that $d(y, A) \geq d(x, A) - d(x, y)$, and hence $d(x, A) - d(y, A) \leq d(x, y)$. Similarly, $d(y, A) - d(x, A) \leq d(x, y)$, so $|d(x, A) - d(y, A)| \leq d(x, y) < \delta = \varepsilon$.
(b) If E is a nonempty compact subset of X. Then $f(E)$ is a compact subset of $\mathbb{R}$, so it is closed and bounded, hence $\inf f(E) = \inf\{d(x, A) : x \in E\} \in f(E)$, so there exists $x_0 \in E$ such that $f(x_0) = \inf\{d(x, A) : x \in E\}$.
(c) Suppose $\inf\{d(x, a) : x \in E, a \in A\} = 0$. Then there exists a sequences (x_n) in E such that $d(x_n, A) < \frac{1}{n}$ for each $n \in \mathbb{N}$. By compactness of E, (x_n) has a subsequence (x_{n_k}) which converges to some $x \in E$. It can be show that $d(x, A) = 0$, so there exists a sequence (a_n) in A which converges to x, which implies $x \in A$ because A is closed. Hence $x \in E \cap A$, which contradicts the assumption that $E \cap A = \emptyset$.
(d) $X = \mathbb{R}$, $E = \{n + \frac{1}{n} : n \in \mathbb{N}\}$, $A = \mathbb{N}$.

7. Let (X, d_X) be a discrete metric space, and let (Y, d_Y) be any metric space. Prove that any function $f : X \to Y$ is continuous.

Solution. Since every set in a discrete metric space is open, the preimage of any open set in Y under any $f : X \to Y$ is open, and hence every function $f : X \to Y$ is continuous.

8. Let (X, d) be a metric space. A *contraction* is a continuous function $f : X \to X$ with the property that there exists $c < 1$ that $d(f(x), f(y)) \leq c \cdot d(x, y)$ for all $x, y \in X$. Prove that if X is complete, then every contraction on X has a unique *fixed point*. (A fixed point of f is an element $x \in X$ such that $f(x) = x$.) (*Hint*: Construct a sequence beginning with some $x_0 \in X$ by repeatedly applying f; then argue that the sequence is Cauchy and hence convergent by completeness of X and verify that the limit is in fact a fixed point. Don't forget to show uniqueness.)

Solution. Let x_0 be any point in X, and let $x_{n+1} = f(x_n)$ for $n \geq 0$. Let $\varepsilon > 0$. There exists N such that $d(x_0, x_1) \sum_{k=N}^{\infty} c^k < \varepsilon$. Then for $m, n \geq N$, $d(x_m, x_n) \leq \sum_{k=N}^{\infty} d(x_k, x_{k+1}) \leq \sum_{k=N}^{\infty} c^k d(x_0, x_1) < \varepsilon$. Hence (x_n) is a Cauchy sequence in X, so by completeness of X, (x_n) converges to some $x \in X$. Let $\varepsilon > 0$. By convergence of (x_n), Cauchy property of (x_n), and continuity of f, there exists $N \in \mathbb{N}$ such that $d(x, x_N) < \varepsilon/3$, $d(x_N, f(x_N)) < \varepsilon/3$, and $d(f(x_N), f(x)) < \varepsilon/3$. Then $d(x, f(x)) \leq d(x, x_N) + d(x_N, f(x_N)) + d(f(x_N), f(x)) < \varepsilon$, and since $\varepsilon > 0$ is arbitrary, we must have $d(x, f(x)) = 0$ so $f(x) = x$. To show uniqueness, suppose x and y are fixed points of f, so $f(x) = x$ and $f(y) = y$. Then $d(x, y) = d(f(x), f(y)) \leq c \cdot d(x, y)$ which implies $d(x, y) = 0$ so $x = y$.

9. Let (X, d) be a metric space, and let $f : X \to \mathbb{R}$ be a continuous function. Define $Z(f) := \{x \in X : f(x) = 0\}$. Prove that $Z(f)$ is a closed subset of X.

Solution. Let $x \in Z(f)^c$, so $f(x) \neq 0$. The by continuity of f, there exists $\delta > 0$ such that for any $y \in X$ for which $|x - y| < \delta$, $|f(x) - f(y)| < |f(x)|$. We have $B_\delta(x) \subseteq Z(f)^c$, so x is an interior point of $Z(f)^c$ and thus $Z(f)^c$ is open.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(U)$ is open for every open set $U \subseteq \mathbb{R}$. Prove that f is monotonic.

Solution. (Contrapositive) Suppose that f is not monotonic. Then without loss of generality there exists $x < y < z$ such that $f(x) < f(y)$ and $f(y) > f(z)$. Since $[x, z]$ is compact, f attains its maximum on $[x, z]$ at some point in $[x, z]$, and since $f(y) > f(x)$ and $f(y) > f(z)$, the maximum is attained at some point $w \in (x, z)$. Then $f(w)$ is not an interior point of $f((x, z))$, so $f((x, z))$ is not open.