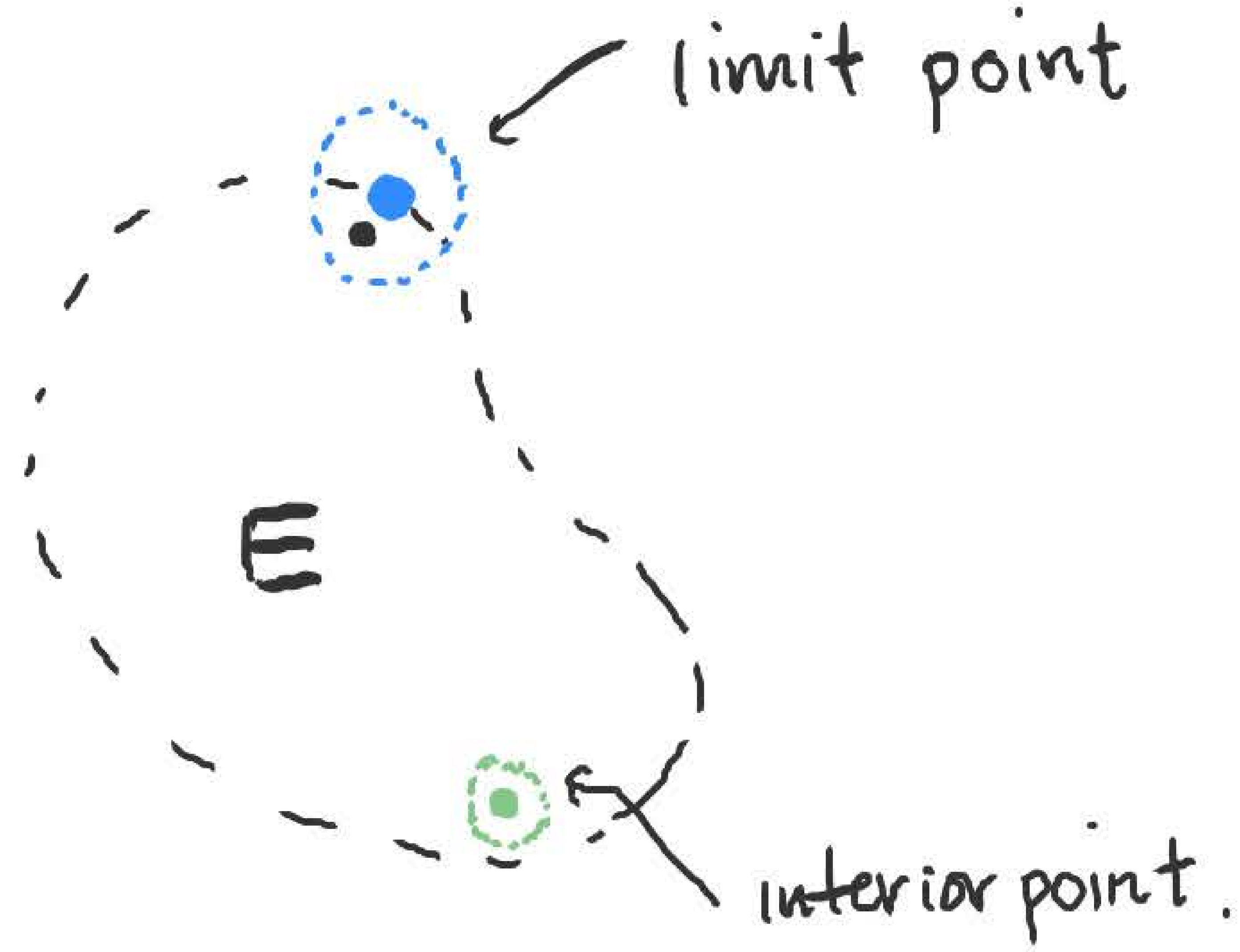
Thursday, July 8

Warm-Up Let (X,d) be a metric space.

(1) Let $x \in X$, $r > 0$. Prove that $B_r(x)$ is an open set in X.

(2) Let $E \subseteq X$. Prove that E' (the set of limit points of E) is closed.

(Useful for HW #9).



Ex. discrete metric space. X

$\emptyset \neq E \subseteq X$.

Every $x \in E$ is an interior point of E.

$B_1(x) = \{x\} \subseteq E$.

Every $x \in X$ is not a limit point of E.

$$(\underbrace{B_1(x) \setminus \{x\}}_{\emptyset}) \cap E = \emptyset.$$

<u>Proposition</u>: Let (X, d) be a metric space.

(i) If $\{U_\alpha\}_{\alpha \in A}$ is any collection of open sets in X, then $\displaystyle\bigcup_{\alpha \in A} U_\alpha$ is open.

(ii) If $\{U_1, \ldots, U_n\}$ is a (finite) collection of open sets in X, then $\displaystyle\bigcap_{i=1}^{n} U_i$ is open.

<u>Proof</u>: (i) Let $x \in \displaystyle\bigcup_{\alpha \in A} U_\alpha$. There exists $\beta \in A$ such that $x \in U_\beta$. Since U_β is open, there exists $r > 0$ such that $B_r(x) \subseteq U_\beta \subseteq \displaystyle\bigcup_{\alpha \in A} U_\alpha$.

(ii) Let $x \in \displaystyle\bigcap_{i=1}^{n} U_i$. So $x \in U_i$ for each i. For each i, there exists $r_i > 0$ such that $B_{r_i}(x) \subseteq U_i$. Set $r = \min(r_1, \ldots, r_n)$. Then $B_r(x) \subseteq \displaystyle\bigcap_{i=1}^{n} U_i$.

$\{(1-\frac{1}{n}, 1+\frac{1}{n})\}_{n \in \mathbb{N}}$.

$\displaystyle\bigcap_{n=1}^{\infty} (1-\tfrac{1}{n}, 1+\tfrac{1}{n}) = \{1\}$

Corollary (i) If $\{E_\alpha\}_{\alpha\in A}$ is any collection of closed set, $\bigcap_{\alpha\in A} E_\alpha$ is closed.

(ii) If $\{E_1, \dots, E_n\}$ is a (finite) collection of closed sets, then $\bigcup_{i=1}^{n} E_i$ is closed.

Proof: $\left(\bigcap_{\alpha\in A} E_\alpha\right)^c = \bigcup_{\alpha\in A} E_\alpha^c$ open sets.

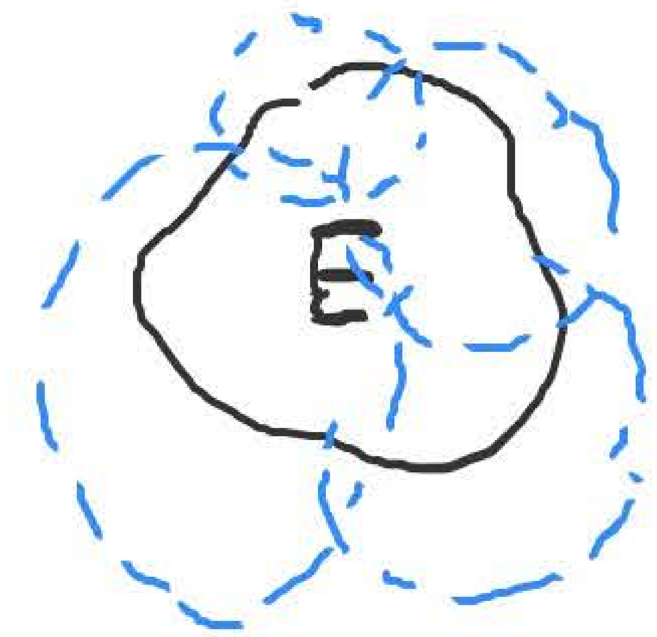
$\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c$.

Ex. $\bigcup_{x\in(0,1)} \underbrace{\{x\}}_{\text{closed.}} = (0,1)$ not closed.

Compactness

Let (X,d) be a metric space, and let $E \subseteq X$.

Def: An open cover of E is a collection of open sets $\{U_\alpha\}_{\alpha \in A}$ such that $E \subseteq \bigcup_{\alpha \in A} U_\alpha$.



Def: A subcover of an open cover $\{U_\alpha\}_{\alpha \in A}$ of E is an open cover $\{U_\alpha\}_{\alpha \in B}$ such that $B \subseteq A$.

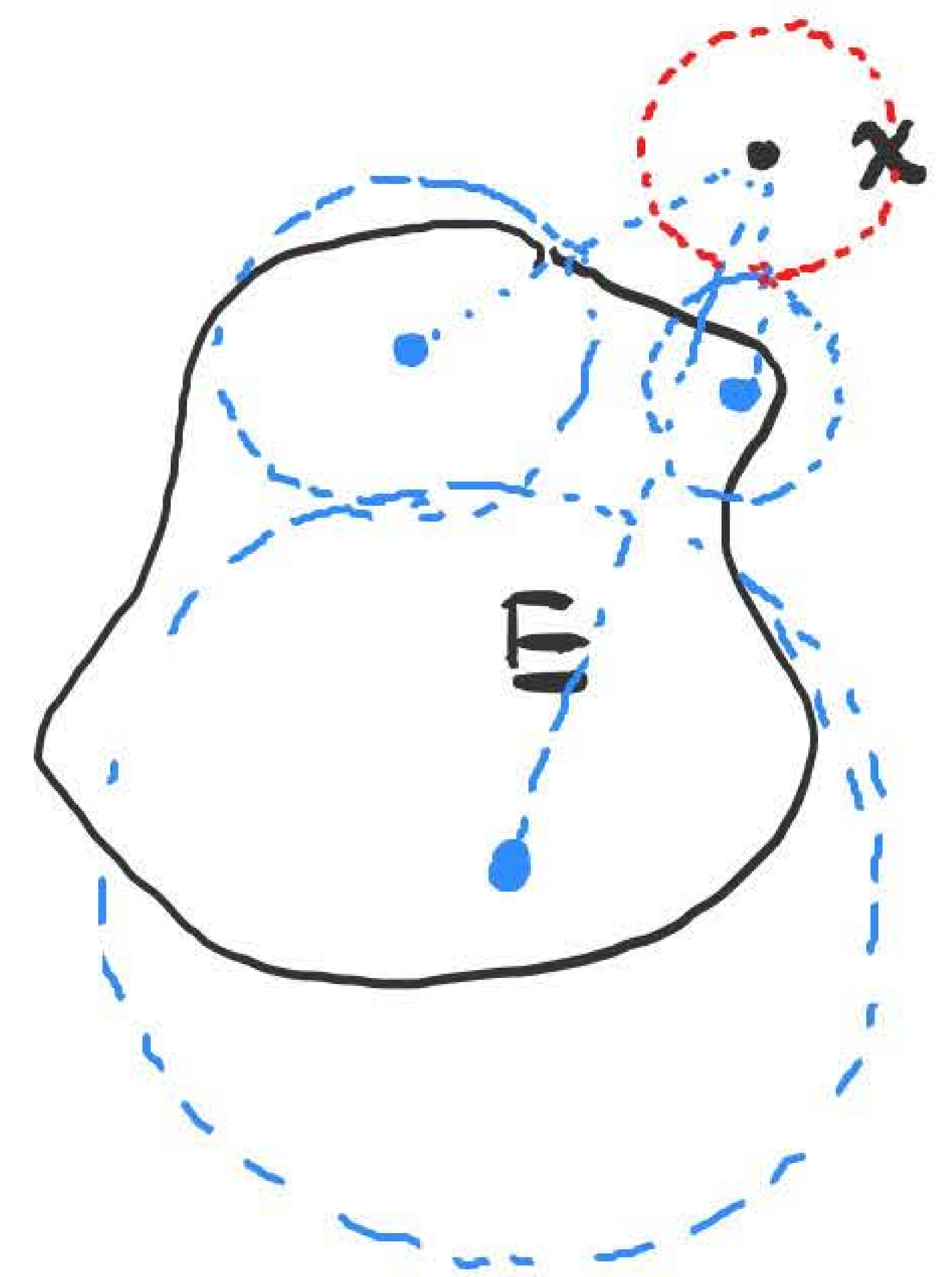
Def: An open cover is finite if it contains finitely many sets. (i.e. $|A| < \infty$).

Def: A set $E \subseteq X$ is compact if every open cover has a finite subcover.

Ex
- Every finite set is compact.
- X infinite discrete metric space. X is not compact. $\{B_1(x)\}_{x \in X}$ is an open cover with no finite subcover.

Ex. $\mathbb{R}$ and $(0,1)$ are not compact.

- $\{(-n,n)\}_{n\in\mathbb{N}}$ for $\mathbb{R}$
- $\{(0,r)\}_{r\in(0,1)}$ for $(0,1)$.



Question: Is $[0,1]$ compact?

Theorem: Compact sets are closed.

Proof: Let $E \subseteq X$ be compact. (Show E^c is open).

Let $x \in E^c$. For each $y \in E$, let $r_y = \frac{1}{2} d(x,y)$.

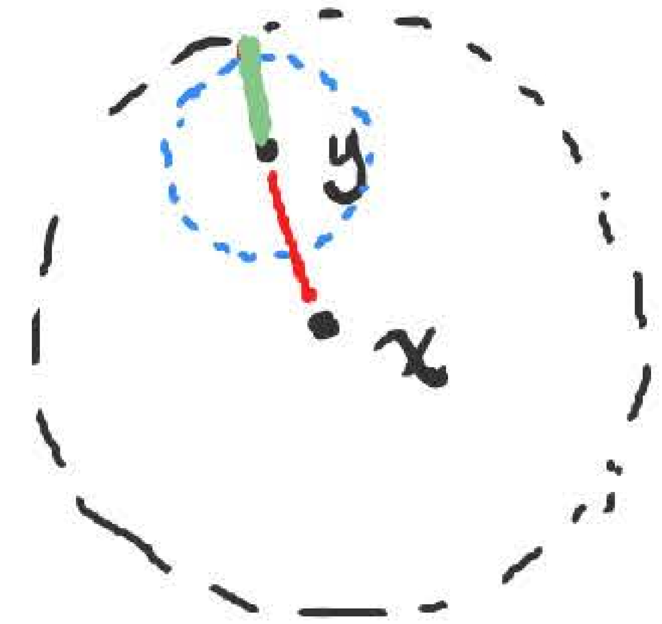
$\{B_{r_y}(y)\}_{y\in E}$ is an open cover of E. Since E is compact, there is a finite subcover $B_{r_{y_1}}(y_1), \dots, B_{r_{y_n}}(y_n)$, so $E \subseteq \bigcup_{i=1}^{n} B_{r_{y_i}}(y_i)$.

Set $r = \min\{r_{y_1}, \dots, r_{y_n}\}$. Then $B_r(x) \cap \underbrace{\left(\bigcup_{i=1}^{n} B_{r_{y_i}}(y_i)\right)}_{E \subseteq \text{ this}} = \emptyset$.

$\Rightarrow B_r(x) \cap E = \emptyset \Rightarrow B_r(x) \subseteq E^c$.

Question: counterexample to converse? $\mathbb{R}$.

(1) Let $y \in B_r(x)$.



Let $s = r - d(x,y)$.

$B_s(y) \subseteq B_r(x)$ because

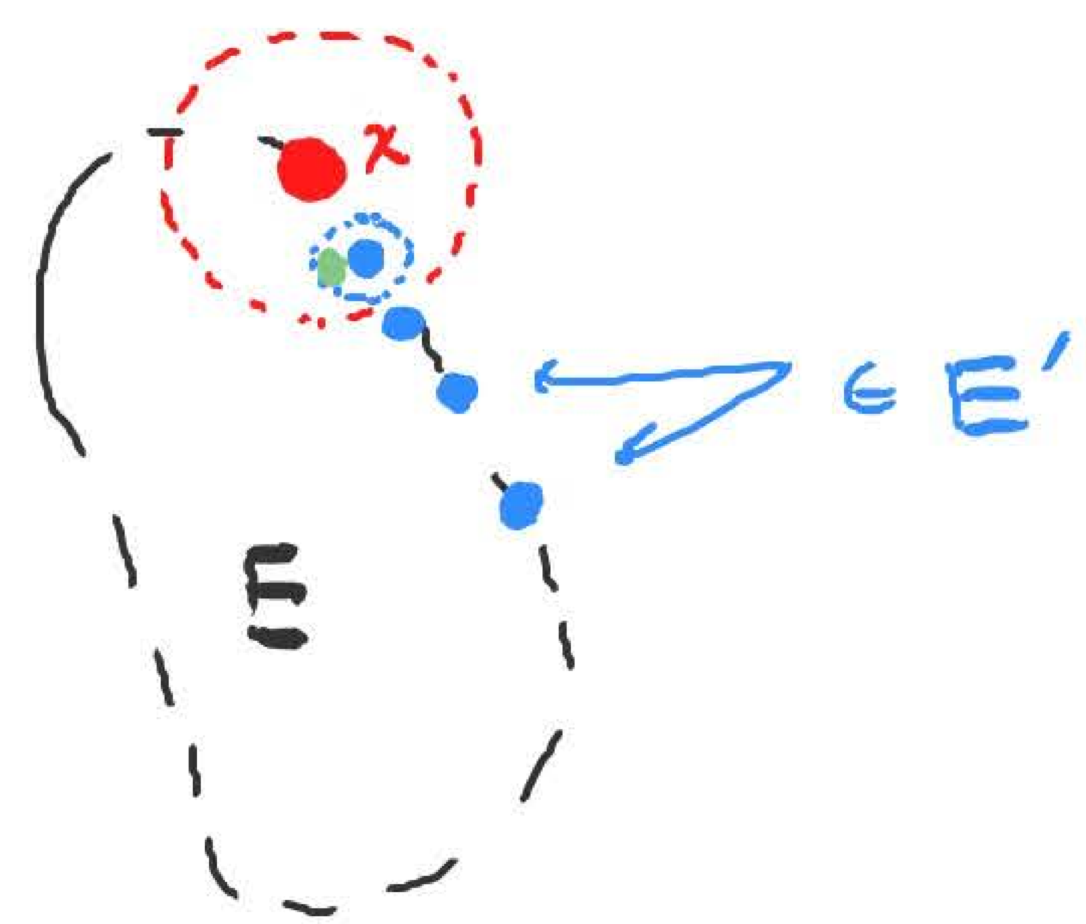
if $z \in B_s(y)$, then $d(y,z) < s$,

then $d(x,z) \leq d(x,y) + \underbrace{d(y,z)}_{< s = r - d(x,y)} < r$, so $z \in B_r(x)$.

(2) $E \subseteq X$.

Let x be a limit point of E' (Show $x \in E'$).

Let $r > 0$. Since x is a limit point of E', there exists $x \neq y \in E'$ such that $y \in B_r(x)$. Let $s = \min\{r - d(x,y), d(x,y)\}$.



Since $y \in E'$, there exists $z \in (B_s(y) \setminus \{y\}) \cap E$.

$\ldots$ $z \in (B_r(x) \setminus \{x\}) \cap E$.

$z \in A \cap E$

$\Rightarrow z \in B \cap E$.

$A \subseteq B$.

More on open and closed sets.

- X and $\emptyset$ are always open and closed.
- NOT opposite notions, i.e. $[0,1)$ is neither open nor closed
- E is open $\Leftrightarrow$ E^c is closed.
  E is closed $\Leftrightarrow$ E^c is open.

Ex: In a discrete metric space, which sets are open? closed?

- all sets are open. $E \subseteq X$. $x \in E$. $B_1(x) = \{x\} \subseteq E$.
- all sets are closed.