# University of California, Berkeley
# Math 104: Introduction to Analysis

Instructor: Theodore Zhu

# Midterm Exam

July 12, 2018

10:10 AM – 11:55 AM

Name: ____________________________

Student ID: ________________________

**Instructions.** This is a closed-book, closed-notes, closed-electronics exam. Please write carefully and clearly in the spaces provided. If you run out of space for a problem, you may continue on the reverse side of the page, or on the extra pages at the end. Cross out any work that you do not want to be graded. Unless otherwise specified, show all work and justify any nontrivial claims. **You may use any results from lecture (including worksheets) and homework problems, but you must clearly state the result that you are using.**

| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 5 | |
| 3 | 5 | |
| 4 | 5 | |
| 5 | 5 | |
| 6 | 5 | |
| Total: | 35 | |

1. **Short answer.** No justification required.

   (a) (2 points) Give an example of a pair of sequences $(s_n)$ and $(t_n)$ of real numbers such that
   $$\limsup(s_n + t_n) \neq \limsup(s_n) + \limsup(t_n).$$

   (b) (2 points) Give an example of an unbounded metric space $(X, d)$ and a nonempty set $E \subsetneq X$ (i.e. $E \subseteq X$ and $E \neq X$) such that $E$ is both open and closed.

   (c) (2 points) Give an example of a metric space which is complete but not compact.

(d) (2 points) Give an example of an open cover of $(0, 1] \subseteq \mathbb{R}$ with no finite subcover.

(e) (2 points) Give an example of a collection of closed sets in $\mathbb{R}$ whose union is not closed.

2. **Definitions and theorems.** Complete sentences are not required.

   (a) (1 point) What does it mean for an ordered field to satisfy the **least upper bound property**?

   (b) (1 point) State the **monotone convergence theorem**.

   (c) (1 point) In a metric space $(X, d)$, what does it mean for a sequence $(x_n)$ to **converge** to $x$?

   (d) (1 point) In a metric space $(X, d)$, what is an **interior point** of a set $E \subseteq X$?

   (e) (1 point) What do we mean when we say that $\mathbb{Q}$ is **dense** in $\mathbb{R}$?

3. (5 points) Let $E$ be a bounded subset of $\mathbb{R}$. Prove that

$$\operatorname{diam}(E) = \sup E - \inf E.$$

Note: $\operatorname{diam}(E) = \sup\{|x - y| : x, y \in E\}$.

4. (5 points) Let $(s_n)$ be a bounded sequence of real numbers such that

$$\limsup s_n - \liminf s_n = M > 0.$$

For each $n \in \mathbb{N}$, let $t_n = |s_{n+1} - s_n|$. Prove that $\limsup t_n \leq M$.

5. (5 points) Let $(X, d)$ be a metric space, and let $(s_n)$ be a Cauchy sequence in $X$. Prove that if $(s_n)$ has a convergent subsequence, then $(s_n)$ converges.

6. (5 points) Let $(X, d)$ be a metric space, and let $K_1, K_2, \ldots, K_n$ be a finite collection of compact sets in $X$. Prove that $\bigcup_{i=1}^{n} K_i$ is compact.

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