Math 104 Final Exam A (Printout Version)

UC Berkeley, Summer 2021

Friday, August 12, 4:10pm - 6:00pm PDT

Problem 1. Short answers. No justification required for examples.

(a) (2 points) Please copy verbatim the following text, followed by your signature. This MUST be handwritten UNLESS you are writing your entire exam electronically.

"As a member of the UC Berkeley community, I will act with honesty, integrity, and respect for others during this exam. The work that I will upload is my own work. I will not collaborate with or contact anyone during the exam, search online for problem solutions, or otherwise violate the instructions for this examination."

(b) (2 points) Give an example of a bounded divergent sequence (s_n) of real numbers such that $\lim(s_{n+1} - s_n) = 0$. (Suggestion: It may be easier to describe an example as opposed to giving an explicit formula.)

Take (s_n) to be the sequence of partial sums of the series

$$1 - 1 + \underbrace{\tfrac{1}{2} + \tfrac{1}{2}} \underbrace{- \tfrac{1}{2} - \tfrac{1}{2}} + \underbrace{\tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{3}} \underbrace{- \tfrac{1}{3} - \tfrac{1}{3} - \tfrac{1}{3}} + \cdots$$

(c) Let f be a function defined on (a,b). Let P be some property that f may or may not satisfy on any given subset of (a,b).

Assertion $[A]$: f satisfies the property P on every closed interval $[s,t] \subseteq (a,b)$.
Assertion $[B]$: f satisfies the property P on (a,b).

(i) (2 points) Give an example of a property P for which $[A]$ implies $[B]$.

continuity

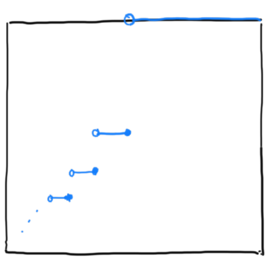
(ii) (2 points) Is it true for any property P that $[A]$ implies $[B]$?
Justify your answer.

No. For example, the function $f(x) = \frac{1}{x}$ satisfies uniform continuity on any $[s,t] \subseteq (0,1)$, but does not on $(0,1)$.

(d) Consider the function f defined on $[0,1]$ given by

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right], n \in \mathbb{N} \\ 0 & \text{if } x = 0 \end{cases}$$

(i) (2 points) Find a partition P of $[0,1]$ such that $U(f,P) - L(f,P) \leq \frac{1}{3}$.



$P = \{0, \frac{5}{12}, \frac{7}{12}, 1\}$

(ii) (2 points) Compute $U(f,P)$ and $L(f,P)$ for your partition P.

$U(f,P) = \frac{1}{2} \cdot \frac{5}{12} + 1 \cdot \frac{2}{12} + 1 \cdot \frac{5}{12} = \frac{19}{24}$

$L(f,P) = 0 \cdot \frac{5}{12} + \frac{1}{2} \cdot \frac{2}{12} + 1 \cdot \frac{5}{12} = \frac{1}{2}$

(e) (1 point) Submit your exam on time via Gradescope, and correctly assign pages to every problem you submit.

Problem 2. (6 points) Suppose the series $\sum_{n=0}^{\infty}(-1)^n a_n$ converges, but not absolutely. Let $M > 0$. What is the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{a_n}{M^n} x^n$? Prove your answer. (Note: There is no assumption that the a_n are nonnegative.)

Claim: The radius of convergence of $\sum \frac{a_n}{M^n} x^n$ is M.

Proof: The two power series $\sum a_n x^n$ and $\sum |a_n| x^n$ have the same radius of convergence, say R_1.

$\sum a_n x^n$ converges at $x=-1 \Rightarrow R_1 \geq 1$

$\sum |a_n| x^n$ does not converge at $x=1 \Rightarrow R_1 \leq 1$

Therefore $R_1 = 1$. It easily follows that $\sum \frac{a_n}{M^n} x^n$ has radius of convergence M.

Problem 3. (6 points) Suppose (f_n) is a sequence of functions defined on $[0,1]$, and suppose f is a function defined on $[0,1]$ such that for each $x \in [0,1]$, there exists $r > 0$ such that $f_n \to f$ uniformly on $(x-r, x+r) \cap [0,1]$ (which is $B_r(x)$ in the metric space $[0,1]$.) Prove that $f_n \to f$ uniformly on $[0,1]$.

Proof: For each $x \in [0,1]$, there exists $r_x > 0$ such that $f_n \to f$ uniformly on $B_{r_x}(x)$ (the open ball of radius r_x about x in the metric space $[0,1]$).

$\{B_{r_x}(x)\}_{x \in [0,1]}$ is an open cover of the compact set $[0,1]$, hence it has a finite subcover $B_{r_{x_1}}(x_1), \ldots, B_{r_{x_m}}(x_m)$.

Let $\varepsilon > 0$. For each $k = 1, \ldots, m$, there exists $N_k \in \mathbb{N}$ such that $n \geq N_k$ implies $|f_n(x) - f(x)| < \varepsilon$ for all $x \in B_{r_{x_k}}(x_k)$. Let $N = \max(N_1, \ldots, N_m)$. Then $n \geq N$ implies $|f_n(x) - f(x)| < \varepsilon$ for all $x \in [0,1]$.

Problem 4. Let $f : [0,1] \to [0,1]$ be a continuous function satisfying $f(0) > 0$. Fix $t \in (0,1]$ and define the set $S_t = \{x \in [0,1] : \frac{x}{f(x)} = t\}$.

(a) (4 points) Prove that S_t is nonempty.

Proof: The function $g(x) = t f(x) - x$ is continuous on $[0,1]$ with $g(0) > 0$ and $g(1) \leq 0$.

Case 1: $g(1) = 0$

Case 2: $g(1) < 0$. In this case, by the intermediate value theorem, there exists $x \in (0,1)$ such that $g(x) = 0$.

In either case, there exists $x^* \in (0,1]$ with $g(x^*) = 0$. Then $x^* \in S_t$.

(b) (4 points) Suppose also that f is differentiable on $(0,1)$ and $|f'(x)| < \frac{1}{t}$ for all $x \in (0,1)$. Prove that S_t contains exactly 1 element.

Proof: (Contradiction) Suppose that $x, y \in S_t$ with $x < y$. So $\frac{x}{f(x)} = t = \frac{y}{f(y)}$, or $f(x) = \frac{x}{t}$ and $f(y) = \frac{y}{t}$.

By the mean value theorem, there exists $z \in (x,y)$ such that

$$\frac{1}{t} > |f'(z)| = \left| \frac{f(y) - f(x)}{y - x} \right| = \left| \frac{\frac{y}{t} - \frac{x}{t}}{y - x} \right| = \frac{1}{t}.$$

Problem 5. Let (s_n) be a sequence of real numbers such that $\lim(s_{n+1} - s_n) = 0$. Suppose there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ such that $s_n = \alpha$ for infinitely many n and $s_n = \beta$ for infinitely many n. Let $s \in (\alpha, \beta)$.

(a) (2 points) Prove that for any $N \in \mathbb{N}$, there exist $N_2 > N_1 \geq N$ such that $s_{N_1} = \alpha$ and $s_{N_2} = \beta$.

Proof: Let $N \in \mathbb{N}$. Since $s_n = \alpha$ for infinitely many n, there exists $N_1 \geq N$ such that $s_{N_1} = \alpha$. Since $s_n = \beta$ for infinitely many n, there exists $N_2 > N_1$ such that $s_{N_2} = \beta$.

(b) (5 points) Inductively construct a subsequence (s_{n_k}) of (s_n) such that $s_{n_k} \to s$.

Let $N^{(1)} \in \mathbb{N}$ such that $n \geq N^{(1)}$ implies $|s_{n+1} - s_n| < 1$.
By (a), there exist $N_2^{(1)} > N_1^{(1)} \geq N^{(1)}$ such that $s_{N_1^{(1)}} = \alpha$ and $s_{N_2^{(1)}} = \beta$.
Let $m_1 = \max\{m < N_2^{(1)} : s_m < s\} \geq N_1^{(1)}$ and let $n_1 = m_1 + 1$.
Then $s \leq s_{n_1} < s_{m_1} + 1 < s + 1$ so $|s_{n_1} - s| < 1$.

Having constructed $n_1, \dots, n_k$, let $N^{(k+1)} \geq n_k$ such that $n \geq N^{(k+1)}$ implies $|s_{n+1} - s_n| < \frac{1}{k+1}$. By (a), there exist $N_2^{(k+1)} > N_1^{(k+1)} \geq N^{(k+1)}$ such that $s_{N_1^{(k+1)}} = \alpha$ and $s_{N_2^{(k+1)}} = \beta$.
Let $m_{k+1} = \max\{m < N_2^{(k+1)} : s_m < s\} \geq N_1^{(k+1)}$ and let $n_{k+1} = m_{k+1} + 1$.
Then $s \leq s_{n_{k+1}} < s_{m_{k+1}} + \frac{1}{k+1} < s + \frac{1}{k+1}$ so $|s_{n_{k+1}} - s| < \frac{1}{k+1}$.

By construction, $s_{n_k} \to s$.