Math 104 Homework 7

UC Berkeley, Summer 2021

Due by Friday, August 6, 11:59pm PDT

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1. (a) Let f be a continuous function on $[0, \infty)$. Prove that if f is uniformly continuous on $[a, \infty)$ for some $a > 0$, then f is uniformly continuous on $[0, \infty)$.
(b) Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$. Note that f' is unbounded on $(0, \infty)$ (cf. Theorem 19.6).
(c) Prove that

$$f(x) = \begin{cases} x \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is uniformly continuous on $\mathbb{R}$. (Hint: Generalize the assertion in part (a) to continuous functions on $\mathbb{R}$.)

2. Let $\sum a_n x^n$ be a power series. Prove that if $0 < \limsup |a_n| < \infty$, then the power series has radius of convergence $R = 1$.

3. (Ross 23.7) For each $n \in \mathbb{N}$, let $f_n(x) = (\cos x)^n$. Note that each f_n is a continuous function. Show that
(a) $\lim f_n(x) = 0$ if x is NOT a multiple of π.
(b) $\lim f_n(x) = 1$ if x is an EVEN multiple of π.
(c) $\lim f_n(x)$ does not exist if x is an ODD multiple of π.

4. Let (X, d_X) and (Y, d_Y) be two metric kspaces, and let f be a function from X to Y. We say f is **Lipschitz** on $E \subseteq X$ if there exists $C > 0$ such that $d_Y(f(x_1), f(x_2)) \leq C \cdot d(x_1, x_2)$ for any $x_1, x_2 \in E$.
(a) Prove that every Lipschitz function on $E \subseteq X$ is uniformly continuous on E.
(b) Prove that $f(x) = x^2$ is not Lipschitz on $\mathbb{R}$.
(c) Let f be a continuous real-valued function on $[a, b]$ which is differentiable on (a, b), and suppose f' is uniformly continuous on (a, b). Prove that f is Lipschitz.
(d) Find an example of a real-valued function on $[0, 1]$ which is uniformly continuous but not Lipschitz.

5. Let f be a real-valued function defined on an open interval (a, b), and suppose that

$$|f(y) - f(x)| \leq (y - x)^2$$

for any $x, y \in (a, b)$. Prove that f is constant on (a, b).

6. Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function with bounded derivative, i.e. there exists $M > 0$ such that $|g'(x)| \leq M$ for all $x \in \mathbb{R}$. Show that there exists $\varepsilon > 0$ such that the function $f(x) = x + \varepsilon g(x)$ is strictly increasing.

7. Let f be a differentiable function on $\mathbb{R}$.
(a) Prove that if $f'(x) \neq 1$ for every $x \in \mathbb{R}$. Prove that f has at most one fixed point.
(b) Show that the function

$$f(x) = x + \frac{1}{1+e^x}$$

has no fixed points.
(c) Prove that if $\sup\{|f'(x)| : x \in \mathbb{R}\} < 1$, then f has a unique fixed point.
(Hint: Start with an arbitrary real number x_1, and construct a sequence with $x_{n+1} = f(x_n)$. Show that $|s_{n+1} - s_n| \leq c|s_n - s_{n-1}|$ where $c = \sup\{|f'(x)| : x \in \mathbb{R}\}$.)

8. (Ross 29.14) Suppose f is differentiable at x. Prove that
(a) $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$;
(b) $\lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h} = f'(x)$.

9. (Ross 29.13) Prove that if f and g are differentiable on $\mathbb{R}$ and $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for all $x \in \mathbb{R}$, then $f(x) \leq g(x)$ for all $x \geq 0$.

10. (a) (Ross 30.6) Let f be a differentiable function on $(0, \infty)$. Suppose $\lim_{x\to\infty}(f(x) + f'(x)) = L \in \mathbb{R}$. Prove that $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} f'(x) = 0$. (Hint: $f(x) = \frac{f(x)e^x}{e^x}$.)
(b) Give an example of a function f which is differentiable on $(0, \infty)$ such that $\lim_{x\to\infty} f(x) = L \in \mathbb{R}$ but $\lim_{x\to\infty} f'(x)$ does not exist.