The following lemma will be used in Q2 and Q4:

Lemma 1. *For sequences $s_n \to s$ and $t_n \to t$, if there exists $N \in \mathbb{N}$ such that $n \geq N \implies s_n \leq t_n$, then $s \leq t$.*

Proof. We will use proof by contradiction. Suppose $s - t = \lim s_n - \lim t_n = \lim(s_n - t_n) > 0$, then let $\epsilon = s - t > 0$, so $\exists N \in \mathbb{N}$ $|s_n - t_n - (s-t)| < s - t \implies s_n - t_n - (s-t) > -(s-t) \implies s_n - t_n > 0 \implies s_n > t_n$. i.e. there are infinitely $n \in \mathbb{N}$ such that $s_n > t_n$. Thus we have a contradiction, completing the proof. $\square$

Q1

We need to show both directions:

$\implies$: We will show the contrapositive of the forward direction which is "If (s_n) does not converge to s, then there exists a subsequence of (s_n) such that all of its subsequences do not converge to s."

Since $s_n \nrightarrow s$, then $\exists \epsilon_0 > 0$ such that $\forall N \in N\ \exists n \geq N\ |s_n - s| \geq \epsilon_0$. Then we can construct a subsequence of (s_n) of which each term is at least ϵ_0 away from s:

Base Case: Let $N = 1$, then there exists $n_1 \in \mathbb{N}$ and $n_1 > 1$ such that $|s_{n_1} - s| \geq \epsilon_0$.

Induction step: Given $n_1 < \cdots < n_k \in \mathbb{N}$ such that $|s_{n_j} - s| \geq \epsilon_0$ for $j = 1, \ldots, k$, there exists $n_{k+1} \in \mathbb{N}$ and $n_{k+1} > n_k$ such that $|s_{k+1} - s| \geq \epsilon_0$ by the condition $s_n \nrightarrow s$.

Now since every term of s_{n_k} is $\epsilon_0 > 0$ away from s, all of its subsequences still have every term at least $\epsilon_0 > 0$ away from s, and hence cannot converge to s.

$\impliedby$: Since $s_n \to s$, then every subsequence (s_{n_k}) of (s_n) converges to s. Since each (s_{n_k}) itself is also a sequence and converges, (s_{n_k}) is bounded. Thus by Bolzano-Weierstrass Theorem, (s_{n_k}) has a convergent subsequence which converges to s since $s_{n_k} \to s$.

Q2

We know for $N \in \mathbb{N}$, $n \geq N$ implies $s_n \leq \sup\{s_n : n \geq N\}$ and $t_n \leq \sup\{t_n : n \geq N\}$, so $s_n + t_n \leq \sup\{s_n : n \geq N\} + \sup\{t_n : n \geq N\}$ and hence $\sup\{s_n + t_n : n \geq N\} \leq \sup\{s_n : n \geq N\} + \sup\{t_n : n \geq N\}$. Then we have

$$\begin{aligned} \lim\sup\{s_n + t_n : n \geq N\} &\leq \lim(\sup\{s_n : n \geq N\} + \sup\{t_n : n \geq N\}) &(1)\\ &= \lim\sup\{s_n : n \geq N\} + \lim\sup\{t_n : n \geq N\}. &(2) \end{aligned}$$

(1) comes from Lemma 1. (2) comes from theorem 9.3 when (s_n) and (t_n) are bounded.

Q3

(a) Let's show both $\sup(-S) \le -\inf S$ and $\sup(-S) \ge -\inf S$:

$\le$: Let $\inf S = u$, then $\forall s \in S$

$$\begin{aligned} s \ge u &\implies -u \ge -s \\ &\implies -u \ge \sup(-S) \quad \text{since } -u \text{ is an upper bound of } -S \\ &\implies \sup(-S) \le -\inf S \end{aligned}$$

Thus $\sup(-S) \le -\inf S$.

$\ge$: Let $\sup(-S) = v$, then $\forall s \in S$

$$\begin{aligned} -s \le v &\implies -v \le s \\ &\implies -v \le \inf S \quad \text{since } -v \text{ is a lower bound of } S, \\ &\implies -\inf S \le v = \sup(-S) \end{aligned}$$

Thus $\sup(-S) \ge -\inf S$, concluding $\sup(-S) = -\inf S$.

(b) If $k = 0$, then $\limsup(0 \cdot s_n) = \limsup(0) = 0 = 0 \cdot \limsup(s_n)$. Thus $\limsup(ks_n) = k \cdot \limsup(s_n)$.

If $k > 0$, let $v'_N = \sup\{ks_n : n \ge N\}$ and $v_N = \sup\{s_n : n \ge N\}$, then we have

$$\begin{aligned} n \ge N &\implies ks_n \le v'_N \\ &\implies s_n \le \frac{v'_N}{k} \\ &\implies v_N \le \frac{v'_N}{k} \\ &\implies k \cdot v_N \le v'_N, \end{aligned}$$

and

$$\begin{aligned} n \ge N &\implies s_n \le v_N \\ &\implies k \cdot s_n \le k \cdot v_N \\ &\implies v'_N \le k \cdot v_N \end{aligned}$$

Thus $v'_N = k \cdot v_N \implies \limsup(ks_n) = k \cdot \limsup(s_n)$, completing the proof.

(c) Since $k < 0$, $-k > 0$. Then we have

$$\begin{aligned} \limsup(ks_n) &= \limsup((-k)(-s_n)) \\ &= (-k) \cdot \limsup(-s_n) \quad \text{by (b)} \\ &= (-k) \cdot \lim -\inf(s_n) \quad \text{by (a)} \\ &= k \cdot \liminf(s_n). \end{aligned}$$

Q4

(a) Consider $N \in \mathbb{N}$, then $n \geq N \implies s_n \leq \sup\{s_n : n \geq N\}$ and $t_n \leq \sup\{t_n : n \geq N\}$. Then we have

$$\begin{aligned} n \geq N \implies s_n t_n &\leq \sup\{s_n : n \geq N\} \cdot t_n \\ &\leq \sup\{s_n : n \geq N\} \cdot \sup\{t_n : n \geq N\} \end{aligned}$$

Thus $\sup\{s_n : n \geq N\} \cdot \sup\{t_n : n \geq N\}$ is an upper bound of $\{s_n t_n : n \geq N\}$ and hence $\sup\{s_n t_n : n \geq N\} \leq \sup\{s_n : n \geq N\} \cdot \sup\{t_n : n \geq N\}$.

Since (s_n) and (t_n) are bounded, we have

$$\begin{aligned} \limsup s_n t_n &\leq \lim_N (\sup\{s_n : n \geq N\} \cdot \sup\{t_n : n \geq N\}) \quad (1) \\ &= \limsup s_n \cdot \limsup t_n \quad (2) \end{aligned}$$

(1) comes from Lemma 1. (2) comes from theorem 9.4 when (s_n) and (t_n) are bounded.

(b) Let $s_n = (-1)^n$ and $t_n = -1$ for $n \in N$. Then $s_n t_n = (-1)^{n+1}$ for $n \in \mathbb{N}$. Thus $\limsup s_n t_n = 1$, $\limsup s_n = 1$, and $\limsup t_n = -1$. Now we have $\limsup s_n t_n = 1 > -1 = (\limsup s_n)(\limsup t_n)$.

Q5

(a) First show the first inequality $\limsup \bar{s}_n \le \limsup s_n$. There are three cases regarding to the value of $\limsup s_n$.

Case 1: If $\limsup s_n = \infty$, then for any value $\limsup \bar{s}_n \in \mathbb{R} \cup \{+\infty, -\infty\}$, $\limsup \bar{s}_n \le \limsup s_n$.

Case 2: If $\limsup s_n = -\infty$, since $\liminf s_n \le \limsup s_n$, we have $\liminf s_n = -\infty = \limsup s_n \implies \lim s_n = -\infty$. Intuitively, $\lim \bar{s}_n = -\infty$. Because $\lim s_n = -\infty$, for $M < 0$ and $M - 1 < 0$, $\exists N \in\ n \ge N \implies s_n < M - 1$, then we have $n \ge N$ implies

$$\begin{aligned}
\bar{s}_n = \frac{s_1 + \cdots + s_{N-1} + s_N + \cdots + s_n}{n} &= \frac{s_1 + \cdots + s_{N-1}}{n} + \frac{s_N + \cdots + s_n}{n} \\
&< \frac{s_1 + \cdots + s_{N-1}}{n} + \frac{(n - N + 1)(M - 1)}{n} \\
&= \frac{s_1 + \cdots + s_{N-1}}{n} + \frac{n}{n}(M - 1) + \frac{-N + 1}{n}(M - 1) \\
&= \frac{s_1 + \cdots + s_{N-1} + (-N + 1)(M - 1)}{n} + (M - 1)
\end{aligned}$$

Since for fixed N and M, $F(n) = \frac{s_1 + \cdots + s_{N-1} + (-N+1)(M-1)}{n} \to 0$, $\exists N' \ge N\ F(N') < 1$. Because $F(n)$ is nonincreasing, we have $n \ge N' \implies F(n) \le F(N') < 1$.

$$n \ge N' \implies \bar{s}_n < \frac{s_1 + \cdots + s_{N-1} + (-N + 1)(M - 1)}{n} + (M - 1) < 1 + (M - 1) = M$$

Thus $\lim \bar{s}_n = -\infty$, completing the case.

Case 3: If $\limsup s_n = \alpha \in \mathbb{R}$, then for each $\frac{\epsilon}{2} > 0$, $\exists N \in \mathbb{N}\ v_N < \alpha + \frac{\epsilon}{2}$. Notice v_N is nonincreasing. Observe that for fixed N, $F(n) = \frac{s_1 + \cdots + s_{N-1} - (N-1)v_N}{n} \to 0$ as $n \to 0$, so for each $\frac{\epsilon}{2} > 0$, $\exists N' \ge N\ n \ge N' \implies F(n) \le F(N') < \frac{\epsilon}{2}$ since $F(n)$ is nonincreasing. Thus for each $\epsilon > 0$, we have $n \ge N' \implies \bar{s}_n \le F(N') + v_{N'} \le F(N') + v_N < \frac{\epsilon}{2} + (\alpha + \frac{\epsilon}{2}) = \alpha + \epsilon \implies \bar{s}_n < \alpha \implies \sup\{\bar{s}_n : n \ge N' \ge N\} \le \alpha$. Thus $\limsup \bar{s}_n \le \lim \alpha = \alpha = \limsup s_n$, completing the proof of the first inequality.

The proof of the second inequality mirrors the proof of the first.

(b) If $\lim s_n$ exists, then $\liminf s_n = \limsup s_n$. It is clear that $\liminf \bar{s}_n \le \limsup \bar{s}_n$, then $\liminf s_n \le \liminf \bar{s}_n \le \limsup \bar{s}_n \le \limsup s_n$ achieves equality every where, so $\liminf \bar{s}_n = \limsup \bar{s}_n$ and hence $\lim \bar{s}_n$ exists. Then $\lim \bar{s}_n = \liminf \bar{s}_n = \liminf s_n = \lim s_n$, completing the proof.

(c) Let $s_n = (-1)^n$. Obviously (s_n) does not converge since its set of subsequential limit has elements -1 and 1. However $\bar{s}_n = \frac{(-1)^n}{n}$ converges to 0.

(d) First such a sequence is not monotonic. Consider a sequence whose terms are in $\{-1, -1\}$. Each group of 1's or -1's is followed by a longer alternative group of -1's or 1's, like $\{1, -1, -1, 1, 1, 1, -1, -1, -1, -1, 1, 1, 1, 1, 1, \dots\}$. Thus, its first n terms average s_n will fluctuate between -1 and 1 though slower and slower but never converge to some point.

Q6

(a) We need to show positive definiteness, symmetry, and triangular inequality of this metric:

- Positive Definiteness: $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ $d(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^k |y_j - x_j| \geq \sum_{j=1}^k 0 = 0$. Also if $\mathbf{x} = \mathbf{y}$, then $\forall j = 1, \ldots, k$ $x_j = y_j \implies y_j - x_j = 0 \implies \sum_{j=1}^k |y_j - x_j| = 0$. If $d(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^k |y_j - x_j| = 0$, then $\forall j = 1, \ldots, k$ $y_j - x_j = 0 \implies x_j = y_j \implies \mathbf{x} = \mathbf{y}$.
- Symmetry: Since $|y_j - x_j| = |(-1)(x_j - y_j)| = |x_j - y_j|$, it is clear that $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ $\sum_{j=1}^k |y_j - x_j| = \sum_{j=1}^k |x_j - y_j| \implies d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$.
- Triangular Inequality: $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^k$

$$
\begin{aligned}
d(\mathbf{x}, \mathbf{z}) = \sum_{j=1}^k |z_j - x_j| &= \sum_{j=1}^k |z_j - y_j + y_j - x_j| \\
&\leq \sum_{j=1}^k (|z_j - y_j| + |y_j - x_j|) \\
&= \sum_{j=1}^k |z_j - y_j| + \sum_{j=1}^k |y_j - x_j| \\
&= d(\mathbf{y}, \mathbf{z}) + d(\mathbf{x}, \mathbf{y})
\end{aligned}
$$

Thus $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$, completing the proof.

(b) Consider a Cauchy sequence $(\mathbf{x}^{(n)}) \in \mathbb{R}^k$. By Lemma 13.3, $\forall j = 1, \ldots, k$ $\mathbf{x}_j^{(n)}$ is a Cauchy sequence in $\mathbb{R}$. By the completeness of $\mathbb{R}$, $\forall j = 1, \ldots, k$ $\mathbf{x}_j^{(n)}$ is convergent in $\mathbb{R}$. Then by Lemma 13.3 again $(\mathbf{x}^{(n)})$ is convergent in $\mathbb{R}^k$ and hence $(\mathbb{R}^k, d)$ is complete.

Q7

We will show both directions:

$\Longrightarrow$: Suppose x is a limit point of E, then $\forall r > 0$ $(B_r(x)\backslash\{x\})\cap E \neq \emptyset$. We will use inductive construction to build a sequence (x_n) of points in $E\backslash\{x\}$ such that (x_n) converges to x:

Base case: Let $r = 1$, then $\exists s \in (B_1(x)\backslash\{x\}) \cap E \implies s \in E\backslash\{x\}$ and $d(x, s) < 1$. Let $s_1 = s$.

Induction Step: Given $s_1, \ldots, s_k \in E\backslash\{x\}$ such that $d(x, s_j) < \frac{1}{j}$ for $j = 1, \ldots, k$. Since x is a limit point of E, $\exists s \in (B_{\frac{1}{k+1}}(x)\backslash\{x\}) \cap E \implies s \in E\backslash\{x\}$ and $d(x, s) < \frac{1}{k+1}$. Let $s_{k+1} = s$.

Thus we've built a (x_n) of points in $E\backslash\{x\}$ such that $d(x, s_n) < \frac{1}{n}$ for $n \in \mathbb{N}$. Since $0 \leq d(x, s_n)$ for $n \in \mathbb{N}$, by Squeeze Lemma $\lim_n d(x, s_n) = 0 \implies x_n \to x$.

$\Longleftarrow$: Suppose there exists a sequence (x_n) of points in $E\backslash\{x\}$ such that (x_n) converges to x. In other words, $\forall r > 0$ $\exists N \in \mathbb{N}$ $n \geq N \implies (x_n \in E\backslash\{x\}) \wedge (d(x, x_n) < r) \implies \forall n \geq N$ $x_n \in (B_r(x)\backslash\{x\}) \cap E \implies (B_r(x)\backslash\{x\}) \cap E \neq \emptyset$. Thus x is a limiting point.

Q8

Consider $x \in E'$. Then we have $\forall r > 0$ $(B_r(x) \backslash \{x\}) \cap E \neq \emptyset$. Now $\forall s \in (B_r(x) \backslash \{x\}) \cap E$

$$
\begin{aligned}
(s \in (B_r(x) \backslash \{x\})) \wedge (s \in E) &\implies (s \in (B_r(x) \backslash \{x\})) \wedge (s \in F) && (1) \\
&\implies s \in (B_r(x) \backslash \{x\}) \cap F && (2)
\end{aligned}
$$

(1) comes from $E \subseteq F$, and (2) comes from the definition of intersection. Thus $(B_r(x) \backslash \{x\}) \cap E \subseteq (B_r(x) \backslash \{x\}) \cap F$, and hence $(B_r(x) \backslash \{x\}) \cap F \neq \emptyset$. This implies x is also a limit point of F, so $x \in F'$. Thus $E' \subseteq F'$.

Q9

(a) If we can show $\overline{E}^{\mathsf{C}}$ is open, then $\overline{E}$ is closed. Consider $x \in \overline{E}^{\mathsf{C}}$, then

$$\begin{aligned}\forall x \in (E \cup E')^{\mathsf{C}} &\implies (x \notin E) \land (x \notin E') \\ &\implies \exists r_1 > 0 \; B_{r_1}(x) \cap E = \emptyset \\ &\implies \exists r_1 > 0 \; B_{r_1}(x) \subseteq E^{\mathsf{C}}\end{aligned}$$

Since $x \notin E'$, $x \in (E')^{\mathsf{C}}$. Also we know E' is closed, so $(E')^{\mathsf{C}}$ is open, and hence $\exists r_2 > 0 \; B_{r_2}(x) \subseteq (E')^{\mathsf{C}}$. Take $r = \min\{r_1, r_2\}$ then

$$(B_r(x) \subseteq E^{\mathsf{C}}) \land (B_r(x) \subseteq (E')^{\mathsf{C}}) \implies B_r(x) \subseteq (E \cup E')^{\mathsf{C}} = \overline{E}^{\mathsf{C}}$$

Since $\forall x \in \overline{E}^{\mathsf{C}} \; \exists r_x > 0 \; B_{r_x}(x) \subseteq \overline{E}^{\mathsf{C}}$,

$$\bigcup_{x \in \overline{E}^{\mathsf{C}}} B_{r_x}(x) \subseteq \overline{E}^{\mathsf{C}}.$$

It is clear that $\overline{E}^{\mathsf{C}} \subseteq \bigcup_{x \in \overline{E}^{\mathsf{C}}} B_{r_x}(x)$ because every point in $\overline{E}^{\mathsf{C}}$ is a center of an open ball. Now since $\overline{E}^{\mathsf{C}} = \bigcup_{x \in \overline{E}^{\mathsf{C}}} B_{r_x}(x)$ and union of open balls (sets) is still open, $\overline{E}^{\mathsf{C}}$ is open.

(b) We will show both directions:

$\implies$: From (a) we know $\overline{E}$ is closed, so E is closed.

$\impliedby$: If E is closed, by definition $E' \subseteq E$. Thus $\overline{E} = E \cup E' = E$.

(c) From (b) we know $\overline{F} = F \cup F' = F$. From Q8 we have $E \subseteq F$ implies $E' \subseteq F'$. Then it is clear that $\overline{E} = E \cup E' \subseteq F \cup F' = \overline{F} = F$, completing the proof.

Q10

(a) $\forall x \in E^\circ\ \exists r > 0\ B_r(x) \subseteq E$. Since $B_r(x)$ itself is open, $\forall y \in B_r(x)$

$$\begin{aligned}\exists s > 0\ B_s(y) \subseteq B_r(x) \subseteq E &\implies y \in E^\circ \\ &\implies B_r(x) \subseteq E^\circ.\end{aligned}$$

Thus $x \in (E^\circ)^\circ$, and hence E° is open by definition.

(b) We will show both directions:

$\implies$: From (a) we know E° is open, so E is open.

$\impliedby$: If E is open, by definition $\forall x \in E\ x \in E^\circ \implies E \subseteq E^\circ$. It is clear that $E^\circ \subseteq E$ since any interior point of a set is in the set. Thus $E = E^\circ$.

(c) Since F is open, by (b) $F^\circ = F$. $\forall x \in F^\circ\ \exists r > 0\ B_r(x) \subseteq F \subseteq E$, so $x \in E^\circ$ and hence $F^\circ \subseteq E^\circ$. Thus $F = F^\circ \subseteq E^\circ$, completing the proof.