Q1

(a) Fix $a > 0$. With f begin continuous on $[0, a+1]$, f is uniformly continuous on $[0, a+1]$. For each $\epsilon > 0$, observe that for each $x, y \in [0, \infty)$ such that $|x-y| < 1$, both x and y are either in $[0, a+1]$ or $[a, \infty)$. Since f is uniformly continuous on $[0, a+1]$, $\exists \delta_1 \; x, y \in [0, a+1]$ and $|x-y| < \delta_1 \implies |f(x) - f(y)| < \epsilon$; since f is uniformly continuous on $[a, \infty)$, $\exists \delta_2 \; x, y \in [a, \infty)$ and $|x-y| < \delta_2 \implies |f(x)-f(y)| < \epsilon$. Then take $\delta = \min\{1, \delta_1, \delta_2\}$, we have $x, y \in [0, \infty)$ and $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. Thus f is uniformly continuous on $[0, \infty)$.

(b) Clearly $\sqrt{x}$ is continuous on $[0, \infty)$. If we can show $\sqrt{x}$ is uniformly continuous on $[1, \infty)$, then by (a) it is uniformly continuous on $[0, \infty)$.

For each $\epsilon > 0$, select $\delta = 2\epsilon$. Then $x, y \in [1, \infty)$ and $|x - y| < \delta = 2\epsilon$ imply

$$\left|\sqrt{x} - \sqrt{y}\right| = \left|\frac{x-y}{\sqrt{x} + \sqrt{y}}\right| \leq \frac{|x-y|}{1+1} < \frac{2\epsilon}{2} = \epsilon.$$

Thus $\sqrt{x}$ is uniformly continuous on $[1, \infty)$, completing the proof.

(c) Since $x, \sin x, \frac{1}{x}$ is continuous on $(-\infty, 0) \cup (0, \infty)$, $f(x)$ is continuous on $(-\infty, 0) \cup (0, \infty)$. At $x_0 = 0$, suppose $(x_n) \subseteq \mathbb{R}\backslash\{0\}$ such that $x_n \to 0$ (if some $x_n = 0$ and $x_n \to 0$, then it follows that $f(x_n)) \to 0$ trivially), i.e. $\forall \epsilon > 0 \; \exists N \in \mathbb{N} \; n \geq N \implies |x_n| < \epsilon$. Since $|f(x_n)| = \left|x_n \sin\left(\frac{1}{x_n}\right)\right| \leq |x_n| \left|\sin\left(\frac{1}{x_n}\right)\right| \leq |x_n|$, $f(x_n) \to 0$ which is equal to $f(0)$, and hence f is continuous at 0. Therefore, f is continuous on $\mathbb{R}$.

Now let's generalize the assertion in part (a) to that "if f is uniformly continuous on $(-\infty, a]$ for some $a < 0$, then f is uniformly continuous on $(-\infty, 0]$". Then we just need to show that $f(x)$ is uniformly continuous on both $(-\infty, -1]$ and $[1, \infty)$.

Since $x, \sin(x), 1/x$ are differentiable on $[1, \infty)$, f is differentiable on $[1, \infty)$. $f'(x)$ is equal to $\sin(1/x) - \cos(1/x)/x$ on $[1, \infty)$. Thus $|f'(x)|$ is bounded by 2 on $[1, \infty)$. By theorem 19.6, f is uniformly continuous on $[1, \infty)$. Thus by the assertion in (a), f is uniformly continuous on $[0, \infty)$. Similarly, we can show f is uniformly continuous on $(-\infty, -1]$, and hence by the generalized assertion, f is uniformly continuous on $(-\infty, 0]$. For any $x, y \in \mathbb{R}$, if both x and y are in $(-\infty, 0]$ or $[0, \infty)$, thenIt follows that f is uniformly continuous on $\mathbb{R}$.

Q2

First show $\limsup |a_n| > 0 \implies R \leq 1$. Let $c \in \mathbb{R}$ such that $0 < c < \limsup |a_n|$, then it follows that $\forall N \in \mathbb{N}\ \sup\{|a_n| : n \geq N\} > c$ since $\sup\{|a_n| : n \geq N\}$ is nonincreasing. It implies that

$$\forall N \in \mathbb{N}\ \exists n \geq N\ |a_n| > c.$$

Take $N = 1$, then we can choose $n_1 \geq 1$ such that $|a_{n_1}| > c$. Having already selected $n_1 < n_2 < \cdots < n_k$ such that $\left|a_{n_j}\right| > c$ for each $j = 1, \ldots, k$, choose $n_{k+1} \geq n_k + 1$ such that $\left|a_{n_{k+1}}\right| > c$. Thus we inductively construct a subsequence (a_{n_k}) such that

$$\forall k \in \mathbb{N}\ |a_{n_k}| > c \implies |a_{n_k}|^{\frac{1}{n_k}} > c^{\frac{1}{n_k}}.$$

It follows that

$$\limsup |a_n|^{\frac{1}{n}} \geq \lim_k \sup |a_{n_k}|^{\frac{1}{n_k}} \geq \lim_k \sup c^{\frac{1}{n_k}} = \lim_k c^{\frac{1}{n_k}} = 1.$$

Thus $R = \frac{1}{\limsup |a_n|^{\frac{1}{n}}} \leq 1$.

Next show $\limsup |a_n| < \infty \implies R \geq 1$. Since $\limsup |a_n| \neq \infty$,

$$\exists M > 0\ \forall N \in \mathbb{N}\ \sup\{|a_n| : n \geq N\} < M.$$

Take $N = 1$, then we have $\forall n \in \mathbb{N}\ |a_n| < M \implies |a_n|^{\frac{1}{n}} < M^{\frac{1}{n}}$. It follows that $\limsup |a_n|^{\frac{1}{n}} \leq \limsup M^{\frac{1}{n}} = \lim M^{\frac{1}{n}} = 1$. Thus $R = \frac{1}{\limsup |a_n|^{\frac{1}{n}}} \geq 1$.

Combining two cases above, we have $0 < \limsup |a_n| < \infty \implies R \leq 1$ and $R \geq 1$, i.e. $R = 1$.

Q3

(a) Observe that

$$\begin{aligned} x \notin \{k\pi : k \in \mathbb{Z}\} &\iff -1 < \cos x < 1 \\ &\iff |\cos x| < 1 \\ &\iff 0 = \lim_n |\cos x|^n = \lim_n |(\cos x)^n| \\ &\iff \lim_n (\cos x)^n = 0, \end{aligned}$$

as desired.

(b) Observe that

$$\begin{aligned} x \in \{2k\pi : k \in \mathbb{Z}\} &\iff \cos x = 1 \\ &\iff \lim_n (\cos x)^n = 1, \end{aligned}$$

as desired.

(c) Observe that

$$\begin{aligned} x \in \{(2k+1)\pi : k \in \mathbb{Z}\} &\iff \cos x = -1 \\ &\iff \lim_n (\cos x)^n = \lim_n (-1)^n \\ &\iff \lim_n (\cos x)^n \text{ does not exists}, \end{aligned}$$

as desired.

Q4

(a) Let f be a Lipschitz function on $E \subseteq X$. Suppose there exists $C > 0$ such that $d_Y(f(x_1), f(x_2)) \leq C \cdot d(x_1, x_2)$ for all $x_1, x_2 \in E$. Then for each $\epsilon > 0$, let $\delta = \frac{\epsilon}{C}$. For any $x_1, x_2 \in E$ such that $d_X(x_1, x_2) < \delta = \frac{\epsilon}{C}$ implies $d_Y(f(x_1), f(x_2)) < C \cdot \delta = C \cdot \frac{\epsilon}{C} = \epsilon$. Thus f is uniformly continuous on E.

(b) Let $\epsilon = 1$. Then for each $\delta > 0$. Consider $x \in \mathbb{R}$. If $x \geq \frac{4-\delta^2}{4\delta}$, then $f(x + \delta/2) - f(x) = x\delta + \frac{\delta^2}{4} \geq \frac{(4-\delta^2)\delta}{4\delta} + \frac{\delta^2}{4} = 1 = \epsilon$. Thus $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$, and hence it's not Lipschitz.

(c) From HW6.2, we know f' is bounded on (a, b), i.e. $\exists M > 0\, \forall x \in (a, b)\, |f'(x)| \leq M$. Then for any $x, y \in [a, b]$, there exists $c \in (a, b)$ such that $M \geq |f'(c)| = |f(x) - f(y)|/|x - y|$, i.e. $|f(x) - f(y)| \leq M \cdot |x - y|$. It follows that f satisfies the definition of Lipschitz function as desired.

(d) Negate the definition of Lipschitz function, we want to find a uniformly continuous real-valued function on $[0, 1]$ but $\forall C > 0\ \exists x, y \in [0, 1] \left| \frac{f(x)-f(y)}{x-y} \right| > C$. Thus $f(x) = \sqrt{x}$ should satisfies the condition.

Q5

Observe that $\forall x, y \in (a,b)$ $0 \leq |f(y) - f(x)| \leq (y-x)^2$ and $\lim_{y\to x} 0 = 0$ and $\lim_{y\to x}(y-x)^2 = 0$. Thus by squeeze lemma, $\lim_{y\to x} |f(y) - f(x)| = 0$. Then since $-|f(y) - f(x)| \leq f(y) - f(x) \leq |f(y) - f(x)|$, by squeeze lemma again, $\lim_{y\to x} f(y) - f(x) = 0$. It follows that $\forall x \in (a,b)$ $f'(x) = \lim_{y\to x} \frac{f(y)-f(x)}{y-x} = 0$, i.e. f is constant on (a,b).

Q6

Select $\epsilon > 0$ with $\epsilon < 1/M$. Thus $\forall x \in \mathbb{R}$ $f'(x) = 1 + \epsilon g'(x) \geq 1 - \epsilon M > 1 - 1/M \cdot M = 0$. Thus f is strictly increasing.

Q7

(a) (Contrapositive) Consider $g(x) = f(x) - x$. Suppose f has two fixed points $a, b \in \mathbb{R}$, i.e. $g(a) = g(b) = 0$. By Rolle's theorem, $\exists x \in (a, b)$ $g'(x) = 0$, i.e. $f'(x) - 1 = 0 \implies f'(x) = 1$.

(b) Consider $g(x) = f(x) - x = \frac{1}{1+e^x}$. Suppose $a \in \mathbb{R}$ is a fixed point of f, then $g(a) = \frac{1}{1+e^a} = 0$, which is clearly impossible. Thus f does not have any fixed point.

(c) By HW6.8, if we can show such f is a contraction, then f has a unique fixed point ($\mathbb{R}$ is complete). Let $C = \sup\{|f'(x)| : x \in \mathbb{R}\} < 1$. Then for any $x, y \in \mathbb{R}$ $\exists c \in (x, y)$ $\left|\frac{f(x)-f(y)}{x-y}\right| = |f'(c)| \leq C \implies |f(x) - f(y)| \leq C \cdot |x - y|$. Thus f is a contraction, completing the proof.

Q8

(a) Let $y = x + h$. Then

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{y-x\to 0}\frac{f(y)-f(x)}{y-x} = \lim_{y\to x}\frac{f(y)-f(x)}{y-x} = f'(x),$$

completing the proof.

(b) From (a), we know

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = f'(a) \text{ and } \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h} = f'(a).$$

Thus

$$\begin{aligned} f'(a) = \frac{2f'(a)}{2} &= \frac{1}{2}\left(\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} + \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}\right) \\ &= \frac{1}{2}\left(\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} - \frac{f(a-h)-f(a)}{h}\right) \\ &= \lim_{h\to 0}\frac{f(a+h)-f(a-h)}{2h}, \end{aligned}$$

as desired.

Q9

$\forall x \geq 0 \ f(x) \leq g(x) \implies (g' - f')(x) = (g - f)'(x) \geq 0$. i.e. $(g - f)$ is non-decreasing on $[0, \infty)$. Since $(g - f)(0) = 0$, it follows that $(g - f)(x) \geq 0$ for all $x \geq 0$, i.e. $f(x) \leq g(x)$ for all $x \geq 0$.

Q10

(a) Since $e^x \neq 0$ for $x \in (0, \infty)$, we can write $f(x) = \frac{f(x)e^x}{e^x}$. Thus

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{f'(x)e^x + f(x)e^x}{e^x} = \lim_{x\to\infty} (f'(x) + f(x)) = L.$$

Thus second equality comes from L'Hospital's Rule. Then clearly it follows that $\lim_{x\to\infty} f'(x) = 0$.

(b) Consider $f(x) = \frac{1}{x}\sin(x^2)$. $f(x) \to 0$ while $\lim_{x\to\infty} f'(x) = \lim_{x\to\infty}\left(2\cos(x^2) - \frac{\sin(x^2)}{x^2}\right)$ does not exists.