Tuesday, June 29

- HW 2 due Friday
- HW 1 Solutions posted

Recall

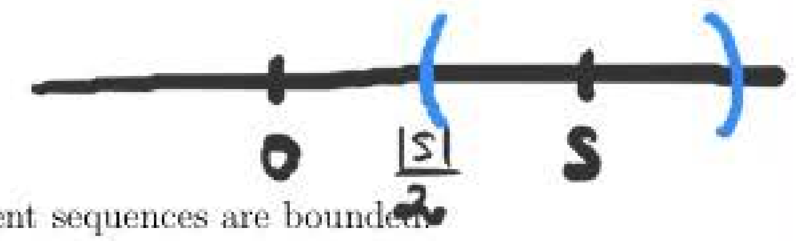
real number

$s_n \to s$ means that

for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|s_n - s| < \varepsilon$.

Also defined notion of divergence to ∞ or $-\infty$.

4. Let (s_n) be a sequence of nonzero real numbers, and suppose (s_n) converges to $s \neq 0$. Then
(a) $\inf\{|s_n| : n \in \mathbb{N}\} > 0$;
(b) The sequence $(1/s_n)$ converges to $1/s$.



Proof.
(a) *Hint*: The proof is similar to the proof that convergent sequences are bounded.

There exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|s_n - s| < \frac{|s|}{2}$ $= \varepsilon$.

So $n \geq N$ implies $|s_n| > \frac{|s|}{2}$.

Let $M = \min\left\{|s_1|, |s_2|, \dots, |s_{N-1}|, \frac{|s|}{2}\right\} > 0$.

M is a lower bound for $\{|s_n| : n \in \mathbb{N}\}$.

Show $\exists N$ s.t. $n \geq N \Rightarrow \left|\frac{1}{s_n} - \frac{1}{s}\right| < \varepsilon$.

(b) Let $\varepsilon > 0$. Goal: ...

$\inf\{|s_n| : n \in \mathbb{N}\} \geq M > 0$.

Let $m = \inf\{|s_n| : n \in \mathbb{N}\}$. By part (a), $m > 0$. Let $N \in \mathbb{N}$ be such that $|s - s_n| <$ $\varepsilon \cdot m|s|$ for all $n \geq \mathbb{N}$. Then

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s - s_n}{s_n s}\right| = \frac{|s - s_n|}{\underbrace{|s_n|}_{\geq m} \cdot |s|} \leq \frac{|s - s_n|}{m|s|} < \frac{\varepsilon \cdot m|s|}{m|s|} = \varepsilon$$

□

5. Suppose that (s_n) converges to s and (t_n) converges to t. If $s \neq 0$ and $s_n \neq 0$ for all n, then (t_n/s_n) converges to t/s.

Proof. Hint: Use two of the previous problems on this worksheet.

Follows from #3 and #4b.

$s_n \to s$

$\therefore \frac{1}{s_n} \to \frac{1}{s}$

$t_n \to t$

$\therefore \frac{t_n}{s_n} \to \frac{t}{s}$.

□

Proposition: If (s_n) converges, then its limit is unique, i.e. if $s_n \to s$ and $s_n \to t$, then $s = t$.

Proof: (Strategy: Show that $|s-t| = 0$ by showing that for any $\varepsilon > 0$, $|s-t| < \varepsilon$.)

If you're not sure — convince yourself by arguing contrapositive.

Suppose $s_n \to s$ and $s_n \to t$.

Let $\varepsilon > 0$.

Since $s_n \to s$, there exists $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies $|s_n - s| < \frac{\varepsilon}{2}$.

Since $s_n \to t$, there exists $N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies $|s_n - t| < \frac{\varepsilon}{2}$.

Let $N = \max\{N_1, N_2\}$. Then $n \geq N$ implies

$$|s-t| = |s - s_n + s_n - t| \leq \underbrace{|s - s_n|}_{< \frac{\varepsilon}{2}} + \underbrace{|s_n - t|}_{< \frac{\varepsilon}{2}} < \varepsilon.$$

Theorem: For (s_n) a sequence of positive real numbers, $\lim_{n\to\infty} s_n = \infty$ if and only if $\lim_{n\to\infty} \frac{1}{s_n} = 0$.

Proof: $\Rightarrow$ Suppose $s_n \to \infty$.

Let $\varepsilon > 0$. Let $M = \frac{1}{\varepsilon}$. Since $s_n \to \infty$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $s_n > M$. Then for $n \geq N$, $\left|\frac{1}{s_n} - 0\right| = \frac{1}{s_n} < \frac{1}{M} = \varepsilon$.

$\Leftarrow$ Suppose $\frac{1}{s_n} \to 0$.

Let $M > 0$. Let $\varepsilon = \frac{1}{M} > 0$. Since $\frac{1}{s_n} \to 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $\left|\frac{1}{s_n} - 0\right| = \frac{1}{s_n} < \varepsilon$. Then for $n \geq N$, $s_n > \frac{1}{\varepsilon} = M$.

Monotone sequences

Def: nondecreasing sequence: $s_{n+1} \geq s_n$ for all n.

nonincreasing sequence: $s_{n+1} \leq s_n$ for all n.

($s_{n+1} > s_n$ for all n: use strictly increasing;

$s_{n+1} < s_n$ for all n: use strictly decreasing.).

Def: A monotone (or monotonic) sequence is a sequence which is either nondecreasing or nonincreasing.

Examples

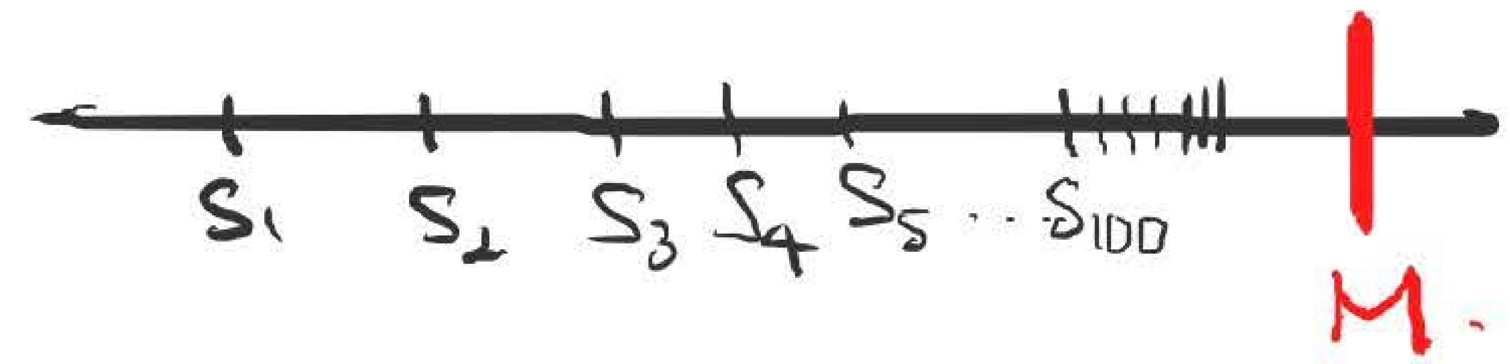
- $(0, 0, \ldots)$ or any constant sequence
- $(1, \frac{1}{2}, \frac{1}{3}, \ldots)$
- $(1, 2, 3, 4, \ldots)$
- ~~$(-1, 1, -1, 1, \ldots)$~~

Theorem: All bounded monotone sequences converge.

(Monotone Convergence Theorem).

Proof: Suppose (s_n) is a bounded nondecreasing sequence.

Let $u = \sup\{s_n : n \in \mathbb{N}\}$.



Goal: Show $s_n \to u$.

Let $\varepsilon > 0$. $u - \varepsilon$ is NOT an upper bound for $\{s_n : n \in \mathbb{N}\}$,

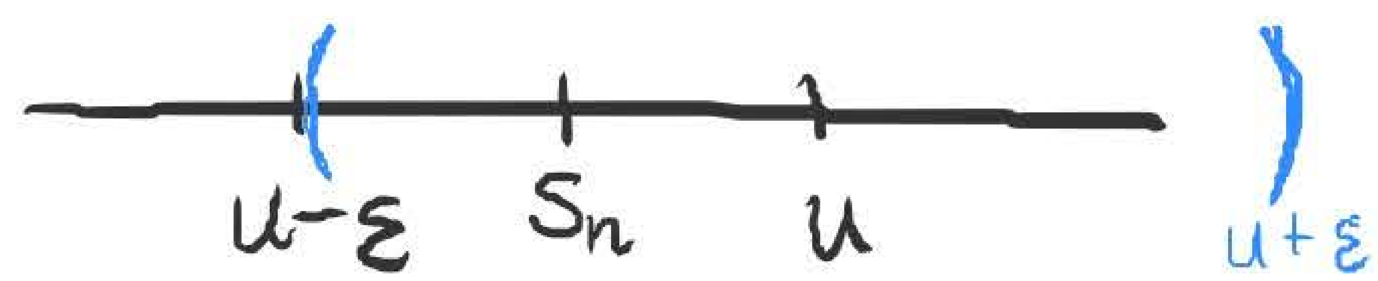
so there exists $N \in \mathbb{N}$ such that $s_N > u - \varepsilon$.

Then for $n \geq N$,

$$u \underset{u=\sup\{s_n:n\in\mathbb{N}\}}{\geq} s_n \underset{(s_n)\text{ is nondecreasing}}{\geq} s_N > u - \varepsilon \implies |s_n - u| < \varepsilon.$$

For (s_n) bounded nonincreasing:

Method 1: do the analogous argument on other side.



Method 2: Apply first result to $(-s_n)$.

$-s_n \to V$. Then $s_n \to -V$.

Corollary: All monotone sequences have a limit.

(requires the easy fact that an unbounded monotone sequence diverges to ∞ or $-\infty$.) — Easy exercise.

The limit of a sequence (s_n) tells us information about the behavior of (s_n) as n gets large. But not all sequences have a limit..

- the limit of a sequence does not depend on any finite collection of term.
- we want a more general notion that describes the limiting behavior of any sequence, even ones that don't converge.

Let (s_n) be a bounded sequence.

Let $u_n = \inf\{s_m : m \ge n\}$ "how small the sequence gets, ignoring the first $n-1$ terms"

$v_n = \sup\{s_m : m \ge n\}$.

Example: $\left(1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{4}, \dots\right)$.

$u_1 = -1$ $\quad u_2 = -1$ $\quad u_3 = -\frac{1}{2}$ $\quad u_4 = -\frac{1}{2}$

$v_1 = 1$ $\quad v_2 = \frac{1}{2}$ $\quad v_3 = \frac{1}{2}$ $\quad v_4 = \frac{1}{3}$ $\quad \dots$

$(u_n) = \left(-1, -1, -\frac{1}{2}, -\frac{1}{2}, \dots\right)$

$(v_n) = \left(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \dots\right)$

Observe: (u_n) is nondecreasing
(v_n) is nonincreasing. (why?)

u_{n+1} is the infimum of a smaller set than u_n. $\Rightarrow u_{n+1} \ge u_n$.

$\boxed{\text{In general, if } A \subseteq B, \text{ then } \inf A \ge \inf B \text{ and } \sup A \le \sup B.}$

This observation implies that

$\lim u_n$ and $\lim v_n$ exist.

$\sup\{s_m : m \geq n\}$

What does it mean for $\lim v_n = v \in \mathbb{R}$?

For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$n \geq N$ implies $\left| \underbrace{\sup\{s_m : m \geq n\}}_{v_n} - v \right| < \varepsilon$.

i.e. $v - \varepsilon < \sup\{s_m : m \geq n\} < v + \varepsilon$. for all $n \geq N$.

$v - \varepsilon$ is never going to be an upper bound for the sequence starting at any point, i.e. $s_n > v - \varepsilon$ $\boxed{\text{infinitely often}}$

$s_n < v + \varepsilon$ $\boxed{\text{eventually}}$

Define: $\limsup s_n \overset{\text{def}}{=} \lim v_n$.

$v_n = \sup\{s_m : m \geq n\}$.

$\liminf s_n \overset{\text{def}}{=} \lim u_n$

$u_n = \inf\{s_m : m \geq n\}$.

(Note that $\limsup s_n, \liminf s_n \in \mathbb{R} \cup \{\infty, -\infty\}$.)

Convention: If (s_n) is not bounded above, then $\limsup s_n = +\infty$;

If (s_n) is not bounded below, then $\liminf s_n = -\infty$.

Try Worksheet 4 – we'll go over it tomorrow.