Math 104 Worksheet 12

UC Berkeley, Summer 2021
Thursday, July 22

Let X and Y be two sets, and let $f : X \to Y$, let $E \subseteq X$, and let $A, B \subseteq Y$.

Exercise 1. Prove the following assertions.

(a) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

Proof. $x \in f^{-1}(A \cap B) \iff f(x) \in A \cap B \iff f(x) \in A$ and $f(x) \in B \iff x \in f^{-1}(A)$ and $x \in f^{-1}(B) \iff x \in f^{-1}(A) \cap f^{-1}(B)$.

(b) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(c) $f^{-1}(A^c) = \big(f^{-1}(A)\big)^c$.

(d) $f^{-1}(A) \subseteq f^{-1}(B)$ if $A \subseteq B$.

(e) $E \subseteq f^{-1}\big(f(E)\big)$

(f) Find a counterexample to show that the statement $E = f^{-1}\big(f(E)\big)$ is not always true.

Let (X, d_X) and (Y, d_Y) be two metric spaces, and let $f : X \to Y$. The following are three definitions of continuity at a point $x_0 \in X$.

1. (ε-δ definition) For any $\varepsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \varepsilon$ whenever $x \in X$ and $d_X(x, x_0) < \delta$.
2. (sequential definition) For any sequence (x_n) in X converging to x_0, the sequence $f(x_n)$ converges to $f(x_0)$.
3. (topological definition) For any open set U in Y such that $f(x_0) \in U$, there exists an open set V in X such that $x_0 \in V \subseteq f^{-1}(U)$.

Theorem. The three definitions above are equivalent.

Exercise 2. Prove the preceding theorem.
(a) Prove (2) $\Rightarrow$ (1).

(b) Prove (1) $\Rightarrow$ (3).

(c) Prove (3) $\Rightarrow$ (2).

Exercise 3. Using the topological definition of continuity at a point, prove that f is continuous (on its domain) if and only if $f^{-1}(U)$ is open in X for every open set U in Y.
(A function is continuous if and only if the preimage of every open set is open.)