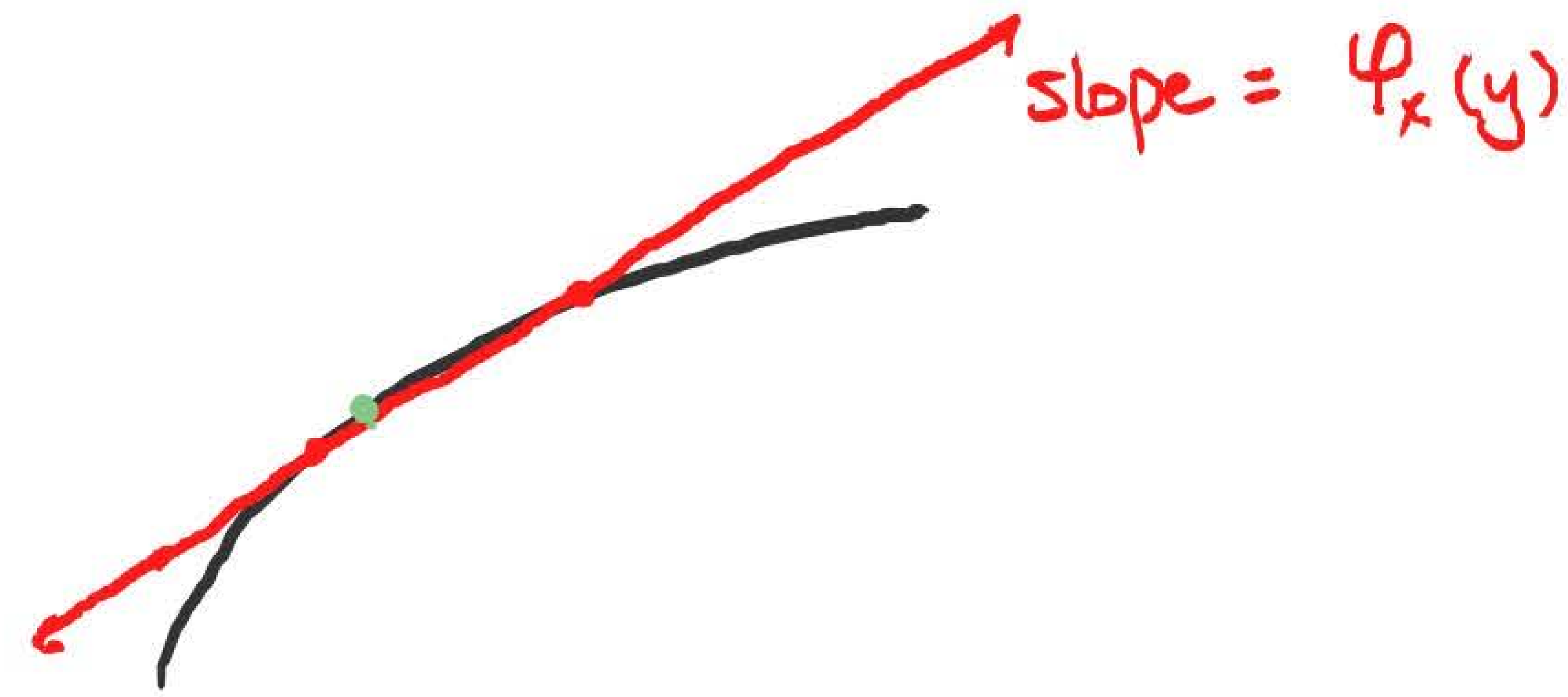
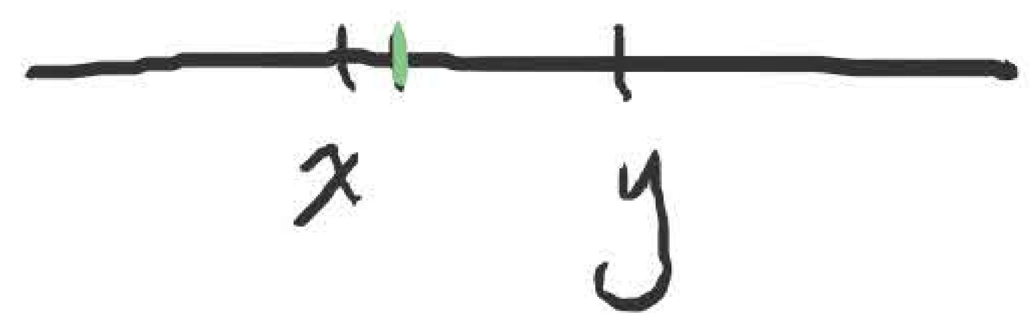
Tuesday, August 3



Recall: $\varphi_x(y) = \dfrac{f(y)-f(x)}{y-x}$

$$\lim_{y\to x} \varphi_x(y) = f'(x)$$

derivative of f at x
(if this limit exists).



- Ex $f(x) = x^2$. $f'(x) = 2x$.

$1 + a + a^2 + \dots + a^{n-1} = \dfrac{1-a^n}{1-a}$.

- Ex $f(x) = x^n$

$$\lim_{y\to x} \frac{f(y)-f(x)}{y-x} = \lim_{y\to x} \frac{y^n - x^n}{y-x} = \lim_{y\to x} \frac{\cancel{(y-x)}(y^{n-1} + y^{n-2}x + y^{n-3}x^2 + \dots + yx^{n-2} + x^{n-1})}{\cancel{y-x}}$$

$$= nx^{n-1}.$$

Theorem: If f is differentiable at x, then f is continuous at x.

Proof: (Goal: Show $\lim_{y \to x} f(y) = f(x)$.)

Observe

$$f(y) = (y-x) \underbrace{\boxed{\frac{f(y)-f(x)}{y-x}}}_{= \varphi_x(y)} + f(x).$$

If f is differentiable at x, then $\lim_{y \to x} \varphi_x(y)$ exists, finite.

$$\Rightarrow \lim_{y \to x} \underbrace{(y-x)}_{\to 0} \underbrace{\varphi_x(y)}_{\to \text{some real number}} = 0.$$

$$\Rightarrow \lim_{y \to x} f(y) = f(x).$$

Theorem: Suppose f, g are differentiable at x. Then cf, $f+g$, fg, $\frac{f}{g}$ (if $g(x) \neq 0$) are differentiable at x and

(i) $(cf)'(x) = c \cdot f'(x)$

(ii) $(f+g)'(x) = f'(x) + g'(x)$

(iii) $(fg)'(x) = f'(x)\, g(x) + f(x)\, g'(x)$

(iv) $\left(\frac{f}{g}\right)'(x) = \dfrac{g(x) f'(x) - f(x) g'(x)}{g(x)^2}$.

Proof: See text (algebraic manipulations, then take limits).

Math 104 Worksheet 16

UC Berkeley, Summer 2021

Tuesday, August 3

Lemma. Let f be defined on an open interval I containing x. If f attains its maximum (or minimum) at x and f is differentiable at x, then $f'(x) = 0$.

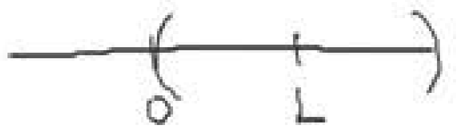
1. Prove the preceding lemma.
(*Hint:* Suppose that f attains its maximum at x. Argue by contradiction: show that if $f'(x) > 0$, then there exists $y \in I$ such that $f(y) > f(x)$, and analogously for $f'(x) < 0$.)

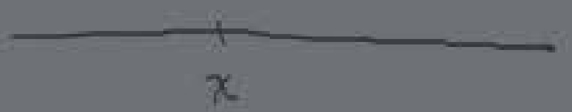
Suppose $f'(x) > 0$. $\lim_{y \to x} \underbrace{\frac{f(y)-f(x)}{y-x}}_{\varphi_x(y)} > 0$.

There exists $\delta > 0$ s.t.

$0 < |y-x| < \delta \Rightarrow \varphi_x(y) > 0$.

$\frac{f(y)-f(x)}{y-x} > 0$





There exists $y \in I$, $0 < |y-x| < \delta$, $y > x$:

$\frac{f(y)-f(x)}{y-x} > 0 \Rightarrow f(y) - f(x) > 0$ i.e. $f(y) > f(x)$.

Contradiction.

Rolle's Theorem. Suppose f is continuous on $[a, b]$ and differentiable on (a, b), and that $f(a) = f(b)$. Then there exists $x \in (a, b)$ such that $f'(x) = 0$.

2. Prove Rolle's Theorem.
(*Hint:* f is a continuous function on the compact set $[a, b]$, so it attains its maximum and minimum in the closed interval. Consider cases depending on whether or not the max/min occurs at the endpoints of the interval.)

f attains its max and min on $[a,b]$.

Case 1: max and min $= f(a) = f(b)$. f is constant, $f'(x) = 0$ at every $x \in (a,b)$.

Case 2: WLOG max $> f(a) = f(b)$. There exists $x \in (a,b)$ s.t. $f(x) = \max_{y \in [a,b]} f(y)$. By lemma, $f'(x) = 0$.

<u>Chain rule</u> : If g is differentiable at x and f is differentiable at $g(x)$, then $f\circ g$ is differentiable at x and $(f\circ g)'(x) = f'(g(x))\cdot g'(x)$.

<u>Proof</u>: $$\frac{f\circ g(y) - f\circ g(x)}{y-x} = \underbrace{\frac{f(g(y)) - f(g(x))}{g(y)-g(x)}}_{\text{as } y\to x,\ g(y)\to g(x).} \cdot \underbrace{\frac{g(y)-g(x)}{y-x}}_{\to g'(x) \text{ at } y\to x}.$$

Issue : What if $g(y) = g(x)$ for y close to x?

$\boxed{\text{Let } y_n \to x}$ $(y_n \neq x)$.

<u>Case 1</u> : There is an open interval I around x such that for all $y \in I\setminus\{x\}$, $g(y)\neq g(x)$.

Then for sufficiently n, $g(y_n) \neq g(x)$, so

$$\lim_{n\to\infty} \frac{f\circ g(y_n) - f\circ g(x)}{y_n - x} = \lim_{n\to\infty} \underbrace{\frac{f\circ g(y_n) - f\circ g(x)}{g(y_n) - g(x)}}_{\to f'(g(x))} \cdot \underbrace{\frac{g(y_n) - g(x)}{y_n - x}}_{\to g'(x)}$$

$$= f'(g(x))\cdot g'(x).$$

<u>Case 2</u>: In every open interval around x, there exists $y \neq x$ in the interval such that $g(y) = g(x)$.

There exists a sequence $z_n \to x$, $z_n \neq x$ such that $g(z_n) = g(x)$ for all n.

Then $g'(x) = \lim\limits_{n\to\infty} \dfrac{g(z_n) - g(x)}{z_n - x} = 0$.

Just need to show $(f \circ g)'(x) = 0$. $\qquad f'(g(x)) \cdot g'(x) = 0$.

Since f is differentiable at $g(x)$, so

$$\lim_{y\to x} \underbrace{\frac{f\circ g(y) - f\circ g(x)}{y - x}}_{\varphi_x^*(y)} \quad \text{exists, finite}$$

so φ_x^* is bounded in some interval J around $g(x)$, i.e.

there exists M: $\left| \dfrac{f(z) - f(g(x))}{z - g(x)} \right| \leq M$ for all $z \in J \setminus \{g(x)\}$.

$(y_n \to x)$ Then for sufficiently large n, y_n can be made close enough to x so that $g(y_n) \in J$.

If $g(y_n) = g(x)$: $\boxed{\left| \dfrac{f\circ g(y_n) - f\circ g(x)}{y_n - x} \right| \leq M \left| \dfrac{g(y_n) - g(x)}{y_n - x} \right|}$ $\quad (0 \leq 0)$.

If $g(y_n) \neq g(x)$: $\left| \dfrac{f\circ g(y_n) - f\circ g(x)}{y_n - x} \right| = \left| \dfrac{f\circ g(y_n) - f\circ g(x)}{g(y_n) - g(x)} \right| \cdot \left| \dfrac{g(y_n) - g(x)}{y_n - x} \right| \leq M \left| \dfrac{g(y_n) - g(x)}{y_n - x} \right|$.

In any case,

$$\left| \frac{f\circ g(y_n) - f\circ g(x)}{y_n - x} \right| \leq M \underbrace{\left| \frac{g(y_n) - g(x)}{y_n - x} \right|}_{\to 0}.$$

$$\lim_{n\to\infty} \frac{f\circ g(y_n) - f\circ g(x)}{y_n - x} = 0, \quad \text{so} \quad (f\circ g)'(x) = 0 = g'(x)$$
$$= f'(g(x)) \cdot g'(x).$$

<u>Ex</u>. $(x^2+5)^3$. $\quad f(x) = x^3$, $\quad g(x) = x^2+5$.

$= f\circ g(x)$.

$$\left[(x^2+5)^3 \right]' = \underbrace{f'(g(x))}_{3[x^2+5]^2} \cdot \underbrace{g'(x)}_{2x} = 6x(x^2+5)^2.$$

<u>Mean value theorem</u>: Let f be a continuous function on $[a,b]$ which is differentiable on (a,b). Then there exists $x \in (a,b)$ such that $f'(x) = \dfrac{f(b)-f(a)}{b-a}$.

<u>Proof</u>: Let $g(x) = \big(f(b)-f(a)\big)x - (b-a)f(x)$.

g cont on $[a,b]$, differentiable on (a,b),

$g(a) = g(b)$.

Apply Rolle's theorem: there exists $x \in (a,b)$,

$$g'(x) = 0.$$

$$g'(x) = f(b)-f(a) - (b-a)f'(x) = 0.$$

$$\Rightarrow f'(x) = \frac{f(b)-f(a)}{b-a}.$$