12. Minimum spanning trees

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Input $G = (V, E)$ undirected, connected, augmented $w: E \to \mathbb{R}^+$

Goal to compute a subgraph $H = (V, E_H)$ to minimize $\sum_{e \in E_H} w(e)$ and H is connected

Spanning tree of G
$T = (V, E_T)$ is a spanning tree of G if it's connected acylic subgraph of G
Define $w(T) = \sum_{e \in E_T} w(e)$, then find the MST min$\{ w(T) \mid T\ be\ a\ spanning\ tree \}$

Generic algorithm
$E_T = \emptyset$
while (V, E_T) is not connected
    Find a ***SAFE*** edge e
    $E_T = E_T \cup \{e\}$
SAFE an edge such that if E_T is contained in some minimizing spanning tree, then the same is true for $E_T \cup \{e\}$

Spanning Forest of G is on acyclic subgraph $H = (V, E_H)$

Connected components of H are trees $T_1, \dots, T_k$

Theorem Let H be a spanning forest of G with connected components $T_1, \dots, T_k, T_i = (V_i, E_i), E_H = \cup_1^k E_i$ such that $E_H \subseteq E_T$ where $T = (V, E_T)$ is a MST
Let $(u, v) \in E$ has the minimum weight among edges with one vertex in V_i and one vertex in V_i^C, then $H' = (V, E_H \cup \{u, v\})$ is contained in some MST of G

Fact Let T be a spanning tree of $G = (V, E)$. If $(u, v) \in E_T^C$ then the graph $T' = (v, E_T \cup \{u, v\})$ has a unique cycle

If $(u, v) \in E_T \Rightarrow$ done !
If not the case, $T' = (V, E_T \cup \{u, v\})$ then unique cycle $C \in T'$ must contains some edge $(u', v') \neq (u, v)$ s.t. $u' \in U_i, v' \notin U_i$
$w(u', v') \geq w(u, v)$ since the way we choose (u, v)