STA 303/1002-Methods of Data Analysis II

Sections L0101& L0201, Winter 2019

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STA 303/1002: Week 4- Two-way analysis of variance

- Review: General Linear Model (GLM)
  - Response, Y is continuous
  - Categorical or continuous predictors, X
  - Y is linear in β's
  - Assumptions: $\epsilon \sim N(0, \sigma^2 \mathbf{I})$
- Review: One-way ANOVA
  - Special case of a GLM
  - One-way classification/ One factor with $G \geq 2$ levels

- What if we have more than one factor?
  - Main and Interaction effect of factors on Y?
  - Assumptions?
  - Visualizations?
  - Analyses?

Factor 1 Factor 2

Judge Gender
1 vs 2

S — 1
S — 2

A — 1
A — 2

— S 1
— A 2
— B 1
— C 1
D 2

Two-way Classification or Two-way Analysis of Variance

- Another special case of a GLM
- Extension of One-way ANOVA
- Two factors, each with at least 2 levels ($G_1 \geq 2, G_2 \geq 2$)
- Uses a maximum of $(G_1 - 1) + (G_2 - 1) + (G_1 - 1)(G_2 - 1)$ indicator variables

Terminology from Design of Experiments:

- Factor- a categorical predictor variable, eg. *Treatment*
- Factors are composed of different class levels, eg. various types of treatments

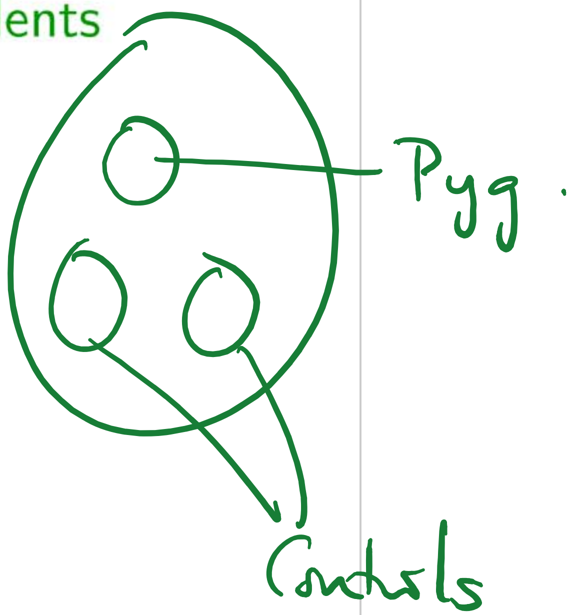
Two types of factors

- FIXED effect: data has been gathered from all the levels of the factor that are of interest
- Random effect: interest is in all possible levels of the factor, but only a random sample of levels is included in the data
- Egs.: Suppose measurements are taken on the yield of a machine operated by each of several operators. We want to compare the mean yields under different operators.
  - Factor: operator
  - Fixed effect: Interest is only in those particular operators (may be all the operators at the plant)
  - Random effect: Operators are a random sample from larger population of all operators.

Case Study II-The Pygmalion Effect

- *Pygmalion effect*- high expectations of a supervisor or teacher translate to improved performance by subordinates or students
- Data:

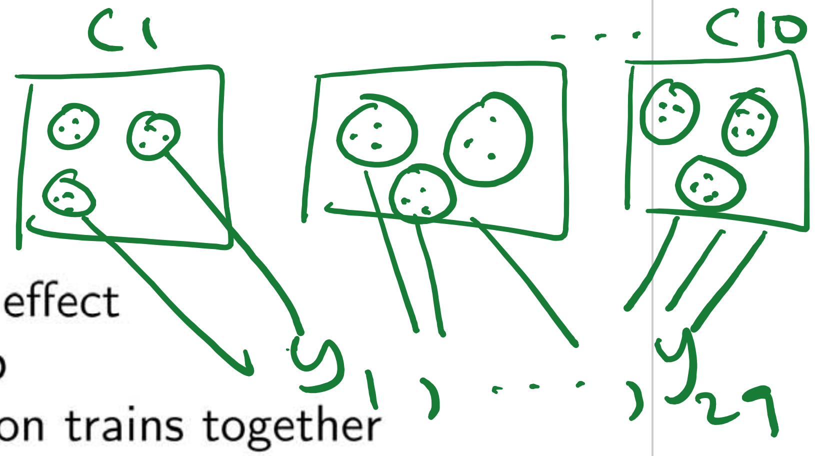
	Treatments		
Company	Pygmalion	Control	
1	80.0	63.2	69.2
2	83.9	63.1	81.5
3	68.2	76.2	
4	76.5	59.5	73.5
5	87.8	73.9	78.5
6	89.8	78.9	84.7
7	76.1	60.6	69.6
8	71.5	67.8	73.2
9	69.5	72.3	73.9
10	83.7	63.7	77.7



Unbalanced design

N = 29 = 10 + 10 + 9

Case Study II-The Pygmalion Effect



- Setup:
  - A randomized experiment to test Pygmalion effect
  - Used 10 companies in an army training camp
  - Most companies have 3 platoons; each platoon trains together under 1 leader (1 leader per platoon).
  - Within each company, 1 platoon leader was told that he an exceptionally good group- this is the pygmalion platoon; the other 2 are control platoons.
  - Each pygmalion platoon was randomly chosen.
- Experimental units: platoons
- Unbalanced design: one company had only two platoons
- y_i Response: score on a basic weapons test per platoon $(0,100)$ $i=1,\ldots,29$
- Factors:
  (1) *Company*- 10 levels (company 1,. . ., company 10)
  (2) *Treatment*- 2 levels (pygmalion, control)

active treatment — inactive treatment

Case Study II Objective

- Aim: Investigate the interaction between *Company* and *Treatment*
- Method: Fit a Two-way ANOVA (a General LM)

Case Study II Variables

- Response: Y_i - score for ith platoon, $i = 1, \ldots, 29$
- Explanatory variables: $\underline{9 + 1 + 9}$ Indicator variables-
  - 9 for *Company* $(\mathbb{1}_{COMP_1,i}, \cdots, \mathbb{1}_{COMP_9,i})$ $(G_1 - 1)$
  - 1 for *Treatment* $(\mathbb{1}_{PYG,i})$ $(G_2 - 1)$
  - 9 for interaction terms $(\mathbb{1}_{PYG,i} \times \mathbb{1}_{COMP_1,i}, \cdots, \mathbb{1}_{PYG,i} \times \mathbb{1}_{COMP_9,i})$ $(G_1 - 1)(G_2 - 1)$

where

$$\mathbb{1}_{PYG,i} = \begin{cases} 1 & \text{if } i\text{th platoon is ``pygmalion''} \\ 0 & i\text{th platoon is ``control''} \end{cases}$$

$$\mathbb{1}_{COMP_1,i} = \begin{cases} 1 & \text{if } i\text{th platoon is from ``company 1''} \\ 0 & i\text{th platoon is NOT from ``company 1''} \end{cases}$$

Case Study II Linear Model

Full Model:

$$
\begin{aligned}
Y_i = \beta_0 + \beta_1 \mathbb{1}_{PYG,i} &+ \beta_2 \mathbb{1}_{COMP_1,i} \\
&+ \beta_3 \mathbb{1}_{COMP_2,i} \\
&+ \cdots \\
&+ \beta_{10} \mathbb{1}_{COMP_9,i} \\
&+ \beta_{11} \mathbb{1}_{PYG,i} \times \mathbb{1}_{COMP_1,i} \\
&+ \beta_{12} \mathbb{1}_{PYG,i} \times \mathbb{1}_{COMP_2,i} \\
&+ \cdots \\
&+ \beta_{19} \mathbb{1}_{PYG,i} \times \mathbb{1}_{COMP_9,i} \\
&+ \epsilon_i
\end{aligned}
$$

1

9

$\mathbb{1}_{comp_1,i} = 1$

$\mathbb{1}_{PYG,i} = 1$

$9(1) = 9$

$E(y_i)$

Case Study II: Expected Response | (Company∗Treatment)

Company	Pygmalion($\mathbb{1}_{PYG,i} = 1$)	Control($\mathbb{1}_{PYG,i} = 0$)	Treatment effect
1	$\beta_0 + \beta_1 + \beta_2 + \beta_{11}$ —	$\beta_0 + \beta_2$ =	$\beta_1 + \beta_{11}$
2	$\beta_0 + \beta_1 + \beta_3 + \beta_{12}$	$\beta_0 + \beta_3$	$\beta_1 + \beta_{12}$
3			
4	$\vdots$	$\vdots$	$\vdots$
5			
6			
7			
8			
9	$\beta_0 + \beta_1 + \beta_{10} + \beta_{19}$	$\beta_0 + \beta_{10}$	$\beta_1 + \beta_{19}$
10	$\beta_0 + \beta_1$	β_0	β_1

9

Question 1: Does mean treatment effect differ with *Company*?

Null Hypothesis, H_0 : $\beta_{11} = \beta_{12} = \beta_{13} = \ldots\ldots = \beta_{19} = 0$

Alternative Hypothesis, H_a : at least one β from the set is not 0.

Overall versus Partial F-tests in Two-way ANOVA

Global

①

- Overall test: $H_0 : \beta_1 = \beta_2 = \cdots = \beta_{dfMODEL} = 0$
- Partial test: H_0 : a subset of $\{\beta_1, \beta_2, \cdots, \beta_{dfMODEL}\} = 0$
- Approach: Fit full model (with all explanatory variables) and reduced (without variables whose coefficients you are testing) model

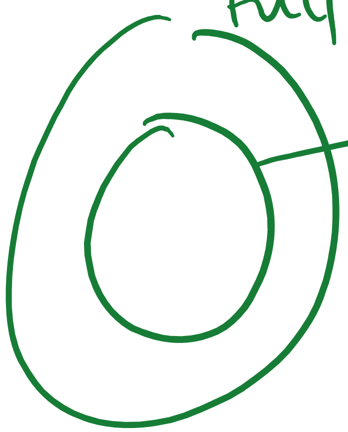
②

- Test statistic:

$$\boxed{\begin{aligned} F &= \frac{(SSReg_{full} - SSReg_{reduced})/(\#\text{ of }\beta\text{'s -being- tested})}{MSE_{full}} \\ &= \frac{(RSS_{reduced} - RSS_{full})/(\#\text{ of }\beta\text{'s -being- tested})}{MSE_{full}} \end{aligned}}$$

Full

Reduced



- If H_0 is true, F is an observation from F distribution with $df = (\#\text{ of }\beta\text{'s being tested}, df\text{ERROR of full model})$

Case II

$dfModel = 19$
$= 9+1+9$
$(G_1-1)+(G_2-1)+(G_2-1)(G_1-1)$

$19-9=10$

$19-10=9$

$DFT = DFR + DFE$

$SST = SSReg + SSE$
$= SSReg + RSS$

$DFT - DFReg = (N-1) - \left[(G_1-1)+(G_2-1) + (G_2-1)(G_1-1)\right]$

Case Study II: Testing interaction

$N-1 = 29-1 = 28$

$H_0: \beta_{11} = \beta_{12} = \cdots = \beta_{19} = 0$

(9 β's being tested)

- FULL:

$$\text{full=lm(score}\sim\text{company*treat)}$$

19 β's

$DFE_{full} = 28 - 19 = 9$

- Reduced:

$$\text{reduced=lm(score}\sim\text{company+treat)}$$

10 β's

- Partial F-test (Refer to R output)
  - Test statistic:

$$F = \frac{(1321.32 - 1009.86)/9}{51.89} = \frac{(778.5^{*} - 467.04^{*})/9}{51.89} = \frac{311.46/9}{51.89} = 0.67$$

MSE_{full}

  - Under H_0, F statistic $\sim$ F distribution with $\boxed{df = (9, 9).}$
  - The resulting p-value is large ($p = 0.7221$), implying that the data are consistent with zero coefficient for the interaction term.
- No evidence that treatment effect differs with *Company*.

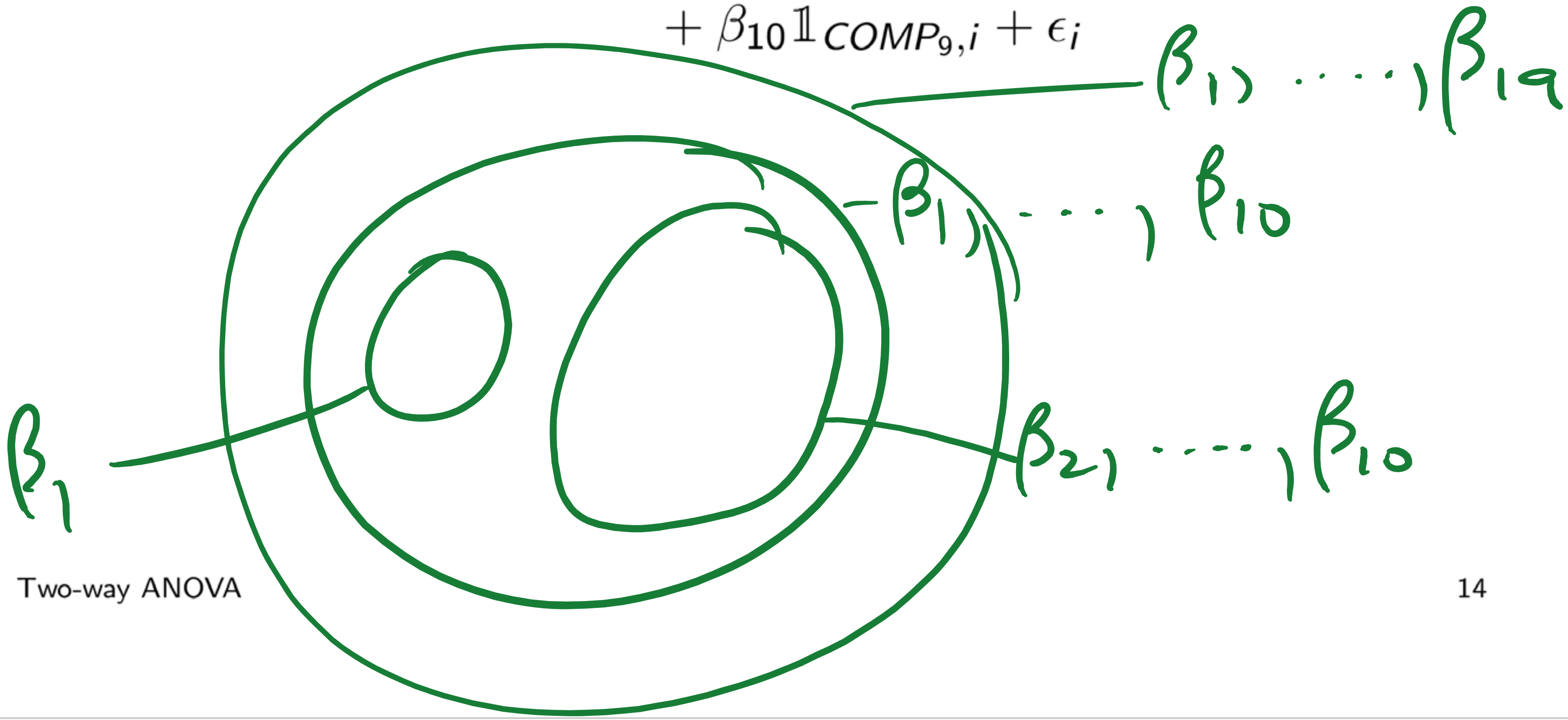
Case Study II: Interaction model summary

```
Call:
lm(formula = Score ~ company * treat)

Coefficients:
                          Estimate Std. Error t value Pr(>|t|)
(Intercept)                 66.200      5.094  12.996 3.89e-07 ***
companyC10                   4.500      7.204   0.625   0.5477
companyC2                    6.100      7.204   0.847   0.4191
companyC3                   10.000      8.823   1.133   0.2863
companyC4                    0.300      7.204   0.042   0.9677
companyC5                   10.000      7.204   1.388   0.1985
companyC6                   15.600      7.204   2.166   0.0585 .
companyC7                   -1.100      7.204  -0.153   0.8820
companyC8                    4.300      7.204   0.597   0.5653
companyC9                    6.900      7.204   0.958   0.3632
treatPygmalion              13.800      8.823   1.564   0.1522
companyC10:treatPygmalion   -0.800     12.477  -0.064   0.9503
companyC2:treatPygmalion    -2.200     12.477  -0.176   0.8639
companyC3:treatPygmalion   -21.800     13.477  -1.618   0.1402
companyC4:treatPygmalion    -3.800     12.477  -0.305   0.7676
companyC5:treatPygmalion    -2.200     12.477  -0.176   0.8639
companyC6:treatPygmalion    -5.800     12.477  -0.465   0.6531
companyC7:treatPygmalion    -2.800     12.477  -0.224   0.8275
companyC8:treatPygmalion   -12.800     12.477  -1.026   0.3317
companyC9:treatPygmalion   -17.400     12.477  -1.395   0.1966

Residual standard error: 7.204 on 9 degrees of freedom
Multiple R-squared:  0.7388,Adjusted R-squared:  0.1875
F-statistic:  1.34 on 19 and 9 DF,  p-value: 0.3358
```

Case Study II: Additive Model

Additive (a reduced) Model:

$$
\begin{aligned}
Y_i = \beta_0 + \beta_1 \mathbb{1}_{PYG,i} &+ \beta_2 \mathbb{1}_{COMP_1,i} \\
&+ \beta_3 \mathbb{1}_{COMP_2,i} \\
&+ \cdots \\
&+ \beta_{10} \mathbb{1}_{COMP_9,i} + \epsilon_i
\end{aligned}
$$



Case Study II: Additive Model Expected Response

$\mu_{PYG} \longrightarrow \mu_{control} \Rightarrow \beta_1$

Company	*Treatment* Pygmalion($\mathbb{1}_{PYG,i} = 1$)	Control($\mathbb{1}_{PYG,i} = 0$)	Treatment effect
1	$(\beta_0 + \beta_1 + \beta_2)$ $-$	$(\beta_0 + \beta_2)$ $=$	$\beta_1 = \mu_{PYG} - \mu_{control}$
2	$\beta_0 + \beta_1 + \beta_3$	$\beta_0 + \beta_3$	β_1
...	...	...	...
8	$\beta_0 + \beta_1 + \beta_9$	$\beta_0 + \beta_9$	β_1
9	$\beta_0 + \beta_1 + \beta_{10}$	$\beta_0 + \beta_{10}$	β_1
10	$\beta_0 + \beta_1$	β_0	β_1

Test 1: Is there a difference in mean score between pygmalion and control group?

$H_0: \beta_1 = \mu_{PYG} - \mu_{control} = 0$

Test 2: Are there differences between companies?

$H_0: \beta_2 = \beta_3 = \cdots = \beta_{10} = 0$

Case Study II: Additive model summary

$\hat{\beta}$ $se(\hat{\beta})$

```
Call:
lm(formula = Score ~ company + treat)

Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)     68.39316    3.89308  17.568 8.92e-13 ***
companyC10       4.23333    5.36968   0.788   0.4407
companyC2        5.36667    5.36968   0.999   0.3308
companyC3        0.19658    6.01886   0.033   0.9743
companyC4       -0.96667    5.36968  -0.180   0.8591
companyC5        9.26667    5.36968   1.726   0.1015
companyC6       13.66667    5.36968   2.545   0.0203 *
companyC7       -2.03333    5.36968  -0.379   0.7094
companyC8        0.03333    5.36968   0.006   0.9951
companyC9        1.10000    5.36968   0.205   0.8400
treatPygmalion   7.22051    2.57951   2.799   0.0119 *

Residual standard error: 6.576 on 18 degrees of freedom
Multiple R-squared:  0.5647,Adjusted R-squared:  0.3228
F-statistic: 2.335 on 10 and 18 DF,  p-value: 0.0564
```

Global test

$H_0: \beta_1 = \beta_2 = \beta_3 = \cdots = \beta_{10}$ (no relevant factors)

Case Study II: Additive model summary

```
Call:
lm(formula = Score ~ treat + company)

Coefficients:
               Estimate Std. Error t value Pr(>|t|)
(Intercept)    68.39316    3.89308  17.568 8.92e-13 ***
treatPygmalion  7.22051    2.57951   2.799   0.0119 *
companyC10      4.23333    5.36968   0.788   0.4407
companyC2       5.36667    5.36968   0.999   0.3308
companyC3       0.19658    6.01886   0.033   0.9743
companyC4      -0.96667    5.36968  -0.180   0.8591
companyC5       9.26667    5.36968   1.726   0.1015
companyC6      13.66667    5.36968   2.545   0.0203 *
companyC7      -2.03333    5.36968  -0.379   0.7094
companyC8       0.03333    5.36968   0.006   0.9951
companyC9       1.10000    5.36968   0.205   0.8400

Residual standard error: 6.576 on 18 degrees of freedom
Multiple R-squared:  0.5647,Adjusted R-squared:  0.3228
F-statistic: 2.335 on 10 and 18 DF,  p-value: 0.0564
```

Case Study II: Additive Model-Testing main effects

	Test 1	Test 2
Null Alt	$H_0 : \beta_1 = 0$ $H_a : \beta_1 \neq 0$	$H_0 : \beta_2 = \beta_3 = \cdots = \beta_{10} = 0$ H_a :at least one $\beta \neq 0$
F statistic F-dist df p-value	7.84 $= \frac{338.88}{43.25}$ (1,18) 0.0119	1.75 $= \frac{\quad}{43.25}$ (9,18) 0.1484
Conc.	Evidence of a difference in mean score between pygmalion and control platoons (over and above difference btw companies)	No evidence of difference between companies.

- On average, pygmalion platoons (mean=78.7) scored higher than control platoons (mean=71.6).

C.I

Case Study II: Model Checking

- Look at diagnostic panel of plots
  - No outliers
  - Normality ok
  - Perhaps decreasing variance
- Independent observations: by assuming that platoons were chosen at random and were not interacting

▷ Model is appropriate (incl. & excl. factors)

In the presence of "Interactions"

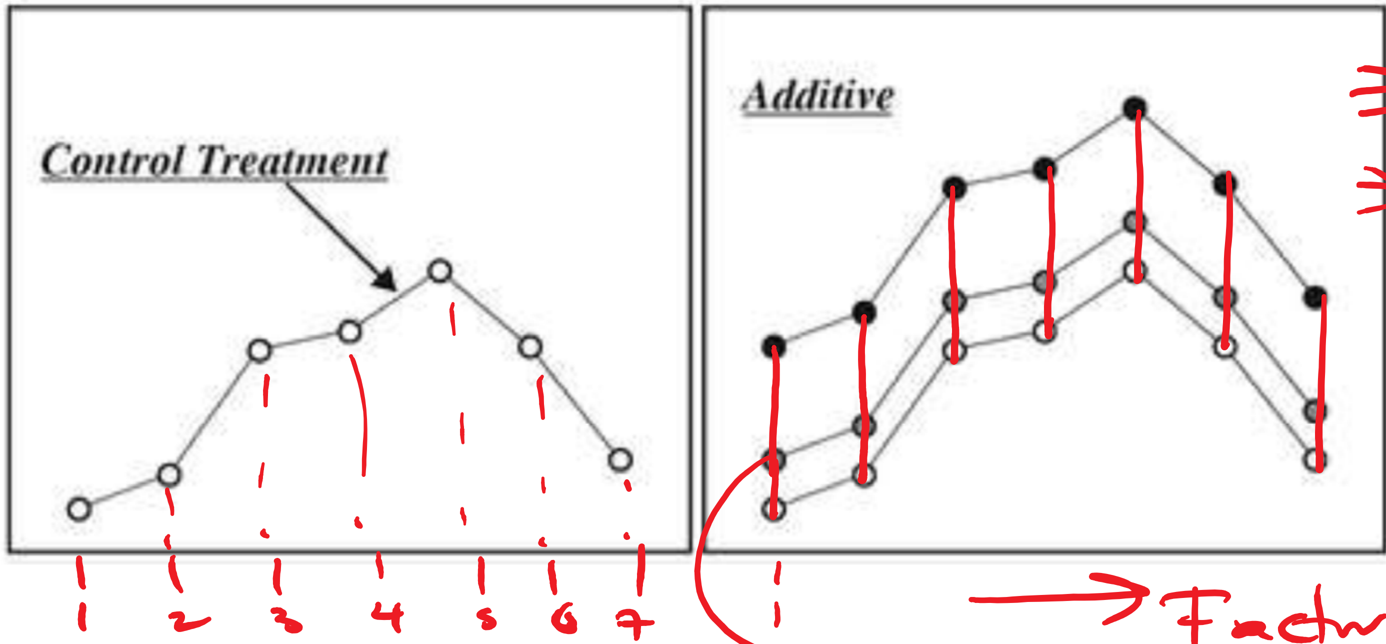
- Hard to answer questions about the main factor effects
- Communicate a table of estimated means
- Have separate models of Y against one factor for the different levels of the other factor

References:

- The Statistical Sleuth, 3rd edition by Ramsey and Schafer
- `https://cran.r-project.org/web/packages/Sleuth3/vignettes/chapter13-HortonMosaic.pdf`

In the presence of "Interactions"

DISPLAY 13.21 Hypothetical treatment curves plotted against another factor, illustrating additive and some nonadditive conditions



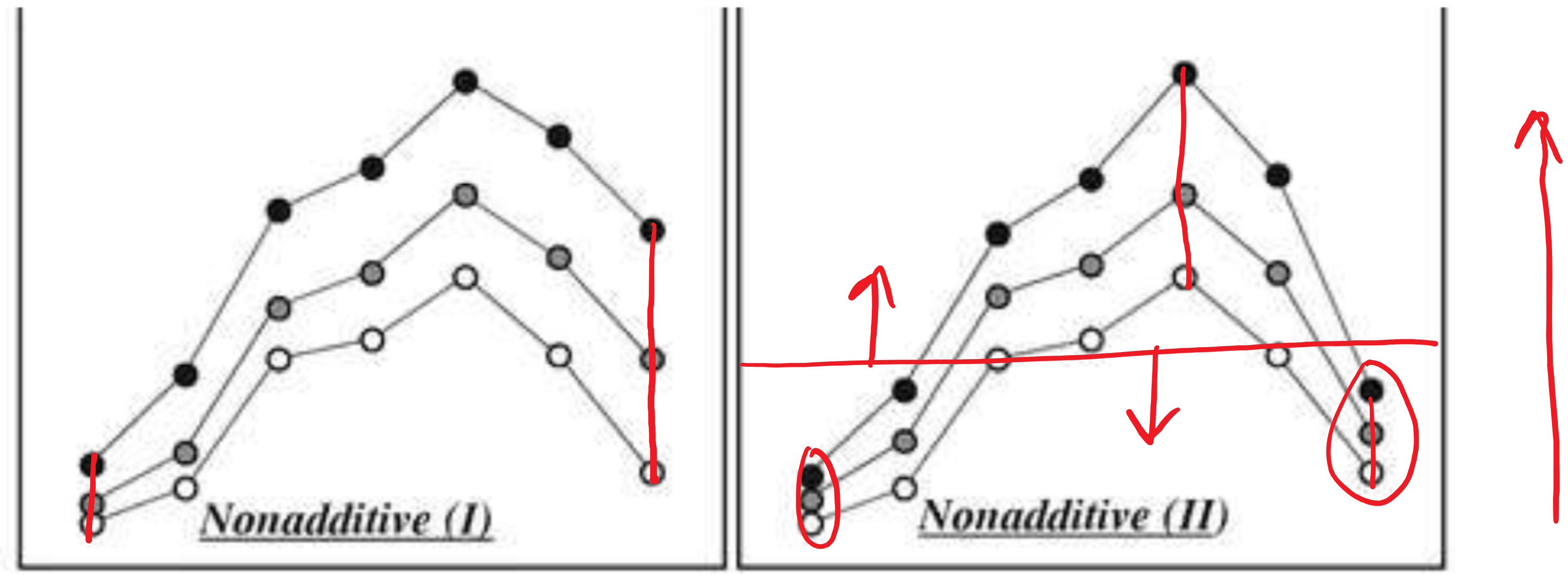
⇒ No interaction

⇒ The treatment effect does not change as the levels of Factor 2 vary.

Factors

1. Treatment (3) : Control O, Treatment 1 ⊘, Treatment ●
2. Factor 2 (7) : 1–7

In the presence of "Interactions"

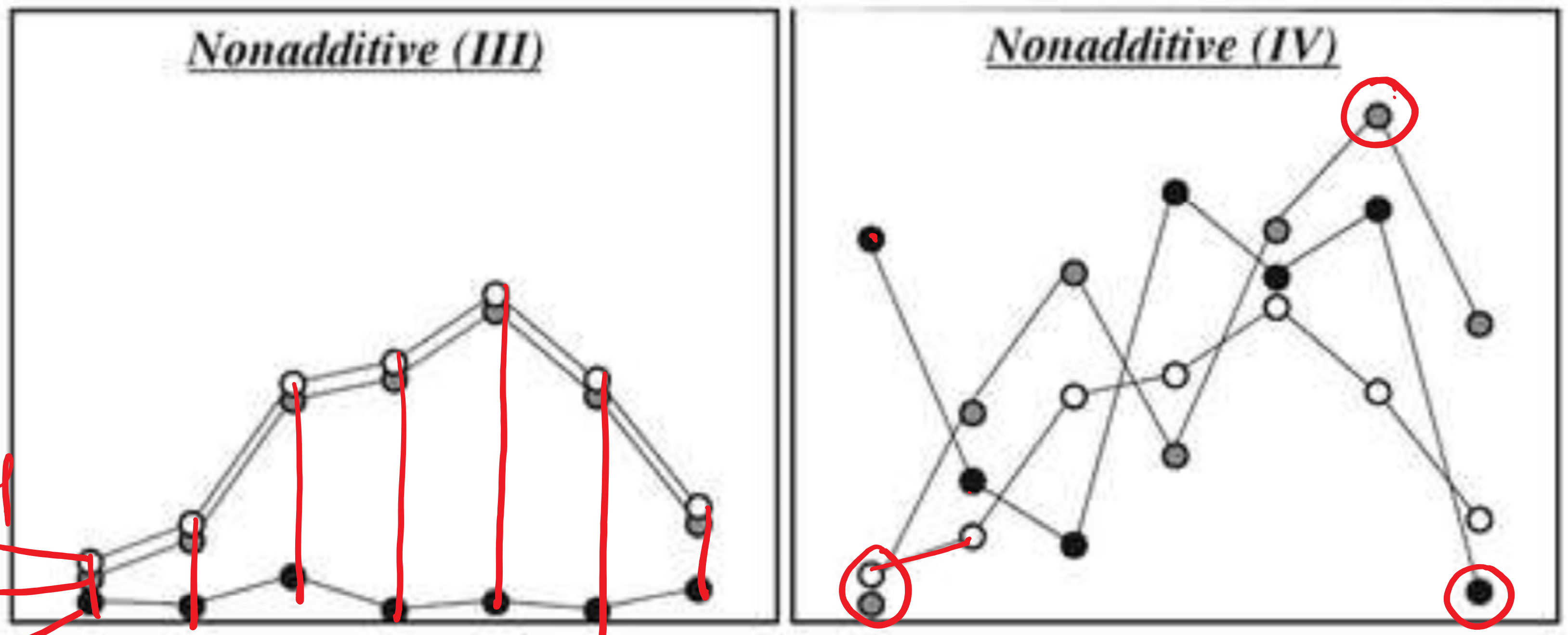


Factor 2

→ The treatment effect increases as we increase the level of Factor 2.

⇒ The treatment effect increases as the mean of Factor 2 increases.

In the presence of "Interactions"



Control
T1
T2

→ Treatment 2's effect <u>differs</u> from that of the Control and/or Treatment 1 as we vary Factor 2.

→ Complicated effects.

→ Consider adding additional factors to the model.

Should insignificant block effects be kept in the model?

Eg, company

- General advice is to drop insignificant terms
- For data from a randomized block experiment, block effects should be maintained
- Ensure that the control exercised by blocking is maintained in the analysis.

Estimated Mean Response from Additive Model

$\hat{\beta}_0 + \hat{\beta}_1$

Company	Pygmalion($\mathbb{1}_{PYG,i} = 1$)	Control($\mathbb{1}_{PYG,i} = 0$)
1	68.39+7.22	68.39 $\hat{\beta}_0$
2	68.39+7.22+5.37	68.39+5.37
3		
4	$\hat{\beta}_0 + \hat{\beta}_1 + \hat{\beta}_2$	
5		
6		
7		
8		
9		
10	68.39+7.22+4.23	68.39+4.23

$\hat{\beta}_0 + \hat{\beta}_1 + \hat{\beta}_{10}$ $\hat{\beta}_0 + \hat{\beta}_{10}$

Observed Group means vs Estimated means

y_i $\hat{y}_i$

	Observed Means			*Estimated* Means	
Company	Pyg	Control	$n_{control}$	Pyg	Control
1	80.0	66.2	2	75.61	68.39
2	83.9	72.3	2	80.98	73.76
3	68.2	76.2	1	75.81	68.59
4	76.5	66.5	2	74.65	67.43
5	87.8	76.2	2	84.88	77.66
6	89.8	81.8	2	89.28	82.06
7	76.1	65.1	2	73.58	66.36
8	71.5	70.5	2	75.65	68.43
9	69.5	73.1	2	76.71	69.49
10	83.7	70.7	2	79.85	72.63
means	78.70	71.63		78.70	71.48

Parameter estimation and Unbalanced design

- Estimated means for treatments are averages over 10 companies
- Observed Means vs Estimated means: Not the same because there are unequal number of control observations per company. Company 3 has 1 control platoon; other companies have 2.
- The design is nearly balanced.
- Affects constant variance assumption and variance estimate
- Consider any evidence as exploratory
- Consider *weighted* least squares regression

Measuring treatment effect

```
> qt(1-0.05/2,df=27)
> sqrt((9*var(Score[treat=="Pygmalion"])+18*var(Score[treat=="Control"]))/27)
> t.test(Score[treat=="Pygmalion"], Score[treat=="Control"], var.equal=T)

[1] 2.051831
[1] 7.356078

Two Sample t-test

data:  Score[treat == "Pygmalion"] and Score[treat == "Control"]
t = 2.4595, df = 27, p-value = 0.0206
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
  1.171707 12.965135
sample estimates:
mean of x mean of y
 78.70000  71.63158
```

Welch (unpooled)

$\bar{x}_1$ $\bar{x}_2$

Case Study II: Conclusions

$\sqrt{MSE} = s_p = \sqrt{\dfrac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}}$

- There is evidence of a difference in mean score between pygmalion and control platoons (p=0.0119). (Consider this as weak evidence since we have some concerns about variance estimates.)
- Confidence Intervals for the difference in mean score between pygmalion and control platoons:

  $n_1 = 10$ $19 = n_2$

  CI $(\mu_{PYG} - \mu_{Control})$

  - Pooled 2-sample t:

    $(\bar{x}_1 - \bar{x}_2) \pm t_{27, 0.025}\, s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$

    $$(78.7 - 71.6) \pm 2.05(7.36)\sqrt{(1/10 + 1/19)} = (1.17, 12.96)$$

  - Least-squares approach (Additive model):

    $se(\hat{\beta}_1)$

    $$\hat{\beta}_1 \; 7.22 \pm \underbrace{2.101}_{t_{18, 0.025}}(2.5795) = (1.8, 12.6)$$

    DFE

- On average, pygmalion platoons (mean=78.7) scored higher than control platoons (mean=71.6).