STA347 Probability I
Assignment #2

Due: October 29, 2018 before class starts

Solve the following and hand in by due date.

1. Solve Problem 3.8.12

Solution. (a) The second axiom implies

$$1 = \int\int c(x^2 - y^2)e^{-x} \cdot 1(0 \le x < \infty, -x \le y < x)\, dy\, dx = \int\int c(x^2 y - y^3/3)|_{-x}^{x} e^{-x} \cdot 1(0 \le x < \infty)\, dx$$
$$= \int (4c/3)x^3 e^{-x} \cdot 1(0 \le x < \infty)\, dx = (4c/3)\Gamma(4) = 8c.$$

Hence $c = 1/8$.
(b) integrating out y or x implies

$$\mathrm{pdf}_X(x) = \int_{-x}^{x} \frac{1}{8}(x^2 - y^2)e^{-x}\, dy = \frac{1}{6}x^3 e^{-x} \sim \mathrm{gamma}(4,1),$$
$$\mathrm{pdf}_Y(y) = \int_{|y|}^{\infty} \frac{1}{8}(x^2 - y^2)e^{-x}\, dx = \frac{1}{8}(-x^2 + y^2 - 2x - 2)e^{-x}|_{|y|}^{\infty} = \frac{1}{4}(|y|+1)e^{-|y|}.$$

(c) The conditional density of X given $Y = y$ is

$$\mathrm{pdf}_{X\,|\,Y}(x\,|\,y) = \frac{\mathrm{pdf}_{X,Y}(x,y)}{\mathrm{pdf}_Y(y)} = \frac{(1/8)(x^2-y^2)e^{-x}1(-x \le y < x, 0 \le x < \infty)}{(1/4)(|y|+1)e^{-|y|}} = \frac{(x^2-y^2)e^{-(x-|y|)}1(x > y \text{ or } x \ge -y)}{2(|y|+1)}$$

The conditional density of Y given $X = x$ is

$$\mathrm{pdf}_{Y\,|\,X}(y\,|\,x) = \frac{\mathrm{pdf}_{X,Y}(x,y)}{\mathrm{pdf}_X(x)} = \frac{(1/8)(x^2-y^2)e^{-x}1(-x \le y < x, 0 \le x < \infty)}{(1/6)x^3 e^{-x}} = \frac{3}{4}\frac{1}{y}(1 - (\frac{y}{x})^2)1(-x \le y < x)$$

2. Solve Problem 3.8.73

Solution. Solution I: First of all, $P(X_i = X_j \text{ for some } i \ne j) = 0$ because the common distribution is continuous and having density. So no two random variables having the same value. Assume $x_1 < x_2 < \cdots < x_n$. Take $\delta = \min(|x_i - x_j|, i \ne j)/3 > 0$. Take a_is and b_is satisfying $x_i - \delta < a_i < x_i$ and $x_i < b < x_i + \delta$ so that $b_i < a_j$ for any $i < j$. Then

$$P(a_i < X_{(i)} \le b_i, i = 1, \ldots, n) = \sum_{\sigma \in V} P(a_i < X_{\sigma(i)} \le b_i, i = 1, \ldots, n) = \sum_{i=1}^{n} P(a_i < X_i \le b_i, i = 1, \ldots, n)$$
$$= n! \prod_{\sigma \in V} P(a_i < X_i \le b_i).$$

Hence the joint density becomes

$$n! \prod_{i=1}^{n} f(x_i).$$

Solution II: The map $(x_1, \ldots, x_n) \to (x_{(1)}, \ldots, x_{(n)})$ is unique, howerver, there are $n!$ sets of tuples which maps to $(x_{(1)}, \ldots, x_{(n)})$. Also on each inverse, the absolute Jacobean determinant is 1 because of change of order. Hence the joint density becomes

$$\text{pdf}_{X_{(1)},\ldots,X_{(n)}}(x_1, \ldots, x_n) = n!\text{pdf}_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = n!f(x_1)\cdots f(x_n).$$

3. Solve Problem.3.8.74

Solution. (a) From #2, $\text{pdf}_{U_{(1)},U_{(2)},U_{(3)}}(u_1, u_2, u_3) = 3!f(u_1)f(u_2)f(u_3) = 6(0 < u_1 \le u_3 < 1)$.
(b) Let U_1, U_2, U_3 be uniform on $(0, 1)$. The probability of interest is $P(|U_i - U_j| > 1/3, i \neq j)$. Or $P(U_{(2)} - U_{(1)} > 1/3, U_{(3)} - U_{(2)} > 1/3)$, that is,

$$\int_0^1 \int_0^1 \int_0^1 1(u_2 - u_1 > 1/3, u_3 - u_2 > 1/3) \cdot 6\, du_3\, du_2\, du_1 = 6 \int_0^1 \int_0^1 1(u_2 - u_1 > 1/3) \max(0, 1 - (u_2 + 1/3))\, du_2\, du_1$$
$$= 6 \int_0^{1/3} \int_{u_1+1/3}^{2/3} (2/3 - u_2)\, du_2\, du_1 = 6 \int_0^{1/3} (1/3 - u_1)^2/2\, du_1 = -(1/3 - u_1)^3|_0^{1/3} = \frac{1}{27}.$$

4. Let $X_1, X_2, \ldots$ be a sequence of random variables. Prove that $\sup X_n$ and $\limsup_{n\to\infty} X_n$ are random variables.

Solution. Both $Y = \sup X_n$ and $Z = \limsup_{n\to\infty} X_n$ are assumed to be finite. Then $Y, Z : S \to \mathbb{R}$ are functions on sample space. For any $r \in \mathbb{R}$,

$$\{Y > r\} = \{\sup X_n > r\} = \bigcup_{n=1}^{\infty} \{X_n > r\}$$

which is a countable union of events, hence, $\{Y \le r\}$ is also an event. Similarly,

$$\{Z > r\} = \{\limsup_{n\to\infty} X_n > r\} = \{\lim_{m\to\infty} \sup_{n\ge m} X_n > r\} = \bigcap_{m=1}^{\infty} \{\sup_{n\ge m} X_n > r\} = \bigcap_{m=1}^{\infty} \bigcup_{n\ge m} \{X_n > r\}$$

which is an event, so is $\{Z \le r\}$. Therefore Y and Z are random variables.

5. Independent random variables $X_1, X_2, \ldots$ are identically distributed from $N(0, \sigma^2)$.
(a) If $\sigma^2 = 1$, then show that $X_1^2 \sim \chi^2(1) \sim \text{gamma}(1/2, 1/2)$.
(b) Find the density of $Z = (X_1 + \cdots + X_k)/\sqrt{k(X_1^2 + \cdots + X_k^2)}$.
(c) Conclude that the density of Z does not contain σ term.

Solution. (a) Let $Y = X_1^2$ and the density of Y is given by

$$\text{pdf}_Y(y) = \frac{d}{dy}P(Y \le y) = \frac{d}{dy}P(X_1^2 \le y) = 2\frac{d}{dy}P(0 \le X_1 \le \sqrt{y}) = 2\text{pdf}_{X_1}(\sqrt{y})\frac{d}{dy}\sqrt{y} = 2 \cdot \frac{1}{\sqrt{2\pi}}e^{-y/2} \cdot \frac{1}{2\sqrt{y}}$$
$$= \frac{(1/2)^{1/2}}{\Gamma(1/2)}y^{1/2-1}e^{-(1/2)y}1(y > 0) \sim \text{gamma}(1/2, 1/2).$$

(b) Let $\bar{X} = (X_1 + \cdots + X_k)/k$ and $U = \sqrt{k}\bar{X} \sim N(0, \sigma^2)$. Then $X_1^2 + \cdots + X_k^2 = \sum_{j=1}^k (X_j - \bar{X} + \bar{X})^2 = \sum_{j=1}^k (X_j - \bar{X})^2 + k\bar{X}^2$. Let $V = (X_1 - \bar{X})^2 + \cdots + (X_k - \bar{X})^2$. The joint density of $(X_1, \ldots, X_k)$ becomes

$$\text{pdf}(x_1, \ldots, x_k) = (2\pi\sigma^2)^{-k/2} \exp(-(x_1^2 + \cdots + x_k^2)/(2\sigma^2)) = (2\pi\sigma^2)^{-k/2} \exp(-(u^2 + v)/(2\sigma^2))$$

By integrating under the condition $(x_1+\cdots+x_k)=u\sqrt{k}$ and $(x_1-u/\sqrt{k})^2+\cdots+(x_k-u/\sqrt{k})^2=v$, the marginal densities of u and v are

$$\begin{aligned}
\mathrm{pdf}_{U,V}(u,v) &\propto (2\pi\sigma^2)^{-k/2}e^{-u^2/(2\sigma^2)}\cdot v^{(k-3)/2}\exp(-v/(2\sigma^2))\\
&\propto (2\pi\sigma^2)^{-1/2}e^{-u^2/(2\sigma^2)}\times 1/((2\sigma^2)^{(k-1)/2}\Gamma((k-1)/2))v^{(k-3)/2}e^{-v/(2\sigma^2)}\\
&\sim N(0,\sigma^2)\times \mathrm{gamma}((k-1)/2,1/(2\sigma^2)).
\end{aligned}$$

Finally, the change of variable $(u,v)\mapsto(z=u/\sqrt{v+u^2},v)$ has inverse $u=z\sqrt{v/(1-z^2)}$ and the Jacobean becomes

$$\begin{pmatrix} \sqrt{v/(1-z^2)^3} & z/(2\sqrt{v(1-z^2)}) \\ 0 & 1 \end{pmatrix}$$

Thus the joint density of (Z,V) is

$$\begin{aligned}
\mathrm{pdf}_{Z,V}(z,v) &= (2\pi\sigma^2)^{-1/2}\exp\Big(-\frac{1}{2\sigma^2}\frac{z^2v}{1-z^2}\Big)\cdot\Big(\frac{1}{2\sigma^2}\Big)^{(k-1)/2}\frac{1}{\Gamma((k-1)/2)}v^{(k-1)/2-1}\exp\Big(-\frac{1}{2\sigma^2}v\Big)\cdot\sqrt{\frac{v}{(1-z^2)^3}}\\
&= (\pi)^{-1/2}\frac{1}{\Gamma((k-1)/2)}(1-z^2)^{-3/2}\Big(\frac{1}{2\sigma^2}\Big)^{k/2}v^{k/2-1}\exp\Big(-\frac{v}{2\sigma^2(1-z^2)}\Big).
\end{aligned}$$

By integrating out V, the marginal density of z is

$$\begin{aligned}
\mathrm{pdf}_Z(z) &= \int_0^\infty (\pi)^{-1/2}\frac{1}{\Gamma((k-1)/2)}(1-z^2)^{-3/2}\Big(\frac{1}{2\sigma^2}\Big)^{k/2}v^{k/2-1}\exp\Big(-\frac{v}{2\sigma^2(1-z^2)}\Big)\,dv\\
&= (\pi)^{-1/2}\frac{1}{\Gamma((k-1)/2)}(1-z^2)^{-3/2}\Big(\frac{1}{2\sigma^2}\Big)^{k/2}(2\sigma^2(1-z^2))^{k/2}\Gamma(k/2)\\
&= \frac{\Gamma(k/2)}{\sqrt{\pi}\Gamma((k-1)/2)}(1-z^2)^{(k-3)/2}.
\end{aligned}$$

The value of Z is defined on $[-1,1]$.
(c) Let $Y_1,Y_2,\dots$ be an i.i.d. sample from $N(0,1)$ so that $(X_1,X_2,\dots,X_k)$ and $(\sigma Y_1,\dots,\sigma Y_k)$ have the same distribution. Then

$$Z=\frac{X_1+\cdots+X_k}{\sqrt{k(X_1^2+\cdots+X_k^2)}}\equiv^d\frac{\sigma(Y_1+\cdots+Y_k)}{\sqrt{\sigma^2k(Y_1^2+\cdots+Y_k^2)}}=\frac{Y_1+\cdots+Y_k}{\sqrt{k(Y_1^2+\cdots+Y_k^2)}}$$

which does not depend on σ^2. Hence the density of Z does not contain σ.