STA347 - Probability I - Term Test - 2018 Fall

Last Name:	First Name and Initials:
Student Number:	

A non-programmable calculator is allowed

Instruction

1. Don't forget writing your name and student number.
2. If you do not have enough space, use the other side and refer it correctly.
3. Some problems could be wrong. If it happens, correct questions by yourself after arguing why the problem is wrong.
4. There are 5 questions on **4 pages**, and the total marks is 100.
5. You can use either pen or pencil for the test. **But please be aware that you are not allowed to dispute any credit after the test is returned if you use pencil.**
6. Pages 3–4 can be used as a scratch space.
7. Please hand in your paper when the time is up.

Useful Information

1. Talyor expansions of exponential and log functions are
$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots \text{ and } \log(1-z) = -\frac{z}{1} - \frac{z^2}{2} - \frac{z^3}{3} - \cdots$$
for $x \in \mathbb{R}$ and $|z| < 1$.
2. The gamma function is defined by $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$ for $\alpha > 0$ and satisfies $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ and $\Gamma(n+1) = n!$ for any positive integer n.
3. $X \sim \text{Poisson}(\mu)$ has probability mass function $P(X = k) = e^{-\mu}\mu^k/k!$ if $k = 0, 1, \ldots$ and zero otherwise.
4. $X \sim \text{binomial}(n, p)$ has probability mass function $P(X = k) = \binom{n}{k}p^k(1-p)^{n-k}$ if $k = 0, 1, \ldots, n$ and zero otherwise.
5. $X \sim \text{beta}(\alpha, \beta)$ has probability density function $\text{pdf}_X(x) = 1(0 < x < 1)[\Gamma(\alpha + \beta)/(\Gamma(\alpha)\Gamma(\beta))]x^{\alpha-1}(1-x)^{\beta-1}$.
6. A random variable Z is normally distributed with parameters μ and σ^2 if its' probability density function is $\text{pdf}_Z(z) = (2\pi\sigma^2)^{-1/2}\exp(-(z-\mu)^2/(2\sigma^2))$, denoted by $Z \sim N(\mu, \sigma^2)$.

Problem 1. Three events A_1, A_2, A_3 satisfy $P(A_i) = i/5$ for $i = 1, 2, 3$ and, for $i \neq j$,

$$P(A_i \cup A_j) = P(A_i) + P(A_j) - P(A_i)P(A_j)$$

(a) [4 marks] Prove A_1, A_2, A_3 are pair-wise independent.
(b) [6 marks] Find the maximum possible value of $P(A_1 \cup A_2 \cup A_3)$.
(c) [6 marks] Are A_1, A_2, A_3 independent when the maximum is achieved in part (b) ?

Solution. (a) From the assumption, $P(A_i \cap A_j) = P(A_i) + P(A_j) - P(A_i \cup A_j) = P(A_i)P(A_j)$ which implies A_i and A_j are independent, in other words, A_i's are pair-wise independent.
(b) Note that $P(A_1 \cup A_2 \cup A_3) = P(A_1 \cup A_2) + P(A_3 \cap (A_1 \cup A_2)^c)$ and expansions imply

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3) &= P(A_1) + P(A_2) - P(A_1 \cap A_2) + P(A_3) - P(A_3 \cap A_1) - P(A_3 \cap A_2) + P(A_1 \cap A_2 \cap A_3) \\ &= 1/5 + 2/5 - (1/5)(2/5) + (3/5) - (3/5)(1/5) - (3/5)(2/5) + P(A_1 \cap A_2 \cap A_3) \\ &= 19/25 + P(A_1 \cap A_2 \cap A_3). \end{aligned}$$

The monotonicity implies $P(A_1 \cap A_2 \cap A_3) \leq \min(P(A_i \cap A_j); i \neq j) = P(A_1 \cap A_2) = 2/25$. Hence the maximum is $19/25 + 2/25 = 21/25$ if and only if $P(A_1 \cap A_2 \cap A_3) = P(A_1 \cap A_2) = 2/25$.
(c) $P(A_1 \cap A_2 \cap A_3) = 2/25 \neq 6/125 = P(A_1)P(A_2)P(A_3)$. Hence A_1, A_2, A_3 are not independent.

Problem 2. Let X and Y be two continuous random variables having F and G as their cumulative distribution functions respectively.
(a) [8 marks] Prove that $H(u, v) = F(u)G(v)[1 + \alpha(1 - F(u))(1 - G(v))]$ is a joint cumulative distribution function where $-1 \leq \alpha \leq 1$.
(b) [8 marks] The joint cumulative distribution function of two random variables (U, V) is H. Find the marginal cumulative distribution functions of U and V.

Solution. (a) Solution I: First of all $1 + \alpha(1 - F(u))(1 - G(v)) \geq 1 - (1 - F(u))(1 - G(v)) \geq 0$ implies $H(u, v) \geq 0$. We prove H is non-decreasing, limit as diverging toward ∞ or $-\infty$, and right continuous for both components. We only prove for v, symmetry holds for u. For non-decreasing property, assume $v \leq w$,

$$\begin{aligned} H(u, w) - H(u, v) &= F(u)[G(w) - G(v)] + F(u)\alpha(1 - F(u))[G(v) - G(w)] \\ &= [G(w) - G(v)]F(u)[1 - \alpha(1 - F(u))] \\ &\geq 0 \end{aligned}$$

since $G(w) \geq G(v)$ and $1 - \alpha(1 - F(u)) \geq 1 - (1 - F(u)) = F(u) \geq 0$. Or $H(u, v) \leq H(u, w)$ for all $v \leq w$. The limits are

$$\begin{aligned} \lim_{v \to \infty} H(u, v) &= F(u)[\lim_{v \to \infty} G(v) + \alpha(1 - F(u)) \lim_{v \to \infty} G(v)(1 - G(v))] \\ &= F(u)[1 + \alpha(1 - F(u)) \cdot 0] = F(u) \\ \lim_{v \to -\infty} H(u, v) &= F(u)[\lim_{v \to -\infty} G(v) + \alpha(1 - F(u)) \lim_{v \to -\infty} G(v)(1 - G(v))] \\ &= F(u)[0 + \alpha(1 - F(u)) \cdot 0] = 0. \end{aligned}$$

The right continuity is derived from

$$\begin{aligned}\lim_{\substack{w\to v\\ w\geq v}} H(u,w) &= F(u)\lim_{\substack{w\to v\\ w\geq v}} G(w)[1+\alpha(1-F(u))(1-G(w))]\\ &= F(u)G(v)[1+\alpha(1-F(u))(1-G(v))] = H(u,v).\end{aligned}$$

Symmetry holds for u. Hence H is a cumulative distribution function.
Solution II: The derivative with respect to both u and v is

$$\begin{aligned}h(u,v) &= \frac{\partial^2}{\partial u\partial v}F(u)G(v)[1+\alpha(1-F(u))(1-G(v))]\\ &= \frac{\partial}{\partial v}[f(u)G(v)(1+\alpha(1-F(u))(1-G(v))) + F(u)G(v)\alpha(-f(u))]\\ &= f(u)g(v)(1+\alpha(1-2F(u))(1-2G(v))) \geq 0.\end{aligned}$$

Moreover $\lim_{u\to-\infty}\lim_{v\to-\infty} H(u,v) = 0$ and $\lim_{u\to\infty}\lim_{v\to\infty} H(u,v) = \lim_{u\to\infty} F(u)\cdot 1\cdot[1+\alpha(1-F(u))(1-1)] = \lim_{u\to\infty} F(u) = 1$ which implies

$$\int\int h(u,v)\,du\,dv = \lim_{u\to\infty}\lim_{v\to\infty} H(u,v) - \lim_{u\to-\infty}\lim_{v\to-\infty} H(u,v) = 1-0 = 1.$$

Thus h is a bivariate density and H is the cumulative distribution function.
(b) The marginalization implies $\mathrm{cdf}_U(u) = \lim_{v\to\infty} H(u,v) = F(u)\cdot 1\cdot[1+\alpha(1-F(u))(1-1)] = F(u)$. Similarly $\mathrm{cdf}_V(v) = \lim_{u\to\infty} H(u,v) = G(v)$.

Problem 3. Assume X is normally distributed with parameters 1 and 1. A random variable Y given $X = x$ follows a normal distribution with parameters x and 3.
(a) [8 marks] Find the marginal density of Y.
(b) [8 marks] Compute expectation of Y.
(c) [8 marks] Compute the conditional probability $P(X > 2\,|\,Y = 5)$.

Solution. (a) The joint density is

$$\mathrm{pdf}_{X,Y}(x,y) = \mathrm{pdf}_X(x)\cdot\mathrm{pdf}_{Y\,|\,X}(y\,|\,x) = \frac{1}{\sqrt{2\pi}}\exp(-(x-1)^2/2)\cdot\frac{1}{\sqrt{2\pi 3}}\exp(-(y-x)^2/6).$$

By integrating out with respect to x gives the marginal density

$$\begin{aligned}\mathrm{pdf}_Y(y) &= \int \mathrm{pdf}_{X,Y}(x,y)\,dx = \int \frac{1}{2\pi\sqrt{3}}\exp(-\frac{1}{8}(y-1)^2 - \frac{2}{3}(x-\frac{3+y}{4})^2)\,dx\\ &= \frac{1}{2\pi\sqrt{3}}\sqrt{2\pi 3/4}\exp(-(y-1)^2/8) = \frac{1}{\sqrt{2\pi 4}}\exp\Big(-\frac{(y-1)^2}{2\cdot 4}\Big).\end{aligned}$$

(b) $\mathbb{E}(Y) = 1+\mathbb{E}(Y-1) = 1+\int(y-1)\cdot\frac{1}{\sqrt{2\pi 4}}\exp(-(y-1)^2/8)\,dy = 1$. In the last equality, $(y-1)\exp(-(y-1)^2/8)$ is an odd function around $y=1$, hence, the integral is zero.

(c) The conditional density of X given $Y = y$ is

$$\mathrm{pdf}_{X\,|\,Y}(x\,|\,y) = \frac{\mathrm{pdf}_{X,Y}(x,y)}{\mathrm{pdf}_Y(y)} = \frac{(2\pi\sqrt{3})^{-1}\exp(-(y-1)^2/8 - 2(x-(3+y)/4)^2/3)}{(2\pi 4)^{-1/2}\exp(-(y-1)^2/8)}$$
$$= \frac{1}{\sqrt{2\pi 3/4}}\exp(-(x-(3+y)/4)^2/(2\cdot 3/4)).$$

Given $Y = 5$, the conditional distribution of X is $N(2, 3/4)$. Since normal distributions are symmetric around the first parameter, $P(X > 2\,|\,Y = 5) = 1/2$, that is,

$$P(X > 2\,|\,Y = 2) = \int_2^\infty \frac{1}{\sqrt{2\pi 3/4}}\exp(-(x-2)^2/(3/2))\,dx$$

Let $z = (x-2)/\sqrt{3/4}$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi}}\exp(-z^2/2)\,dz = \frac{1}{2}.$$

Problem 4. Two random variables X and Y have a joint density given by

$$\mathrm{pdf}_{X,Y}(x,y) = c\cdot xy^2(1-x-y)1(x>0, y>0, x+y<1).$$

(a) [7 marks] Find the value of constant c.
(b) [7 marks] Prove or disprove that X and $Y/(1-X)$ are independent.
(c) [7 marks] Find the conditional density of X given $Y = y$.
(d) [7 marks] Compute the expectation of X.

Solution. (a) The second axiom of probability implies

$$1 = \int\int c\cdot xy^2(1-x-y)1(x>0, y>0, x+y<1)\,dx\,dy$$
$$= c\int_0^1 y^2\int_0^{1-y} x(1-y-x)\,dx\,dy$$
$$= c\int_0^1 y^2\frac{1}{6}(1-y)^3\,dy = c/360.$$

Hence $c = 360$.
(b) Let $Z = Y/(1-X)$ so that $Y = Z(1-X)$. The Jacobean matrix is $\begin{pmatrix} 1 & 0 \\ -z & 1-x \end{pmatrix}$ which has determinant $1-x$. The change of variable formula gives

$$\mathrm{pdf}_{X,Z}(x,z) = 360x(z(1-x))^2(1-x-z(1-x))1(x>0, z(1-x)>0, x+z(1-x)<1)\cdot(1-x)$$
$$= 360x(1-x)^4z^2(1-z)1(0<x<1, 0<z<1)$$
$$= 30x(1-x)^4 1(0<x<1)\times 12z^2(1-z)1(0<z<1).$$

The joint density decomposed into x and z parts separately. Hence X and $Z = Y/(1-X)$ are independent.
(c) The conditional density function of X given $Y = y$ is

$$\mathrm{pdf}_{X\,|\,Y}(x\,|\,y) = \frac{\mathrm{pdf}_{X,Y}(x,y)}{\mathrm{pdf}_Y(y)} = \frac{cxy^2(1-x-y)}{cy^2(1-y)^3/6} = \frac{6}{1-y}\frac{x}{1-y}\Big(1-\frac{x}{1-y}\Big)1(0<x<1-y).$$

(d) Solution I: From the joint density,

$$\begin{aligned}\mathbb{E}(X) &= \int_0^1\int_0^{1-x} x\cdot 360xy^2(1-x-y)\;dy\;dx = \int_0^1 360x^2\Big[(1-x)\frac{y^3}{3}\Big|_0^{1-x} - \frac{y^4}{4}\Big|_0^{1-x}\Big]\;dx \\ &= 30\int_0^1 x^2(1-x)^4\;dx = 30\cdot\frac{1}{105} = \frac{2}{7}.\end{aligned}$$

Solution II: From part (b), the marginal density of X is $\mathrm{pdf}_X(x) = 30x(1-x)^4 1(0<x<1)$. Hence the expectation is

$$\mathbb{E}(X) = \int_0^1 x\cdot 30x(1-x)^4\;dx = 30\int_0^1 x^2(1-x)^4\;dx = \frac{2}{7}.$$

Problem 5. Two independent random variables X and Y have Poisson distributions with parameters α and β respectively.
(a) [8 marks] Show that $U = X+Y$ have a Poisson distribution, specify parameter.
(b) [8 marks] Show that the conditional distribution of X given $X+Y=m$ is a binomial distribution, specify parameters.

Solution. (a) For any integer $k \geq 0$,

$$\begin{aligned}P(U=k) &= \sum_{x=0}^{k} P(X=x, Y=k-x) = \sum_{x=0}^{k} e^{-\alpha}\frac{\alpha^x}{x!}\cdot e^{-\beta}\frac{\beta^{k-x}}{(k-x)!} \\ &= e^{-(\alpha+\beta)}\frac{(\alpha+\beta)^k}{k!}\sum_{x=0}^{k}\frac{k!}{x!(k-x)!}\Big(\frac{\alpha}{\alpha+\beta}\Big)^x\Big(\frac{\beta}{\alpha+\beta}\Big)^{k-x} \\ &= e^{-(\alpha+\beta)}\frac{(\alpha+\beta)^k}{k!}\Big(\frac{\alpha}{\alpha+\beta}+\frac{\beta}{\alpha+\beta}\Big)^k \\ &= e^{-(\alpha+\beta)}\frac{(\alpha+\beta)^k}{k!} \sim \mathrm{Poisson}(\alpha+\beta).\end{aligned}$$

(b) The conditional probability $X = k$ given $X + Y = m$ is

$$
\begin{aligned}
P(X = k \,|\, X + Y = m) &= \frac{P(X = k, X + Y = m)}{P(X + Y = m)} = \frac{P(X = k, Y = m - k)}{P(X + Y = m)} \\
&= \frac{e^{-\alpha}\alpha^k/k! \cdot e^{-\beta}\beta^{m-k}/(m-k)!}{e^{-(\alpha+\beta)}(\alpha+\beta)^m/m!} \\
&= \binom{m}{k}\Big(\frac{\alpha}{\alpha+\beta}\Big)^k\Big(1 - \frac{\alpha}{\alpha+\beta}\Big)^{m-k} \sim \text{binomial}(m, \alpha/(\alpha+\beta)).
\end{aligned}
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