STA347 Probability I
Assignment #1

Due: October 1, 2018 before class starts

Solve the following and hand in by due date.

1. Solve Problem 1.8.24

Solution. Each ball has k possible urn positions. Among them choose j of them and put them at the last urn, hence, those j balls have just one possibility. For all others can be placed any of urn 1 through urn $k-1$. That is, the possible combinations are $\binom{n}{j}1^j \cdot (k-1)^{n-j}$ out of k^n. Hence the probability is $\binom{n}{j}(k-1)^{n-j}/k^n$.

2. Solve Problem 1.8.54

Solution. a. From $p = P(R_0)$, the probability of rain tomorrow is $P(R_1) = P(R_1 \,|\, R_0)P(R_0)+P(R_1 \,|\, R_0^c)P(R_0^c) = \alpha p + (1-\beta)(1-p) = 1-\beta+(\alpha+\beta-1)p$.
b. $P(R_2) = P(R_2 \,|\, R_1)P(R_1) + P(R_2 \,|\, R_1^c)P(R_1^c) = \alpha P(R_1) + (1-\beta)P(R_1^c) = 1-\beta+(\alpha+\beta-1)P(R_1) = 1-\beta+(\alpha+\beta-1)(1-\beta)+(\alpha+\beta-1)^2 p$.
c. Let $a_n = P(R_n)$. Then $a_n = P(R_n) = P(R_n \,|\, R_{n-1})P(R_{n-1}) + P(R_n \,|\, R_{n-1}^c)P(R_{n-1}^c) = 1-\beta+(\alpha+\beta-1)P(R_{n-1}) = 1-\beta+(\alpha+\beta-1)a_{n-1} = (1-\beta)(1+(\alpha+\beta-1)+\cdots+(\alpha+\beta-1)^{n-1}) + (\alpha+\beta-1)^n a_0 = (1-\beta)((\alpha+\beta-1)^n-1)/(\alpha+\beta-2)+(\alpha+\beta-1)^n p$.

3. Solve Problem 1.8.60

Solution. Let D be an event of a product being defective. Let S_1, S_2, S_3 be events of a product being produced from first, second, and third shifts. Since each shift have the same productivity, $P(S_1) = P(S_2) = P(S_3)$ and $P(S_1) + P(S_2) + P(S_3) = 1$ imply $P(S_i) = 1/3$. The depective rates are $P(D \,|\, S_i) = p_i$. The overall defective rate is $P(D) = \sum_{i=1}^3 P(D \,|\, S_i)P(S_i)$ using the law of total probability, that is, $P(D) = 0.01P(S_1) + 0.02P(S_2) + 0.05P(S_3) = 0.08/3 = 0.02667$. The second question is the probability of a product was produced in third shift when it is defective, that is, $P(S_3 \,|\, D) = P(D \,|\, S_3)P(S_3)/P(D) = (0.05/3)/(0.08/3) = 5/8 = 0.675$.

4. Let $S = (0, 1)$ and define a set function Q such that $Q(E) = 1$ if there exists $0 < \epsilon < 1$ satisfying $(0, \epsilon) \subset E$ and $Q(E) = 0$ otherwise.
(a) Show that Q satisfies Axiom 1 and 2.
(b) Prove that Q satisfies finite additivity.
(c) Find a counterexample showing Q is not countably additivy.

Solution. (a) Axiom 1. By definition $Q(E) = 0$ or 1 both of them are non-negative. Axiom 2. $S = (0, 1)$ is of form $(0, \epsilon)$. Hence $Q(S) = 1$.
(b) Let A, B are rational and irrational number in S. Then $Q(S) = 1 \neq 0 = Q(A) + Q(B)$. Hence Q is not finite additive if all Borel sets are considered. If Q is defined on finite unions of intervals, then Q is finite additive. Assume $E_1, \ldots, E_n$ are disjont. If there exists E_k, $\epsilon > 0$ such that $(0, \epsilon) \subset E_k$, then $(0, \epsilon) \cap E_j = \emptyset$ for $j \neq k$, thus, $Q(E_k) = 1$ and $Q(E_j) = 0$ for $j \neq k$. Also $\bigcup_{j=1}^n E_j$ contains $(0, \epsilon)$ and $Q(\bigcup_{j=1}^n E_j) = 1 = \sum_{j=1}^n Q(E)j)$. If $Q(E_j) = 0$ for all j, then there exists $\epsilon > 0$ such that $(0, \epsilon) \cap E_j = \emptyset$ for all j and $(0, \epsilon) \cap \bigcup_{j=1}^n E_j = \bigcup_{j=1}^n \{(0, \epsilon) \cap E_j\} = \emptyset$ and $Q(\bigcup_{j=1}^n E_j) = 0 = \sum_{j=1}^n Q(E_j)$. Therefore Q is finitely additive.
(c) Let $E_j = [1/(j+1), 1/j)$ so that $E_i \cap E_j = \emptyset$ for $i \neq j$ and $Q(E_j) = 0$. Then $\bigcup_{j=1}^\infty E_j = \bigcup_{j=1}^\infty [1/(j+1), 1/j) =$

$(0,1) = S$. Then

$$Q(\overset{\infty}{\underset{j=1}{\cup}} E_j) = Q(S) = 1 > 0 = \sum_{j=1}^{\infty} Q(E_j).$$

which violates countable additivity.

5. Let $E_1, \ldots, E_n$ be n arbitrary events. Show that the probability that exactly one of these n events will occur, say E, is

$$\sum_{i=1}^{n} P(E_i) - 2\sum_{i<j} P(E_i \cap E_j) + \cdots + (-1)^{n-1} nP(E_1 \cap \cdots \cap E_n).$$

Note $E = \overset{n}{\underset{i=1}{\cup}} [E_i - (\underset{j \neq i}{\cup} E_j)]$.

Solution. From the pool of all sets, any intersection of sets must be removed. For example, $A_i \cap A_j$ are added twice and its probability must be subtracted. Hence we need terms like

$$\sum P(A_i) - 2\sum P(A_i \cap A_j).$$

In general the answer is of form

$$Q = \sum_i P(A_i) - 2\sum_{i<j} P(A_i \cap A_j) + c_3 \sum_{i<j<k} P(A_i \cap A_j \cap A_k) + \cdots + c_n P(A_1 \cap \cdots \cap A_n).$$

For simplicity let $c_1 = 1$ and $c_2 = -2$.

Consider a k set intersection $A = A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}$. The set A is added $\binom{k}{1}$ times in the first term of Q and subtracted $-c_2\binom{k}{2}$ times in the second term,..., and added $c_{k-1}\binom{k}{k-1}$ times in the $(k-1)$st term. Hence $c_1\binom{k}{1} + c_2\binom{k}{2} + \cdots + c_k = 0$ solves

$$c_k = -[c_1\binom{k}{1} + c_2\binom{k}{2} + \cdots + c_{k-1}\binom{k}{k-1}].$$

We will prove $c_k = (-1)^{k-1}k$ using induction. It is true for $k = 1, 2$. Assume it is still true for $k = 1, \ldots, m$. Then

$$\begin{aligned} c_{m+1} &= -\sum_{j=1}^{m} c_j \binom{m+1}{j} = -\sum_{j=1}^{m} (-1)^{j-1} j \frac{m+1}{j} \binom{m}{j-1} = -(m+1)[(1-1)^m - (-1)^m \binom{m}{m}] \\ &= (-1)^m (m+1). \end{aligned}$$

Hence the equation holds.