Graph Search: Depth First Search (DFS)

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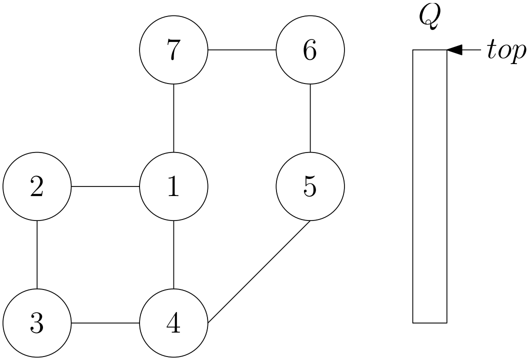
CSC 265: Enriched Data Structures and Analysis

27 November 2018

Outline

BFS

$G = (V, E)$ is a graph with with $|V| = n$ and $|E| = m$.



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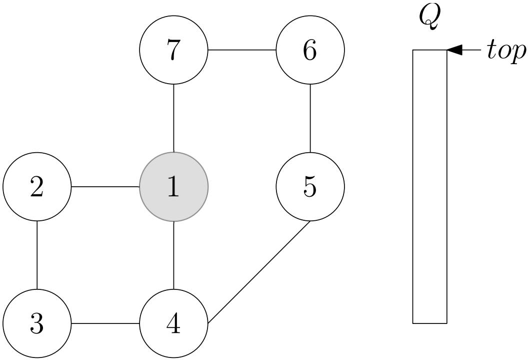

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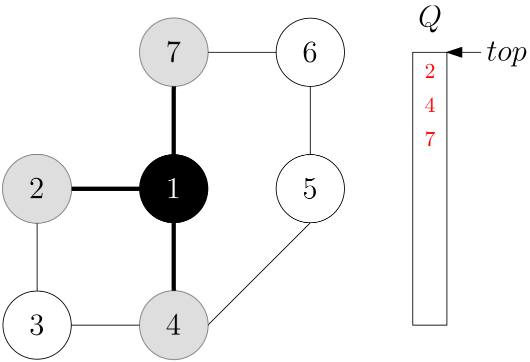

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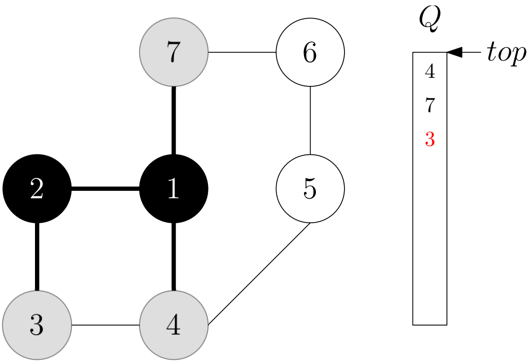

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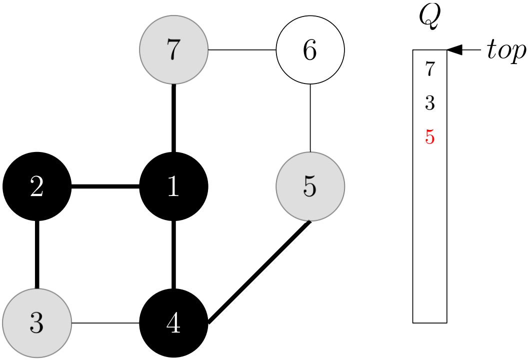

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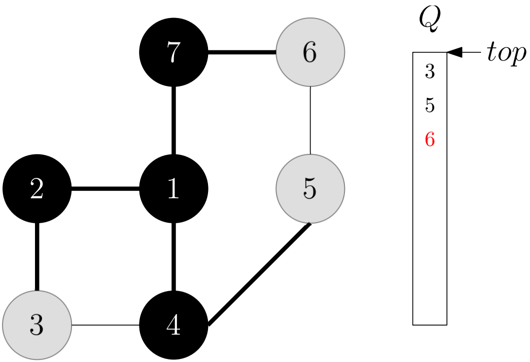

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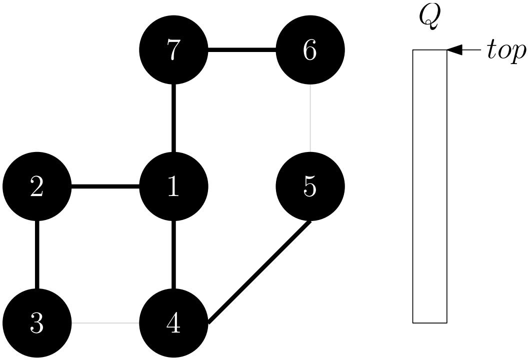

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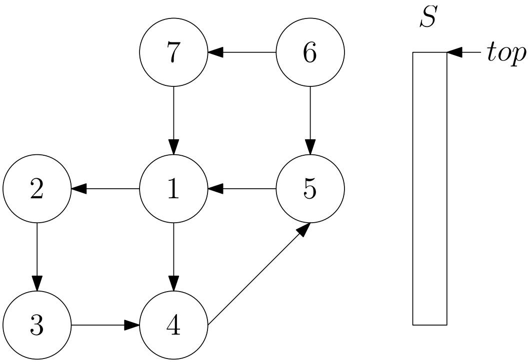

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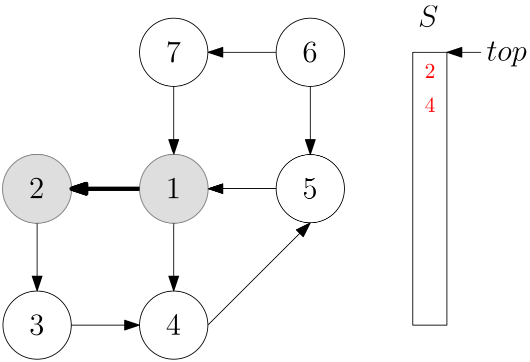

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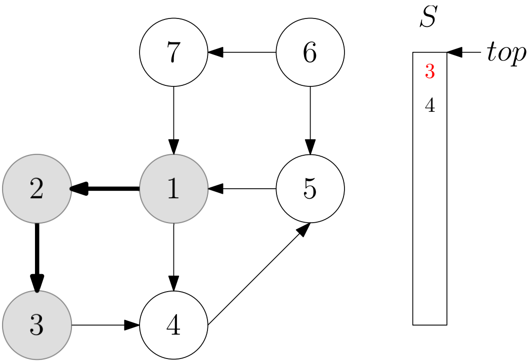

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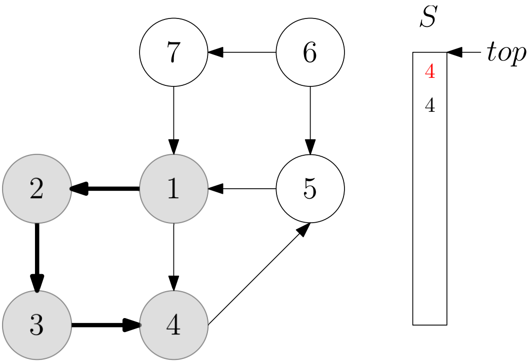

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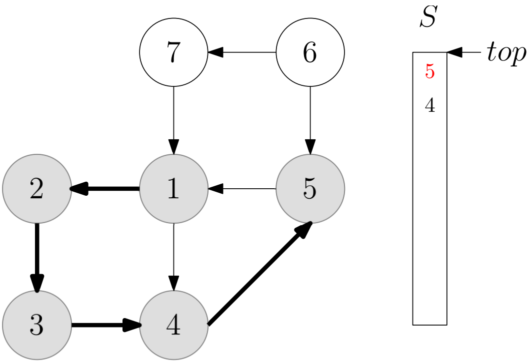

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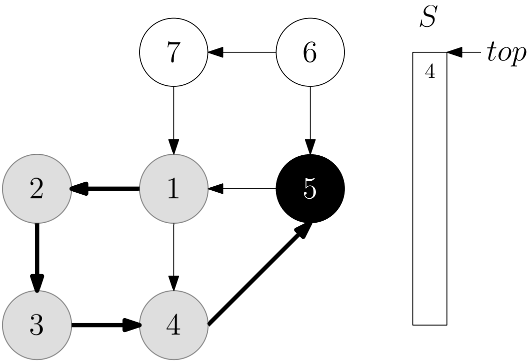

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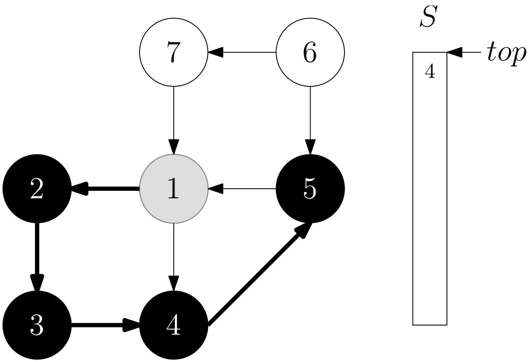

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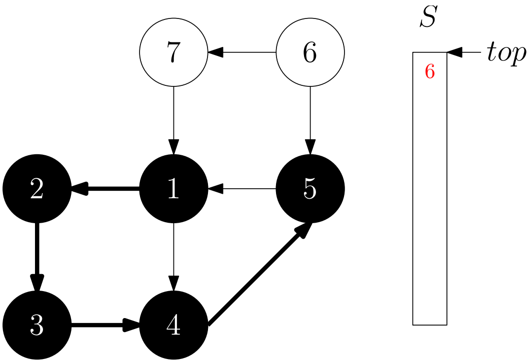

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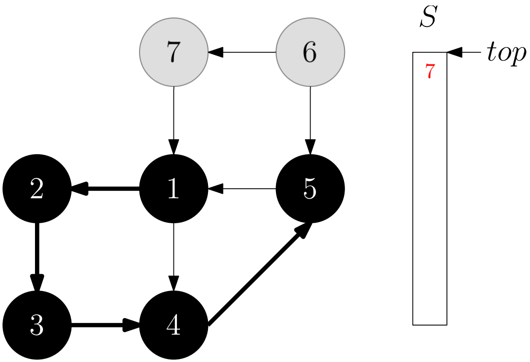

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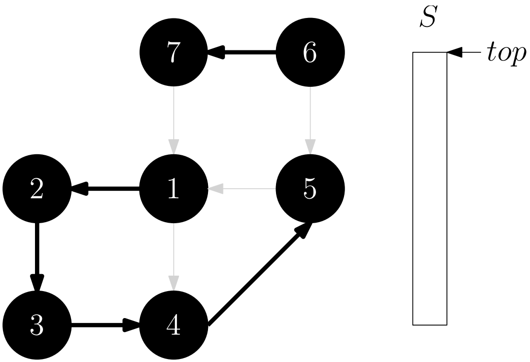

DFS: Basics

(Special) Parameters of Vertex v:

- $d[v]$: the discovery time of v. *Different from* $v.d$ *of BSF.*
- $f[v]$: the finish time of v.
- $v.colour$: indicates state of v; v is white before $d[v]$, gray between $d[v]$ and $f[v]$, and black after $f[v]$.

DFS: Pseudo-code

DFS(G)

$t \leftarrow 0$
for $v \in V$
    $v.colour =$ white
for $v \in V$
    if $v.colour =$ white
        DFS-VISIT(v)

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```

```
DFS-Visit(v)
v.colour ← gray
t ← t + 1
d[v] ← t
for u ∈ N(v)
    if u.colour = white
        u.parent = v
        DFS-Visit(u)
v.colour ← black
f[v] ← t
t ← t + 1
```

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Q: What is the running time?

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v.colour ← black
f[v] ← t
t ← t + 1
```
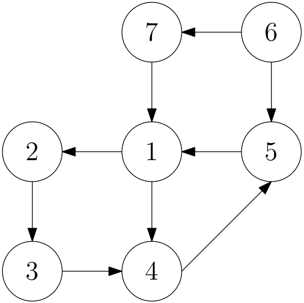

Q: What is the running time?

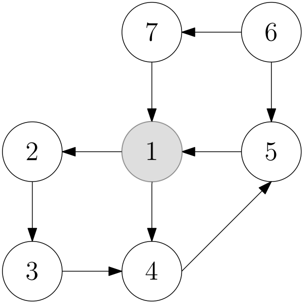
A: $O(m+n)$.

DFS: Example



		t	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$d[v]$	$f[v]$	v														

DFS: Example



$d[v]$ $f[v]$

t	1	2	3	4	5	6	7	8	9	10	11	12	13	14
v	1													

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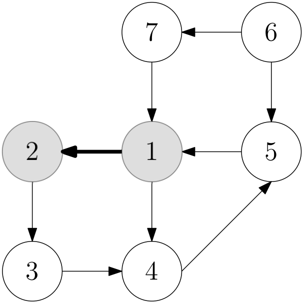
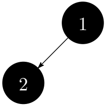
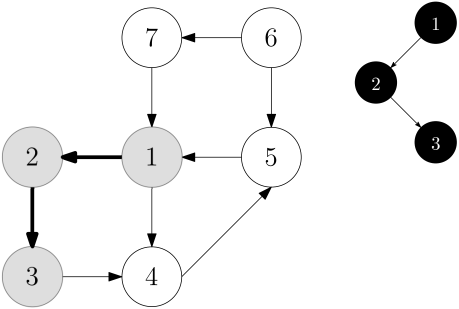




$d[v]$ $f[v]$

t	1	2	3	4	5	6	7	8	9	10	11	12	13	14
v	1	2												

DFS: Example



$d[v]$	$f[v]$	t	1	2	3	4	5	6	7	8	9	10	11	12	13	14
		v	1	2	3											

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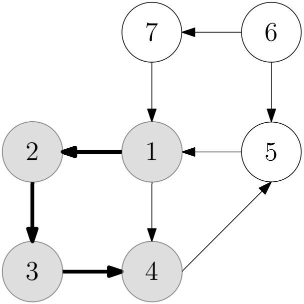

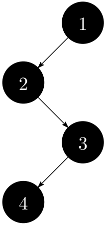

		t	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$d[v]$	$f[v]$	v	1	2	3	4										

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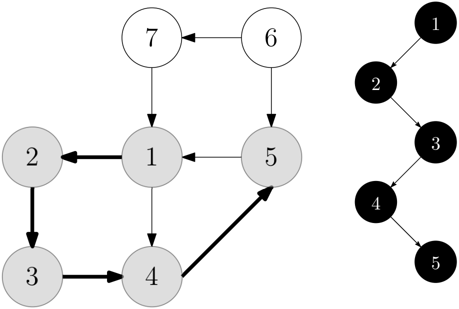

$d[v]$	$f[v]$	t	1	2	3	4	5	6	7	8	9	10	11	12	13	14
		v	1	2	3	4	5									

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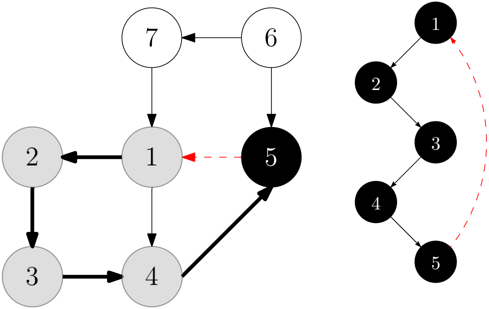

			1	2	3	4	5	6	7	8	9	10	11	12	13	14
		t	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$d[v]$	$f[v]$	v	1	2	3	4	5	5								

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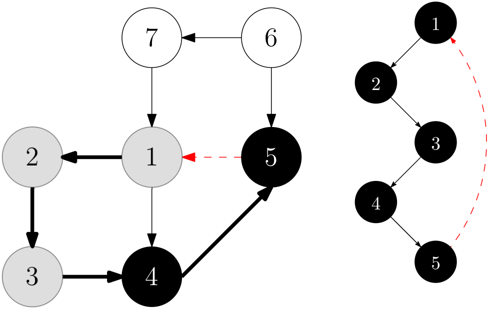

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t	1	2	3	4	5	6	7	8	9	10	11	12	13	14
v	1	2	3	4	5	5	4							

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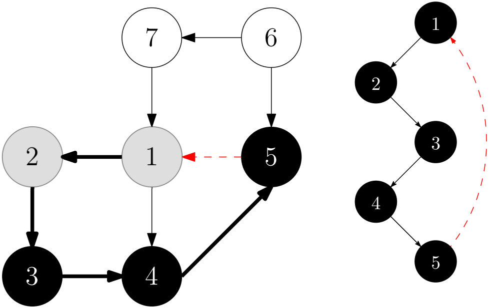

$d[v]$	$f[v]$	t	1	2	3	4	5	6	7	8	9	10	11	12	13	14
		v	1	2	3	4	5	5	4	3						

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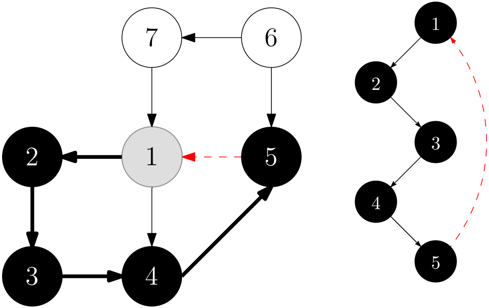

		t	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$d[v]$	$f[v]$	v	1	2	3	4	5	5	4	3	2					

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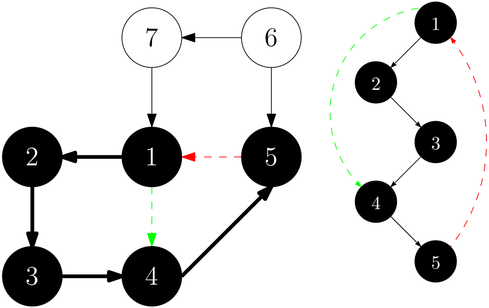

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$d[v]$	$f[v]$	v	1	2	3	4	5	5	4	3	2	1				

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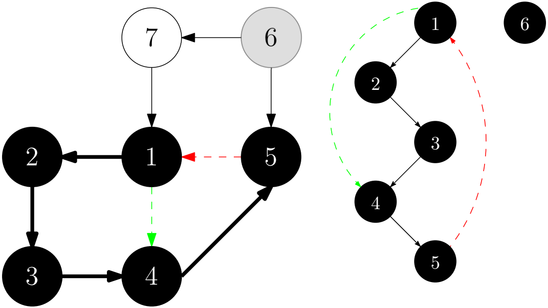

t	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$d[v]$ $f[v]$ v	1	2	3	4	5	5	4	3	2	1	6			

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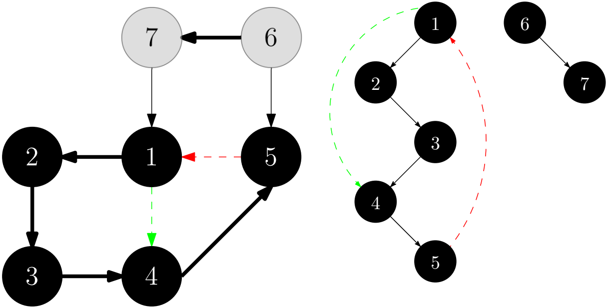

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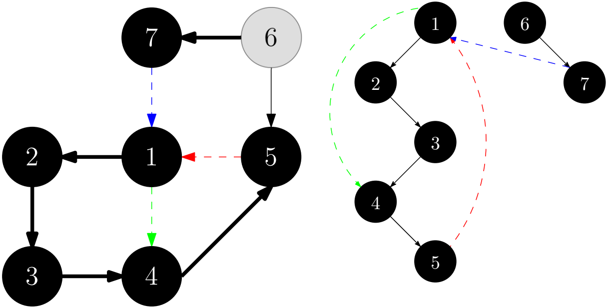

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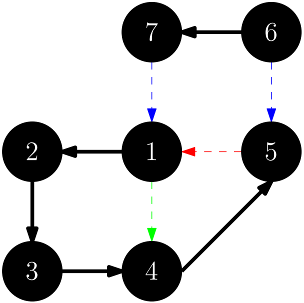

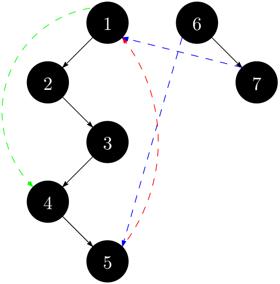

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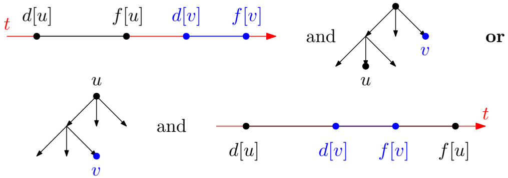
$d[v]$	$f[v]$	v	1	2	3	4	5	5	4	3	2	1	6	7	7	6
			(	(	(	(	(	)	)	)	)	)	(	(	)	)

Parenthesis Structure

Theorem (Parenthetical Property)

Let $G = (V, E)$ and $u, v \in V$. Then the following could occur:



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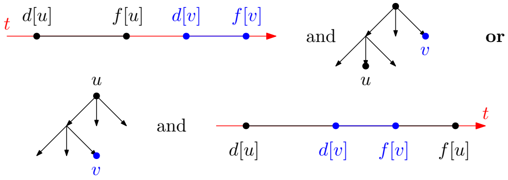

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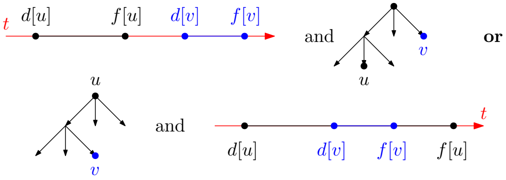

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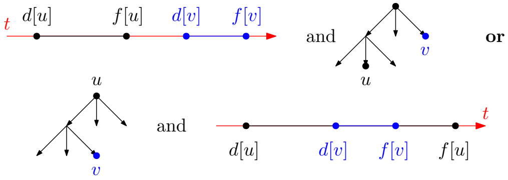

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2. $f[u] > d[v]$: u was gray when v was discovered. Thus v is a descendant of u and finish exploring v before u ($f[v] < f[u]$).

$\square$

White Path

Theorem (White Path Property)

In a depth-first forest, v is a descendant of u if and only if at $d[u]$ there exists a path $u \rightsquigarrow v$ of only white nodes.

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Claim: all vertices w on this path satisfy $f[w] < f[u]$.

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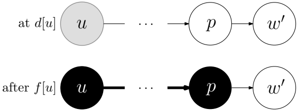
Claim: all vertices w on this path satisfy $f[w] < f[u]$.

Proof: Suppose not and let w' be the first vertex on the path such that $f[w'] > f[u]$. Then the parent p of w' on the path finished before exploring the edge pw'. This is not possible.



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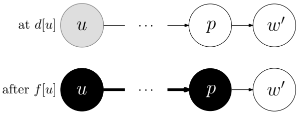

By Paren. Prop. every vertex on the path, including v, is a descendant of u.

$\square$

DFS: Exercise

G is an undirected graph. Running BFS on G at vertex u produces a tree T. Running DFS on the u produces the same tree T.

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G is an undirected graph. Running BFS on G at vertex u produces a tree T. Running DFS on the u produces the same tree T.
Show that $G = T$ i.e. all edges in G belong to T.

Topological Sort: Setup

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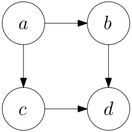

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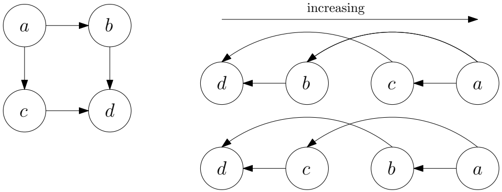

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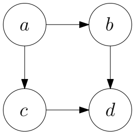

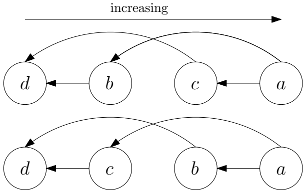

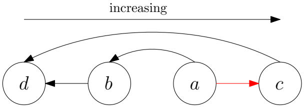

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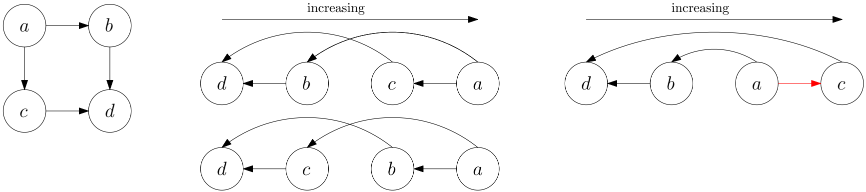

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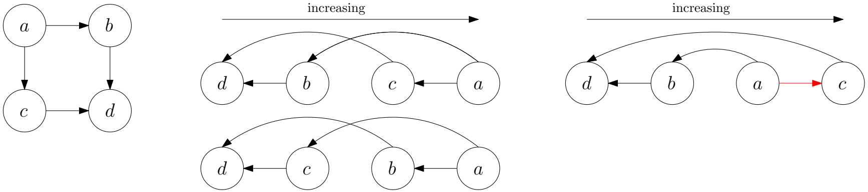

Q: Do all directed graph have a topological order? How would you perform topological sort on a graph with a topological order.
A: G has a topological order if and only if it is acyclic (DAG).

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Using Sinks

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Algorithm:

1. While graph G is not empty find a sink v.
2. Remove v and its adjacent edges.
3. Go back to step 1.

Topological Sort: Algorithm 2

Using DFS

Algorithm: run DFS and sort vertices by increasing $f[v]$.

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Two cases to consider:

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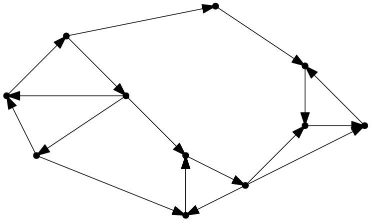
2. DFS visits v before u: Since the graph is acyclic, there does not exists a path $v \rightsquigarrow u$. Thus DFS on v must finish before even discovering u.

$\square$

SCC: Setup

Consider this graph.



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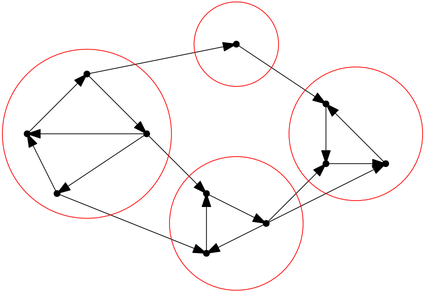

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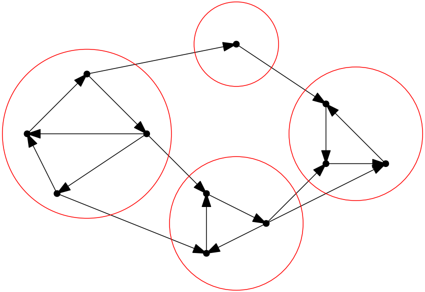

In the language of equivalence relations:

$$R: \text{ "is in the same SCC as"}$$

then uRv if there exists paths $u \rightsquigarrow v$ and $v \rightsquigarrow u$.

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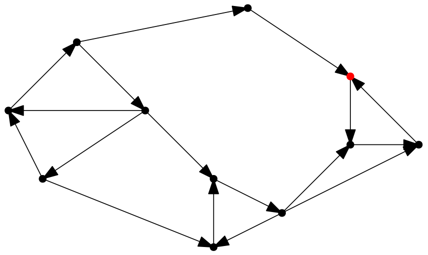

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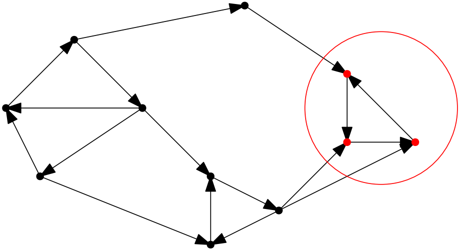

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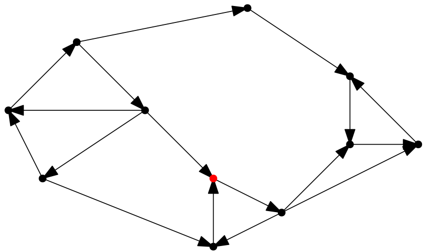

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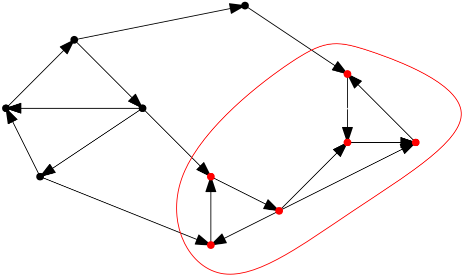
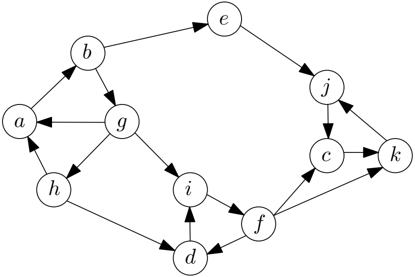


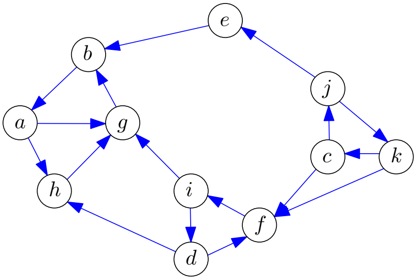
SCC: Example

0. Begin with a graph G.



SCC: Example

1. Reverse all edges in G and call this graph G_R.



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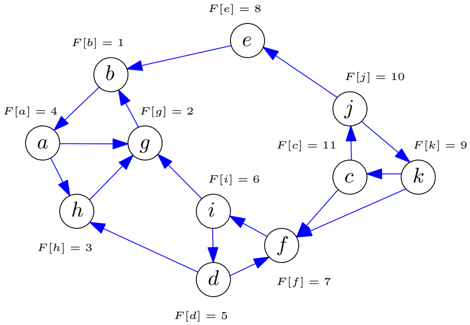
2. Run $\mathrm{DFS}(G_R)$. Record the finish times of the vertices.



Heuristic: break ties lexicographically (first a, then b, etc.)

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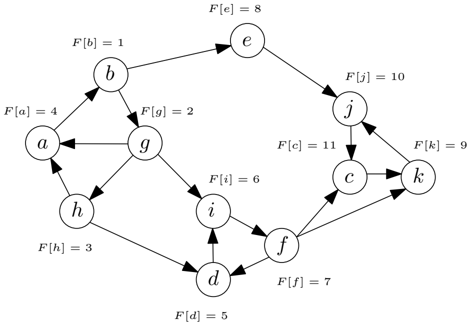

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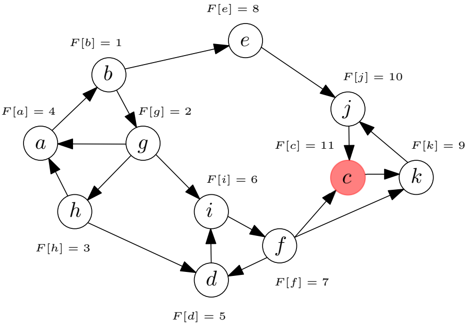

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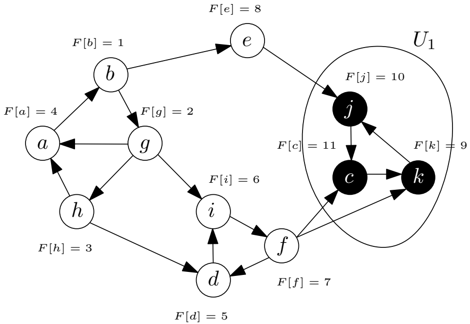

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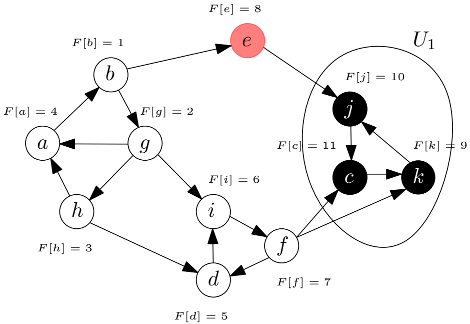

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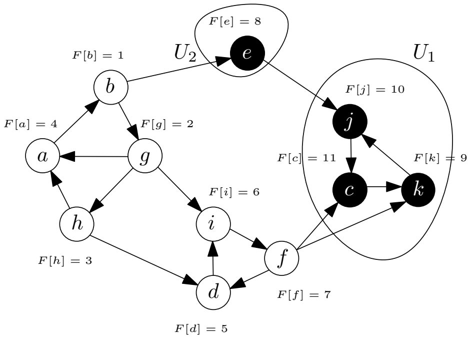

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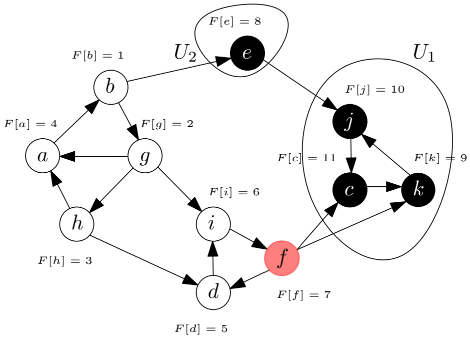

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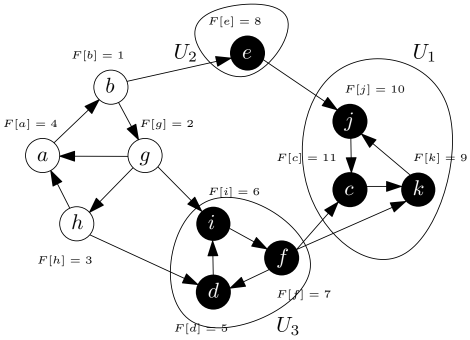

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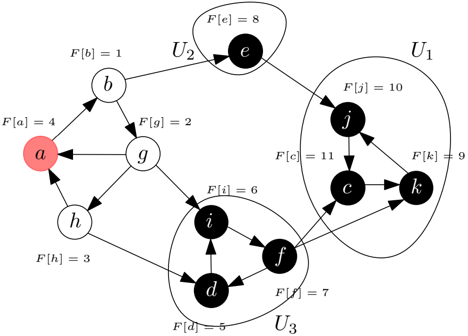

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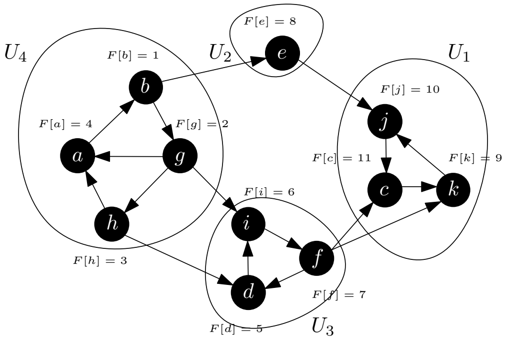

SCC: Proof of Correctness

Proof.

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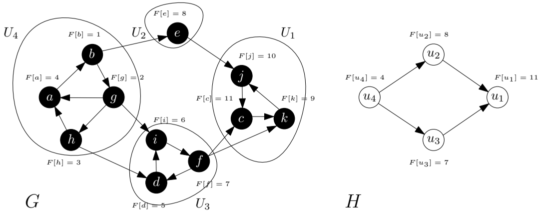

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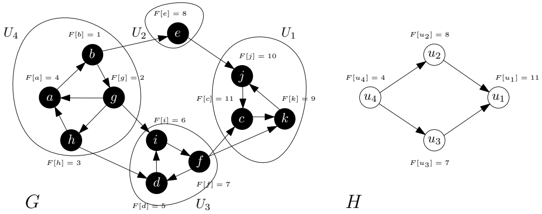

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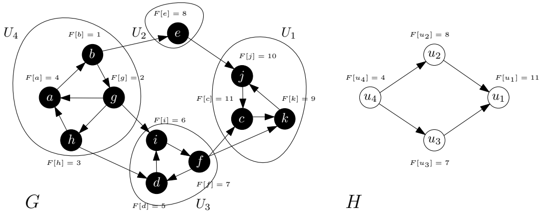

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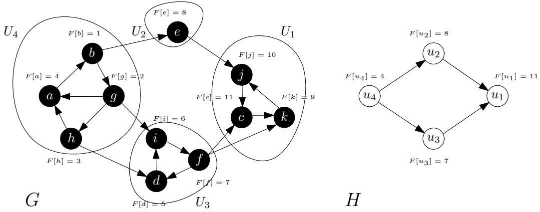

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4. **Perform DFS on** G in decreasing order of finish times. Show that this discovers the SCCs by induction.

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$\mathrm{DFS}(G)$ *in decreasing order of finish times discovers the SCCs.*

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By induction on the number of calls of $\mathrm{DFS\text{-}V{\scriptsize ISIT}}(v)$ in $\mathrm{DFS}(G)$.

1. **Base Case:** By the claim, $v \in V$ with the largest $F[v]$ is in a sink SCC, U_1. $\mathrm{DFS\text{-}V{\scriptsize ISIT}}(v)$ will find all vertices in U_1.

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□

SCC: Proof of Claim

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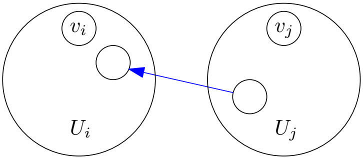

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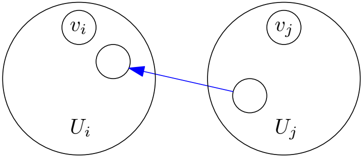

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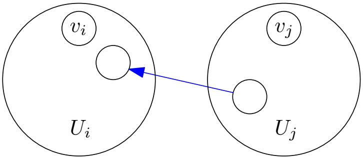

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2. **v_j is explored before v_i**: There exists a white path from v_j to v_i, so by White Path Prop. and Paren. Prop. $f[v_i] < f[v_j]$.

$\square$