Homework 2

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Problem 1

- a

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$
$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

  Given above equation:

$$P(B|A)P(A) = P(A|B)P(B) \Rightarrow P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

- b

$$P(A,B,C) = P(A)P(B|A)P(C|A,B)$$

- c
  By defination, $\mathbb{E}[x] = P(A) * X_1 + (1 - P(A)) * X_2$, where $X_1 = 1$ and $X_2 = 0$.
  Therefore, $\mathbb{E}[x] = P(A)$
- d
  - i
    From the table:
    $P(X = 1) = \frac{7}{18}, P(X = 1|Y = 0) = \frac{3}{8}$
    $P(X = 1) \neq P(X = 1|Y = 0)$
    Not independent.
  - ii
    $P(X = 1|Y = 1, Z = 1) = \frac{1}{3} = P(X = 1|Y = 0, Z = 1) = P(X = 1|Z = 1)$
    $P(X = 1|Y = 1, Z = 0) = \frac{1}{2} = P(X = 1|Y = 0, Z = 0) = P(X = 1|Z = 0)$
    Independent.
  - iii
    $P(X = 0|X + Y > 0) = \frac{5}{12}$

Problem 2

- a

$$L(\hat{\theta}) = \prod_{i=1}^{n} \theta^{x_i} * (1-\theta)^{(1-x_i)}$$

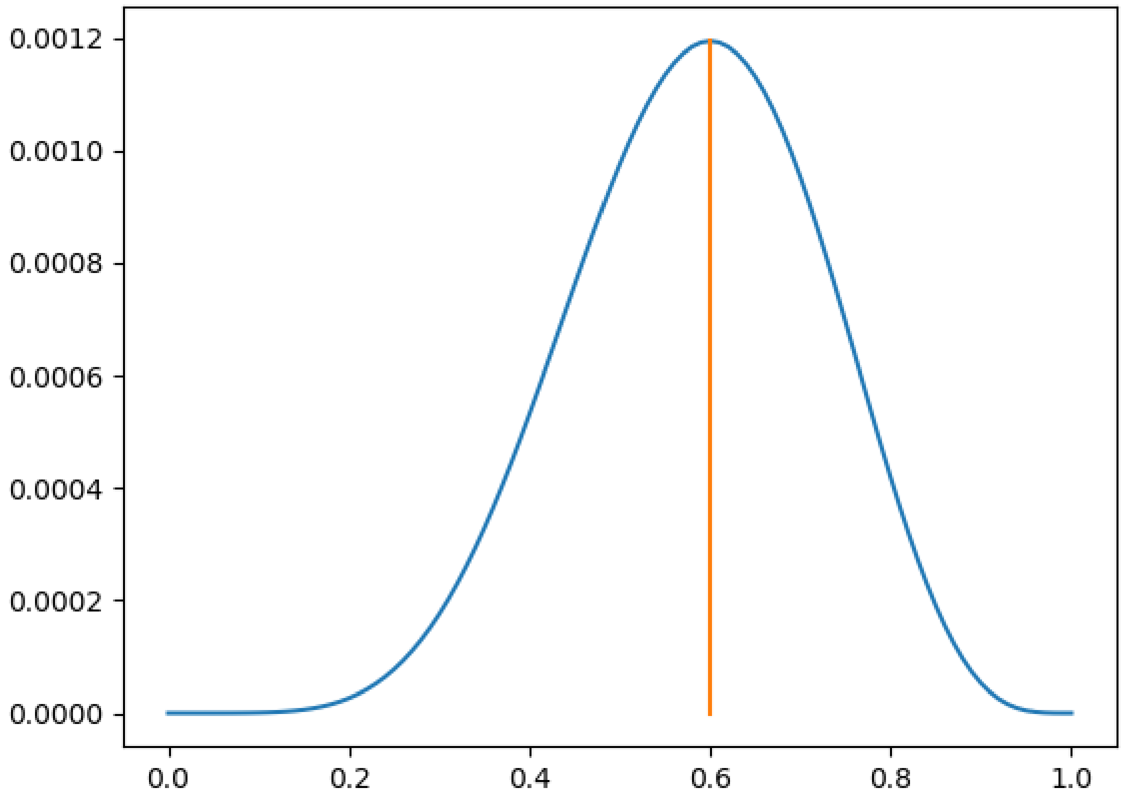
  Not depend on the order.

- b

```
import matplotlib.pyplot as plot

def mle(D, theta):
    res = 1
    for d in D:
        res *= ((theta ** d) * ((1 - theta) ** (1 - d)))
    return res

if __name__ == "__main__":
    D = [1] * 6 + [0] * 4
    init = 0
    xAxis = []
    yAxis = []
    maxX = 0
    maxY = 0
    while init <= 1.01:
        xAxis.append(init)
        yAxis.append(mle(D, init))
        if yAxis[-1] > maxY:
            maxY = yAxis[-1]
            maxX = xAxis[-1]
        init += .01
    plot.plot(xAxis, yAxis)
    plot.plot([maxX, maxX], [0, maxY])
    plot.show()
```



- c

  We define $n_1 = \# of X_i = 1$ and $n_2 = \# of X_i = 0$.

$$L(\hat{\theta}) = \prod_{i=1}^{n} \theta^{x_i} * (1-\theta)^{(1-x_i)} \Rightarrow L(\hat{\theta}) = \theta^{n_1} * (1-\theta)^{n_2}$$

  By putting `ln` on the equation:

$$\ln L(\hat{\theta}) = n_1 \ln \theta + n_2 \ln(1-\theta)$$

  To find the argument that makes the function max, we derive it:

$$\frac{d \ln L(\hat{\theta})}{d\theta} = \frac{n_1}{\theta} - \frac{n_2}{1-\theta} = 0$$
$$\theta = \frac{n_1}{n_1 + n_2}$$

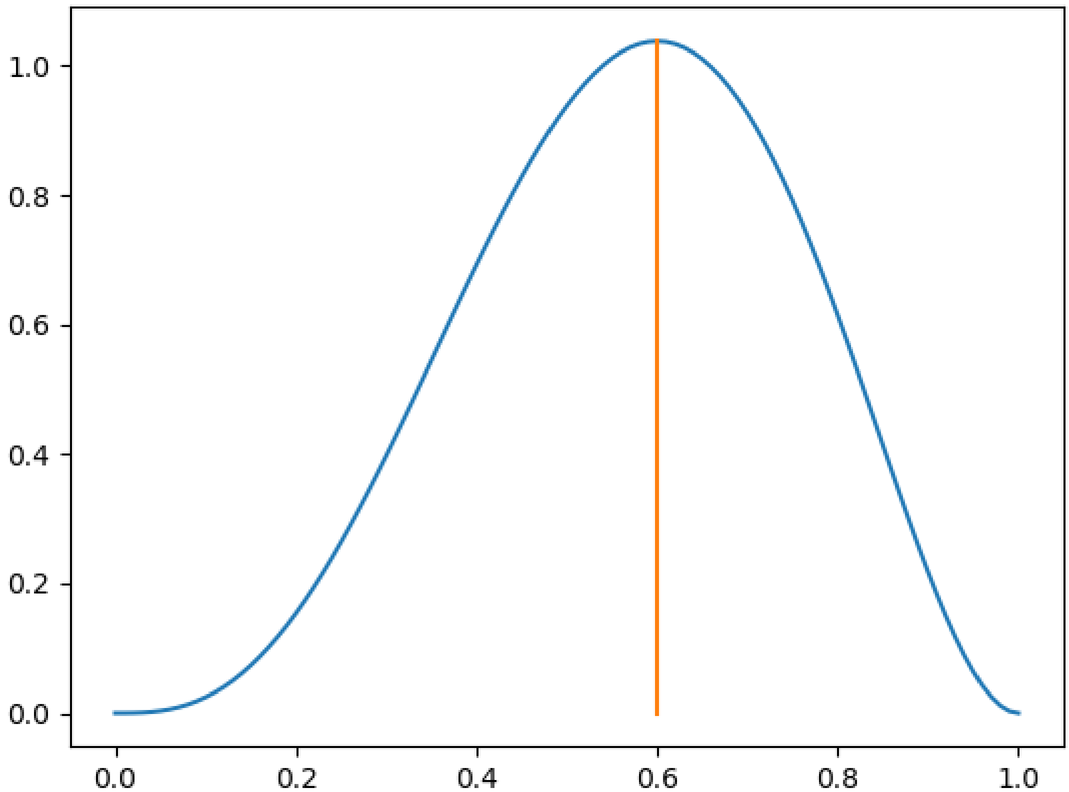
  It agrees with the plot.

- d

```python
import matplotlib.pyplot as plot

def mapFunc(D, theta):
    res = 1
    for d in D:
        res *= ((theta) ** d) * ((1 - theta) ** (1 - d))
        beta = ((theta ** 2)*(1 - theta) ** 2) / .0333
        return res * beta

if __name__ == "__main__":
    D = [1] * 6 + [0] * 4
    init = 0
    xAxis = []
    yAxis = []
    maxX = 0
    maxY = 0
    while init <= 1.01:
        xAxis.append(init)
        yAxis.append(mapFunc(D, init))
        if yAxis[-1] > maxY:
            maxY = yAxis[-1]
            maxX = xAxis[-1]
        init += .01
    plot.plot(xAxis, yAxis)
    plot.plot([maxX, maxX], [0, maxY])
    plot.show()
```



- e

$$L(\hat{\theta})p(\hat{\theta}) = \frac{\theta^{n_1+2}(1-\theta)^{n_2+2}}{B(3,3)}$$
$$\frac{d\ln L(\hat{\theta})p(\hat{\theta})}{d\theta} = \frac{n_1+2}{\theta} - \frac{n_2+2}{1-\theta}$$
$$\theta = \frac{n_1+2}{n_1+n_2+4}$$

  It agreed with the plot.
- f
  MLE is a specific case of MLA where its prior $p(\theta)$ is 1
- g
  As $n \to \infty, \theta^{MAP} = \theta^{MLE}$. That is mainly becaseu that the prior distributation becomes less significant.