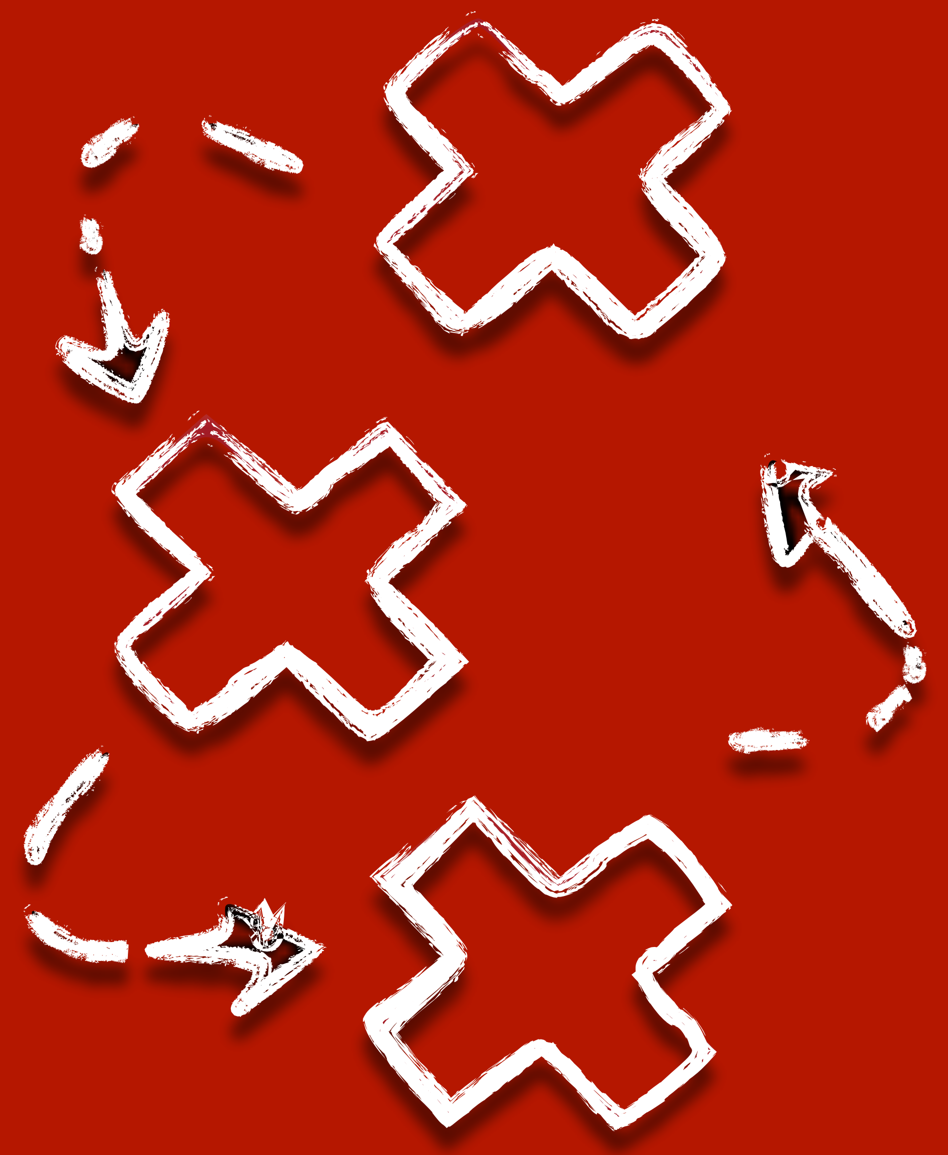


Causal Data Science

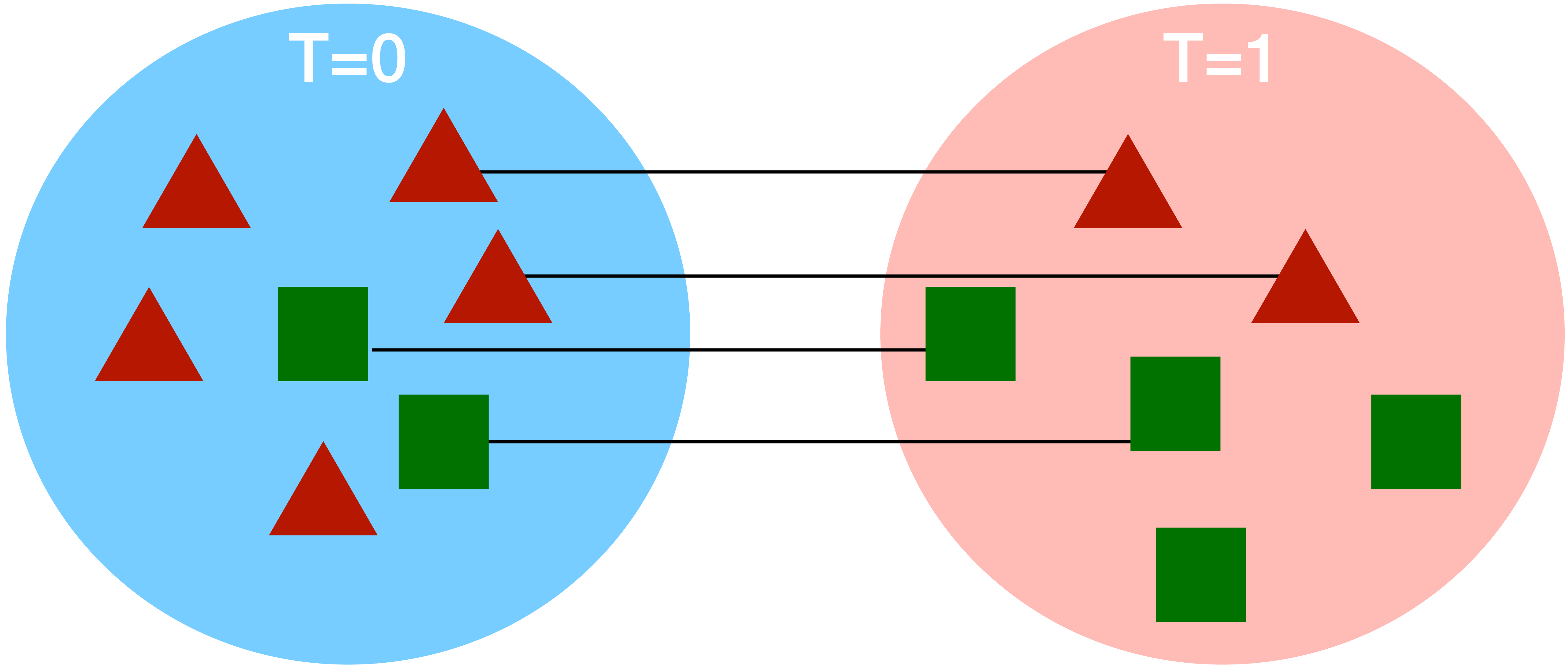
Lecture 8.1: Estimation methods 2

Lecturer: Sara Magliacane

UvA - Spring 2024

Last class: Exact matching (simplified)

- **Usually for ATT,** sometimes for ATE
- **Intution:** find the most similar couple of units in terms of covariates $\mathbf{X}$, such that one is in the treatment and the other in the control group

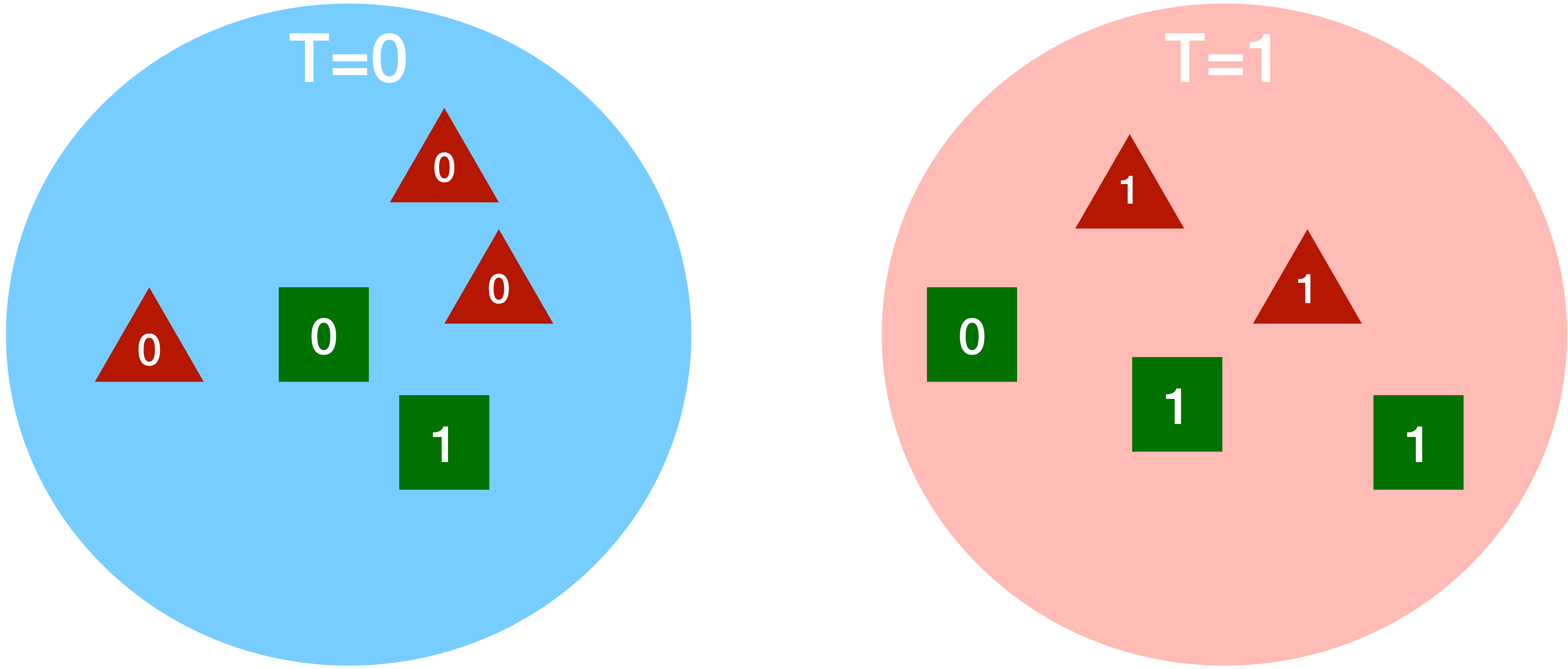


Advanced: Exact matching (slightly less simplified)

- **Usually for ATT,** with M multiple matches

$$\hat{\text{ATT}} = \frac{1}{n_t}\sum_{i=1}^{n_t}(Y_i - \frac{1}{M}\sum_{j=1}^{M} Y_{m_j(i)})$$

$Y_{m_j(i)}$ **match j for i**



Advanced: Exact matching (slightly less simplified)

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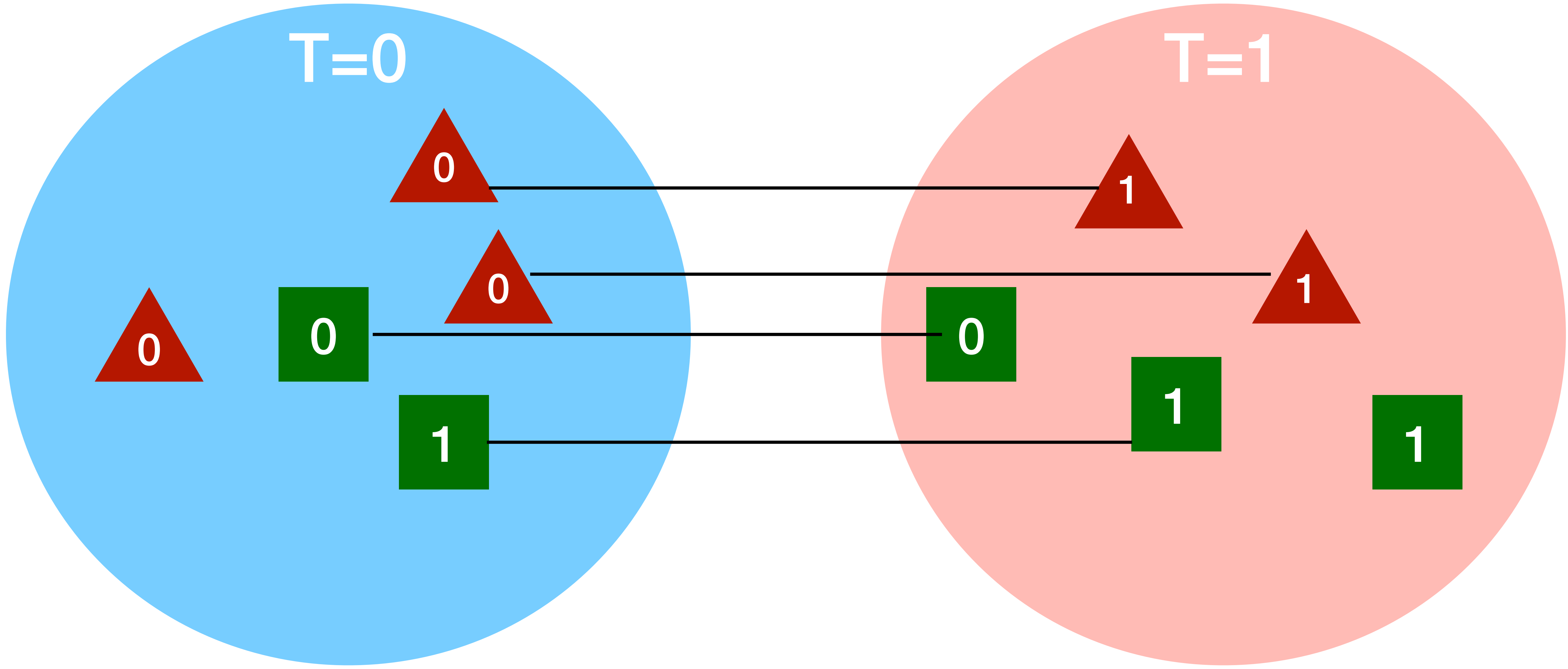


$$M = 1$$

$$\hat{\text{ATT}} = \frac{1}{5} \sum_{i=1}^{5} (Y_i - Y_{m(i)})$$

$$\hat{\text{ATT}} = \frac{1}{5}[1 + 1 + 0 + 0] = \frac{2}{5}$$

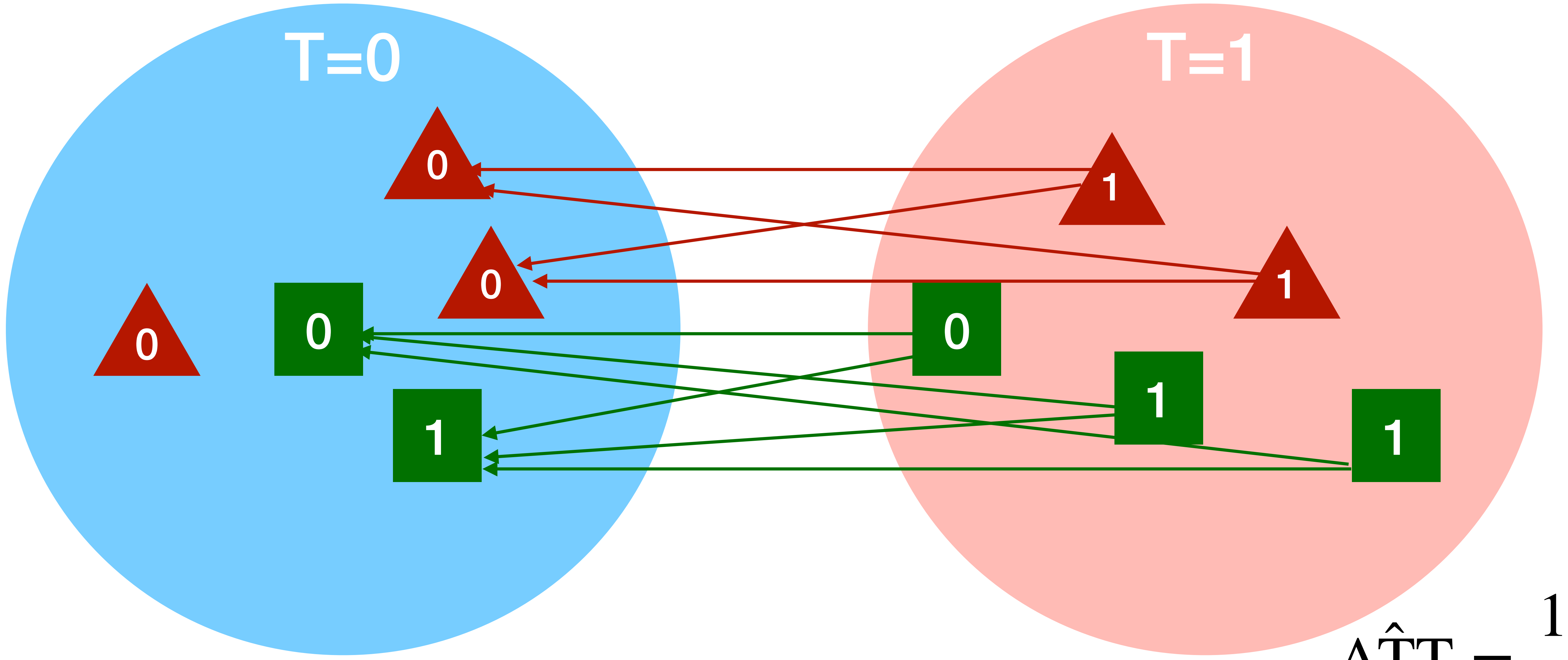
Advanced: Exact matching (slightly less simplified)

- **Usually for ATT,** with M multiple matches

$$\hat{\text{ATT}} = \frac{1}{n_t}\sum_{i=1}^{n_t}(Y_i - \frac{1}{M}\sum_{j=1}^{M} Y_{m_j(i)})$$

$Y_{m_j(i)}$ **match j for i**

$$M = 2$$



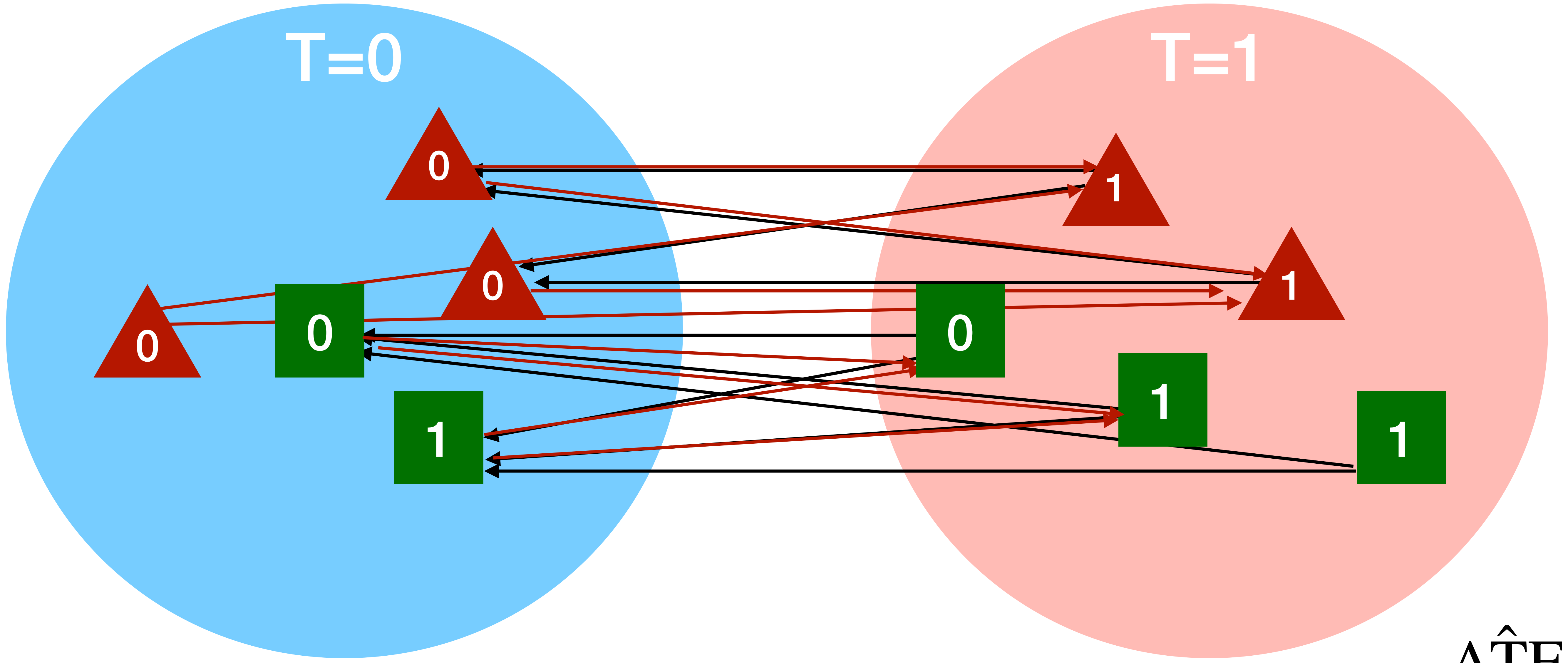
$$\hat{\text{ATT}} = \frac{1}{5}\sum_{i=1}^{5}(Y_i - \frac{1}{2}\sum_{j=1}^{M} Y_{m_j(i)})$$

$$\hat{\text{ATT}} = \frac{1}{5}[1 + 1 - \frac{1}{2} + \frac{2}{2}] = \frac{1}{5}\cdot\frac{5}{2} = \frac{1}{2}$$

Advanced: Exact matching (slightly less simplified)

- **ATE** with M multiple matches (e.g. M=2, can be random):

$$\hat{\text{ATE}} = \frac{1}{n_t + n_c}[\sum_{i=1}^{n_t}(Y_i - \frac{1}{M}\sum_{j=1}^{M} Y_{m_j(i)}) + \sum_{j=1}^{n_c}(\frac{1}{M}\sum_{i=1}^{M} Y_{m_i(j)} - Y_j)]$$



$$\hat{\text{ATE}} = \frac{1}{10}[\frac{5}{2} + 3 + \frac{1}{2} - \frac{1}{2}] = \frac{11}{20}$$

Last class: Propensity score matching (PSM)

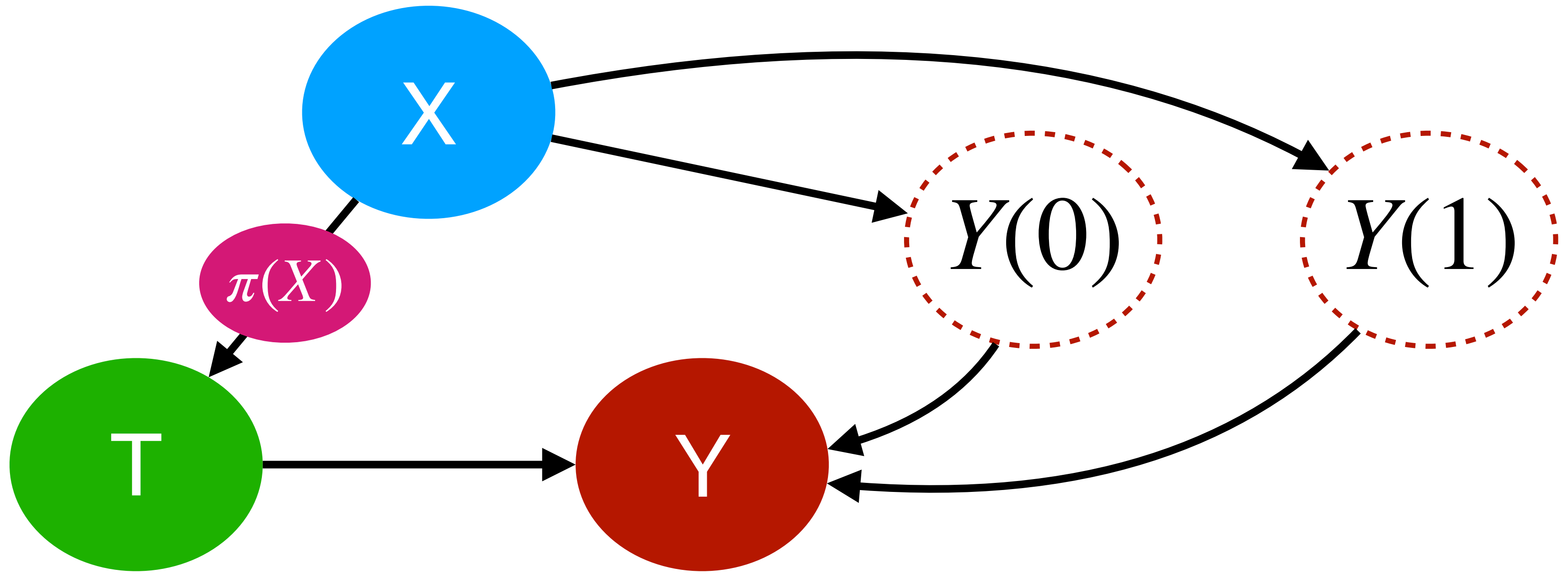
- **Assumptions**: binary treatment T, $\mathbf{X}$ is valid adjustment set
- **Propensity score:** the probability of getting assigned the treatment

$$e(x) \quad \pi(x) := P(T = 1 \mid \mathbf{X} = x)$$

Conditional ignorability

- We can show that $T \perp\!\!\!\perp \mathbf{X} \mid \pi(\mathbf{X})$ and that if $Y(0), Y(1) \perp\!\!\!\perp T \mid \mathbf{X}$ then

$$Y(0), Y(1) \perp\!\!\!\perp T \mid \pi(\mathbf{X})$$



e.g. with **logistic regression**

- We can estimate π from data and use it to match
  - If $\mathbf{X}$ has a lot of covariates, it is easier to match since it's a single number

Estimation method: Inverse probability weighting (IPW)

- **Idea:** rather than match **(and discard some samples)**, reweight (downweight or upweight) samples

- **Inverse probability (of treatment) weighting:** weight by inverse of probability of treatment **received**:
  - For treated $T = 1$: weight by the inverse of $\pi = P(T = 1 \,|\, \mathbf{X})$
  - For untreated $T = 0$: weight by the inverse of $1 - \pi = P(T = 0 \,|\, \mathbf{X})$

Inverse probability weighting (IPW) - derivation

- We can estimate the average causal effect/**average treatment effect**

$$\text{ATE} = \mathbb{E}[Y(t=1) - Y(t=0)] = \mathbb{E}[Y \mid \text{do}(T=1)] - \mathbb{E}[Y \mid \text{do}(T=0)]$$

- $\mathbf{X}$ is a valid adjustment set for the causal effect of T on Y, so:

$$P(Y=y \mid \text{do}(T=1)) = \sum_{\mathbf{x}} P(Y=y \mid \mathbf{X}=\mathbf{x}, T=1) P(\mathbf{X}=\mathbf{x})$$

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- $\mathbf{X}$ is a valid adjustment set for the causal effect of T on Y, so:

$$P(Y=y \mid \text{do}(T=t)) = \sum_{\mathbf{x}} P(Y=y \mid \mathbf{X}=\mathbf{x}, T=t) P(\mathbf{X}=\mathbf{x})$$

$$\mathbb{E}[Y \mid \text{do}(T=t)] = \sum_{y} y \sum_{\mathbf{x}} P(Y=y \mid \mathbf{X}=\mathbf{x}, T=t) P(\mathbf{X}=\mathbf{x})$$

Inverse probability weighting (IPW) - derivation

$$\mathbb{E}[Y \mid \mathrm{do}(T=t)] = \sum_y \sum_{\mathbf{x}} y \cdot P(Y=y \mid \mathbf{X}=\mathbf{x}, T=t) P(\mathbf{X}=\mathbf{x})$$

Assuming $P(T=t \mid \mathbf{X}=\mathbf{x}) \neq 0$

$$= \sum_y \sum_{\mathbf{x}} y \cdot P(Y=y \mid \mathbf{X}=\mathbf{x}, T=t) P(\mathbf{X}=\mathbf{x}) \frac{P(T=t \mid \mathbf{X}=\mathbf{x})}{P(T=t \mid \mathbf{X}=\mathbf{x})}$$

$$= P(Y=y, \mathbf{X}=\mathbf{x}, T=t)$$

$$= \sum_y \sum_{\mathbf{x}} y \cdot P(Y=y \mid \mathbf{X}=\mathbf{x}, T=t) P(\mathbf{X}=\mathbf{x}) \frac{P(T=t \mid \mathbf{X}=\mathbf{x})}{P(T=t \mid \mathbf{X}=\mathbf{x})}$$

$$= \sum_y \sum_{\mathbf{x}} \frac{y \cdot P(Y=y, \mathbf{X}=\mathbf{x}, T=t)}{P(T=t \mid \mathbf{X}=\mathbf{x})}$$

π for $\mathrm{t}=1$, $(1-\pi)$ for $\mathrm{t}=0$

Estimation method: Inverse probability weighting (IPW)

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  - For untreated $T = 0$: weight by the inverse of $1 - \pi = P(T = 0 \,|\, \mathbf{X})$

$$\hat{\mathbb{E}}(Y(t=1)) = \frac{1}{n}\sum_{i=1}^{n} Y_i \cdot 1\{T=1\} \cdot \frac{1}{\boxed{P(T=1 \,|\, X_i)}} \qquad \pi$$

$$\hat{\mathbb{E}}(Y(t=0)) = \frac{1}{n}\sum_{i=1}^{n} Y_i \cdot 1\{T=0\} \cdot \frac{1}{\boxed{P(T=0 \,|\, X_i)}} \qquad (1-\pi)$$

IPW Example

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ T \arrow[rr] & & Y \end{tikzcd}$$

$P(T=1 \mid X=1) = 0.1$

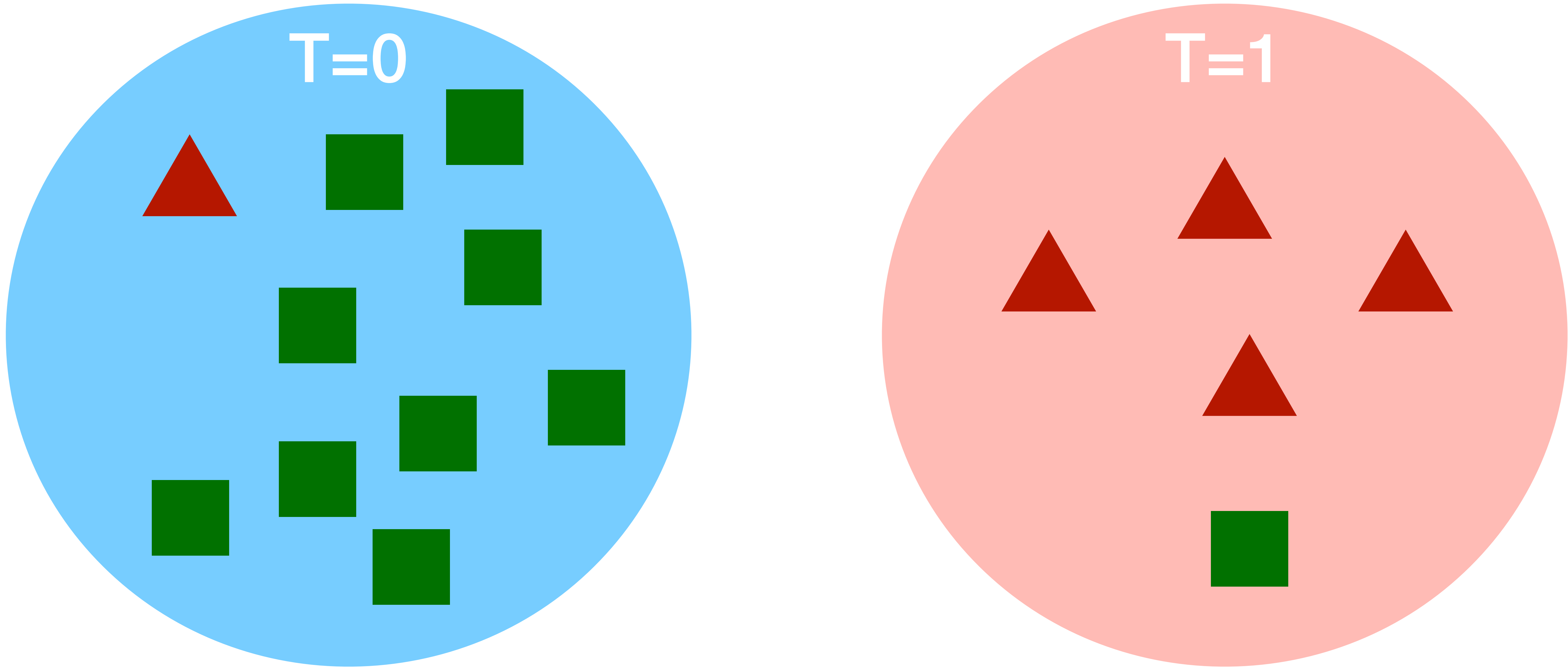
$P(T=1 \mid X=0) = 0.8$

$P(T=0 \mid X=1) = 0.9$

$P(T=0 \mid X=0) = 0.2$

Population:

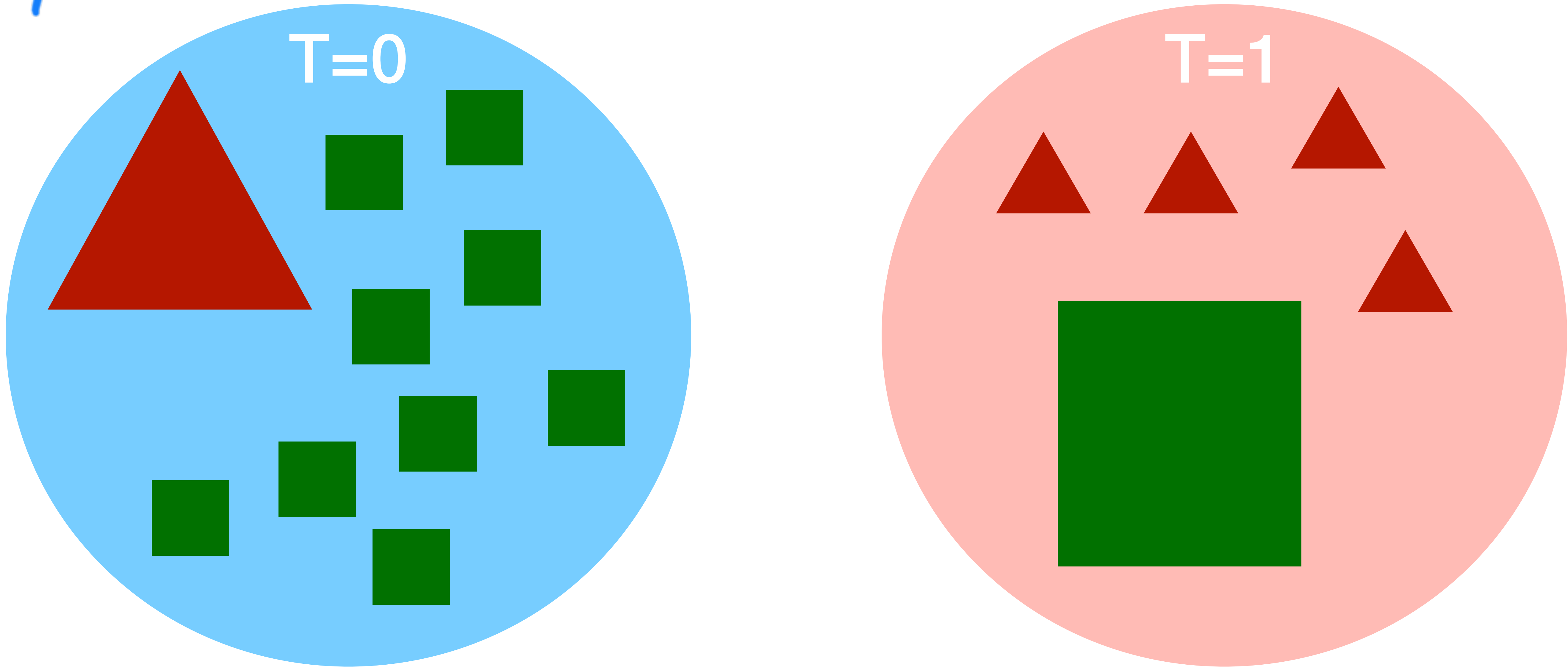
	$X=0$	$X=1$
$T=0$	1	9
$T=1$	4	1



IPW Example

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ T & \longrightarrow & Y \end{array}$$

Population:



$P(T=1 \mid X=1) = 0.1$

$P(T=1 \mid X=0) = 0.8$

$P(T=0 \mid X=1) = 0.9$

$P(T=0 \mid X=0) = 0.2$

	X=0	X=1
T=0	1/0.2	9/0.9
T=1	4/0.8	1/0.1

	X=0	X=1
T=0	5	10
T=1	5	10

Reweight by $\dfrac{1}{P(T_i \mid X_i)}$

$T \perp\!\!\!\perp \mathbf{X}$ **in pseudo-population**

Estimating (conditional) ATEs -S/X learners

- We can estimate the average causal effect/**average treatment effect**

$$\text{ATE} = \mathbb{E}[Y(t=1) - Y(t=0)] = \mathbb{E}_{\text{X}}[\mathbb{E}[Y(t=1)\,|\,\text{X}] - \mathbb{E}[Y(t=0)\,|\,\text{X}]]$$

Outcome model $\hat{\mu}(1,\mathbf{X})$ $\hat{\mu}(0,\mathbf{X})$

We still assume $\mathbf{X}$ is a valid adjustment set!

Estimating (conditional) ATEs -S/X learners

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$$\hat{\text{ATE}} = \frac{1}{n}\sum_{i=1}^{n} \hat{\mu}(1,\mathbf{x}_i) - \hat{\mu}(0,\mathbf{x}_i)$$

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We still assume $\mathbf{X}$ is a valid adjustment set!

- We can also estimate the **conditional average treatment effect:**

$$\text{CATE}(\text{w}) = \mathbb{E}[Y(t=1) - Y(t=0)\,|\,\text{W} = \text{w}]$$

Estimating (conditional) ATEs -S/X learners

- We can estimate the average causal effect/**average treatment effect**

$$\mathrm{ATE} = \mathbb{E}[Y(t=1) - Y(t=0)] = \mathbb{E}_{\mathrm{X}}[\underbrace{\mathbb{E}[Y(t=1)\,|\,\mathrm{X}]}_{\hat{\mu}(1,\mathbf{X})} - \underbrace{\mathbb{E}[Y(t=0)\,|\,\mathrm{X}]}_{\hat{\mu}(0,\mathbf{X})}]$$

$$\hat{\mathrm{ATE}} = \frac{1}{n}\sum_{i=1}^{n} \hat{\mu}(1,\mathbf{x}_i) - \hat{\mu}(0,\mathbf{x}_i)$$

We assume $\mathbf{X} \cup W$ is a valid adjustment set!

- We can also estimate the **conditional average treatment effect:**

$$\mathrm{CATE}(\mathrm{w}) = \mathbb{E}[Y(t=1) - Y(t=0)\,|\,\mathrm{W}=\mathrm{w}]$$

$$= \mathbb{E}_{\mathrm{X}}[\underbrace{\mathbb{E}[Y(t=1)\,|\,\mathrm{X},\mathrm{W}=\mathrm{w}]}_{\hat{\mu}(1,\mathbf{x}_i,w)} - \underbrace{\mathbb{E}[Y(t=0)\,|\,\mathrm{X},\mathrm{W}=\mathrm{w}]}_{\hat{\mu}(0,\mathbf{x}_i,w)}]$$

S-learners [Küntzel et al 2019]

- We learn a single model to predict the both potential outcomes $Y_i(0), Y_i(1)$

$$\hat{\text{ATE}} = \frac{1}{n}\sum_{i=1}^{n} \hat{\mu}(1,\mathbf{x}_i) - \hat{\mu}(0,\mathbf{x}_i)$$

$$\hat{\text{CATE}}(\text{w}) = \frac{1}{n_w}\sum_{i=1}^{n} 1(W = w)[\hat{\mu}(1,\mathbf{x}_i, w) - \hat{\mu}(0,\mathbf{x}_i, w)]$$

- **Issue:** for high-dimensional $\mathbf{X}$, S-learners can ignore the treatment

https://www.pnas.org/doi/10.1073/pnas.1804597116

X-learners [Küntzel et al 2019]

1. Learn two separate models $\hat{\mu}_1(\mathbf{x}_i)$ (only treated) and $\hat{\mu}_0(\mathbf{x}_i)$ (only control)
2. We impute the treatment effect per unit (*individual treatment effect)*

$Y(0)$

Treatment group

$$\hat{\tau}_{i,1} = Y_i - \hat{\mu}_0(\mathbf{x}_i)$$

Estimated from control

Control group

$$\hat{\tau}_{i,0} = \hat{\mu}_1(\mathbf{x}_i) - Y_i$$

Estimated from treated

Unit	Y(0)	Y(1)	T	X
1	?	1	1	1
2	1	?	0	1
3	?	0	1	0
4	0	?	0	0
5	?	1	1	1
6	?	0	1	0

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Control group

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Estimated from treated

3. Learn two separate models $\hat{\tau}_1(\mathbf{x}_i)$ (only treated) and $\hat{\tau}_0(\mathbf{x}_i)$ (only control)
4. The final estimator is a weighted average where $g(\mathbf{x}) : \mathcal{X} \rightarrow [0,1]$

$$\hat{\tau}(\mathbf{x}) = g(\mathbf{x}_i)\hat{\tau}_1(\mathbf{x}_i) + (1 - g(\mathbf{x}_i))\hat{\tau}_0(\mathbf{x}_i)$$

Issues: Inverse probability weighting (IPW)

- **Inverse probability (of treatment) weighting:** weight by inverse of estimated probability of treatment **received**:
  - For treated $T = 1$: weight by the inverse of $\hat{\pi}(X_i)$
  - For untreated $T = 0$: weight by the inverse of $1 - \hat{\pi}(X_i)$

$$\hat{\mathbb{E}}(Y(t=1)) = \frac{1}{n}\sum_{i=1}^{n} Y_i \cdot T_i \cdot \frac{1}{\hat{\pi}(X_i)}$$

$$\hat{\mathbb{E}}(Y(t=0)) = \frac{1}{n}\sum_{i=1}^{n} Y_i \cdot (1 - T_i) \cdot \frac{1}{1 - \hat{\pi}(X_i)}$$

We estimate $\hat{\pi}$ e.g. with logistic regression

What if the estimated $\hat{\pi}(X_i)$ is biased?

Advanced: Augmented Inverse probability weighting (AIPW)

- We assume we can estimate in an **unbiased way at least one of the two:**

-> we say this is a doubly robust method

1. Propensity scores $\hat{\pi}(\mathbf{x}_i)$
2. S-learner (outcome model) $\hat{\mu}(t_i, \mathbf{x}_i) \approx y_i$

Then:

$$\hat{\text{ATE}}_{S-learn} = \frac{1}{n}\sum_{i=1}^{n} \hat{\mu}(1,\mathbf{x}_i) - \hat{\mu}(0,\mathbf{x}_i)$$

Adjustment on the residuals of the S-learner with IPW

$$\hat{\text{Adj}}_{S-learm} = \frac{1}{n}\sum_{i=1}^{n} \frac{T_i}{\hat{\pi}(x_i)}(Y_i - \hat{\mu}(1,x_i)) - \frac{1-T_i}{1-\hat{\pi}(x_i)}(Y_i - \hat{\mu}(0,x_i))$$

$$\hat{\text{ATE}}_{AIPW} = \hat{\text{ATE}}_{S-learn} + \hat{\text{Adj}}_{S-learn}$$

This is unbiased if either propensity score or S-learner are unbiased

Advanced: Missing data (very briefly)

- **Typical approaches in practice (depending on the assumptions):**
  - Remove all samples with a missing feature (listwise deletion), or
  - Ignore the problem and use the non-missing features of all samples, or
  - Impute (predict) the missing values

Advanced: Missing data (very briefly)

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  - Impute (predict) the missing values

- **Typical assumptions (see talk by Karthika Mohan):**
  - R_X is an indicator variable that is 0 if X is missing and 1 otherwise
  - Missing completely at random (MCAR): $R_X \perp\!\!\!\perp \mathbf{X_V}$ (all variables)
  - Missing at random (MAR): $R_X \perp\!\!\!\perp X \,|\, \mathbf{X_V} \setminus \{X\}$
  - Missing not at random (MNAR) - anything else

Advanced: Missing at random (MAR)

- Missing completely at random (MCAR): coin toss, quite unrealistic

- **Missing at random (MAR):** missing at random given the completely observed (not missing) variables
  - Similar to **ignorability/unconfoundedness**
  - Imputation with EM
  - Multiple imputation (Rubin 1987) - impute m datasets, analyse, combine
    - (Augmented) IPW can be used to analyse/estimate ATE of each dataset

  - See https://scikit-learn.org/stable/modules/impute.html#impute, http://juliejosse.com/wp-content/uploads/2018/07/LectureNotesMissing.html