This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TEVC.2021.3078441, IEEE Transactions on Evolutionary Computation

Weighted indicator-based evolutionary algorithm for multimodal multi-objective optimization

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***Abstract*—Multimodal multi-objective problems (MMOPs) arise frequently in the real world, in which multiple Pareto optimal solution (PS) sets correspond to the same point on the Pareto front. Traditional multi-objective evolutionary algorithms (MOEAs) show poor performance in solving MMOPs due to a lack of diversity maintenance in the decision space. Thus, recently, many multimodal multi-objective evolutionary algorithms (MMEAs) have been proposed. However, for most existing MMEAs, the convergence performance in the objective space does not meet expectations. In addition, many of them cannot always obtain all equivalent Pareto solution sets. To address these issues, this study proposes a multimodal multi-objective evolutionary algorithm based on a weighted indicator, termed MMEA-WI. The algorithm integrates the diversity information of solutions in the decision space into an objective space performance indicator to maintain the diversity in the decision space and introduces a convergence archive to ensure a more effective approximation of the Pareto optimal front (PF). These strategies can readily be applied to other indicator-based MOEAs. The experimental results show that MMEA-WI outperforms some state-of-the-art MMEAs on chosen benchmark problems in terms of inverted generational distance (IGD) and IGD in the decision space (IGDX) metrics.**

***Index Terms*—Multimodal multi-objective optimization, Indicator-based algorithms, Evolutionary computation, Diversity-preserving mechanisms**

I. Introduction

Multiobjective optimization problems (MOPs) that require the simultaneous optimization of multiple objectives are common in real-world applications. In an MOP, the objectives are often in conflict with each other. Without a loss of generality, an MOP can be formulated as follows:

$$\begin{aligned} \text{Minimize} \quad & F(\mathbf{x}) = \{f_1(\mathbf{x}), f_2(\mathbf{x}), \cdots, f_m(\mathbf{x})\}, \\ s.t. \quad & \mathbf{x} = (x_1, x_2, \dots, x_n) \in \Omega, \end{aligned} \tag{1}$$

where Ω denotes the search space, m is the number of objectives, and $\mathbf{x}$ is a decision vector that consists of n decision variables x_i. A solution $\mathbf{x_a}$ is considered to Pareto dominate another solution $\mathbf{x_b}$ iff $\forall i = 1, 2, ..., m, f_i(\mathbf{x_a}) \leq f_i(\mathbf{x_b})$ and $\exists j = 1, 2, ..., m, f_j(\mathbf{x_a}) < f_j(\mathbf{x_b})$. Furthermore, a Pareto optimal solution is a solution that is not Pareto dominated by any other solution. The images of all Pareto optimal solution (PS) sets in the objective space are termed the Pareto optimal front (PF). Generally, the goal in MOPs is to obtain a set of solutions that are near the PF and spread uniformly along the entire PF.

Multimodal multi-objective problems (MMOPs) arise frequently in the real world, for example, diet design problems [1], space mission design problems [2], rocket engine design problems [3], and functional brain imaging problems [4]. A notable feature of MMOPs is that multiple PS sets correspond to equivalent PF. Fig. 1 illustrates the concept of an MMOP, where the left subfigure and right subfigure represent the decision space and objective space, respectively. From the figure, we can observe that although the three points $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are distant in the decision space, the corresponding objective vectors are close in the objective space. Moreover, $\mathbf{A}$ and $\mathbf{B}$ correspond to the same point $\mathbf{b}$ in the objective space. These problems often arise in real-world engineering problems. For example, in the multi-objective vehicle routing problem (MVRP), a decision maker needs to obtain solutions that have a minimum route distance and maximum cover area. Currently, normal multi-objective evolutionary algorithms (MOEAs) can adequately solve the MVRP and provide a set of solutions. However, these problems are proven to be multimodal [5], which means that for one certain objective vector, there is more than one corresponding solution. It is necessary to obtain all optimal solutions so the decision maker can better understand the problem; on the other hand, it is easier to transfer to another route if some constraints arise.

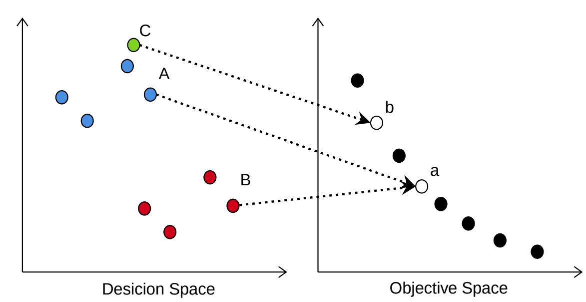

Fig. 1. Illustration of a two-objective multimodal problem.

Over the past two decades, many MOEAs have been proposed. Representative MOEAs include i) Pareto dominance-based algorithms, e.g., nondominated sorting genetic algorithm II (NSGA-II) [6] and strength Pareto evolutionary algorithm

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Manuscript received August 30, 2020;

2 (SPEA2) [7], ii) indicator-based algorithms, e.g., indicator-based evolutionary algorithm (IBEA) [8] and S-metric selection evolutionary multi-objective optimization algorithm (SMS-EMOA) [9], and iii) decomposition-based algorithms, e.g., MOEA based on decomposition (MOEA/D) [10], [11]. MOEAs rarely consider the diversity of solutions in the decision space; thus, they are often found to be ineffective on MMOPs. Effectively, there are two tasks in solving MMOPs: i) approximating the true PF of an MMOP with high accuracy and ii) maintaining the diversity of solutions in the decision space. To develop effective multimodal multi-objective evolutionary algorithms (MMEAs), the two tasks must be addressed simultaneously.

With respect to MMEAs in the literature, the omni-optimizer [12], [13], which is extended from NSGA-II [6], is perhaps the most representative MMEA. In the omni-optimizer, several strategies are proposed to obtain multiple PSs, including a Latin hypercube sampling-based population, restricted mating selection and alternative crowding distance. Based on the omni-optimizer, the double-niched evolutionary algorithm (DNEA) [14] and decision space-based niching NSGA-II (DN-NSGA-II) [15] are further proposed. Specifically, two sharing functions are introduced in the DNEA, and a solution distance-based mating selection method is proposed in DN-NSGA-II. Since the two algorithms belong to Pareto dominance-based MOEAs, it is not surprising that they perform poorly on MMOPs with many objectives [16].

Indicator-based algorithms have shown competitive performance on many-objective optimization problems (MaOPs) [17]. It is thus reasonable to embed diversity-preserving strategies in the decision space into IBEA [8] to solve multimodal many-objective problems (MMaOPs). As a convergence-first algorithm, the IBEA selects the individuals with the best fitness indicator values in the current population as parents to generate offspring. However, since an MMEA must maintain the diversity of solutions in the decision space, a convergence-first selection scheme cannot be directly applied. Motivated by this limitation, we propose a weighted indicator-based multi-objective evolutionary algorithm for multimodal problems, termed MMEA-WI.

To address the imbalance between the diversity in the decision space and the convergence in the objective space, we propose an approach to evaluate the potential convergence of solutions. Specifically, for each individual, we use the sum of all solutions' indicator values that are inversely weighted by the Euclidean distance in the decision space from the individual to each solution. The summed result, termed the weighted indicator (I^w), indicates the potential convergence quality of a solution. The effect of I^w on the diversity in the decision space will be explained in Section III-B. By adopting this indicator, solutions distribute over multiple PSs rather than converging to a single PS.

In addition, to achieve high-quality convergence in the objective space, an archive is introduced to store a prespecified number of nondominated solutions that have been obtained. To update the archive, a modified crowding distance is introduced to balance the number of solutions for each equivalent PS. Moreover, to enhance the search ability, a mating selection strategy is proposed for global and local search, in which both the diversity of the population and the convergence of the archive solutions are considered.

To demonstrate the performance of MMEA-WI, the algorithm is compared to six state-of-the-art MMEAs, i.e., the omni-optimizer [12], [13], multi-objective particle swarm optimization using ring topology and special crowding distance (MO_Ring_PSO_SCD) [18], multi-objective particle swarm optimization for multimodal multi-objective problems (MO_PSO_MM) [19], MOEA using two-archive and recombination strategies (TriMOEA-TA&R) [20], and convergence-penalized density evolutionary algorithm (CPDEA) [21], and DN-NSGA-II [15]. In addition, multi-objective differential evolution based on decomposition (MDEA/D-DE) [22] and NSGA-II [6] are chosen to represent the traditional MOEAs. Our experimental results show that MMEA-WI outperforms, or is at least comparable to, the compared algorithms in terms of the Pareto set proximity (PSP), inverted generational distance (IGD) and IGD in the decision space (IGDX) performance metrics for most of the test problems, especially for MMaOPs. Moreover, several user-defined parameters are introduced to control the performance of MMEA-WI. The influence of these parameters is also discussed. The main contributions of this study can be concluded as follows: 1) A weighted indicator is proposed to evaluate the potential convergence ability of an individual, which is applied as the criterion to select parents. 2) We have integrated the weighted indicator to an evolutionary algorithm, resulting in MMEA-WI, to solve MMOPs. The parameter sensitivity of MMEA-WI is clearly analyzed. 3) We have performed a rigorous comparison of MMEA-WI with state-of-the-art MOEAs and MMEAs, which shows that MMEA-WI is effective and efficient for solving MMOPs.

The remainder of this paper is organized as follows: Section II briefly reviews related studies of multimodal multi-objective optimization, including representative algorithms and benchmark problems. Section III elaborates the proposed algorithm, MMEA-WI. Section IV describes the experimental settings. In Section V, experimental results are reported and analyzed. The sensitivity of the search ability of the proposed algorithm with respect to parameter specifications is also discussed. The paper is concluded in Section VI.

II. Related studies

A. Indicator-based MOEAs

Indicator-based MOEAs use performance metrics or indicators to directly guide the evolution of solutions, that is, an optimization task with multiple objectives is transformed into the optimization of a certain performance indicator. Thus, multi-objective optimization is converted into single-objective optimization. In the literature, various performance metrics have been proposed for population evaluation; for example, the generational distance (GD) [23] measures convergence, and the IGD [24], [25] and hypervolume (HV) evaluate both convergence and diversity. Indicator-based MOEAs have been demonstrated to be effective compared to Pareto dominance-based and decomposition-based MOEAs in solving many MOPs.

The IBEA [8] is the earliest indicator-based MOEA, in which the additive ϵ indicator ($I_{\epsilon+}$) and the hypervolume indicator (I_{HD}) are utilized to guide the evolution process, resulting in $I_{\epsilon+}$-IBEA and I_{HD}-IBEA, respectively. Notably, the R2 indicator I_{R2} is also often employed in the IBEA [26]. The R2 indicator maps the solution set to a scalar value by using a utility function with a weight vector set.

Since $I_{\epsilon+}$-IBEA often results in a poorly distributed PF approximation [27], [17], several variants of $I_{\epsilon+}$ have been proposed to address this issue. In Two_Arch2 [28], the authors use the norm-based crowding distance L_p to update the diversity archive. In the stochastic ranking algorithm (SRA) [29], shift-based density estimation I_{SDE} is adopted to sort individuals by stochastic ranking [30]. In addition, the bicriterion evolution (BCE) framework proposed in [31] uses an online bounded external archive to maintain the solution distribution in the objective space.

The SMS-EMOA [9] is another representative indicator-based MOEA, in which the individuals are sorted by their contribution to the hypervolume indicator. The individual with the smallest hypervolume contribution is removed from the population. SMS-EMOA, which has a steady-state strategy, can address MOPs with two and three objectives. However, since the hypervolume computational complexity grows rapidly as the number of objectives increases, the calculation of the hypervolume contribution in MaOPs becomes extremely expensive. To address this issue, instead of calculating the exact hypervolume values, hypervolume estimation (HypE) [32] applies a Monte Carlo simulation method to approximate the hypervolume value and has been shown to be effective for MaOPs. Importantly, modified algorithms that are based on the SMS-EMOA, such as the R2-EMOA [26], many objective differential evolution with mutation restriction (MyO-DEMR) [33] and reference point adaptation MOEA (AR-MOEA) [34], have been shown to be effective for MaOPs with a maximum of 50 objectives. In addition to SMS-EMOA, the generational distance MOEA (GD-MOEA) [35], IGD$^+$-EMOA [36] and discrete differential evolution (DDE) [37] have employed the GD indicator, IGD$^+$ indicator and Hausdorff distance (Δp) indicator, respectively.

B. Multimodal Multiobjective Algorithms (MMEAs)

An MMOP is a kind of MOP in which multiple Pareto optimal solution sets correspond to equivalent Pareto fronts. Although the idea of MMOPs was proposed in 2005, their formal definition is still controversial. In this study, the definition of MMOPs given by Rudolph and Preuss in [38] is adopted; it can be described as follows:

Definition 1. An MMOP involves obtaining all solutions that are equivalent to Pareto optimal solutions.

Definition 2. Two different solutions $\mathbf{x_1}$ and $\mathbf{x_2}$ are said to be equivalent iff $\|F(\mathbf{x_1}) - F(\mathbf{x_2})\| \leq \sigma$, where $\|\mathbf{a}\|$ is an arbitrary norm of $\mathbf{a}$.

In this study, we only consider the case $\sigma = 0$. Specifically, an MMOP is an MOP that has multiple Pareto solution sets corresponding to the same Pareto front. Thus, an MMEA aims to locate multiple optimal (or near-optimal) solutions in a single simulation run.

Due to the simplicity and robustness of NSGA-II [6], many modified algorithms have been proposed. The omni-optimizer is considered one of the most representative MMEAs. In the algorithm, a new crowding distance is proposed, which takes into account the solution diversity in both the objective space and the decision space. Specifically, the crowding distance $c_{i,j}^{\text{obj}}$ in the objective space is calculated in the same way as that in NSGA-II, while the "variable-wise" crowding distance of x_i in the j-th decision variable is calculated as follows:

$$c_{i,j}^{\text{dec}} = \begin{cases} 2\left(\frac{x_{i+1,j}-x_{i,j}}{x_j^{\max}-x_j^{\min}}\right) & \text{if } x_{i,j} = x_j^{\min}, \\ 2\left(\frac{x_{i,j}-x_{i-1,j}}{x_j^{\max}-x_j^{\min}}\right) & \text{else if } x_{i,j} = x_j^{\max}, \\ \frac{x_{i+1,j}-x_{i-1,j}}{x_j^{\max}-x_j^{\min}} & \text{otherwise ,} \end{cases} \tag{2}$$

where $x_j^{\max}$ and $x_j^{\min}$ are the maximum value and minimum value, respectively, of the j-th variable. We assume that all solutions are sorted according to their j-th decision variable values in descending order. Unlike the crowding distance in the objective space, infinitely large values are not assigned to the boundary solutions.

Motivated by NSGA-II, other MMEAs based on Pareto dominance have been proposed with different diversity-maintenance strategies, e.g., DNEA [14] and DN-NSGA-II [15]. In the niching-covariance matrix adaptation (CMA) approach [39], an aggregate distance metric in the objective and solution spaces is employed to group individuals into multiple niches. Notably, although genetic variation operators, e.g., SBX crossover and polynomial mutation, are popular and utilized in many MMEAs, other approaches have advantages in MMOPs. MO_PSO_MM [19] and MO_Ring_PSO_SCD [18] use a PSO variation operator to evolve solutions with special crowding distances.

Decomposition-based MOEAs usually outperform Pareto dominance-based algorithms in MaOPs. It is therefore easy to know that decomposition-based MMEAs may show excellent performance in MMaOPs, e.g., MOEA/D-AD [40]. The algorithm assigns one or more individuals to each subproblem to address multimodality. These strategies are also discussed in [41]. Moreover, the multimodal multi-objective differential evolution optimization algorithm (MMODE) [42] introduces a modified differential evolution (DE) operator to promote diversity in the decision space. In addition, Tanabe et al. [43] proposed a niching indicator-based multimodal many-objective optimizer (NIMMO). By using niching information, NIMMO is shown to work well for MMOPs with many objectives.

III. The MMEA-WI algorithm

This section explains the procedure of the proposed MMEA-WI algorithm. The main novelties of MMEA-WI are described as follows: i) adopting a diversity-maintaining mechanism instead of a convergence-first selection strategy; ii) coevolving two archives: the first archive is to maintain the diversity of solutions in the decision space; and the second archive is to improve the convergence toward the true PF in the objective space. By adopting these strategies, the numbers of individuals located in different PS areas can be well balanced. The

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detailed process of MMEA-WI is illustrated in the following subsections.

A. Framework of MMEA-WI

The framework of the proposed MMEA-WI is illustrated in Algorithm 1. Similar to most MOEAs, MMEA-WI consists of the following parts: population initialization, mating selection, offspring generation and environmental selection. Borrowing the idea from TriMOEA-TA&R [20], note that a convergence archive, termed the convergence archive, is introduced to store the nondominated solutions obtained at each generation. The effect of the convergence archive is discussed later in this section.

There are two phases in MMEA-WI. In the first phase, while the number of individuals in the convergence archive is less than N, we only select parents from the current population, which helps in exploring the whole decision space. Once the convergence archive size reaches N, the evolution moves to the second phase. The convergence archive is incorporated to generate offspring, aiming to obtain better convergence performance. Parents are probabilistically selected from the convergence archive and the current population. In this phase, the number of solutions in different PSs can be automatically adjusted. Mating selection is further explained later in this section. The convergence archive is presented as the final result of the algorithm. Fig. 2 and Fig. 3 illustrate the distribution of the solutions in the main population and convergence archive, respectively, where the red lines represent two different PSs in the decision space and the black circles are the obtained solutions. Intuitively, in the final stage of evolution, individuals in the main population will be gathered around multiple PSs, while the solutions in the convergence archive will be uniformly located on different PSs.

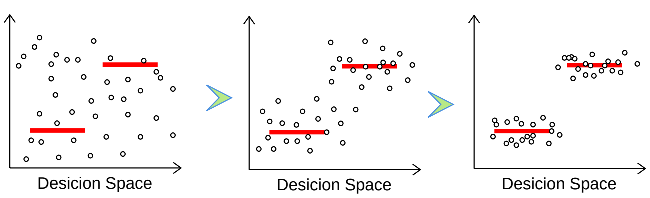

Fig. 2. Distribution of solutions in the main population during the evolutionary process of MMEA-WI, where the red line and points represent the Pareto optimal region and solutions, respectively.

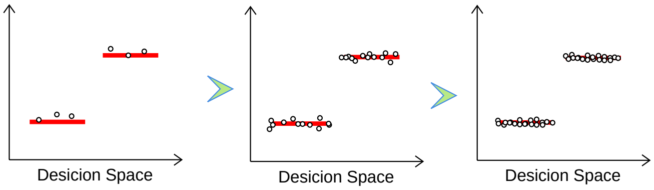

Fig. 3. Distribution of solutions in the convergence archive during the evolutionary process of MMEA-WI, where the red line and points represent the Pareto optimal region and solutions, respectively.

Theoretically, the size of the convergence archive can be infinite [44]. Specifically, in each generation of the evolution, the offspring are integrated into the convergence archive without discarding any solution. At the end of the evolution, the size of the archive is truncated to a certain value. This mechanism can be found in the artificial immune network for omni-optimization (omni-aiNet) [45], $P_{Q,\epsilon}$-MOEA [2], and MOEA/D-AD [40] and is regarded as a potential strategy for improving algorithm performance [16]. Another way to update the archive is to maintain the size of the archive, which means that in each generation, the archive separately discards the worst individual until the size reaches a certain value. Intuitively, keeping a limitless archive is competitive since the potential optimal solutions will not be discarded. However, a limitless archive means that there is a high demand for space and high computational consumption. Thus, we adopted an online convergence archive with a certain size.

Algorithm 1 General Framework

Input: Maximum generations $MaxGen$, population size N
Output: Convergence archive Arc

```
Pop ← Initialization(N)
Arc ← Nondominated(Pop) /*select nondominated solutions to form the convergence archive*/
while gen ≤ MaxGen do
    if SizeOf(Arc) < N then
        MatingPool ← TournamentSelection(Pop)
    else
        if rand < p then
            MatingPool ← TournamentSelection(Arc)
        else
            MatingPool ← TournamentSelection(Pop)
        end if
    end if
    Offspring ← Variation(MatingPool)
    Pop ← EnvironmentalSelection(Pop, Offspring, N)
    Arc ← UpdateArc(Arc, Offspring, N)
end while
```

B. Construction of weighted indicators

1) $I_{\epsilon+}$ indicator: In the process of updating the population, a modified indicator based on $I_{\epsilon+}$ [8] is proposed to evaluate the potential convergence of each individual. Other indicators, such as I_{HD} and R_2 can be easily embedded into MMEA-WI in the same way. The assigned fitness of $I_{\epsilon+}$ can be described as follows:

$$F(x^1) = \sum_{x^2 \in P \backslash x^1} exp\left(-\frac{I(x^2, x^1)}{\kappa I^{max}}\right) \tag{3}$$

where x^1 and x^2 are solutions in population P and I is a Pareto dominance–preserving binary indicator. $I(x^2, x^1) < I(x^1, x^2)$ iff x^2 dominates x^1. I^{max} refers to the maximum absolute indicator value, which is defined as $I^{max} = max_{x^1, x^2 \in P}|I(x^1, x^2)|$. κ denotes the scale factor, which is usually set to 0.05. The smaller the fitness value is, the better the solution is.

The binary additive ϵ indicator $I_{\epsilon+}$ has been shown to be effective for MaOPs, e.g., Two_Arch2 [28] and SRA

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[30]. Specifically, $I_{\epsilon+}$ gives the minimum distance on each dimension in the objective space, which can be expressed as follows:

$$I_{\epsilon+}(x^2, x^1) = \max_{i\in\{1,...,m\}} \{f_i^0(x^2) - f_i^0(x^1)\}, \tag{4}$$

where $f_i^0(x)$ denotes the normalized objective value of the solution x, and m is the number of objectives. It is reported in [46] that the normalization methods could cast different effects in MMOPs. As a result, we use the same normalization method to normalize the objective vector and decision variables, which can be expressed as follows:

$$f_i^0(x) = \frac{f_i(x) - f_i^{min}}{f_i^{max} - f_i^{min}}, \tag{5}$$

where f_i^{min} and f_i^{max} are the minimum value and maximum value, respectively, for the i-th objective.

2) Calculation of the weighted indicator: Prematurity will lead to poor solution diversity in the decision space. However, for traditional MOEAs with a convergence-first selection strategy, solutions will directly move towards the Pareto optimal without widely searching the whole decision space [21], [20]. Inspired by the local convergence quality applied in the CPDEA, we introduce a weighted indicator I^w to evaluate the potential convergence quality of each individual:

$$I_i^w = \sum_{j=1}^{N} w_{i,j} F(x^j),\ i = 1, 2, ..., N, \tag{6}$$

where N is the population size and I_i^w is the weighted indicator of individual i. I_i^w is the weighted sum of $F(x^j)$ with weights $w_{i,j}$. The weight $w_{i,j}$ is calculated according to the Euclidean distance between solutions i and j in the decision space.

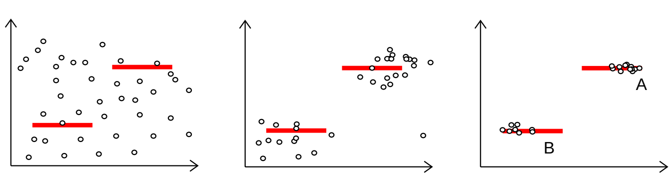

Fig. 4. Convergence process in the decision space of traditional MOEAs for multimodal multi-objective problems, where the red lines represent two equivalent PSs.

Since $I_{\epsilon+}$ is a convergence indicator, by using this indicator as a selection criterion, the obtained individuals usually converge to the Pareto optimal without considering searching the whole decision space. For example, in Fig. 4, once the algorithm obtains the solution with better convergence, other solutions will quickly converge to this region. In this circumstance, the distribution of solutions in the decision space is apparently nonuniform, and the whole space is hardly explored. To solve this issue, I^w sums the indicator values of its neighboring individuals with different weights to measure the weighted-sum indicator value of a certain area.

Via the transformation, solutions located exactly on one of the true PSs may have the same fitness values as solutions located near the same true PS. By using the modified crowding distance, which is explained in the next subsection, individuals will not directly converge to the true PS but gather around it. Therefore, diversity in the decision space can be well maintained. It is worth mentioning that the final distribution of solutions in the decision space is presented in Section V-B, which can help to understand the effect of the proposed weighted indicator. We introduce the weighted indicator to maintain the diversity of solutions in the main population and utilize the convergence archive to improve the convergence ability of the algorithm.

The weight $w_{i,j}$ is calculated by the Euclidean distance $d_{i,j}$ between solutions i and j in the decision space as follows:

$$w_{i,j} = f(d_{i,j}) = \frac{1}{1 + exp(\frac{d_{i,j}}{\beta})}, \tag{7}$$

where $f(x)$ is a monotonically decreasing function modified from the sigmoid activation function. The smaller the Euclidean distance is, the larger the contribution from its neighbor is. Namely, Eq. (7) emphasizes the fitness of an individual's neighbors by assigning a large value to $w_{i,j}$ if the Euclidean distance between x_i and x_j is small, and vice versa. Other monotonically decreasing functions can also be utilized, e.g., the probability density function of the normal distribution applied in [21]. β is a parameter to control the decreasing rate of $f(x)$, which is defined as follows:

$$\beta = \frac{1}{exp(\frac{gen}{maxgen})} \left(\frac{\prod_{i=1}^{N}(x_i^{upper} - x_i^{lower})}{N} \right)^{1/M}, \tag{8}$$

where x_i^{upper} and x_i^{lower} represent the upper bound and lower bound, respectively, of the i-th decision variable. The second part of β means the average distance of the solutions in the decision space. If we assume the second part of β to be 1, then

$$\beta = \frac{1}{exp(\frac{gen}{maxgen})}, \tag{9}$$

the graph of $f(x)$ can be seen in Fig. 5.

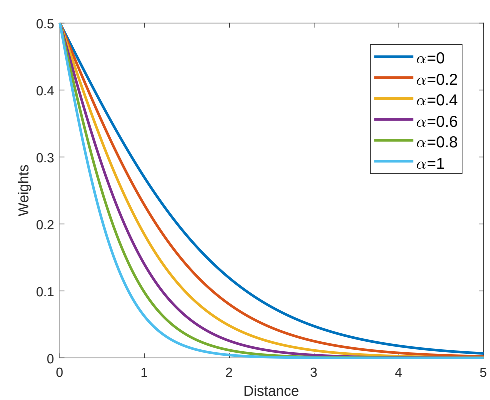

Fig. 5. Relationship between the solution distance in the decision space $d_{i,j}$ and the value of weight $w_{i,j}$, where $\alpha = \frac{gen}{maxgen}$. According to Eq. (9), when $\alpha = 0$ and 1, $\beta = 1$ and e^{-1}, respectively.

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As we can observe in Fig. 5, the value of $w_{i,j}$, which means the contribution of the j-th solution, decreases as $d_{i,j}$ increases. In addition, as the evolution progresses, the decreasing rate of $w_{i,j}$ increases continuously. Therefore, the calculation of $w_{i,j}$ will be less influenced by the distant solutions in the later stage of evolution, which implicitly helps improve the convergence ability of MMEA-WI.

C. Updating the Convergence Archive

A convergence archive is employed to store nondominated solutions obtained during the evolution. The detailed updating process can be found in Algorithm 2. Note that when the number of solutions in the archive exceeds N, the modified crowding distance is applied to prune the number of solutions in the archive into N, see $lines\ 4-8$. The modified crowding distance is expressed as follows:

$$cd_i^x = \frac{1}{1 + \frac{\sum_{k=1}^{K} d_{i,k}^x}{d_{mean}^x}}, \tag{10}$$

$$d_{mean}^x = \frac{1}{N} \frac{\sum_{i=1}^{N}\sum_{k=1}^{K} d_{i,k}^x}{K}, \tag{11}$$

where $d_{i,k}^x$ and d_{mean}^x are the Euclidean distances between x_i and its kth nearest neighbors ($k = 1, 2, ..., K$) and the mean value of these distances, respectively. The range of cd_i^x is restricted to $(0, 1]$ by Eq. (10).

Algorithm 2 Updating the Archive

Input: Convergence archive Arc, offspring $Offspring$, population size N

Output: Convergence archive Arc, modified crowding distance cd^x

1: $JointArc \leftarrow Arc \cup Offspring$
2: $FrontNo \leftarrow NondonminatedSort(JointArc)$ /* get the Pareto rank of the JointArc */
3: $Next \leftarrow IndexOf(FrontNo == 1)$ /* $Next$ is a binary array, where 1 means $FrontNo == 1$ /*
4: **while** $N \leq sum(Next)$ **do**
5: $\quad cd^x \leftarrow CalModifiedDis(JointArc(Next))$
6: $\quad del \leftarrow FindMin(cd^x)$ /* get the minimum index of cd^x */
7: $\quad Next(del) \leftarrow false$
8: **end while**
9: $Arc \leftarrow JointArc(Next)$

In [21], the authors use the double k-nearest neighbor method to evaluate the fitness of solutions in both the objective and decision spaces. However, the diversity in the objective and decision spaces is usually conflicting, which means that extra solutions may be incorrectly discarded, resulting in poor convergence quality. We adopt the weighted $I_{\epsilon+}$ indicator to guide the evolution, which takes into account the spread of the solutions in the objective space. Compared to the double k-nearest neighbor method, only the crowding distance in the decision space is calculated in the proposed MMEA-WI. Therefore, the convergence in the objective space could be better compared to that of taking the double $k-$nearest neighbor method. Specifically, once the convergence archive size exceeds N, we calculate the modified crowding distance of each solution in the convergence archive. An individual with the smallest distance value is discarded. This process is repeated until the convergence archive size decreases to N.

D. Environmental Selection

The update process for the population is based on the framework of the IBEA [8]. The main differences are listed as follows:

i. A two-stage mating selection scheme is employed to choose parents.
ii. The weighted indicator is utilized to maintain the population diversity.

As mentioned in Algorithm 1, there are two stages of the evolution process. The first stage, termed the preliminary exploration, focuses on detecting all potential PSs in the entire decision space. In this stage, we only select parents from the main population. Since I^w evaluates the potential convergence quality of solutions, the solution diversity in the decision space can be well maintained in this stage.

In the second stage, termed the multiple Pareto solution balancing stage, the user-defined probability p is applied to select parents from the convergence archive. Intuitively, a larger probability of selecting parents from the convergence archive can lead to better convergence but worse solution diversity. In contrast, if a smaller p is employed, the diversity of solutions in the population and the convergence archive can be well maintained, but the convergence will be poor. Here, $p = 0.4$ is selected. The effect of this parameter is discussed in Section V.

Algorithm 3 Environmental Selection

Input: Population Pop, population size N

Output: Population Pop, weighted fitness WF

1: $[Fitness, I, C] \leftarrow CalFitness(Pop)$
2: $WF \leftarrow CalWeightedFitness(Pop, Fitness)$
3: $Choose \leftarrow 0$ /* $Choose$ a binary array, where 1 means being selected */
4: **while** $Choose \leq N$ **do**
5: $\quad del \leftarrow FindMin(WF)$ /* find the index of minimum WF */
6: $\quad Choose(del) \leftarrow true$
7: $\quad WF \leftarrow UpdateWF(WF)$
8: **end while**
9: $Pop \leftarrow Pop(Choose)$

The detailed environmental selection process can be illustrated in Algorithm 3. When updating the population, first, the fitness of each individual is calculated according to Eq. (6). Second, in each iteration, the individual with the lowest weighted fitness is discarded; refer to $line\ 4$. Last, a population of size N is returned.

E. Empirical Computational Complexity Analysis

In this section, we provide the computational runtime complexity analysis of MMEA-WI in the worst situation.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TEVC.2021.3078441, IEEE Transactions on Evolutionary Computation

For each generation, four main steps should be executed: 1) mating selection; 2) individual variation; 3) environmental selection and 5) archive updating. We assume the population size and number of objectives to be N and M, respectively. For each generation, the mating selection needs a runtime of $O(MN^2)$. We adopt the simulated binary crossover (SBX) [47] to perform the variation operation, which needs a runtime of $O(DN)$, where D is the number of decision variables. In addition, for the environmental selection, we use $I_{\epsilon+}$ as the basic indicator to construct the weighted indicator, which needs a runtime of $O(MN^2)$. Second, the construction of the weighted indicator needs $O(NM^2)$. Third, MMEA-WI needs $O(N \log N)$. Therefore, the total runtime complexity of environmental selection is $O(MN^2)$. For the updating process of the convergence archive, first, we need to perform nondominated sorting, which needs a runtime of $O(MN^2)$ in the worst case. Second, we need to calculate the crowding distance and obtain the index of the individual with the minimized crowding distance, which needs a runtime of $O(N^2 \log N)$. Thus, the final runtime of the updating archive is $max\{O(MN^2), O(N^2 \log N)\}$. Generally, we have $\log N > M$. As a result, we can determine that the runtime complexity of MMEA-WI is $O(GMN^2)$, where G is the number of generations.

For comparison, the runtime complexities of Two_Arch2, IBEA and NSGA-III are $max\{O(N \log^{m-2} N), O(MN^2)\}$, $O(N^2)$ and $max\{O(N \log^{m-2} N), O(MN^2)\}$ [28], respectively. From the theoretical analysis, MMEA-WI is computationally efficient. In addition, we will empirically compare the runtime performance of MMEA-WI with other state-of-the-art MMEAs, which will be illustrated in the following section.

IV. Experimental Setting

A. Benchmark Problems

In the literature, many benchmark MMOPs have been proposed. Among these test problems, the two-objective and two-variable symmetrical parts (SYM-PART) [39] are one of the most representative test problems, which can be expressed as $f_1(\mathbf{y}) = (y_1 + a)^2 + y_2^2$ and $f_1(\mathbf{y}) = (y_1 - a)^2 + y_2^2$, where $y_1 = x_1 - t_1(c + 2a)$ and $y_2 = x_2 - t_2 b$. Specifically, the parameter a controls the region of the Pareto optimal solutions, while b and c control the positions of the subsets. In addition, eight min-max fairness (MMF) problems are designed by mirroring the original Pareto optimal solution to form multiple equivalent subsets. The MMF problems are bi-objective. In [20], imbalanced distance minimization problems (IDMPs) are proposed to benchmark the algorithm performance for problems of varying levels of difficulty in finding different PSs. Polygon problems [48] are designed to test the performance of MMEAs on MaOPs with an adjustable number of objectives. However, the number of decision variables is small and not scalable. To address this issue, multipolygon problems [49], [50] are proposed with adjustable dimensions of decision variables and objectives. Overall, the MMF, multipolygon and IDMP benchmarks are selected as test problems. The parameters for each benchmark problem are set to the suggested values in the original paper.

B. Performance Metrics

To evaluate the performance of MOEAs, many metrics have been proposed, e.g., the hypervolume, GD, and IGD indicators [24], [25]. However, these indicators are inappropriate to evaluate MMEAs since these metrics only consider the quality of obtained solutions in the objective space [16]. Thus, new indicators have been proposed to qualify the obtained solutions in the decision space. Among the proposed performance indicators for MMEAs, the IGDX [51] is perhaps the most popular index; it evaluates how well the obtained solution set approximates the true solution set in the decision space. In addition, the PSP [18] and inverted generational distance–multimodal (IGDM) [20] measure the similarity between the obtained PSs and the true PSs, which is a combination indicator.

Generally, these performance metrics evaluate the performance of MMEAs mainly by comparing the distance between the obtained solution sets and the true PSs in the decision space, which only measures the convergence and diversity of obtained solutions in the decision space. In addition, the quality in the objective space is also important for an MOEA. Thus, in addition to the mentioned metrics, we also need to employ traditional indicators, such as the IGD, to measure the performance in the objective space. In this study, the IGD, IGDX and PSP are employed to evaluate the quality of the obtained solutions. The three metrics are expressed here. Note that $\mathbf{X}$ and $\mathbf{X}^*$ denote the obtained solution set and a set of a finite number of Pareto optimal solutions uniformly sampled from the true PS, respectively.

$$IGD(\mathbf{X}) = \frac{1}{|\mathbf{X}^*|} \sum_{y \in \mathbf{X}^*} \min_{x \in \mathbf{X}} \{ED(F(x), F(y))\}, \tag{12}$$

where $ED(x, y)$ is the Euclidean distance between x and y.

Similar to IGD, the IGDX calculates the convergence and spread of $\mathbf{X}$ in the decision space, which can be written as follows:

$$IGDX(\mathbf{X}) = \frac{1}{|\mathbf{X}^*|} \sum_{y \in \mathbf{X}^*} \min_{x \in \mathbf{X}} \{ED(x, y)\}. \tag{13}$$

The PSP indicator is a combination indicator that is written as follows:

$$PSP(\mathbf{X}) = \frac{CR(\mathbf{X})}{IGDX(\mathbf{X})}, \tag{14}$$

$$CR(\mathbf{X}) = (\prod_{i=1}^{D} \sigma_i)^{\frac{1}{2D}}, \tag{15}$$

$$\sigma_i = (\frac{min(x_i^{*,max}, x_i^{max}) - max(x_i^{*,min}, x_i^{min})}{x_i^{*,max} - x_i^{*,min}})^2, \tag{16}$$

where $x_i^{*,max}$ is the maximum value of the i-th variable in the PS. $\sigma_i = 1$ when $x_i^{*,max} = x_i^{*,min}$; $\sigma_i = 0$ when $x_i^{*,max} \leq x_i^{*,min}$ or $x_i^{*,max} \leq x_i^{*,min}$.

A small IGD value indicates that the solution set $\mathbf{X}$ has both reasonable convergence and diversity in the objective space, which also reflects that the MMEA is an effective MOEA. An MMEA is recognized as a high-performance MMEA if

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the obtained $\mathbf{X}$ has a small IGDX and a large PSP value. In [43], the authors analyzed the merits and demerits of the performance metrics. The results show that a solution set with a satisfactory IGDX usually has an acceptable IGD, while a reasonable IGD does not naturally produce a reasonable IGDX.

C. Compared Algorithms

To verify the effectiveness of MMEA-WI in solving MMOPs, the omni-optimizer [12], [13], MO_Ring_PSO_SCD [18], MO_PSO_MM [19], TriMOEA-TA&R [20], CPDEA [21], DN-NSGA-II [15], MDEA/D-DE [22] and NSGA-II [6] are chosen as competitor algorithms. The omni-optimizer and DN-NSGA-II are Pareto dominance-based representative MMEAs. CPDEA and TriMOEA-TA&R are two state-of-the-art MMEAs. MOEA/D-DE and NSGA-II are classic MOEAs. For all the algorithms, we set the population size N to 200, 210, 210, 156, 210, 230, and 135 when the number of objectives M is 2, 3, 5, 8, 9, 10 and 15, respectively. Since the population size for the omni-optimizer must be a multiple of four, we set N slightly larger than its value in the other algorithms to satisfy this constraint. The simulated binary crossover (SBX) and polynomial mutation (PM) operators are employed to generate offspring except for MO_Ring_PSO_SCD. The maximum number of function evaluations N_E is set to 10,000, 12,000, and 15,000 for the two-objective problem, three-objective problem and four-objective problem, respectively. For problems with more than four objectives, N_E is set to 20,000. Note that the specific parameters in each algorithm are set as follows: For MO_Ring_PSO_SCD, $C_1 = C_2 = 2.05$ and $W = 0.7298$. For TriMOEA-TA&R, p_{con}, σ_{niche} and ϵ_{peak} are set to 0, 0.5 and 0.01, respectively.

V. Experimental Results

A. Performance of MMEA-WI

This section systematically compares MMEA-WI and the competitor algorithms. The results are reported in TABLE II, TABLE III and TABLE I, where the mean of the IGDX, IGD and PSP values over 31 independent runs are listed. The best result is highlighted with a gray background. In each table, we use the Wilcoxon rank-sum test with $p < 0.05$ to compare each algorithm with MMEA-WI. In the last column of each table, the symbols '+' and '-' indicate the number of test problems where the compared algorithm shows significantly better performance or worse performance, respectively, than MMEA-WI. In addition, the symbol '=' indicates the number of test problems where there is no significant difference between MMEA-WI and the compared algorithms. Parameters m and d in IDMPmT and Poly-m-d indicate the number of objectives and the number of decision variables, respectively.

TABLE II shows the IGDX comparison results, from which we can observe that MMEA-WI outperforms the other algorithms on 14 test instances over 26 chosen benchmark problems. MO_PSO_MM, MO_Ring_PSO_SCD and CPDEA perform the best on 4 instances, 2 instances, and 6 instances, respectively. Moreover, the DN-NSGA-II and omni-optimizer do not win on any test instance, although they are designed for multimodal multi-objective optimization. As we expected, NSGA-II and MOEA/D-DE show poor performance in solving these MMOPs in terms of the IGDX indicator, because these two algorithms are designed for general MOPs without considering the solution diversity in the decision space. Notably, to examine the ability of MMEAs on large-scale MMOPs, we set the number of variables to 10 and 100 in multipolygon problems. MMEA-WI wins for five of the six multipolygon problems, which shows that it has a strong ability to solve large-scale problems with more than 100 decision variables.

The comparison results based on the PSP indicator are very similar to those of the IGDX. Specifically, for the PSP indicator, MMEA-WI wins on 15 instances, while MO_PSO_MM and CPDEA win on 2 instances and 7 instances, respectively.

For the IGD results in TABLE I, MOEA/D-DE and NSGA-II have significantly better results than the algorithms designed for multimodal multi-objective optimization because NSGA-II and MOEA/D-DE are designed for general multi-objective optimization, which has not considered the diversity in the decision space. It is difficult (or almost impossible) to obtain a set of nondominated solutions that minimizes both the IGD and IGDX metrics. In addition, for problems with more than three objectives, MOEA/D-DE achieves better performance than NSGA-II, and vice versa. Surprisingly, unlike other MMEAs, MMEA-WI wins on 4 test instances in terms of the IGD indicator, which indicates that MMEA-WI has a better ability to approximate the true PF than the other competitor MMEAs.

As mentioned in Section IV-B, the IGDX indicator was proposed to evaluate the diversity and convergence of a solution set to the true Pareto set in the decision space, while the PSP indicator was proposed to measure the spread performance in the decision space. Generally, if an algorithm performs well in terms of the IGD, it is an eligible MOEA, while a high-performance MMEA should obtain reasonable results in terms of the IGDX and PSP. According to these three tables, MMEA-WI obtained significantly better results on average than the other compared algorithms in terms of the IGDX and PSP. Although MMEA-WI cannot outperform traditional MOEAs such as MOEA/D-DE and NSGA-II, it has the ability to converge to the true PF. In addition, for problems with more than 5 objectives, MMEA-WI shows a strong ability to obtain equivalent PS sets. Thus, it can be concluded that the proposed MMEA-WI algorithm is better than the chosen MMEAs for the benchmark problems in terms of the IGDX and PSP metrics.

B. Results Obtained by MMEA-WI

To visually demonstrate the operational results of MMEA-WI, the final obtained results are plotted. We use the results that approximate the average best result for each algorithm in terms of the IGDX. The MMF test suites can examine the performance of the algorithms in various ways in terms of maintaining convergence and diversity. Fig. 6 shows the PSs obtained by six MMEAs on selected MMF test suites. Notably, for each MMF test problem with 2 objectives, there are two Pareto optimal solution sets, which can also be observed in

TABLE I
IGD RESULTS OF THE NINE ALGORITHMS ON MULTIMODAL PROBLEMS, WHERE THE BEST MEAN FOR EACH TEST INSTANCE IS SHOWN WITH A GRAY BACKGROUND

Problems	MMEA-WI	DN-NSGA-II	MO_PSO_MM	MO_Ring_PSO_SCD	Omni- optimizer	CPDEA	TriMOEA-TA&R	MOEA/D-DE	NSGA-II
MMF1	2.76E-3	4.56E-3 -	2.69E-3 +	3.69E-3 -	4.17E-3 -	2.62E-3 +	4.79E-3 -	3.73E-3 -	1.68E-3 +
MMF2	1.20E-2	2.64E-2 -	1.45E-2 ≈	2.02E-2 -	2.11E-2 -	1.22E-2 -	2.75E-2 ≈	8.44E-3 +	2.00E-2 -
MMF3	1.20E-2	2.31E-2 -	1.23E-2 ≈	1.44E-2 -	1.87E-2 +	1.25E-2 -	2.38E-2 -	5.44E-3 +	1.86E-2 ≈
MMF4	9.99E-4	3.21E-3 -	2.63E-3 -	3.65E-3 -	2.84E-3 -	2.55E-3 -	8.97E-3 -	3.56E-3 -	2.55E-3 -
MMF5	2.75E-3	3.72E-3 -	2.71E-3 +	3.69E-3 -	3.16E-3 -	2.60E-3 -	4.81E-3 -	5.05E-3 -	1.69E-3 +
MMF6	2.65E-3	3.63E-3 -	2.58E-3 +	2.55E-3 +	3.13E-3 -	2.53E-3 -	4.17E-3 -	4.54E-3 -	1.53E-3 +
MMF7	2.59E-3	3.90E-3 -	2.64E-3 ≈	3.81E-3 -	3.11E-3 -	2.57E-3 -	3.76E-3 ≈	3.46E-3 ≈	8.94E-4 +
MMF8	1.97E-3	3.83E-3 -	3.60E-3 -	4.77E-3 -	3.25E-3 -	3.15E-3 -	4.52E-3 -	5.56E-3 -	2.63E-3 ≈
IDMPM2T1	4.91E-4	6.40E-4 -	2.68E-3 -	3.12E-3 -	6.49E-4 ≈	6.95E-4 -	6.32E-4 -	4.69E-4 +	4.82E-4 +
IDMPM2T2	3.60E-4	5.83E-4 -	1.55E-3 -	1.53E-3 -	5.67E-4 -	5.32E-4 -	6.48E-4 -	4.03E-4 -	4.59E-4 ≈
IDMPM2T3	6.41E-4	6.01E-4 -	1.93E-3 -	1.67E-3 -	5.71E-4 ≈	6.91E-4 -	6.01E-3 -	6.28E-4 ≈	4.61E-4 +
IDMPM2T4	4.66E-4	3.42E-3 -	2.05E-3 -	3.29E-3 -	3.21E-3 -	5.96E-4 -	6.38E-4 -	4.20E-4 ≈	4.01E-4 +
IDMPM3T1	5.85E-3	8.95E-3 -	1.41E-2 -	1.39E-2 -	8.46E-3 -	5.64E-3 -	1.03E-2 -	6.74E-3 ≈	3.65E-3 +
IDMPM3T2	5.77E-3	8.85E-3 -	1.14E-2 -	1.12E-2 -	8.93E-3 -	5.26E-3 ≈	1.07E-2 -	6.90E-3 -	3.58E-3 +
IDMPM3T3	5.77E-3	7.81E-3 -	1.16E-2 -	1.10E-2 -	7.30E-3 -	5.57E-3 ≈	1.01E-2 -	7.32E-3 -	4.29E-3 +
IDMPM3T4	5.86E-3	2.49E-2 -	1.84E-2 -	1.94E-2 -	1.60E-2 -	5.54E-3 -	1.07E-2 -	6.71E-3 ≈	4.25E-3 +
IDMPM4T1	1.1E-2	2.22E-2 -	4.28E-2 -	3.33E-2 -	2.13E-2 -	8.17E-3 +	2.49E-2 -	7.92E-3 +	9.44E-3 +
IDMPM4T2	9.83E-3	3.82E-2 -	4.44E-2 -	2.58E-2 -	4.32E-2 -	7.37E-3 ≈	2.55E-2 -	6.28E-3 +	9.42E-3 +
IDMPM4T3	9.70E-3	2.58E-2 -	4.59E-2 -	2.90E-2 -	1.75E-2 -	7.47E-3 ≈	2.26E-2 -	6.31E-3 +	9.61E-3 +
IDMPM4T4	9.63E-3	3.00E-1 -	7.75E-2 -	4.25E-2 -	2.51E-1 -	7.74E-3 ≈	2.53E-2 -	5.78E-3 +	9.61E-3 ≈
Poly-3-10	2.41E-1	2.80E-1 -	2.55E-1 ≈	2.87E-1 ≈	2.79E-1 ≈	5.50E-1 -	5.77E-1 -	2.06E-1 +	4.14E-1 -
Poly-3-100	3.77E+0	4.45E+0 -	4.18E+0 -	4.13E+0 -	4.49E+0 -	4.84E+0 -	4.23E+0 ≈	4.05E+0 -	4.21E+0 ≈
Poly-5-10	3.72E-1	3.81E-1 -	4.43E-1 -	4.31E-1 -	4.21E-1 -	6.60E-1 -	8.62E-1 -	2.93E-1 +	5.14E-1 ≈
Poly-5-100	5.45E+0	6.45E+0 -	6.51E+0 -	5.61E+0 ≈	5.49E+0 ≈	6.19E+0 ≈	6.19E+0 -	4.90E+0 +	6.39E+0 -
Poly-10-10	5.66E-1	6.37E-1 -	5.84E-1 -	6.14E-1 ≈	6.62E-1 -	8.64E-1 -	1.24E+0 -	4.80E-1 +	6.89E-1 -
Poly-10-100	8.15E+0	8.31E+0 -	9.73E+0 -	9.64E+0 -	9.67E+0 -	8.58E+0 -	9.14E+0 ≈	6.05E+0 +	9.13E+0 ≈
+/-/=		0/24/2	6/11/4	1/22/3	1/19/6	8/12/6	0/21/4	13/6/7	15/4/7

Fig. 6. Almost all of the chosen algorithms can obtain two PS sets. For the MMF2 problem, MMEA-WI, MO_PSO_MM and MO_Ring_PSO_SCD obviously outperform the other algorithms. Although the MOEA/D-DE and NSGA-II are not designed for multimodal problems, they obtained better results than the omni-optimizer and DN-NSGA-II. On the other hand, all algorithms except the omni-optimizer and the MOEA/D-DE can effectively obtain the PS of the MMF4 problem. However, the result obtained by MMEA-WI has better convergence and diversity.

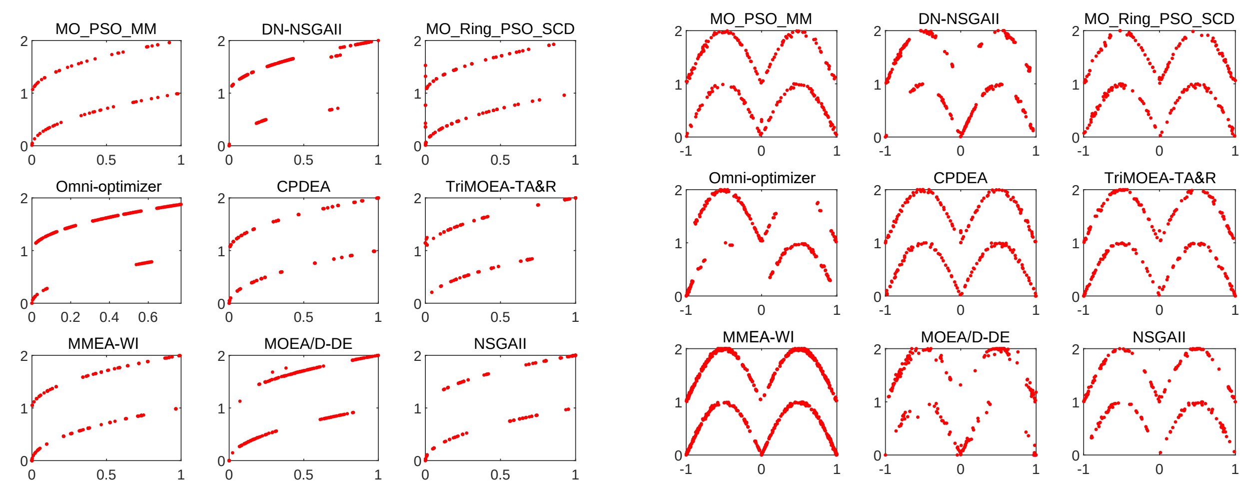
IDMP test suites are proposed to examine the ability of MMEAs to address the imbalance difficulty in the convergence process. They can more accurately reflect the performance of an MMEA. As we observe in Fig. 7, in terms of the IDMP test problems, MMEA-WI and CPDEA can identify all PS regions while maintaining the balance of each region. Notably, MO_PSO_MM and MO_Ring_PSO_SCD can identify different PS regions to some extent on the IDMP test suites. However, these two algorithms cannot always obtain equivalent PS sets; refer to Fig. 7. Further investigation of these results shows that for different regions, there is an obvious difference in the number of obtained solutions among different PSs. Moreover, other algorithms fail to obtain all the PS sets. Due to the lack of diversity-maintenance mechanisms, MOEA/D-DE and NSGA-II show a poor ability to preserve the equivalent PS regions. For problems with the same difficulty in searching different PSs, traditional MOEAs such as MOEA/D-DE and NSGA-II, can obtain a well-distributed PF, whereas the distribution of solutions in the decision space is slightly random. For problems with different difficulties in searching different Pareto optimal sets, traditional MOEAs will quickly converge to the PS, which is the easiest to search. In some circumstances, MOEAs will determine more than one PS. However, it is difficult for normal MOEAs to obtain all PSs.

In Section III-C, we introduce a convergence archive to better approximate the true PF. For the main evolving population archive, which is also known as the diversity archive, we use a weighted indicator to continuously explore the whole

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TEVC.2021.3078441, IEEE Transactions on Evolutionary Computation

TABLE II
IGDX RESULTS OF THE NINE ALGORITHMS ON MULTIMODAL PROBLEMS, WHERE THE BEST MEAN FOR EACH TEST INSTANCE IS SHOWN WITH A GRAY BACKGROUND

Problems	MMEA-WI	DN-NSGA-II	MO_PSO_MM	MO_Ring_PSO_SCD	Omni-optimizer	CPDEA	TriMOEA-TA&R	MOEA/D-DE	NSGA-II
MMF1	3.24E-2	9.47E-2 -	3.97E-2 -	4.89E-2 -	9.56E-2 -	3.73E-2 ≈	7.15E-2 -	1.01E-1 -	6.85E-2 ≈
MMF2	3.32E-2	1.07E-1 -	2.60E-2 +	4.22E-2 ≈	1.17E-1 -	2.85E-2 -	8.00E-2 -	7.63E-2 -	7.10E-2 -
MMF3	5.43E-2	8.82E-2 -	2.18E-2 +	2.87E-2 +	8.13E-2 -	4.43E-2 ≈	7.95E-2 -	8.00E-2 -	8.33E-2 -
MMF4	1.40E-2	8.77E-2 -	2.36E-2 -	2.78E-2 -	7.94E-2 ≈	1.94E-2 -	4.71E-2 -	8.60E-2 -	5.02E-2 -
MMF5	6.23E-2	1.74E-1 -	7.20E-2 -	8.42E-2 -	1.71E-1 -	6.52E-2 ≈	1.15E-1 -	1.44E-1 -	1.24E-1 -
MMF6	5.13E-2	1.39E-1 -	6.38E-2 -	7.29E-2 -	1.42E-1 -	5.71E-2 -	9.25E-2 -	1.12E-1 -	1.10E-1 -
MMF7	1.93E-2	5.46E-2 ≈	2.10E-2 ≈	2.66E-2 ≈	4.88E-2 ≈	1.12E-2 +	3.74E-2 -	4.96E-2 -	4.04E-2 -
MMF8	7.20E-2	3.52E-1 -	5.38E-2 -	6.66E-2 +	2.94E-1 -	5.20E-2 +	3.38E-1 -	2.37E-1 -	4.93E-1 -
IDMPM2T1	6.76E-4	2.32E-1 -	5.66E-3 -	1.21E-1 -	2.47E-1 -	1.13E-3 -	5.66E-1 -	6.52E-1 -	6.09E-1 -
IDMPM2T2	7.40E-4	3.36E-1 -	4.40E-3 -	8.00E-3 -	2.24E-1 -	9.63E-4 -	5.24E-1 -	6.52E-1 -	5.28E-1 -
IDMPM2T3	3.07E-3	7.11E-2 -	4.44E-3 ≈	3.32E-3 ≈	6.70E-2 -	2.34E-3 +	2.37E-1 -	5.42E-2 -	5.27E-1 -
IDMPM2T4	1.96E-1	5.49E-1 -	4.66E-3 +	8.25E-2 +	5.93E-1 -	2.25E-2 -	5.87E-1 -	6.73E-1 -	6.73E-≈ -
IDMPM3T1	5.81E-3	5.58E-1 -	3.20E-2 -	1.55E-1 -	5.38E-1 -	8.79E-3 -	6.78E-1 -	8.60E-1 -	8.03E-1 -
IDMPM3T2	1.45E-2	6.30E-1 -	8.75E-2 -	9.76E-2 -	7.40E-1 -	1.68E-2 -	6.55E-1 -	6.42E-1 -	7.55E-1 -
IDMPM3T3	2.49E-2	5.60E-1 -	3.86E-2 ≈	2.05E-2 +	4.65E-1 -	1.91E-2 +	5.21E-1 -	2.52E-1 -	6.33E-1 -
IDMPM3T4	2.75E-1	8.28E-1 -	1.71E-1 -	3.44E-1 ≈	8.79E-1 ≈	2.49E-2 +	8.77E-1 ≈	6.74E-1 -	8.28E-1 ≈
IDMPM4T1	5.20E-2	8.48E-1 -	5.66E-1 -	7.63E-1 -	8.99E-1 -	7.41E-1 -	1.18E+0 -	1.19E+0 -	1.18E+0 -
IDMPM4T2	6.79E-1	9.84E-1 ≈	5.75E-1 +	6.28E-1 +	1.04E+0 -	5.96E-1 -	1.09E+0 -	1.09E+0 -	1.11E+0 -
IDMPM4T3	5.66E-1	9.14E-1 -	2.21E-1 -	5.05E-2 +	8.17E-1 ≈	4.53E-1 ≈	9.06E-1 ≈	6.65E-1 ≈	1.02E+0 -
IDMPM4T4	9.30E-1	1.13E+0 -	6.53E-1 +	5.12E-1 +	1.11E+0 -	5.44E-1 -	1.09E+0 -	1.02E+0 -	1.09E+0 -
Poly-3-10	8.35E+0	9.48E+0 -	9.53E+0 -	9.51E+0 -	9.63E+0 -	8.65E+0 -	8.67E+0 -	8.45E+0 -	8.60E+0 -
Poly-3-100	2.82E+1	3.25E+1 -	2.84E+1 -	3.00E+1 -	3.27E+1 -	2.88E+1 -	2.88E+1 -	2.83E+1 -	2.88E+1 -
Poly-5-10	8.2E+0	9.21E+0 -	8.46E+0 -	9.54E+0 -	8.32E+0 -	8.28E+0 -	8.40E+0 -	8.17E+0 -	8.19E+0 -
Poly-5-100	2.78E+1	3.00E+1 -	2.81E+1 -	2.80E+1 -	3.05E+1 -	2.77E+1 -	2.80E+1 -	2.76E+1 -	2.76E+1 -
Poly-10-10	7.91E+0	8.95E+0 -	8.07E+0 -	8.61E+0 -	8.62E+0 -	8.26E+0 -	8.58E+0 -	8.18E+0 -	8.16E+0 -
Poly-10-100	2.75E+1	2.84E+1 -	3.20E+1 -	2.96E+1 -	3.10E+1 -	2.78E+1 -	2.81E+1 -	2.77E+1 -	2.76E+1 -
+/-/=		0/23/3	5/16/5	7/13/6	0/20/6	6/14/6	1/21/3	0/23/3	0/22/4



(a) MMF2 (b) MMF4

Fig. 6. PSs obtained by MOEAs on MMF problems.

TABLE III
PSP RESULTS OF THE NINE ALGORITHMS ON MULTIMODAL PROBLEMS, WHERE THE BEST MEAN FOR EACH TEST INSTANCE IS SHOWN WITH A GRAY BACKGROUND

Problems	MMEA-WI	DN-NSGA-II	MO_PSO_MM	MO_Ring_PSO_SCD	Omni-optimizer	CPDEA	TriMOEA-TA&R	MOEA/D-DE	NSGA-II
MMF1	3.09E+1	1.05E+1 -	2.51E+1 -	2.03E+1 -	1.05E+1 -	2.68E+1 -	1.39E+1 -	1.05E+1 -	1.47E+1 -
MMF2	3.14E+1	9.58E+0 -	3.79E+1 +	2.52E+1 ≈	9.76E+0 -	3.65E+1 +	1.39E+1 -	1.76E+1 -	1.49E+1 -
MMF3	1.89E+1	1.17E+1 ≈	4.51E+1 +	3.59E+1 -	1.42E+1 -	2.28E+1 ≈	1.24E+1 ≈	1.42E+1 ≈	1.24E+1 ≈
MMF4	7.29E+1	1.17E+1 -	4.22E+1 -	3.58E+1 -	1.32E+1 -	5.16E+1 -	2.89E+1 -	1.22E+1 -	2.05E+1 ≈
MMF5	1.61E+1	5.71E+0 -	1.38E+1 -	1.18E+1 ≈	5.78E+0 -	1.53E+1 ≈	8.72E+0 -	6.95E+0 -	8.09E+0 -
MMF6	1.96E+1	7.15E+0 -	1.56E+1 -	1.36E+1 -	7.02E+0 -	1.75E+1 -	1.08E+1 ≈	8.94E+0 -	9.27E+0 -
MMF7	9.10E+1	1.83E+1 -	4.74E+1 -	3.75E+1 -	2.07E+1 -	5.17E+1 -	2.62E+1 ≈	1.94E+1 -	2.51E+1 -
MMF8	1.46E+1	3.18E+0 -	1.85E+1 +	1.50E+1 ≈	3.87E+0 -	1.92E+1 +	3.05E+0 -	4.27E+0 -	1.91E+0 -
IDMPM2T1	1.51E+3	8.00E+1 -	2.28E+2 -	7.49E+1 -	1.46E+2 -	1.15E+3 -	6.77E+1 -	1.17E+0 -	3.61E+1 ≈
IDMPM2T2	1.39E+3	9.32E+1 -	3.12E+2 -	1.51E+2 -	2.44E+2 -	1.05E+3 -	4.70E+1 -	4.28E+0 -	1.04E+1 -
IDMPM2T3	4.56E+2	4.60E+2 ≈	2.32E+2 -	3.03E+2 ≈	7.29E+2 +	3.47E+2 -	6.75E+1 -	1.27E+2 ≈	3.12E+1 -
IDMPM2T4	1.28E+3	1.07E+1 -	2.90E+2 -	6.78E+1 -	3.87E+0 -	1.22E+3 ≈	1.14E+2 -	1.33E-2 -	9.66E-3 -
IDMPM3T1	1.73E+2	1.59E+0 -	4.09E+1 -	2.06E+1 -	4.43E+0 -	1.14E+2 -	4.30E+0 -	2.73E-1 -	2.49E+0 -
IDMPM3T2	1.09E+2	9.60E-1 -	3.19E+1 -	2.78E+1 -	7.61E-1 -	1.50E+2 +	1.45E+0 -	7.86E-1 -	6.04E-1 -
IDMPM3T3	8.58E+1	3.90E+0 -	4.28E+1 -	4.97E+1 -	5.57E+0 -	1.02E+2 +	1.85E+0 -	1.42E+1 -	1.23E+0 -
IDMPM3T4	3.33E+1	4.65E-1 -	1.54E+1 ≈	7.03E+0 -	3.08E-1 -	1.02E+2 +	2.58E-1 -	1.98E+0 -	1.54E+0 -
IDMPM4T1	9.53E+1	6.77E-1 -	2.15E+0 -	8.52E-1 ≈	4.34E-1 -	9.49E-1 -	2.32E-2 -	1.25E-2 -	1.09E-2 -
IDMPM4T2	1.57E+0	2.29E-1 -	1.89E+0 ≈	1.62E+0 +	1.21E-1 -	4.28E+0 +	9.11E-2 -	1.08E-1 -	3.08E-2 -
IDMPM4T3	9.29E+0	4.95E-1 -	9.00E+0 ≈	2.33E+1 +	5.94E-1 -	4.23E+0 -	3.94E-1 -	8.90E-1 -	1.03E-1 -
IDMPM4T4	4.65E-1	3.33E-2 -	2.14E+0 -	2.84E+0 -	9.27E-2 -	4.20E+0 +	5.56E-2 -	2.10E-1 -	1.22E-2 -
Poly-3-10	2.92E-2	2.30E-2 -	2.82E-2 ≈	2.26E-2 -	2.44E-2 -	2.21E-2 ≈	1.70E-2 -	2.74E-2 -	2.15E-2 ≈
Poly-3-100	6.60E-3	5.10E-3 -	5.61E-3 -	6.09E-3 -	6.32E-3 -	3.06E-3 -	1.03E-3 -	4.37E-3 -	3.19E-3 -
Poly-5-10	3.53E-2	3.24E-2 ≈	2.52E-2 -	3.00E-2 ≈	3.38E-2 ≈	3.09E-2 -	2.32E-2 ≈	3.26E-2 -	2.93E-2 -
Poly-5-100	7.12E-3	5.67E-3 -	6.40E-3 ≈	5.63E-3 -	6.57E-3 ≈	5.30E-3 -	3.35E-3 ≈	5.58E-3 -	4.33E-3 -
Poly-10-10	3.80E-2	3.05E-2 -	3.13E-2 ≈	2.78E-2 -	2.84E-2 -	3.27E-2 ≈	2.24E-2 -	3.40E-2 -	3.22E-2 -
Poly-10-100	7.57E-3	7.21E-3 -	7.07E-3 -	5.39E-3 -	7.00E-3 ≈	7.63E-3 -	3.51E-3 -	5.63E-3 -	4.41E-3 -
+/-/=		0/23/3	3/17/6	2/18/6	2/21/3	8/13/5	1/20/5	0/24/2	0/22/4

decision space. Thus, MMEA-WI can sufficiently balance convergence and diversity. We illustrate the updating process of the convergence archive and the diversity archive. The main purpose of the diversity archive is to keep the algorithm continuously searching for potential PS regions. Thus, the population in this archive will not directly converge but will continue circling the true PS. Fig. 8 shows the final population distributions of the diversity archive obtained by MMEA-WI for chosen benchmark problems. It is worth mentioning that the multipolygon problem has multiple decision variables, which is hard to present with figures. Thus, we adopt the normal polygon problem with 2 decision variables to better present the distribution of solutions in the decision space, which is known as Polygon-3-2.

As we can observe in Fig. 8, solutions in the diversity archive will not move directly to the true PS, although their fitness or Pareto ranking is poor. Compared to the $I_{\epsilon+}$ indicator, which only evaluates the convergence and spread quality of an individual, the proposed weighted indicator considers the crowding distance in the decision space. Thus, if all individuals move toward the same region, they will incur a penalty. As a result, the proposed MMEA-WI has the ability to maintain solution diversity in the decision space and convergence in the objective space.

In Section III-E, we theoretically analyze the computational complexity of MMEA-WI. Fig. 9 presents the runtime of several algorithms on multipolygon problems with a different number of objectives. We note that the runtime increases as the number of objectives increases. As we mentioned in Section III-E, the computational complexity of MMEA-WI is slightly worse than that of traditional MOEAs. In this comparison, we can determine that the runtime of MMEA-WI on multipolygon problems is very similar to that of MOEA/D-DE, which means that MMEA-WI is an efficient MOEA. It is worth mentioning that the runtime of CPDEA on the chosen benchmark problems is apparently larger than that of the other algorithms because CPDEA adopts the steady-state strategy, which employs a single individual to perform the variation

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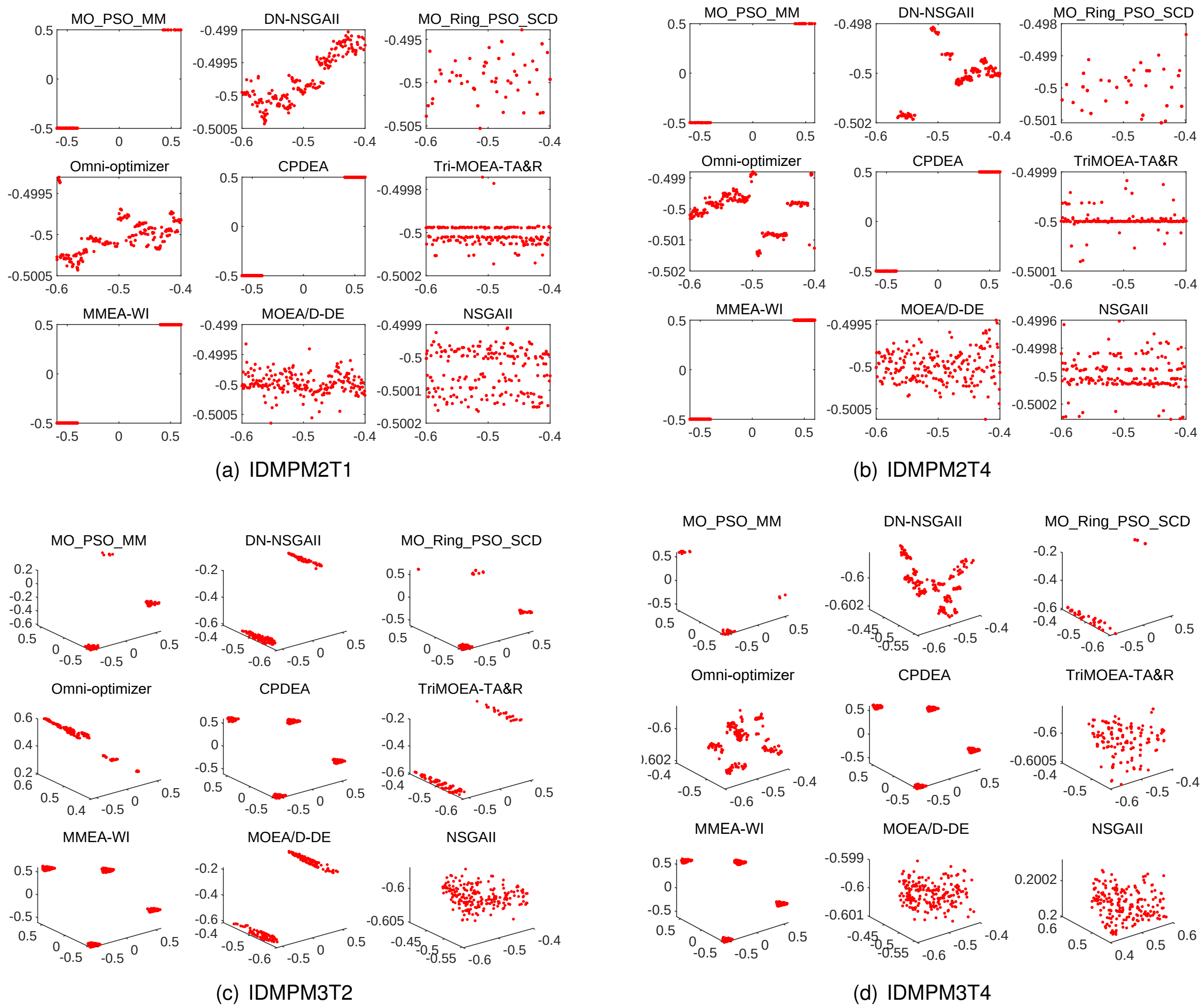

Fig. 7. PSs obtained by MOEAs on IDMP benchmark problems.

and selection operations on each generation. This strategy is generally time consuming.

C. The Effects of Probability p

We introduce a user-predefined parameter p in Section III-D to set a probability that selects parents from the convergence archive. As we expect, a larger p means that there is a higher probability of selecting parents from the convergence archive, resulting in better convergence. However, this result also leads to less diversity in the decision space. Intuitively, we need to consider both convergence and solution diversity to obtain better performance for an MMEA. Setting the exact value for this parameter is currently an open issue. To examine the effect and proper value of parameter p, we vary p between 0 and 1. All parameters are equal to those of Section IV-C, except parameter p. The results are obtained from TABLE IV and Fig. 10, which show the average rank of the IGDX, IGD and PSP on 26 selected test problems.

Fig. 10 shows that a larger value of parameter p will yield better IGDX and PSP results, while the IGD values degenerate, and vice versa. As mentioned previously, parameter p is introduced to set the probability of selecting parents from the convergence archive. Thus, as p increases, it is more likely that parents will be chosen from the convergence archive, resulting in a better IGD value. However, this operation will also lead to poor solution diversity in the decision space since the parents mostly originate from the convergence archive. In particular, when $p = 1$, MMEA-WI performs similarly to traditional IBEA. Notably, there is a great decrease in the average rank of IGDX and PSP when $p > 0.4$. The IGDX and PSP indicators, which evaluate the performance of MMEAs, are insensitive when $p \in [0, 0.4]$. When $p > 0.4$, the performance of the MMEA-WI degenerates rapidly as p increases. As a result, we recommend setting $p = 0.4$ to obtain a better trade-off between convergence and diversity.

VI. Conclusion

Multimodal multi-objective evolutionary algorithms aim to obtain all equivalent Pareto solutions for MMOPs. Although the importance of MMEAs has been emphasized in many studies, they have not attracted enough attention compared to other types of algorithms in the evolutionary multi-objective optimization community.

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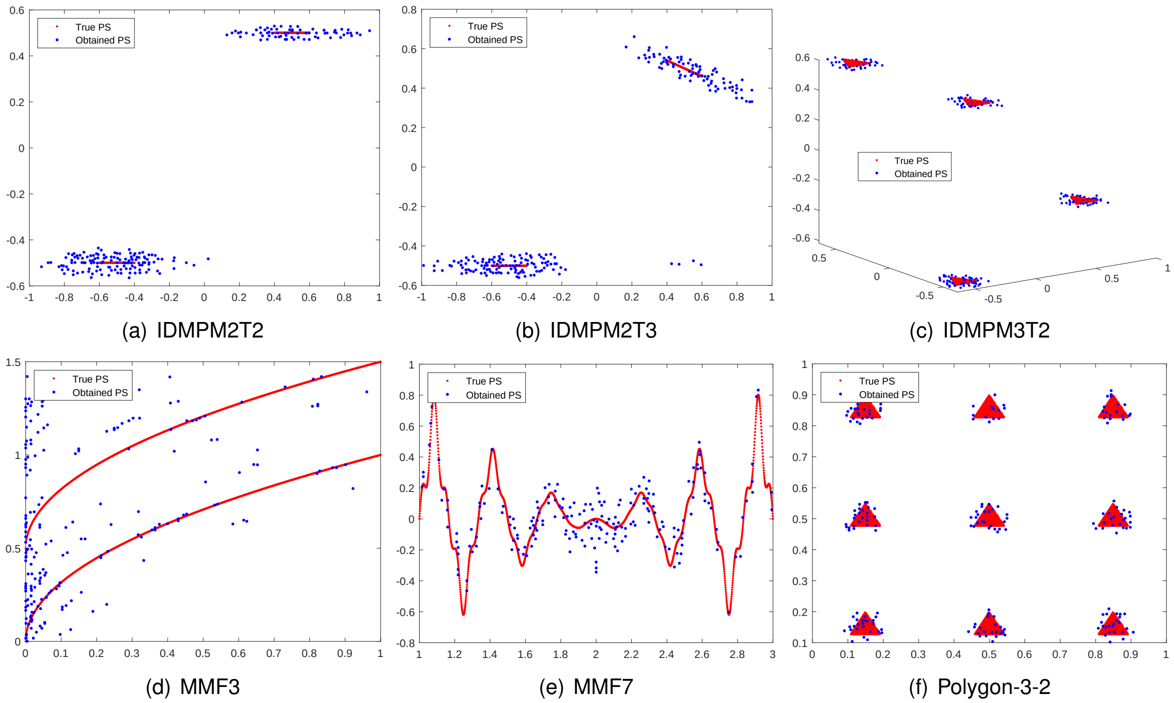

Fig. 8. Distribution in the decision space of the archive obtained by the MMEA-WI.

TABLE IV
AVERAGE RANK OF THE IGDX, IGD AND PSP WITH DIFFERENT VALUES OF p OBTAINED BY THE MMEA-WI

p	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
PSP	5.32	5.28	4.48	4.72	4.40	5.32	6.56	6.12	7.28	8.00	8.52
IGD	7.84	7.72	7.36	6.88	5.84	5.60	5.40	4.92	5.28	4.80	4.36
IGDX	4.04	4.36	4.04	4.68	4.56	6.00	6.92	6.68	7.76	8.04	8.92

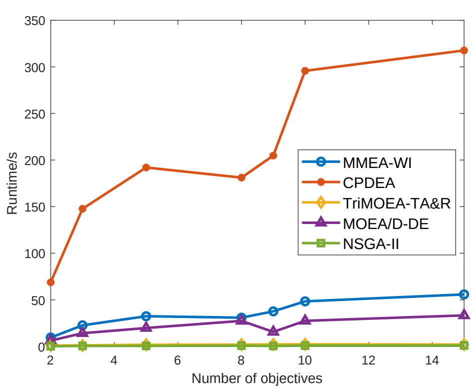

Fig. 9. Runtime(s) of the five algorithms on multipolygon problems, where the runtime of an algorithm on M objectives is obtained by averaging 31 algorithm runs.

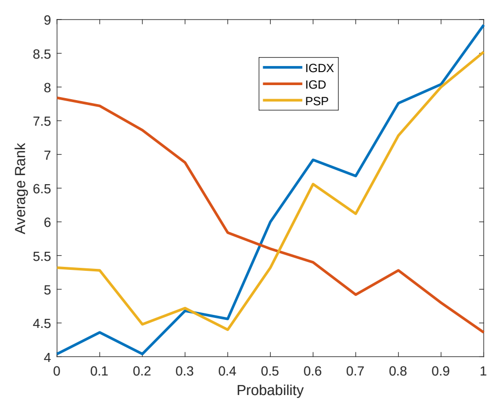

Fig. 10. Average rank of IGDX, IGD and PSP with different values of p obtained by MMEA-WI.

In this study, related studies on MMOPs were reviewed. We proposed a weighted indicator-based evolutionary algorithm for multimodal problems, termed MMEA-WI. The weighted method for constructing an indicator can be extended to other indicator-based algorithms. Specifically, the weighted method can measure both the convergence and the diversity of a population. Thus, other equivalent Pareto solutions are preserved during the evolution. To examine the performance of MMEA-WI, extensive experiments were performed. The experimental results showed that MMEA-WI can obtain all equivalent solutions with high robustness. In addition, compared to other multimodal algorithms, MMEA-WI can achieve better convergence. The effects of the parameters proposed in our algorithms were also discussed.

The proposed MMEAs have aimed to obtain equivalent Pareto solution sets that have the same optimal objective values. However, for most real-world problems, the objective values for different PSs may not be exactly equivalent. In general, two solutions that are located far from each other in the decision space may have similar objective values. A solution with slightly worse objective values also needs to be considered a potential solution. Unfortunately, no work has aimed to obtain PSs under these circumstances, which deserves further study.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China (61773390, 7207010433), the Hunan Youth elite program (2018RS3081), the scientific key research project of National University of Defense Technology (ZK18-02-09, ZZKY-ZX-11-04) and the key project of 193-A11-101-03-01.

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This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TEVC.2021.3078441, IEEE Transactions on Evolutionary Computation

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