ECSE 597: Circuit Simulations and Modeling

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1 Question 1

1.a Step Size of Forward Euler

To use forward Euler, if the pole λ is real, we use the formula

$$|1 + h\lambda| < 1 \quad \rightarrow \quad -2 < h\lambda < 0 \tag{1}$$

to find the appropriate size of each step, leading to the following equations:

- $\lambda = -5$, (1) $\rightarrow 0 < h < \frac{2}{5}$
- $\lambda = -3$, (1) $\rightarrow 0 < h < \frac{2}{3}$
- $\lambda = -2$, (1) $\rightarrow 0 < h < 1$
- $\lambda = -7$, (1) $\rightarrow 0 < h < \frac{2}{7}$
- $\lambda = -8$, (1) $\rightarrow 0 < h < \frac{1}{4}$

Choose the smallest value here, and $(0, \frac{1}{4})$ is an appropriate interval of h.

1.b Step Size of Backward Euler

Due to the stability of Backward Euler method, the system is guaranteed to be stable as long as $\mathrm{Re}\{\lambda\} < 0$. Since all poles given in the question have negative real values, the Backward Euler system is stable regardless of the step size.

1.c Trapezoidal Rule

The system solved by Trapezoidal Rule is stable when

$$Re\{h\lambda\} < 0$$

Since all poles given by the question are negative, this system with step size of $h > 0$ is guaranteed to be stable.

2 Question 2

Use the Adam-Moulton integration method shown in Equation 2 to derive a difference equation of MNA.

$$x_n = x_{n-1} + \frac{5h}{12}\dot{x}_n + \frac{8h}{12}\dot{x}_{n-1} + \frac{h}{12}\dot{x}_{n-2} \tag{2}$$

From Equation 2, change the form:

$$5\dot{x}_n + 8\dot{x}_{n-1} + \dot{x}_{n-2} = \frac{12}{h}(x_n - x_{n-1}) \tag{3}$$

From the MNA equation

$$\boldsymbol{G}\boldsymbol{x}_n + \boldsymbol{C}\dot{\boldsymbol{x}}_n + \boldsymbol{f}(\boldsymbol{x}_n) = \boldsymbol{b}_n$$

for every $\boldsymbol{x}_n$. Replace n by $n-1$ and $n-2$, and integrate the terms of $\boldsymbol{G}$, $\boldsymbol{C}$, and $\boldsymbol{b}$:

$$\boldsymbol{G}(\boldsymbol{x_n} + \boldsymbol{x_{n-1}} + \boldsymbol{x_{n-2}}) + \boldsymbol{C}(\dot{\boldsymbol{x}}_{\boldsymbol{n}} + \dot{\boldsymbol{x}}_{\boldsymbol{n-1}} + \dot{\boldsymbol{x}}_{\boldsymbol{n-2}}) + \boldsymbol{F} = \boldsymbol{b}_n + \boldsymbol{b}_{n-1} + \boldsymbol{b}_{n-2} \tag{4}$$

where

$$\boldsymbol{F} = \boldsymbol{f}(\boldsymbol{x}_n) + \boldsymbol{f}(\boldsymbol{x}_{n-1}) + \boldsymbol{f}(\boldsymbol{x}_{n-2})$$

Plug Equation 3 into 4, and we can arrive at:

$$\boldsymbol{G}(\boldsymbol{x_n} + \boldsymbol{x_{n-1}} + \boldsymbol{x_{n-2}}) + \frac{12C}{h}(\boldsymbol{x}_n + \boldsymbol{x}_{n-1}) + \boldsymbol{F} = 5\boldsymbol{b}_n + 8\boldsymbol{b}_{n-1} + \boldsymbol{b}_{n-2}$$

$$(5\boldsymbol{G} + \frac{12\boldsymbol{C}}{h})\boldsymbol{x}_n + (8\boldsymbol{G} - \frac{12\boldsymbol{C}}{h})\boldsymbol{x}_{n-1} + \boldsymbol{G}\boldsymbol{x}_{n-2} + \boldsymbol{F} = 5\boldsymbol{b}_n + 8\boldsymbol{b}_{n-1} + \boldsymbol{b}_{n-2}$$

which is the solution difference equation for this question.

3 Question 3

For simplicity, use LU to denote the LU decomposition and F/B to denote the forward / backward substitution. Table below shows the computational cost of each method, with steps followed.

	Sparse Ordering	LU	F/B
Perturbation	1	4000	4000
Differentiation	1	1000	4000
Adjoint	1	1000	6000

3.a Perturbation Method

- 1 sparse ordering for the entire process;
- 1 LU and 1 F/B to find the frequency response;

- According to the equation

$$(A + \Delta A)(x + \Delta x) = b$$

  A is changing for each sensitivity test, therefore, 1 additional LU and F/B is required to find the sensitivity with respect to every additional parameter.

Therefore, the Perturbation method will need 1 sparse ordering. 1000 frequency points result in 1000 LU and F/B, and the three sensitivity parameters lead to an additional 3000 LU and F/B calculations.

3.b Differentiation Method

- 1 sparse ordering for the entire process;
- Matrix $\boldsymbol{A}$ is reused at each frequency point, thus only 1 LU is needed. But the sensitivity with respect to the frequency requires an LU and F/B at each frequency points.
- An additional F/B is needed to compute sensitivity with respect to each parameter.

Therefore, the Differentiation method requires 1 sparse ordering, 1000 LU and F/B for overall sensitivity and frequency response, and an additional 3000 F/B for each sensitivity parameter.

3.c Adjoint Method

- 1 sparse ordering for the entire process;
- 1 LU and 1 F/B for sensitivity at each frequency point;
- 1 additional F/B for sensitivity analysis with respect to all output nodes.

Therefore, the Adjoint method requires 1 sparse ordering, 1000 LU and F/B for overall sensitivity and frequency response, and an additional 5000 F/B for 5 output nodes.