

ECSE 543: Numerical Methods
Assignment 3 Report

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Introduction

This assignment explored the use of linear interpolations and other mathematical methods. The programs are programmed and compiled using Python 3.6, and the plots are generated using package matplotlib. Listing 1 shows the implementations of polynomials including their possible maneuvers. The object classes included in this file will be used for the interpolations.

1 Linear Interpolation of BH Points

1.a Lagrange Full Domain Interpolation of First Six-Point Set

Listing 2 shows the implementation of various interpolation methods. For the first six points, the Lagrange interpolation shows an interpolated polynomial

$$B(h) = 9.275 \times 10^{-12}h^5 - 5.951 \times 10^{-9}h^4 + 1.469 \times 10^{-6}h^3 - 1.849 \times 10^{-4}h^2 + 1.603 \times 10^{-2}h$$

whose plot is shown in Figure 1.

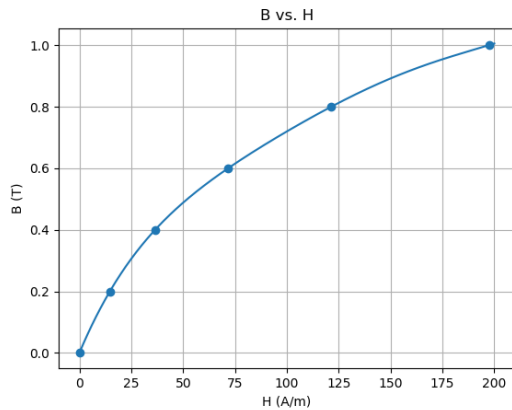


Figure 1: Interpolation of the First Six Data Points

From the figure, the interpolation has returned a plot with a **plausible** result over this range.

1.b Lagrange Full Domain Interpolation of the Second Six-Point Set

Select a second data point set, the Lagrange interpolation returned a polynomial of

$$B(h) = 7.467 \times 10^{-19}h^5 - 3.505 \times 10^{-14}h^4 + 5.3 \times 10^{-10}h^3 - 2.864 \times 10^{-6}h^2 + 3.804 \times 10^{-3}h$$

whose plot is shown in Figure 2.

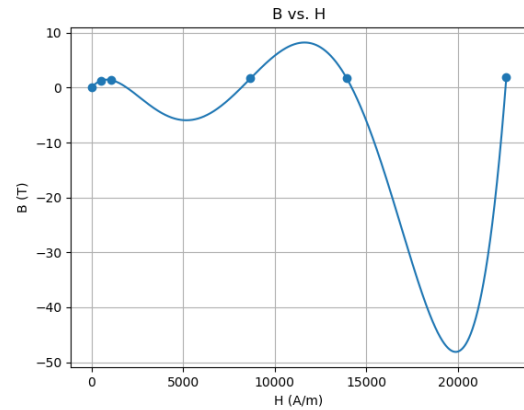


Figure 2: Interpolation of the Second Six Data Points

From this plot, we can see that the interpolation using the second set of data points is **not plausible** as the graph fluctuates violently as the value of B goes to negative at some ranges.

1.c Cubit Hermite Polynomial Interpolation

1.d Nonlinear Equation of the Magnetic Circuit

Consider the magnetic circuit shown in Figure 3.

The Magnetomotive force (MMF) can be calculated by Equation 1,

$$M = (R_g + R_c)\psi \quad (1)$$

where R_g and R_c are the reluctance of the air gap and the coil, respectively. Plug in the variables from the problem, we can transform Equation 1 to

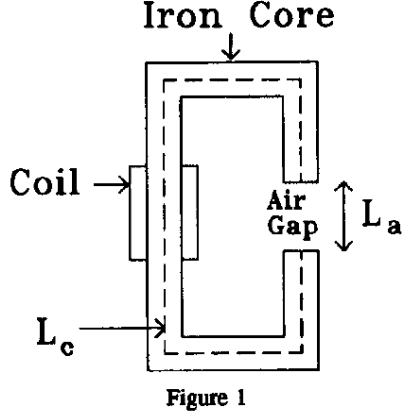


Figure 1

Figure 3: The Magnetic Circuit Discussed About

the equation as follows:

$$M = \left(\frac{l_g}{\mu_0 A} + \frac{l_c}{\mu A} \right) \psi$$

$$NI = \left(\frac{l_g}{\mu_0 A} + \frac{l_c H(\psi)}{AB} \right) \psi$$

$$NI = \left(\frac{l_g}{\mu_0 A} + \frac{l_c H(\psi)}{\psi} \right) \psi$$

Simplify the equation by bringing NI to the right of the equation, and the equation will be the final formula of $f(\psi)$, as is shown in Equation

$$f(\psi) = \frac{l_g \psi}{\mu_0 A} + l_c H(\psi) - NI = 0 \quad (2)$$

Plug in the numbers, we can finalize the equation by calculating all the coefficients of the polynomial, shown in Equation 3.

$$f(\psi) = 3.979 \times 10^7 \psi + 0.3H(\psi) - 8000 \quad (3)$$

1.e Newton Raphson Method

This part of the problem implements the algorithm of Newton Raphson to solve the non-linear equation. The equation is shown in the previous section in Equation 3.

In the equation, there are two factors affecting the result of $f(\psi)$. One is the flux ψ , and the other one is the magnitude of the magnetic field $H(\psi)$. To find the magnetic field, construct a piece-wise linear interpolation shown in Figure 4.

Note that the figure is plotted with respect to H vs. B . and B is calculated as follows:

$$B = \frac{\psi}{A} \quad (4)$$

where A denotes the cross-sectional area. In this case, the area is $1 \times 10^{-4} m^2$.

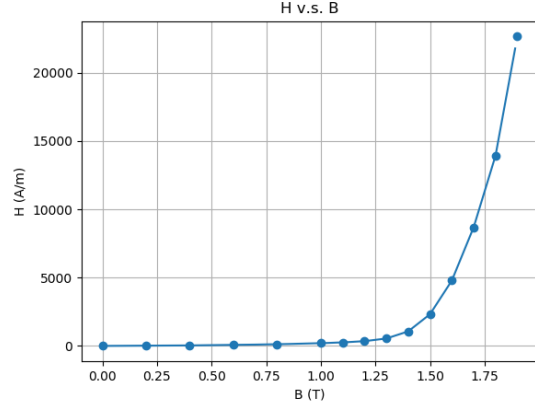


Figure 4: Plot of the Piecewise Polynomial

Using this plot, the magnetic field magnitude can be found and $f(\psi)$ can be calculated.

Listing 3 shows the implementation of the Newton-Raphson method. Run the main script of the assignment, Newton-Raphson returns with four iterations and a final flux of $\psi = 1.613 \times 10^{-4} Wb$, shown in Figure 5.

```
===== Q1, Part e =====
Printing the piecewise polynomials...
Plot of the polynomial has been stored to /home/wenjie/f18/numerical_method/a3/data/Piecewise_Polynomial
number of iterations = 4, final flux = 0.00016127
```

Figure 5: Result of Newton-Raphson Run

1.f Successive Substitution

Listing 3 shows the implementation of successive substitution as well. The successive substitution turns out to be that the method is diverging to infinity. The reason is that the step has been too large. Therefore, I have reduced the step with a factor of 5×10^{-9} . Therefore, the method will run with smaller steps and does not miss the target point.

After the modification, the method returns with an iteration step of 483 and a flux of $1.161 \times 10^{-4} Wb$, which is similar to the result returned by Newton-Raphson, but with a much larger number of iterations, shown in Figure 6.

```
number of iterations = 4, final flux = 0.00016127
===== Q1, Part f =====
number of iterations = 483, final flux = 0.00016127
```

Figure 6: Result of Successive Substitution Run

2 The Problem of the Diode Circuit

2.a Derivation of Circuit Equation

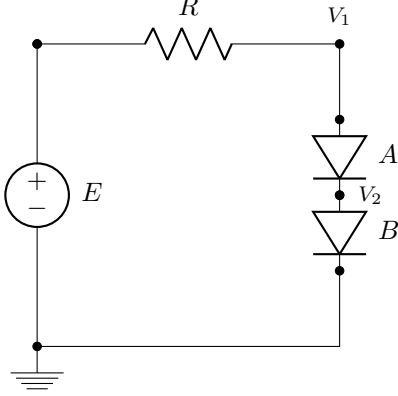


Figure 7: The Diode Circuit to be Investigated

Figure 7 shows the diode circuit to be investigated in this problem. Define the current flowing in the circuit to be I , and the current is expressed with:

$$I = \frac{E - V_1}{R} \quad (5)$$

In the circuit, the current flowing through the two diodes are identical to the current flowing through the resistor. Therefore, using the diode characteristic current, the following relations can be derived:

$$I = I_{s,A}(e^{q(V_1-V_2)/(kT)} - 1) \quad (6)$$

and

$$I = I_{s,B}(e^{qV_2/(kT)} - 1) \quad (7)$$

From the above equations, we can derive the following two entries for the \mathbf{f} matrix, represented explicitly in terms of the variables:

$$\begin{aligned} f_1 &= (5) - (6) \\ &= \frac{E - V_1}{R} - I_{s,A}(e^{q(V_1-V_2)/(kT)} - 1) \end{aligned}$$

$$\begin{aligned} f_2 &= (6) - (7) \\ &= I_{s,A}(e^{q(V_1-V_2)/(kT)} - 1) - I_{s,B}(e^{qV_2/(kT)} - 1) \end{aligned}$$

The \mathbf{f} matrix is then expressed as follows:

$$\vec{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2.b Solution using Newton-Raphson

Since this equation has output a multi-variable vector, the first step will be finding the Jacobian matrix \mathbf{F} and the multi-variable Newton Raphson update formula will be changed to:

$$\mathbf{V}_n^{(k+1)} = \mathbf{V}_n^{(k)} - \mathbf{F}^{-1(k)} \mathbf{f}^{(k)}$$

The Jacobian matrix will be calculated following Equation as follows:

$$\mathbf{F} = \begin{bmatrix} \frac{\partial f_1}{\partial V_1} & \frac{\partial f_1}{\partial V_2} \\ \frac{\partial f_2}{\partial V_1} & \frac{\partial f_2}{\partial V_2} \end{bmatrix} \quad (8)$$

From the previous calculations for f_1 and f_2 , we can derive the following expression for the four entries in the \mathbf{F} matrix:

$$\frac{\partial f_1}{\partial V_1} = -\frac{1}{R} - I_{s,A} \frac{q}{kT} \exp\left(\frac{q(V_1 - V_2)}{kT}\right)$$

$$\frac{\partial f_1}{\partial V_2} = I_{s,A} \frac{q}{kT} \exp\left(\frac{q(V_1 - V_2)}{kT}\right)$$

$$\frac{\partial f_2}{\partial V_1} = I_{s,A} \frac{q}{kT} \exp\left[\frac{q(V_1 - V_2)}{kT}\right]$$

$$\frac{\partial f_2}{\partial V_2} = -I_{s,A} \frac{q}{kT} \exp\left[\frac{q(V_1 - V_2)}{kT}\right] - I_{s,B} \frac{q}{kT} \exp\left(\frac{qV_2}{kT}\right)$$

As the Jacobian matrix is a 2-by-2 matrix, its inverse can be easily calculated by:

$$\mathbf{F}^{-1} = \det(\mathbf{F}) \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (9)$$

where $\det(\mathbf{F})$ is calculated by:

$$\det(\mathbf{F}) = \frac{1}{ad - bc}$$

The code in the main script shows the implementation of the Newton Raphson update. The error measurement is selected to be $\varepsilon_k = 1 \times 10^{-6}$, and the program three iterations to converge. By running the main script, the detailed information during the iterations are shown in Figure 8.

```

===== Q2, Part b =====
k   V_1      V_2      f_1      f_2      err
0  0.00000000 0.00000000 0.00039063 0.00000000 1.00000000
1  0.19812073 0.08341926 -0.00007417 0.00004800 0.05115547
2  0.18413995 0.08433690 -0.00001156 0.00001154 0.00174811
3  0.18230749 0.08710807 -0.00000069 0.00000049 0.00000469

```

Figure 8: Values During Newton Raphson Iterations

To inspect if the convergence is quadratic, make a plot of the four error points shown in Figure 9.

From the plot, we can show the convergence to be quadratic.

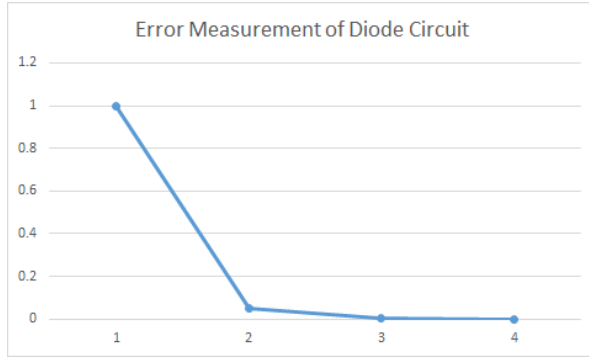


Figure 9: Plot of Error Measurement

3 Gauss-Legendre Integration

The implementation of the one-point Gauss-Legendre Integration is shown in Listing 4.

The implementation of the algorithm is straight forward. The function requires the target function, the lower and upper limit, and number of rectangular segments to be used.

3.a One-Point Gauss-Legendre Integration of $\sin(x)$

A Code Listings

Listing 1: Polynomials Implementation (polynomial.py).

```
1  import math
2
3
4  class Polynomial(object):
5      def __init__(self, coeff):
6          self._coeff = coeff
7          self._order = len(coeff) - 1
8
9      def calculate(self, value):
10         """
11         This function calculates the result of the polynomial.
12
13         :param value: value of x
14         :return: value of y
15         """
16         result = 0
17         for i in range(len(self._coeff)):
18             result += self._coeff[i] * math.pow(value, i)
19
20         return result
21
22     def derive(self, der_order):
23         result_coeff = []
24         counter = 0
25
26         for i in range(1, len(self._coeff)):
27             result_coeff.append(i * self[i])
28         result_poly = Polynomial(result_coeff)
29         counter += 1
30
31         if counter < der_order:
32             return result_poly.derive(der_order - 1)
33         else:
34             return result_poly
35
36     def __getitem__(self, item):
37         return self._coeff[item]
38
39     def __add__(self, other):
40         result_coeff = []
41
42         if isinstance(other, int):
43             result_coeff = self._coeff
44             result_coeff[0] += other
45         else:
46             self_has_higher_order = (max(self.order, other.order) == self.order)
47
48             if self_has_higher_order:
49                 big_coeff = self.coefficient
50                 small_coeff = other.coefficient
51             else:
52                 big_coeff = other.coefficient
53                 small_coeff = self.coefficient
54
55             for i in range(len(small_coeff), len(big_coeff)):
56                 small_coeff.append(0)
57
58             for i in range(len(big_coeff)):
59                 result_coeff.append(small_coeff[i] + big_coeff[i])
60
61             return Polynomial(result_coeff)
62
63     def __sub__(self, other):
64         result_coeff = []
65         if isinstance(other, int):
```

```

66         result_coeff = self._coeff
67         result_coeff[0] -= other
68
69     else:
70         self_has_higher_order = (max(self.order, other.order) == self.order)
71
72         if self_has_higher_order:
73             for i in range(len(other.coefficient), len(self.coefficient)):
74                 other.coefficient.append(0)
75         else:
76             for i in range(len(self.coefficient), len(other.coefficient)):
77                 self.coefficient.append(0)
78
79         for i in range(len(self.coefficient)):
80             result_coeff.append(self.coefficient[i] - other.coefficient[i])
81
82     return Polynomial(result_coeff)
83
84 def __mul__(self, other):
85     result_coefficients = []
86
87     result_order = self.order + other.order
88
89     for i in range(result_order + 1):
90         coefficient = 0
91         for j in range(self.order + 1):
92             for k in range(other.order + 1):
93                 if j + k == i:
94                     coefficient += self[j] * other[k]
95
96         result_coefficients.append(coefficient)
97
98     return Polynomial(result_coefficients)
99
100 def toString(self):
101     print("y = ", end="")
102     for i in range(self.order, 0, -1):
103         if self[i] != 1 and self[i] != -1 and self[i] != 0:
104             if self[i] >= 0:
105                 print("+ " + str(self[i]) + "x^" + str(i), end=" ")
106             else:
107                 print("- " + str(-self[i]) + "x^" + str(i), end=" ")
108         elif self[i] == 1:
109             print("+ x^" + str(i), end=" ")
110         elif self[i] == -1:
111             print("- x^" + str(i), end=" ")
112         else:
113             pass
114
115     if self[0] < 0:
116         print("- " + str(-self[0]))
117     else:
118         print("+ " + str(self[0]))
119
120 def modify_const(self, value):
121     self._coeff[0] = value
122
123 @property
124 def order(self):
125     return self._order
126
127 @property
128 def coefficient(self):
129     return self._coeff
130
131
132 class LagrangePolynomial(object):
133     def __init__(self, n, xr, j, xj):
134         """
135         Construct a Lagrange polynomial.

```



```

136
137     :param n: how many points are on the x axis
138     :param xr: the values of x
139     :param j: the position of the current x
140     :param xj: the value of x at position j
141     """
142     self._order = n
143     self._j = j
144     self._xr = []
145     self._xj = xj
146
147     self._x = 0
148
149     for i in range(len(xr)):
150         self._xr.append(-xr[i])
151
152     self._numerator = self._create_numerator()
153     self._denominator = self._create_denominator(xj)
154
155     self._polynomial = self._create_polynomial()
156
157 def _create_numerator(self):
158     """
159     This method creates the list of the parameters x_r.
160
161     :return: no return value
162     """
163     i = 0
164     result_numerator = Polynomial([1])
165
166     while i < self._order:
167         if i == self._j:
168             i += 1
169
170         if i >= self._order:
171             break
172
173         result_numerator *= Polynomial([self._xr[i], 1])
174         i += 1
175
176     return result_numerator
177
178 def _create_denominator(self, x):
179     """
180     This method calculates the numerical result of the denominator.
181
182     :return: the value in decimal of the denominator.
183     """
184
185     return self._numerator.calculate(x)
186
187 def _create_polynomial(self):
188     """
189     This method creates the general form of the lagrange polynomial.
190     :return:
191     """
192     denom = Polynomial([1 / self._denominator])
193
194     return denom * self._numerator
195
196 def set_x(self, value):
197     self._x = value
198
199 @property
200 def j(self):
201     return self._j
202
203 @property
204 def xj(self):
205     return self._xj

```

```

206
207     @property
208     def denominator(self):
209         return self._denominator
210
211     @property
212     def numerator(self):
213         return self._numerator
214
215     @property
216     def poly(self):
217         return self._polynomial
218
219
220 if __name__ == "__main__":
221     coeff1 = Polynomial([2])
222     coeff2 = Polynomial([4, 5, 7, 8])
223
224     coeff2.toString()
225     (coeff2 - 3).toString()

```

Listing 2: Lagrange Interpolation Implementation (*interpolation.py*).

```

1  from polynomial import Polynomial, LagrangePolynomial
2
3
4  TOLERANCE = 1e-6
5
6  def lagrange_full_domain(xr, y, points=None):
7      """
8      This is the method for the lagrange full domain interpolation.
9      X is the variable that varies.
10     Y is the variable that varies with respect to X.
11
12     :param X: X vector of type Matrix
13     :param Y: Y vector of type Matrix
14     :param points: select the range of data to be interpolated if needed.
15     :return: Polynomial expression for y(x)
16     """
17     result_polynomial = Polynomial([0])
18
19     if points is None:
20         for j in range(len(xr)):
21             xj = xr[j]
22             aj = y[j]
23
24             temp_lagrange_poly = LagrangePolynomial(len(xr), xr, j, xj)
25
26             result_polynomial += Polynomial([aj]) * temp_lagrange_poly.poly
27
28     else:
29         pass
30
31     return result_polynomial
32
33
34 def cubic_hermite(xr, y, slopes):
35     result = Polynomial([0])
36
37     for j in range(len(xr)):
38         xj = xr[j]
39         aj = y[j]
40         bj = slopes[j]
41
42         temp = LagrangePolynomial(len(xr), xr, j, xj).poly
43         lagrange_backup = LagrangePolynomial(len(xr), xr, j, xj).poly
44
45         # Calculate the polynomial u(x)
46         temp = (temp.derive(1) * Polynomial([-xj, 1])) * Polynomial([-2])

```

```

47     temp = temp + 1
48
49     square = lagrange_backup * lagrange_backup
50     uj = temp * square
51
52     # Calculate the polynomial v(x)
53     vj = Polynomial([-xj, 1]) * square
54
55     aj_poly = Polynomial([aj])
56     bj_poly = Polynomial([bj])
57
58     result += uj * aj_poly + vj * bj_poly
59
60     return result
61
62 def piecewise_linear_interpolate(xr, y):
63     polynomials = []
64
65     for i in range(1, len(xr)):
66         a = (y[i] - y[i - 1]) / (xr[i] - xr[i - 1])
67         b = y[i] - a * xr[i]
68
69         temp_poly = Polynomial([b, a])
70         polynomials.append(temp_poly)
71
72     return polynomials

```

Listing 3: Newton Raphson (nonlinear.py).

```

1  from polynomial import Polynomial
2  from matrix import Matrix
3  import math
4
5  TOLERANCE = 1e-6
6  MAX_ITERATIONS = 1000
7  r = 512
8  e = 0.2
9  isa = 0.8e-6
10 isb = 1.1e-6
11 ktq = 0.025
12
13
14 def calc_newton_raphson(equation, data_x, data_y):
15     """
16     calculates the newton raphson
17     :param equation: either a polynomial or a list of piecewise linear polynomials
18     :param data_x: the list of data on the x axis
19     :param data_y: the list of data on the y axis
20     :return: number of iterations and the final result
21     """
22     if isinstance(equation, list):
23         """
24         This condition will be taken if equation is a list of linear polynomials.
25         """
26         area = 1e-4
27         flux_list = []
28         coefficients = [3.9790e7, 0.3, -8000]
29
30         # Calculate f(0)
31         k = 0
32         flux = 0
33         convergent = False
34         fk = -8000
35         prev_fk = -8000
36
37         while not convergent:
38             if abs(fk / prev_fk) < TOLERANCE or k >= MAX_ITERATIONS:
39                 break
40             prev_fk = fk

```

```

41     # Find the piecewise polynomial segment of the current flux
42     for i in range(1, len(data_x)):
43         if data_x[i - 1] <= (flux / area) < data_x[i]:
44             temp_poly = equation[i - 1]
45             start_H = data_y[i - 1]
46             start_B = data_x[i - 1]
47             break
48         else:
49             temp_poly = equation[len(equation) - 1]
50             start_H = data_y[len(equation)]
51             start_B = data_x[len(equation)]
52
53     # The polynomial segment is located at location i - 1
54     # Calculating stuff at k
55     slope = temp_poly[1]
56     H = slope * (flux - (start_B * area)) / area + start_H
57     fk = coefficients[0] * flux + coefficients[1] * H + coefficients[2]
58     fk_prime = coefficients[0] + coefficients[1] * temp_poly[1] / area
59
60     k += 1
61     flux = flux - fk / fk_prime
62     flux_list.append(flux)
63
64     return k, flux_list
65
66
67 def calc_successive_subs(equation, data_x, data_y):
68     if isinstance(equation, list):
69         area = 1e-4
70         coefficients = [3.979e7, 0.3, -8000]
71         flux_list = []
72
73     # Calculate f(0)
74     k = 0
75     flux = 0
76     convergent = False
77     f0 = -8000 * 5e-9
78     fk = -8000 * 5e-9
79
80     while not convergent:
81         if abs(fk / f0) < TOLERANCE or k >= MAX_ITERATIONS:
82             break
83         # Find the piecewise polynomial segment of the current flux
84         for i in range(1, len(data_x)):
85             if data_x[i - 1] <= (flux / area) < data_x[i]:
86                 temp_poly = equation[i - 1]
87                 start_H = data_y[i - 1]
88                 start_B = data_x[i - 1]
89                 break
90             else:
91                 temp_poly = equation[len(equation) - 1]
92                 start_H = data_y[len(equation)]
93                 start_B = data_x[len(equation)]
94
95         # The polynomial segment is located at location i - 1
96         # Calculating stuff at k
97         slope = temp_poly[1]
98         H = slope * (flux - (start_B * area)) / area + start_H
99         fk = coefficients[0] * flux + coefficients[1] * H + coefficients[2]
100        fk /= 5e9
101
102        k += 1
103        flux -= fk
104        flux_list.append(flux)
105
106    return k, flux_list
107
108
109 def calc_jacobian(voltages):
110     if not isinstance(voltages, Matrix):

```

```

111         raise ValueError("The input must be the list of V1 and V2.")
112
113     j_vec = [[0, 0], [0, 0]]
114     jacobian = Matrix(j_vec, 2, 2)
115
116     jacobian[0][0] = - 1 / r - (isa / ktq * math.exp((voltages[0][0] - voltages[1][0]) / ktq))
117     jacobian[0][1] = isa / ktq * math.exp((voltages[0][0] - voltages[1][0]) / ktq)
118     jacobian[1][0] = jacobian[0][1]
119     jacobian[1][1] = - isa / ktq * math.exp((voltages[0][0] - voltages[1][0]) / ktq) \
120         - isb / ktq * math.exp(voltages[1][0] / ktq)
121
122     inv_jacobian = jacobian.inv()
123
124     return jacobian, inv_jacobian
125
126
127 def calc_f1(voltages):
128     return (e - voltages[0][0]) / r - isa * (math.exp((voltages[0][0] - voltages[1][0]) / ktq) - 1)
129
130
131 def calc_f2(voltages):
132     return isa * (math.exp((voltages[0][0] - voltages[1][0]) / ktq) - 1) - isb * (math.exp(voltages[1][0] / ktq) - 1)
133
134
135 def calc_norm_vec(vector):
136     if vector.cols > 1:
137         raise ValueError("The vector must be a one-column one!")
138
139     result = 0
140     for i in range(vector.rows):
141         result += pow(vector[i][0], 2)
142
143     return result

```

Listing 4: Gauss-Legendre Integration (*integration.py*).

```

1  import math
2  from math import log
3
4
5  def gauss_legendre_integration(function, lim_low, lim_high, n, real_value):
6      width = (lim_high - lim_low) / n
7
8      result = 0
9      x = lim_low
10     for i in range(n):
11         result += function(x + width / 2) * width
12         x += width
13
14     error = abs(result - real_value)
15     return result, error
16
17
18 def nested_integration(lim_low, lim_high, n, real_value):
19     width = (lim_high - lim_low) / n
20
21     result = 0
22     x = lim_low
23     for i in range(n):
24         x_0 = x + width / 2
25         result += (log(x_0) / math.sqrt(0.04 - x_0 ** 2)) * width
26         x += width
27
28     error = abs(result - real_value)
29     return result, error
30
31
32 def modified_width(function, lim_low, lim_high, n, real_value):

```

```

33     width = []
34     for i in range(n):
35         temp = (lim_high - lim_low) * (i + 1) / (5.5 * n)
36         width.append(temp)
37
38     result = 0
39     x = lim_low
40     for i in range(n):
41         result += function(x + width[i] / 2) * width[i]
42         x += width[i]
43
44     error = abs(result - real_value)
45     return result, error
46
47
48 def modified_width_nested_integration(lim_low, lim_high, n, real_value):
49     width = []
50     for i in range(n):
51         temp = (lim_high - lim_low) * (i + 1) / (5.5 * n)
52         width.append(temp)
53
54     result = 0
55     x = lim_low
56     for i in range(n):
57         x_0 = x + width[i] / 2
58         result += (log(x_0) / math.sqrt(0.04 - x_0 ** 2)) * width[i]
59         x += width[i]
60
61     error = abs(result - real_value)
62     return result, error

```