

ECSE 543: Numerical Methods
Assignment 2 Report

Wenjie Wei
260685967

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Introduction

In this assignment, three numerical methods discussed in class were explored. The interpreter used for the Python codes is Python 3.6.

First Order Finite Element Problem

Figure 1 shows an illustration of the first order triangular finite element problem to be solved.

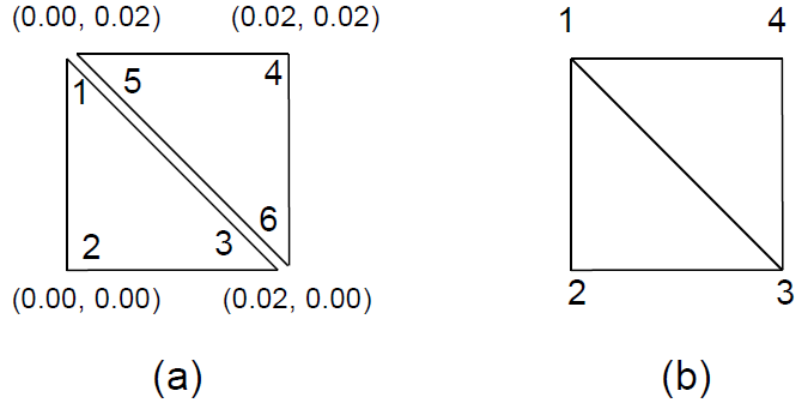


Figure 1: 1st Order Triangular FE Problem

Take the triangle with nodes 1, 2, and 3 as the beginning step. Firstly, interpolate the potential U as:

$$U = a + bx + cy$$

and at vertex 1, we can write an equation of potential as:

$$U_1 = a + bx_1 + cy_1$$

Thus, we can have a vector of potentials for vertex 1, 2, and 3 as follows:

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

and the terms a, b, c are acquired following:

$$U = \sum_{i=1}^3 U_i \alpha_i(x, y) \quad (1)$$

and we can derive a general formula for α_i :

$$\nabla \alpha_i = \nabla \frac{1}{2A} [(x_{i+1}y_{i+2} - x_{i+2}y_{i+1}) + (y_{i+1} - y_{i+2})x + (x_{i+2} - x_{i+1})y] \quad (2)$$

where A holds the value of the area of the triangle.

Following Equation 2, when the index i exceeds the top limit 3, it is wrapped around to 1. Now we can get the following calculations for α_1, α_2 and α_3 :

$$\begin{aligned} \nabla \alpha_1 &= \nabla \frac{1}{2A} [(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y] \\ \nabla \alpha_2 &= \nabla \frac{1}{2A} [(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y] \end{aligned}$$

$$\nabla\alpha_3 = \nabla \frac{1}{2A}[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y]$$

With the expressions for α derived, we now go ahead and calculate the $S_{ij}^{(e)}$ matrices. The general formula below is used to calculate the \mathbf{S} matrix:

$$S_{ij}^{(e)} = \int_{\Delta_e} \nabla\alpha_i \nabla\alpha_j dS \quad (3)$$

Using the equation above, plug in the values provided in Figure 1, we can have the following calculations:

$$S_{11} = \frac{1}{4A}[(y_2 - y_3)^2 + (x_3 - x_2)^2] = \frac{1}{4 \times 2 \times 10^{-4}}[0 + 0.02^2] = 0.5$$

$$S_{12} = \frac{1}{4A}[(y_2 - y_3)(y_3 - y_1) + (x_3 - x_2)(x_1 - x_3)] = -0.5$$

$$S_{13} = \frac{1}{4A}[(y_2 - y_3)(y_1 - y_2) + (x_3 - x_2)(x_2 - x_1)] = 0$$

Before performing the calculation for the next row, we inspect the calculation rules of the entries of the \mathbf{S} matrix, we can easily discover that $S_{ij} = S_{ji}$, since the flip of the orders of the operands in the parenthesis results in the same sign of the result. Therefore, the following statements can be made:

$$S_{21} = S_{12} = -0.5$$

$$S_{31} = S_{13} = 0$$

$$S_{22} = \frac{1}{4A}[(y_3 - y_1)^2 + (x_1 - x_3)^2] = 1$$

$$S_{23} = S_{32} = \frac{1}{4A}[(y_3 - y_1)(y_1 - y_2) + (x_1 - x_3)(x_2 - x_1)] = -0.5$$

$$S_{33} = \frac{1}{4A}[(y_1 - y_2)^2 + (x_2 - x_1)^2] = 0.5$$

From the calculation results above, we can come up with the \mathbf{S} matrix for vertices 1, 2, and 3:

$$S^{(1)} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 1 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix}$$

Use the similar approach for the other triangle and obtain S_{456} :

$$S^{(2)} = \begin{bmatrix} S_{44} & S_{45} & S_{46} \\ S_{54} & S_{55} & S_{56} \\ S_{64} & S_{65} & S_{66} \end{bmatrix} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0 \\ -0.5 & 0 & 0.5 \end{bmatrix}$$

Add the triangles to get the energy of the whole system shown in (b) of Figure 1:

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{bmatrix}_{dis} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ 1 & & & & 1 & \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}_{joint}$$

which is also denoted as:

$$U_{dis} = CU_{joint}$$

Use \mathbf{S}_{dis} to denote a 6×6 matrix to represent the disjoint matrix:

$$S_{dis} = \begin{bmatrix} S^{(1)} & \\ & S^{(2)} \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 & 0 & & & \\ -0.5 & 1 & -0.5 & & & \\ 0 & -0.5 & 0.5 & & & \\ & & & 1 & -0.5 & -0.5 \\ & & & -0.5 & 0.5 & 0 \\ & & & -0.5 & 0 & 0.5 \end{bmatrix}$$

Now the global \mathbf{S} matrix will be calculated as:

$$\begin{aligned} S_{joint} &= C^T S_{dis} C \\ &= \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} 0.5 & -0.5 & 0 & & & \\ -0.5 & 1 & -0.5 & & & \\ 0 & -0.5 & 0.5 & & & \\ & & & 1 & -0.5 & -0.5 \\ & & & -0.5 & 0.5 & 0 \\ & & & -0.5 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -0.5 & 0 & -0.5 \\ -0.5 & 1 & -0.5 & 0 \\ 0 & -0.5 & 1 & -0.5 \\ -0.5 & 0 & -0.5 & 1 \end{bmatrix} \end{aligned}$$

which is the final result of this problem.

Coaxial Cable Electrostatic Problem

Use the triangular finite element model for the analysis of the coaxial cable problem seen in the previous assignment. We take the third quadrant for the analysis.

The Finite Element Mesh

Listing 1 shows the implementation of the construction of the finite element mesh and the creation of the MATLAB input file.

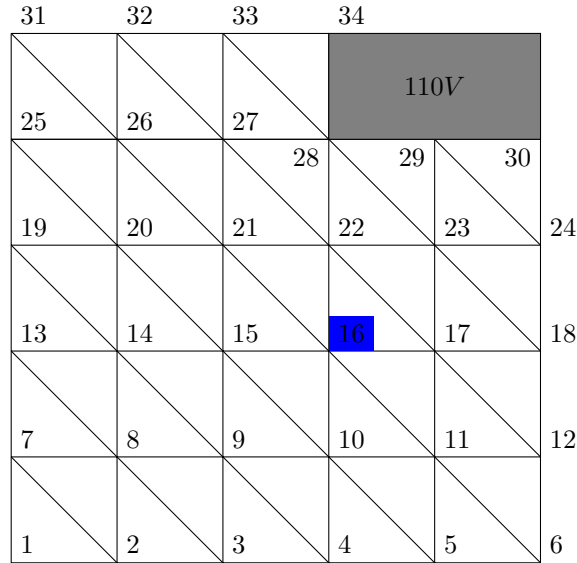


Figure 2: Organization of the Finite Element Mesh

Figure 2 shows the organization of the finite element mesh constructed by the program. The input file written by this program is shown in Listing 2. Note that the first number at the beginning of the lines are not an input to the MATLAB file, as it is the line number which is provided by the *minted* package in L^AT_EX.

Potential Solved by SIMPLE2D.m

Use the input file generated in the previous section, we are able to use the MATLAB file to calculate the potential at every node we have specified. The output of the SIMPLE2D.m file is shown in Listing 3. The target node (0.06, 0.04) is highlighted in blue as Node 16, and from Listing 3 shows that the potential at this node is 40.527V.

Capacitance per Unit Length

To compute the capacitance, apply the fundamental Equation 4:

$$E = \frac{1}{2}CV^2 \quad (4)$$

Now apply the finite element method used in the previous section. Use U_{joint} to denote the potential vector shown in Listing 3. Use the S_{joint} calculated in the first question, we derive an equation to calculate the total capacitance:

$$W = \frac{1}{2}U_{joint}^T S_{joint} U_{joint} \quad (5)$$

The value of the entries in the U matrix are the potential on the four corners of the square defined by the two finite element triangles.

Use the Python function implemented in Listing 1, we are able to calculate the energy contained in a single quadrant using Equation 5. The following calculation is performed to find the total capacitance per unit length:

$$C = \frac{2E}{V^2} = \frac{2\varepsilon_0 W}{V^2} = 5.2137 \times 10^{-11} F/m$$

which equals to 52.137pF/m.

Code Listings

Listing 1: Finite Element Mesh Implementation (finite_element.py).

```
1  from matrix import Matrix
2  from finite_difference import Node
3
4  EPSILON = 8.854188e-12
5  HIGH_VOLTAGE = 110
6  LOW_VOLTAGE = 0
7  SPACING = 0.02
8  s_vec = [[1, -0.5, 0, -0.5], [-0.5, 1, -0.5, 0], [0, -0.5, 1, -0.5], [-0.5, 0, -0.5, 1]]
9  S = Matrix(s_vec, 4, 4)
10 f = open('SIMPLE2Dinput.dat', 'w')
11
12
13 class two_element(object):
14     def __init__(self, x, y, bl_node, id):
15         """
16         This is the constructor of a two-triangle finite element
17         the vertices are numbered from 0 to 5, replacing 1 - 6 in question 1
18
19         :param x: x coord for the bottom-left corner
20         :param y: y coord for the bottom-left corner
21         """
22
23         # vertices are put in the array
24         # vertices 2&5, vertices 0&4 have the same properties
25         self._vertex_array = [Node(0) for _ in range(6)]
26         self._vertex_array[5] = self._vertex_array[2]
27         self._vertex_array[4] = self._vertex_array[0]
28         self._bl_x = x
29         self._bl_y = y
30
31         self._bl_node = bl_node
32         self._tl_node = bl_node + 6
33         self._br_node = bl_node + 1
34         self._tr_node = bl_node + 7
35
36         self._id = id
37
38         if (self._bl_x + SPACING) > 0.1 or (self._bl_y + SPACING) > 0.1:
39             raise ValueError("The finite elements cannot exceed the third quadrant!")
40
41         if self._bl_y == 0:
42             # configure node 1
43             self._vertex_array[1].set_fixed()
44             self._vertex_array[1].set_value(LOW_VOLTAGE)
45
46             # configure node 2 and 5
47             self._vertex_array[2].set_fixed()
48             self._vertex_array[2].set_value(LOW_VOLTAGE)
49
50             # configure node 3
51             self._vertex_array[3].set_free()
52
53             if self._bl_x == 0:
54                 # configure node 0 and 4
55                 self._vertex_array[0].set_fixed()
56                 self._vertex_array[0].set_value(LOW_VOLTAGE)
57             else:
58                 self._vertex_array[0].set_free()
59         elif self._bl_x >= 0.06 and self._bl_y == 0.06:
60             # configure node 1
61             self._vertex_array[1].set_free()
62
63             # configure node 2 and 5
64             self._vertex_array[2].set_free()
65
```

```

66         # configure node 0 and 4
67         self._vertex_array[0].set_fixed()
68         self._vertex_array[0].set_value(HIGH_VOLTAGE)
69
70         # configure node 3
71         self._vertex_array[3].set_fixed()
72         self._vertex_array[3].set_value(HIGH_VOLTAGE)
73     elif self._bl_x == 0.04 and self._bl_y == 0.06:
74         # configure node 1
75         self._vertex_array[1].set_free()
76
77         # configure node 2 and 5
78         self._vertex_array[2].set_free()
79
80         # configure node 0 and 4
81         self._vertex_array[0].set_free()
82
83         # configure node 3
84         self._vertex_array[3].set_fixed()
85         self._vertex_array[3].set_value(HIGH_VOLTAGE)
86     elif self._bl_x == 0.04 and self._bl_y == 0.08:
87         # configure node 1
88         self._vertex_array[1].set_free()
89
90         # configure node 2 and 5
91         self._vertex_array[2].set_fixed()
92         self._vertex_array[2].set_value(HIGH_VOLTAGE)
93
94         # configure node 0 and 4
95         self._vertex_array[0].set_free()
96
97         # configure node 3
98         self._vertex_array[3].set_fixed()
99         self._vertex_array[3].set_value(HIGH_VOLTAGE)
100     elif self._bl_x == 0:
101         # configure node 1
102         self._vertex_array[1].set_fixed()
103         self._vertex_array[1].set_value(LOW_VOLTAGE)
104
105         # configure node 0 and 4
106         self._vertex_array[0].set_fixed()
107         self._vertex_array[0].set_value(LOW_VOLTAGE)
108
109         # configure node 3
110         self._vertex_array[3].set_free()
111
112         # configure node 2 and 5
113         self._vertex_array[2].set_free()
114     else:
115         for i in range(6):
116             self._vertex_array[i].set_free()
117
118     def print_two_element(self):
119         for i in range(6):
120             print("Vertex " + str(i) + " has value " + str(self._vertex_array[i].value) + ", free node: "
121                   + str(self._vertex_array[i].is_free))
122
123     @property
124     def bl_x(self):
125         return self._bl_x
126
127     @property
128     def bl_y(self):
129         return self._bl_y
130
131     @property
132     def bl_node(self):
133         return self._bl_node
134
135     @property

```



```

136     def tl_node(self):
137         return self._tl_node
138
139     @property
140     def br_node(self):
141         return self._br_node
142
143     @property
144     def tr_node(self):
145         return self._tr_node
146
147     @property
148     def vertex(self, i):
149         return self._vertex_array[i]
150
151     @property
152     def id(self):
153         return self._id
154
155
156 def calc_energy(fe_matrix):
157     file = open('potentials.dat', mode='r', encoding='utf-8-sig')
158     lines = file.readlines()
159     file.close()
160     potentials = [0 for _ in range(len(lines))]
161
162     energy = 0
163
164     count = 0
165     for line in lines:
166         line = line.split(' ')
167         line = [i.strip() for i in line]
168         potentials[count] = line[3]
169         count += 1
170
171     for i in range(4, -1, -1):
172         for j in range(5):
173             temp_two_element = fe_matrix[i][j]
174
175             if temp_two_element is not None:
176                 u_vec = [[0] for _ in range(4)]
177                 U = Matrix(u_vec, 4, 1)
178
179                 U[0][0] = float(potentials[temp_two_element.bl_node - 1])
180                 U[1][0] = float(potentials[temp_two_element.bl_node])
181                 U[2][0] = float(potentials[temp_two_element.bl_node + 5])
182                 U[3][0] = float(potentials[temp_two_element.bl_node + 6])
183
184                 energy += 0.5 * EPSILON * U.T.dot_product(S.dot_product(U))[0][0]
185
186     return energy
187
188
189 def write_mesh(fe_matrix):
190     y_coord = 0
191     count = 0
192
193     print("Creating the mesh of the finite elements...")
194     node_count = 0
195     row = 1
196     for i in range(4, -1, -1):
197         x_coord = 0
198         for j in range(5):
199             if x_coord >= 0.06 and y_coord == 0.08:
200                 break
201             else:
202                 temp_two_element = two_element(x_coord, y_coord, node_count + row, node_count)
203                 fe_matrix[i][j] = temp_two_element
204                 count += 1
205                 node_count += 1

```

```

206         x_coord += SPACING
207         y_coord += SPACING
208         row += 1
209
210
211     print("Finite elements created: " + str(count * 2))
212
213     # Now write the input file for SIMPLE2D.m
214     print("Writing node information...")
215     # write the bottom row
216     i = 4
217     for j in range(5):
218         temp_two_element = fe_matrix[i][j]
219         f.write('%d %.3f %.3f\n' % (temp_two_element.bl_node, temp_two_element.bl_x,
↪ temp_two_element.bl_y))
220         if j == 4:
221             f.write('%d %.3f %.3f\n' % (temp_two_element.br_node,
222                                     temp_two_element.bl_x + SPACING, temp_two_element.bl_y))
223
224     # write the general rows
225     for i in range(4, -1, -1):
226         for j in range(5):
227             temp_two_element = fe_matrix[i][j]
228             if temp_two_element is not None:
229                 if i != 0 and j != 4:
230                     f.write('%d %.3f %.3f\n' %
231                             (temp_two_element.tl_node, temp_two_element.bl_x, temp_two_element.bl_y
↪ + SPACING))
232                 elif i != 0 and j == 4:
233                     f.write('%d %.3f %.3f\n' %
234                             (temp_two_element.tl_node, temp_two_element.bl_x, temp_two_element.bl_y
↪ + SPACING))
235                     f.write('%d %.3f %.3f\n' %
236                             (temp_two_element.tr_node, temp_two_element.bl_x + SPACING,
237                             temp_two_element.bl_y + SPACING))
238                 else:
239                     if j != 2:
240                         f.write('%d %.3f %.3f\n' %
241                                 (temp_two_element.tl_node, temp_two_element.bl_x, temp_two_element.bl_y
↪ + SPACING))
242                     else:
243                         f.write('%d %.3f %.3f\n' %
244                                 (temp_two_element.tl_node,
245                                 temp_two_element.bl_x, temp_two_element.bl_y + SPACING))
246                         f.write('%d %.3f %.3f\n' %
247                                 (temp_two_element.tr_node, temp_two_element.bl_x + SPACING,
248                                 temp_two_element.bl_y + SPACING))
249                 else:
250                     break
251
252     f.write('\n')
253     # Now write the triangle connection
254     print("Writing triangle information...")
255     for i in range(4, -1, -1):
256         for j in range(5):
257             temp_two_element = fe_matrix[i][j]
258             if temp_two_element is not None:
259                 f.write('%d %d %d %.3f\n' %
260                         (temp_two_element.bl_node, temp_two_element.br_node, temp_two_element.tl_node, 0))
261             else:
262                 break
263         for j in range(5):
264             temp_two_element = fe_matrix[i][j]
265             if temp_two_element is not None:
266                 f.write('%d %d %d %.3f\n' %
267                         (temp_two_element.tr_node, temp_two_element.tl_node, temp_two_element.br_node, 0))
268             else:
269                 break
270
271     f.write('\n')

```

```

272
273     print("Writing boundary conditions")
274     for i in range(4, -1, -1):
275         for j in range(5):
276             temp_two_element = fe_matrix[i][j]
277             if temp_two_element is not None:
278                 if i == 4 and j != 4:
279                     f.write('%d %.3f\n' % (temp_two_element.bl_node, LOW_VOLTAGE))
280                 elif i == 4 and j == 4:
281                     f.write('%d %.3f\n' % (temp_two_element.bl_node, LOW_VOLTAGE))
282                     f.write('%d %.3f\n' % (temp_two_element.br_node, LOW_VOLTAGE))
283                 elif i == 3 and j == 0:
284                     f.write('%d %.3f\n' % (temp_two_element.bl_node, LOW_VOLTAGE))
285                     f.write('%d %.3f\n' % (temp_two_element.tl_node, LOW_VOLTAGE))
286                 elif j == 0 and i != 3 and i != 4:
287                     f.write('%d %.3f\n' % (temp_two_element.tl_node, LOW_VOLTAGE))
288                 elif j >= 2 and i <= 1:
289                     f.write('%d %.3f\n' % (temp_two_element.tr_node, HIGH_VOLTAGE))
290             else:
291                 break
292
293
294 if __name__ == "__main__":
295     fe_vec = [[None for _ in range(5)] for _ in range(5)]
296     fe_matrix = Matrix(fe_vec, 5, 5)
297
298     write_mesh(fe_matrix)
299
300     energy = 4 * calc_energy(fe_matrix)
301     capacitance = 2 * energy / (HIGH_VOLTAGE * HIGH_VOLTAGE)
302     print(energy)
303     print(capacitance)

```

Listing 2: Finite Element Mesh Input File

```
1 1 0.000 0.000
2 2 0.020 0.000
3 3 0.040 0.000
4 4 0.060 0.000
5 5 0.080 0.000
6 6 0.100 0.000
7 7 0.000 0.020
8 8 0.020 0.020
9 9 0.040 0.020
10 10 0.060 0.020
11 11 0.080 0.020
12 12 0.100 0.020
13 13 0.000 0.040
14 14 0.020 0.040
15 15 0.040 0.040
16 16 0.060 0.040
17 17 0.080 0.040
18 18 0.100 0.040
19 19 0.000 0.060
20 20 0.020 0.060
21 21 0.040 0.060
22 22 0.060 0.060
23 23 0.080 0.060
24 24 0.100 0.060
25 25 0.000 0.080
26 26 0.020 0.080
27 27 0.040 0.080
28 28 0.060 0.080
29 29 0.080 0.080
30 30 0.100 0.080
31 31 0.000 0.100
32 32 0.020 0.100
33 33 0.040 0.100
34 34 0.060 0.100
35
36 1 2 7 0.000
37 2 3 8 0.000
38 3 4 9 0.000
39 4 5 10 0.000
40 5 6 11 0.000
41 8 7 2 0.000
42 9 8 3 0.000
43 10 9 4 0.000
44 11 10 5 0.000
45 12 11 6 0.000
46 7 8 13 0.000
47 8 9 14 0.000
48 9 10 15 0.000
49 10 11 16 0.000
50 11 12 17 0.000
51 14 13 8 0.000
52 15 14 9 0.000
53 16 15 10 0.000
54 17 16 11 0.000
55 18 17 12 0.000
56 13 14 19 0.000
57 14 15 20 0.000
58 15 16 21 0.000
59 16 17 22 0.000
60 17 18 23 0.000
61 20 19 14 0.000
62 21 20 15 0.000
63 22 21 16 0.000
64 23 22 17 0.000
65 24 23 18 0.000
66 19 20 25 0.000
67 20 21 26 0.000
```

```
68 21 22 27 0.000
69 22 23 28 0.000
70 23 24 29 0.000
71 26 25 20 0.000
72 27 26 21 0.000
73 28 27 22 0.000
74 29 28 23 0.000
75 30 29 24 0.000
76 25 26 31 0.000
77 26 27 32 0.000
78 27 28 33 0.000
79 32 31 26 0.000
80 33 32 27 0.000
81 34 33 28 0.000
82
83 1 0.000
84 2 0.000
85 3 0.000
86 4 0.000
87 5 0.000
88 6 0.000
89 7 0.000
90 13 0.000
91 19 0.000
92 25 0.000
93 28 110.000
94 29 110.000
95 30 110.000
96 31 0.000
97 34 110.000
```

Listing 3: MATLAB File Outputs

```
1 1 0.000000 0.000000 0.000000
2 2 0.020000 0.000000 0.000000
3 3 0.040000 0.000000 0.000000
4 4 0.060000 0.000000 0.000000
5 5 0.080000 0.000000 0.000000
6 6 0.100000 0.000000 0.000000
7 7 0.000000 0.020000 0.000000
8 8 0.020000 0.020000 7.018554
9 9 0.040000 0.020000 13.651929
10 10 0.060000 0.020000 19.110684
11 11 0.080000 0.020000 22.264306
12 12 0.100000 0.020000 23.256867
13 13 0.000000 0.040000 0.000000
14 14 0.020000 0.040000 14.422288
15 15 0.040000 0.040000 28.478477
16 16 0.060000 0.040000 40.526503
17 17 0.080000 0.040000 46.689671
18 18 0.100000 0.040000 48.498858
19 19 0.000000 0.060000 0.000000
20 20 0.020000 0.060000 22.192122
21 21 0.040000 0.060000 45.313189
22 22 0.060000 0.060000 67.827178
23 23 0.080000 0.060000 75.469018
24 24 0.100000 0.060000 77.359224
25 25 0.000000 0.080000 0.000000
26 26 0.020000 0.080000 29.033010
27 27 0.040000 0.080000 62.754981
28 28 0.060000 0.080000 110.000000
29 29 0.080000 0.080000 110.000000
30 30 0.100000 0.080000 110.000000
31 31 0.000000 0.100000 0.000000
32 32 0.020000 0.100000 31.184936
33 33 0.040000 0.100000 66.673724
34 34 0.060000 0.100000 110.000000
```