ECSE 543: Numerical Methods

Assignment 2 Report

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Contents

Introduction

In this assignment, three numerical methods discussed in class were explored. The interpreter used for the Python codes is Python 3.6.

1 First Order Finite Difference Problem

Figure 1 shows an illustration of the first order triangular finite element problem to be solved.

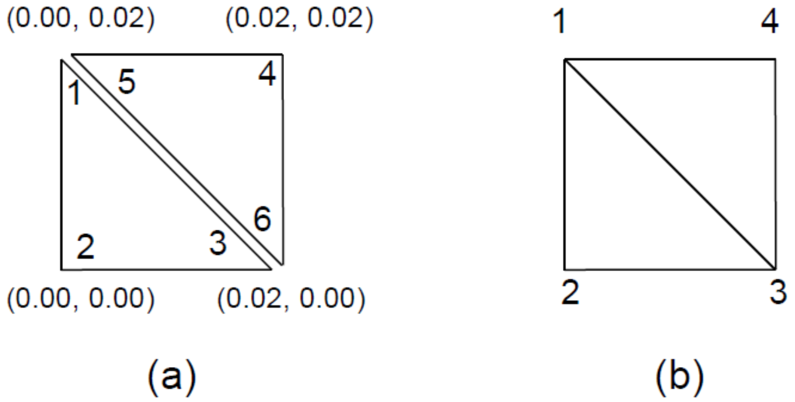

Figure 1: 1st Order Triangular FE Problem

Take the triangle with nodes 1, 2, and 3 as the beginning step. Firstly, interpolate the potential U as:

$$U = a + bx + cy$$

and at vertex 1, we can write an equation of potential as:

$$U_1 = a + bx_1 + cy_1$$

Thus, we can have a vector of potentials for vertex 1, 2, and 3 as follows:

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

and the terms a, b, c are acquired following:

$$U = \sum_{i=1}^{3} U_i \alpha_i(x, y) \tag{1}$$

and we can derive a general formula for α_i:

$$\nabla \alpha_i = \nabla \frac{1}{2A}[(x_{i+1}y_{i+2} - x_{i+2}y_{i+1}) + (y_{i+1} - y_{i+2})x + (x_{i+2} - x_{i+1})y] \tag{2}$$

where A holds the value of the area of the triangle.

Following Equation 2, when the index i exceeds the top limit 3, it is wrapped around to 1. Now we can get the following calculations for α_1, α_2 and α_3:

$$\nabla \alpha_1 = \nabla \frac{1}{2A}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y]$$

$$\nabla \alpha_2 = \nabla \frac{1}{2A}[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y]$$

$$\nabla\alpha_3 = \nabla\frac{1}{2A}[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y]$$

With the expressions for α derived, we now go ahead and calculate the $S_{ij}^{(e)}$ matrices. The general formula below is used to calculate the $\boldsymbol{S}$ matrix:

$$S_{ij}^{(e)} = \int_{\Delta e} \nabla\alpha_i \nabla\alpha_j dS \tag{3}$$

Using the equation above, plug in the values provided in Figure 1, we can have the following calculations:

$$S_{11} = \frac{1}{4A}[(y_2 - y_3)^2 + (x_3 - x_2)^2] = \frac{1}{4 \times 2 \times 10^{-4}}[0 + 0.02^2] = 0.5$$

$$S_{12} = \frac{1}{4A}[(y_2 - y_3)(y_3 - y_1) + (x_3 - x_2)(x_1 - x_3)] = -0.5$$

$$S_{13} = \frac{1}{4A}[(y_2 - y_3)(y_1 - y_2) + (x_3 - x_2)(x_2 - x_1)] = 0$$

Before performing the calculation for the next row, we inspect the calculation rules of the entries of the $\boldsymbol{S}$ matrix, we can easily discover that $S_{ij} = S_{ji}$, since the flip of the orders of the operands in the parenthesis results in the same sign of the result. Therefore, the following statements can be made:

$$S_{21} = S_{12} = -0.5$$

$$S_{31} = S_{13} = 0$$

$$S_{22} = \frac{1}{4A}[(y_3 - y_1)^2 + (x_1 - x_3)^2] = 1$$

$$S_{23} = S_{32} = \frac{1}{4A}[(y_3 - y_1)(y_1 - y_2) + (x_1 - x_3)(x_2 - x_1)] = -0.5$$

$$S_{33} = \frac{1}{4A}[(y_1 - y_2)^2 + (x_2 - x_1)^2] = 0.5$$

From the calculation results above, we can come up with the $\boldsymbol{S}$ matrix for vertices 1, 2, and 3:

$$S^{(1)} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 1 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix}$$

Use the similar approach for the other triangle and obtain S_{456}:

$$S^{(2)} = \begin{bmatrix} S_{44} & S_{45} & S_{46} \\ S_{54} & S_{55} & S_{56} \\ S_{64} & S_{65} & S_{66} \end{bmatrix} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0 \\ -0.5 & 0 & 0.5 \end{bmatrix}$$

Add the triangles to get the energy of the whole system shown in (b) of Figure 1:

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{bmatrix}_{dis} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \\ & & 1 & \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}_{joint}$$

which is also denoted as:

$$U_{dis} = CU_{con}$$

Use $\mathbf{S}_{dis}$ to denote a 6×6 matrix to represent the disjoint matrix:

$$S_{dis} = \begin{bmatrix} S^{(1)} & \\ & S^{(2)} \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 & 0 & & & \\ -0.5 & 1 & -0.5 & & & \\ 0 & -0.5 & 0.5 & & & \\ & & & 1 & -0.5 & -0.5 \\ & & & -0.5 & 0.5 & 0 \\ & & & -0.5 & 0 & 0.5 \end{bmatrix}$$

Now the global $\boldsymbol{S}$ matrix will be calculated as:

$$\begin{aligned} S_{con} &= C^T S_{dis} C \\ &= \begin{bmatrix} 1 & & & & 1 & \\ & 1 & & & & \\ & & 1 & & & 1 \\ & & & 1 & & \end{bmatrix} \begin{bmatrix} 0.5 & -0.5 & 0 & & & \\ -0.5 & 1 & -0.5 & & & \\ 0 & -0.5 & 0.5 & & & \\ & & & 1 & -0.5 & -0.5 \\ & & & -0.5 & 0.5 & 0 \\ & & & -0.5 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \\ & & 1 & \end{bmatrix} \\ &= \begin{bmatrix} 1 & -0.5 & 0 & -0.5 \\ -0.5 & 1 & -0.5 & 0 \\ 0 & -0.5 & 1 & -0.5 \\ -0.5 & 0 & -0.5 & 1 \end{bmatrix} \end{aligned}$$

which is the final result of this problem.

2 Coaxial Cable Electrostatic Problem

Use the mesh constructed in Figure 1, we construct a finite element mesh for a quarter of the coaxial cable.