RATIONAL PRICING OF INTERNET COMPANIES

September 1999

Revised January 2000

Eduardo S. Schwartz

Anderson School at UCLA

Mark Moon

Fuller & Thaler Asset Management

ABSTRACT

In this article, we apply real options theory and modern capital budgeting to the problem of valuing an Internet company. We formulate the model in continuous time, form a discrete time approximation, estimate the model parameters, solve the model by simulation, and then perform sensitivity analysis. Depending on the parameters chosen, we find that the value of an Internet stock may be rational given high enough growth rates in revenues. Even with a very real chance that a firm may go bankrupt, if the initial growth rates are sufficiently high, and if there is enough volatility in this growth rate over time, then valuations can be what would otherwise appear to be dramatically high. In addition, we find a large sensitivity of the valuation to initial conditions and exact specification of the parameters. This is also consistent with the observation that the returns of Internet stocks have been strikingly volatile.

RATIONAL PRICING OF INTERNET COMPANIES

1. INTRODUCTION

It is difficult to imagine a recent investment topic that elicits stronger feelings than Internet stocks. The skyrocketing valuations among these companies have made millionaires and billionaires out of many Internet entrepreneurs, while the actual companies have generated significant and often growing losses. Interestingly, as the Internet has grown, so has the means by which individuals can trade easily over the Internet with relatively low transaction costs.

The conventional view among some traditional money managers is that Internet stocks have been bid upward irrationally by individual "day traders", sitting at home at their computers, buying any stock that begins with "e-" or ends with ". com". Such managers see the current frenzy as a spectacular example of a market bubble, the likes of which many will witness only once in their lifetimes. These traditionalists fear significant negative consequences to the real economy after this bubble bursts. In contrast, others see the Internet as dramatically transforming the way in which businesses is transacted. These investors believe that some of the upstart Internet companies will rapidly grow to dominate and even make irrelevant their traditional "bricks-and-mortar" competitors.

In this paper, we apply real options theory and modern capital budgeting to the problem of valuing an Internet stock. We formulate the model in continuous time, form a discrete time approximation, estimate the model parameters, solve the model by simulation, and then perform sensitivity analysis. We find that, depending on the parameters chosen, the value of an Internet stock may be rational, given high enough growth rates in revenues. Even with a very real chance that a firm may go bankrupt, if the initial growth rates are sufficiently high, and if there is enough volatility in this growth over time, then valuations can be what would otherwise appear to be dramatically high. In addition, we find a large sensitivity of the valuation to initial conditions and exact specification of the parameters. This is also consistent with the observation that the returns of Internet stocks have been strikingly volatile.

In a recent paper Berger, Ofek and Swary (1996) empirically investigate whether investors price the option to abandon a firm at its exit value, and conclude that firm value does increase in exit value, after controlling for other variables. In our model, even though the exit value is assumed to be zero, the abandonment option can be very valuable.

In Section 2, we develop the continuous time model, and we describe its discrete version in Section 3. The estimation of the parameters of the model are discussed in Section 4. In Section 5, we apply the model to Amazon.com and report simulation results. We also analyze the sensitivity of the firm value to the most critical parameters. The procedure for determining the share value is described in Section 6. Section 7 discusses important extensions of the model, and Section 8 provides some concluding comments.

2. CONTINUOUS TIME MODEL

In this section, we develop a simple model to value Internet stocks. For simplicity, we initially describe the model in continuous time, although its implementation will be in discrete time using the quarterly accounting data available from Internet companies.

Consider an Internet company with instantaneous rate of revenues (or sales) at time t given by R(t). Assume that the dynamics of these revenues are given by the stochastic differential equation:

$$(1)\quad \frac{dR(t)}{R(t)} = m(t)dt + s(t)dz_1$$

Where $m(t)$, the drift, is the expected rate of growth in revenues and is assumed to follow a mean reverting process with a long-term average drift $\bar{m}$. That is, the initial very high growth rates of the Internet firm are assumed to converge stochastically to a more reasonable and sustainable rate of growth for the industry to which the firm belongs.

$$(2)\quad dm(t) = k(\bar{m} - m(t))dt + h(t)dz_2$$

The mean-reversion coefficient (k) describes the rate at which the growth rate is expected to converge to its long term average, and $\ln(2)/k$ can be interpreted as the "half-life" of the deviations in that any deviation m is expected to be halved in this time period. The unanticipated changes in revenues are also assumed to converge (deterministically) to a more normal level and the unanticipated changes in the drift are assumed to converge (also deterministically) to zero.

$$(3)\quad ds(t) = k_1(\bar{s} - s(t))dt$$

$$(4)\quad dh(t) = -k_2 h(t)dt$$

The unanticipated changes in the growth rate of revenues and the unanticipated changes in its drift may be correlated:

$$(5)\quad dz_1 dz_2 = r\,dt$$

The net after tax rate of cash flows to the firm, Y(t), is then given by:

$$(6)\ \ Y(t) = (R(t) - Cost(t))(1 - t_c)$$

The costs at time t have two components. The first is the cost of goods sold, which are assumed to be proportional to the revenues. The second is other expenses, which are assumed to have a fixed component and a variable component proportional to the revenues:

Cost(t) = Cost of Goods Sold(t) + Other Expenses(t)

$$Cost(t) = \alpha R(t) + (F + \beta R(t))$$

$$(7)\ \ Cost(t) = (\alpha + \beta)R(t) + F$$

More complicated cost structures can be easily accommodated in the model. In particular, the cost function could be stochastic reflecting, for example, the uncertainty about future potential competitors, market share, or technological developments[1]. The corporate tax rate t_c in (6) is only paid if there is no loss-carry-forward (i.e. if the loss-carry-forward is positive the tax rate is zero).

For simplicity in the framework above, we have neglected the depreciation tax shields in the computation of the after tax cash flows. These could, however, be easily incorporated in the analysis.

The dynamics of the loss-carry-forward is given by:

(8) dL(t) = - Y(t)dt if L(t) > 0

dL(t) = Max [- Y(t)dt , 0] if L(t) = 0

Finally, the firm is assumed to have an amount of cash available given by X(t) which evolves according to:

[1] In future research we plan to introduce uncertainty into the cost function.

$$(9)\ dX(t) = Y(t)dt$$

The firm is assumed to go bankrupt when the amount of cash available reaches zero. That is, bankruptcy in the model is defined as the first passage time of X(t) to zero. This bankruptcy condition is clearly a simplification of reality. It does not take into account the possibility of additional financing in the future. In particular, the firm could run out of cash but have good enough prospects to be able to raise cash, sell all its equity, or merge with another company. In Section 7, we discuss a more realistic alternative bankruptcy condition which addresses some of these issues.

If future financing is planned, the cash raised could be added to the cash balances available at the time of issue. The possible future financing could even be state dependent; that is, it could be a function of the revenues and the expected rate of growth in revenues at the time of issue. To keep things simple, we assume that there will be no additional financing in the future.

To avoid having to define a dividend policy in the model, we assume that the cash flow generated by the firm's operations remains in the firm and earns the risk free rate of interest, and will be available for distribution to the shareholders at an arbitrary long-term horizon T, by which time the firm will have reverted to a "normal" firm. This assumption may induce an underestimation of the probability of bankruptcy. However, since this type of firm is unlikely to start paying dividends until the cash flows are reliably positive, this underestimation is likely to be small. Then, the interest earned on the cash available has to be added to the revenues in equation (6).

The objective of the model is to determine the value of the Internet firm at the current time (assumed to be time zero). According to standard theory this value is obtained by discounting the expected net cash flows to the firm under the risk neutral measure (the equivalent martingale measure) at the risk free rate, which for simplicity is assumed to be constant[2].

$$(10) \qquad V(0) = E_Q[X(T)e^{-rT}]$$

An implicit assumption in Equation (10) is that the firm is liquidated at the horizon T and all cash flows are distributed. In most cases it might be more appropriate to use a terminal value for the firm which is related to the net cash flows at the horizon (given by equation (6)). For example, the value of the firm at the horizon can be assumed to be a multiple (e.g. ten times) of the EBITDA. This would make the value of the firm less sensitive to the horizon chosen.

The model has two sources of uncertainty. First, there is uncertainty about the changes in revenues and second, there is uncertainty about the expected rate of growth in revenues. Under some simplifying assumptions (see for example Brennan and Schwartz (1982)), the risk adjusted processes for the state variables can be obtained from the true processes as in:

$$(11) \ \frac{dR(t)}{R(t)} = [m(t) - l_1 s\,(t)]dt + s\,(t)dz_1^*$$

$$(12) \ dm(t) = [k\,(\overline{m} - m(t)) - l_2 h(t)]dt + h(t)dz_2^*$$

$$(13) \ dz_1^* dz_2^* = r\,dt$$

where the market prices of factor risks, l_1 and l_2, are constant.

The expectation in (10) is taken with respect to these risk-adjusted processes. Note that since the cash flow in (10) is discounted at the risk free rate and it is also

[2] It would be easy to incorporate stochastic interest rates into our framework.

assumed to earn the risk free rate if retained in the firm, if the probability of bankruptcy is negligible, the timing of the cash flow does not affect V(0).

Implicit in the model described above is that the value of the firm at any point in time is a function of the value of the state variables (revenues, expected growth in revenues, loss-carry-forward and cash balances) and time. That is, the value of the firm can be written as:

$$(14) \qquad V \equiv V(R, m, L, X, t)$$

Applying Ito's Lemma to this expression we can obtain the dynamics of the value of the firm as

$$(15) \quad dV = V_R dR + V_m dm + V_L dL + V_X dX + V_t dt + \tfrac{1}{2} V_{RR} dR^2 + \tfrac{1}{2} V_{mm} dm^2 + V_{Rm} dRdm$$

The volatility of the firm value can be derived directly from this expression

$$(16) \quad s_V^2 = \frac{1}{dt} \operatorname{var}(\tfrac{dV}{V}) = (\tfrac{V_R}{V} sR)^2 + (\tfrac{V_m}{V} h)^2 + 2 \tfrac{V_R V_m}{V^2} R s h r$$

The model can then be used, not only to determine the value of the firm, but also its volatility.

3. DISCRETE VERSION OF THE MODEL

The model developed in the previous section is path-dependent. The cash available at any point in time, which determines when bankruptcy is triggered, depends on the whole history of past cash flows. Similarly, the loss-carry-forward, which determines when the firm has to pay corporate taxes, is also path-dependent. In a more general model that also includes depreciation tax shields, which would affect the after tax cash flows, path dependencies would become even more complex.

These path-dependencies can easily be taken into account by using Monte Carlo simulation to solve for the value of the Internet company. To implement the simulation we use the discrete version of the risk-adjusted process (11)-(13)[3]:

$$(17)\ R(t+\Delta t) = R(t)e^{\{[m(t) - l_1 s(t) - \frac{s(t)^2}{2}]\Delta t + s(t)\sqrt{\Delta t}e_1\}}$$

$$(18)\ m(t+\Delta t) = e^{-k\Delta t}m(t) + (1-e^{-k\Delta t})(\overline{m} - \frac{l_2 h(t)}{k}) + \sqrt{\frac{1-e^{-2k\Delta t}}{2k}}h(t)\sqrt{\Delta t}e_2$$

where

$$(19)\ s(t) = s_0 e^{-k_1 t} + \overline{s}(1-e^{-k_1 t})$$

$$(20)\ h(t) = h_0 e^{-k_2 t}$$

Equations (19) and (20) are obtained by integrating (3) and (4), with initial values s_0 and h_0. e_1 and e_2 are standard normal variates with correlation r.

The net after tax cash flows is still given by equation (6) where both revenues and costs are measured over the period Δt. The discrete versions of the dynamics of the loss-carry-forward and the amount of cash available are immediate from equations (8) and (9), respectively.

4. ESTIMATION OF THE PARAMETERS OF THE MODEL

Even the simple model described in the previous section requires more than 20 parameters for its implementation. Some of these parameters are easily observable, while

[3] For a discrete version similar to equation (18) see Schwartz and Smith (1997).

others can be estimated with the quarterly data available for most Internet firms. The determination of some parameters, however, would require the use of judgement which can only come from a deep knowledge of the specific situation. In this section, we will give some insights into the estimation of the parameters of the model. This is probably the most critical step in the analysis, and the one that requires the most expertise in the particular Internet firm being valued as well as the industry to which the firm belongs.

It should also be kept in mind that at this stage we are valuing the whole firm. To obtain the value of the stock, we need to get into the details of the capital structure and of the options that most of these firms grant generously to their employees. We do this in Section 6.

In Table 1, we describe the parameters of the model and give some suggestions of possible ways to estimate these parameters. For the actual implementation of the approach a more detailed study would be required. Since there is a limited past history of the firm to estimate the parameters, it is essential to use judgement and the knowledge of the firm's industry and characteristics to infer the parameters.

5. SIMULATION RESULTS

In this section, we illustrate the methodology developed for valuing Internet companies by applying it to one of the best known firms in the sector: Amazon.com. The basic data we are working with is given in Table 2 and includes quarterly sales, cost of goods sold, and other expenses for the last fifteen quarters. In addition to this data, we have also used balance sheet data to estimate the loss-carry-forward and the amount of cash available. We perform the evaluation with the information available as of December

31, 1999. This includes financial statements from the third quarter of 1999, as well as supplementary current analyst projections.

Figure 1 shows that sales grow dramatically at the beginning of the sample period, but then begin to slow. Figure 2 presents the growth rate over the sample period; it starts out very high and then declines. Figures 3 and 4 show the relation between cost of goods sold and sales (COGS vs. Sales), and selling, general, and administrative expenses and sales (SG&A vs. Sales). While the relation between cost of goods sold seems to be very stable, the relation between SG&A and sales is more erratic. This is due in part to extraordinary expenses in building infrastructure, some of which do not reflect actual cash outlays. Figure 5 shows the stock price from May 1997 through December 1999. As can be seen from this figure the stock price has grown dramatically up to December 1998, after which it has exhibited great volatility without an apparent trend.

In Table 3, we present the parameters used in our basic valuation of Amazon.com. Some of these parameters come from the financial statements or are otherwise directly observable and thus do not require further explanation. Others, however, that have been estimated using past data and/or future projections should be discussed further. We will perform sensitivity analysis to determine the sensitivity of the firm value to changes in such estimated parameters.

As the initial expected rate of growth in revenues we took the average growth rate over the last two quarters and the analysts expectations from IBES of the rate of growth over the next four quarters. The standard deviation of past percentage changes in revenue was used as the initial volatility of revenues. The initial volatility of the expected rate of growth in revenues was inferred from the observed stock price volatility. We assumed that the changes in revenues and changes in expected growth rates were uncorrelated [4]. For the long-term rate of growth in revenues for the industry we choose 0.015 per quarter (0.06 per year) and for the long-term volatility of revenues we choose 0.05 per quarter (0.10 per year). To obtain the three speed-of-adjustment or mean-reversion coefficients, we assumed that the "half-life" of the deviations is approximately 10 quarters.

The cost parameters, which we have simplified in this illustration, are a critical part of the analysis. We assumed that the costs of goods sold are 75% of the revenues, very much in accordance with the data available. We used a higher fixed component of other expenses ($75 million per quarter) and a lower variable component as a proportion of revenues (19%) than the historical past to reflect some recent extraordinary expenses. Had we used the cost parameters estimated from the historical data, the firm would have never made any profits, since the profit margins were negative.

To estimate the two market prices of risk, we used as the standard deviation for aggregate wealth 0.05 per quarter (or 0.10 per year). We assumed a correlation of 0.2 between the percentage changes is revenue and the return on aggregate wealth, but we assumed that the changes in growth rates were uncorrelated with aggregate wealth. Finally, we took 25 years as the horizon of the estimation and a quarter as the time increment since all the data we had was provided quarterly. For a terminal value at this horizon we assumed the value of the firm was equal to ten times pre-tax operating profit (EBITDA), as frequently employed by practitioners.

For all the valuations, we used 100,000 simulations. For the base valuation using the parameters of Table 3 the total value of Amazon was $5,457 million. This value was obtained even though in 27.9% of the simulations the firm went bankrupt. Table 4

[4] It is difficult to get contemporaneous values of these variables to compute the correlation.

reports the proportion of bankruptcies per year. Note that bankruptcies only start in year 5 when the cash available has been exhausted and that there are no bankruptcies after year 18. The majority of the bankruptcies occur in year 6 (9%) and decrease slowly up to year 18 (0.1%).

Table 5 reports the sensitivity of the total value of the firm to the most critical parameters. The numbers in this table have been obtained by valuing the firm using a perturbation (usually a ten-percent higher value) for the indicated parameter while leaving all the other parameters the same as the base valuation.

Table 5 shows that there are two sets of parameters that have a significant effect on the value of the firm. First and most obviously, is the variable component of the cost function, which is proportional to the revenues. From equation (7), we see that an increase in ether a or b has the same effect on the cost function and therefore also on the value of the firm. In the base example, the sum of these two variable costs is 94% of sales, leaving a profit margin of only 6% of sales. If we increase any of these variable cost by 1% as in Table 5, the profit margin decreases to 5% of sales and the value of the firm decreases from $5.5 billion to $4.3 billion (a 22% decrease, in line with the decrease in profit margin). This discussion emphasizes the importance of correctly assessing the variable components of the cost function. It should be pointed out, however, that this would be the case for any method of analysis.

The second and not so obvious parameters that have a significant effect on the value of the firm are the parameters for the stochastic process of the changes in the growth rate in revenues (equation (2)), in particular those that affect the future distribution of rates of growth in revenues. An increase in the initial volatility of this rate of growth (h_0) from 0.03 per quarter to 0.033 (a 10% increase) increases the value of the firm from 5.5 billion to 6.6 billion. Similarly, but in the opposite direction, an increase in the mean reversion coefficient (k) from 0.07 to 0.077 decreases the value of the firm from 5.5 to 4.3 billion. The deterministic mean reversion coefficient for the volatility of this process (k_2) also has a significant effect, but not as large as the other two. These three parameters affect the distribution of future rates of growth in revenues. Increases in the first of these parameters will increase the variance of this distribution, and increases in the other two will reduce this variance.

The variance of the distribution of future growth rates is important in the valuation because it determines the *option value* of the Internet firm. High variance of future rates of growth implies that there is a higher probability of very high rates of growth and of very low (or even negative) rates of growth. For individual paths of the growth rate over time, higher growth rates lead to larger cash flows, which imply a more valuable firm. In contrast, if growth rates are sufficiently low, the firm may go bankrupt. However, if the firm goes bankrupt, it will be worth zero if growth rates are just low enough for the firm to go bankrupt or even if growth rates are far lower than that critical level. Limited liability for the shareholders of the firm implies a non-linearity in the valuation function, which results in a more valuable firm given a more variable distribution of future growth rates. Figure 6 shows the probability density of rates of growth in revenues five years and ten years into the future for the parameters of the base valuation. Because the variance of this distribution is important to the valuation, parameters should be jointly chosen to give what is believed to be a reasonable distribution of future rates of growth (and of future revenues) for the Internet firm.

In addition to the option effect, the variance of the rate of growth in revenues has an effect on the mean of revenue distribution. Higher volatility implies higher mean revenues. This effect is due to Jensen's inequality and can be inferred from equation (17).

Table 6 shows the quarterly distribution of revenues one, three, five, seven and ten years in the future. The means for one and three years are approximately consistent with analysts' forecasts. Note that the mean quarterly revenues grow substantially over time reaching \$3.8 billion in ten years.

6. DETERMINING THE SHARE VALUE

In the preceding sections, we have developed and implemented a model to value an Internet firm. To obtain the share price, we need to look in more detail at the capital structure of the firm. We need to know how many shares are currently outstanding and how many shares are likely to be issued to current employee stock option holders and convertible bondholders. We also need to know how much of the cash flows will be available to the shareholders after coupon and principal payment to the bondholders.

To simplify the analysis, we assume that options will be exercised and convertible bonds will be converted into shares whenever the firm survives. This means that in the no-default paths of the simulations, we adjust the number of shares to reflect the exercise of options and convertibles. To obtain the cash flows available to shareholders from the cash flows available to all security-holders (which determine the total value of the firm), we subtract the principal and after-tax coupon payments on the debt and add the payments by option holders at the exercise of the options. Since we are assuming that the firm pays no dividends, the exercise of the options and convertibles occurs at their maturity. If all option holders exercise their options optimally, the above procedure overvalues the stock by undervaluing the options and convertibles, since there might be some states of the world where the firm survives but it is not optimal to exercise the options or convert the convertible.

In addition, it is well known, that for diversification purposes, employee stock options are frequently exercised before maturity, if they are exercisable, to allow for the sale of the underlying stock. Also, even if they are in the money, not all the options will be exercised since many of the employees will leave the firm before their options become vested. If the number of shares to be issued at exercise and conversion is small relative to the total number of shares outstanding, the impact of these issues on the share value is likely to be very small. In Section 7 we discuss an extension which takes into account the optimal exercise of these options.

At the valuation date, Amazon.com had 339 million shares outstanding. In addition to equity, the capital structure consists of a convertible bond, a discount note and employee options. The convertible debt issue has a face value of $1,250 million with a coupon rate of 4.75%; it matures in February of 2009 and is convertible into 16 million shares. The senior discount notes have a face value of $265 million and matures in May of 2008. The employee stock options outstanding as of December 31, 1999 were obtained from the company’s 10-K and have been adjusted for a subsequent stock split. In total there are 76 million options outstanding, of which 60 million (more that 78%) have average exercise prices below $7.5. Since the current stock price is of the order of $75 per share, it is very likely that, if the firm survives, these options will be exercised.

We modified the simulation program to take into account the shares issued at the exercise of the options and conversion of the convertible, and to compute the part of the cash flows which belong to the shareholders. The stock value obtained for the base valuation is $12.42. This value is strikingly lower than the market price of $76.125 at the close of the year 1999. Below we discuss this large discrepancy.

The above analysis has implicitly assumed that the total cash flows available to all security-holders are independent of the capital structure. Recall that bankruptcy occurs when the cash balances are driven to zero. This means that, for example, when a debt matures and it is paid, an equal amount of debt is issued to keep the cash balances the same. Alternative financing assumptions can easily be incorporated into the analysis if they are judged to be more reasonable.

The volatility of the firm can be obtained from Equation (16) and the volatility of the equity can be obtained from an identical equation where we substitute the equity value for the firm value. The partial derivatives of firm (and equity) value with respect to the level of revenues and to the expected rate of growth in revenues are obtained by simulation[5]. With the parameters used in the base valuation we obtain a volatility for the equity of 106% per year. This model volatility is consistent with observed historical volatility of Amazon.com equity over the past year. (Recall that we chose the volatility of the expected growth rate in revenues to give this result.)

Since the volatility of the expected growth rate of revenues (η_0) is the most critical parameter in the valuation model, in Figures 7 and 8 we show its effect on the stock price and its volatility, respectively. As can be seen from the figures, the stock price increases

[5] The initial value of the revenues (the rate of growth in revenues) is perturbed to obtain new values of the firm and the equity from which these derivatives are computed.

dramatically with η_0, whereas the volatility of the stock price increases linearly. Further, to obtain a model stock price consistent with the market price, a value of 0.06 for η_0 would be required. However, such a value would also produce a model volatility of 182% which is almost double the market volatility. In addition, the revenue distribution implied by this parameter would seem to be unrealistic.

This analysis suggests that given the profitability assumed in the base valuation (through the cost function), Amazon.com equity is overpriced or its volatility is too low. We would need substantially higher profitability to obtain model prices and volatilities that are consistent with those observed in the market. The profit margin before taxes would have to increase from 6% to 30% to attain this result.

7. EXTENSIONS

In the model developed in this article we have made some simplifying assumptions about the optimal exercise of American-type options. We have assumed that bankruptcy depends only on the level of the cash balances and that when this level goes to zero, the value of the firm also goes to zero. The value of the firm, however, depends not only on the level of cash balances but also on all the other state variables of the problem: the level of revenues, the expected rate of growth in revenues, their volatilities, and the amount of loss-carry-forward. It could very well be the case that the cash balances go to zero, but that the prospects of the firm are good enough for the firm to be able to raise additional cash or merge with another company.

To determine the value of the common equity, we have also assumed that options are exercised and convertibles are converted whenever the firm survives, whereas the optimal exercise of these options again depends of the value of the firm at the decision date. For example, at the maturity of the convertible debt, the face value of the debt could be larger than its conversion value and the bondholders would optimally not convert and receive the face value instead.

Very recently, Longstaff and Schwartz (1998) developed a least-squares Monte Carlo approach (LSM) to value American-type options by simulation, which can be easily adapted to deal with the issues above. In the case of American-type options, the issue is to compare the value of immediate exercise with the conditional expected value (under the risk neutral measure) of continuation. The conditional expected value of continuation, for each path at each point in time, is obtained from the fitted value of the linear regression of the discounted value (at the risk free rate) of the cash flows obtained from the simulation following the optimal policy in the future, on a set of basis functions of the state variables. Since this is a recursive procedure starting from the maturity of the option, the outcome is the optimal stopping time for each path in the simulation. Knowing the optimal stopping time for each path, the American option can be easily valued.

In our case, the objective is to determine the conditional expected value of the firm (under the risk neutral measure) at each point in time. We start from the horizon T, where the value of the firm will be equal to the maximum of the cash balances or zero. Note that now we do not stop a particular path when the value of the cash balances are zero, since we want to optimally determine the stopping time (bankruptcy). Then we move back to time T-Δt. To determine the conditional expected value of the firm at this point in time, we regress the discounted (at the risk free rate) cash flow (firm value) in period T on a set of basis functions of the state variables (revenues, rate of growth in revenues, volatilities, cash balances and loss-carry-forward) at time T-Δt.[6] The fitted value of this regression is the conditional expected value of the firm. If this value is less or equal to zero, the firm is bankrupt and the value of the firm is zero. We proceed recursively in the same manner up to the present time. This procedure will give us the optimal stopping time for each path from which now we can calculate the current value of the firm.

To determine the optimal exercise of the options and convertibles, we can follow a similar procedure. Knowing the value of the firm at each possible exercise date allows us to determine whether the exercise value of the options is larger than the exercise price, or whether the conversion value larger than the face value of the convertible bonds. This approach allows us to accurately keep track of the number of shares outstanding and of the part of total cash flows that belong to the shareholders, and therefore to estimate a more accurate share value.

7. SUMMARY AND CONCLUSIONS

The valuation of Internet companies has been a subject of much discussion in the financial press and among financial economists. In this article, we developed a simple model to value these companies which is based fundamentally on assumptions about the expected growth rate of revenues and on expectations about the cost structure of the firm. Since these expectations are likely to change continuously as new information becomes available, it is not surprising that the model generates firm values and stock prices which

[6] See Longstaff and Schwartz (1998) for details on the selection of the basis functions.

are highly volatile. The model, however, gives a systematic way to think about the drivers of value of Internet companies and directs the attention to the parameters that are more critical in the valuation.

To be able to implement the model, we have had to make many assumptions about possible future financing, about future cash distributions to shareholders and bondholders, about the horizon of the estimation, etc. Alternative assumptions are possible and easily incorporated into the analysis. We expect that potential users of a model such as the one presented in this article would have a deeper knowledge of the firm and the industry in which the firm operates to make (perhaps!) more reasonable assumptions.

One issue which we have not addressed in this article is seasonality. Revenues of firms in certain industries exhibit seasonality. If this is not taken into account when estimating parameters, the effect will be to overestimate the volatility of the growth rate in revenues. When this issue is of significance it should be accounted for in the estimation process by using seasonally adjusted revenues.

One of the more challenging issues in this analysis has been the estimation of the parameters to use in the model. To illustrate the application of the model, we have estimated these parameters using data from only one firm, and made some judgement calls on the parameters for which we had no data. A more thorough analysis could use cross sectional data from a larger sample of Internet companies to estimate the parameters. The cross sectional data could also be used testing the model. We plan to pursue these issues in future research.

Acknowledgment

We thank Michael Brennan and Russ Fuller for helpful suggestions and Marc Kelly for editorial comments. This article was written while the first author was a Visiting Scholar at the University of British Columbia during the summer of 1999.

References

Berger, P.G., E. Ofek and I. Swary. (1996), “Investors Valuation of the Abandonment Option”, Journal of Financial Economics 42:2, 257-287.

Brennan, M.J. and E.S. Schwartz, (1982), "Consistent Regulatory Policy Under Uncertainty," The Bell Journal of Economics, 13,, 2, 507-521 (Autumn).

Longstaff, F.A. and E.S. Schwartz, (1998), “Valuing American Options by Simulation: A Simple Least-Square Approach”, UCLA Working Paper (August.)

Schwartz, E.S. and J.E. Smith, (1997), “Short-Term Variations and Long-Term Dynamics in Commodity Prices”, UCLA Working Paper (June).

Parameter		Proposed estimation procedure
Initial revenue	R_0	Observable from current income statement
Initial loss-carry-forward	L_0	Observable from current balance sheet
Initial cash balance available	X_0	Observable from current balance sheet
Initial expected rate of growth in revenues	m_0	From past income statements and projections of future growth
Initial volatility of revenues	s_0	Standard deviation of percentage change in revenues over the recent past
Initial volatility of expected rates of growth in revenues	h_0	Inferred from market volatility of stock price
Correlation between change in revenue (%) and change in expected rate of growth	r	Estimated from past company or cross-sectional data
Long-term rate of growth in revenues	$\overline{m}$	Rate of growth in revenues for a stable firm in the same industry as the firm
Long-term volatility of the rate of growth in revenues	$\overline{s}$	Volatility of percentage changes in revenues for a stable firm in the same industry as the firm
Corporate tax rate for the firm	t_c	Observable from tax code
Risk free interest rate	r	One year Treasury-bill rate
Speed of adjustment for the rate of growth process	k	Estimated from assumptions about the “half life” of the process to $\overline{m}$
Speed of adjustment for the volatility of revenue process	k_1	Estimated from assumptions about the “half life” of the process to $\overline{s}$
Speed of adjustment for the volatility of the rate of growth process	k_2	Estimated from assumptions about the “half life” of the process to zero
Cost of goods sold as a percentage of revenues	a	Analysts’ future projections
Fixed component of other expenses	F	Analysts’ future projections
Variable component of other expenses	b	Analysts’ future projections
Market price of risk for the revenue factor	l_1	Obtained from the product of the correlation between % changes in revenues and return on aggregate wealth, and the standard deviation of aggregate wealth
Market price of risk for the expected rate of growth in revenues factor	l_2	Obtained from the product of the correlation between changes in growth rates in revenues and return on aggregate wealth, and the standard deviation of aggregate wealth
Horizon for the estimation	T	An arbitrary long term horizon at which the firm is deemed to become a “normal” firm
Time increment for the discrete version of the model	Δt	Chosen according to the data availability, which is usually quarterly

Table 1

Key Parameters of the Model

	Sales	Cost of Goods Sold	Gross Profit	SG&A	Operating Profit Before Taxes (EBITDA)
Mar-96	0.875	0.678	0.197	0.516	-0.319
Jun-96	2.230	1.725	0.505	1.253	-0.748
Sep-96	4.173	3.172	1.001	3.383	-2.382
Dec-96	8.468	6.426	2.042	4.286	-2.244
Mar-97	16.005	12.484	3.521	6.623	-3.102
Jun-97	27.855	22.641	5.214	13.067	-7.853
Sep-97	37.887	30.717	7.170	17.486	-10.316
Dec-97	66.040	53.127	12.913	24.237	-11.324
Mar-98	87.361	66.222	21.139	29.283	-8.144
Jun-98	116.044	89.793	26.251	44.651	-18.400
Sep-98	153.698	118.823	34.875	76.381	-41.506
Dec-98	252.893	199.476	53.417	95.486	-42.069
Mar-99	293.643	223.629	70.014	95.386	-25.372
Jun-99	314.377	246.846	67.531	190.005	-122.474
Sep-99	355.8	285.3	70.500	260.945	-190.445

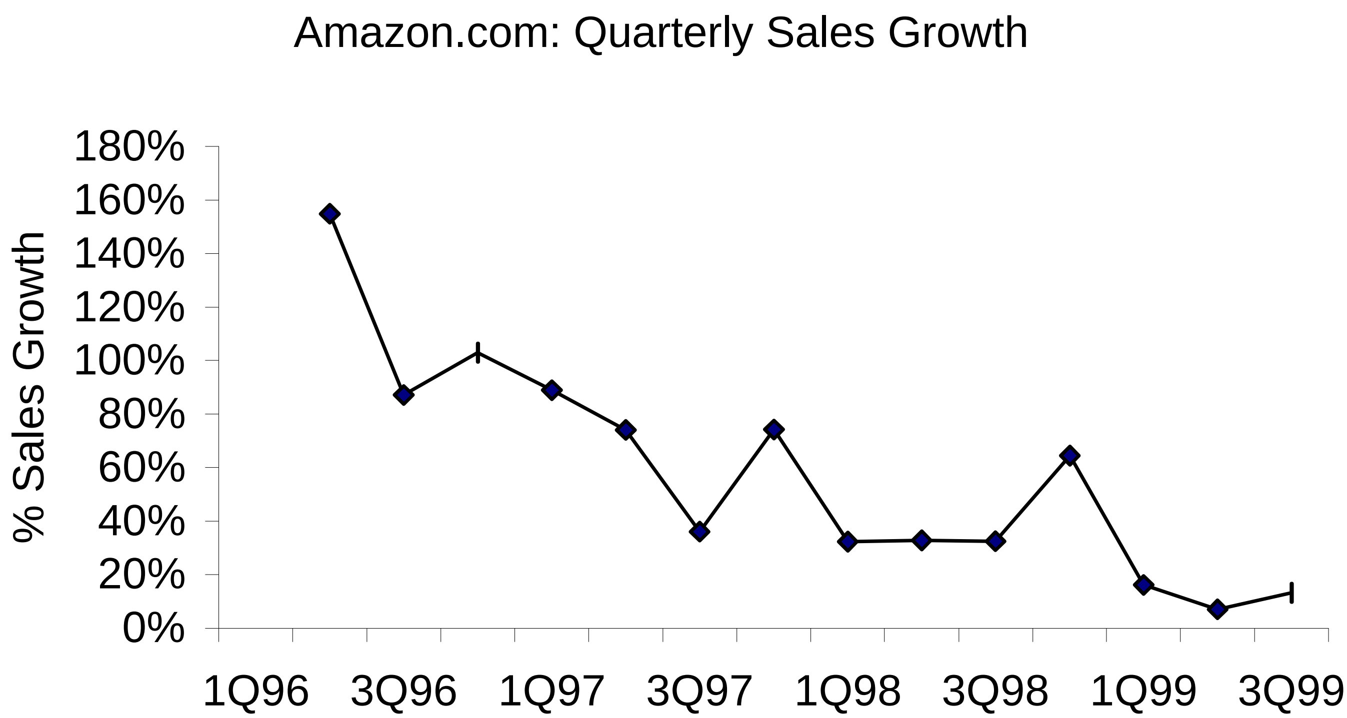
Table 2

Quarterly Sales and Costs for Amazon.com from March 96 to September 99

(In million of dollars)


Amazon.com: Quarterly Sales
Sales ($MM)
$400
$300
$200
$100
$0
1Q96
3Q96
1Q97
3Q97
1Q98
3Q98
1Q99
3Q99

Figure 1



Amazon.com: Quarterly Sales Growth
% Sales Growth
180%
160%
140%
120%
100%
80%
60%
40%
20%
0%
1Q96
3Q96
1Q97
3Q97
1Q98
3Q98
1Q99
3Q99

Figure 2

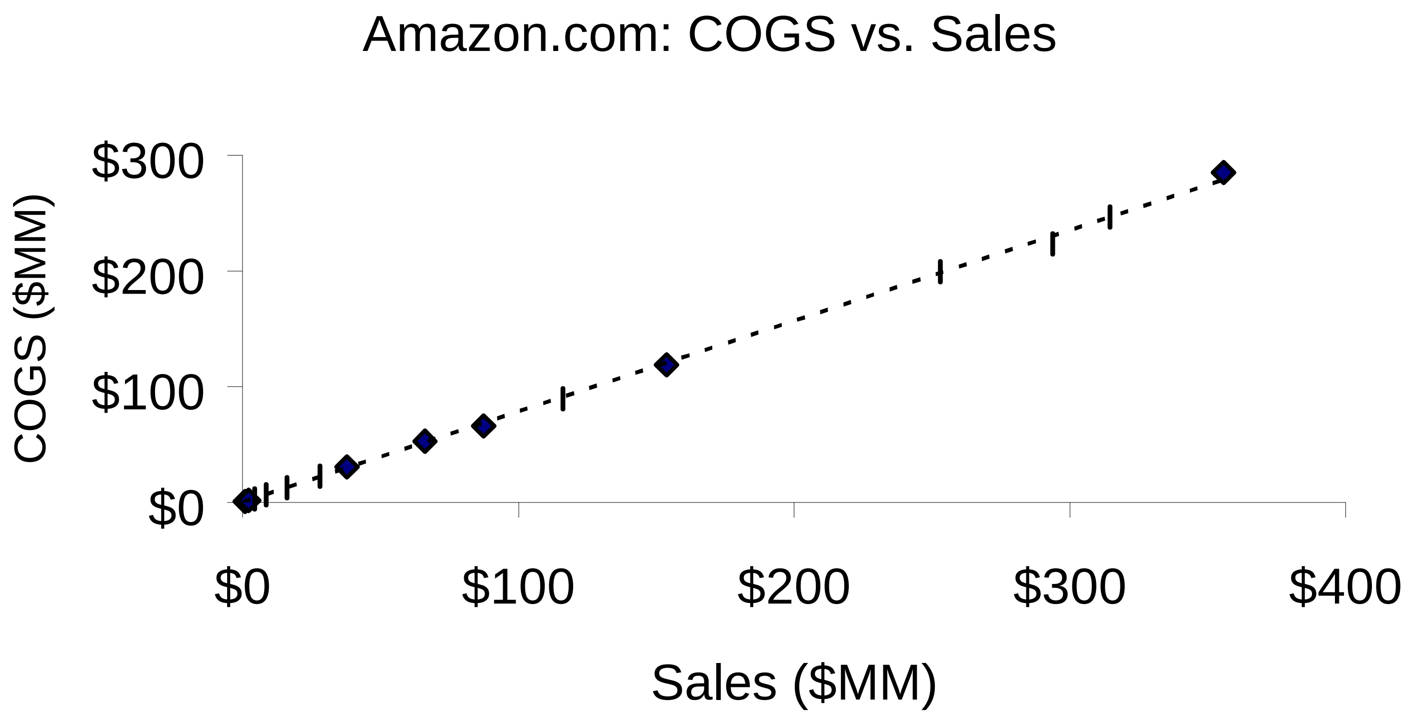
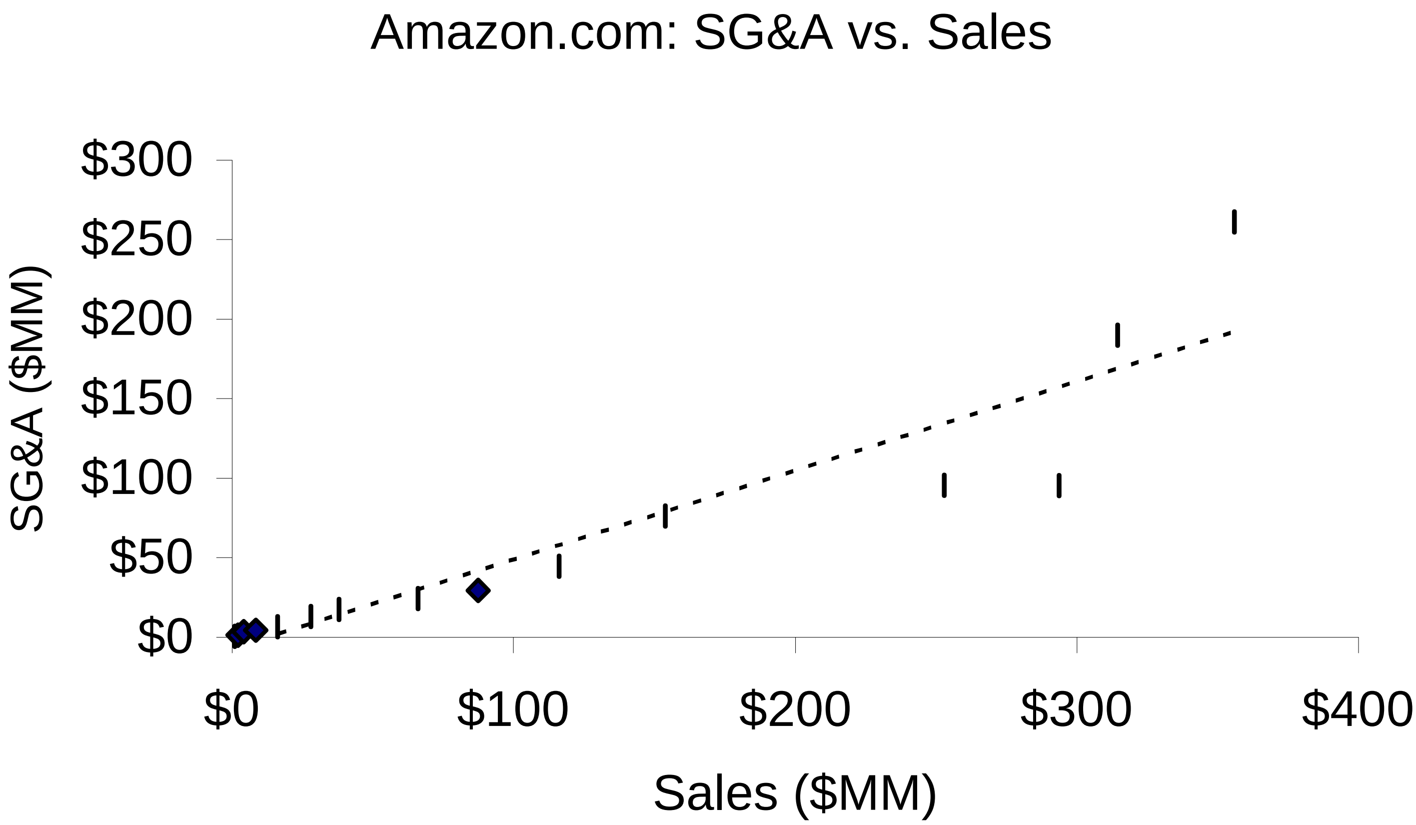

Amazon.com: COGS vs. Sales
$300
$200
$100
$0
COGS ($MM)
$0
$100
$200
$300
$400
Sales ($MM)

Figure 3



Amazon.com: SG&A vs. Sales
SG&A ($MM)
$300
$250
$200
$150
$100
$50
$0
$0
$100
$200
$300
$400
Sales ($MM)

Figure 4


Amazon.com Share Price
(split-adjusted)
$100
$80
$60
$40
$20
$0
5/97
8/97
11/97
2/98
5/98
8/98
11/98
2/99
5/99
8/99
11/99

Figure 5

Initial revenue	R_0	\$356 million/quarter
Initial loss-carry-forward	L_0	\$559 million
Initial cash balance available	X_0	\$906 million
Initial expected rate of growth in revenues	m_0	0.11 per quarter
Initial volatility of revenues	s_0	0.10 per quarter
Initial volatility of expected rates of growth in revenues	h_0	0.03 per quarter
Correlation between change in revenue (%) and change in expected rate of growth	r	0.0
Long-term rate of growth in revenues	$\bar{m}$	0.015 per quarter
Long-term volatility of revenues	$\bar{s}$	0.05 per quarter
Corporate tax rate for the firm	t_c	0.35
Risk free interest rate	R	0.05 per year
Speed of adjustment for the rate of growth process	k	0.07 per quarter
Speed of adjustment for the volatility of revenue process	k_1	0.07 per quarter
Speed of adjustment for the volatility of the rate of growth process	k_2	0.07 per quarter
Cost of goods sold as a percentage of revenues	a	0.75
Fixed component of other expenses	F	\$75 million per quarter
Variable component of other expenses	b	0.19
Market price of risk for the revenue factor	l_1	0.01 per quarter
Market price of risk for the expected rate of growth in revenues factor	l_2	0.0 per quarter
Horizon for the estimation	T	25 years
Time increment for the discrete version of the model	Δt	1 quarter

Table 3

Parameters Used in the Base Valuation of Amazon:com

Year	% Bankruptcy
1	0.0%
2	0.0%
3	0.0%
4	0.0%
5	3.9%
6	9.0%
7	6.2%
8	3.5%
9	2.0%
10	1.1%
11	0.7%
12	0.5%
13	0.3%
14	0.2%
15	0.2%
16	0.1%
17	0.1%
18	0.1%
19	0.0%
20	0.0%
21	0.0%
22	0.0%
23	0.0%
24	0.0%
25	0.0%
Total	27.9%

Table 4

Probability of Bankruptcy per Year for Base Valuation

Valuation	Perturbed Parameter	Total Firm Value (million dollars)	St. Dev.	Probability of Bankruptcy
Base Valuation		5,457	34	27.9%
m_0	0.121 per quarter	6,558	39	22.8%
s_0	0.11 per quarter	5,446	34	28.7%
h_0	0.033 per quarter	6,256	44	29.6
r	0.01	5,483	34	28.0%
$\overline{m}$	0.0165 per quarter	6,064	14	26.9%
$\overline{s}$	0.055 per quarter	5,437	34	28.5%
k	0.077 per quarter	4,282	24	29.9%
k_1	0.077 per quarter	5,461	33	27.8%
k_2	0.077 per quarter	5,134	30	27.2%
a	0.76	4,349	28	37.1%
F	\$82.5 million per quarter	5,253	34	35.6%
b	0.20	4,349	28	37.1%
l_1	0.011 per quarter	5,429	33	28.1%
l_2	0.001 per quarter	5,423	33	28.1%
T	26 years	5,620	35	28.2%

Table 5

Sensitivity of Firm Value to Different Parameters

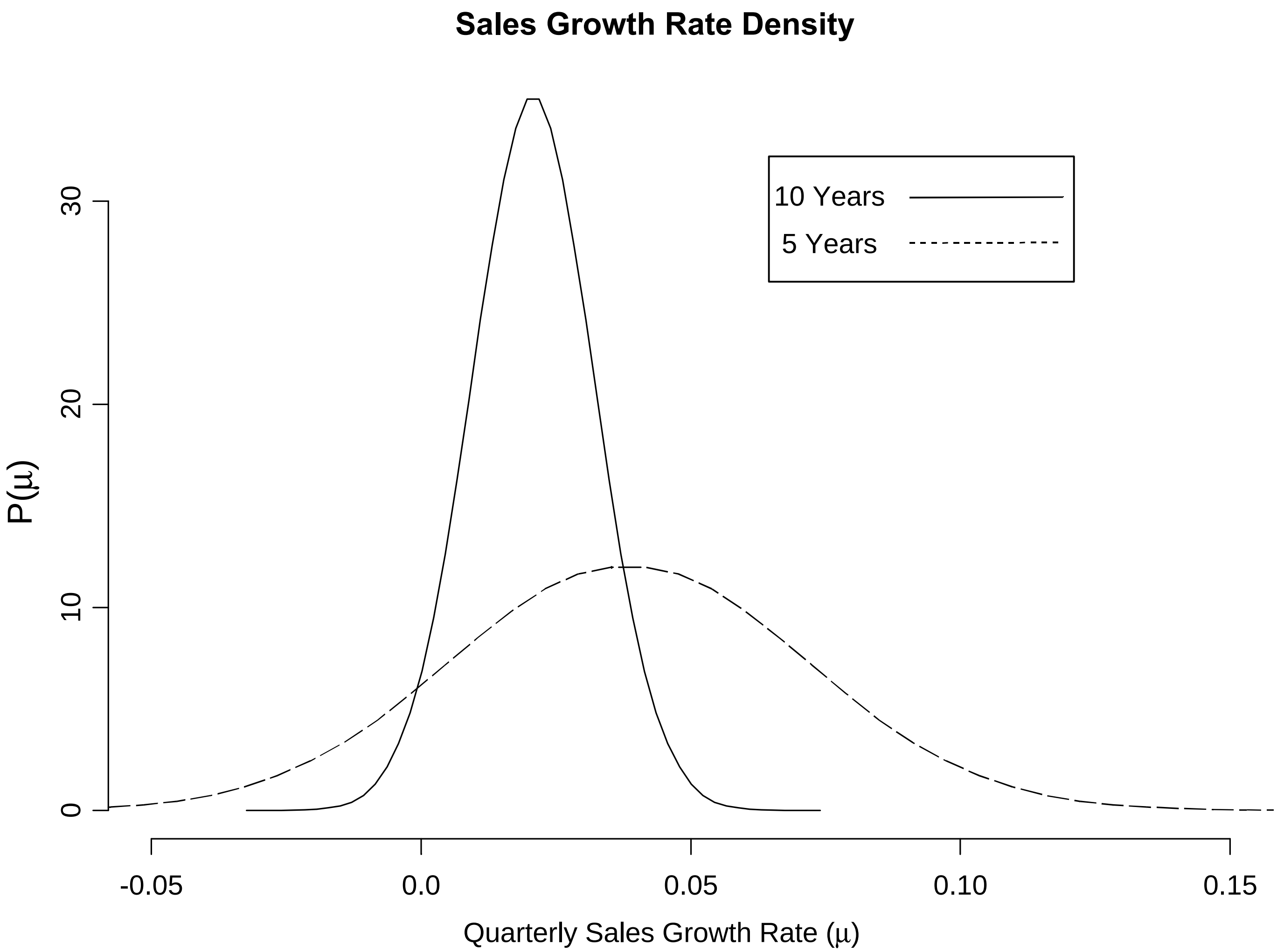

Sales Growth Rate Density
10 Years
5 Years
P(μ)
30
20
10
0
-0.05
0.0
0.05
0.10
0.15
Quarterly Sales Growth Rate (μ)

Figure 6

Amazon.com
Quarterly Revenue Distributions
($ MM)

Percentile	1 yr Fwd.	3yr Fwd.	5yr Fwd.	7yr Fwd.	10yr Fwd.
5%	370	398	379	366	374
10%	399	476	495	511	547
15%	421	538	597	641	715
20%	438	593	692	766	879
25%	453	643	782	893	1,051
30%	468	693	873	1,024	1,234
35%	482	743	967	1,161	1,427
40%	495	794	1,066	1,311	1,648
45%	508	846	1,172	1,472	1,887
50%	522	899	1,286	1,651	2,158
55%	535	956	1,411	1,850	2,468
60%	550	1,019	1,550	2,078	2,827
65%	565	1,088	1,709	2,346	3,265
70%	581	1,166	1,893	2,661	3,775
75%	600	1,257	2,114	3,053	4,431
80%	621	1,365	2,388	3,559	5,300
85%	646	1,503	2,770	4,254	6,510
90%	681	1,700	3,337	5,332	8,521
95%	735	2,030	4,363	7,444	12,448
Mean	533	1,017	1,692	2,507	3,810

Table 6

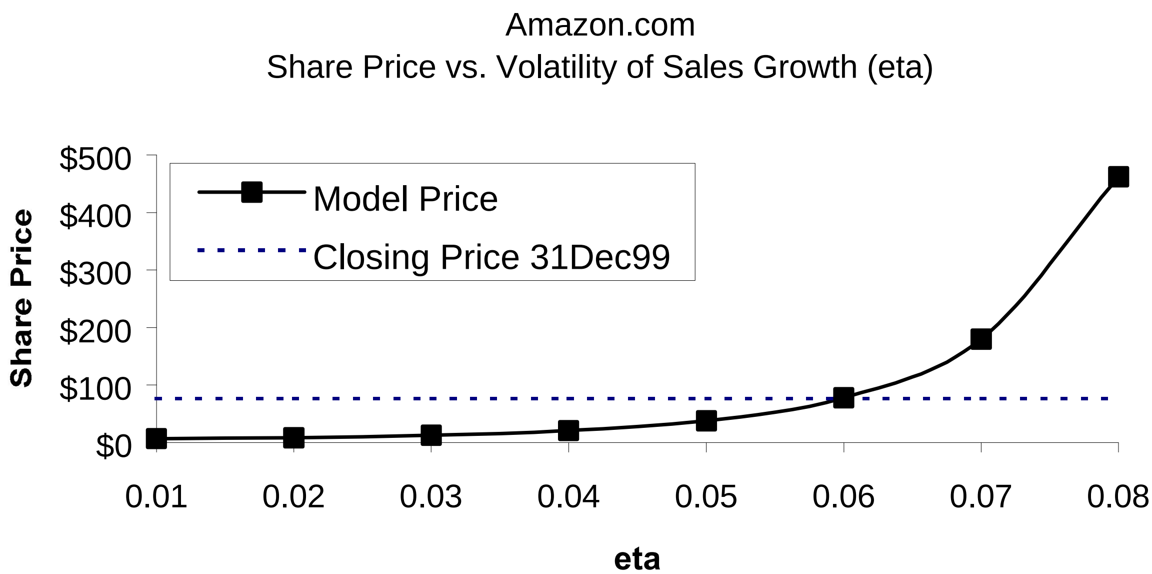

Amazon.com
Share Price vs. Volatility of Sales Growth (eta)
Share Price
$500
$400
$300
$200
$100
$0
Model Price
Closing Price 31Dec99
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
eta

Figure 7

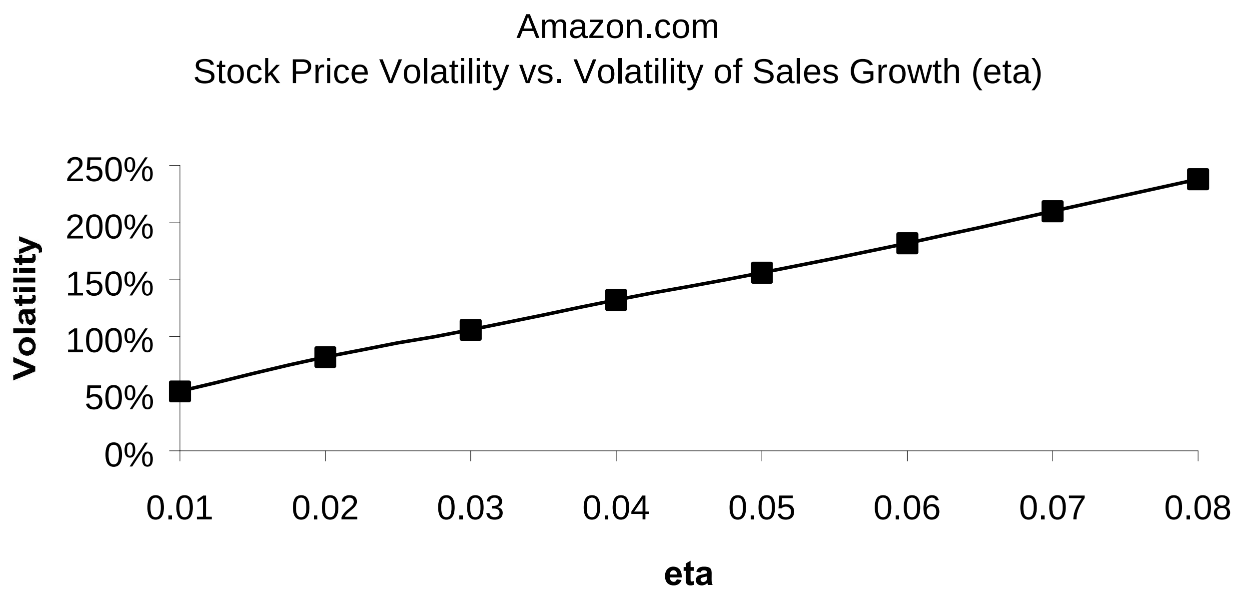

Amazon.com
Stock Price Volatility vs. Volatility of Sales Growth (eta)
Volatility
250%
200%
150%
100%
50%
0%
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
eta

Figure 8