In the proof of Theorem 7.1 we use the fact that certain systems of equations are of full rank. In this appendix we write down the corresponding matrices and show that their determinants are nonzero. The computer program does the calculations involved. For example the matrix below is constructed using the function `construct_matrix_a()`. See the description of the program in the readme file for the other matrices.

We have

$$\begin{aligned}\Theta_{j,2}(a,b) = &-(j+1)(j+2)\left\langle a\omega^{j}, b\omega^{4}\right\rangle + 4j(j+2)\left\langle a\omega^{j+1}, b\omega^{3}\right\rangle \\ &- 6(j^2-j-4)\left\langle a\omega^{j+2}, b\omega^{2}\right\rangle + \frac{4j(j+1)(j+2)}{(j+3)}\left\langle a\omega^{j+3}, b\omega\right\rangle.\end{aligned}$$

We present the matrices from the proof of Lemma 7.6 and their determinants. For the first matrix the rows correspond to the relations

$$\Phi_1(a\omega^i,\omega,\omega), \quad \Theta_{1,2}(1,\omega)\left\langle a\omega^i\right\rangle, \quad \Theta_{i,2}(a,1)\left\langle \omega^2\right\rangle.$$

Remember from (9) that Φ_k is not a single relation but a system of relations parametrized by distributions of the integer k. The rows for $\Phi_1(a\omega^i,\omega,\omega)$ in order are given by the distrubtions $[1,0,0,0,0,0]$ and $[0,1,0,0,0,0]$. The columns correspond to

$$\left\langle a\omega^{i+2},\omega^2,\omega^2\right\rangle, \left\langle a\omega^{i+1},\omega^3,\omega^2\right\rangle, \left\langle a\omega^{i},\omega^4,\omega^2\right\rangle, \left\langle a\omega^{i},\omega^3,\omega^3\right\rangle.$$

$$\begin{pmatrix} -30 & 10 & 0 & 0 \\ 0 & -20 & 4 & 4 \\ 0 & 0 & 0 & 24 \\ 6(i^2-i-4) & 4i(i+2) & -(i+1)(i+2) & 0 \end{pmatrix}$$

This matrix has determinant $-2880(i^2-9i-18)$, which has no integer roots.

For the next matrix the rows correspond to the relations

$$\Phi_2(a\omega^i,b,\omega), \quad (-1)^{|a||b|}\Theta_{1,2}(\omega,b)\left\langle a\omega^i\right\rangle, \quad \Theta_{i+1,2}(a,1)\left\langle b\omega\right\rangle, \quad \Theta_{i,2}(a,1)\left\langle b\omega^2\right\rangle.$$

Here the ordering of the rows for Φ_2 is given by the folliwing distributions of 2.

$$[2,0,0,0,0,0], [1,1,0,0,0,0], [1,0,1,0,0,0], [0,2,0,0,0,0],$$
$$[0,1,1,0,0,0], [0,0,2,0,0,0], [0,0,1,1,0,0].$$

The columns correspond to -3cm-3cm

$$\left\langle a\omega^{i+3},b\omega,\omega^2\right\rangle, \left\langle a\omega^{i+2},b\omega^2,\omega^2\right\rangle, \left\langle a\omega^{i+2},b\omega,\omega^3\right\rangle, \left\langle a\omega^{i+1},b\omega^3,\omega^2\right\rangle, \left\langle a\omega^{i+1},b\omega^2,\omega^3\right\rangle$$
$$\left\langle a\omega^{i+1},b\omega,\omega^4\right\rangle, \left\langle a\omega^{i},b\omega^4,\omega^2\right\rangle, \left\langle a\omega^{i},b\omega^3,\omega^3\right\rangle, \left\langle a\omega^{i},b\omega^2,\omega^4\right\rangle, \left\langle a\omega^{i},b\omega,\omega^5\right\rangle.$$

$$
\left(\begin{matrix}
-24 & 6 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & -24 & 4 & 6 & 4 & -2 & 0 & 0 & 0 \\
-3 & -3 & -12 & 1 & 4 & 3 & 0 & 0 & 0 \\
0 & 6 & 0 & -24 & 4 & 0 & 6 & 4 & -2 \\
1 & -3 & 4 & -3 & -12 & 3 & 1 & 4 & 3 \\
-6 & 0 & 3 & -6 & 3 & -12 & 3 & -1 & 3 \\
4 & 0 & -8 & 4 & -8 & -4 & -3 & 4 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & -6 & 12 & 12 \\
6(i^2+i-4) & 0 & 4(i+1)(i+3) & 0 & 0 & -(i+2)(i+3) & 0 & 0 & 0 \\
0 & 6(i^2-i-4) & 0 & 0 & 4i(i+2) & 0 & 0 & 0 & -(i+1)(i
\end{matrix}\right.
$$

This matrix has determinant $60825600(i^4+22i^3+56i^2-33i-121)$, which does not have any integer roots.

We have

$$\Phi_0(a,b,\omega) \sim -30 \left\langle a\omega, b\omega, \omega^2 \right\rangle .$$

The system of relations $\Phi_1(a,b,\omega)$ is given by the following matrix where the rows correspond to

$$[1,0,0,0,0,0],[0,1,0,0,0,0],[0,0,1,0,0,0]$$

and the columns correspond to

$$\left\langle a\omega^2, b\omega, \omega^2 \right\rangle, \left\langle a\omega, b\omega^2, \omega^2 \right\rangle, \left\langle a\omega, b\omega, \omega^3 \right\rangle$$

$$\begin{pmatrix} -24 & 6 & 4 \\ 6 & -24 & 4 \\ -3 & -3 & -12 \end{pmatrix},$$

which has nonzero determinant.

Consider the equations

$$\Phi_0(a,b,c), \quad \Phi_1(a,b,c), \quad \Phi_2(a\omega^i,b,c),$$

$$\Theta_{i,2}(a,b)\left\langle c\omega \right\rangle, \quad (-1)^{|b||c|}\Theta_{i,2}(a,c)\left\langle b\omega \right\rangle .$$

Together these equations imply $\left\langle a\omega^i, b\omega^j, c\omega^k \right\rangle \sim 0$, for $i \geq 0$, $0 \leq j \leq 2$, and $0 \leq k \leq 2$. The system of 11 relations $\Psi_2(a,b,c)$ has rank 8. So we only take 8 relations corresponding to the distributions

$$[2,0,0,0,0,0],[1,1,0,0,0,0],[1,0,1,0,0,0],[1,0,0,1,0,0],$$

$$[0,2,0,0,0,0],[0,1,1,0,0,0],[0,1,0,1,0,0],[0,0,2,0,0,0].$$

These form our rows together with

$$\Theta_{i,2}(a,b)\left\langle c\omega\right\rangle, \quad (-1)^{|b||c|}\Theta_{i,2}(a,c)\left\langle b\omega\right\rangle.$$

The columns correspond to -3cm-3cm

$\left\langle a\omega^{i+3}, b\omega, c\omega\right\rangle, \left\langle a\omega^{i+2}, b\omega^2, c\omega\right\rangle, \left\langle a\omega^{i+2}, b\omega, c\omega^2\right\rangle, \left\langle a\omega^{i+1}, b\omega^3, c\omega\right\rangle, \left\langle a\omega^{i+1}, b\omega^2, c\omega^2\right\rangle$

$\left\langle a\omega^{i+1}, b\omega, c\omega^3\right\rangle, \left\langle a\omega^{i}, b\omega^4, c\omega\right\rangle, \left\langle a\omega^{i}, b\omega^3, c\omega^2\right\rangle, \left\langle a\omega^{i}, b\omega^2, c\omega^3\right\rangle, \left\langle a\omega^{i}, b\omega, c\omega^4\right\rangle.$

$$\left(\begin{array}{cccccccccc}
-12 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & \\
3 & -12 & 3 & 3 & 3 & -2 & 0 & 0 & 0 & \\
3 & 3 & -12 & -2 & 3 & 3 & 0 & 0 & 0 & \\
-4 & -4 & -4 & 2 & 2 & 2 & 0 & 0 & 0 & \\
0 & 3 & 0 & -12 & 3 & 0 & 3 & 3 & -2 & \\
-2 & 3 & 3 & 3 & -12 & 3 & -2 & 3 & 3 & \\
2 & -4 & 2 & -4 & -4 & 2 & 2 & 2 & 2 & \\
0 & 0 & 3 & 0 & 3 & -12 & 0 & -2 & 3 & \\
\frac{4i(i+1)(i+2)}{i+3} & 6(i^2-i-4) & 0 & 4i(i+2) & 0 & 0 & -(i+1)(i+2) & 0 & 0 & \\
\frac{4i(i+1)(i+2)}{i+3} & 0 & 6(i^2-i-4) & 0 & 0 & 4i(i+2) & 0 & 0 & 0 & -(
\end{array}\right.$$

This matrix has determinant $19008000\frac{(i^2-3)(i^3+3i^2-2i-7)}{(i+3)}$, which does not have any nonnegative integer zeros or poles.

We have

$$\Phi_0(a,b,c) \sim -12\left\langle a\omega, b\omega, c\omega\right\rangle.$$

The system of relations $\Phi_1(a,b,c)$ is given by the following matrix where the rows correspond to

$$[1,0,0,0,0,0],[0,1,0,0,0,0],[0,0,1,0,0,0]$$

and the columns correspond to

$$\left\langle a\omega^2, b\omega, c\omega\right\rangle \; \left\langle a\omega, b\omega^2, c\omega\right\rangle \; \left\langle a\omega, b\omega, c\omega^2\right\rangle$$

$$\begin{pmatrix} -12 & 3 & 3 \\ 3 & -12 & 3 \\ 3 & 3 & -12 \\ -4 & -4 & -4 \end{pmatrix},$$

which has rank 3.