AN UPPER BOUND FOR HESSIAN MATRICES OF POSITIVE SOLUTIONS OF HEAT EQUATIONS ON KÄHLER MANIFOLD

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1. Introduction

In [3], Peter Li and S.-T. Yau developed the fundamental gradient estimate, which is now widely called the Li-Yau estimate, for any positive solution of the heat equation on a Riemannian manifold and showed how the classical Harnack inequality can be derived from their gradient estimate.

Later in [2], Richard Hamilton extended the Li-Yau estimate to the full matrix version of the Hessian estimate under the stronger assumptions that manifold is Ricci parallel and of nonnegative sectional curvature.In [1] Huai-Dong Cao and Lei Ni found in Kähler manifold,the restriction on Gradient Ricci curvature can be removed and the assumption of the sectional curvature in Hamilton's result can be replaced by that of the holomorphic bisectional curvature.

In [5] Qing Han and Qi S. Zhang derived an upper bound of the Hessian matrices of log solutions of the heat equation on manifolds. They proved global and local version of the bounds ,which are generally sharp in respective cases, in fixed metric and under Ricci flow respectively. In fixed metric case, their results need bound on Gradient of Ricci curvature. Motivated by [1], We can remove the bound of Gradient Ricci and replace Riemanian curvature by holomorphic bisectional curvature on Kähler Manifold. We mainly follow [5] in this paper.

Notations: In this paper, (M, g) is a compact Kähler manifold. In local coordinate $(z^1, \ldots, z^n)$:

Christoffel symbol: $\Gamma^k_{ij} = g^{k\bar{p}}\partial_i g_{j\bar{p}}$

Connection:

$$\nabla_i x^j = \partial_i x^j + \Gamma^j_{ik}x^k \quad \nabla_{\bar{i}} x^j = \partial_{\bar{i}} x^j$$

$$\nabla_i a_j = \partial_i a_j - \Gamma^k_{ij}a_k \quad \nabla_{\bar{i}} a_j = \partial_{\bar{i}} a_j$$

Curvature:

$$R^l_{i\bar{j}k} = -\partial_{\bar{j}}\Gamma^l_{ik}$$

$$R_{i\bar{j}k\bar{e}} = R_m(\partial_i, \partial_{\bar{j}}, \partial_k, \partial_{\bar{l}}) = g^{p\bar{l}}R^p_{i\bar{j}k}$$

$$R_{i\bar{j}} = R_{i\bar{j}k\bar{l}}g^{k\bar{l}}$$

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Hermitian paring:

$$S = S_i dz^i + S_{\bar{j}} dz^{\bar{j}}$$
$$T = T_k dz^k + T_{\bar{l}} dz^{\bar{l}}$$
$$\langle S, T \rangle = g^{i\bar{l}} S_i T_{\bar{l}} + g^{k\bar{j}} T_k S_{\bar{j}}$$

The Hessian of a function u is written as $u_{i\bar{j}}$. If V is a $(1,1)$-form on M and ξ is a $(1,0)$-vector field, then in local coordinates, we use $\bar{\xi}^T V \xi$ to denote $V(\xi, \xi)$. The distance function is denoted by $d(x, y)$. A geodesic ball is denoted by $B(x, r)$ where x is a point in M and r is the radius. For $R > 0$ and $T > 0$, a parabolic cube is defined by

$$Q_{R,T}(x_0, t_0) = B(x_0, R) \times (t_0 - T, t_0].$$

A positive constant is denoted by c and C with or without index, which may change from line to line.

Theorem 1.1. *Let M be a Kähler manifold with a metric g.*

(a) Suppose u is a solution of

$$\partial_t u - \Delta u = 0 \quad \text{in } M \times (0, T].$$

Assume $0 < u \leq A$. Then,

$$t(u_{ij}) \leq u(5 + Bt)\left(1 + \log \frac{A}{u}\right) \quad \text{in } M \times (0, T],$$

where B is a nonnegative constant depending only on the bound of holomorphic bisectional curvature.

(b) Suppose u is a solution of

$$\partial_t u - \Delta u = 0 \quad \text{in } Q_{R,T}(x_0, t_0).$$

Assume $0 < u \leq A$. Then,

$$(u_{ij}) \leq Cu\left(\frac{1}{T} + \frac{1}{R^2} + B\right)\left(1 + \log \frac{A}{u}\right)^2 \quad \text{in } Q_{\frac{R}{2},\frac{T}{2}}(x_0, t_0).$$

where C is a universal constant and B is a nonnegative constant depending only on the bound of holomorphic bisectional curvature.

2. Heat Equations on Kähler manifold

Let M be a Kähler manifold with metric g and Δ be the Laplace-Beltrami operator. We consider a positive solution u of the heat equation

$$u_t = \Delta u \quad \text{in } M \times (0, \infty).$$

We assume

$$0 < u \leq A.$$

Set

$$f = \log \frac{u}{A}.$$

Let $\{z^1, \ldots, z^n\}$ be a local holomorphic normal frame at a point, say $p \in M$. Then

$$f_i = \frac{u_i}{u}, \quad f_{i\bar{j}} = \frac{u_{i\bar{j}}}{u} - \frac{u_i u_{\bar{j}}}{u^2},$$

and hence

$$f_t = \Delta f + \frac{1}{2}|\nabla f|^2.$$

The following lemma is the key point of the paper, in [5] the Ricci flow can be used to cancel the Gradient Ricci term. In fact, in kähler case these terms can vanish, since the symmetry of Cuvrature tensors in kähler manifold.

Lemma 2.1. *(of Lemma 2.1 in [6]) Let u satifies the heat equation $(\partial_t - \Delta)u = 0$, the hessian u satifies the following formula:*

$$\left(\frac{\partial}{\partial t} - \Delta\right) u_{i\bar{j}} = R_{i\bar{j}l\bar{k}} u_{k\bar{l}} - \frac{1}{2}\left(R_{i\bar{m}} u_{m\bar{j}} + R_{m\bar{j}} u_{i\bar{m}}\right).$$

Proof. In holomorphic normal coordinate system

$$\begin{aligned}
\left(\frac{\partial}{\partial t} u_{i\bar{j}}\right) = (\Delta u)_{i\bar{j}} = \left(g^{k\bar{l}} u_{k\bar{l}}\right)_{i\bar{j}} &= g^{k\bar{l}} u_{k\bar{l}i\bar{j}} + R_{i\bar{j}l\bar{k}} u_{k\bar{l}} \\
\Delta u_{i\bar{j}} &= \frac{1}{2}\left(\nabla_{\bar{m}} \nabla_m u_{i\bar{j}} + \nabla_m \nabla_{\bar{m}} u_{i\bar{j}}\right) \\
\nabla_{\bar{l}} \nabla_k u_{i\bar{j}} &= \nabla_{\bar{l}}\left(\partial_k u_{i\bar{j}} - \Gamma^m_{ik} u_{m\bar{j}}\right) \\
&= u_{i\bar{j}k\bar{l}} + R_{i\bar{l}k\bar{m}} u_{m_{\bar{j}}},
\end{aligned}$$

Ricci identity implies

$$\nabla_{\bar{m}} \nabla_m u_{i\bar{j}} - \nabla_m \nabla_{\bar{m}} u_{i\bar{j}} = R_{i\bar{p}} u_{p\bar{j}} - R_{p\bar{j}} u_{i\bar{p}},$$

so we get

$$\Delta u_{i\bar{j}} = u_{ijm\bar{m}} + \frac{1}{2}\left(R_{i\bar{l}} u_{l_{\bar{j}}} + R_{l_{\bar{j}}} u_{i\bar{l}}\right).$$

∎

Lemma 2.2. *Set*

$$v_{i\bar{j}} = \frac{u_{i\bar{j}}}{u(1-f)}, \quad v_{ij} = \frac{u_{ij}}{u(1-f)},$$

then

$$\begin{aligned}
\left[-\partial_t + \Delta - \frac{1}{u(1-f)} \nabla f \cdot \nabla\right] v_{i\bar{j}} =& \frac{1}{2}\frac{|\nabla f|^2}{1-f} v_{i\bar{j}} + \frac{1}{u(1-f)} \\
&+ \left(-R_{i\bar{j}l\bar{k}} u_{k\bar{l}} + \frac{1}{2} R_{i\bar{l}} u_{l\bar{j}} + \frac{1}{2} R_{l\bar{j}} u_{i\bar{l}}\right).
\end{aligned}$$

Proof. By noting:

$$\begin{aligned}
\partial_t(u(1-f)) &= -u_t f \\
\partial_k(u(1-f)) &= -u_k f\text{'}
\end{aligned}$$

we have

$$\partial_t v_{i\bar{j}} = \frac{u_{i\bar{j}t}}{u(1-f)} + \frac{u_{i\bar{j}} f u_t}{u^2(1-f)^2}$$
$$\partial_k v_{i\bar{j}} = \frac{u_{i\bar{j}k}}{u(1-f)} + \frac{u_{i\bar{j}} f u_k}{u^2(1-f)^2}.$$

Note that

$$\begin{aligned}\Delta v_{i\bar{j}} =& \frac{1}{2}\left(\nabla_k \nabla_{\bar{k}} v_{i\bar{j}} + \nabla_{\bar{k}} \nabla_k v_{i\bar{j}}\right)\\ =& \frac{\Delta u_{i\bar{j}}}{u(1-f)} + \frac{u_{i\bar{j}} f \Delta u}{u^2(1-f)^2} + \frac{f\left\langle \nabla u_{i\bar{j}}, \nabla u\right\rangle}{u^2(1-f)^2}\\ &+ \frac{1}{2}\frac{u_{i\bar{j}}\langle \nabla f, \nabla u\rangle}{u^2(1-f)^2} + \frac{u_{i\bar{j}} f^2 \left|\nabla u\right|^2}{u^3(r-f)^3}\end{aligned}$$

with $u_k = u f_k$ and Lemma 2.1, we have

$$\begin{aligned}(\Delta - \partial_t) v_{i\bar{j}} =& \frac{f\left\langle \nabla u_{i\bar{j}}, \nabla f\right\rangle}{u(1-f)^2} + \frac{1}{2}\frac{u_{i\bar{j}}\left|\nabla f\right|^2}{u(1-f)^2}\\ &+ \frac{u_{i\bar{j}} f^2 |\nabla f|^2}{u(1-f)^3} + \frac{1}{u(1-f)}\left(-R_{i\bar{j}l\bar{k}} u_{k\bar{l}} + \frac{1}{2} R_{i\bar{l}} u_{l\bar{j}} + \frac{1}{2} R_{l\bar{j}} u_{i\bar{l}}\right)\\ =& \frac{f}{1-f}\left\langle \nabla f, \frac{\nabla u_{i\bar{j}}}{u(1-f)} + \frac{u_{i\bar{j}} f \nabla f}{u(1-f)^2}\right\rangle\\ &+ \frac{1}{2}\frac{u_{i\bar{j}}}{u(1-f)}\frac{\left|\nabla f\right|^2}{(1-f)} + \frac{1}{u(1-f)}\left(-R_{i\bar{j}l\bar{k}} u_{k\bar{l}} + \frac{1}{2} R_{i\bar{l}} u_{l\bar{j}} + \frac{1}{2} R_{l\bar{j}} u_{i\bar{l}}\right),\end{aligned}$$

with help of the expression for $\partial_k v_{i\bar{j}}$, we obtain

$$\begin{aligned}(\Delta - \partial_t) v_{i\bar{j}} =& \frac{f}{1-f}\langle \nabla f, \nabla v_{i\bar{j}}\rangle + \frac{1}{2}\frac{|\nabla f|^2}{1-f} v_{i\bar{j}}\\ &+ \frac{1}{u(1-f)}\left(-R_{i\bar{j}l\bar{k}} u_{k\bar{l}} + \frac{1}{2} R_{i\bar{l}} u_{l\bar{j}} + \frac{1}{2} R_{l\bar{j}} u_{i\bar{l}}\right). \quad \square\end{aligned}$$

Lemma 2.3. *Set*

$$w_{ij} = \frac{u_i u_{\bar{j}}}{u^2(1-f)^2}, \quad w_{ij} = \frac{u_i u_j}{u^2(1-f)^2},$$

then

$$\begin{aligned}\left(-\partial t + \Delta - \frac{f}{1-f}\nabla f \cdot \nabla\right) w_{i\bar{j}} =& \frac{|\nabla f|^2}{1-f} w_{i\bar{j}} + \left(v_{i\bar{k}} + f w_{i\bar{k}}\right)\left(v_{k\bar{j}} + f w_{k\bar{j}}\right)\\ &+ \left(v_{ik} + f w_{ik}\right)\overline{\left(v_{kj} + f w_{kj}\right)} + \frac{1}{2} R_{i\bar{k}} u_{k\bar{j}} + \frac{1}{2} R_{k\bar{j}} w_{\bar{k}i}.\end{aligned}$$

Proof. We proceed similarly as in proof of Lemma 2.2. First,

$$\partial_t w_{i\bar{j}} = \frac{u_{it} + u_i u_{\bar{j}t}}{u^2(1-f)^2} + \frac{2u_i u_i u_{\bar{j}} f}{u^3(1-f)^3}$$
$$\partial_k w_{i\bar{j}} = \frac{u_{ik} u_{\bar{j}} + u_i u_{\bar{j}k}}{u^2(1-f)^2} + \frac{2u_k u_i u_{\bar{j}} f}{u^3(1-f)^3}.$$

Second, by Bochner's formula, we have

$$
\begin{aligned}
\Delta w_{i\bar{j}} =& \frac{(\Delta u)_i u_{\bar{j}} + \left\langle \nabla u_i, \nabla u_{\bar{j}} \right\rangle + u_i \left(\Delta u\right)_{\bar{j}}}{u^2(1-f)^2} \\
&+ \frac{1}{2} R_{i\bar{k}} \frac{u_k u_{\bar{j}}}{u^2(1-f)^2} + \frac{1}{2} R_{k\bar{j}} \frac{u_{\bar{k}} u_i}{u^2(1-f)^2} + \frac{2u_i u_{\bar{j}} \Delta f}{u^3(1-f)^3} \\
&+ \frac{u_i u_{\bar{j}} \left\langle \nabla u, \nabla f \right\rangle}{u^3(1-f)^3} + \frac{2f \left(\left\langle \nabla u_i, \nabla u \right\rangle u_{\bar{j}} + \left\langle \nabla u_{\bar{j}}, \nabla u \right\rangle u_i \right)}{u^3(1-f)^3} + \frac{3|\nabla u|^2 u_i u_{\bar{j}} f^2}{u^4(1-f)^4}
\end{aligned}
$$

hence

$$
(\Delta - \partial_t) w_{i\bar{j}} = H + \frac{1}{2} R_{i\bar{k}} w_{k\bar{j}} + \frac{1}{2} R_{k\bar{j}} w_{i\bar{k}},
$$

where

$$
\begin{aligned}
H =& \frac{\left(\left\langle \nabla u_i, \nabla u \right\rangle u_{\bar{j}} + \left\langle \nabla u_{\bar{j}}, \nabla u \right\rangle u_i \right) f}{u^3(1-f)^3} + \frac{2|\nabla u|^2 u_i u_{\bar{j}} f^2}{u^4(1-f)^4} + \frac{u_i u_{\bar{j}} \left\langle \nabla u, \nabla f \right\rangle}{u^3(1-f)^3} + \frac{\left\langle \nabla u_i, \nabla u_{\bar{j}} \right\rangle}{u^2(1-f)^2} \\
&+ \frac{\left(\left\langle \nabla u_i, \nabla u \right\rangle u_{\bar{j}} + \left\langle \nabla u_{\bar{j}}, \nabla u \right\rangle u_i \right) f}{u^3(1-f)^3} + \frac{u_i u_{\bar{j}} |\nabla u|^2 f^2}{u^4(1-f)^4} \\
=& \frac{f}{u(1-f)} \left\langle \nabla u, \frac{\nabla u_i u_{\bar{j}} + u \nabla u_i}{u^2(1-f)^2} + \frac{2u_i u_{\bar{j}} f \nabla u}{u^3(1-f)^3} \right\rangle \\
&+ \frac{\left\langle \nabla u, \nabla f \right\rangle}{u(1-f)} \frac{u_i u_{\bar{j}}}{u^2(1-f)^2} + \left\langle \frac{\nabla u_i}{u(1-f)} + \frac{u_i f \nabla u}{u^2(1-f)^2}, \frac{\nabla u_{\bar{j}}}{u(1-f)} + \frac{u_{\bar{j}} f \nabla u_{\bar{j}}}{u^2(1-f)^2} \right\rangle
\end{aligned}
$$

with $u_k = u f_k$ and the expression for $\partial_k w_{i\bar{j}}$, we get

$$
\begin{aligned}
H =& \frac{f}{u(1-f)} \langle \nabla u, \nabla w_{i\bar{j}} \rangle + \frac{|\nabla f|^2}{1-f} w_{i\bar{j}} \\
&+ \left(v_{i\bar{k}} + f w_{i\bar{k}} \right) \left(v_{k\bar{j}} + f w_{k\bar{j}} \right) + \left(v_{ik} + f w_{ik} \right) \left(\overline{v_{kj} + f w_{kj}} \right). \quad \square
\end{aligned}
$$

Remark 2.4. *We define the trace w of $\left(w_{i\bar{j}}\right)$ by*

$$
w = \operatorname{tr}\left(w_{i\bar{j}}\right) = \frac{|\nabla u|^2}{u^2(1-f)^2}.
$$

We also have

$$
\begin{aligned}
\left(-\partial_t + \Delta - \frac{f}{1-f} \nabla f \cdot \nabla\right) v_{i\bar{j}} =& \frac{1}{2}(1-f) w v_{i\bar{j}} \\
&+ \frac{1}{u(1-f)} \left(-R_{i\bar{j}l\bar{k}} u_{k\bar{l}} + \frac{1}{2} R_{i\bar{l}} u_{l\bar{j}} + \frac{1}{2} R_{l\bar{j}} u_{i\bar{l}} \right), \\
\left(-\partial_t + \Delta - \frac{f}{1-f} \nabla f \cdot \nabla\right) w_{i\bar{j}} =& (1-f) w w_{i\bar{j}} \\
&+ \left(v_{i\bar{k}} + f w_{i\bar{k}} \right) \left(v_{k\bar{j}} + f w_{k\bar{j}} \right) + \left(v_{ik} + f w_{ik} \right) \left(\overline{v_{kj} + f w_{kj}} \right) + \frac{1}{2} R_{i\bar{k}} w_{k\bar{j}} + \frac{1}{2} R_{k\bar{j}} w_{i\bar{k}}
\end{aligned}
$$

We now prove Theorem 1.1.

Proof of Theorem 1.1. **Part (a).** We first perform some important calculations. Let p be a point on the manifold and $\{z^1, \ldots, z^n\}$ be a local holomorphic normal coordinates system. In this coordinate and using the same notations as in Lemma 2.1 and Lemma 2.2, the $(1,1)$ tensor fields v_{ij} and w_{ij} can be regarded as $n \times n$ matrices. Set $V = \left(v_{i\bar{j}}\right), W = \left(w_{i\bar{j}}\right), V' = \left(v_{ij}\right), W' = (w_{ij})$, $w = \operatorname{tr}(W)$, and

$$L = -\partial_t + \Delta - \frac{f}{1-f}\nabla f \cdot \nabla.$$

Then, by Lemma 2.1 and Lemma 2.2,

$$LV = \frac{1}{2}(1-f)wV + P, \tag{2.1}$$

$$LW = (1-f)wW + (V+fW)(\overline{V+fW})^T + (V'+fW')(\overline{V'+fW'})^T + Q, \tag{2.2}$$

where P and Q are matrices whose (i,j)-th components are

$$P_{i\bar{j}} = -R_{i\bar{j}l\bar{k}}v_{k\bar{l}} + \frac{1}{2}R_{i\bar{l}}v_{l\bar{j}} + \frac{1}{2}R_{l\bar{j}}v_{i\bar{l}} \tag{2.3}$$

and

$$Q_{i\bar{j}} = \frac{1}{2}R_{i\bar{k}}w_{k\bar{j}} + \frac{1}{2}R_{k\bar{j}}w_{i\bar{k}}. \tag{2.4}$$

For a constant $\alpha \in \mathbb{R}$ to be determined, we have

$$\begin{aligned} L(\alpha V + W) = &\frac{\alpha}{2}(1-f)wV + (1-f)wW + (V+fW)(\overline{V+fW})^T \\ &+ (V'+fW')(\overline{V'+fW'})^T + \alpha P + Q. \end{aligned}$$

Let $\xi \in T_p^{(1,0)}M$ be a unit tangent vector at the point p. We use parallel translation along geodesics emanating from p to extend ξ to a smooth vector field in the local coordinate neighborhood. We still denote the vector field by ξ. Since V and W are $(1,1)$-tensor fields, the function

$$\lambda = \bar{\xi}^T(\alpha V + W)\xi \equiv (\alpha V + W)(\xi, \xi)$$

is a well-defined smooth function in a neighborhood of p. Then

$$L\lambda = H + \bar{\xi}^T(\alpha P + Q)\xi$$

where

$$H = \frac{\alpha}{2}(1-f)w\bar{\xi}^T V\xi + (1-f)w\bar{\xi}^T W\xi + |(V+fW)\xi|^2 + |(V'+fW')\xi|^2. \tag{2.5}$$

By $\alpha\bar{\xi}^T V\xi = \lambda - \bar{\xi}^T W\xi$, we have

$$\begin{aligned} H &= \frac{1}{2}(1-f)w\left(\lambda - \bar{\xi}^T W\xi\right) + (1-f)w\bar{\xi}^T W\xi + |(V+fW)\xi|^2 + |(V'+fW')\xi|^2 \\ &= \frac{(1-f)}{2}w\lambda + \frac{(1-f)}{2}w\bar{\xi}^T W\xi + |(V+fW)\xi|^2 + |(V'+fW')\xi|^2. \end{aligned}$$

To simplify the last term further, we fix the point p and assume ξ is the vector field generated, via parallel translation through geodesics emanating from p, by an eigenvector of $\alpha V + W$ at p, i.e., at p

$$(\alpha V + W)\xi = \lambda\xi.$$

Then

$$(V + fW)\xi = \frac{\lambda}{\alpha}\xi - \frac{1}{\alpha}W\xi + fW\xi = \frac{\lambda}{\alpha}\xi - \left(\frac{1}{\alpha} - f\right)W\xi,$$

and hence

$$|(V + fW)\xi|^2 = \frac{\lambda^2}{\alpha^2} - \frac{2\lambda}{\alpha}\left(\frac{1}{\alpha} - f\right)\bar{\xi}^T W\xi + \left(\frac{1}{\alpha} - f\right)^2 |W\xi|^2.$$

Hence

$$\begin{aligned} H =& \frac{\lambda^2}{\alpha^2} + \frac{\lambda}{2}\left(w - \frac{4}{\alpha^2}\bar{\xi}^T W\xi\right) - \frac{f\lambda}{2}\left(w - \frac{4}{\alpha}\bar{\xi}^T W\xi\right) \\ &+ \frac{(1-f)}{2} w\bar{\xi}^T W\xi + \left(\frac{1}{\alpha} - f\right)^2 |W\xi|^2 + |(V' + fW')\xi|^2. \end{aligned}$$

The last two terms are independent of λ and nonnegative. Hence

$$H \geq \frac{\lambda^2}{\alpha^2} + \frac{\lambda}{2}\left(w - \frac{4}{\alpha^2}\bar{\xi}^T W\xi\right) - \frac{f\lambda}{2}\left(w - \frac{4}{\alpha}\bar{\xi}^T W\xi\right). \tag{2.6}$$

For the last term, we note W is a rank-one matrix and hence

$$\bar{\xi}^T W\xi \leq w.$$

With $f < 0$ and choosing $\alpha \geq 4$, the third term in nonnegative, if $\lambda \geq 0$. If $\lambda \geq 0$, the second term is also nonnegative. Hence

$$H \geq \frac{\lambda^2}{\alpha^2}. \tag{2.7}$$

In summary, if $\lambda \geq 0$, then at the point p

$$L\lambda \geq \frac{\lambda^2}{\alpha^2} + \bar{\xi}^T(\alpha P + Q)\xi. \tag{2.8}$$

Now we proceed to prove Part (a). Let τ be a universal constant to be fixed later. With $\alpha = 4$, suppose the 2 form

$$\alpha V + W - \frac{\tau}{t}g$$

assumes its largest nonnegative eigenvalue at a space-time point (p_1, t_1), with $t_1 > 0$. Let ξ be a unit eigenvector at p_1. We use parallel translation along geodesics emanating from p_1 to extend ξ to a smooth vector field which is still denoted by ξ. Set, in a local coordinate

$$\mu = \bar{\xi}^T\left(\alpha V + W - \frac{\tau}{t}g\right)\xi \tag{2.9}$$

and

$$\lambda = \bar{\xi}^T(\alpha V + W)\xi. \tag{2.10}$$

Then, both μ and λ are smooth functions in a space time neighborhood of (x_1, t_1). Also

$$L\mu = L\left(\lambda - \frac{\tau}{t}\right) = L\lambda - \frac{\tau}{t^2} = H - \frac{\tau}{t^2} + \bar{\xi}^T(\alpha P + Q)\xi$$

Here H is given by (2.5). We now evaluate at (p_1, t_1). Since $\lambda - \tau/t$ has its nonnegative maximum at (p_1, t_1), we have, by (2.8),

$$0 \geq L\left(\lambda - \frac{\tau}{t}\right) \geq \frac{\lambda^2}{\alpha^2} - \frac{\tau}{t^2} - \left|\bar{\xi}^T(\alpha P + Q)\xi\right| \quad \text{at } (p_1, t_1)$$

or

$$\frac{\lambda^2}{\alpha^2} \leq \frac{\tau}{t^2} + \left|\bar{\xi}^T(\alpha P + Q)\xi\right| \quad \text{at } (p_1, t_1)\,. \tag{2.11}$$

In order to bound μ and λ from above, we need to find an upper bound for $\left|\bar{\xi}^T(\alpha P + Q)\xi\right|$ at (p_1, t_1).

Let $\xi = (\xi_1, \ldots, \xi_n)$. By (2.3) and (2.4), we obtain

$$\begin{aligned}
\left|\bar{\xi}^T(\alpha P + Q)\xi\right| &\leq \left|\alpha\bar{\xi}^T P\xi\right| + \left|\bar{\xi}^T Q\xi\right| \\
&\leq \left|\bar{\xi}_i\xi_j\alpha\left(-R_{i\bar{j}l\bar{k}}v_{k\bar{l}} + \frac{1}{2}R_{i\bar{l}}v_{l\bar{j}} + \frac{1}{2}R_{l\bar{j}}v_{i\bar{l}}\right)\right| \\
&\quad + \left|\bar{\xi}_i\xi_j\left(\frac{1}{2}R_{i\bar{k}}w_{k\bar{j}} + \frac{1}{2}R_{k\bar{j}}w_{i\bar{k}}\right)\right| \\
&\leq \left|\bar{\xi}_i\xi_j\alpha\left(-R_{i\bar{j}l\bar{k}}v_{k\bar{l}} + \frac{1}{2}R_{i\bar{l}}v_{l\bar{j}} + \frac{1}{2}R_{l\bar{j}}v_{i\bar{l}}\right)\right| + C|Ric||W|.
\end{aligned}$$

Writing $\alpha v_{kl} = \alpha v_{k\bar{l}} + w_{k\bar{l}} - w_{k\bar{l}}$ etc in the last line, we deduce

$$\begin{aligned}
&\left|\bar{\xi}^T(\alpha P + Q)\xi\right| \\
&\leq \left|\bar{\xi}_i\xi_j\left[-R_{i\bar{j}l\bar{k}}\left(\alpha v_{k\bar{l}} + w_{k\bar{l}}\right) + \frac{1}{2}R_{i\bar{l}}\left(\alpha v_{l\bar{j}} + w_{l\bar{j}}\right) + \frac{1}{2}R_{l\bar{j}}\left(\alpha v_{i\bar{l}} + w_{i\bar{l}}\right)\right]\right| \\
&\quad + \left|\bar{\xi}_i\xi_j\left(-R_{i\bar{j}l\bar{k}}w_{k\bar{l}} + \frac{1}{2}R_{i\bar{l}}w_{l\bar{j}} + \frac{1}{2}R_{l\bar{j}}w_{i\bar{l}}\right)\right| + C|Ric||W|.
\end{aligned} \tag{2.12}$$

At the point p_1, we can choose a coordinate system so that the matrix $\alpha V + W$ is diagonal. Let $\mu_1, \cdots, \mu_n$ be the eigenvalues of the matrix $\alpha V + W - \frac{\tau}{t}I$, listed in the increasing order. Without loss of generality, we assume $\mu_1 < 0$ and $\mu_n > 0$.

Let K be the bound of holomorphic bisectional cuvrature, note that holomorphic bisectioal cuvrature determines Ricci cuvrature on Kähler manifold.

$$\begin{aligned}
|R_{i\bar{j}l\bar{k}}(\alpha v_{k\bar{l}} + w_{k\bar{l}})| &\leq \left|R_{i\bar{j}l\bar{k}}\left(\alpha v_{k\bar{l}} + w_{k\bar{l}} - \frac{\tau}{t}\delta_{ij}\right)\right| + |R_{i\bar{j}l\bar{k}}\delta_{ij}|\frac{\tau}{t} \\
&\leq \left|R_{i\bar{j}k\bar{k}}\left(\alpha v_{k\bar{k}} + w_{k\bar{k}} - \frac{\tau}{t}\right)\right| + \frac{CK\tau}{t} \\
&\leq CK(\mu_n + |\mu_1|) + \frac{CK\tau}{t}.
\end{aligned}$$

Similarly

$$\left|R_{i\bar{l}}\left(\alpha v_{l\bar{j}}+w_{l\bar{j}}\right)\right| \leq C|Ric|\left(\mu_n+|\mu_1|\right)+\frac{C|Ric|\tau}{t} \leq CK(\mu_n+|\mu_1|)+\frac{CK\tau}{t}.$$

Combining the last few inequalities, we deduce

$$\left|\bar{\xi}^T(\alpha P+Q)\xi\right| \leq C|Rm|\left(\mu_n+|\mu_1|\right)+C|Rm||W|+C|Rm|\frac{\tau}{t}.$$

Then

$$\left|\bar{\xi}^T(\alpha P+Q)\xi\right| \leq CK\left(\mu_n+|\mu_1|+\frac{\tau}{t}\right)+CK|W|.$$

Observe that

$$\begin{aligned}\mu_1+(n-1)\mu_n &\geq \mu_1+\cdots+\mu_n\\ &= \operatorname{tr}\left(\frac{\alpha u_{i\bar{j}}}{u(1-f)}+\frac{u_i u_{\bar{j}}}{u^2(1-f)^2}-\frac{\tau}{t}\delta_{ij}\right) \geq \frac{\alpha\Delta u}{u(1-f)}-n\frac{\tau}{t}.\end{aligned}$$

Hence,

$$|\mu_1| \leq (n-1)\mu_n-\frac{\alpha\Delta u}{u(1-f)}+n\frac{\tau}{t}$$

Therefore

$$\left|\bar{\xi}^T(\alpha P+Q)\xi\right| \leq CK\left(\mu_n-\frac{\alpha\Delta u}{u(1-f)}+\frac{\tau}{t}\right)+CK|W|.$$

By [3], we have

$$\frac{|\nabla u|^2}{u^2}-2\frac{u_t}{u} \leq \frac{c_1}{t}+K$$

where $Ric \geq -CK$. With $u_t=\Delta u$, we get

$$-\frac{\Delta u}{u} \leq \frac{c_1}{t}+K.$$

Since $0 \leq u < A$, we have

$$\frac{1}{1-f} \leq 1$$

Therefore, at (p_1,t_1), it holds

$$\left|\bar{\xi}^T(\alpha P+Q)\xi\right| \leq CK\left(\mu_n+\frac{1+\tau}{t}\right)+CK^2+CK|W|.$$

By the definition of $W=\left(w_{i\bar{j}}\right)$, we have

$$|W| \leq \frac{|\nabla u|^2}{u^2(1-f)^2}.$$

By Theorem 1.1 in [2], we obtain

$$|W| \leq C\left(\frac{1}{t}+K\right),$$

and hence

$$\left|\bar{\xi}^T(\alpha P+Q)\xi\right| \leq CK\left(\mu_n+\frac{1+\tau}{t}\right)+CK^2.$$

Since $\mu = \mu_n < \lambda$ at (p_1, t_1), this shows, by (2.11), that

$$\frac{\lambda^2}{\alpha^2} \le \frac{\tau}{t^2} + CK\left(\lambda + \frac{1+\tau}{t}\right) + CK^2 \quad \text{at } (p_1, t_1).$$

A simple application of the Cauchy inequality yields

$$\frac{\lambda}{\alpha} \le \frac{\sqrt{\tau+1}}{t} + B \quad \text{at } (p_1, t_1)$$

where B is a nonnegative constant depending only on K with the property that $B = 0$ if $K = 0$. Then

$$\lambda - \frac{\tau}{t} \le (\alpha\sqrt{\tau+1} - \tau)\frac{1}{t} + \alpha B \le \alpha B \quad \text{at } (p_1, t_1),$$

by choosing τ sufficiently large. In fact, for $\alpha = 4$, we can take $\tau = 8 + 4\sqrt{5}$.

By definition, $\mu = \lambda - \frac{\tau}{t}$ at (p_1, t_1) is the largest eigenvalue of the $(1,1)$-form $\alpha V + W - \frac{\tau}{t} g$ on $M \times (0, T]$. Therefore, given any unit tangent vector $\eta \in T_x^{(1,0)} M, x \in M$, it holds

$$\bar{\eta}^T(\alpha V + W)\eta - \frac{\tau}{t} g(\eta, \eta) \le \left(\lambda - \frac{\tau}{t}\right)\Big|_{(p_1,t_1)} \le \alpha B \quad \text{in } M \times (0, T).$$

Thus

$$t\bar{\eta}^T V \eta \le \frac{\tau}{\alpha} + Bt.$$

This proves part (a) of the theorem.

Part (b). Now we localize the result in part (a).

Let ψ be a cutoff function which will be specified later. Then, for any smooth function η, we have

$$\begin{aligned}
\partial_t(\psi\eta) &= \partial_t\psi\eta + \psi\partial_t\eta, \\
\nabla(\psi\eta) &= \nabla\psi\eta + \psi\nabla\eta, \\
\Delta(\psi\eta) &= \Delta\psi\eta + \nabla\psi\nabla\eta + \psi\Delta\eta.
\end{aligned}$$

Hence

$$\begin{aligned}
\psi L\eta =& - \psi\partial_t\eta + \psi\Delta\eta - \psi\frac{f}{1-f}\nabla f \cdot \nabla\eta \\
=& - (\partial_t(\psi\eta) - \eta\partial_t\psi) + \Delta(\psi\eta) - \eta\Delta\psi - \nabla\psi \cdot \nabla\eta - \frac{f}{1-f}\nabla f(\nabla(\psi\eta) - \eta\nabla\psi) \\
=& - \partial_t(\psi\eta) + \Delta(\psi\eta) - \frac{f}{1-f}\nabla f \cdot \nabla(\psi\eta) + \eta\partial_t\psi - \eta\Delta\psi \\
&+ \frac{f}{1-f}\eta\nabla f \cdot \nabla\psi - \nabla\psi \cdot \nabla\eta.
\end{aligned}$$

For the last term, we write

$$\begin{aligned}
\nabla\psi \cdot \nabla\eta &= \frac{\nabla\psi}{\psi}\psi\nabla\eta = \frac{\nabla\psi}{\psi}(\nabla(\psi\eta) - \nabla\psi\eta) \\
&= \frac{\nabla\psi}{\psi}\nabla(\psi\eta) - \frac{|\nabla\psi|^2}{\psi}\eta.
\end{aligned}$$

Hence

$$\psi L\eta = -\partial_t(\psi\eta) + \Delta(\psi\eta) - \frac{f}{1-f}\nabla f \cdot \nabla(\psi\eta) - \frac{\nabla\psi}{\psi}\nabla(\psi\eta)$$
$$+\eta\partial_t\psi - \eta\Delta\psi + \eta\frac{f}{1-f}\nabla f \cdot \nabla\psi + \frac{|\nabla\psi|^2}{\psi}\eta.$$

Set

$$L_1 = -\partial_t + \Delta - \frac{f}{1-f}\nabla f \cdot \nabla - \frac{\nabla\psi}{\psi}\nabla. \tag{2.13}$$

Then

$$\psi L\eta = L_1(\eta\psi) - \eta L_1\psi,$$

or

$$L_1(\eta\psi) = \psi L\eta + \eta L_1\psi.$$

With λ introduced before in (2.10), we have

$$L_1(\psi\lambda) = \psi L\lambda + \lambda L_1\psi = \psi\left[H + \bar{\xi}^T(\alpha P + Q)\xi\right] + \lambda L_1\psi. \tag{2.14}$$

Here H is given by (2.5). Now we analyze $L_1\psi$. We write

$$L_1\psi = -\partial_t\psi + \Delta\psi - \frac{|\nabla\psi|^2}{\psi} - \frac{f}{1-f}\nabla f \cdot \nabla\psi.$$

The first three terms are obviously bounded by choosing suitable ψ. For the last one, we write

$$-\frac{f}{1-f}\nabla f \cdot \nabla\psi = -\frac{f}{1-f}\sqrt{\psi}\nabla f \cdot \frac{\nabla\psi}{\sqrt{\psi}}.$$

Note $\frac{-f}{1-f} < 1$ and $\frac{\nabla\psi}{\sqrt{\psi}}$ is bounded. We need to control

$$\sqrt{\psi}\nabla f.$$

To this end, we recall the equation for f

$$-\partial_t f + \Delta f = -\frac{|\nabla f|^2}{2}.$$

Then

$$Lf = -\partial_t f + \Delta f - \frac{f}{1-f}|\nabla f|^2 = -\frac{|\nabla f|^2}{2} - \frac{f}{1-f}|\nabla f|^2 = \frac{-1-f}{2(1-f)}|\nabla f|^2.$$

Note

$$\frac{-1-f}{1-f} \geq \frac{1}{2} \text{ if } f \leq -3.$$

Then

$$Lf \geq \frac{1}{4}|\nabla f|^2.$$

Hence, we obtain

$$L_1 f = Lf - \frac{\nabla\psi}{\psi}\nabla f = Lf - \frac{\nabla\psi}{\psi\sqrt{\psi}}\sqrt{\psi}\nabla f,$$

and then

$$\begin{aligned}\psi L_1 f &= \psi L f - \frac{\nabla\psi}{\sqrt{\psi}}\sqrt{\psi}\nabla f \geq \frac{1}{4}\psi|\nabla f|^2 - \frac{\nabla\psi}{\sqrt{\psi}}\sqrt{\psi}\nabla f \\ &\geq \frac{1}{8}\psi|\nabla f|^2 - C\frac{|\nabla\psi|^2}{\psi}.\end{aligned} \tag{2.15}$$

Now we consider, for some constant $\beta \in \mathbb{R}^+$to be determined,

$$\psi L_1(\psi\lambda + \beta f) = \psi^2 H + \psi^2\bar{\xi}^T(\alpha P + Q)\xi + \psi\lambda L_1\psi + \beta\psi L_1 f.$$

In the following, we consider eigenvalues of the $(1,1)$-form

$$\psi(\alpha V + W) + \beta f g.$$

If ξ is an eigenvector of $\psi(\alpha V + W) + \beta f g$ at some point (x,t), then

$$[\psi(\alpha V + W) + \beta f g]\xi = \mu\xi,$$

or in local coordinates,

$$\psi(\alpha V + W)\xi = (\mu - \beta f)\xi.$$

If $\psi(x,t) \neq 0$, then ξ is also an eigenvector of $\alpha V + W$. Hence

$$\mu = \psi\lambda + \beta f.$$

Again we extend ξ to a vector field around x by parallel transporting along geodesics starting from x. The vector field is still denoted by ξ.

Let Ω be the parabolic cube given by

$$\Omega = Q_{R,T}\left(x_0, t_0\right) = B\left(x_0, R\right) \times \left(t_0 - T, t_0\right].$$

Let

$$\mu = \bar{\xi}^T[\psi(\alpha V + W) + \beta f g]\xi = \psi\lambda + \beta f.$$

Then

$$\mu|_{\partial_p\Omega} = \beta f|_{\partial_p\Omega} < 0.$$

We will estimate μ from above. Recall from (2.14) and $\mu = \lambda + \beta f$ that

$$\psi L_1\mu = \psi^2 H + \psi^2\bar{\xi}^T(\alpha P + Q)\xi + \psi\lambda L_1\psi + \beta\psi L_1 f. \tag{2.16}$$

We first have, from (2.7), that

$$\psi^2 H \geq \frac{(\psi\lambda)^2}{\alpha^2} + \frac{\psi^2\lambda}{2}\left(w - \frac{4}{\alpha^2}\bar{\xi}^T W\xi\right) - \frac{f\psi^2\lambda}{2}\left(w - \frac{4}{\alpha}\bar{\xi}^T W\xi\right)$$

At points where $\mu \geq 0$, we have

$$\psi\lambda + \beta f \geq 0,$$

and hence $\psi\lambda \geq 0$. Then

$$\psi^2 H \geq \frac{(\psi\lambda)^2}{\alpha^2}.$$

By this, (2.15) and (2.16), we deduce,

$$\psi L_1\mu \geq \frac{(\psi\lambda)^2}{\alpha^2} + \psi^2\bar{\xi}^T(\alpha P+Q)\xi + \beta\left[\frac{1}{8}\psi|\nabla f|^2 - C\frac{|\nabla\psi|^2}{\psi}\right]$$
$$-\left[|\partial_t\psi| + |\Delta\psi| + \frac{|\nabla\psi|^2}{\psi} + \sqrt{\psi}|\nabla f|\cdot\frac{\nabla\psi}{\sqrt{\psi}}\right]\psi\lambda.$$

For the last term, we use the Cauchy inequality to control the $\psi\lambda$ factor by the first term to get

$$\psi L_1\mu \geq \frac{1}{2\alpha^2}(\psi\lambda)^2 + \psi^2\bar{\xi}^T(\alpha P+Q)\xi + \beta\left(\frac{1}{8}\psi|\nabla f|^2 - C\frac{|\nabla\psi|^2}{\psi}\right)$$
$$-C\left(|\partial_t\psi| + |\Delta\psi| + \frac{|\nabla\psi|^2}{\psi}\right)^2 - C\psi|\nabla f|^2\frac{|\nabla\psi|^2}{\psi}$$

We now take

$$\beta = c\sup\frac{|\nabla\psi|^2}{\psi}. \tag{2.17}$$

If c is sufficiently large, we have

$$\beta\left(\frac{1}{8}\psi|\nabla f|^2 - C\frac{|\nabla\psi|^2}{\psi}\right) - C\psi|\nabla f|^2\frac{|\nabla\psi|^2}{\psi} \geq -C'\sup\frac{|\nabla\psi|^4}{\psi^2},$$

and hence

$$\psi L_1\mu \geq \frac{1}{2\alpha^2}(\psi\lambda)^2 + \psi^2\bar{\xi}^T(\alpha P+Q)\xi + \frac{1}{16}\beta\psi|\nabla f|^2$$
$$-C\left[|\partial_t\psi| + |\Delta\psi| + \frac{|\nabla\psi|^2}{2\psi}\right]^2 - C\sup\frac{|\nabla\psi|^4}{\psi^2}.$$

Let ψ be a cut off function supported in the space-time cube $Q_{R,T}(x_0,t_0)$ such that $\psi = 1$ in the cube of half the size $Q_{R/2,T/2}(x_0,t_0)$. We also require that

$$|\nabla\psi| \leq \frac{C}{R}, \quad |\Delta\psi| \leq C\frac{K+1}{R^2}, \quad \frac{|\partial_t\psi|}{\sqrt{\psi}} \leq C\frac{1}{T}, \quad \frac{|\nabla\psi|^2}{\psi} \leq \frac{2C}{R^2}.$$

Also the quotients are regarded as 0 when $\psi = 0$ somewhere. Without loss of generality we just take $t_0 = T$. We also require that ψ is supported in the slightly shorter space time cube $Q_{R,3T/4}(x_0,t_0)$. As usual, the cutoff function can be constructed from the distance function, which is not always smooth. One can either mollify the distance by convoluting with a smooth kernel or use the well known trick by Calabi to get around singular points of the distance.

Let μ_0 be the maximal eigenvalues of $\psi(\alpha V+W)+\beta fg$ with a unit eigenvector ξ. Assume μ_0 is taken at the space-time point (p_1,t_1). Again, by parallel translation, we extend ξ to a vector field in a neighborhood of p_1, which is still denoted by ξ. We are interested only in the case $\mu_0 > 0$. Since $f \leq 0$ and $\psi = 0$ on the parabolic boundary of Ω, we know that p_1 must lie in the interior of $B(p,R)$. Define a function $\mu = \mu(x,t)$ around (p_1,t_1) by

$$\mu = \bar{\xi}^T(\psi(\alpha V+W)+\beta fg)\xi.$$

Since (p_1, t_1) is a maximum point of μ, we have

$$\begin{aligned}0 \geq \psi L_1 \mu \geq &\frac{1}{2\alpha^2}(\psi\lambda)^2 + \psi^2 \bar{\xi}^T(\alpha P + Q)\xi \\ &- C\left[|\partial_t \psi| + |\Delta\psi| + \frac{|\nabla\psi|^2}{2\psi}\right]^2 - C\sup\frac{|\nabla\psi|^4}{\psi^2}.\end{aligned}$$

Hence, at (p_1, t_1), it holds

$$\frac{1}{2\alpha^2}(\psi\lambda)^2 \leq \psi^2\left|\bar{\xi}^T(\alpha P + Q)\xi\right| + C\left(\frac{1}{T} + \frac{1}{R^2}\right)^2. \tag{2.18}$$

From (2.12),

$$\begin{aligned}&\psi\left|\bar{\xi}^T(\alpha P + Q)\xi\right| \\ \leq &\left|\bar{\xi}_i\xi_j\left[-R_{i\bar{j}l\bar{k}}\psi\left(\alpha v_{k\bar{l}} + w_{k\bar{l}}\right) + \frac{R_{i\bar{l}}\psi}{2}\left(\alpha v_{l\bar{j}} + w_{l\bar{j}}\right) + \frac{R_{l\bar{j}}\psi}{2}\left(\alpha v_{i\bar{l}} + w_{i\bar{l}}\right)\right]\right| \\ &+ \psi\left|\bar{\xi}_i\xi_j\left[-R_{i\bar{j}l\bar{k}}w_{kl} + \frac{R_{i\bar{l}}}{2}w_{l\bar{j}} + \frac{R_{l\bar{j}}}{2}w_{i\bar{l}}\right]\right| + C\psi|Ric||W| \\ \leq &\left|\bar{\xi}_i\xi_j\left[-R_{i\bar{j}l\bar{k}}\psi\left(\alpha v_{k\bar{l}} + w_{k\bar{l}}\right) + \frac{R_{i\bar{l}}\psi}{2}\left(\alpha v_{l\bar{j}} + w_{l\bar{j}}\right) + \frac{R_{l\bar{j}}\psi}{2}\left(\alpha v_{i\bar{l}} + w_{i\bar{l}}\right)\right]\right| \\ &+ CK\psi|W|.\end{aligned}$$

By splitting off a term βf, we obtain

$$\begin{aligned}&\psi\left|\bar{\xi}^T(\alpha P + Q)\xi\right| \\ \leq &\left|\bar{\xi}_i\xi_j\left[-R_{i\bar{j}l\bar{k}}\left\{\psi\left(\alpha v_{k\bar{l}} + w_{k\bar{l}}\right) - \beta f\delta_{kl}\right\} + \frac{R_{i\bar{l}}}{2}\left\{\psi\left(\alpha v_{l\bar{j}} + w_{l\bar{j}}\right) - \beta f\delta_{kl}\right\}\right.\right. \\ &\left.\left.+\frac{R_{l\bar{j}}}{2}\left\{\psi\left(\alpha v_{i\bar{l}} + w_{i\bar{l}}\right) - \beta f\delta_{kl}\right\}\right]\right| + CK\beta\psi|f|.\end{aligned}$$

Let $\mu_1, \ldots, \mu_n$ be the eigenvalues of the $(1,1)$-form $\psi(\alpha V + W) + \beta f g$ at (p_1, t_1), which are listed in increasing order. We assume without loss of generality that $\mu_1 < 0$. Then the above inequality implies

$$\psi\left|\bar{\xi}^T(\alpha P + Q)\xi\right| \leq CK\left(\mu_n + |\mu_1|\right) + CK\beta|f| + CK\psi|W|.$$

Observe that

$$\begin{aligned}\mu_1 + (n-1)\mu_n &\geq \mu_1 + \cdots + \mu_n \\ &= \operatorname{tr}\left[\psi\left(\frac{\alpha u_{i\bar{j}}}{u(1-f)} + \frac{u_i u_{\bar{j}}}{u^2(1-f)^2}\right) + \beta f\delta_{ij}\right] \\ &\geq \psi\frac{\alpha\Delta u}{u(1-f)} + n\beta f.\end{aligned}$$

Hence,

$$|\mu_1| \leq (n-1)\mu_n - \psi\frac{\alpha\Delta u}{u(1-f)} + n\beta|f|.$$

Therefore,

$$\psi\left|\bar{\xi}^T(\alpha P+Q)\xi\right| \le CK\left(\mu_n-\psi\frac{\alpha\Delta u}{u(1-f)}\right)+CK\psi|W|+CK\beta|f|. \tag{2.19}$$

Since $\mathrm{Ric} \ge -CK$, by [3] and our choice of the cutoff function ψ, we have, for any $a>1$,

$$\psi^2\left(\frac{|\nabla u|^2}{u^2}-a\frac{u_t}{u}\right) \le C\left(\frac{1}{T}+\frac{1}{R^2}+K\right).$$

We note that the time $1/T$ is actually $1/t$ in [3]. However, since our cutoff function is supported in a shorter cube, these two terms are equivalent. With $u_t=\Delta u$ and $\mu=\psi\lambda+\beta f\le\psi\lambda$, we get, at (p_1,t_1),

$$\psi^2\left|\bar{\xi}^T(\alpha P+Q)\xi\right| \le CK\psi^2\lambda+CK\left(\frac{1}{T}+\frac{1}{R^2}\right)+CK^2+CK\beta|f|.$$

From the definition of $W=(w_{ij})$, we have

$$|W|\le\frac{|\nabla u|^2}{u^2(1-f)^2}.$$

By Theorem 1.1 in [4], we obtain

$$\psi^2|W|\le C\left(\frac{1}{T}+\frac{1}{R^2}+K\right),$$

and hence

$$\psi^2\left|\bar{\xi}^T(\alpha P+Q)\xi\right| \le CK\psi^2\lambda+CK\left(\frac{1}{T}+\frac{1}{R^2}\right)+CK^2+CK\beta|f|.$$

Substituting this to (2.18), we find, at (p_1,t_1), that

$$\frac{1}{2\alpha^2}(\psi\lambda)^2\le CK\psi^2\lambda+CK\left(\frac{1}{T}+\frac{1}{R^2}\right)+C\left(\frac{1}{T}+\frac{1}{R^2}\right)^2+CK^2+CK\beta|f|,$$

and hence

$$\psi\lambda\le C\left(\frac{1}{T}+\frac{1}{R^2}+B\right)+C\sqrt{K\beta|f|},$$

where B is a nonnegative constant depending only on K with the property that $B=0$ if $K=0$. This implies

$$\mu=\psi\lambda+\beta f\le C\left(\frac{1}{T}+\frac{1}{R^2}+B\right)+C\sqrt{K\beta|f|}+\beta f.$$

Since $f<0$, we know that $C\sqrt{K\beta|f|}+\beta f\le CK$. Thus

$$\mu_0=\mu|_{(p_1,t_1)}=(\psi\lambda+\beta f)|_{(p_1,t_1)}\le C\left(\frac{1}{T}+\frac{1}{R^2}+B\right).$$

Here B may have changed from the last line. Therefore

$$\mu\le C\left(\frac{1}{T}+\frac{1}{R^2}+B\right)\qquad\text{in } Q_{R,T}.$$

Hence, for any unit tangent vector ξ at x with $(x,t)\in Q_{R,T}$, it holds

$$\psi\bar{\xi}^T(\alpha V+W)\xi+\beta f\leq C\left(\frac{1}{T}+\frac{1}{R^2}+B\right)\quad\text{ in } Q_{R,T},$$

or

$$\psi\bar{\xi}^T(\alpha V+W)\xi\leq C\left(\frac{1}{T}+\frac{1}{R^2}+B\right)+\beta|f|,\quad\text{ in } Q_{R,T}.$$

Recall from (2.17) that $\beta=\frac{C}{R^2}$, we then have

$$\psi\bar{\xi}^T V\xi\leq C\left(\frac{1}{T}+\frac{1}{R^2}+B\right)(1-f),$$

and hence

$$\psi\frac{u_{i\bar{j}}\bar{\xi}_i\xi_j}{u}\leq C\left(\frac{1}{T}+\frac{1}{R^2}+B\right)(1-f)^2.$$

This implies the desired estimate. $\square$

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