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1 Introduction

Let (M^n, g) be a compact Riemannian manifold without boundary. Assume M has nonnegative Barky Emery Ricci curvature ie. Ric+Hessf≥ 0, where f is a smooth function on M. We consider "isotropic" equations of the following form:

$$\left[\alpha(u,|Du|)\frac{D_iuD_ju}{|Du|^2}+\beta(u,|Du|)\left(\delta_{ij}-\frac{D_iuD_ju}{|Du|^2}\right)\right]D_iD_ju+q(u,|Du|)-\beta(u,|Du|)\langle Du,Df\rangle=0. \tag{1}$$

We assume that Equation (1) is nonsingular, i.e. the left hand side of (1) is continuous on $\mathbb{R}\times TM\times \mathrm{Sym}^2\left(T^*M\right)$, α and β are nonnegative functions, and that $\beta(s,t)>0$ for $t>0$. Our main result is the following estimate:

Theorem 1. *Let (M^n, g) be a compact Riemannian manifold with nonnegative Barky Emery Ricci curvature $\mathrm{Hess}+\mathrm{Ric}\, f\geq 0$, and let u be a viscosity solution of Equation (1). Suppose $\varphi:[a,b]\to[\inf u,\sup u]$ is a C^2 solution of*

$$\alpha\left(\varphi,\varphi'\right)\varphi''+q\left(\varphi,\varphi'\right)=0 \quad \text{on } [a,b]; \tag{2}$$

$$\varphi(a)=\inf u;\quad \varphi(b)=\sup u;\quad \varphi'>0 \quad \text{on } [a,b]. \tag{3}$$

Moreover let ψ be the inverse of φ, i.e. $\psi(\varphi(z))=z$. Then we have

$$\psi(u(y))-\psi(u(x))-d(x,y)\leq 0, \forall x,y\in M$$

By allowing y to approach x, if u is smooth we deduce the following gradient estimate:

Corollary 1. *Under the assumptions of Theorem 1,$|Du(x)|\leq\varphi'(\psi(u(x)))$ for all $x\in M$.*

2 Preliminaries

2.1 **Definition of viscosity solutions on manifolds.** Let M be a Riemannian manifold. We use the following notations:

$$USC(M)=\{u:M\to\mathbb{R}\mid u \text{ is upper semicontinuous }\},$$
$$LSC(M)=\{u:M\to\mathbb{R}\mid u \text{ is lower semicontinuous }\}.$$

Next we introduce the semijets on manifolds.

Definition 2.1. *For a function $u \in USC(M)$, the second order superjet of u at a point $x_0 \in M$ is defined by*

$$\mathcal{J}^{2,+}u\left(x_0\right) := \left\{\left(D\varphi\left(x_0\right), D^2\varphi\left(x_0\right)\right) : \varphi \in C^2(M), \text{ such that } u-\varphi \right.$$
$$\left. \text{attains a local maximum at } x_0\right\} .$$

For $u \in \mathrm{LSC}(M)$, the second order subjet of u at $x_0 \in M$ is defined by

$$\mathcal{J}^{2,-}u\left(x_0\right) := -\mathcal{J}^{2,+}(-u)\left(x_0\right).$$

We also define the closures of $\mathcal{J}^{2,+}u\left(x_0\right)$ and $\mathcal{J}^{2,-}u\left(x_0\right)$ by $\overline{\mathcal{J}}^{2,+}u\left(x_0\right) = \left\{(p,X) \in T_{x_0}M \times \mathrm{Sym}^2\left(T^*_{x_0}M\right) \mid \text{there is a sequence } \left(x_j,p_j,X_j\right) \text{ such that } \left(p_j,X_j\right) \in \mathcal{J}^{2,+}u\left(x_j\right) \text{ and } \left(x_j,u\left(x_j\right),p_j,X_j\right) \to \left(x_0,u\left(x_0\right),p,X\right) \text{ as } j \to \infty\right\}$.

$$\overline{\mathcal{J}}^{2,-}u\left(x_0\right) := -\overline{\mathcal{J}}^{2,+}(-u)\left(x_0\right).$$

Now we can define the viscosity solution for the general equation

$$F\left(x,u,Du,D^2u\right) = 0 \tag{4}$$

on M : Assume $F \in C\left(M \times \mathbb{R} \times TM \times \mathrm{Sym}^2\left(T^*M\right)\right)$ is degenerate elliptic, i.e.

$$F(x,r,p,X) \le F(x,r,p,Y), \text{ whenever } X \le Y.$$

Definition 2.2. *(1) A function $u \in USC(M)$ is a viscosity subsolution of (4) if for all $x \in M$ and $(p,X) \in \mathcal{J}^{2,+}u(x)$,*

$$F(x,u(x),p,X) \ge 0.$$

(2) A function $u \in LSC(M)$ is a viscosity supersolution of (4) if for all $x \in M$ and $(p,X) \in \mathcal{J}^{2,-}u(x)$,

$$F(x,u(x),p,X) \le 0.$$

(3) A viscosity solution of (4) is a continuous function which is both a viscosity subsolution and a viscosity supersolution of (4).

2.2. Maximum principle for semicontinuous functions.

Theorem 2 (Theorem 3.2 in [?]). *Let $M_1^{n_1}, \ldots, M_k^{n_k}$ be Riemannian manifolds, and $\Omega_i \subset M_i$ open subsets. Let $u_i \in USC\left(\Omega_i\right)$ and $\varphi \in C^2\left(\Omega_1 \times \cdots \times \Omega_k\right)$. Suppose the function*

$$w\left(x_1,\ldots,x_k\right) := u_1\left(x_1\right) + \cdots + u_k\left(x_k\right) - \varphi\left(x_1,\ldots,x_k\right)$$

attains a maximum at $\left(\hat{x}_1,\ldots,\hat{x}_k\right)$ on $\Omega_1 \times \cdots \times \Omega_k$. Then for each $\lambda > 0$ there exists $X_i \in \mathrm{Sym}^2\left(T^*_{\hat{x}_i}M_i\right)$ such that

$$\left(D_{x_i}\varphi\left(\hat{x}_1,\ldots,\hat{x}_k\right), X_i\right) \in \overline{\mathcal{J}}^{2,+}u_i\left(\hat{x}_i\right) \text{ for } i = 1,\ldots,k,$$

and the block diagonal matrix with entries X_i satisfies

$$-\left(\frac{1}{\lambda}+\|A\|\right) I \leq \begin{pmatrix} X_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_k \end{pmatrix} \leq A+\lambda A^2,$$

where $A = D^2\varphi\left(\hat{x}_1, \ldots, \hat{x}_k\right).$

2.3. **First and second variation formulae for arclength.** Let $\gamma_0 : [0, l] \to M$ be a geodesic in M parametrised by arc length. Suppose $\gamma(\varepsilon, s)$ is any smooth variation of $\gamma_0(s)$ with $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Then the first variation formula for arclength is

$$\left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} L(\gamma(\varepsilon, \cdot)) = \left.\langle \gamma_s, \gamma_\varepsilon \rangle\right|_0^l,$$

where γ_s is the unit tangent vector of γ_0 and $\gamma_\varepsilon = \frac{\partial}{\partial \varepsilon}\gamma$ is the variational vector field. Furthermore, the second variation formula is given by

$$\left.\frac{\partial^2}{\partial \varepsilon^2}\right|_{\varepsilon=0} L(\gamma(\varepsilon, \cdot)) = \int_0^l \left(\left|\nabla_{\gamma_s}\left(\gamma_\varepsilon^{\perp}\right)\right|^2 - R\left(\gamma_s, \gamma_\varepsilon, \gamma_\varepsilon, \gamma_s\right)\right) ds + \left.\langle \gamma_s, \nabla_{\gamma_\varepsilon} \gamma_\varepsilon \rangle\right|_0^l,$$

where $\gamma_\varepsilon^{\perp}$ means the normal part of the variational vector. Here and in the sequel we use the convention on the Riemannian curvature tensor R such that $\mathrm{Ric}(X, Y) = \mathrm{tr}_g(R(X, \cdot, \cdot, Y))$ for $X, Y \in T_x M$.

3 Close Riemannian Manifold with Nonnegative Barry Emery Ricci curature.

Lemma 1. *Let u be a continuous function. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a C^2 function with $\varphi' > 0$. Let ψ be the inverse of φ , so that*

$$\varphi(\psi(u(x))) = u(x).$$

(1) Suppose $(p, X) \in \mathcal{J}^{2,+}(\psi \circ u)(x_0)$. Then

$$\left(\varphi' p, \varphi'' p \otimes p + \varphi' X\right) \in \mathcal{J}^{2,+} u\left(x_0\right)$$

where all derivatives of φ are evaluated at $\psi \circ u\left(x_0\right)$.

(2) Suppose $(p, X) \in \mathcal{J}^{2,-}(\psi \circ u)(x_0)$. Then

$$\left(\varphi' p, \varphi'' p \otimes p + \varphi' X\right) \in \mathcal{J}^{2,-} u\left(x_0\right),$$

where all derivatives of φ are evaluated at $\psi \circ u\left(x_0\right)$.

(3) The same holds if we replace the semijets by their closures.

Proof. (1) Recall the definition of the superjet:

$$\mathcal{J}^{2,+}u\,(x_0) := \left\{ \left(D\varphi\,(x_0)\,, D^2\varphi\,(x_0) \right) : \varphi \in C^2(M),\ \text{such that } u - \varphi$$

attains a local maximum at $x_0\}$.

Assume $(p, X) \in \mathcal{J}^{2,+}(\psi \circ u)\,(x_0)$. Let h be a C^2 function such that $\psi(u(x)) - h(x)$ has a local maximum at x_0 and $(Dh, D^2h)\,(x_0) = (p, X)$. Since φ is increasing, we know $u(x) - \varphi(h(x)) = \varphi(\psi(u(x))) - \varphi(h(x))$ has a local maximum at x_0 . So it follows that

$$(\varphi' p, \varphi'' p \otimes p + \varphi' X) \in \mathcal{J}^{2,+}u\,(x_0)\,.$$

(2) is similar. (3) follows by approximation. $\square$

We prove the following modulus of continuity estimate, which implies Theorem 1 immediately.

Theorem 3. *Let (M^n, g) be a compact Riemannian manifold with nonnegative Barky Emery Ricci curvature Ric+Hessf≥ 0 and u be a viscosity solution of Equation (1). Suppose the barrier $\varphi : [a,b] \to [\inf u, \sup u]$ satisfies*

$$\varphi' > 0, \tag{5}$$

$$\tfrac{d}{dz}\left(\tfrac{q(\varphi,\varphi') + \varphi''\alpha(\varphi,\varphi')}{\varphi'\beta(\varphi,\varphi')}\right) < 0. \tag{6}$$

Moreover let ψ be the inverse of φ, i.e. $\psi(\varphi(z)) = z$. Then we have

$$\psi(u(y)) - \psi(u(x)) - d(x,y) \leq 0, \forall x, y \in M. \tag{7}$$

Proof. The proof is by contradiction. Assume there exists some $\varepsilon_0 > 0$ such that

$$\psi(u(y)) - \psi(u(x)) - d(x,y) \leq \varepsilon_0,$$

for any $x, y \in M$ and with equality for some $x_0 \neq y_0$.

Next we want to replace $d(x,y)$ by a smooth function $\tilde{d}(x,y)$ on a neighbourhood of (x_0, y_0) . To construct this we let γ_0 be the unit speed lengthminimizing geodesic joining x_0 and y_0 , with length $l = L\,(\gamma_0)$. Let $\{e_i(s)\}_{i=1}^n$ be parallel orthonormal vector fields along $\gamma_0 with e_n(s) = \gamma_0'(s)$ for each s . Then in small neighbourhoods U_{x_0} about x_0 and U_{y_0} about y_0 , there are mappings $x \mapsto (a_1(x), \ldots, a_n(x))$ and $y \mapsto (b_1(y), \ldots, b_n(y))$ defined by

$$x = \exp_{x_0}\left(\sum_{i=1}^n a_i(x) e_i(0)\right), \quad y = \exp_{y_0}\left(\sum_{i=1}^n b_i(y) e_i(l)\right).$$

We can define a smooth function $\tilde{d}(x,y)$ in $U_{x_0} \times U_{y_0}$ to be the length of the curve

$$\exp_{\gamma_0(s)}\left(\frac{l-s}{l}\sum_{i=1}^n a_i(x) e_i(s) + \frac{s}{l}\sum_{i=1}^n b_i(y) e_i(s)\right), s \in [0,l].$$

It is easy to see that $d(x,y) \leq \tilde{d}(x,y)$ in $U_{x_0} \times U_{y_0}$ and with equality at (x_0, y_0) . Therefore we have

$$\psi(u(y)) - \psi(u(x)) - \tilde{d}(x,y) \leq \varepsilon_0,$$

for any $(x,y) \in U_{x_0} \times U_{y_0}$ and with equality at (x_0, y_0) .

Thus we can apply the maximum principle to conclude that for each $\lambda > 0$, there exist $X \in \mathrm{Sym}^2\left(T^*_{x_0}M\right)$ and $Y \in \mathrm{Sym}^2\left(T^*_{y_0}M\right)$ such that

$$\begin{aligned} \left(D_y\tilde{d}\left(x_0,y_0\right), Y\right) &\in \overline{\mathcal{J}}^{2,+}\left(\psi\circ u\right)\left(y_0\right), \\ \left(D_x\tilde{d}\left(x_0,y_0\right), -X\right) &\in \overline{\mathcal{J}}^{2,+}\left(-\psi\circ u\right)\left(x_0\right), \\ \text{i.e. } \left(-D_x\tilde{d}\left(x_0,y_0\right), X\right) &\in \overline{\mathcal{J}}^{2,-}\left(\psi\circ u\right)\left(x_0\right), \end{aligned}$$

and

$$\begin{pmatrix} -X & 0 \\ 0 & Y \end{pmatrix} \leq M + \lambda M^2,$$

where $M = D^2\tilde{d}\left(x_0,y_0\right)$.

Note that $D_y\tilde{d}\left(x_0,y_0\right) = e_n(l)$ and $D_x\tilde{d}\left(x_0,y_0\right) = -e_n(0)$. By Lemma 1 , we have

$$\begin{aligned} \left(\varphi'\left(z_{y_0}\right)e_n(l), \varphi'\left(z_{y_0}\right)Y + \varphi''\left(z_{y_0}\right)e_n(l)\otimes e_n(l)\right) &\in \overline{\mathcal{J}}^{2,+}u\left(y_0\right), \\ \left(\varphi'\left(z_{x_0}\right)e_n(0), \varphi'\left(z_{x_0}\right)X + \varphi''\left(z_{x_0}\right)e_n(0)\otimes e_n(0)\right) &\in \overline{\mathcal{J}}^{2,-}u\left(x_0\right), \end{aligned}$$

where $z_{x_0} = \psi\left(u\left(x_0\right)\right)$ and $z_{y_0} = \psi\left(u\left(y_0\right)\right)$.

On the other hand, since u is both a subsolution and a supersolution of (1), we have

$$\begin{aligned} &\mathrm{tr}\left(\varphi'\left(z_{y_0}\right)A_2Y + \varphi''\left(z_{y_0}\right)A_2e_n(l)\otimes e_n(l)\right) + q\left(\varphi\left(z_{y0}\right), \varphi'\left(z_{y_0}\right)\right) \\ &\qquad -\beta(\varphi,\varphi')(\varphi(z_{y_0}),\varphi'(z_{y_0}))\varphi'(z_{y_0})\langle e_n(l), Df(y_0)\rangle \geq 0 \end{aligned}$$

and

$$\begin{aligned} &\mathrm{tr}\left(\varphi'\left(z_{x_0}\right)A_1X + \varphi''\left(z_{x_0}\right)A_1e_n(0)\otimes e_n(0)\right) + q\left(\varphi\left(z_{x_0}\right), \varphi'\left(z_{x_0}\right)\right) \\ &\qquad -\beta(\varphi,\varphi')(\varphi(z_{x_0}),\varphi'(z_{x_0}))\varphi'(z_{x_0})\langle e_n(0), Df(x_0)\rangle \leq 0, \end{aligned}$$

where

$$A_1 = \begin{pmatrix} \beta\left(\varphi\left(z_{x_0}\right), \varphi'\left(z_{x_0}\right)\right) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \beta\left(\varphi\left(z_{x_0}\right), \varphi'\left(z_{x_0}\right)\right) & 0 \\ 0 & \cdots & 0 & \alpha\left(\varphi\left(z_{x_0}\right), \varphi'\left(z_{x_0}\right)\right) \end{pmatrix},$$

and

$$A_2 = \begin{pmatrix} \beta\left(\varphi\left(z_{y_0}\right), \varphi'\left(z_{y_0}\right)\right) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \beta\left(\varphi\left(z_{y_0}\right), \varphi'\left(z_{y_0}\right)\right) & 0 \\ 0 & \cdots & 0 & \alpha\left(\varphi\left(z_{y_0}\right), \varphi'\left(z_{y_0}\right)\right) \end{pmatrix}.$$

Therefore, first we have

$$0 \le q\left(\varphi\left(z_{y_0}\right), \varphi'\left(z_{y_0}\right)\right) + \varphi''\left(z_{y_0}\right) \alpha\left(\varphi\left(z_{y_0}\right), \varphi'\left(z_{y_0}\right)\right) - \beta(\varphi, \varphi')(\varphi(z_{y_0}), \varphi'(z_{y_0}))\varphi'(z_{y_0})\langle e_n(l), Df(y_0)\rangle$$
$$+\varphi'\left(z_{y_0}\right) \operatorname{tr}\left(\begin{pmatrix} 0 & C \\ C & A_2 \end{pmatrix}\begin{pmatrix} -X & 0 \\ 0 & Y \end{pmatrix}\right)$$

where C is an $n \times n$ matrix to be determined. Multiplying by $\frac{1}{\varphi'\left(z_{y_0}\right)\rho\left(\varphi\left(z_{y_0}\right), \varphi'\left(z_{y_0}\right)\right)}$ gives

$$0 \le \frac{1}{\varphi'\left(z_{y_0}\right)\beta\left(\varphi\left(z_{y_0}\right), \varphi'\left(z_{y_0}\right)\right)} \cdot$$
$$\left(q\left(\varphi\left(z_{y_0}\right), \varphi'\left(z_{y_0}\right)\right) + \varphi''\left(z_{y_0}\right) \alpha\left(\varphi\left(z_{y_0}\right), \varphi'\left(z_{y_0}\right)\right) - \beta(\varphi, \varphi')(\varphi(z_{y_0}), \varphi'(z_{y_0}))\varphi'(z_{y_0})\langle e_n(l), Df(y_0)\rangle\right)$$
$$+ \frac{1}{\beta\left(\varphi\left(z_{y_0}\right), \varphi'\left(z_{y_0}\right)\right)} \operatorname{tr}\left(\begin{pmatrix} 0 & C \\ C & A_2 \end{pmatrix}\begin{pmatrix} -X & 0 \\ 0 & Y \end{pmatrix}\right).$$

Similarly, for the inequality at x_0 we get

$$0 \ge \frac{1}{\varphi'\left(z_{x_0}\right)\beta\left(\varphi\left(z_{x_0}\right), \varphi'\left(z_{x_0}\right)\right)}$$
$$\left(q\left(\varphi\left(z_{x_0}\right), \varphi'\left(z_{x_0}\right)\right) + \varphi''\left(z_{x_0}\right) \alpha\left(\varphi\left(z_{x_0}\right), \varphi'\left(z_{x_0}\right)\right) - \beta(\varphi(z_{x_0}), \varphi'(z_{x_0}))\varphi'(z_{x_0})\langle e_n(0), Df(x_0)\rangle\right)$$
$$- \frac{1}{\beta\left(\varphi\left(z_{x_0}\right), \varphi'\left(z_{x_0}\right)\right)} \operatorname{tr}\left(\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -X & 0 \\ 0 & Y \end{pmatrix}\right).$$

Combining them we obtain

$$\begin{aligned}
0 \le & \frac{1}{\varphi'\beta\left(\varphi, \varphi'\right)}\left(q\left(\varphi, \varphi'\right) + \varphi''\alpha\left(\varphi'\right) - \beta(\varphi, \varphi')\varphi'\langle e_n(l), Df(y_0)\right) \cdot \\
& \Big|_{z_{y_0}} - \frac{1}{\varphi'\beta\left(\varphi, \varphi'\right)}\left(q\left(\varphi, \varphi'\right) + \varphi''\alpha\left(\varphi, \varphi'\right)\right) - \beta(\varphi, \varphi')\varphi'\langle e_n(0), Df(x_0)\rangle\Big|_{z_{x_0}} \\
& + \frac{1}{\beta\left(\varphi\left(z_{y_0}\right), \varphi'\left(z_{y_0}\right)\right)} \operatorname{tr}\left(\begin{pmatrix} 0 & C \\ C & A_2 \end{pmatrix}\begin{pmatrix} -X & 0 \\ 0 & Y \end{pmatrix}\right) \\
& + \frac{1}{\beta\left(\varphi\left(z_{x_0}\right), \varphi'\left(z_{x_0}\right)\right)} \operatorname{tr}\left(\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -X & 0 \\ 0 & Y \end{pmatrix}\right).
\end{aligned}$$

Letting

$$C = \begin{pmatrix} \beta\left(\varphi\left(z_{y_0}\right), \varphi'\left(z_{z_0}\right)\right) & & & \\ & \ddots & & \\ & & \beta\left(\varphi\left(z_{y_0}\right), \varphi'\left(z_{y_0}\right)\right) & \\ & & & 0 \end{pmatrix},$$

then the matrix

$$\begin{aligned}
W &= \frac{1}{\beta\left(\varphi\left(z_{y_0}\right), \varphi'\left(z_{y_0}\right)\right)}\begin{pmatrix} 0 & C \\ C & A_2 \end{pmatrix} + \frac{1}{\beta\left(\varphi\left(z_{x_0}\right), \varphi'\left(z_{x_0}\right)\right)}\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} I_{n-1} & 0 & I_{n-1} & 0 \\ 0 & \frac{\alpha(\varphi,\varphi')}{\beta(\varphi,\varphi')}\Big|_{z_{x_0}} & 0 & 0 \\ I_{n-1} & 0 & I_{n-1} & 0 \\ 0 & 0 & 0 & \frac{\alpha(\varphi,\varphi')}{\beta(\varphi,\varphi')}\Big|_{z_{p_0}} \end{pmatrix}
\end{aligned}$$

is positive semidefinite.

So we can use

$$\begin{pmatrix} -X & 0 \\ 0 & Y \end{pmatrix} \leq M + \lambda M^2$$

to get

$$0 \leq \frac{1}{\varphi' \beta(\varphi, \varphi')} \left(q(\varphi, \varphi') + \varphi'' \alpha(\varphi, \varphi') - \beta(\varphi, \varphi')(\varphi(z), \varphi'(z)) \varphi'(z) \langle e_n(l), Df(y_0) \rangle \right)$$
$$\Big|_{z_{y_0}} - \frac{1}{\varphi' \beta(\varphi, \varphi')} \left(q(\varphi, \varphi') + \varphi'' \alpha(\varphi, \varphi') \right) - \beta(\varphi, \varphi')(\varphi(z), \varphi'(z)) \varphi'(z) \langle e_n(0), Df(x_0) \rangle \Big|_{z_{z_0}}$$
$$+ \operatorname{tr}(WM) + \lambda \operatorname{tr}\left(WM^2\right).$$

Now we compute $\operatorname{tr}(WM)$ as follows.

$$\begin{aligned} \operatorname{tr}(WM) = & \sum_{i=1}^{n-1} D^2 \tilde{d} \left((e_i(0), e_i(l)), (e_i(0), e_i(l)) \right) \\ & + \left. \frac{\alpha(\varphi, \varphi')}{\beta(\varphi, \varphi')} \right|_{z_{z_0}} D^2 \bar{d} \left((e_n(0), 0), (e_n(0), 0) \right) \\ & + \left. \frac{\alpha(\varphi, \varphi')}{\beta(\varphi, \varphi')} \right|_{z_{y_0}} D^2 \bar{d} \left((0, e_n(l)), (0, e_n(l)) \right). \end{aligned}$$

Note that

$$\begin{aligned} & D^2 \bar{d} \left((e_i(0), e_i(l)), (e_i(0), e_i(l)) \right) \\ = & \left. \frac{d^2}{dt^2} \right|_{t=0} \tilde{d} \left(\exp_{x_0}(t e_i(0)), \exp_{y_0}(t e_i(l)) \right) \\ = & \left. \frac{d^2}{dt^2} \right|_{t=0} L \left(\exp_{\tau_{\uparrow 0}(s)} (t e_i(s))_{s \in [0,l]} \right) \\ = & \int_0^l \left[-R(e_n, e_i, e_i, e_n) \right] ds \end{aligned}$$

which implies

$$\begin{aligned} & \textstyle\sum_{i=1}^{n-1} D^2 \tilde{d} \left((e_i(0), e_i(l)), (e_i(0), e_i(l)) \right) \\ & = \operatorname{Ric}(e_n, e_n) \, ds \end{aligned}$$

Similarly we get

$$\begin{aligned} D^2 \tilde{d} \left(e_n(0), 0 \right), \left(e_n(0), 0 \right)) = 0, \\ D^2 \tilde{d} \left((0, e_n(l)), (0, e_n(l)) \right) = 0. \end{aligned}$$

Moreover $\gamma_0'(0) = e_n(0)$,$\gamma_0'(l) = e_n(l)$

$$
\begin{aligned}
&\langle Df(y_0), e_n(l)\rangle - \langle Df(x_0), e_n(0)\rangle \\
=&\langle Df(\gamma_0(l)), \gamma_0'(l)\rangle - \langle Df(\gamma_0(0)), \gamma_0'(0)\rangle \\
=&\int_0^l \operatorname{Hess} f(\gamma_0'(s), \gamma_0'(s))ds \\
=&\int_0^l \operatorname{Hess} f(e_n, e_n)ds
\end{aligned}
$$

In summary, we have

$$
\begin{aligned}
0 \leq & \frac{1}{\varphi'\beta(\varphi,\varphi')}\left(q(\varphi,\varphi') + \varphi''\alpha(\varphi,\varphi')\right)\Big|_{z_{y_0}} - \frac{1}{\varphi'\beta(\varphi,\varphi')}\left(q(\varphi,\varphi') + \varphi''\alpha(\varphi,\varphi')\right)\Big|_{z_{z_0}} \\
& - \int_0^l \operatorname{Ric}(e_n(s), e_n(s)) + \operatorname{Hess} f(e_n(s), e_n(s))ds + \lambda \operatorname{tr}\left(WM^2\right).
\end{aligned}
$$

And letting $\lambda \to 0$, we have

$$
0 \leq \frac{q(\varphi,\varphi') + \varphi''\alpha(\varphi,\varphi')}{\varphi'\beta(\varphi,\varphi')}\Big|_{z_{x_0}}^{z_{y_0}}.
$$

Now taking Condition (6) into account, since $z_{y_0} = z_{x_0} + d(x_0, y_0) + \varepsilon_0 > z_{x_0}$, we get a contradiction. Then we must have

$$
Z(x,y) = \psi(u(y)) - \psi(u(x)) - d(x,y) \leq 0,
$$

which is the desired result. □

Now we use Theorem 3 prove Theorem 1

Proof. Let φ satisfy (2) and (3) in Theorem1 Then for sufficiently small $\delta > 0$, we can solve

$$
\begin{gathered}
\alpha(\varphi_\delta, \varphi_\delta')\varphi_\delta'' + q(\varphi_\delta, \varphi_\delta') = -\delta z \cdot \varphi_\delta' \cdot \beta(\varphi_\delta, \varphi_\delta'), \\
\varphi_\delta(a) = \varphi(a), \quad \varphi_\delta'(a) = \varphi'(a),
\end{gathered}
$$

to get φ_δ which satisfies (5) and (6). So by Theorem 3 we have (7) for φ_δ. Letting $\delta \to 0^+$, we finish the proof of Theorem 1 .

□