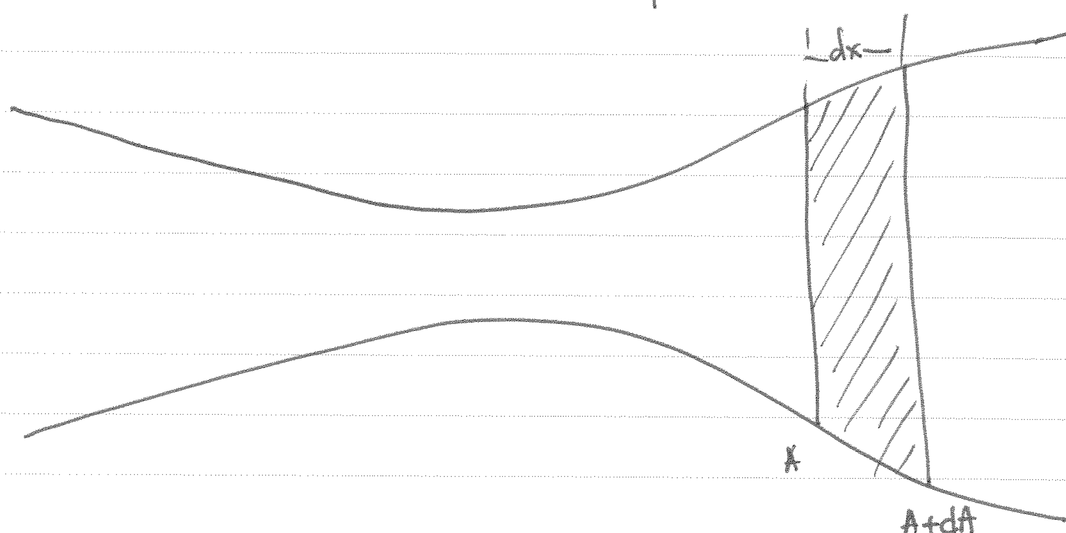
Quasi-1D Euler Equations

Derivation of conitinuity equation,

$$\frac{\partial}{\partial t}\int_{CV} \rho \, d\mathcal{V} + \int_{CS} \rho(\vec{V}\cdot\vec{n})\, dA = 0 .$$

The control volume for a quasi-1D flow,



$$\frac{\partial}{\partial t}\int_{CV} \rho \, d\mathcal{V} = \frac{\partial}{\partial t}\int_{CV} (\rho A \, dx)$$

where $A = A(x)$

$$\begin{aligned}
\int_{CS} \rho(\vec{V}\cdot\vec{n})\, dA &= \rho(-u)A + (\rho+d\rho)(u+du)(A+dA) \\
&= -\rho u A + (\rho+d\rho)(uA + udA + duA + \cancel{dudA}) \\
&= -\rho u A + \rho u A + \rho u dA + \rho du \cdot A \\
&\quad + d\rho\, uA + d\rho\, \cancel{udA} + A\, d\rho\, du
\end{aligned}$$

In the limit of $dx \longrightarrow 0$, products of $d\rho$, du and dA will go to zero faster.

$$\int_{CS} \rho(\vec{V}\cdot\vec{n})\, dA = \int \rho u\, dA + \rho A\, du + uA\, d\rho = \int d(\rho u A)$$

Therefore if the equations are written for a single discrete control volume instead of the entire nozzle, then

$$\frac{\partial}{\partial t}(\rho A dx) + d(\rho A u) = 0$$

Assume that $A = A(x)$ only and not $A = A(x,t)$, then,

$$\frac{\partial \rho}{\partial t}(A dx) + d(\rho A u) = 0$$

$$\frac{d\rho}{\partial t} + \frac{1}{A}\frac{d}{dx}(\rho A u) = 0$$

– Derivation of momentum equation,

$$\frac{\partial}{\partial t}\int_{CV} \rho u \, d\forall + \int_{CS} \rho u (\vec{V}\cdot\vec{n}) dA = -\int_{CS} p \, dA$$

$$\rightarrow \frac{\partial}{\partial t}\int_{CV} \rho u \, d\forall = \frac{\partial}{\partial t}\int \rho u \, A(x) dx$$

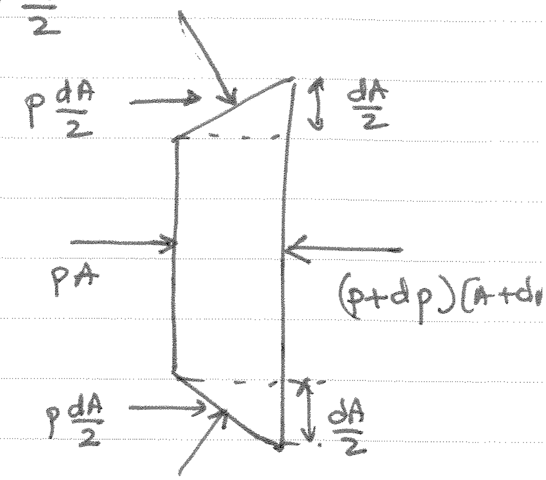
$$= \left[\frac{\partial}{\partial t}(\rho u)\right] A dx$$

$$\rightarrow \int_{CS} \rho u \, \vec{V}\cdot\vec{n} \, dA = \rho u(-u)A + (\rho + d\rho)(u + du)^2 (A + dA)$$

$$\approx d(\rho u^2 A)$$

$\rightarrow -\int p\,dA = pA - (p+dp)(A+dA) + 2p\frac{dA}{2}$

$= pA - pA - p\,dA - A\,dp - dA\,dp + 2p\frac{dA}{2}$



$= -p\,dA - A\,dp + p\,dA$

$= -A\,dp$

Collecting all terms, we have for a single discrete volume

$$\left[\frac{\partial}{\partial t}(\rho u)\right]A\,dx + d(\rho u^2 A) = -A\,dp = -d(pA) + p\,dA$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{1}{A}\frac{d}{dx}(\rho u^2 A + pA) = \frac{p}{A}\frac{dA}{dx}$$

→ Derivation of energy equation,

Similarly from above,

$$\frac{\partial e}{\partial t} + \frac{1}{A}\frac{d}{dx}\left[(e+p)uA\right] = 0$$

Then the complete differential form can be written as,

$$\frac{\partial W}{\partial t} + \frac{1}{A}\frac{\partial}{\partial x}(FA) = Q.$$

where $W = \begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix}$, $F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (e+p)u \end{bmatrix}$, $Q = \begin{bmatrix} 0 \\ \frac{p}{A}\frac{dA}{dx} \\ 0 \end{bmatrix}$

$$p = (\gamma - 1)\rho\left(\frac{e}{\rho} - \frac{u^2}{2}\right) \quad \text{and} \quad A = A(x)$$

where A is the cross-sectional area as a function of x.

In integral form, we can write the equation as,

$$\frac{\partial W}{\partial t} + \frac{1}{V}\int_{cs} F\,dA = \frac{1}{V}\int_{cv} Q\,dV$$

Boundary Conditions for Euler and Navier-Stokes Equations

- To solve for values along inlet and outlet boundaries as well as the farfied, the method of Characteristics can be employed.
- It can be employed for solving hyperbolic sets of equations in two coordinates systems but not three.