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Quasi One-Dimensional Problem

1. Discretization of the Quasi One-Dimensional Euler Equation.
2. Formulation of the Adjoint Equation.

Discretization of the Quasi 1D Problem

Consider the quasi one-dimensional Euler equations for calculating the flow within a channel in which the cross-sectional area change is small

$$\begin{aligned}\frac{\partial w}{\partial t} + \frac{1}{S}\frac{\partial FS}{\partial x} &= Q \\ \frac{\partial}{\partial t}(wS) + \frac{\partial FS}{\partial x} &= Q\end{aligned}$$

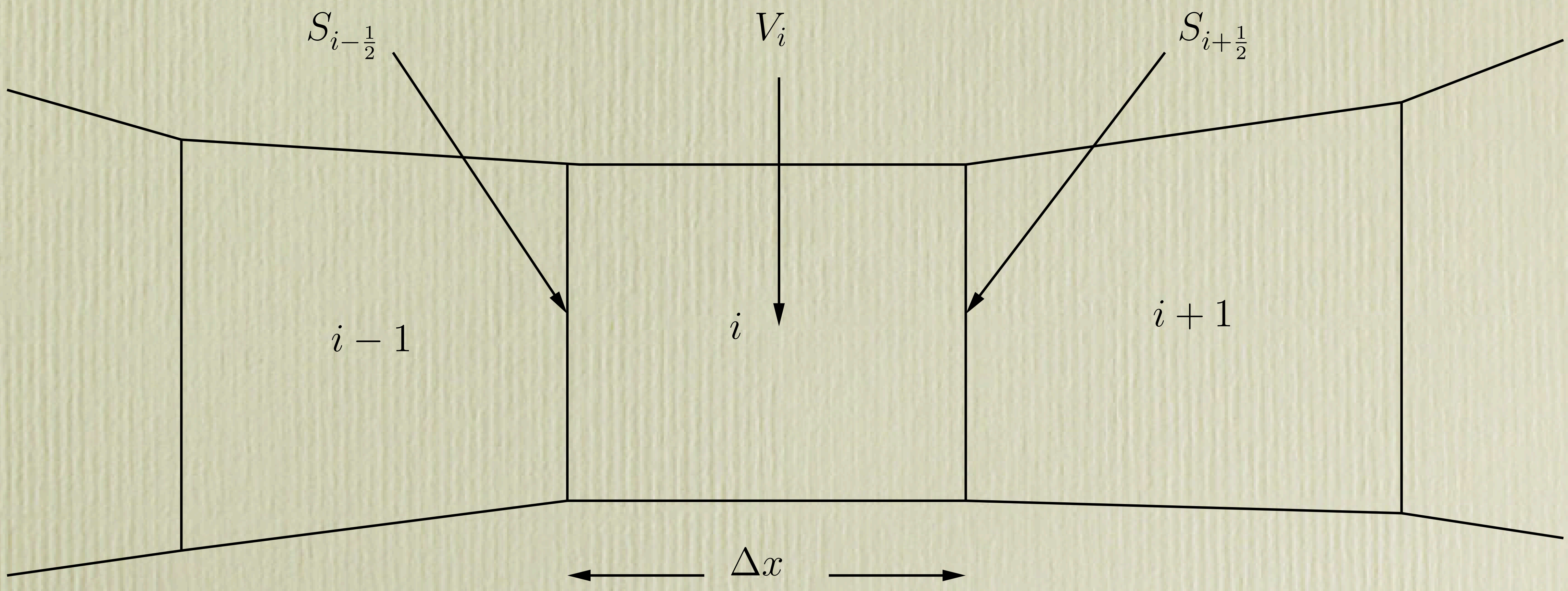
where,

$$w = \begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix}, \quad F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (e+p)u \end{bmatrix}, \text{and} \quad Q = \begin{bmatrix} 0 \\ \frac{p}{S}\frac{\partial S}{\partial x} \\ 0 \end{bmatrix}$$

$$p = (\gamma - 1)\left[e - \frac{\rho u^2}{2}\right], S = S(x),$$

and S represents the cross-sectional area of the channel as a function of x.

Discretization of the Quasi 1D Problem



Discretization of the Quasi 1D Problem

The quasi one-dimensional Euler equations can then be written in computational space as

$$\frac{\partial}{\partial t}(wS) + R(w) = 0 \qquad \text{where,} \quad R(w) = \frac{\partial}{\partial x}(FS) - \begin{bmatrix} 0 \\ p\frac{\partial S}{\partial x} \\ 0 \end{bmatrix}$$

When this equation is formulated for each computational cell, a system of first order ordinary differential equations is obtained. The equation can then be written for each computational cell as

$$\frac{(w_i^{n+1} - w_i^n)}{\Delta t} V_i + R(w)_i = 0 \qquad \text{where,} \quad V_i = S_i dx_i.$$

where,

$$R(w)_i = \left[F_{i+\frac{1}{2}} S_{i+\frac{1}{2}} - F_{i-\frac{1}{2}} S_{i-\frac{1}{2}}\right] - \begin{bmatrix} 0 \\ p_i(S_{i+\frac{1}{2}} - S_{i-\frac{1}{2}}) \\ 0 \end{bmatrix}$$

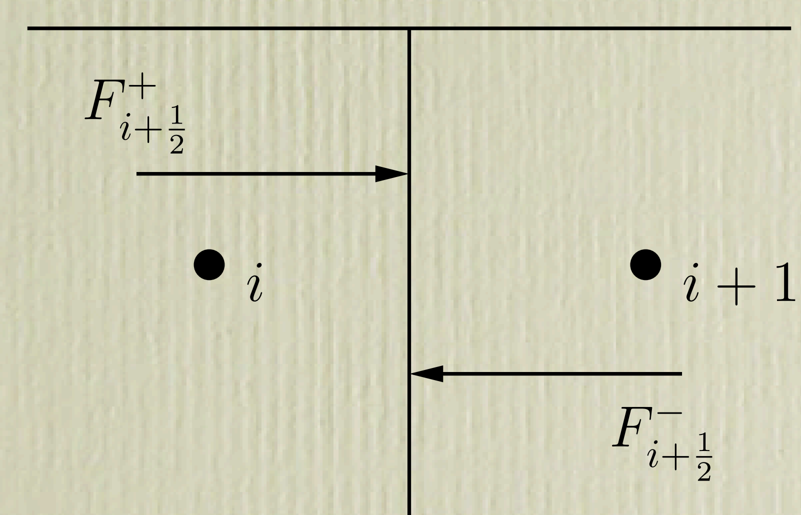
Discretization of the Quasi 1D Problem

Steger and Warming formulated characteristics theory into conservation law form by splitting the flux vectors according to the signs of the characteristic speeds of the flow.

The total flux at the $(i+\frac{1}{2})$ can then be defined as

$$\begin{aligned} \boldsymbol{F}_{i+\frac{1}{2}} &= \boldsymbol{F}^{+}_{i+\frac{1}{2}} + \boldsymbol{F}^{-}_{i+\frac{1}{2}} \\ &= \boldsymbol{A}^{+}_{i}\boldsymbol{w}_{i} + \boldsymbol{A}^{-}_{i+1}\boldsymbol{w}_{i+1} \end{aligned}$$

where $\boldsymbol{A}$ is the Jacobian of the Euler flux vectors. The subscripts $\boldsymbol{i}$ and $\boldsymbol{i+1}$ indicate that the Jacobians were computed using the flow variables from these cells and the superscripts $+$ and $-$ indicate that either the positive or negative eiqenvalues were used in the computation.



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Quasi 1D Problem

Formulation of the Adjoint Equation

To formulate the discrete adjoint equations, we first take a variation of the residual

$$\begin{aligned}\delta \boldsymbol{R}(\boldsymbol{w})_i \;&=\; \delta \boldsymbol{F}_{i+\frac{1}{2}} \boldsymbol{S}_{i+\frac{1}{2}} + \boldsymbol{F}_{i+\frac{1}{2}} \delta \boldsymbol{S}_{i+\frac{1}{2}} - \delta \boldsymbol{F}_{i-\frac{1}{2}} \boldsymbol{S}_{i-\frac{1}{2}} - \boldsymbol{F}_{i-\frac{1}{2}} \delta \boldsymbol{S}_{i-\frac{1}{2}} \\ &\quad - \begin{bmatrix} 0 \\ \delta \boldsymbol{p}_i (\boldsymbol{S}_{i+\frac{1}{2}} - \boldsymbol{S}_{i-\frac{1}{2}}) \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \boldsymbol{p}_i (\delta \boldsymbol{S}_{i+\frac{1}{2}} - \delta \boldsymbol{S}_{i-\frac{1}{2}}) \\ 0 \end{bmatrix}\end{aligned}$$

Rearrange the equation to group terms that are variation of the flow field variables and variation of the metric terms separately

$$\begin{aligned}\delta \boldsymbol{R}(\boldsymbol{w})_i \;&=\; \delta \boldsymbol{F}_{i+\frac{1}{2}} \boldsymbol{S}_{i+\frac{1}{2}} - \delta \boldsymbol{F}_{i-\frac{1}{2}} \boldsymbol{S}_{i-\frac{1}{2}} - \begin{bmatrix} 0 \\ \delta \boldsymbol{p}_i (\boldsymbol{S}_{i+\frac{1}{2}} - \boldsymbol{S}_{i-\frac{1}{2}}) \\ 0 \end{bmatrix} \\ &\quad + \boldsymbol{F}_{i+\frac{1}{2}} \delta \boldsymbol{S}_{i+\frac{1}{2}} - \boldsymbol{F}_{i-\frac{1}{2}} \delta \boldsymbol{S}_{i-\frac{1}{2}} - \begin{bmatrix} 0 \\ \boldsymbol{p}_i (\delta \boldsymbol{S}_{i+\frac{1}{2}} - \delta \boldsymbol{S}_{i-\frac{1}{2}}) \\ 0 \end{bmatrix}\end{aligned}$$

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Quasi 1D Problem

Formulation of the Adjoint Equation

Next, define the variation of the fluxes across the flux boundaries

$$
\begin{aligned}
\delta F_{i+\frac{1}{2}} &= A_i^{+}\delta w_i + A_{i+1}^{-}\delta w_{i+1} \\
\delta F_{i-\frac{1}{2}} &= A_{i-1}^{+}\delta w_{i-1} + A_i^{-}\delta w_i
\end{aligned}
$$

Substitute the variation of the fluxes into the equation for the variation for the total residual for cell i (only terms related to the variation of the flow field variables are written)

$$
\begin{aligned}
\delta R(w)_i = & \left[A_i^{+}\delta w_i + A_{i+1}^{-}\right] S_{i+\frac{1}{2}} - \left[A_{i-1}^{+}\delta w_{i-1} + A_i^{-}\delta w_i\right] S_{i-\frac{1}{2}} \\
& - \begin{bmatrix} 0 \\ \delta p_i (S_{i+\frac{1}{2}} - S_{i-\frac{1}{2}}) \\ 0 \end{bmatrix}
\end{aligned}
$$

Quasi 1D Problem

Formulation of the Adjoint Equation

Next, we premultiply the variation of the residual with the Lagrange Multiplier, ψ_i^T and sum of the domain to produce

$$\begin{aligned}
\sum_{i=2}^{N-1} \psi_i^T \delta R(w)_i &= \cdots + \psi_{i-1}^T \delta R(w)_{i-1} + \psi_i^T \delta R(w)_i + \psi_{i+1}^T \delta R(w)_{i+1} + \cdots \\
&= \cdots + \psi_{i-1}^T A_i^- \delta w_i S_{i-\frac{1}{2}} + \psi_i^T \left[A_i^+ \delta w_i S_{i+\frac{1}{2}} - A_i^- \delta w_i S_{i-\frac{1}{2}} \right] \\
&\quad - \psi_{i+1}^T A_i^+ \delta w_i S_{i+\frac{1}{2}} + \cdots
\end{aligned}$$

Reordering the terms in the equation leads to the following equation

$$\sum_{i=2}^{N-1} \psi_i^T \delta R(w)_i = \cdots - \left[A_i^+ \left(\psi_{i+1}^T - \psi_i^T \right) S_{i+\frac{1}{2}} + A_i^- \left(\psi_i^T - \psi_{i-1}^T \right) S_{i-\frac{1}{2}} \right] \delta w_i + \cdots$$

Take a transpose of the equation and the adjoint convective flux can then be written as

$$\mathcal{R}(\psi) = {A_i^+}^T \left(\psi_{i+1} - \psi_i \right) S_{i+\frac{1}{2}} + {A_i^-}^T \left(\psi_i - \psi_{i-1} \right) S_{i-\frac{1}{2}}$$

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Quasi 1D Problem

Formulation of the Adjoint Equation

We can then introduce $\sum_{i=2}^{N-1} \psi_i^T \delta R(w)_i$ as a constraint into the optimal problem by adding it to the discrete cost function

$$\delta I = \delta I_c + \sum_{i=2}^{N-1} \psi_i^T \delta R(w)_i$$

For an inverse design problem, define the cost function as

$$I_c = \frac{1}{2} \int (p - p_t)^2 ds$$

Take a variation of the cost function,

$$\begin{aligned} \delta I_c &= \int (p - p_t)\delta p ds + \frac{1}{2} \int (p - p_t)^2 \delta ds \\ &= \sum_{2}^{N-1} (p_i - p_t)\delta p_i dx_i + \frac{1}{2} \sum_{2}^{N-1} (p_i - p_t)^2 \delta dx_i \end{aligned}$$

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Quasi 1D Problem

Formulation of the Adjoint Equation

Sum the variation of the residual to the variation of the cost function to produce

$$
\begin{aligned}
\boldsymbol{\delta I} &= \sum_{2}^{N-1} (\boldsymbol{p_i} - \boldsymbol{p_t})\boldsymbol{\delta p_i dx_i} + \frac{1}{2}\sum_{2}^{N-1} (\boldsymbol{p_i} - \boldsymbol{p_t})^2 \boldsymbol{\delta dx_i} + \sum_{i=2}^{N-1} \boldsymbol{\psi_i^T \delta R(w)_i} \\
&= \sum_{2}^{N-1} (\boldsymbol{p_i} - \boldsymbol{p_t})\boldsymbol{\delta p_i dx_i} + \frac{1}{2}\sum_{2}^{N-1} (\boldsymbol{p_i} - \boldsymbol{p_t})^2 \boldsymbol{\delta dx_i} \\
&\quad + \sum_{2}^{N-1} - \left[\boldsymbol{A_i^+}\left(\boldsymbol{\psi_{i+1}^T} - \boldsymbol{\psi_i^T}\right)\boldsymbol{S_{i+\frac{1}{2}}} + \boldsymbol{A_i^-}\left(\boldsymbol{\psi_i^T} - \boldsymbol{\psi_{i-1}^T}\right)\boldsymbol{S_{i-\frac{1}{2}}}\right]\boldsymbol{\delta w_i} \\
&\quad - \begin{bmatrix} 0 \\ \boldsymbol{\psi_i^T \delta p_i}(\boldsymbol{S_{i+\frac{1}{2}}} - \boldsymbol{S_{i-\frac{1}{2}}}) \\ 0 \end{bmatrix} \\
&\quad + \boldsymbol{\psi_i^T F_{i+\frac{1}{2}} S_{i+\frac{1}{2}}} - \boldsymbol{\psi_i^T F_{i-\frac{1}{2}} S_{i-\frac{1}{2}}} - \begin{bmatrix} 0 \\ \boldsymbol{\psi_i^T p_i}(\boldsymbol{\delta S_{i+\frac{1}{2}}} - \boldsymbol{\delta S_{i-\frac{1}{2}}}) \\ 0 \end{bmatrix}
\end{aligned}
$$

Quasi 1D Problem

Formulation of the Adjoint Equation

Rearrange the equation to yield,

$$
\begin{aligned}
\delta I \;=\; \sum_{2}^{N-1} \Big[&- \Big[A_i^{+} \left(\psi_{i+1}^T - \psi_i^T \right) S_{i+\frac{1}{2}} + A_i^{-} \left(\psi_i^T - \psi_{i-1}^T \right) S_{i-\frac{1}{2}} \Big] \\
&+ (p_i - p_t)\delta p_i dx_i - \psi_i^T \delta p_i (S_{i+\frac{1}{2}} - S_{i-\frac{1}{2}}) \\
&+ \frac{1}{2}(p_i - p_t)^2 \delta dx_i + \psi_i^T F_{i+\frac{1}{2}} S_{i+\frac{1}{2}} - \psi_i^T F_{i-\frac{1}{2}} S_{i-\frac{1}{2}} - \left[\begin{array}{c} 0 \\ \psi_i^T p_i (\delta S_{i+\frac{1}{2}} - \delta S_{i-\frac{1}{2}}) \\ 0 \end{array} \right] \Big]
\end{aligned}
$$

From here, the adjoint equation can be formed from the first line for each computational cell

$$
\frac{\partial \psi_i}{\partial t} - A_i^{+^T} \left(\psi_{i+1} - \psi_i \right) S_{i+\frac{1}{2}} - A_i^{-^T} \left(\psi_i - \psi_{i-1} \right) S_{i-\frac{1}{2}} = 0.
$$

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Quasi 1D Problem

Formulation of the Adjoint Equation

The adjoint boundary condition can computed as

$$(p_i - p_t)dx_i = -\psi_i^T(S_{i+\frac{1}{2}} - S_{i-\frac{1}{2}})$$

And the gradient can be calculated as

$$\delta I = \frac{1}{2}(p_i - p_t)^2 \delta dx_i + \psi_i^T F_{i+\frac{1}{2}} S_{i+\frac{1}{2}} - \psi_i^T F_{i-\frac{1}{2}} S_{i-\frac{1}{2}} - \begin{bmatrix} 0 \\ \psi_i^T p_i(\delta S_{i+\frac{1}{2}} - \delta S_{i-\frac{1}{2}}) \\ 0 \end{bmatrix}$$

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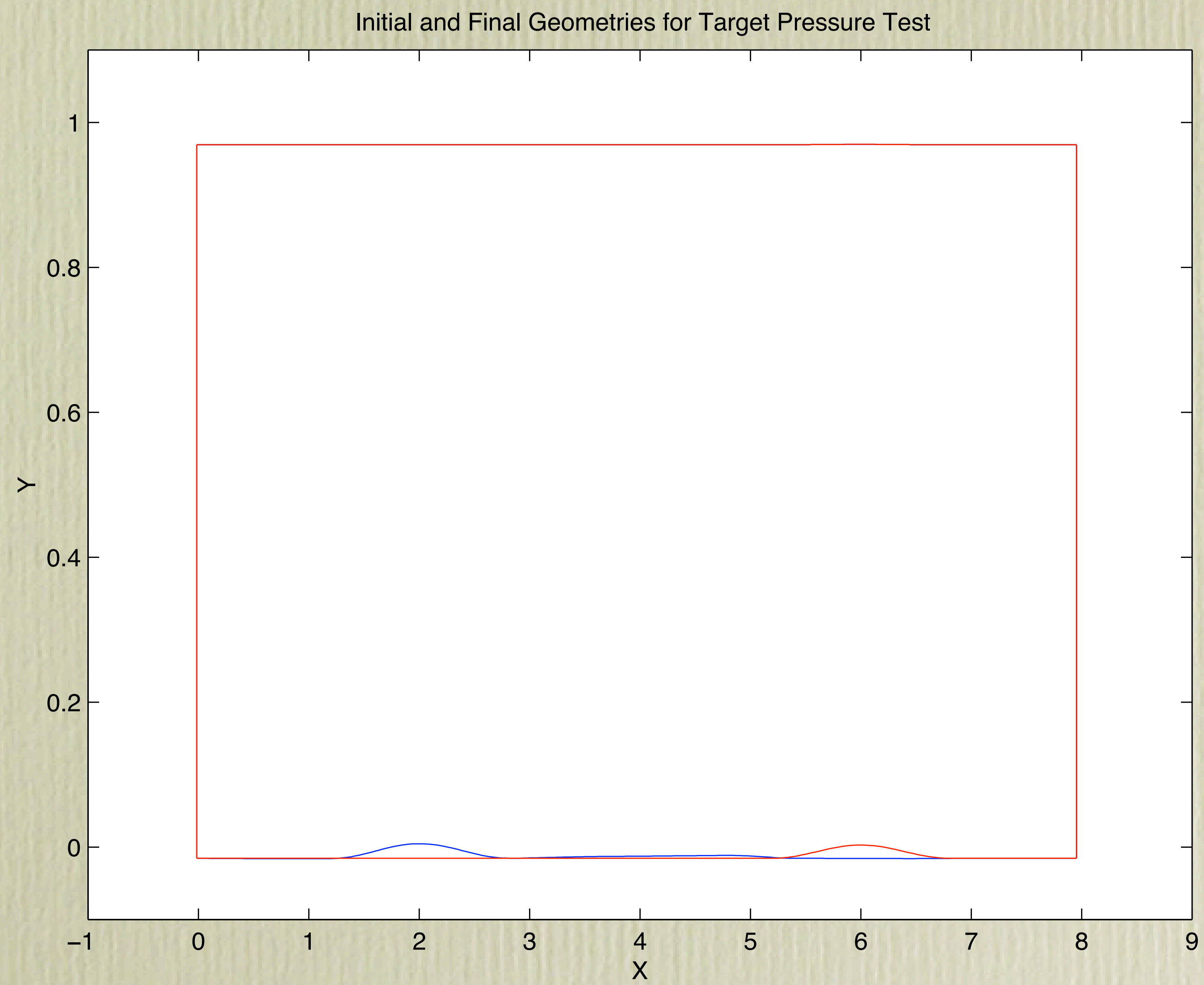
Quasi 1D Problem

Test Case

- Ni-bump geometry with a longer downstream portion of the channel.
- A channel of unit height and length, $l = 8.0$.
- A 1.8% thick Ni-bump of unit chord is centered about $x = 6.0$
- Along the upper wall, a target pressure corresponding to the presence of the same Ni-bump centered about $x = 2.0$ is specified, and the geometry of the complete lower surface of the channel is allowed to move so that the target pressure is obtained.

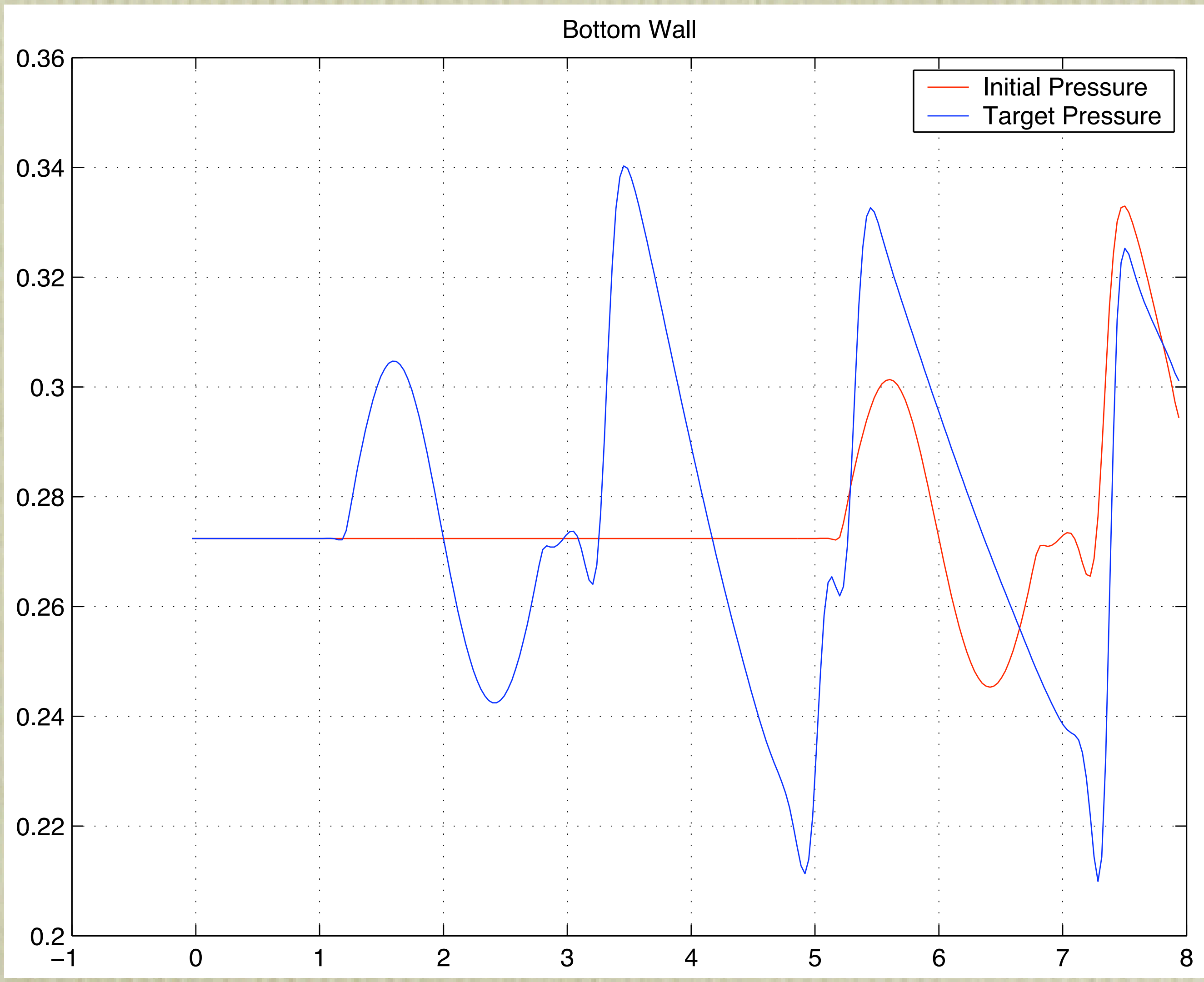
Quasi 1D Problem

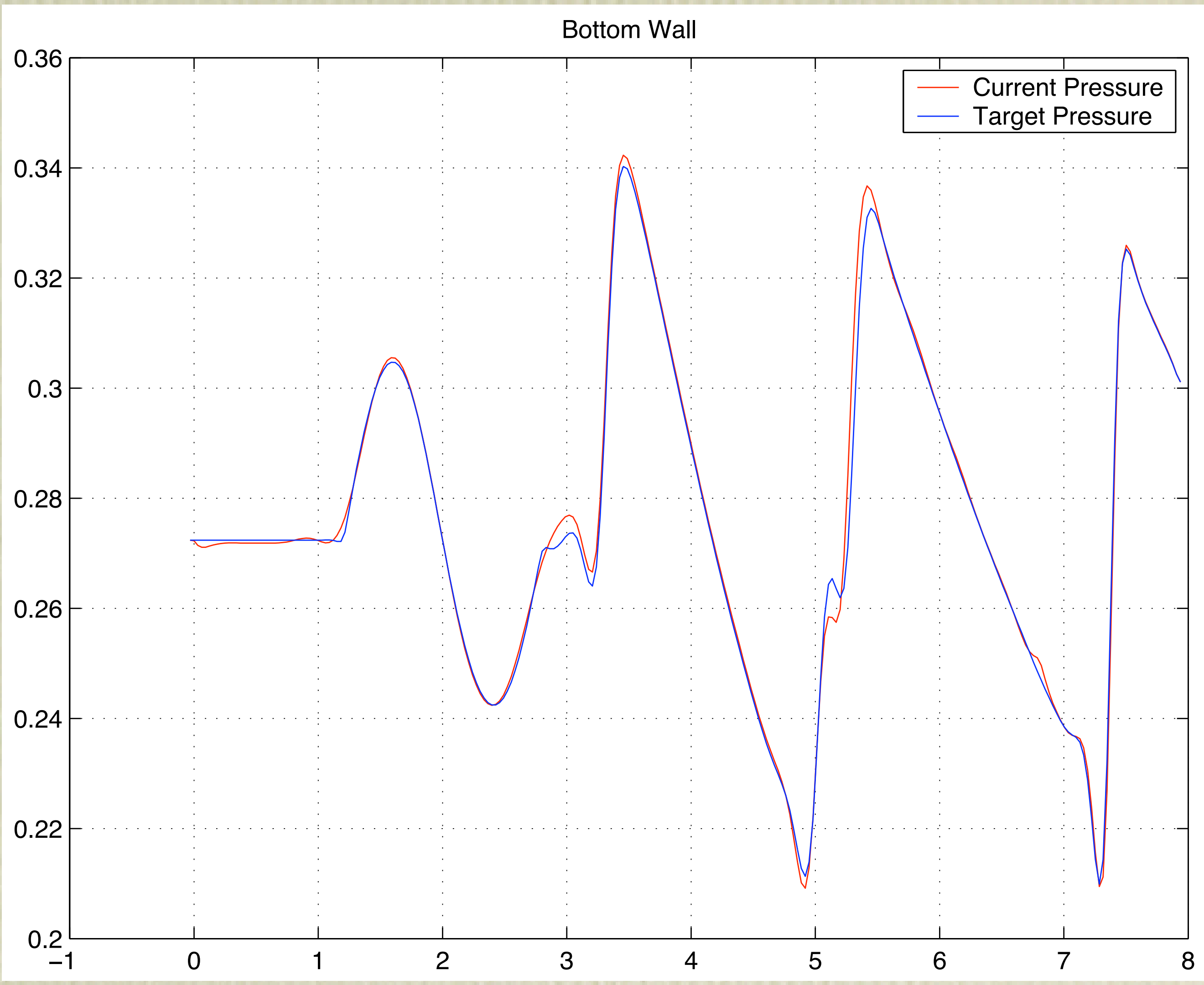
Test Case



Initial and Final Geometries

Quasi 1D Problem
Test Case





Pressure Distribution along the Bottom Wall Before and After Optimization

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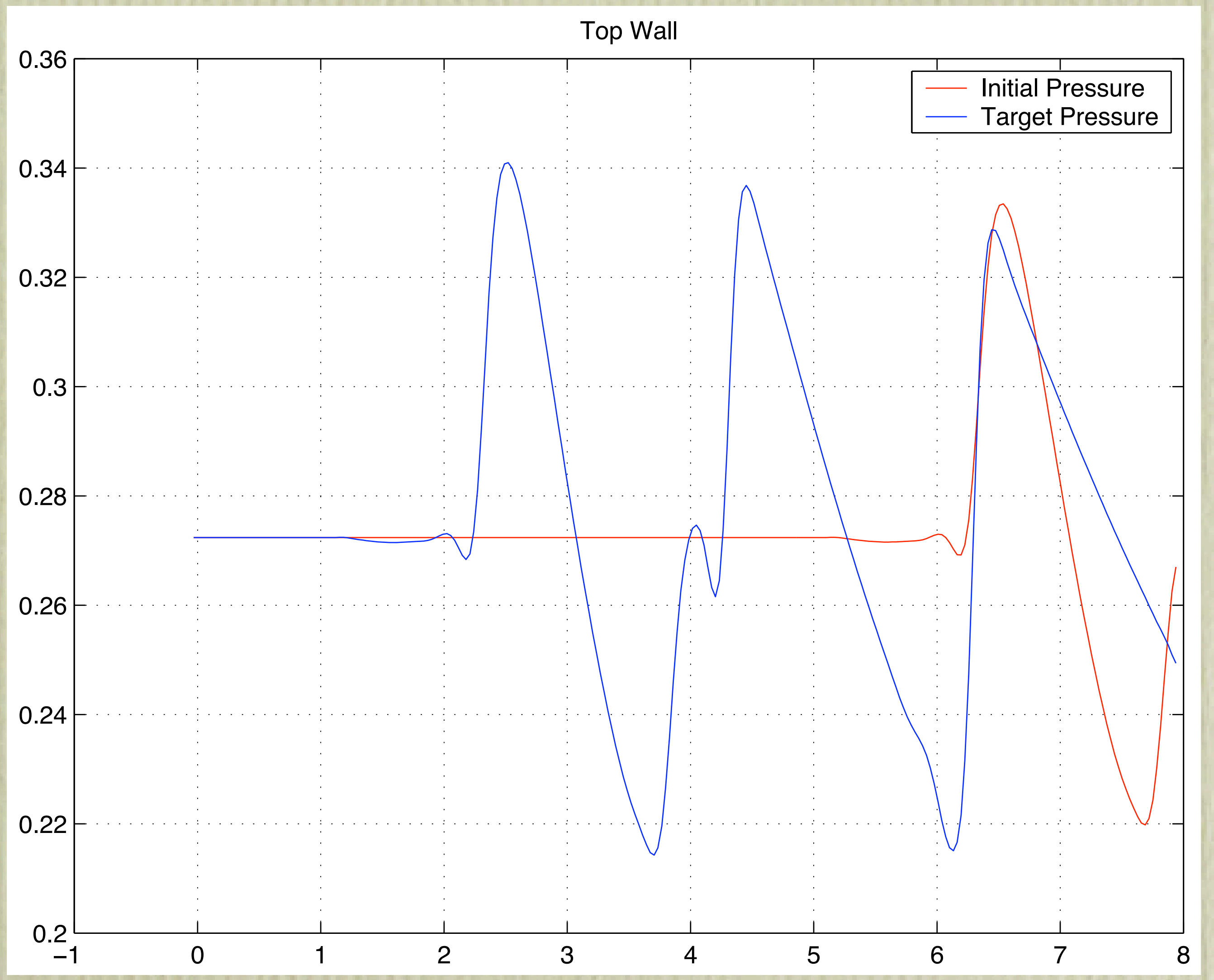
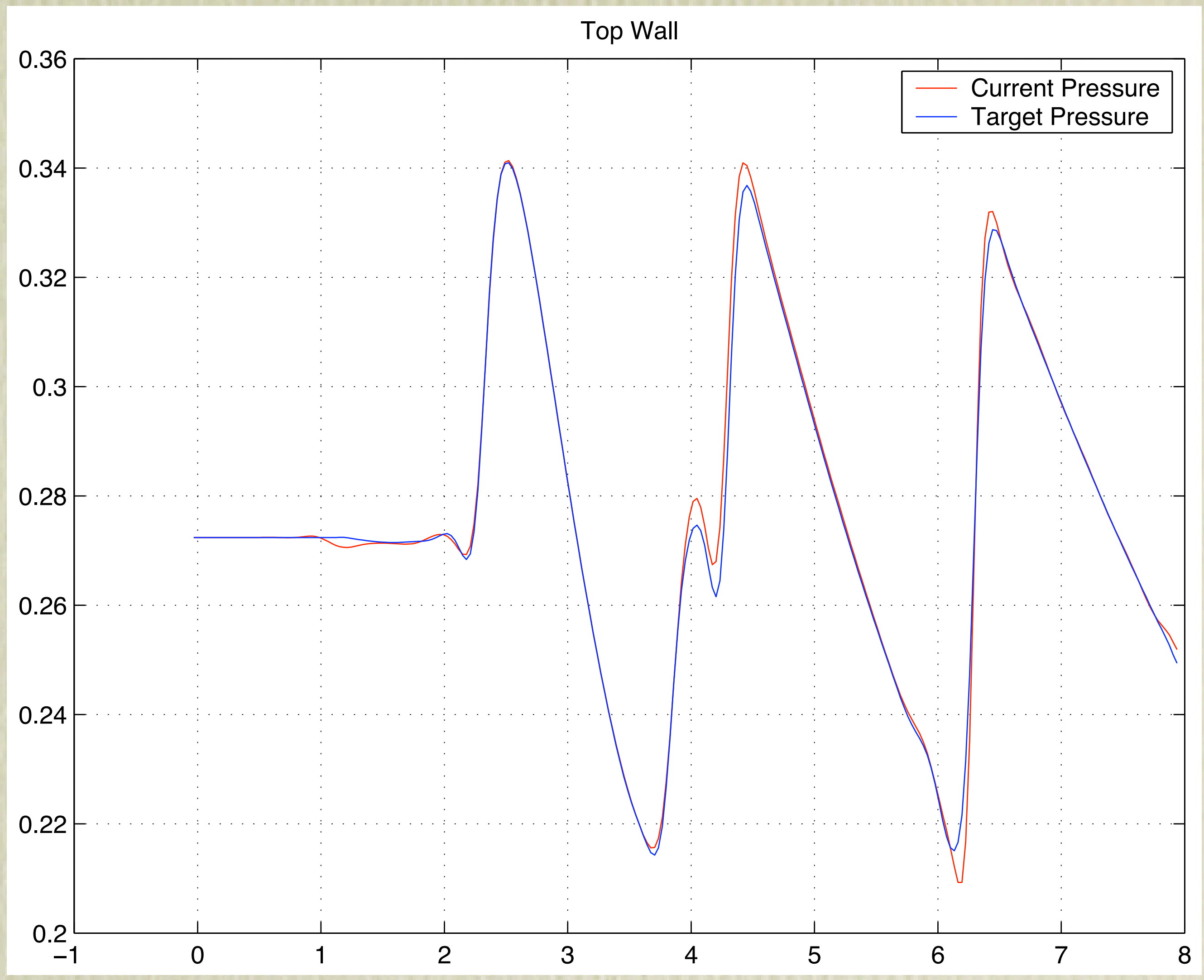
Quasi 1D Problem
Test Case





Pressure Distribution along the Top Wall Before and After Optimization

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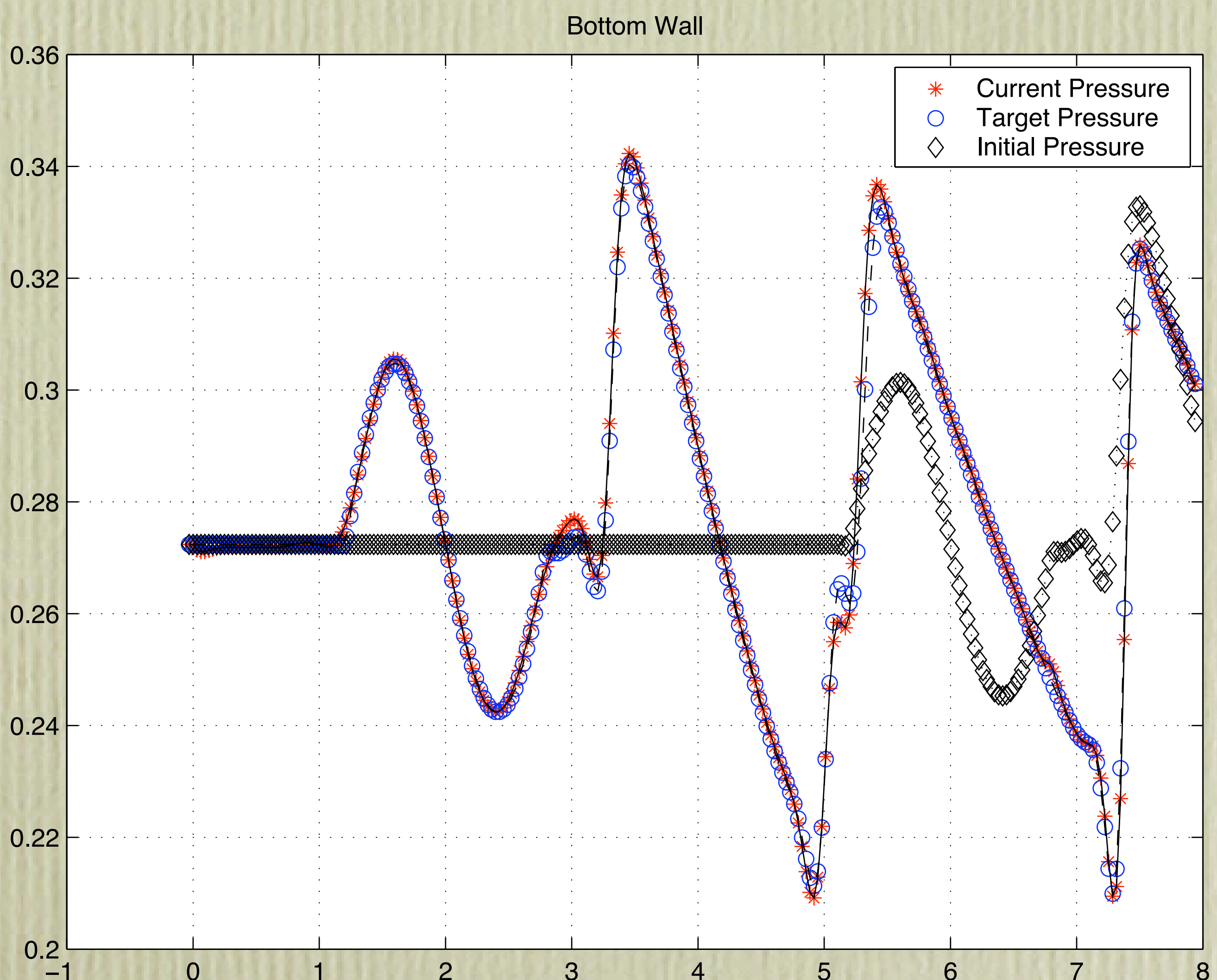
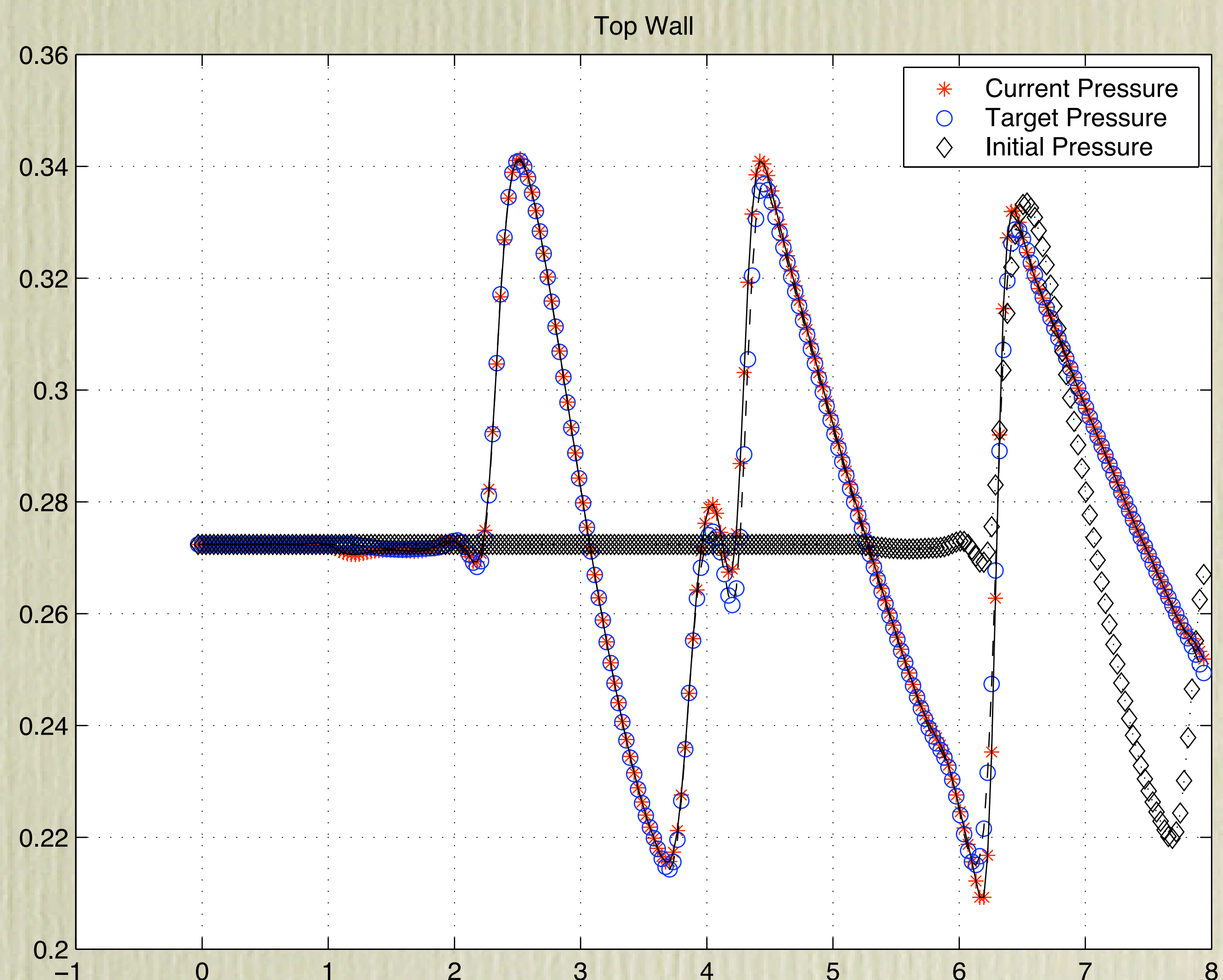
Quasi 1D Problem

Test Case





Pressure Distribution along the
Bottom and Top Walls Before and After Optimization

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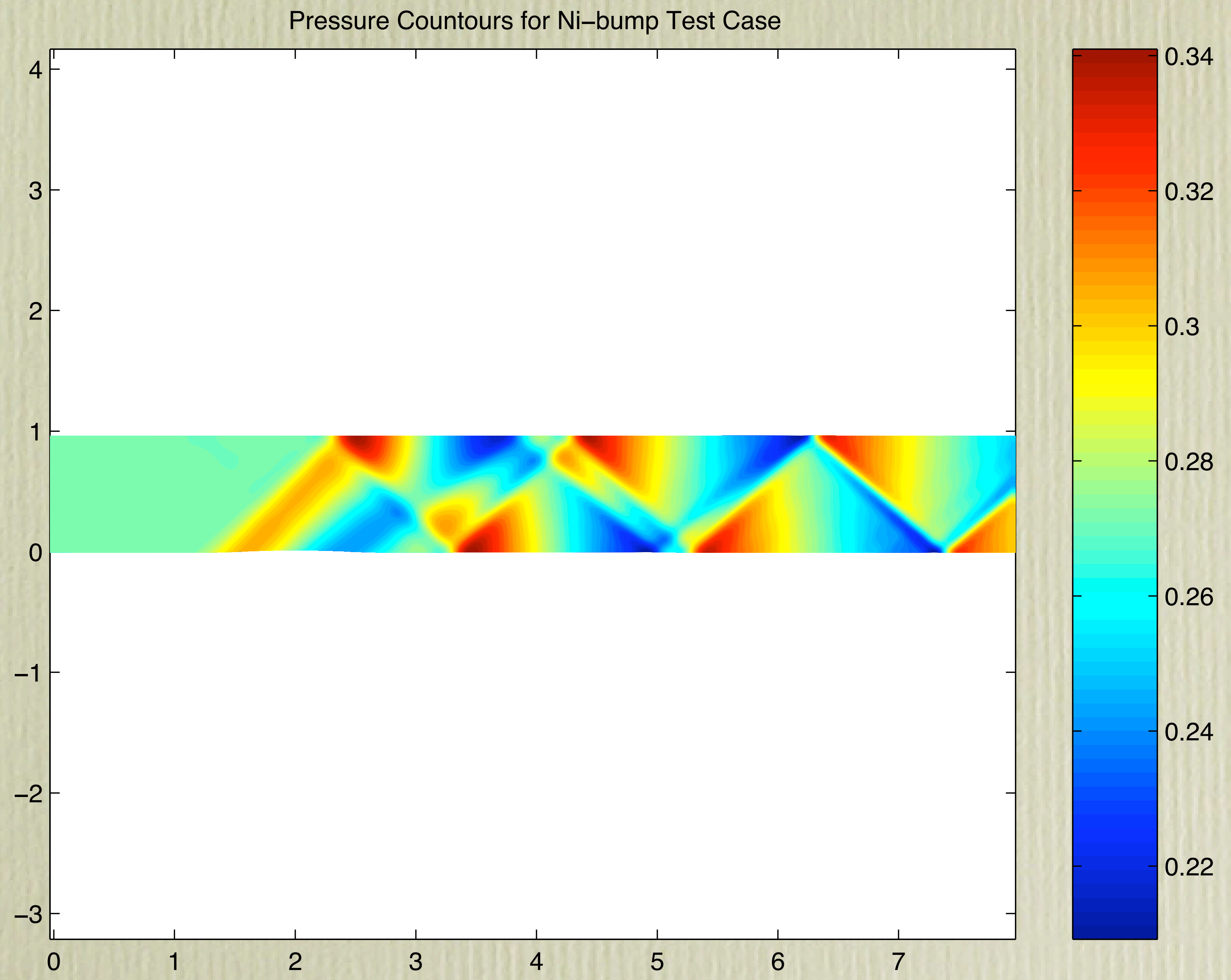
Quasi 1D Problem

Test Case



Pressure Contour