HW 2

Due: Fri, Feb 12

1: Building blocks Assume $A, B \in \mathbb{R}^{n\times n}$ and let `x = solveA(b)` be a function that efficiently solves the system $Ax = b$. Write an efficient MATLAB code to solve the system

$$\begin{bmatrix} A & B \\ 0 & A \end{bmatrix} x = c$$

function x = hw2blockSolve(solveA, B, c)
% Given a solver for systems with a nonsingular matrix A of size
% n x n, a matrix B, also of size n x n, and a column
*% vector c of length 2*n, this function solves*
% the system of equations [A B ; zeros(n) A] x = c

2: Numerical stability For $x > 1$, the equation $x = \cosh(y)$ can be solved as

$$y = -\log\left(x - \sqrt{x^2 - 1}\right).$$

Test this formula in MATLAB, Octave, or Python for $x = 10^9$; what happens? Rearrange the formula to retain accuracy for $x \gg 1$.

3: Norms and complements Consider the two-by-two block matrix

$$Z = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where D and A are square and invertible. For any of the standard operator norms (1-norm, 2-norm, ∞-norm), show that if

$$\|A^{-1}\|\|B\|\|C\|\|D^{-1}\| < 1$$

then Z is nonsingular and

$$\|Z^{-1}\| \leq \frac{\max(\|A^{-1}\|, \|D^{-1}\|)(1 + \max(\|C\|\|A^{-1}\|, \|B\|\|D^{-1}\|))}{1 - \|A^{-1}\|\|B\|\|C\|\|D^{-1}\|}.$$

Hint: If Z^{-1} exists, it can be written as

$$\begin{bmatrix} (I - A^{-1}BD^{-1}C)^{-1} & 0 \\ 0 & (I - D^{-1}CA^{-1}B)^{-1} \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} I & -BD^{-1} \\ -CA^{-1} & I \end{bmatrix}$$

Use the Neumann series to show that the first term is well defined under the hypothesis (there is no trouble with the others). You may also want to use the fact that for the standard operator norms,

$$\left\| \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \right\| = \max(\|A\|, \|D\|) \qquad \left\| \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right\| = \max(\|B\|, \|C\|).$$

This last is not hard to prove, but you do not need to provide the argument.