Homework 1: Problem 2

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March 11, 2019

Part A:

First 31 Factorial operations (with 5 digits in mantissa):

$n = 0$, $0! = 1$

$n = 1$, $1! = (1 \cdot 0!) = (1 \cdot 1.00000e+00) = 1.00000e+00$

$n = 2$, $2! = (2 \cdot 1!) = (2 \cdot 1.00000e+00) = 2.00000e+00$

$n = 3$, $3! = (3 \cdot 2!) = (3 \cdot 2.00000e+00) = 6.00000e+00$

$n = 4$, $4! = (4 \cdot 3!) = (4 \cdot 6.00000e+00) = 2.40000e+01$

$n = 5$, $5! = (5 \cdot 4!) = (5 \cdot 2.40000e+01) = 1.20000e+02$

$n = 6$, $6! = (6 \cdot 5!) = (6 \cdot 1.20000e+02) = 7.20000e+02$

$n = 7$, $7! = (7 \cdot 6!) = (7 \cdot 7.20000e+02) = 5.04000e+03$

$n = 8$, $8! = (8 \cdot 7!) = (8 \cdot 5.04000e+03) = 4.03200e+04$

$n = 9$, $9! = (9 \cdot 8!) = (9 \cdot 4.03200e+04) = 3.62880e+05$

$n = 10$, $10! = (10 \cdot 9!) = (10 \cdot 3.62880e+05) = 3.62880e+06$

$n = 11$, $11! = (11 \cdot 10!) = (11 \cdot 3.62880e+06) = 3.99170e+07$

$n = 12$, $12! = (12 \cdot 11!) = (12 \cdot 3.99170e+07) = 4.79000e+08$

$n = 13$, $13! = (13 \cdot 12!) = (13 \cdot 4.79000e+08) = 6.22700e+09$

$n = 14$, $14! = (14 \cdot 13!) = (14 \cdot 6.22700e+09) = 8.71780e+10$

$n = 15$, $15! = (15 \cdot 14!) = (15 \cdot 8.71780e+10) = 1.30770e+12$

$n = 16$, $16! = (16 \cdot 15!) = (16 \cdot 1.30770e+12) = 2.09230e+13$

$n = 17$, $17! = (17 \cdot 16!) = (17 \cdot 2.09230e+13) = 3.55690e+14$

$n = 18$, $18! = (18 \cdot 17!) = (18 \cdot 3.55690e+14) = 6.40240e+15$

$n = 19$, $19! = (19 \cdot 18!) = (19 \cdot 6.40240e+15) = 1.21650e+17$

$n = 20$, $20! = (20 \cdot 19!) = (20 \cdot 1.21650e+17) = 2.43300e+18$

$n = 21$, $21! = (21 \cdot 20!) = (21 \cdot 2.43300e+18) = 5.10930e+19$

$n = 22$, $22! = (22 \cdot 21!) = (22 \cdot 5.10930e+19) = 1.12400e+21$

$n = 23$, $23! = (23 \cdot 22!) = (23 \cdot 1.12400e+21) = 2.58520e+22$

$n = 24$, $24! = (24 \cdot 23!) = (24 \cdot 2.58520e+22) = 6.20450e+23$

$n = 25$, $25! = (25 \cdot 24!) = (25 \cdot 6.20450e+23) = 1.55110e+25$

$n = 26$, $26! = (26 \cdot 25!) = (26 \cdot 1.55110e+25) = 4.03290e+26$

$n = 27$, $27! = (27 \cdot 26!) = (27 \cdot 4.03290e+26) = 1.08890e+28$

$n = 28$, $28! = (28 \cdot 27!) = (28 \cdot 1.08890e+28) = 3.04890e+29$

$n = 29$, $29! = (29 \cdot 28!) = (29 \cdot 3.04890e+29) = 8.84180e+30$

$n = 30$, $30! = (30 \cdot 29!) = (30 \cdot 8.84180e+30) = 2.65250e+32$

(Generated with MATLAB script: factorialFormat.m)

First 31 Exponential operations (with 5 digits in mantissa):
$n = 0$, $5.5^0 = 1.0$
$n = 1$, $5.5^1 = (5.5 \cdot 5.5^0) = (5.5 \cdot 1.00000e+00) = 5.50000e+00$
$n = 2$, $5.5^2 = (5.5 \cdot 5.5^1) = (5.5 \cdot 5.50000e+00) = 3.02500e+01$
$n = 3$, $5.5^3 = (5.5 \cdot 5.5^2) = (5.5 \cdot 3.02500e+01) = 1.66380e+02$
$n = 4$, $5.5^4 = (5.5 \cdot 5.5^3) = (5.5 \cdot 1.66380e+02) = 9.15090e+02$
$n = 5$, $5.5^5 = (5.5 \cdot 5.5^4) = (5.5 \cdot 9.15090e+02) = 5.03300e+03$
$n = 6$, $5.5^6 = (5.5 \cdot 5.5^5) = (5.5 \cdot 5.03300e+03) = 2.76820e+04$
$n = 7$, $5.5^7 = (5.5 \cdot 5.5^6) = (5.5 \cdot 2.76820e+04) = 1.52250e+05$
$n = 8$, $5.5^8 = (5.5 \cdot 5.5^7) = (5.5 \cdot 1.52250e+05) = 8.37380e+05$
$n = 9$, $5.5^9 = (5.5 \cdot 5.5^8) = (5.5 \cdot 8.37380e+05) = 4.60560e+06$
$n = 10$, $5.5^{10} = (5.5 \cdot 5.5^9) = (5.5 \cdot 4.60560e+06) = 2.53310e+07$
$n = 11$, $5.5^{11} = (5.5 \cdot 5.5^{10}) = (5.5 \cdot 2.53310e+07) = 1.39320e+08$
$n = 12$, $5.5^{12} = (5.5 \cdot 5.5^{11}) = (5.5 \cdot 1.39320e+08) = 7.66260e+08$
$n = 13$, $5.5^{13} = (5.5 \cdot 5.5^{12}) = (5.5 \cdot 7.66260e+08) = 4.21440e+09$
$n = 14$, $5.5^{14} = (5.5 \cdot 5.5^{13}) = (5.5 \cdot 4.21440e+09) = 2.31790e+10$
$n = 15$, $5.5^{15} = (5.5 \cdot 5.5^{14}) = (5.5 \cdot 2.31790e+10) = 1.27480e+11$
$n = 16$, $5.5^{16} = (5.5 \cdot 5.5^{15}) = (5.5 \cdot 1.27480e+11) = 7.01140e+11$
$n = 17$, $5.5^{17} = (5.5 \cdot 5.5^{16}) = (5.5 \cdot 7.01140e+11) = 3.85630e+12$
$n = 18$, $5.5^{18} = (5.5 \cdot 5.5^{17}) = (5.5 \cdot 3.85630e+12) = 2.12100e+13$
$n = 19$, $5.5^{19} = (5.5 \cdot 5.5^{18}) = (5.5 \cdot 2.12100e+13) = 1.16660e+14$
$n = 20$, $5.5^{20} = (5.5 \cdot 5.5^{19}) = (5.5 \cdot 1.16660e+14) = 6.41630e+14$
$n = 21$, $5.5^{21} = (5.5 \cdot 5.5^{20}) = (5.5 \cdot 6.41630e+14) = 3.52900e+15$
$n = 22$, $5.5^{22} = (5.5 \cdot 5.5^{21}) = (5.5 \cdot 3.52900e+15) = 1.94100e+16$
$n = 23$, $5.5^{23} = (5.5 \cdot 5.5^{22}) = (5.5 \cdot 1.94100e+16) = 1.06760e+17$
$n = 24$, $5.5^{24} = (5.5 \cdot 5.5^{23}) = (5.5 \cdot 1.06760e+17) = 5.87180e+17$
$n = 25$, $5.5^{25} = (5.5 \cdot 5.5^{24}) = (5.5 \cdot 5.87180e+17) = 3.22950e+18$
$n = 26$, $5.5^{26} = (5.5 \cdot 5.5^{25}) = (5.5 \cdot 3.22950e+18) = 1.77620e+19$
$n = 27$, $5.5^{27} = (5.5 \cdot 5.5^{26}) = (5.5 \cdot 1.77620e+19) = 9.76910e+19$
$n = 28$, $5.5^{28} = (5.5 \cdot 5.5^{27}) = (5.5 \cdot 9.76910e+19) = 5.37300e+20$
$n = 29$, $5.5^{29} = (5.5 \cdot 5.5^{28}) = (5.5 \cdot 5.37300e+20) = 2.95520e+21$
$n = 30$, $5.5^{30} = (5.5 \cdot 5.5^{29}) = (5.5 \cdot 2.95520e+21) = 1.62540e+22$
(Generated with MATLAB script: expFormat.m)

First 31 Terms (with 5 digits in mantissa):
$n = 0$, $(5.5^0/0!) = 1.0$
$n = 1$, $(5.5^1/1!) = (5.50000e+00/1.00000e+00) = 5.50000e+00$
$n = 2$, $(5.5^2/2!) = (3.02500e+01/2.00000e+00) = 1.51250e+01$
$n = 3$, $(5.5^3/3!) = (1.66380e+02/6.00000e+00) = 2.77300e+01$
$n = 4$, $(5.5^4/4!) = (9.15090e+02/2.40000e+01) = 3.81290e+01$
$n = 5$, $(5.5^5/5!) = (5.03300e+03/1.20000e+02) = 4.19420e+01$
$n = 6$, $(5.5^6/6!) = (2.76820e+04/7.20000e+02) = 3.84470e+01$
$n = 7$, $(5.5^7/7!) = (1.52250e+05/5.04000e+03) = 3.02080e+01$
$n = 8$, $(5.5^8/8!) = (8.37380e+05/4.03200e+04) = 2.07680e+01$
$n = 9$, $(5.5^9/9!) = (4.60560e+06/3.62880e+05) = 1.26920e+01$
$n = 10$, $(5.5^{10}/10!) = (2.53310e+07/3.62880e+06) = 6.98050e+00$
$n = 11$, $(5.5^{11}/11!) = (1.39320e+08/3.99170e+07) = 3.49020e+00$
$n = 12$, $(5.5^{12}/12!) = (7.66260e+08/4.79000e+08) = 1.59970e+00$
$n = 13$, $(5.5^{13}/13!) = (4.21440e+09/6.22700e+09) = 6.76790e-01$
$n = 14$, $(5.5^{14}/14!) = (2.31790e+10/8.71780e+10) = 2.65880e-01$
$n = 15$, $(5.5^{15}/15!) = (1.27480e+11/1.30770e+12) = 9.74840e-02$
$n = 16$, $(5.5^{16}/16!) = (7.01140e+11/2.09230e+13) = 3.35100e-02$
$n = 17$, $(5.5^{17}/17!) = (3.85630e+12/3.55690e+14) = 1.08420e-02$
$n = 18$, $(5.5^{18}/18!) = (2.12100e+13/6.40240e+15) = 3.31280e-03$
$n = 19$, $(5.5^{19}/19!) = (1.16660e+14/1.21650e+17) = 9.58980e-04$
$n = 20$, $(5.5^{20}/20!) = (6.41630e+14/2.43300e+18) = 2.63720e-04$
$n = 21$, $(5.5^{21}/21!) = (3.52900e+15/5.10930e+19) = 6.90700e-05$
$n = 22$, $(5.5^{22}/22!) = (1.94100e+16/1.12400e+21) = 1.72690e-05$
$n = 23$, $(5.5^{23}/23!) = (1.06760e+17/2.58520e+22) = 4.12970e-06$
$n = 24$, $(5.5^{24}/24!) = (5.87180e+17/6.20450e+23) = 9.46380e-07$
$n = 25$, $(5.5^{25}/25!) = (3.22950e+18/1.55110e+25) = 2.08210e-07$
$n = 26$, $(5.5^{26}/26!) = (1.77620e+19/4.03290e+26) = 4.40430e-08$
$n = 27$, $(5.5^{27}/27!) = (9.76910e+19/1.08890e+28) = 8.97150e-09$
$n = 28$, $(5.5^{28}/28!) = (5.37300e+20/3.04890e+29) = 1.76230e-09$
$n = 29$, $(5.5^{29}/29!) = (2.95520e+21/8.84180e+30) = 3.34230e-10$
$n = 30$, $(5.5^{30}/30!) = (1.62540e+22/2.65250e+32) = 6.12780e-11$
(Generated with MATLAB script: divisionFormat.m)

Part B:

Sum of first 31 terms (with 5 digits in mantissa):

$n = 0$, $0 + 1.0 = 1.0$

$n = 1$, $1.00000e+00 + 5.50000e+00 = 6.50000e+00$

$n = 2$, $6.50000e+00 + 1.51250e+01 = 2.16250e+01$

$n = 3$, $2.16250e+01 + 2.77300e+01 = 4.93550e+01$

$n = 4$, $4.93550e+01 + 3.81290e+01 = 8.74840e+01$

$n = 5$, $8.74840e+01 + 4.19420e+01 = 1.29430e+02$

$n = 6$, $1.29430e+02 + 3.84470e+01 = 1.67880e+02$

$n = 7$, $1.67880e+02 + 3.02080e+01 = 1.98090e+02$

$n = 8$, $1.98090e+02 + 2.07680e+01 = 2.18860e+02$

$n = 9$, $2.18860e+02 + 1.26920e+01 = 2.31550e+02$

$n = 10$, $2.31550e+02 + 6.98050e+00 = 2.38530e+02$

$n = 11$, $2.38530e+02 + 3.49020e+00 = 2.42020e+02$

$n = 12$, $2.42020e+02 + 1.59970e+00 = 2.43620e+02$

$n = 13$, $2.43620e+02 + 6.76790e-01 = 2.44300e+02$

$n = 14$, $2.44300e+02 + 2.65880e-01 = 2.44570e+02$

$n = 15$, $2.44570e+02 + 9.74840e-02 = 2.44670e+02$

$n = 16$, $2.44670e+02 + 3.35100e-02 = 2.44700e+02$

$n = 17$, $2.44700e+02 + 1.08420e-02 = 2.44710e+02$

$n = 18$, $2.44710e+02 + 3.31280e-03 = 2.44710e+02$

$n = 19$, $2.44710e+02 + 9.58980e-04 = 2.44710e+02$

$n = 20$, $2.44710e+02 + 2.63720e-04 = 2.44710e+02$

$n = 21$, $2.44710e+02 + 6.90700e-05 = 2.44710e+02$

$n = 22$, $2.44710e+02 + 1.72690e-05 = 2.44710e+02$

$n = 23$, $2.44710e+02 + 4.12970e-06 = 2.44710e+02$

$n = 24$, $2.44710e+02 + 9.46380e-07 = 2.44710e+02$

$n = 25$, $2.44710e+02 + 2.08210e-07 = 2.44710e+02$

$n = 26$, $2.44710e+02 + 4.40430e-08 = 2.44710e+02$

$n = 27$, $2.44710e+02 + 8.97150e-09 = 2.44710e+02$

$n = 28$, $2.44710e+02 + 1.76230e-09 = 2.44710e+02$

$n = 29$, $2.44710e+02 + 3.34230e-10 = 2.44710e+02$

$n = 30$, $2.44710e+02 + 6.12780e-11 = 2.44710e+02$

(Generated with MATLAB script: sumFormat_LtR.m)

The final value of $e^{5.5}$ computed using a mantissa of 5 digits is 244.71. The value converged to 5 significant digits at $n = 17$. The true value of $e^{5.5}$ is 244.69193226..., and the relative error is $7.3839 \cdot 10^{-5}$.

Part C:

Summing from right to left gave the same value (244.71) as summing left to right. (Generated with MATLAB script: sumFormat_RtL.m)

Part D:

(i). Adding left to right gave the value: $3.8363 \cdot 10^{-3}$. It took until $n = 24$ to converge and the relative error is 0.0613.

(ii). Adding right to left gave the value: $4.0 \cdot 10^{-3}$. It took until $n = 0$ to converge and the relative error is 0.0212.

(iii). Adding left to right, positive and negative values separately gave the value: 0. It took until $n = 16$ for the positive set and $n = 17$ for the negative set to converge and the relative error was 1.

(iv). Adding right to left, positive and negative values separately gave the value: 0.01. It took until $n = 0$ for the positive set and $n = 1$ for the negative set to converge and the relative error was 1.4469.

The method that converges the quickest is adding from left to right. Adding positive and negative values separately (left to right) converges faster within the separate positive and negative sums, however the total number of summing operations is greater than adding left to right at the convergence point. The lowest error is when you add right to left. When the exponential was positive the relative errors were the same as the sum was the same.

(Generated with MATLAB script: partD.m)

Part E:

To fix the large error when computing e^{-x}, you can instead compute $1/e^x$, as division does not have the same rounding error as subtraction when values are close together. For $e^{-5.5} = 1/e^{5.5}$ the computed value (with 5 digits in the mantissa is: $4.0865 \cdot 10^{-3}$. The relative error is $6.6419 \cdot 10^{-5}$

(Generated with MATLAB script: partE.m)