Homework 1: Problem 8

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Part A

The reversed recurrence relation is:

$$y_n = \frac{e - y_{n+1}}{n+1}$$

The first few terms in the map $g_k(y_N) = y_k$ are:

$$\begin{aligned}
g_N(y_N) &= y_N \\
g_{N-1}(y_N) &= \frac{e - g_N}{N} = \frac{e - y_N}{N} \\
g_{N-2}(y_N) &= \frac{e - g_{N-1}}{N} = \frac{e(N-1) + y_N}{N(N-1)} \\
g_{N-3}(y_N) &= \frac{e - g_{N-2}}{N} = \frac{e(N-1)^2 + y_N}{N(N-1)(N-2)}
\end{aligned}$$

After a few terms we observe that $g_k(y_N)$ is linear in terms of y_N. The derivative can be written as:

$$g_k'(y_N) = \frac{(-1)^{N-k}k!}{N!}$$

And the condition number $(\text{cond } g_k)(y_N)$ is:

$$(\text{cond } g_k)(y_N) = \left|\frac{y_N k!}{y_k N!}\right|$$

The ratio y_N/y_k is upper bounded by 1 for all $k < N$, so:

$$(\text{cond } g_k)(y_N) = \frac{k!}{N!}$$

Part B

To achieve a relative error of ϵ in y_k given a relative error of $\epsilon_{y_N} = 1$ one has to solve the following equation for N:

$$\begin{aligned}
\epsilon &= (\text{cond } g_k)(y_N)\epsilon_{y_N} \\
\epsilon &= \frac{k!}{N!} \\
N! &= \frac{k!}{\epsilon}
\end{aligned}$$

N can be determined through trial and error in a while loop.

Part C

For eps $= 2^{-53}$ and $k = 20$ the value of N is 32. (Code used: partC.m)

Part D

Using the code in partD.m the value computed using the backwards recurrence relation was 0.1238038307625699..., and the value computed using MATLAB's 'integral' function was 0.1238038307625699... . The relative error between the two values is 0.