Homework 1: Problem 3

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Part A

If x is a machine number then using method (i) to calculate x^n results in the following error:

$$\begin{aligned}
s_1 &= \text{fl}(x \cdot x) = (x \cdot x)(1+\epsilon_1) \\
s_2 &= \text{fl}(x \cdot s_1) = (x \cdot x^2)(1+\epsilon_1+\epsilon_2) \\
s_3 &= \text{fl}(x \cdot s_3) = (x \cdot x^3)(1+\epsilon_1+\epsilon_2+\epsilon_3) \\
s_n &= \text{fl}(x \cdot s_{n-1}) = (x \cdot x^{n-1})(1+\epsilon_1+\epsilon_2+\epsilon_3+...+\epsilon_n) = x^n(1+\epsilon_{net(i)})
\end{aligned}$$

where $|\epsilon_i| \leq \text{eps}$ for $i = 1, 2, ..., n$. The upper bound on the error is: $\epsilon_{net(i)} \leq n \cdot \text{eps}$. Using method (ii) to calculate x^n results in the following error:

$$\begin{aligned}
\text{fl}(\ln(x)) &= \ln(x)(1+\epsilon_{log}) \quad |\epsilon_{log}| \leq \text{eps} \\
\text{fl}(n \cdot \text{fl}(\ln(x))) &= n \cdot \text{fl}(\ln(x))(1+\epsilon_{mult}) \approx n\ln(x)(1+\epsilon_{mult}+\epsilon_{log}) \quad |\epsilon_{mult}| \leq \text{eps} \\
\text{fl}(e^{\text{fl}(n\cdot\text{fl}(\ln(x)))}) &= e^{\text{fl}(n\cdot\text{fl}(\ln(x)))}(1+\epsilon_{exp}) = e^{n\ln(x)(1+\epsilon_{mult}+\epsilon_{log})}(1+\epsilon_{exp}) \\
&\approx e^{n\ln(x)}(1+n\ln(x)(\epsilon_{mult}+\epsilon_{log}))(1+\epsilon_{exp}) \\
&\approx e^{n\ln(x)}(1+n\ln(x)(\epsilon_{mult}+\epsilon_{log})+\epsilon_{exp}) \quad |\epsilon_{exp}| \leq \text{eps}
\end{aligned}$$

The upper bound on the total error, $\epsilon_{net(ii)} = n\ln(x)(\epsilon_{mult}+\epsilon_{log})+\epsilon_{exp}$, is: $\epsilon_{net(ii)} \leq (2n\ln(x)+1)\cdot\text{eps}$. If $2n\ln(x)+1 > n$ then use method (i), otherwise use method (ii).

Part B

(i). First let $\text{fl}(x) = x$ and $\text{fl}(a) = a(1+\epsilon_a)$

$$\begin{aligned}
\text{fl}(\ln x) &= \ln x(1+\epsilon_{log}) \quad |\epsilon_{log}| \leq \text{eps} \\
\text{fl}(\text{fl}(a)\cdot\text{fl}(\ln x)) &= a\cdot\text{fl}(\ln x)(1+\epsilon_{mult})(1+\epsilon_a) \approx a\ln x(1+\epsilon_{mult}+\epsilon_{log}+\epsilon_a) \quad |\epsilon_{mult}| \leq \text{eps} \\
\text{fl}(e^{\text{fl}(\text{fl}(a)\cdot\text{fl}(\ln x))}) &= e^{\text{fl}(\text{fl}(a)\cdot\text{fl}(\ln x))}(1+\epsilon_{exp}) = e^{a\ln x(1+\epsilon_{mult}+\epsilon_{log}+\epsilon_a)}(1+\epsilon_{exp}) \\
&\approx e^{a\ln(x)}(1+a\ln x(\epsilon_{mult}+\epsilon_{log}+\epsilon_a))(1+\epsilon_{exp}) \\
&\approx e^{a\ln(x)}(1+a\ln x(\epsilon_{mult}+\epsilon_{log}+\epsilon_a)+\epsilon_{exp}) \quad |\epsilon_{exp}| \leq \text{eps} \\
\epsilon_{total} &= (\epsilon_{mult}+\epsilon_{log}+\epsilon_a)a\ln x+\epsilon_{exp}
\end{aligned}$$

(ii). Now if $\mathrm{fl}(x) = x(1+\epsilon_x)$ and $\mathrm{fl}(a) = a$

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\begin{aligned}
\mathrm{fl}(\ln \mathrm{fl}(x)) &= \ln x(1+\epsilon_{log}+\epsilon_x) \quad |\epsilon_{log}| \le \text{eps} \\
\mathrm{fl}(a \cdot \mathrm{fl}(\ln \mathrm{fl}(x))) &= a \cdot \mathrm{fl}(\ln \mathrm{fl}(x))(1+\epsilon_{mult}) \approx a \ln x(1+\epsilon_{mult}+\epsilon_{log}+\epsilon_x) \quad |\epsilon_{mult}| \le \text{eps} \\
\mathrm{fl}(e^{\mathrm{fl}(a \cdot \mathrm{fl}(\ln \mathrm{fl}(x)))}) &= e^{\mathrm{fl}(a \cdot \mathrm{fl}(\ln \mathrm{fl}(x)))}(1+\epsilon_{exp}) = e^{a \ln x(1+\epsilon_{mult}+\epsilon_{log}+\epsilon_x)}(1+\epsilon_{exp}) \\
&\approx e^{a \ln(x)}(1+a \ln x(\epsilon_{mult}+\epsilon_{log}+\epsilon_x))(1+\epsilon_{exp}) \\
&\approx e^{a \ln(x)}(1+a \ln x(\epsilon_{mult}+\epsilon_{log}+\epsilon_x)+\epsilon_{exp}) \quad |\epsilon_{exp}| \le \text{eps} \\
\epsilon_{total} &= (\epsilon_{mult}+\epsilon_{log}+\epsilon_x) a \ln x + \epsilon_{exp}
\end{aligned}
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In either case the error is proportional to $a \ln x$ so if $a \ln x$ is large, the error will also be large.