Homework 1: Problem 6

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March 12, 2019

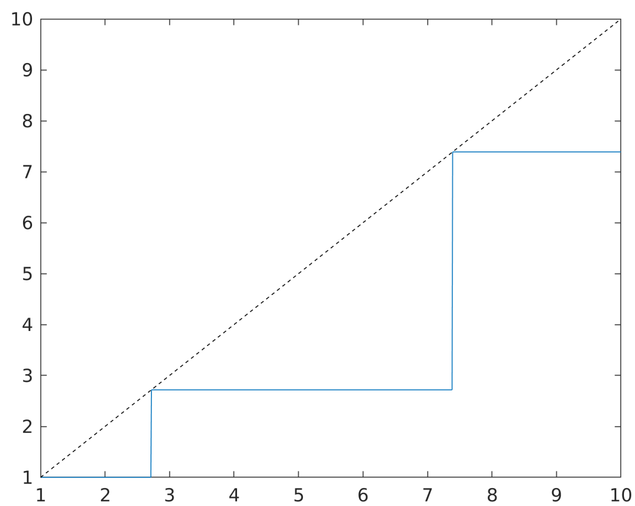

Figure 1: Plot of vector x after 52 square roots and 52 square operations (blue) and the initial values of vector x (dashed)

After 52 repeated square roots, the value of a number (between 1 and 10) becomes very close to $1 \pm \epsilon$ with $\epsilon \ll 1$. To figure out what ϵ is we can rewrite $x^{2^{-n}}$ as $e^{2^{-n}\ln(x)}$. Now using a Taylor expansion we write the previous expression as:

$$e^{2^{-n}\ln(x)} = 1 + 2^{-n}\ln(x) + \cancelto{\approx 0}{\frac{(2^{-n}\ln(x))^2}{2!}} + ...$$

For large n the higher order terms can be neglected. Now we write ϵ as:

$$\epsilon = 2^{-n}\ln(x)$$

When representing the number, $x^{2^{-n}}$, in a finite number of bits rounding error plays a large role when n is large. As n approaches 53, ϵ reaches the order of machine epsilon, eps $= 2^{-53}$, for double precision floating point numbers. This means that ϵ will be rounded to the nearest integer multiple of eps. When x is a power of e, $x = e^p$, then $\epsilon = p \cdot 2^{-n}$. When

$n = 52$ then $x^{2^{-n}}$ will be stored as $1 + p \cdot \text{eps}$. Then after n repeated squaring operations the floating point numbers (with rounding error) will resemble a staircase. Only starting values close to e^p will be preserved.