Homework 1: Problem 7

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Part A

Wilkinson's polynomial, with $n = 20$ is:

$$\begin{aligned} w(x) = x^{20} &- 210x^{19} + 20615x^{18} - 1256850x^{17} + 53327946x^{16} - 1672280820x^{15} + \\ &40171771630x^{14} - 756111184500x^{13} + 11310276995381x^{12} - 135585182899530x^{11} + \\ &1307535010540395x^{10} - 10142299865511450x^{9} + 63030812099294896x^{8} - \\ &311333643161390640x^{7} + 1206647803780373360x^{6} - 3599979517947607200x^{5} + \\ &8037811822645051776x^{4} - 12870931245150988800x^{3} + 13803759753640704000x^{2} - \\ &8752948036761600000x + 2432902008176640000 \end{aligned}$$

Part B

Using MATLAB's 'fzero' function with an initial guess of $x = 21$, the closest root it found was: 20.000001304... . Using AMTALB's 'roots' function to find all roots the largest root it found was: 20.000244798... . 'fzero' gave a better estimate of the root than 'roots', however it was still off by $1.304 \cdot 10^{-6}$.

Part C

The largest root changes with delta as follows:

δ	Root
10^{-8}	$20.6476 + 1.1869i$
10^{-4}	$23.1490 + 2.7410i$
10^{-4}	$28.4002 + 6.5104i$
10^{-2}	$38.4782 + 20.834i$

Part D

The roots 16 and 17 change from 16.0411 and 16.9743 to $16.7308 + 2.8127i$ and $16.7308 - 2.8127i$ respectively.

Part E

(i)

$$p(x) = a_0 + ... + a_i x^i + ... + a_{n-1} x^{n+1} + x^n$$

Let a perturbation in $p(x)$ from a perturbation in a_i be defined as:

$$\bar{p}(x) = a_0 + ... + (a_i + \delta a_i) x^i + ... + a_{n-1} x^{n+1} + x^n$$

Then from a perturbation in x:

$$\begin{aligned}\bar{p}(x+\delta x) &= a_0 + ... + (a_i + \delta a_i)(x+\delta x)^i + ... + a_{n-1}(x+\delta x)^{n+1} + (x+\delta x)^n \\ &= p(x+\delta x) + \delta a_i (x+\delta x)^i\end{aligned}$$

After subtracting and dividing by δx we get:

$$\begin{aligned}\left|\frac{p(x+\delta x)-p(x)}{\delta x}\right| &= \left|\frac{a_i(x+\delta x)^i}{\delta x}\right| \\ \implies |p'(x)| &= \left|\frac{a_i x^i}{\delta x}\right|\end{aligned}$$

We want to find the condition number κ_i in:

$$\left|\frac{\delta x}{x}\right| = \kappa_i \left|\frac{\delta a_i}{a_i}\right|$$

After rearranging we get:

$$\kappa_i = \frac{|a_i x^{i-1}|}{|p'(x)|}$$

Thus the condition number at the root Ω_k is:

$$(\text{cond } \Omega_k)(a_i) = \frac{|a_i \Omega_k^{i-1}|}{|p'(\Omega_k)|}$$

And $(\text{cond } \Omega_k)(\vec{a})$ is:

$$(\text{cond } \Omega_k)(\vec{a}) = \sum_{i=0}^{n-1} \frac{|a_i \Omega_k^{i-1}|}{|p'(\Omega_k)|}$$

(ii) The condition numbers for $\Omega_k = 14$, 16, 17, and 20 are: 5.401185e+13, 3.544043e+13, 1.812241e+13, and 1.378468e+11 respectively.

(iii) Because the condition numbers for the 'true' function are so large an algorithm cannot be any better. There would not be any improvement by trying to come up with a clever algorithm without introducing some arbitrarily precise floating point representation of a number.