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a) $y_{k+1} = e - (k+1)\,y_k \Rightarrow \boxed{y_k = \frac{e - y_{k+1}}{k+1}}$

$\frac{1}{(k+1)(k+2)(k+3)}\left(e - y_{k+4}\right)$ ...

$g_k(y_N) = y_k$

↳ iterate from y_N back to y_k

$$y_k = \frac{e}{k+1} - \frac{y_{k+1}}{k+1}$$

need to go $N-k$ steps back from k to N

$$= \frac{e}{k+1} - \frac{1}{k+1}\left(\frac{e}{k+2} - \frac{y_{k+2}}{k+2}\right)$$

$$= \frac{e}{k+1} - \frac{e}{(k+1)(k+2)} + \frac{y_{k+2}}{(k+1)(k+2)}$$

repeat iteratively

$$y_k = \sum_{l=1}^{N-k} \frac{(-1)^{l+1} e}{\prod_{j=1}^{l}(k+j)} + (-1)^{N-k}\frac{y_N}{\prod_{j=1}^{N-k}(k+j)}$$

Changes sign with each step

$$\therefore \boxed{g_k(y_N) = y_k = \sum_{l=1}^{N-k} \frac{(-1)^{l+1} e}{\prod_{j=1}^{l}(k+j)} + (-1)^{N-k}\frac{y_N}{\prod_{j=1}^{N-k}(k+j)}}$$

$\text{cond } g_k(y_N) = \left|\frac{g_k'\, y_N}{y_k}\right|$ now $|g_k'| = \frac{1}{\prod_{j=1}^{N-k}(k+j)} = \frac{k!}{N!}$

→ also we have $|y_k| > |y_N|$ for $k > N$ $\therefore \boxed{\left|\frac{y_k}{y_N}\right| < 1}$

$$\therefore \boxed{(\text{cond } g_k)(y_N) = \left|\frac{k!}{N!}\right|\left|\frac{y_N}{y_k}\right| \le \left|\frac{k!}{N!}\right|}$$

An upper limit on $(\text{cond } g_k)(y_N)$ is $\frac{k!}{N!}$ as required

b) → $(\text{cond}(g_k))(y_N)\,\varepsilon_{y_N} = \varepsilon_{y_k}$ (cond. number relates error in input to error in output)

→ 100% error in y_N → $|\varepsilon_{y_N}| = 1$, also let $|\varepsilon_{y_k}| = \varepsilon$

$\varepsilon = |(\text{cond } g_k)(y_N)| \le \left|\frac{k!}{N!}\right|$ $\therefore$ Need $\boxed{N! = \frac{k!}{\varepsilon}}$ to achieve target ε in worst case

In terms of Gamma function

→ [illegible]

$\boxed{\Gamma(N+1) = \frac{\Gamma(k+1)}{\varepsilon}}$ gives the minimum N