2 See Notebook

these are technically independent #'s, but since they all have $\varepsilon_i \leq eps$, I will treat them as the same in all that follows

3 a) (i) → recall that $fl(x \cdot x) = (x \cdot x)(1+\varepsilon) = x^2(1+\varepsilon)$

$\therefore fl(x^3) = fl(fl(x^2) \cdot x) = fl(x^2(1+\varepsilon) \cdot x)$

$= (x^2(1+\varepsilon)x)(1+\varepsilon)$

$= x^3(1+2\varepsilon+\cancel{\varepsilon^2})$

$fl(x^3) = x^3(1+2\varepsilon)$

repeating this pattern ..

$fl(x^n) = x^n(1+\varepsilon)^{n-1} \approx x^n(1+(n-1)\varepsilon)$

$\therefore$ since $\varepsilon \leq eps$, repeated multiplication gives max error $\boxed{(n-1)\,eps}$

(ii) $fl(\ln(x)) = \ln(x)(1+\varepsilon)$

$\cancel{fl(\ldots)} = fl(n\,fl(\ln(x))) = n\ln(x)(1+\varepsilon)(1+\varepsilon)$

$fl(n\,fl(\ln x)) = n\ln(x)(1+2\varepsilon)$

$fl(\exp(fl(n\,fl(\ln(x))))) = fl(\exp(n\ln(x)(1+2\varepsilon)))$

$= \exp(n\ln(x)(1+2\varepsilon))(1+\varepsilon)$

$= \exp(n\ln(x))\exp(n\ln(x)(2\varepsilon))(1+\varepsilon)$

$= e^{n\ln(x)}(1+n\ln(x)2\varepsilon+\cancel{\frac{(n\ln(x)2\varepsilon)^2}{2}}+\ldots)(1+\varepsilon)$

$= e^{n\ln(x)}(1+2n\ln(x)\varepsilon)(1+\varepsilon)$

$fl(\exp(fl(n\,fl(\ln(x))))) = e^{n\ln(x)}(1+(2n\ln(x)+1)\varepsilon)$

$\therefore e^{n\ln(x)}$ gives max error $\boxed{(2n\ln(x)+1)\,eps}$

Repeat multiplication more accurate if $n-1 < 2n\ln(x)+1$

$n < 2(n\ln(x)+1)$

$n(\frac{1}{2}-\ln(x)) < 1$

(x^0 is trivial)

↳ n is a positive integer ∴ satisfied if $\frac{1}{2} - \ln(x) < 0$

$n\left(\frac{1}{2} - \ln(x)\right) < 1$

$\frac{1}{2} < \ln(x)$

$x > e^{\frac{1}{2}}$

∴ Repeated multiplication is more accurate when $x > e^{\frac{1}{2}}$

b) → need to use $e^{a\ln(x)}$ (i) x is exact, a has error ε_a $\quad fl(a) = a(1+\varepsilon_a)$

$fl(\cdot)$

$fl(\ln(x)) = \ln(x)(1+\varepsilon)$

$fl(fl(a)\ln(x)) = a\ln(x)(1+2\varepsilon+\varepsilon_a)$

$fl(\exp(fl(fl(a)fl(\ln(x))))) = fl(\exp(a\ln(x)(1+2\varepsilon+\varepsilon_a)))$

$= \exp(a\ln(x)(1+2\varepsilon+\varepsilon_a))(1+\varepsilon)$

$= e^{a\ln(x)}(1+a\ln(x)(2\varepsilon+\varepsilon_a))(1+\varepsilon)$

$fl(e^{a\ln(x)}) = e^{a\ln(x)}(1+(a\ln(x)+1)\varepsilon + a\ln(x)\varepsilon_a)$

The propagated error is $a\ln(x)\varepsilon_a$ (additional maximum error $(a\ln(x)+1)\varepsilon$)

$\ln(1+\varepsilon_x) = \varepsilon_x - \frac{\varepsilon_x^2}{2} + \dots$

ii) a is exact, $fl(x) = x(1+\varepsilon_x)$

$fl(\ln(fl(x))) = fl(\ln(x(1+\varepsilon_x))) = \ln(x(1+\varepsilon_x))(1+\varepsilon)$

$= (\ln(x) + \ln(1+\varepsilon_x))(1+\varepsilon)$

$fl(\ln(fl(x))) = \ln(x)(1+\varepsilon)+\varepsilon_x$ $\quad = (\ln(x)+\varepsilon_x)(1+\varepsilon)$

→ As before

→ $fl(a\ln(x)) = fl(a\ln(x)(1+\varepsilon)+a\varepsilon_x) = a\ln(x)(1+2\varepsilon) + a\varepsilon_x$

↳ $fl(\exp(a\ln(x))) = fl(\exp(a\ln(x)(1+2\varepsilon)+a\varepsilon_x)) = \exp(a\ln(x)(1+2\varepsilon))\exp(a\varepsilon_x)(1+\varepsilon)$

$= \exp(a\ln(x))(1+2\varepsilon)^{a\ln x} a\varepsilon_x(1+\varepsilon)$

$fl(\exp a\ln(fl(x))) = e^{a\ln(x)}(1+a\varepsilon_x+(2a\ln(x)+1)\varepsilon)$

∴ The propagated error is $a\varepsilon_x$ (maximum additional error $(2a\ln(x)+1)\varepsilon$)

The propagated error can be substantial in case (i) if x is small, since then $|a\ln(x)|$ is large