4. Conditionals

$$(\mathrm{cond}f)(x) = \left|\frac{x f'(x)}{x}\right|$$

a) $f(x) = 1 - e^{-x}$ $\quad f'(x) = e^{-x}$

$$(\mathrm{cond}f)(x) = \left|\frac{x e^{-x}}{1-e^{-x}}\right| = \frac{x}{e^x - 1}$$

Now $e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}$, so $e^x - 1 \geq x$ on $[0,1]$ so $(\mathrm{cond}f)(x) \leq 1$ on $[0,1]$

also $(\mathrm{cond}f)(x) = \left(\frac{e^x-1}{x}\right)^{-1} = \left(\sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}\right)^{-1} = \frac{1}{1 + \frac{x}{2} + \frac{x^2}{9}} < 1$ on $[0,1]$

b) $f\ell(e^{-x}) = e^{-x}(\varepsilon + 1)$

$$f\ell(1 - f\ell(e^{-x})) = f\ell(1 - e^{-x}(1+\varepsilon)) = f\ell\left((1-e^{-x})\left(1 - \frac{e^{-x}}{1-e^{-x}}\varepsilon\right)\right) =$$

$$= (1-e^{-x})\left(1 - \frac{e^{-x}}{1-e^{-x}}\varepsilon\right)(1+\varepsilon) = (1-e^{-x})\left(1 - \frac{1}{e^x - 1}\varepsilon\right)$$

$$f_A(x) = f\ell(1 - f\ell(e^{-x})) = (1-e^{-x})\left(1 - \frac{e^x}{e^x - 1}\varepsilon\right)$$

$$f(x_A) = 1 - e^{-x_A} = (1-e^{-x})\left(1 - \frac{e^x}{e^x-1}\varepsilon\right)$$

$$1 - e^{-x_A} = 1 - e^{-x} - \varepsilon \to e^{-x_A} = e^{-x} + \varepsilon = e^{-x}(1 + \varepsilon e^x)$$

$$-x_A = -x + \ln(1 + \varepsilon e^x)$$

$$x_A = x - \ln(1 + \varepsilon e^x)$$

$$|x_A - x| = \ln(1 + \varepsilon e^x)$$

so absolute error $|x_A - x| \leq e^x \varepsilon$

$$(\mathrm{cond}A)(x) = \frac{1}{e^x}\frac{|x_A - x|}{x}$$

$$(\mathrm{cond}A)(x) = \frac{e^x}{x}$$

$$(\mathrm{cond}A)(x) = \frac{e^x}{x} = \sum_{n=0}^{\infty} \frac{x^{n-1}}{n!}$$

$(\mathrm{cond}A)(x) > 1$ on $[0,1]$ since the first term is $\frac{1}{x}$, which is always ≥ 1 on $[0,1]$, and all others are positive

→ from b) $f_A(x) = (1-e^{-x})\left(1 - \frac{\varepsilon}{1-e^{-x}}\right)$

∴ lose bits: $\frac{\varepsilon}{1-e^{-x}}$ → ie lose at most b bits if $\frac{1}{1-e^{-x}} \leq 2^b$

since this is the # multiplying ε and thus causing errors, first bit is at 2ε

∴ if we are willing to lose at most b bits, the minimum x satisfies:

$$\frac{1}{1-e^{-x}} = 2^b$$

$$1-2^{-b} = e^{-x}$$

$$\boxed{x_{min} = -\ln(1-2^{-b})}$$

f) → avoid subtraction by evaluating

$f(x) = 1-e^{-x} = \frac{e^x - 1}{e^x} = \frac{\sum_{n=1}^{\infty} \frac{x^n}{n!}}{\sum_{n=0}^{\infty} \frac{x^n}{n!}}$

→ this way all we do is add, so should not pose particular problems for small x

∴ for 1 bit: $x_{min}^{(b=1)} = -\ln(1-\frac{1}{2}) = \ln(2)$

∴ for 1 bit of precision left, the minimum $x = \ln(2)$

2 bits: $x_{min}(b=2) = \ln((1-\frac{1}{4})^{-1}) = \ln(\frac{4}{3})$

3 bits: $x_{min}(b=3) = \ln((1-\frac{1}{8})^{-1}) = \ln(\frac{8}{7})$

4 bits: $x_{min}(b=4) = \ln((1-\frac{1}{16})^{-1}) = \ln(\frac{16}{15})$

e) relative error ≡ $\tilde{\varepsilon} \leq \frac{|f_A(x) - f(x)|}{f(x)} \leq \left|\frac{eps}{1-e^{-x}}\right| \leq \left|\frac{eps}{1-\exp(\ln(1-2^{-b}))}\right| \leq \left|\frac{eps}{2^{-b}}\right|$ (sub in $x = x_{min}$)

∴ the relative error in each case $\tilde{\varepsilon}$ is bounded by $\boxed{\tilde{\varepsilon} \leq 2^b eps}$

$x = \ln(2), b=1$: $\tilde{\varepsilon} \leq 2eps$
$x = \ln(\frac{4}{3}), b=2$: $\tilde{\varepsilon} \leq 4eps$
$x = \ln(\frac{8}{7}), b=3$: $\tilde{\varepsilon} \leq 8eps$
$x = \ln(\frac{16}{15}), b=4$: $\tilde{\varepsilon} \leq 16eps$

} as required