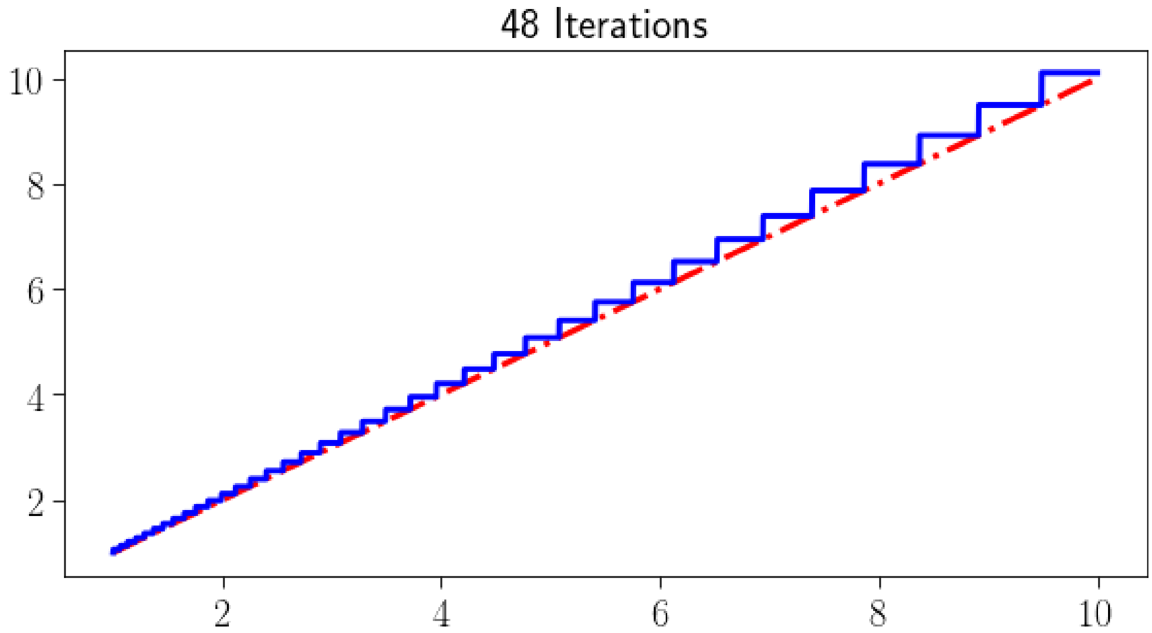
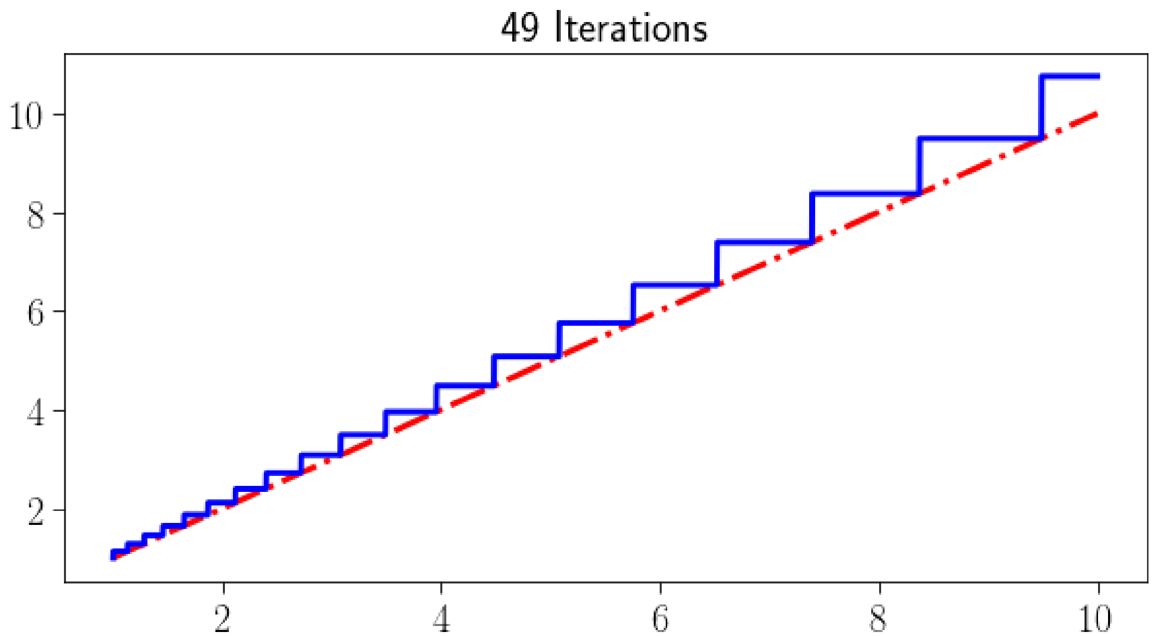
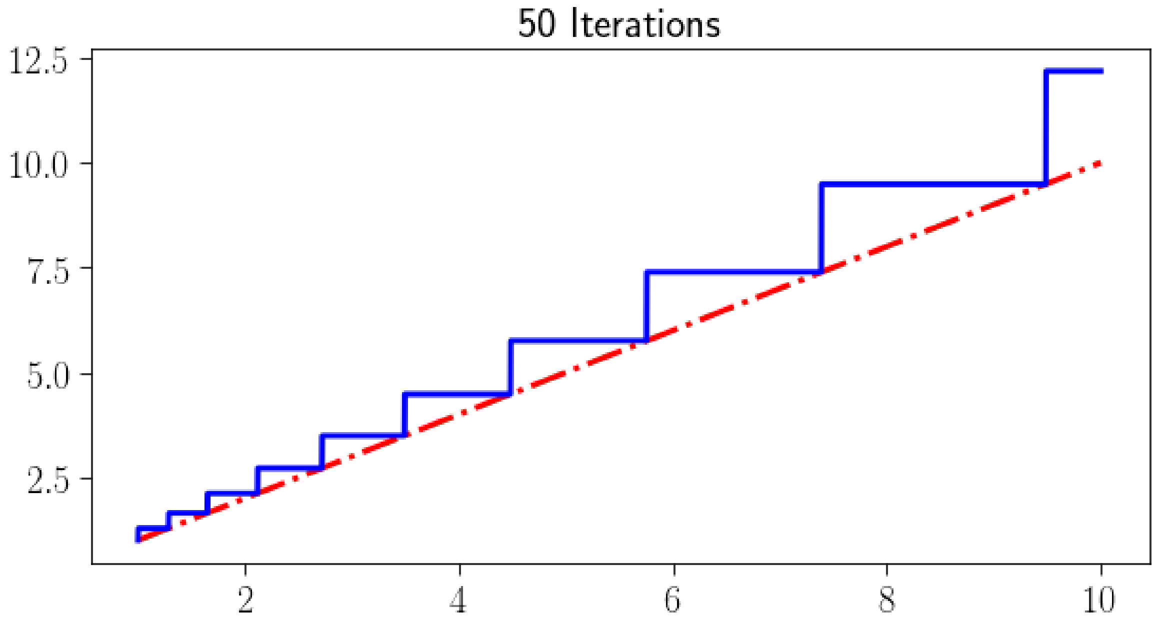
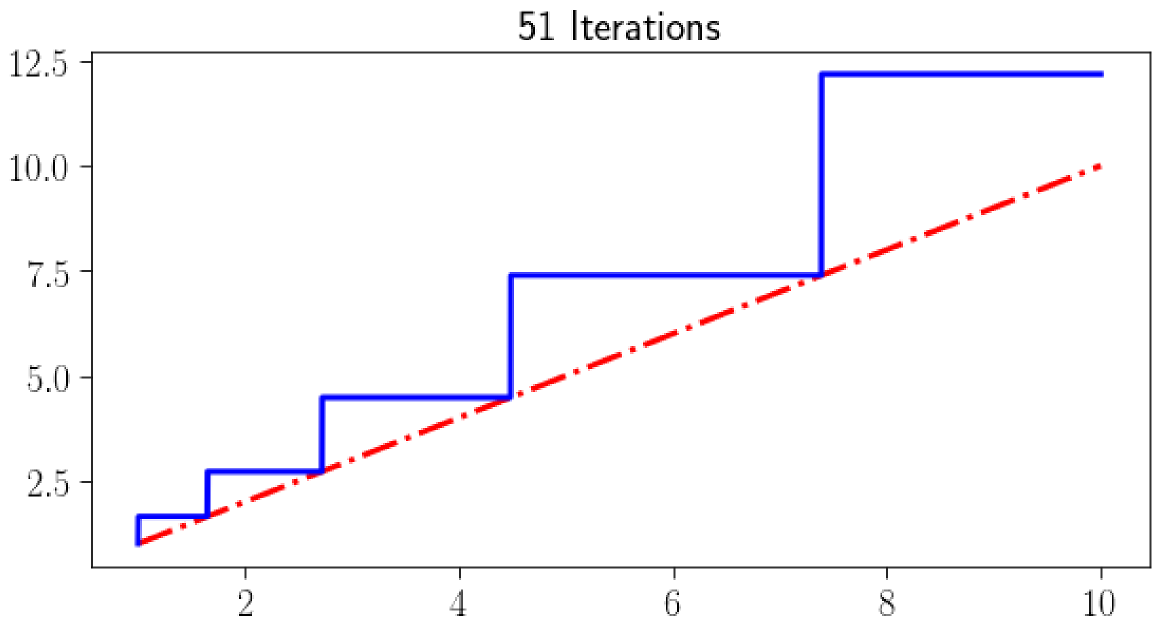
Problem 6

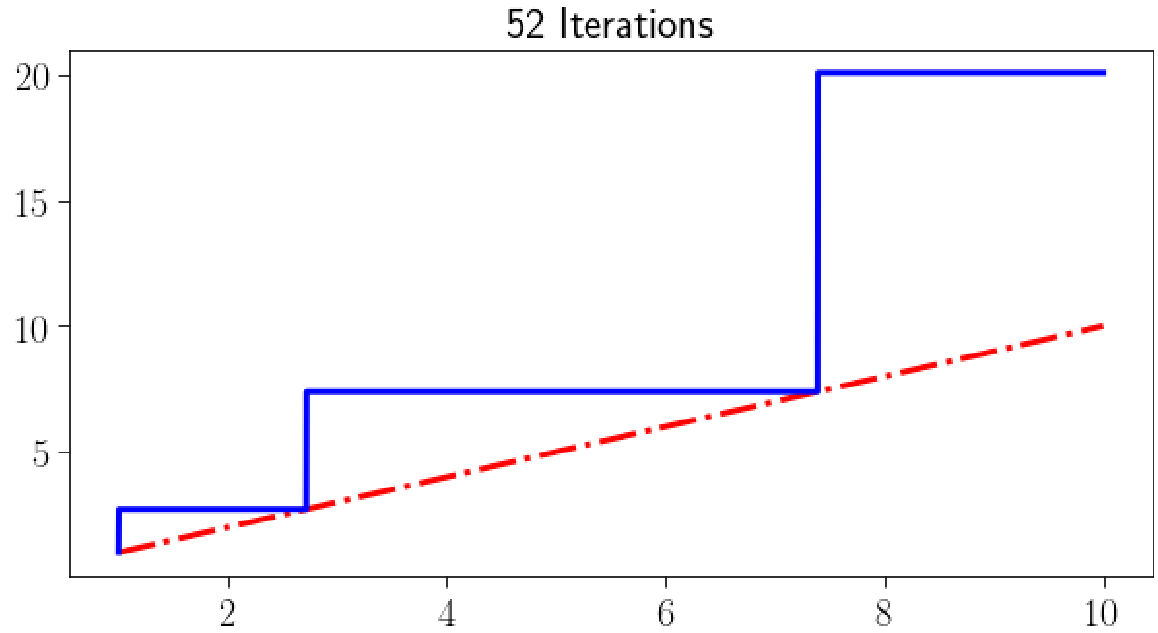
In [5]:
```
import numpy as np
import matplotlib.pyplot as plt
import sys



npoints=10001
nplots=7
counter=0
iterations=np.linspace(48, 48+nplots-1, nplots, dtype=int)
x = np.linspace(1,10,npoints)
y = np.zeros((nplots, npoints))
for its in iterations:
    ytemp = np.linspace(1,10,npoints)

    for ind in range(its):
        ytemp = np.sqrt(ytemp)

    for ind in range(its):
        ytemp = np.square(ytemp)

    y[counter, :]=ytemp
    counter=counter+1

for its in range(nplots):
    plt.figure(figsize=(10,5))
    plt.plot(x, x, '-.r')
    plt.plot(x, y[its, :], '-b')
    plt.title('{} Iterations'.format(iterations[its]))
```

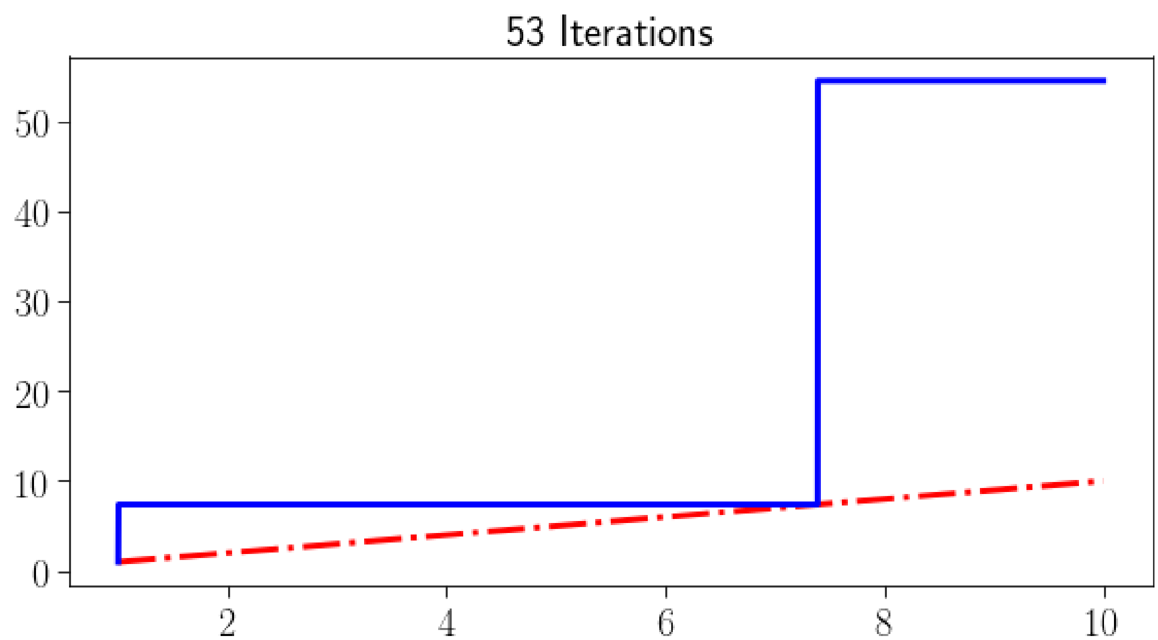

48 Iterations
10
8
6
4
2
2
4
6
8
10

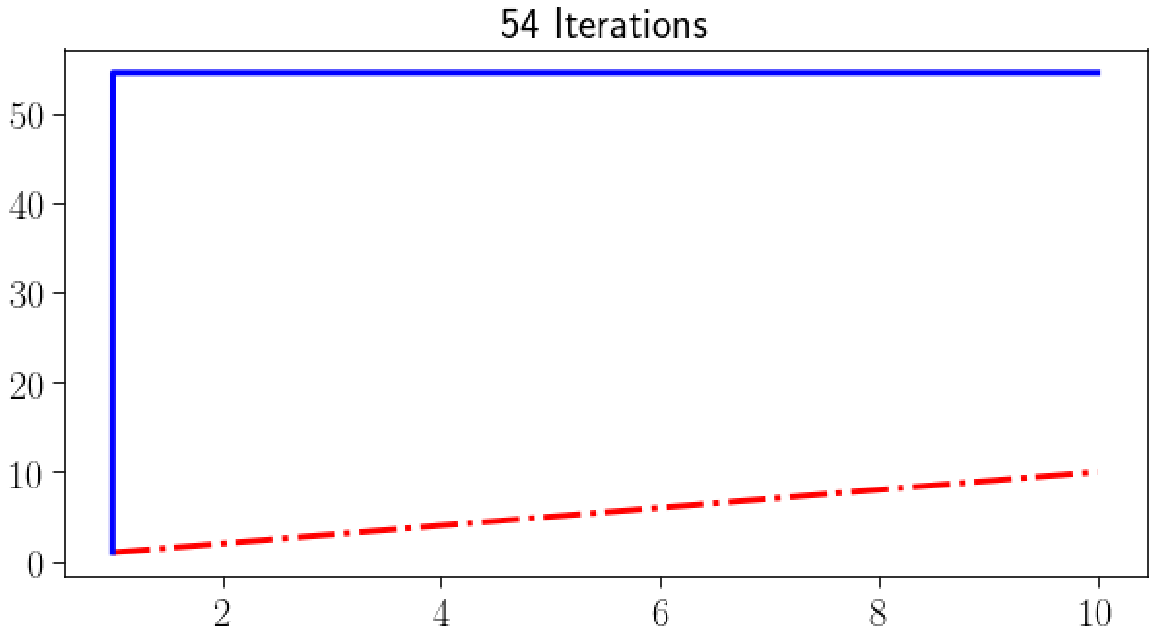

49 Iterations
10
8
6
4
2
2
4
6
8
10


50 Iterations
12.5
10.0
7.5
5.0
2.5
2
4
6
8
10


51 Iterations
12.5
10.0
7.5
5.0
2.5
2
4
6
8
10


52 Iterations
20
15
10
5
2
4
6
8
10


53 Iterations
50
40
30
20
10
0
2
4
6
8
10



In [2]:
```
epsilon=sys.float_info.epsilon
print('machine epsilon is {}, which = 2^-52'.format(epsilon))
iterations[4]
print('first plateau for 52 iterations is {}=e'.format(y[4, 1]))
min_number=1+epsilon
print('Note that (1+epsilon)^2^52={}'.format(min_number**(2**52)))
inde=(np.abs(x - y[4, 1])).argmin() #index of x=e
print('x and y agree at x={:.6}=> y({:.6})={:.6}'.format(x[inde], x[i
nde],y[4, inde]))
print('next point is start of next plateau y({:.6})={:.6}=e^2'.format
(x[inde+1],y[4, inde+1]))
```

```
machine epsilon is 2.220446049250313e-16, which = 2^-52
first plateau for 52 iterations is 2.718281808182473=e
Note that (1+epsilon)^2^52=2.718281828459045
x and y agree at x=2.7181=> y(2.7181)=2.71828
next point is start of next plateau y(2.719)=7.38906=e^2
```

When computing $(\cdots(x^{1/2})^{1/2}\cdots)^{1/2} = x^{1/2^n} = x^{2^{-n}}$, xs differing by less than $\epsilon = 2^{-52}$ will be rounded to a common value. In particular, a window of numbers closest to 1 (varies depending on n will be mapped to $1+\epsilon$, the smallest binary number larger than 1. We then compute $(\cdots((1+\epsilon^2)^2\cdots)^2 = (1+\epsilon)^{2^n}$, resulting in a plateau at finite number.

Similarly, there is a set of numbers $1+\epsilon < x^{2^{-n}} < 1+2\epsilon$ map to $1+2\epsilon$ when the repeated roots are taken, resulting in the next plateau at the larger finite number $(1+2\epsilon)^{2^n}$, and this repeats, resulting in the series of m plateaus at $(1+m\epsilon)^{2^n}$. Note that right at the end of each plateau we have $y = x$ (ie the repeated roots gives the correct number), since this is when $x^{2^{-n}} = 1+m\epsilon$ and there is no rounding so repeated squaring returns the original number. The plateaus grow in width with increasing n since 2^{-n} becomes smaller and thus a wider range of x are within ϵ when $x^{2^{-n}}$ is calculated

Looking at the plot for 52 iterations, we see that the first plateau has an amplitude of $e = (1+\epsilon^{2^{52}}$ (see code above), and then y jumps once x>e to $(1+2\epsilon)^{2^52}$ =(1+\epsilon)^{{2}^{53}}=e^2. It can be quickly numerically verified that this pattern continues, with $y = e^m$ for $e^{m-1} < x \leq e^m$

In [3]:
```
print('first plateau for 53 iterations is {:.6}=e^2=(1+epsilon)^2^53={:.6}'.format(y[5, 1], min_number**(2**53)))
inde2=(np.abs(x - y[5, 1])).argmin() -1#index of x=e
print('x and y agree at x={:.6}=> y({:.6})={:.6}'.format(x[inde2], x[inde2],y[5, inde2]))
print('next point is start of next plateau y({:.6})={:.6}=e^4'.format(x[inde2+1],y[5, inde2+1]))
# np.exp(1)**4
```

```
first plateau for 53 iterations is 7.38906=e^2=(1+epsilon)^2^53=7.38906
x and y agree at x=7.3882=> y(7.3882)=7.38906
next point is start of next plateau y(7.3891)=54.5981=e^4
```

Moving on to 53 iterations, we see that the first plateau has an amplitude of $e^2 = (1+\epsilon)^{2^{53}}$ as expected, with plateaus $y = e^{2m}$ for $e^{2(m-1)} < x \leq e^{2m}$

Similarly for 52 iterations, we have the first plateau of $e^{1/2} = (1+\epsilon)^{2^{51}}$ and plateaus of $y = e^{m/2}$ for $(e^{(m-1)/2} < x \leq e^{m/2}$. (numerically below)

In [4]:
```
print('first plateau for 51 iterations is {:.6}=e^2=(1+epsilon)^2^51={:.6}'.format(y[3, 1], min_number**(2**51)))
inde3=(np.abs(x - y[3, 1])).argmin() -1#index of x=e
print('x and y agree at x={:.6}=> y({:.6})={:.6}'.format(x[inde2], x[inde3],y[3, inde3]))
print('next point is start of next plateau y({:.6})={:.6}=e'.format(x[inde3+1],y[3, inde3+1]))
# np.exp(1)**0.5
```

```
first plateau for 51 iterations is 1.64872=e^2=(1+epsilon)^2^51=1.64872
x and y agree at x=7.3882=> y(1.648)=1.64872
next point is start of next plateau y(1.6489)=2.71828=e
```

The general pattern can thus be described as follows (and verified via the same methods as above, or using plots): For n iterations, there are plateaus of $y = exp(m * 2^{n-52})$ for $y = exp((m-1) * 2^{n-52}) < x < exp(m * 2^{n-52})$