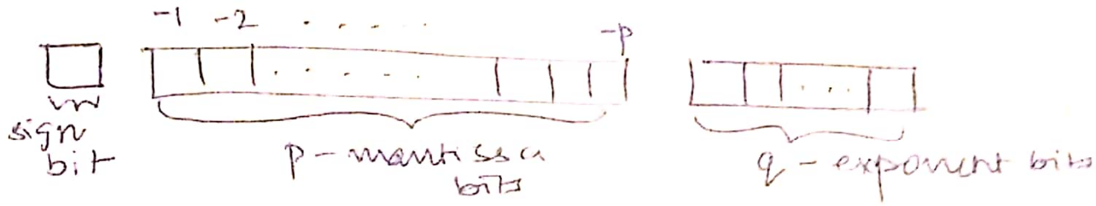
SRIJAN BHARATI DAS   APC523

1. To show:- $\left| \frac{x - rd(x)}{x} \right| \leq 2^{-p}$.

Let the x be denoted as:-



~~If~~ We need to find the maximum error on rounding the 2^{-p} th entry in the mantissa.

$\Rightarrow$ Let the mantissa of number x be:- $b_{-1}\, b_{-2} \cdots b_{-p}\, b_{-(p+1)}\, b_{-(p+2)}$

which is accurate to ∞ precision.

Now; b_{-p} can be changed from 0 to 1 if $b_{-(p+1)}$ is 1.

So; the ~~maximum error can happen if~~ number after ~~to~~ rounding would essentially be:-

$X_{round} = \boxed{b_{-1} \,|\, b_{-2} \,|\, \cdots \,|\, b_{-(p-1)} \,|\, 1} \; 000000$

not in mantissa but this is what it means if the number had infinite accuracy representation.

So; error can be ~~maximum~~ minimum if the actual number was:-

$\boxed{b_{-1} \,|\, b_{-2} \,|\, \cdots \,|\, b_{-p+1} \,|\, 0} \; 1111111\cdots$

However; the error is max when the actual number is:-

$X = \boxed{b_{-1} \,|\, b_{-2} \,|\, \cdots \,|\, b_{-p+1} \,|\, 0} \; 10000\cdots$

$\hookrightarrow$ responsible for rounding.

So; the error is:- $X - X_{round}$

$= 2^{-p} - 2^{-p-1} = 2^{-p-1}(2-1) = 2^{-p-1}$.

$(\times 2^{e})$

Now; $\epsilon = \left|\frac{x - rd(x)}{x}\right|$. To see an upper bound; we need to minimize x.

Now; unless $x=0$; the smallest value of

$$x = \boxed{1 \mid 0\,0 \cdots 0 \mid 0} \times 2^{e}$$

mantissa

So; $x_{min} = 2^{-1} \cdot 2^{e}$.

So; $\epsilon \leq \frac{2^{-p-1}\; 2^{e}}{2^{-1}\; 2^{e}} = 2^{-p}$

So; maximizing the numerator and minimizing the denominator we have:-

$\boxed{\epsilon \leq 2^{-p}}$ for rounding errors.

<u>A similar analysis can be made for rounding down;-</u>

$x_{rounded} = \boxed{b_{-1} \mid b_{-2} \mid \cdots \mid b_{-p+1} \mid 0}\; 0\,0\,0 \ldots$ (this is what it becomes after rounding down b_{-p} to 0)

<u>Maximum error when;-</u>

$x_{actual} = \boxed{b_{-1} \mid b_{-2} \mid \cdots \mid b_{-p+1} \mid 0} \mid 0\; 1\,1\,1\,1\,1 \ldots$

So; error $= \sum_{l=p+2}^{\infty} 2^{-l} = \frac{1}{2^{p+2}}\left(1 + \frac{1}{2} + \cdots\right)$

$= \frac{1}{2^{p+2}} \cdot \frac{1}{1-\frac{1}{2}} = \frac{1}{2^{p+1}}$ $(\times 2^{e})$

So; $\left|\frac{x - rd(x)}{x}\right|$ (Similarly be taking $x_{min} = 2^{-1} 2^{e}$)

$$\frac{2^{-p-1} \cdot 2^{e}}{2^{-1} \cdot 2^{e}} = 2^{-p}$$

So; $\boxed{\left|\frac{x - x_{round}}{x}\right| \leq 2^{-p}}$