3. (a) (i) $fl(x\, fl(x\, fl(x \ldots fl(x\, x))) \ldots)$

Now, $fl(x \cdot x) = x^2(1+\epsilon_1)$

So; multiplying $(n-1)$ times to get x^n will give:-

$$x^n(1+\epsilon_{n-1})(1+\epsilon_{n-2}) \ldots (1+\epsilon_1)$$

$$\equiv x^n(1+\epsilon_{n-1}+\epsilon_{n-2}+\ldots+\epsilon_1) \quad \text{(first order in } \epsilon_i)$$

$$\leq x^n(1+\overline{n-1}\,\epsilon)$$

bounded by machine epsilon.

So; $\boxed{\epsilon_{\text{repeated multiplication}} = (n-1)\epsilon}$ (this is the upper bound)

* we have assumed x is a perfect machine number.

(ii) again, assuming x is a perfect machine number;

$fl(\exp(fl(n\, fl(\ln x))))$

$= fl(\exp(n\, fl(n \ln x(1+\epsilon_{ln}))))$ due to logarithm

$= fl(\exp(n \ln x(1+\epsilon_{ln})(1+\epsilon_{mult})))$ due to multiplication of $\ln x$ with n.

$= \exp(n \ln x(1+\epsilon_{ln})(1+\epsilon_{mult}))(1+\epsilon_e)$ due to exponential

$$= e^{n\ln x(1+\epsilon_{ln})(1+\epsilon_m)}(1+\epsilon_e)$$

$$\Rightarrow e^{n\ln x(1+\epsilon_{ln}+\epsilon_m)}(1+\epsilon_e) \quad \text{upto 1st order in } \epsilon.$$

$$= e^{n\ln x}\, e^{n\ln x(\epsilon_{ln}+\epsilon_m)}(1+\epsilon_e)$$

$$\Rightarrow e^{n\ln x}(1+n\ln x(\epsilon_{ln}+\epsilon_m)+\ldots)(1+\epsilon_e)$$

$$\Rightarrow e^{n\ln x}(1+(n\ln x)(\epsilon_{ln}+\epsilon_m)+\epsilon_e) \quad \text{to 1st order in } \epsilon.$$

so; $e^{n \ln x}\{1+\epsilon\}$ where $\boxed{\epsilon = \epsilon_e + n\ln x(\epsilon_{ln} + \epsilon_m)}$

Now; assuming an upper bound of ϵ (machine epsilon) on all our relative errors (ϵ_e, & ϵ_{ln} & ϵ_m):-

$\epsilon_{\text{taking logarithm method}} = \boxed{\epsilon + 2\epsilon \ln x^n}$

Now; repeated multiplication is better if :-

$$(n-1)\epsilon < \epsilon + 2\epsilon n \ln x$$

$$\Rightarrow \quad n-1 < 1 + 2n\ln x$$

$$\Rightarrow \quad 2n\ln x > n-2$$

$$\Rightarrow \quad \ln x > \frac{n-2}{2n}$$

or $\boxed{x > e^{\frac{n-2}{2n}}}$

or equivalently when :-

$$n(1-2\ln x) < 2$$

or; $\boxed{n < \frac{2}{1-2\ln x}}$

(b) $x^a = e^{a\ln x}$. (for this to be just a mathematical tool for analysis; we assume no error in computing exponential).

(i) x is an exact machine number and a has a relative error ϵ_a.

so; ~~$fl(e^{a \ln x})$~~ $fl(\exp(fl(a)\, fl(\ln x))$ | Not considering error from product.

$= \exp(a(1+\epsilon_a)\ln x(1+\epsilon_{ln}))$

$a\ln x + a\epsilon_a \ln x + a\epsilon_{ln}\ln x + O(\epsilon^2)$ error due to logarithm

$\Rightarrow e$

$\Rightarrow e^{a\ln x}(1 + a\epsilon_a \ln x + a\epsilon_{ln}\ln x)$ | to first order in ϵ's.

So; relative error $= a\epsilon_a \ln x + a\epsilon_{ln} \ln x$.

Now, in our manipulation, $x^a \rightarrow e^{a\ln x}$ we assumed that exponential and logarithm would be carried out flawlessly. In that case;

$$\boxed{\text{relative error} = a\epsilon_a \ln x.}$$

(ii) For a being an exact machine # but not x:-

$$x^a \longrightarrow e^{a\ln x}$$

so; $fl(\exp(fl(a\, fl(\ln x))))$

$\Rightarrow \exp(a \ln(x(1+\epsilon_x)))$ | Not considering error from product or exponential or logarithm

If we consider all errors

then; $\exp(a\ln(x(1+\epsilon_x))(1+\epsilon_m))(1+\epsilon_e)$.

ϵ_x → from x not being a machine #.

ϵ_m → multiplication

ϵ_e → exponential

But because solution is needed only in terms of ϵ_x, ϵ_a, a & x; we do not consider ϵ_m and ϵ_e.

$$\exp(a\ln(x(1+\epsilon_x))) = e^{a\ln(x+x\epsilon_x)}$$

$$\Rightarrow e^{a\ln x + a\ln(1+\epsilon_x)} = e^{a\ln x}\{1+a\ln(1+\epsilon_x)+\dots\}.$$

$$\Rightarrow e^{a\ln x}\{1+a(\epsilon_x+O(\epsilon_x^2))\}$$

$$\Rightarrow e^{a\ln x}(1+a\epsilon_x)$$

so; $\boxed{\text{relative error} = a\epsilon_x}$.

Assuming ϵ_a and ϵ_a are usually very small; comparable to machine precision; the relative ~~rec~~ errors in either scenario could become significant if a is very large.

The effect of $\ln x$ in case of part (i) should not be problematic as x is scaled down by a logarithm.