8. $y_n = \int_0^1 dx \, . \, x^n e^x$ and $y_{n+1} = e - (n+1) y_n$.

↳ recurrence relation from integration by parts of y_n.

(a) $\boxed{y_n = \frac{e - y_{n+1}}{(n+1)}}$.

$g_K : y_N \longrightarrow y_K$ so; $g_K(y_N) = y_K$.

Now, $(\text{cond}(g_K))(y_N) = \left\| \frac{y_N \, g_K'(y_N)}{g_K(y_N)} \right\|$

Now: $g_K'(y_N) = \frac{\partial g_K}{\partial y_N} = \frac{\partial g_K}{\partial g_{K+1}} \frac{\partial g_{K+1}}{\partial g_{K+2}} \cdots \frac{\partial g_{N-1}}{\partial g_N}$

and; $\boxed{g_N(y_N) = y_N}$ $\Rightarrow \frac{\partial g_K}{\partial g_{K+1}} \cdots \frac{\partial g_{N-1}}{\partial y_N}$.

Now $\frac{\partial g_K}{\partial g_{K+1}} = -\frac{1}{K+1}$ (from recurrence relation)

So, $g_K'(y_N) = \left(-\frac{1}{K+1}\right)\left(-\frac{1}{K+2}\right) \cdots \left(-\frac{1}{N}\right)$

$= (-1)^{N-K-1} \frac{K!}{N!}$

So: $(\text{cond } g_K)(y_N) = \left\| \frac{y_N}{g_K} (-1)^{N-K-1} \frac{K!}{N!} \right\|$

Now, to ~~maximize~~ set an upper bound we use the fact that $\frac{y_N}{g_K} \leq 1$ always as y_n gets smaller as n increases, (because $x \in [0,1]$).

So, $(\text{cond } g_K)(y_N) \leq \left\| (-1)^{N-K-1} \frac{K!}{N!} \right\|$

(b) Now, $(\text{cond } g_K)(y_N) = \frac{\epsilon}{\epsilon_{y_N}}$

so; $(\text{cond } g_K)(y_N) = \frac{\epsilon}{-1}$

and taking the upper bound:-

$$\frac{\epsilon}{-1} \leq \frac{K!}{N!}$$

or, ~~$N! \leq \frac{K!}{\epsilon}$~~

or; $\boxed{N! \geq \frac{K!}{\epsilon}}$

and, $\epsilon_{y_N} = 1$ as

$y_N^* = y_N(1+\epsilon_{y_N})$

and for $y_N^* = 0$;

~~$|\epsilon| = |1|$ or $\epsilon = 1$~~

$\epsilon_{y_N} = -1$ as a 100% error.

↓

100% error coming from assuming $y_N^* = 0$.

(c) If $k = 20$ and $\epsilon = 2^{-52}$ (machine epsilon).

then, $\frac{k!}{\epsilon} \simeq \frac{20!}{2^{-52}} \simeq 1.09 \times 10^{34}$

~~and, $54! \simeq 10^{71}$ and $53! \simeq 10^{69}$~~

~~so; $\boxed{N \text{ has to be atleast } 54}$~~

So, to compute y_k upto a precision of $\epsilon = 2^{-52}$ we need to start with a y_N where $N = \cancel{54}\ 32$ and compute backwards recursively. (assuming we start with $y_N^* = 0$ and still are able to achieve a very highly precise value of y_{20}).

So, $N! \geq \frac{k!}{\epsilon} \Rightarrow \ln N! \geq \ln\left(\frac{k!}{\epsilon}\right) \Rightarrow \boxed{\ln N! \geq 34}$

Now, using Sterling's approx and then trial and error, we find $N = 32$. ~~$N! \sim 10^{35}$ and~~

$32! \sim 10^{35}$ and $31! \sim 10^{33}$. So, the minimum $\boxed{N = 32}$.

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