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Research paper

Price predictability at ultra-high frequency: Entropy-based randomness test

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ABSTRACT

We use the statistical properties of Shannon entropy estimator and Kullback–Leibler divergence to study the predictability of ultra-high frequency financial data. We develop a statistical test for the predictability of a sequence based on empirical frequencies. We show that the degree of randomness grows with the increase of aggregation level in transaction time. We also find that predictable days are usually characterized by high trading activity, i.e., days with unusually high trading volumes and the number of price changes. We find a group of stocks for which predictability is caused by a frequent change of price direction. We study stylized facts that cause price predictability such as persistence of order signs, autocorrelation of returns, and volatility clustering. We perform multiple testing for sub-intervals of days to identify whether there is predictability at a specific time period during the day.

1. Introduction

One of the fundamental questions in finance is the predictability of asset prices. Currently, there is a set of tools designed to predict prices in the markets. For instance, traders rely on technical analysis of stocks to make a profit. The prediction power of technical analysis has been the object of many investigations [1,2]. Various trading methods can be developed in order to increase profits starting from prediction of future prices which are even only slightly more accurate than a martingale [3–6].

Randomness of financial data aggregated to daily frequency was investigated in the literature [7–9]. Aggregating data to daily frequencies reduces the amount of available information and may obscure predictable patterns that occur on an intraday level. Studies analyzing the predictability of intraday prices were conducted using hourly [10,11] and minute [12,13] frequencies. We examined predictability occurring at a one-minute frequency level [14,15] by the Shannon entropy, a measure of uncertainty. Analyzing data at this frequency is more complex, as it introduces predictability from both profitable strategies and data microstructure. Prices at millisecond and second frequencies were analyzed in [16]. Our research progresses further toward a microscopic examination of financial time series. Ultra-high frequency data is defined as the full record of transactions and their associated characteristics [17].

We outline several objectives for this research. We analyze the predictability of ultra-high frequency data using Shannon entropy, shifting our focus from well-studied predictability in calendar time to tick time. Our goal is to investigate how quickly the market digests information and how prices are whitened[1] with larger aggregation of prices. We are particularly interested in the price characteristics that distinguish between predictable and unpredictable price time series. To draw conclusions from statistical tests,

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[1] In sense of a white noise also known as Brownian motion.

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we determine the asymptotic distribution for the Kullback–Leibler divergence between time series and a series under the assumption of symbol randomness. The objective of constructing this statistical test is to enable quick conclusions about predictability based on the available data.

We devise a test for randomness of data starting from the estimation of Shannon entropy. Entropy is defined as an averaged measure of uncertainty about a symbol appearing in a sequence generated by a random source [18]. Maximum uncertainty arises when all symbols from a finite alphabet are independently generated with equal probabilities. A common method for entropy estimation is calculating empirical frequencies of blocks of symbols [19]. Previous instances where Shannon entropy served as a measure for assessing randomness in financial time series include [7,9,13]. The entropy of price returns of assets traded on the Moscow Stock Exchange was studied in [14]. The time-varying entropy of meme stocks, which experienced sudden price surges driven by social media communities, was investigated in [15]. Such instabilities in a market and deviations from a fundamental price value was investigated in [20]. In case of independent and identically distributed symbols, the difference between entropy estimation and its maximum follows χ^2-distribution [21]. We make a step forward and use the estimation of relative entropy that allows us to test uncertainty for a symbolic sequence in case when symbols can be not equiprobable. The test statistics is based on all overlapping blocks of symbols and takes into account their dependence. Calculating the statistic is straightforward, requiring only the estimation of frequencies for preselected block lengths. In a financial setting, a significantly high predictability implies that a future price change depends on the price history and thus is not a completely random. Empirical frequencies used in test statistics refer to patterns in sequences of price returns, some of which may be more likely to occur [14].

In a recent work [22], randomness of financial prices at daily frequency was tested using conditional probabilities. The authors tested if probabilities of price increase and decrease are the same conditionally on price history. We provide the test for predictability that considers that probabilities that a price moves up or down can be different. We test if past price changes are helpful in predicting future price movement. We utilize discretization that distinguishes between positive and negative returns. A test for independence related to permutation entropy of price increments was introduced in [23]. Considering blocks of k prices, frequencies of $k!$ events need to be calculated. In contrast, our statistic requires the calculation of only 2^k frequencies. Approximate entropy was used in [24,25] to measure irregularity of financial time series. The authors fixed the length of blocks $k = 3$. We use a larger length for the analysis based on a given sequence length. Entropy of a singular value decomposition was applied to test market efficiency in [26]. Monte Carlo simulations were employed by the authors to establish confidence intervals for low entropy values. In our test, we evaluate whether new outputs remain independent of the sequence history, leveraging statistics with known asymptotic distributions. Applying the test based on empirical frequencies of price directions, we get rid of Monte-Carlo simulations. An alternative method for measuring predictability is estimating the Hurst exponent [27,28]. The authors of [29] determined the value of the Hurst exponent by fitting ultra-high frequency data using relative entropy. Additionally, the predictability of price time series can be assessed by employing machine learning techniques to predict price changes [30,31].

This research is dedicated to exploring the predictability of ultra-high-frequency data. The rapid arrival of new orders makes it challenging to forecast the next price before it appears in such a short period of time. Without taking into account transaction costs, we cannot ensure that predictability at ultra-high frequency indicates the presence of profitable trading strategies. Moreover, a long memory of price returns at ultra-high frequency is a stylized fact that incorporates predictability into the time series but does not contradict the efficient market hypothesis [32,33]. We study the relation between other stylized facts and predictability of financial time series at ultra-high frequency. This study complements the work [13] on stylized facts that add predictability to time series at a one-minute frequency. We investigate the level of predictability as a function of the length of steps in transaction time. By aggregating by the number of transactions, we increase the time between considered changes in price, dividing days into predictable and unpredictable days. We examine the periods in which the predictability of prices is presented. For instance, we observe for most stocks a high probability of consecutive price directions across several transactions, aligning with the long-memory characteristics of price return signs.

This article introduces three key contributions. First, we propose the test for randomness of a symbolic sequence. For conducting the test, the individual probabilities associated with the appearance of each symbol do not need to be equal. The test can be applied with varying numbers of distinct symbols, presuming the sequence length is sufficiently large. Second, we investigate the predictability of tick-by-tick data. The experiments conducted shows that the degree of predictability decreases when we aggregate simulated and real data by a number of transactions. We conduct the test for the intervals with duration less than one trading day to localize the predictable time series. Third, we investigate the empirical properties of price returns and the difference in the properties of returns for not predictable and predictable time series. Some of the properties are stylized facts of price returns such as fat tails, autocorrelation of returns, and price jumps. Other properties indicative of predictable days encompass heightened daily trading volumes and a substantial number of price changes. We reveal that the probability of consecutive returns displaying the same sign (persistence) serves as a feature for predictable sequences. For a group of stocks, this probability is significantly higher during predictable days. On the other hand, we also observe that the probability of repeating symbols is statistically low for predictable days for another group of stocks.

In Section 2, we propose a statistical test for investigating the predictability of symbols of a sequence. In Section 3, we present a chosen group of diverse assets with a set of various characteristics. Section 4 is dedicated to applying the predictability test to simulated and real data. We discuss the results in Section 5. Section 6 concludes the paper.

2. Statistical tests for randomness of symbolic sequence

2.1. Problem formulation

We introduce a statistical test designed to evaluate the predictability of a sequence. We begin this section with a problem formulation. The input for a statistical test is a realization X of a stationary random process $\mathcal{X}$ with symbols from a finite alphabet $A = \{0, 1, \dots, s-1\}$. Symbols of an alphabet with size s can be denoted by integers from 0 to $s-1$ without loss of generality.

$$X = \{x_1, x_2, \dots, x_n\}$$

The goal of the test presented in this section is to verify if symbols of the sequence are independently distributed. Our null and alternative hypotheses are the following.

$\mathcal{H}_0$: The occurrence of a new symbol in sequence X is *independent* of the sequence's prior history.

$\mathcal{H}_a$: Appearance of a new symbol *depends* on past observations of the sequence X.

If the null hypothesis is rejected, the probability of guessing the next symbol of a sequence based on the last symbols, is higher than for a random guess. The test is based on empirical frequencies of blocks of symbols introduced by Marton and Shields [19]. These empirical frequencies serve to estimate Shannon entropy, which is utilized as a measure of uncertainty. The asymptotic distribution of the Shannon entropy estimation is known and presented in Lemma 1. The asymptotic distribution of the Neyman–Pearson statistics proposed to test predictability is given in Lemma 2.

We provide a statistical test exploiting the distribution of the Shannon entropy. To obtain an asymptotic χ^2-distribution with known degrees of freedom, the sequence of symbols is divided into non-overlapping blocks of symbols. The statistical test we propose further in Section 2.3 is based on Kullback–Leibler divergence. It makes the following contribution to the problem of randomness estimation: (i) The test uses overlapping blocks and thus enriches the data to make a conclusion on randomness and (ii) It does not require unique symbols in sequence to be equiprobable to determine a complete randomness of the sequence. We refer to Fig. 1 for the flowchart describing both methods based on the Shannon entropy and the Kullback–Leibler divergence.

2.2. Randomness by Shannon entropy

First, we divide a sequence by *non-overlapping blocks* with length $k \in [2, n-1]$,

$$\hat{X} = \hat{x}_1, \hat{x}_2, \dots, \hat{x}_{n_b},$$

where $\hat{x}_t = \{x_{(t-1)k+1}, x_{(t-1)k+2}, \dots, x_{tk}\}$, $n_b = \lfloor \frac{n}{k} \rfloor$. $\hat{x}_t$ represents a number from 0 to $s^k - 1$ in the numerical system with base s. We calculate s^k empirical frequencies $\hat{f}_j$ of blocks from $\hat{X}$:

$$\hat{f}_j = \sum_{t=1}^{n_b} I(\hat{x}_t = a_j), a_j \in A^k.$$

Here and further I is an indicator function and $a_j \neq a_l$ when $j \neq l$. The value of k is user-defined. The upper bound of the length k of symbolic patterns considered to find predictably is validated theoretically and empirically. We take $k = \lceil 0.5 \log_s n \rceil$ as the length of blocks that is proved to be admissible and suggested to use for estimating empirical frequencies in [34].[2]

Then, entropy estimation is defined as

$$\hat{H} = -\sum_{j=0}^{s^k-1} \frac{\hat{f}_j}{n_b} \ln \frac{\hat{f}_j}{n_b}, \tag{1}$$

where $\ln()$ is the natural logarithm with the convention $0 \ln 0 = 0$.

Lemma 1 (*Asymptotic Distribution of the Shannon Entropy*). *Let us consider a stationary process with symbols from a finite alphabet $A = \{0, 1, \dots, s-1\}$ and its realizations $\{x_t\}_{t=1}^n$. Then, the entropy bias (scaled distance from the maximum of entropy) of a realization defined by Eqs. (1) and (2) follows χ^2-distribution with $s^k - 1$ degrees of freedom if the probabilities of all blocks of symbols, $\hat{x}_t$, are equal.*

$$B = 2n_b(k \ln s - \hat{H}) \tag{2}$$

The proof [21] of the Lemma is made by using a characteristic function. When all probabilities of blocks of symbols are equal, the process is fully unpredictable and the entropy attains its maximum, $k \ln s$. We can test unpredictability of a sequence using the entropy bias and the distribution under the null hypothesis $\chi^2_{s^k-1}$. If the probabilities of appearing symbols (blocks with length 1) are not equal, then entropy estimation $\hat{H}$ has asymptotically normal distribution [21]. The mean and standard deviation of the normal distribution depend on the entropy value and probabilities of blocks, as shown in [35,36] by the Taylor series of the entropy function.

[2] See 2.3d and 3.2 or [19]

2.3. Randomness by Kullback–Leibler divergence

Now, we propose a statistics with *overlapping blocks* of symbols that we apply for testing predictability. For the test, we use the Neyman–Pearson (NP) criterion [37] that contains the likelihood of the sequence under the hypothesis of unpredictability. We use *all* blocks with length $k-1$, $k \in [2, n-1]$,

$$\bar{X} = \bar{x}_1, \bar{x}_2, \dots, \bar{x}_{n-k+2},$$

where $\bar{x}_t = \{x_t, x_{t+1}, \dots, x_{t+k-2}\}$.

We construct statistics in Eq. (3) which has χ^2-distribution under $\mathcal{H}_0$. The similar test statistics for Markov chains was considered by Billingsley [38]:

$$D = 2\sum_{ij} f_{ij} \ln \frac{(n-k+1) f_{ij}}{f_{\cdot j} f_{i\cdot}}, \tag{3}$$

where

$$f_{ij} = \sum_{t=1}^{n-k+1} I\left(\bar{x}_t = a_i\right) I\left(x_{t+k-1} = a_j\right), a_i \in A^{k-1}, a_j \in A \tag{4}$$

is the frequency of the event when a block with the last symbol $j \in A$ follows by the block $a_i \in A^{k-1}$ in the sequence $\bar{X}$. We note that f_{ij} are also empirical frequencies of blocks of k symbols. $f_{\cdot j} = \sum_i f_{ij}$ and $f_{i\cdot} = \sum_j f_{ij}$, $0 \le i \le M-1$, where $M = s^{k-1}$ is the amount of blocks of $k-1$ symbols. The convention is $0 \ln 0 = 0$ and $0 \ln (0/0) = 0$. The asymptotic distribution of the statistics D is described in the Lemma provided below.

Lemma 2 (*Asymptotic Distribution of the NP-Statistics*). *Let us consider a stationary process with symbols from a finite alphabet* $A = \{0, 1, \dots, s-1\}$ *and its realization* $\{x_t\}_{t=1}^n$. *If the hypothesis of unpredictability,* $\mathcal{H}_0$, *holds true, then the Neyman–Pearson (NP) statistics Eq.* (3), (4) *converges in distribution to* χ^2 *with* $(s^{k-1}-1)(s-1)$ *degrees of freedom, i.e.,*

$$2\sum_{ij} f_{ij} \ln \frac{(n-k+1) f_{ij}}{f_{\cdot j} f_{i\cdot}} \xrightarrow[n\to\infty]{d} \chi^2\left((s^{k-1}-1)(s-1)\right).$$

We provide the sketch of proof for the distribution of D from Eq. (3) and subsequently verify it using Q-Q plots in Appendix A. We show that the proposed test statistics is valid even if the probabilities of appearing symbols are not equal. It is possible because of the denominator in Eq. (3) that mitigates differences in the probabilities of symbols inside the logarithm.

Significance of values of entropy estimation and the Neyman–Pearson statistics are defined by χ^2-distributions with suitable degrees of freedom. We calculate p-values associated with values of B and D and degrees of freedom of χ^2-distribution. P-value is the survival function of χ^2-distribution calculated at the test statistics. If a p-value is less than 0.01, we conclude that the sequence does not exhibit significant predictability with the significance level $\alpha = 0.01$. We reject $\mathcal{H}_0$ when p-value < 0.01. We call intervals where $\mathcal{H}_0$ is rejected *predictable*. The algorithm for conducting statistical tests for the randomness of symbolic sequences is summarized in a flowchart, as shown in Fig. 1.

While statistics B is based on Shannon entropy estimation, statistics D is scaled Kullback–Leibler (KL) divergence [39] between empirical probabilities $\frac{f_{ij}}{(n-k+1)}$ of blocks of symbols and the probabilities obtained under $\mathcal{H}_0$:

$$\begin{aligned} \mathrm{KL}\left(\frac{f_{ij}}{(n-k+1)}, \frac{f_{\cdot j} f_{i\cdot}}{(n-k+1)^2}\right) &= \sum_{ij} \frac{f_{ij}}{n-k+1} \ln \frac{(n-k+1) f_{ij}}{f_{\cdot j} f_{i\cdot}} \\ &= \frac{1}{(n-k+1)} \sum_{ij} f_{ij} \ln \frac{(n-k+1) f_{ij}}{f_{\cdot j} f_{i\cdot}} = \frac{D}{2(n-k+1)}. \end{aligned} \tag{5}$$

Asymptotic distribution of D can be rewritten in form of Gamma distribution, $D \xrightarrow{d} \text{Gamma}\left((s^{k-1}-1)(s-1)/2, 2\right)$. Then, KL from Eq. (5) converges as follows,

$$\mathrm{KL} \xrightarrow{d} \text{Gamma}\left(\frac{(s^{k-1}-1)(s-1)}{2}, \frac{1}{n-k+1}\right).$$

3. Limit order book

3.1. Dataset

We explore limit order book data downloaded from LOBSTER (www.lobsterdata.com). In a limit order book, each trader can post buy and sell orders. When a buy (or sell) order is submitted, it can be immediately matched with a previously submitted sell (or buy) order, resulting in an execution. If no match occurs, the order remains active until it is either matched or canceled. Orders submitted without disclosing the order quantity are referred to as hidden [40].

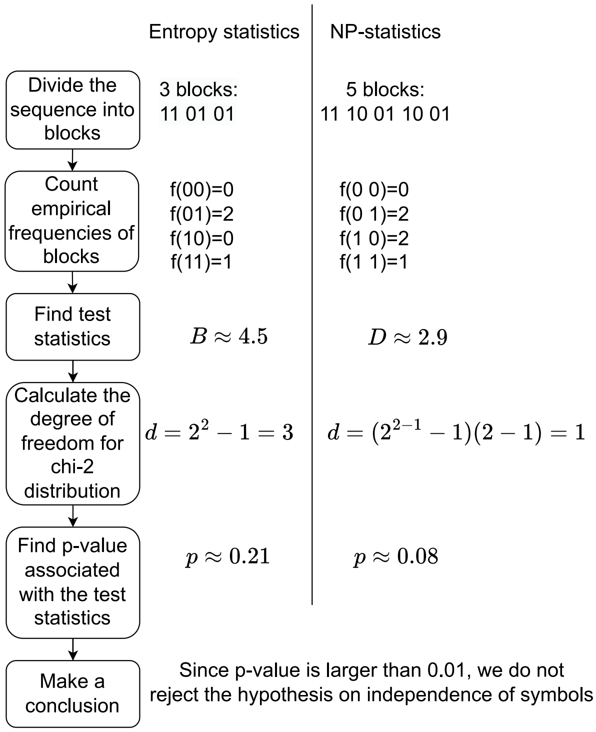


Fig. 1. Flowchart of the statistical test and the example on sequence 110101 with the length of 6 and alphabet size of 2. The block length is 2.

The timeframe under consideration spans from 01.08.2022 to 21.11.2022, encompassing a total of 80 trading days. For each day, the considered daily time interval is from 9:30 to 16:00, that is 390 min in total. Table 1 presents the tickers of all selected assets along with their respective properties. The chosen stocks represent diverse industries, differing in characteristics such as average price, volatility, number of transactions, waiting times between transactions, and trading volumes. Additionally, the analysis includes the ETF SPY, designed to track the S&P 500 Index. Times of transactions are recorded with the precision of one nanosecond. Mean prices of assets range from 9 to 400 dollars, the daily trading volume varies from 0.3 to 12 millions. For the majority of assets, the average time between transactions is less than one second.

For each asset, we download the message file from the lobster data. The files contain 6 columns: order id, time, price, size (trading volume), event type and direction (buy or sell). We focus on two event types: the execution of visible limit orders and the execution of hidden limit orders. We omit the binary sequence of buy and sell orders from the column “direction”, and consider the signs of returns as shown in Eq. (6) instead.

Considering each occurred transaction, we work with data in transaction time: prices are sampled with every transaction as opposed to calendar or tick time sampling [41,42]. Discretization is made by distinguishing between positive and negative returns: 0 corresponds to price decreasing, 1 corresponds to price increasing. Thus, alphabet A is $\{0,1\}$ and a symbolic sequence is obtained according to binary discretization from Eq. (6):

$$s_t^{(2)} = \begin{cases} 0, r_t < 0, \\ 1, r_t > 0, \end{cases} \tag{6}$$

where $r_t = \ln\left(\frac{P_t}{P_{t-1}}\right)$ are price returns and P_t is the price at time t. All 0-returns are removed.

We continue with a review stylized facts about ultra-high frequency data.

3.2. Stylized facts

Lillo and Farmer [32] conducted a statistical test, concluding that both the signs of market orders and executed limit orders exhibit long-memory processes. They attributed this long memory of order signs to news arrivals and order splitting, which is offset by fluctuations in market liquidity. Therefore, the long memory does not contradict efficiency of markets when prices incorporate all available information about future values [43]. Bouchaud et al. [44] investigated statistical properties of the limit order books of several stocks. In particular, the authors stated that the distribution of changes in limit order prices exhibits a power-law tail. Engle and Russell [45] noted that the longest duration between transactions appeared in the middle of trading days. According to the authors, clustering of transactions appears because of the size of bid–ask spread and gathering of informed traders. U-shapes of frequencies of large trades, small trades, and market orders were also discovered in [46]. In another paper by Engle [17], he

Table 1
Assets and characteristics of prices.

Asset	Ticker	Mean price	Standard deviation of price	Daily trading volume	Daily number of transactions	Average time between transactions
Apple Inc.	AAPL	153.47	0.93	12,184,032	136,136	0.165
Microsoft Corporation	MSFT	251.78	1.37	4,529,093	84,342	0.269
Tesla Inc.	TSLA	388.02	3.81	8,686,354	178,704	0.127
Intel Corporation	INTC	30.15	0.20	7,055,642	38,255	0.595
Eli Lilly and Company	LLY	327.33	1.73	370,050	11,404	2.086
Snap Inc.	SNAP	10.67	0.14	4,967,779	18,521	1.358
Ford Motor Company	F	13.93	0.10	4,468,175	12,954	1.815
Carnival Corporation & plc	CCL	9.24	0.12	5,874,376	15,372	1.518
SPDR S&P 500 ETF	SPY	390.52	1.56	9,136,137	95,181	0.246

Mean price, its standard deviation, trading volume, number of transactions, and average time between transactions are calculated for each day and then are averaged over 80 days. Trading volume is summed up for each day. Average time is given in seconds.

showed that intraday volatility has the similar pattern and attains its minimum in the middle of a trading day. Moreover, significant coefficients of ARMA(1,1) model were detected. Highly dependent microstructure noise was stated by [47]. According to [48], changes in stock prices between transactions are associated with trading volumes. Some stylized facts including fat tails of price returns, volatility clustering, and leverage effect were discussed in [49]. For a more comprehensive understanding and detailed discussions regarding market and limit orders, we refer to works [40,46,50].

4. Predictability of limit order book

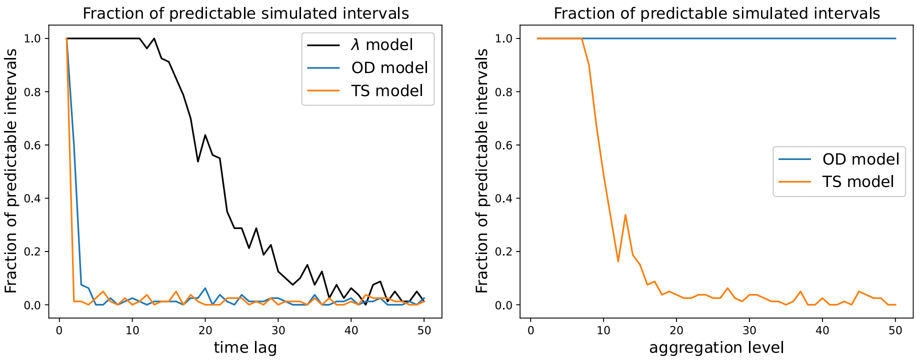
4.1. Modeling ultra-high frequency data

We first apply the proposed method for testing predictability on simulated data. We select models of limit orders exhibiting predictability patterns. We study how the predictability level changes with the aggregation of prices. Executed orders and trade signs are generated using models that replicate order splitting, herd or chartist behavior, and mean-reverting process [33,51,52]. According to Kyle's model [53] including informed traders, a price return linearly depends on the trading volume. In this model, the volume traded by an informed trader depends on the difference between current price and its future fundamental value. According to a behavioral stock market model proposed in [20], prices may deviate from their fundamental value due to the behavior of sentiment traders, who buy stocks in rising markets. Positive autocorrelation in order flow was previously observed in [32,33], where the authors demonstrated that autocorrelation diminishes with increasing aggregation by the number of orders. The λ model proposed in [51] reproduces the property of the positive autocorrelation decreasing with the larger time lags. The λ model introduces a fluctuating number of hidden orders that are divided into equal pieces and submitted gradually. We simulate 80 sequences of signs according to the λ model with the length 10^5 and parameters calibrated in the paper.[3] We present the fraction of predictable sequences for time lags from 1 to 50 in Fig. 2(a) for the λ model and two models described below. For small time lags, all sequences are predictable, then the fraction decreases but not monotonically.

Another explanation of the persistence of the signs is herd behavior [54] when traders execute their orders according to the price trend, sometimes against their private information. The model of herd behavior is presented in [55]. The parameters of the calibrated model of this behavior suggest that the information obtained by the traders are noisy, which creates uncertainty about the events that occurred. The order driven (OD) model where traders rely on both a fundamental price value and the history of trades is proposed in [52]. The predictability for order signs (buy/sell orders) of the OD model[4] drops significantly for time lags larger than 1. However, the predictability of price directions is quite persistent for increasing aggregation level in the amount of transactions as shown in Fig. 2(b). An *aggregation level* is a number of transactions taken as one time step. All 80 sequences generated by the

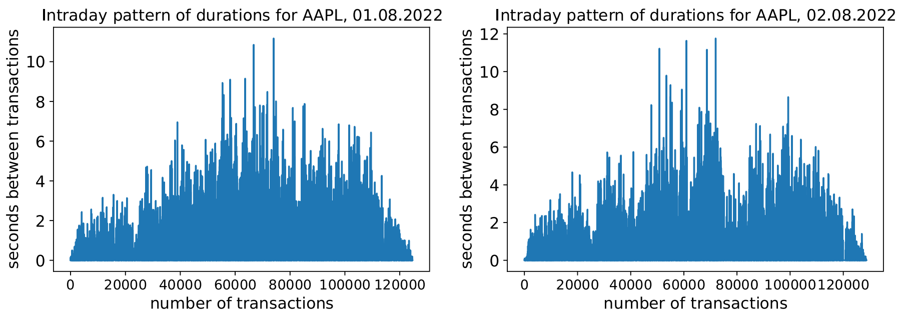
[3] The number of hidden orders is $N = 21$, the parameter of Pareto distribution describing volumes is $\alpha = 1.63$, and the probability of a new hidden order is $\lambda = 0.38$.

[4] The parameters are taken from the article [56]. These parameters decrease the impact of noisy traders on the deviations of a price from its fundamental value. The authors of [56] introduced a modified order flow model incorporating traders' learning and adaptation. It was demonstrated by these authors that price changes within their model display the characteristic of long memory.



(a) predictability of order signs (b) predictability of price returns

Fig. 2. Fraction of predictable days for the λ, OD, TS model from [33,51,52]. The x-axis for (a) is the time lag, which signifies the transaction time between order signs. The x-axis for (b) is the aggregation level, denoting the number of transactions considered as one step to compute price returns.



(a) August 1 (b) August 2

Fig. 3. Time in seconds between each transaction during two trading days, 01–02.08.2022.

OD model are predictable for aggregation levels from 1 to 50 due to the high probability that the price changes the direction in the next time step.

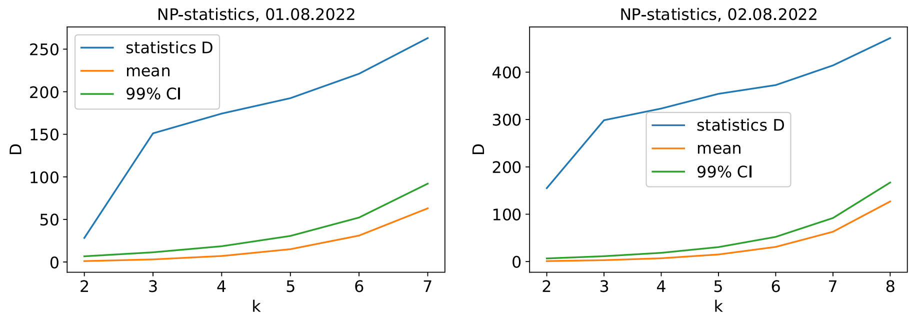
Lastly, we consider the trade superposition (TS) model proposed in [33]. This model posits that each price change results from previous trades, with a specific trade's impact defined by a propagator that diminishes over time.[5] Figs. 2(a) and 2(b) display the fractions of predictable sequences for order signs and price returns, respectively, based on the TS model. Prices generated by the TS model become more random and less predictable as the aggregation level rises, whereas prices simulated by the OD model remain predictable even at an aggregation level of 50. As we will see later in Fig. 6 for real data analysis, the predictability of aggregated prices of frequently traded stocks is similar to the predictability of the TS model. Thus, the TS model catches the behavior of real-word tick-by-tick prices regarding predictability at various aggregation levels.

4.2. Analysis of real data

Our analysis of real data commences by examining the limit order book of the AAPL stock. We focus on August 1 and August 2 of the year 2022 to demonstrate the behavior of the NP-statistics D and entropy bias B concerning various block lengths. Fig. 3 illustrates the duration in seconds before each transaction for these two specific days. Transactions occur less often in the middle of a trading day. In other words, we observe a U-shape of frequencies of trades.

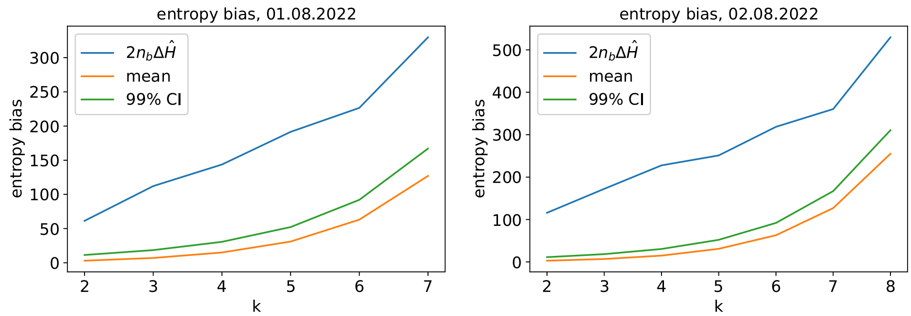
Fig. 4 illustrates the NP-statistics with 99% confidence bounds associated with $\mathcal{H}_0$. In Fig. 5, entropy bias with corresponding confidence intervals is given. The mean value of the χ^2-distribution of the NP-statistics under $\mathcal{H}_0$ represents the degrees of freedom. The values of statistics are larger than corresponding confidence bounds. For all values of k, sequences are predictable according to the statistics B and D. On August 1, the most frequent block with length $2 \le k \le 5$ is a sequence of 0s. However, for $k = 6, 7$ the most frequent block is a repetition of 1. On August 2, the most frequent blocks for corresponding values of k are 00 and 000. When $4 \le k \le 8$ the most frequent event is the repetition of 1 k times. This observation aligns with the well-documented long memory associated with return signs.

[5] The model's parameters, sourced from the paper, specify the propagator as $\frac{2.8\times10^{-3}}{(l+20)^{0.42}}$. The logarithm of volumes follows a normal distribution $N(5.5, 1.8)$ and the noise terms have a standard deviation of 0.01.



(a) August 1 (b) August 2

Fig. 4. NP-statistics for different k for AAPL stock.



(a) August 1 (b) August 2

Fig. 5. Entropy bias for different k for AAPL stock.

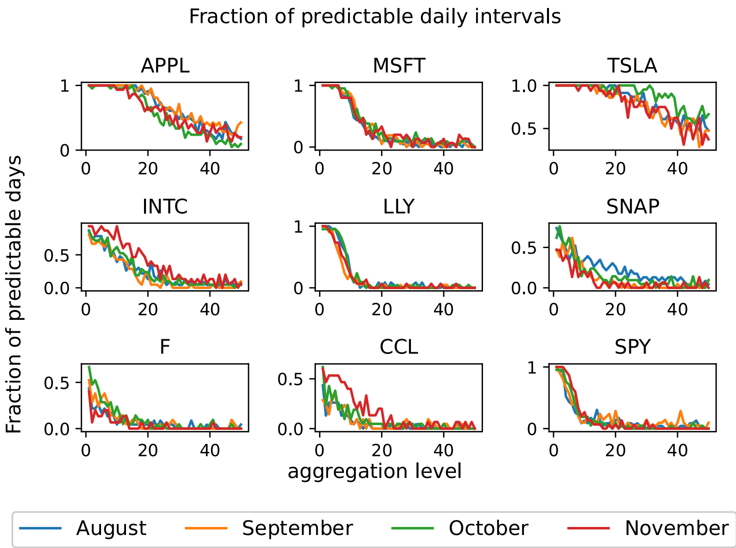
Our interest lies in identifying additional factors contributing to predictability beyond symbol repetition due to price persistence. Consequently, we proceed by analyzing aggregated high-frequency data. We use transaction time, that is, we aggregate by a number of transactions. We record the last available price for each time step. We examine the predictability of assets over several months and with different aggregation levels. We plot the fraction of days which is classified as predictable for the four months under consideration in Fig. 6. There is a noticeable amount of days without predictable patterns even without data aggregation for a group of stocks (INTC, SNAP, F, CCL). As the aggregation level increases, there is a decrease in the fraction of predictable days, although not in a strictly monotonous manner. We could not establish a clear correlation between the predictability of assets and the trading months. For instance, a higher level of aggregation is required in August to diminish the predictability of SNAP price. That is, August displays a greater number of days with significant predictability in price returns compared to the autumn months. October stands out as the month with the highest number of predictable days for TSLA. The results obtained from the NP-statistics and entropy bias suggest that the largest fraction of predictable days for INTC and CCL is observed in November.

With a given precision of nanoseconds, some transactions happen at the same time. These simultaneous events might represent scenarios where the volume of a market order surpasses the volume of the best buy or sell limit order. Another scenario involves automatic execution of market orders from different traders at a specific price. To mitigate the impact of such events on the analysis, we aggregate volumes and consider the final available price for each nanosecond showcasing trading activity.

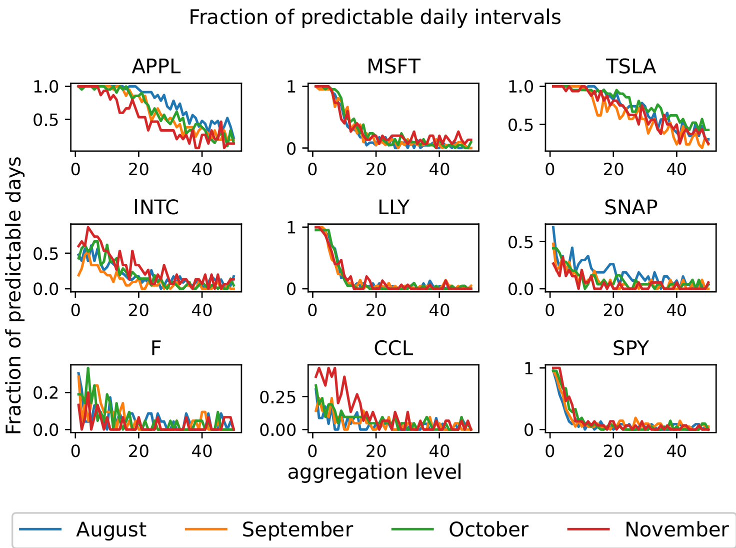
Fig. 7 presents the fraction of predictable days for two statistics and two datasets: full record of transactions and with aggregated transactions in each nanosecond. The exclusion of simultaneous transactions leads to a reduction in the predictability of price return signs. These simultaneous transactions are one of the sources of the predictability at ultra-high frequency. We examine other stylized facts associated with predictability in the next section. The empirical findings from Figs. 6 and 7 indicate a noticeable decrease in the degree of predictability as the aggregation level of transactions increases. The decay of predictability over aggregation level fits with results obtained for the simulated data by the TS model by Bouchaud et al. [33].

4.3. *Statistics of predictable time intervals*

In the previous section, we showed that some assets exhibit unpredictable prices for a part of days. In this section, we explore what price characteristics distinguish days with predictability from others. Now, we consider different parameters of price returns time series and check if there is a dependence between them and predictability. The list of parameters follows. These parameters are based on known stylized facts reviewed in Section 3.2.



(a) NP-statistics



(b) entropy bias

Fig. 6. Fraction of predictable days for different months and aggregation levels according to (a) NP-statistics and (b) entropy bias.

- First, we calculate the amount of non-zero returns that is the length of the symbolized sequence n and the fraction of 0-returns. Then, we record lengths of blocks, $k = \left[0.5 \log_s n\right]$. The amount of price returns used as past observations to test the independence is $k-1$.
- We compute empirical probabilities of observing blocks with the same symbols ($\hat{p}(0\dots 0)+\hat{p}(1\dots 1)$) to determine if predictability appears because the price moves in the same direction. We multiply the sum of two empirical probabilities by 2^k to vanish the difference for different values of k.
- We calculate $|\hat{p}(1)-\hat{p}(0)|$ to check if predictability is caused by the difference in the amount of price increases and decreases during a trading day. We also write down the magnitude of daily changes in a price to determine if predictability appears when the price significantly changes. For the same purpose, we record the mean value of price returns.
- We are interested in autocorrelation with lag 1 of non-zero returns as well as in autocorrelation of their magnitude values. The autocorrelation of magnitudes is a proxy for volatility clustering.

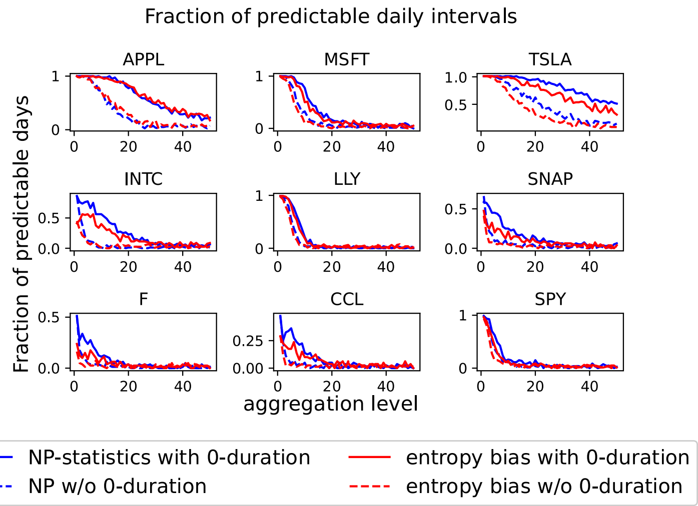


Fig. 7. Fraction of predictable days for different assets with and without simultaneous transactions.

- Then, we consider the distribution of price returns. We fit a Student's distribution of returns distribution and record the degrees of freedom ν, scale, and shift parameters.[6] The smaller value of ν, the fatter tails of price returns.
- We are interested in whether there is a significant difference in trading volumes.
- We also compare the fraction of jumps detected among all price returns. For the detection of jumps, we use the method described by [57]. To employ this test at ultra-high frequency data, we average price returns as suggested by [58].[7]

We conduct a statistical test to examine the differences in mean values between predictable and unpredictable days. A p-value below 0.05 indicates a significant difference with a 95% confidence level. We aggregate prices to collect both sets of predictable and not predictable days. The difference in characteristics may vary depending on the level of aggregation. For example, the persistence of the return sign is more typical for higher frequencies. Mean values of the specified parameters for AAPL stock are presented in Table 2, while Table 3 displays the parameter comparison between predictable and unpredictable days for all assets. Table 2 is provided to illustrate the results that need to be collected at the asset level in order to construct Table 3. To collect days where the hypothesis of unpredictability cannot be rejected, we aggregate price returns. The level of aggregation is chosen to balance between predictable and unpredictable days. For instance, setting the aggregation level to 15 means that we consider only every 15th transaction to construct a symbolic series of price increases or decreases.

In the case of *AAPL stock* without aggregation, all 80 days are determined as predictable using the NP-statistics, while 79 predictable days are detected via entropy estimation. We start from the aggregation level $a = 15$. Predictable days exhibit significantly higher amounts of price movement and trading volumes. Additionally, autocorrelation values for non-zero returns and their magnitudes are notably higher on predictable days. Moreover, the probability of repeating the same symbol is significantly elevated during days with predictable price returns.

To mitigate predictability resulting from the persistence of signs, we increase the aggregation level. With an aggregation level of $a = 30$ only three days (02.08, 20.09, 05.10) are identified as predictable. Then, we divide days into two groups after aggregating by $a = 30$ using the test-statistics B derived from entropy estimation from Eq. (2). There are 14 predictable days, the results are given in Table 5 in Appendix B. Even after aggregation by 30 transactions, the predictable days have a higher probability of repeating the direction of the price. They are also characterized by the larger difference in frequencies of increasing and decreasing of price. To mitigate the difference, the median of price returns could be used instead of zero during the discretization process by Eq. (6). Further, we use NP-statistics D that takes into consideration the difference between frequencies of symbols by design.

For the *Microsoft stock*, we discover 26 predictable days after aggregating data by $a = 10$ transactions. According to the test for the difference in mean value of $(\hat{p}(0\dots 0) + \hat{p}(1\dots 1))2^k$, the price direction is more persistent for predictable days. With a slightly larger aggregation level of $a = 15$, this difference is eliminated and only four predictable days remain. These days are listed in Table 8 in Appendix B.

For the *stock TSLA*, we identify 35 predictable days with aggregation $a = 25$. On these days, we observe a higher number of price changes, a stronger autocorrelation of non-zero returns, and increased probabilities of blocks repeating a symbol. Increasing the aggregation level by 5, we observe that the probability of repeating a symbol is significantly high even when $a = 60$. However, by further increasing the aggregation level to $a = 65$, only 4 predictable days remained (05.08, 14.09, 15.09, 08.11), and the previously significant differences disappeared. Regarding the influence of news on price predictability, it is worth noting that on August 5, Tesla

[6] Here, we use scipy.stats.t.fit in Python.

[7] We use the square root of the amount of price returns as the window size used in the method [57]. The method [58] requires pre-averaging of price returns. We use the same number of transactions for aggregation and pre-averaging. Jumps are defined with the significance level of 1%.

Table 2
Statistics for predictable and not predictable days of AAPL with $a = 15$.

Parameter	Mean for predictable days	Mean for not predictable days	p-value
Sample size (number of days)	35	45	
Number of non-zero returns	5225	4370	0.009**
Fraction of 0-returns	0.142	0.138	0.320
k	6	5.889	0.118
$(\hat{p}(0\dots0) + \hat{p}(1\dots1))2^k$	3.084	2.635	3.152×10^{-5}**
$\lvert\hat{p}(1) - \hat{p}(0)\rvert$	0.051	0.050	0.874
Magnitude of daily log-price increment	0.017	0.012	0.093
Mean price returns	2.185×10^{-7}	-6.673×10^{-7}	0.315
Magnitude of autocorrelation of non-zero returns	0.044	0.027	0.002**
Magnitude of autocorrelation of absolute values	0.115	0.088	0.007**
ν of t-distribution	2.220	2.694	0.31
Scale of t-distribution	1.367×10^{-4}	1.464×10^{-4}	0.195
Magnitude of shift of t-distribution	9.082×10^{-6}	9.991×10^{-6}	0.519
Daily volume	13,367,901	11,257,199	0.026*
Fraction of jumps	1.934×10^{-4}	1.367×10^{-4}	0.262

* Is rejection of equal means with 0.05 significance, in the last column.

** Stands for 0.01 significance, in the last column.

Table 3
Statistics for predictable and not predictable days.

Parameter	AAPL		MSFT	TSLA	INTC	LLY	SNAP	F	CCL	SPY
Aggregation level	15	30	10	25	2	5	1	1	1	5
Number of predictable days	35	3	26	35	44	44	53	41	38	29
Number of non-zero returns	>**		>**	>*	>**	>**	>**	>**	>*	>*
Fraction of 0-returns							>*	>**	>**	
k		>**				>*		>**	>*	
$(\hat{p}(0\dots0) + \hat{p}(1\dots1))2^k$	>**		>**	>**	>**	>**				>**
Magnitude of daily log-price increment								>**		
Mean price returns							>*		>*	
Magnitude of autocorrelation of non-zero returns	>**			>**	>**	>**	>**	>**	>**	>**
Magnitude of autocorrelation of absolute values	>**	<**					>**	>*	>**	
ν of t-distribution		<*						>*	>**	<*
Scale of t-distribution		<*				<*	<*			
Daily volume	>*		>**		>*	>**	>**	>**	>**	>*
Fraction of jumps		<**							>*	

For each stock and aggregation level in columns and price characteristics in rows, we compare means of the statistics between predictable and not predictable days. Predictability of days are defined by NP-statistics. > stands for larger mean for predictable days, < stands for larger mean for not predictable days. * and ** stand for the rejection of the hypothesis that characteristics in two types of days have the same mean. There is no significant difference found for parameters $\lvert\hat{p}(1) - \hat{p}(0)\rvert$ and magnitude of shift of t-distribution. For AAPL stock, the results for two aggregation levels are shown.

* Is for 0.05 level of significance.

** Stands for 0.01 significance.

shareholders approved a 3–1 stock split. On September 14, Tesla faced a lawsuit for false advertising of its autopilot technology. Additionally, on November 8, Tesla recalled over 40 thousand cars.[8] These instances of significant events or news might be associated with predictable price behavior at high frequencies. Traders tend to react more actively to new information, leading to increased

[8] The news are taken from edition.cnn.com and reuters.com.

Table 4
Probabilities of repeating symbols for predictable and not predictable days.

Stock	N. of predictable days	$\hat{p}(00)+\hat{p}(11)$ for predictable days	$\hat{p}(00)+\hat{p}(11)$ for unpredictable days	p-value for difference in mean values
SNAP	36	0.469	0.501	0.0014
F	25	0.460	0.493	2.671×10^{-10}
CCL	27	0.478	0.5	0.044

trading activity reflected in price changes and higher trading volumes. In general, high trading activity observed during predictable days across all considered assets suggests a possible connection between predictability and the arrival of news or significant events. For a more in-depth exploration of how prices react to news, we refer to a review [59].

For the *stock INTC*, we identify 44 predictable days using the aggregation level of $a = 2$ transactions. These days notably differ in terms of daily changes and autocorrelation of returns. For this stock, predictability declines rapidly as the level of aggregation increases. As we increase the aggregation level to $a = 7$ transactions, only 8 predictable days remained, characterized by significantly higher amounts of non-zero returns. At an aggregation level of $a = 10$, the only predictable day is October 13. For the *stock LLY* when $a = 5$, the set of different characteristics is similar to the case of INTC . At a higher aggregation level of $a = 10$ transactions, only three predictable days (03.08, 29.09, 28.10) are found with significantly high probability of repeating symbols.

For the *stock of Snapchat*, we find 53 predictable days without employing aggregation. These predictable days do not exhibit a large probability of repeating symbols. The distinguishing features of predictable days are a larger fraction of 0-returns, an increased mean return, and a high autocorrelation of returns and their absolute values. The similar results are obtained for *stocks CCL and F*. The case of the *stock CCL* is the only one where the percentage of price jumps is significant and higher for predictable days. We present the results for the stock CCL in Table 6 in Appendix B. Last, we consider the price of the *ETF SPY* with $a = 5$ in Table 7 in Appendix B. We discover that price returns of predictable days have fatter tails than returns of unpredictable days. Using the aggregation level $a = 10$, we remain with 9 predictable days presented in Table 8 with no significant difference in probabilities of repeating symbols.

4.4. *Predictability of pairs of signs*

Here, we explore the predictability of pairs of price changes, that is the randomness of future price movement observing the last price change. We set the user-defined parameter k to be equal to 2. For the majority of considered assets, we find that predictability is associated with a high frequency of blocks with the same price direction. However, for three stocks, SNAP, F, CCL, we fail to detect the high frequency of blocks $0\dots0$ and $1\dots1$ without aggregation of prices. Here, we consider the three stocks to test if a price direction depends on the previous recorded decrease or increase. In other words, we test $\mathcal{H}_0$ setting $k = 2$ and estimate $p(00)+p(11)$ for predictable and not predictable days. A summary table is Table 4. For all three stocks we detect a significant difference in $\hat{p}(00)+\hat{p}(11)$ for the two groups of days. However, the sum of two probabilities is less than 0.5 for predictable days. That is, the probability of changing price direction is significantly high for predictable days. Therefore, predictable days are more likely to have a pattern of changing symbols indicating an increase or decrease in price.

The number of predictable days is fewer when considering $k = 2$ compared to the value of $k = \lceil 0.5\log_s n \rceil$, where n represents the length of a symbolic sequence. As the number of symbols used to test dependence on past history increases, more days exhibit predictability. The probability of repeating the same symbol is approximately 0.5 for unpredictable days, whereas it is notably smaller for predictable days. However, this characteristic of predictable days for the three stocks diminishes as k increases to around 5 or 6 as indicated in Table 6. That is, a pattern of switching direction with each price movement tends to last for less than 5 price changes on average.

4.5. *Localization of predictable intervals*

In this section, we consider the length of the interval used to detect predictability. In previous sections, we investigate daily time intervals. The key question here is whether there exists a smaller time interval within a predictable day where significant predictability can be identified. The motivation for searching for the smaller interval is to localize the period when price predictability occurs. Additionally, using a smaller time interval necessitates less historical data for computing entropy-based estimations. Therefore, employing a smaller rolling window enables quicker detection of price predictability.

Since rolling window inside a trading day implies multiple tests, the Šidák correction [60] of the significance level is used. Moreover, to ensure independence among the conducted tests, non-overlapping intervals within a trading day are considered. The maximum value of considered partitions is $S_{max} = \lfloor (n-k+1)/1000 \rfloor$, so that 1000 is the minimum length of intervals. For each trading day, we aim to detect predictability for at least 1 from S partitions with significance level $1 - 0.99^{1/S}$. We record the maximum value of such $S \leq S_{max}$. Smaller intervals require a smaller k for analysis. We present the results in Table 8 in Appendix B.

There are three days for AAPL stock where predictability is detected when the data is aggregated. For all three days predictability is detected only for daily time intervals ($S = 1$). For the MSFT stock, there are three days where predictability is detected only for the part of the day ($S > 1$). Regularity patterns are detected at the end of the day for August 3 and October 26. For August 5, predictability is presented for the first half of the day. It disappears at the next subsequent non-overlapping interval. For each predictable day of the ETF SPY, there is only one from S intervals where predictability is found. Predictability disappears from the time interval to the next subsequent non-overlapping interval. For instance, predictability was noticeable in the middle of the trading day on August 17. Regarding TSLA stock, detectable predictability was identified in both halves of the day on August 5, indicating the potential to reduce the sequence length for predictability analysis by half and still observe significant patterns throughout the day. For 13 out of 30 days where $S_{max} > 1$ for the Ford stock, the predictability can be found in the second half of a day. Similarly, there are 14 days in the sample of 31 days with $S_{max} > 1$ where the predictability is detected only in the second part of a trading day for CCL.

For the stock SNAP, we detect days when predictable time intervals occur several times during a day. We notice that some predictable intervals cluster together and highlight such intervals in the Table 8. For instance, we observe 7 predictable intervals going in a row on October 21 and 24. However, we do not find evidence that predictive intervals follow each other for the other stocks considered.

5. Discussion

We have studied the predictability of asset prices at ultra-high frequency. Considering signs of price changes, we construct binary sequences for all recorded executed transactions. The signs of trades have a long memory in such a microscopic view of transaction data [32,33]. We have shown that the degree of predictability decreases with the increase of aggregation level. We apply aggregation by the number of transactions and work with transaction time. Transaction times have an uneven time intervals in seconds, which has been empirically shown by [45,46]. We have shown that the significant predictability level decreasing with larger aggregation level can be explained by splitting hidden orders into pieces as modeled by [51] or by mean reverting as modeled in [33]. According to [32,33], such type of predictability does not lead to arbitrage opportunities because it is compensated by fluctuations in transaction costs and liquidity and by interaction between market makers and informed traders. We have also demonstrated that transactions appearing simultaneously with the precision of a nanosecond contribute to a larger predictability level.

Our findings from Figs. 6 and 7 imply that almost all days at ultra-high frequency can be determined as predictable. However, as the level of aggregation increases, the degree of predictability diminishes. These results align with the studies of [30,31], which reached similar conclusions using machine learning techniques. In particular, [31] found that predictability in high-frequency returns is systematic and significant over short time horizons. They applied deep learning methods to investigate the predictability of the limit order book and concluded that ultra-high-frequency predictability is widespread. Our results also broadly agree with the findings of [29]. In that study, the authors analyzed the efficiency of US stock market indexes by aggregating tick-by-tick data and using relative cluster entropy to compare the empirical distribution with surrogate data generated by fractional Brownian motions. These processes exhibit memory for any Hurst value except for $H = 0.5$ (this case corresponds to classical Brownian motion). When $H > 0.5$, consecutive increments tend to have the same sign so that these processes are persistent, whereas for $H < 0.5$ consecutive increments are more likely to have opposite signs, and consequently these processes are anti-persistent. In both the $H > 0.5$ and $H < 0.5$ cases the Shannon entropy of the first differences (fractional Gaussian processes) will be smaller than for the Brownian motion [27,28]. Carbone and Porta [29] used the variance of their cluster relative entropy to obtain values of H ranging from 0.55 to 0.63 for different stock indexes (S&P500 and NASDAQ). Moreover, they observed that the divergence between the distributions become negligible as the cluster duration increases, in agreement with our findings.

Applying the test for randomness, we distinguish predictable days from not predictable days. We have shown that the probability that the price of an asset has several subsequent movements in the same direction is one of the factors affecting the predictability of the prices. For a group of assets, predictable days are characterized by repeating signs of price returns. Except LLY, trades on these assets appear at extremely high frequencies, i.e., less than one second on average. The repetition of price direction is explained by the appearance of news, the reaction to them, and the splitting of one order into parts [61,62]. When the level of aggregation increases to the point where repeating symbols are no longer a decisive factor in identifying predictability, only a few predictable days remain. The predictability that persists after aggregation in tick time may indicate potential profitability, assuming there are no transaction costs.

Conversely, another group of stocks (SNAP, F, CCL) demonstrated a lower percentage of predictable days before aggregation. In these cases, days featuring predictable time intervals were characterized by a significantly reduced probability of price moving in the same direction twice. This group differs by relatively low prices and their standard deviations. The pattern of changing price direction can be explained by a bid–ask bounce and fluctuations of the price around a low mean value. We presume that the occurrence of this behavior depends on the frequency of transactions.

For 8 out of 9 assets under consideration, non-zero price returns during predictable days have high autocorrelation. Highly significant coefficients of an AR model for tick-by-tick data were empirically demonstrated by [17,47]. We check volatility clustering by measuring the autocorrelation of the absolute values of returns. The autocorrelation is significantly greater during predictable days for the stocks AAPL, SNAP, F, and CCL. Bouchaud [49] discussed other stylized facts typical for price returns at ultra-high frequency including fat tails of return distribution and volatility clustering. We have compared corresponding characteristics for predictable and not predictable days. To explore fat tails of price returns, we estimate degrees of freedom of the fitted t-distribution of the price returns. For the ETF SPY, we discover that price returns of predictable days have fatter tails than returns recorded during not predictable days. However, we have the opposite result for the stocks of Ford and Carnival Corporation, where not predictable days are described by price returns with fatter tails. There is no strong evidence of a correlation between some of the price characteristics under consideration and price predictability. For instance, detected jumps [57] are associated with predictable days for only one stock, CCL.

We notice that predictable days of AAPL, MSFT, INTC, LLY, F, CCL, and SPY are characterized by larger trading volumes and the larger amount of non-zero price changes in comparison with not predictable days. A dynamic model that concerns both price changes and trading volumes depending on an insider information was introduced in the seminal paper of Kyle [53]. The distinction in the volumes and price changes suggests a correlation between predictability and trading activity, particularly in response to market news events. This assumption aligns with existing research demonstrating the influence of news on stock prices. Stock prices react to announcements about stock dividends and splits [63]. Weekly price returns react to attention to news and their tone [64]. Public news affect monthly price returns [65].

Analysis of the characteristics of predictability intervals and their duration for each stock provides insights into the price generation process. We localize intervals that we call predictable and find the duration of periods of predictability. We demonstrate that typically there exists a single predictable interval within a predictable day. It is observed that two predictable intervals generally do not occur consecutively. In other words, following an interval with detected predictability, the subsequent interval does not display a significant level of predictability. However, considering all transactions of the SNAP stock (Snap Inc.), we are able to detect several predictable time intervals going in a row.

6. Conclusions

We have applied a statistical test for the randomness of a symbolized sequence. A short summary of the method for detecting predictability is as follows. First, we estimate the Shannon entropy using empirical frequencies of blocks of symbols as suggested in [19]. Using empirical frequencies obtained by rolling a window with a certain length, we calculate the NP-statistics. The NP-statistics has χ^2-distribution according to [38,66]. We have found degrees of freedom of the χ^2-distribution that depends on the length of blocks and the size of the alphabet. The statistics is a scaled KL-divergence [39] that measures difference between empirical probabilities and theoretical probabilities under the null hypothesis of unpredictability.

The test based on the KL-divergence does not require an assumption that all symbols in the alphabet have the same probability of appearing. Relaxing this assumption makes the method more appropriate for testing dependency on past history in scenarios where price movements up or down are not equally probable, as compared to using conditional entropy or Shannon entropy [14,22]. The method is computationally efficient since it requires only the empirical frequencies of blocks of symbols. Predictability is determined using p-values from the asymptotic distribution in contrast with using Monte-Carlo simulations as in e.g. [26]. Another measure of uncertainty based on Kullback–Leibler divergence was introduced in [29], however, the degree of randomness in that study was empirically validated by simulating fractional Brownian motions.

Moreover, the proposed method counts frequencies for overlapping blocks of symbols thus collecting more statistics in comparison to non-overlapping blocks. The length of considered sequences of prices are limited by theoretical justifications, allowing us to consider a longer dependencies between price changes. In opposite, the approaches applying permutation entropy and approximate entropy [23–25] are usually limited to the length of blocks equal to 3.

The limitations of our approach are summarized as follows. We rely on the asymptotic distribution of the test statistics and thus require relatively long time series for the analysis. Our simulation study in Appendix A implies that the length of 10,000 is sufficient, while the length of 1000 may be not enough. The second limitation is data discretization. The approach works with discrete data and thus we consider only coarse-grained representation of the process e.g. price direction.

We have applied the correction proposed in [60] to make multiple tests for predictability for short intervals during predictable days. In most cases, a single predictable interval is identifiable by partitioning a day into uniform intervals based on transaction time. In such a way, we determine both the position of this interval relative to the time of day and its duration. For the stock SNAP, we have found several groups of such predictable intervals following each other.

Aggregating the data so that the time between the transactions under consideration increases, we allocate a smaller group of days with predictability. Various methods exist for aggregating data, including calendar time, transaction time, and tick time. Alternative approaches involve counting volumes traded and price changes by a certain amount. A direction for future exploration involves comparing the predictability observed in data aggregated through diverse methodologies.

CRediT authorship contribution statement

Andrey Shternshis: Writing – original draft, Visualization, Software, Methodology, Investigation, Formal analysis, Conceptualization. **Stefano Marmi:** Writing – review & editing, Supervision, Methodology, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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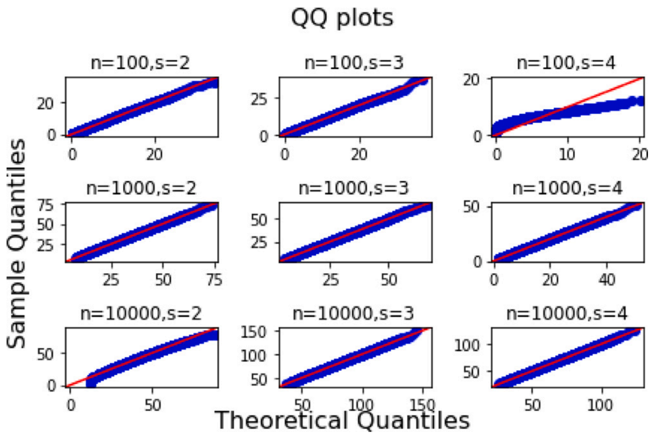

Fig. 8. QQ plot for entropy bias when probabilities of symbols are equal.

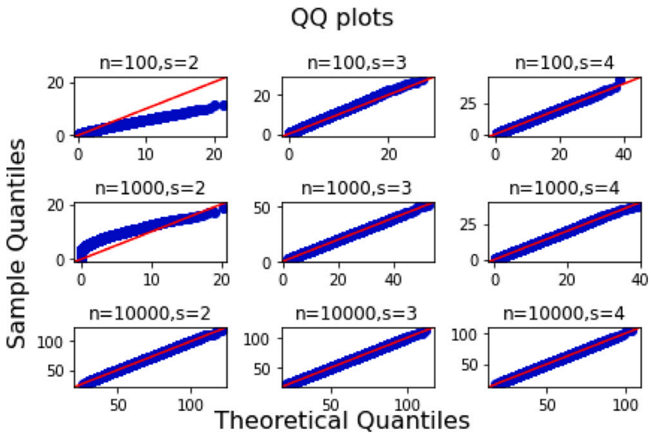

Fig. 9. QQ plot for NP-statistics when probabilities of symbols are equal.

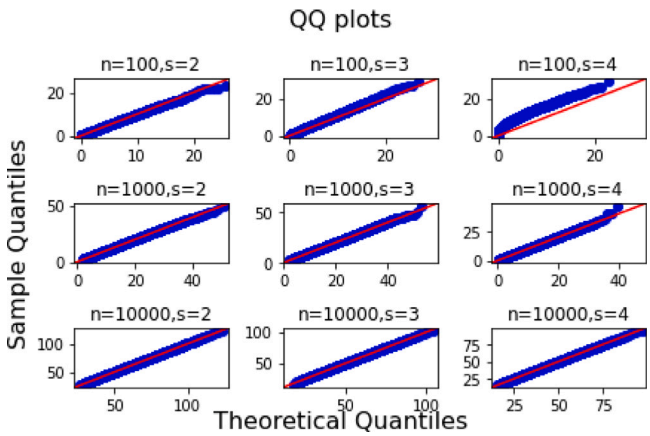

Fig. 10. QQ plot for NP-statistics when probability of symbol 0 is twice that of another symbol.

Appendix A. Asymptotic distribution of the Neyman–Pearson statistics

We outline the proof of Lemma 2. The lemma gives the asymptotic distribution of the statistics D in Eq. (3) based on empirical frequencies of blocks of symbols of a sequence. If symbols of sequence $X = \{x_t\}_{t=1}^{n}$ are independently distributed, then the sequence

Table 5
Statistics for predictable and not predictable days of AAPL determined by entropy bias with $a = 30$.

Parameter	Mean for predictable days	Mean for not predictable days	p-value
Sample size (number of days)	14	66	
Number of non-zero returns	2685	2454	0.177
Fraction of 0-returns	0.100	0.094	0.143
k	6	5.758	7.594×10^{-5}**
$(\hat{p}(0\dots0) + \hat{p}(1\dots1))2^k$	3.033	2.462	0.013*
$\lvert\hat{p}(1) - \hat{p}(0)\rvert$	0.068	0.039	0.003**
Magnitude of daily log-price increment	0.015	0.014	0.952
Mean price returns	-3.124×10^{-7}	-6.043×10^{-7}	0.874
Magnitude of autocorrelation of non-zero returns	0.0299	0.0297	0.966
Magnitude of autocorrelation of absolute values	0.079	0.093	0.208
ν of t-distribution	2.656	3.426	0.435
Scale of t-distribution	1.843×10^{-4}	2.156×10^{-4}	0.04*
Magnitude of shift of t-distribution	1.609×10^{-5}	1.204×10^{-5}	0.11
Daily volume	12,788,633	12,048,587	0.438
Fraction of jumps	2.093×10^{-4}	1.685×10^{-4}	0.682

* Is rejection of equal means with 0.05 significance, in the last column.

** Stands for 0.01 significance, in the last column.

$\bar{X}$ of blocks $\bar{x}_t = \{x_t, x_{t+1}, \dots, x_{t+k-2}\}$ follows the first order Markov process. The transition matrix of the Markov chain has dimension $M \times M$, where $M = s^{k-1}$. However, there are only s non-zero probabilities whose sum is equal to one in each row of the matrix. Thus, the transition matrix can be converted into $M \times s$ matrix with only positive entries. This transformation is feasible since an output $\bar{x}_t$ in sequence $\bar{X}$ differs from $\bar{x}_{t-1}$ solely by a single new symbol $x_{t+k-2} \in A$.

We denote by p_{ij} transition probability from the block of symbols $i \in \{0, 1, \dots, M-1\}$ to the block with the last symbol $j \in A$. The null hypothesis $\mathcal{H}_0$ can be restated as $p_{ij} = p_j$, indicating that the probability of obtaining a new symbol does not rely on the previous $k-1$ symbols. Thus, the hypothesis $\mathcal{H}_0$ assumes the transition matrix with the following structure.

$$T = \begin{bmatrix} p_0 & p_1 & \cdots & p_{s-1} \\ p_0 & p_1 & \cdots & p_{s-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_0 & p_1 & \cdots & p_{s-1} \end{bmatrix}$$

The asymptotic distribution of the Neyman–Pearson statistics Eq. (3) is χ^2-distribution that can be shown using a characteristic function [66]. There are s probabilities with the sum of 1 in each of M rows in the matrix T and it contains s parameters with one constraint. Therefore, the degrees of freedom of the NP-statistics are $M(s-1) - (s-1) = (M-1)(s-1)$ according to Theorem 4.1 of [38].

Then, we empirically show that the entropy bias Eq. (2) and the NP-statistics Eq. (3) follow χ^2-distribution with $s^k - 1$ and $(s^{k-1} - 1)(s-1)$, respectively. As defined in the main text, s is the size of the alphabet and k is the length of blocks. We consider alphabets with 2,3,4 symbols and take sequences with three different lengths, $\log_{10} n = 2, 3, 4$. The length of blocks depends on the sequence length, $k = \lfloor 0.5 \log_s n \rfloor$. For each plot we simulate $N = 10^5$ sequences. We provide the QQ-plot for entropy bias in Fig. 8.

The next two Figures show the QQ plots for calculated NP-statistics. For Fig. 9, sequences have equal probabilities of appearing symbols. In contrast, Fig. 10 displays the scenario where the probability of symbol 0 is twice as great as the probabilities of other symbols. These figures demonstrate the convergence of the empirical distribution of D to the theoretical χ^2-distribution.

Appendix B. Statistics and partitions of predictable days

This section comprises results pertaining to statistics derived from predictable and not predictable days. The results for the stock AAPL using entropy bias are in Table 5. Detailed results for the stock CCL and ETF SPY are provided in Tables 6 and 7, respectively. Finally, Table 8 shows partitions based on the test of randomness conducted for intervals of predictable days.

Table 6
Statistics for predictable and not predictable days of CCL with $a = 1$.

Parameter	Mean for predictable days	Mean for not predictable days	p-value
Sample size (Number of days)	38	42	
Number of non-zero returns	2626	2195	0.018*
Fraction of 0-returns	0.695	0.660	0.0007**
k	5.816	5.595	0.04*
$(\hat{p}(0\dots0)+\hat{p}(1\dots1))2^k$	2.5	2.25	0.118
$\lvert\hat{p}(1)-\hat{p}(0)\rvert$	0.0205	0.0203	0.956
Magnitude of daily log-price increment	0.037	0.027	0.1
Mean price returns	1.735×10^{-6}	-1.491×10^{-6}	0.016*
Magnitude of autocorrelation of non-zero returns	0.132	0.081	0.001**
Magnitude of autocorrelation of absolute values	0.286	0.234	0.001**
ν of t-distribution	1.816	1.356	0.002**
Scale of t-distribution	1.281×10^{-5}	2.868×10^{-5}	0.226
Magnitude of shift of t-distribution	1.672×10^{-7}	4.625×10^{-7}	0.194
Daily volume	7,087,714	4,776,277	0.0001**
Fraction of jumps	0.018	0.016	0.049*

* Is rejection of equal means with 0.05 significance, in the last column.
** Stands for 0.01 significance, in the last column.

Table 7
Statistics for predictable and not predictable days of SPY with $a = 5$.

Parameter	Mean for predictable days	Mean for not predictable days	p-value
Sample size (number of days)	29	51	
Number of non-zero returns	12764	10778	0.04*
Fraction of 0-returns	0.170	0.168	0.669
k	6.862	6.745	0.217
$(\hat{p}(0\dots0)+\hat{p}(1\dots1))2^k$	2.881	2.539	8.075×10^{-5}**
$\lvert\hat{p}(1)-\hat{p}(0)\rvert$	0.015	0.013	0.417
Magnitude of daily log-price increment	0.011	0.008	0.214
Mean price returns	-1.464×10^{-7}	1.080×10^{-7}	0.196
Magnitude of autocorrelation of non-zero returns	0.026	0.018	0.009**
Magnitude of autocorrelation of absolute values	0.086	0.089	0.76
ν of t-distribution	1.988	2.245	0.031*
Scale of t-distribution	5.531×10^{-5}	5.467×10^{-5}	0.792
Magnitude of shift of t-distribution	9.444×10^{-7}	8.551×10^{-7}	0.544
Daily volume	10,155,452	8,554,678	0.037*
Fraction of jumps	4.187×10^{-4}	5.052×10^{-4}	0.147

* Is rejection of equal means with 0.05 significance, in the last column.
** Stands for 0.01 significance, in the last column.

Data availability

The data were downloaded from lobsterdata.com. The authors do not have permission to share the data. The codes are available at the Github link below github.com/AndreyShternshis/predictability-at-ultra-high-frequency.

Table 8
Partition of predictable days.

Stock	Day	S_{max}	S	N. $\in [1, S]$
AAPL($a = 30$)	02.08, 05.10	2	1	1
	20.09	3	1	1
MSFT($a = 15$)	03.08	4	4	4
	05.08	3	2	1
	26.10	8	2	2
	28.10	4	1	1
TSLA($a = 65$)	05.08	2	2	**1, 2**
	14.09,15.09, 08.11	1	1	1
SNAP($a = 1$)	23.09,26.09,27.09,03.11	2	1	1
	15.09,08.11	2	2	1
	03.08,05.08,08.08,12.08,22.08, 14.09,03.10,12.10,28.10,14.11	2	2	**1, 2**
	01.08,04.08,09.08,10.08,19.08, 12.09,28.09,17.10,31.10	2	2	2
	09.09,15.11	3	1	1
	21.09	3	3	1
	06.09	3	3	**1, 2**
	16.08,10.11,11.11	3	3	1, 3
	18.08	3	3	**2, 3**
	02.08,27.10	3	3	**1, 2, 3**
	15.08,06.10	3	3	3
	11.08	4	2	1
	17.08	4	3	2
	01.11	4	4	**3, 4**
	07.09,26.10	4	4	4
	08.09	5	5	2
	20.10	5	5	**2, 3, 4, 5**
	25.10	5	5	3, 5
	01.09	5	5	**4, 5**
	21.10,24.10	7	7	**1, 2, 3, 4, 5, 6, 7**
	31.08	10	10	1, **4, 5**, 10
SPY($a = 10$)	17.08	5	5	3
	30.08	6	1	1
	26.08,18.10	7	1	1
	01.09,02.09	7	2	2
	29.09	8	1	1
	10.11	9	1	1
	13.10	14	10	10

Consecutive intervals for one day are in bold. For the stock SNAP, the results where days with $S_{max} = 1$ are omitted.

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