Random Matrix Theory & The Moment Method

Mark NG

Undergraduate Research Opportunity Programme in Science (UROPS)
Department of Mathematics
National University of Singapore

markngsayyao@gmail.com

August 18, 2016

Introduction

- MA3288 Advanced UROPS in Mathematics I, AY2015/2016 Sem 2 under Assistant Professor WANG Dong.
- Project Title:

  "*The Moment Method in Determining Limiting Densities of Singular Values of Powers of Random Matrices*"
- UROPS paper and today's slides available at my homepage:

  `http://mollymr305.github.io`

Outline

1. Random Matrix Theory
2. The Moment Method.
3. Main Results
4. Remarks
5. Selected References
6. Questions & Answers

Random Matrix Theory: What are random matrices?

- Matrices with random variable entries, e.g.

$$X_n := \begin{bmatrix} \xi_{1,1} & \xi_{1,2} & \cdots & \xi_{1,n-1} & \xi_{1,n} \\ \xi_{2,1} & \xi_{2,2} & \cdots & \xi_{2,n-1} & \xi_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \xi_{n-1,1} & \xi_{n-1,2} & \cdots & \xi_{n-1,n-1} & \xi_{n-1,n} \\ \xi_{n,1} & \xi_{n,2} & \cdots & \xi_{n,n-1} & \xi_{n,n} \end{bmatrix},$$

  where each entry ξ_{ij}, $1 \leq i, j \leq n$ are well-defined random variables.
- Several applications in Physics, Statistics etc.

Random Matrix Theory: Motivation.

- **Universality**: a phenomenon describing an overall effect of independent variables in a system.
- For example:
  - Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with mean and variance $\mu, \sigma^2 < \infty$ respectively.
  - Law of large numbers (LLN). As $n \to \infty$,

$$\frac{X_1 + X_2 + \cdots + X_n}{n} \xrightarrow{p} \mu.$$

  - Central limit theorem (CLT). As $n \to \infty$,

$$\frac{\left(\sum_{i=1}^{n} X_i\right) - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, 1).$$

Random Matrix Theory: Research.

What about random matrices?

- Most progress involves only a certain class of random matrix models, also known as *ensembles*.
- We can study various statistics of interest taken from random matrices.
- For example:
  - Concerning the smallest/largest eigenvalue.
  - 'Gaps' between eigenvalues.
  - **The limiting distribution of eigenvalues, as the dimension of the matrix grows larger.**

The Moment Method.

- A method of (mathematically) proving convergence in distribution, i.e.

$$X_n \xrightarrow{d} X.$$

- By proving that for each k-th *moment*,

$$\mathbf{E}X_n^k \to \mathbf{E}X^k.$$

- *Remark*. Some assumptions are required ("The Moment Problem").

Main Results: Wigner's Semicircle Law.

A *Wigner random matrix* is defined to be a random Hermitian matrix $X_n \in M_n(\mathbb{C})$ with independent upper triangular entries ξ_{ij}, $1 \leq i \leq j \leq n$, satisfying the following conditions:

1. For all $i = j$, we have $\mathbf{E}\xi_{ij} = 0$ and $\mathbf{E}\xi_{ij}^2 = 1$.
2. For all $i < j$, we have $\mathbf{E}\xi_{ij} = 0$ and $\mathbf{E}|\xi_{ij}|^2 = 1$.
3. For any $k \in \mathbb{N}$, there exists $M_k \in \mathbb{R}$ such that $\mathbf{E}|\xi_{ij}|^k \leq M_k$ for all $1 \leq i, j \leq n$.

Main Results: Wigner's Semicircle Law.

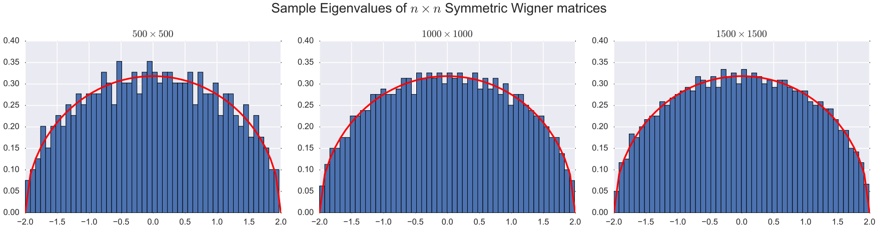
A Wigner matrix X_n will always have n real-valued eigenvalues, $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ we can construct an empirical spectral measure (a "histogram")

$$\mu_n(x) := \frac{1}{n} \sum_{i=1}^{n} \delta_{\frac{\lambda_i}{\sqrt{n}}} (-\infty, x].$$

Then we show that this *random* measure converges weakly, in probability to the semicircle density

$$\sigma(x) = \frac{1}{2\pi} \sqrt{4 - x^2} \cdot \chi_{[-2,2]}.$$

Main Results: Wigner's Semicircle Law.

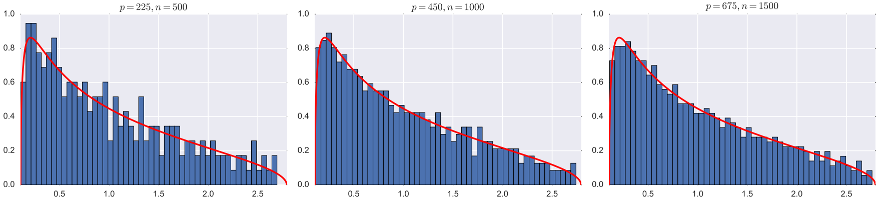


A Rough Outline:

- Wishart random matrices; of the form XX^*, where X is not necessarily a square matrix. Here, X^* denotes the conjugate transpose of X.
- Similar assumptions with regards to matrix entries.
- Construct a *similar* empirical spectral measure of eigenvalues.
- The limiting distribution is known as the *Marchenko-Pastur distribution*.

Main Results: Marchenko-Pastur Law.

Sample Eigenvalues of $p \times p$ Wishart matrices.



A Rough Outline

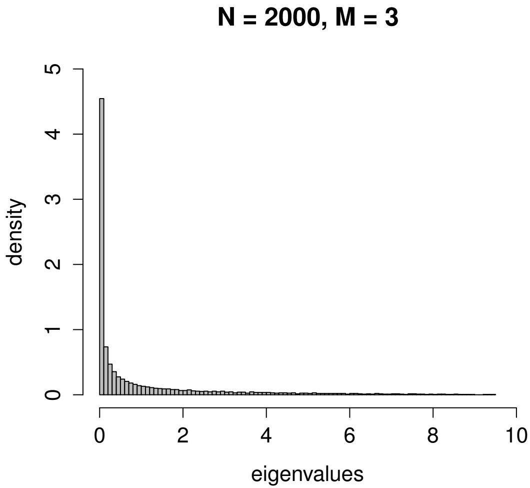
- Last result concerns matrices of the form

$$X^{(m)}X^{*(m)} := \underbrace{XX\cdots X}_{m \text{ times}} \cdot \underbrace{X^*X^*\cdots X^*}_{m \text{ times}},$$

  where m is some fixed natural number and X is an $n \times n$ complex-valued matrix.
- The limiting distribution is rather complicated, but is *uniquely defined* by its k-th moments,

$$C_{m,k} = \frac{1}{mk+1}\binom{mk+k}{k}.$$

Main Results: Powers of Random Matrices.



Remarks: Core Mathematical Ideas.

- Moment Method, in the context of random matrices:
    - The empirical spectral measure of eigenvalues (the “histogram”) is a *random variable*, or *random measure*.
    - Dependendent on the entries of the random matrix.
    - Therefore, we prove that the *expected* moments of the empirical spectral measure converge to the moments of the limiting density.
- The moments of the limiting densities are nonnegative, integer-valued.
    - For example, the even moments of the semicircle distribution correspond to the Catalan numbers.
- Not surprisingly, the proofs for all three random matrix ensembles are ultimately reduced to graph and combinatorial arguments!

Remarks: Contributions.

- Wigner's semicircle law and the Marchenko-Pastur law are well-known.
- For the last ensemble; 'Powers of Random Matrices':
  - The original proof uses the notion of (m, p)-regular graphs. See **[4]**.
  - Managed to supply an alternative proof without using this notion; by using a counting argument.

Selected References.

[1] Greg W. Anderson, Alice Guionnet, Ofer Zeitouni, *An Introduction to Random Matrices*. Cambridge University Press.

[2] Jinho Baik, *Topics in Analysis: Random Matrices, Preliminary Lecture Notes*. Department of Mathematics, University of Michigan, 2009.

[3] Benedek Valkó, *Lectures 6-7: The Marchenko-Pastur Law*. Department of Mathematics, University of Wisconsin-Madison, 2009.

[4] N. Alexeev, F. Götze, A. Tikhomirov. *Asymptotic Distribution of Singular Values of Powers of Random Matrices*. Lithuanian Mathematical Journal, Vol. 50, No. 2, 2010, pp. 121-132.

[5] Ronald L. Graham, Donald E. Knuth, Oren Patashnik. *Concrete Mathematics*. Addison-Wesley Publishing Company, 1994.

Selected References.

Useful Link:

Terence Tao, *Minerva Lectures 2013 - Terence Tao Talk 3: Universality for Wigner random matrices.*
https://www.youtube.com/watch?v=tihxQGGrMcc

Questions & Answers.

Thank you very much for your attention.