Large Deviations for Random Graphs
Honours Year Project Introductory Talk

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General information about this talk.

- Introduce fundamentals and motivation for chosen topic.

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- **Large deviation theory**

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- Slides available at:

    http://mollymr305.github.io

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Large Deviations Theory

Let's begin with *soft* introduction to large deviations theory.

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Large deviation theory is a branch of probability theory which, roughly speaking, concerns the study of two things:

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- **Conditional probabilities of events, *given* that a rare event has occurred.**

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The theory of large deviations is also applied in many fields such as statistical physics, statistical hypothesis testing, etc.

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A Coin Tossing Example

Let $X_1, \ldots, X_n$ be IID random variables with law

$$\mathbf{P}(X_i = 0) = \mathbf{P}(X_i = 1) = \frac{1}{2}.$$

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$$\mathbf{P}(X_i = 0) = \mathbf{P}(X_i = 1) = \frac{1}{2}.$$

Denote their sum by $S_n = X_1 + \cdots + X_n$.

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The Strong Law of Large Numbers (LLN) tells us that

$$\frac{S_n}{n} \xrightarrow{a.s.} \frac{1}{2}, \text{ as } n \longrightarrow \infty.$$

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The Strong Law of Large Numbers (LLN) tells us that

$$\frac{S_n}{n} \xrightarrow{a.s.} \frac{1}{2}, \text{ as } n \longrightarrow \infty.$$

i.e. "The *empirical mean* converges to $\frac{1}{2}$ with probability 1."

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We can then ask about the *asymptotic behaviour* of the probabilities rare events, for example

$$\mathbf{P}\left(\frac{S_n}{n} \geq \frac{2}{3}\right)$$

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$$\mathbf{P}\left(\frac{S_n}{n} \geq \frac{2}{3}\right) = \mathbf{P}\left(S_n \geq \frac{2}{3}n\right)$$

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It turns out that,

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbf{P}\left(S_n \geq \frac{2}{3}n\right)$$

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It turns out that,

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbf{P}\left(S_n \geq \frac{2}{3}n\right) = -\left(\log 2 + \frac{2}{3}\log\frac{2}{3} - \frac{1}{3}\log\frac{1}{3}\right)$$

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$$= -\log\left(\frac{2^{5/3}}{3}\right).$$

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In other words: the probability of the rare event

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In other words: the probability of the rare event

$$\mathbf{P}\left(S_n \geq \frac{2}{3}n\right),$$

decays exponentially as $n \longrightarrow \infty$, with rate

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More generally, for any $\alpha > \frac{1}{2}$ we have

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More generally, for any $\alpha > \frac{1}{2}$ we have

$$\lim_{n \to \infty} \mathbf{P}(S_n \geq \alpha n) = -I(\alpha),$$

where $I : \mathbb{R} \to \overline{\mathbb{R}}$, called the *rate function* is defined by

$$I(z) = \begin{cases} \log 2 + z \log z + (1 - z) \log(1 - z) & \text{if } z \in [0, 1], \\ +\infty & \text{otherwise}. \end{cases}$$

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Remark 1. We set $0 \log 0 = 0$ by convention.

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Remark 1. We set $0 \log 0 = 0$ by convention.

Remark 2. Some properties of this rate function include:

- $I(z) \geq 0$ for all $z \in \mathbb{R}$.
- $I(z) = 0$ if and only if $z = \frac{1}{2}$.

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Cramer's Theorem for the Empirical Average

A further generalisation of the Coin Tossing Example.

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Cramer's Theorem for the Empirical Average

A further generalisation of the Coin Tossing Example.

Theorem (Cramer)

Let $\{X_i\}_{i \in \mathbb{N}}$ be IID $\mathbb{R}$-valued random variables such that

$$\varphi(t) = \mathbf{E}e^{tX_i} < \infty.$$

Let $S_n = \sum_{i=1}^{n} X_i$. Then for any $\alpha > \mathbf{E}X_i$,

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbf{P}(S_n \geq \alpha n) = -I(\alpha),$$

where

$$I(z) = \sup_{t \in \mathbb{R}} \{zt - \log \varphi(t)\}.$$

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⋆ Starting point for the *general* large deviations theory.

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The Large Deviation Principle

$\star$ Starting point for the *general* large deviations theory.

Let $(\mathcal{X}, d)$ be a Polish space (i.e. separable and complete).

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Definition (Good Rate Function)

A function $I : \mathcal{X} \to [0, \infty]$ is said to be a *good rate function* if

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D2 I is lower semi-continuous.

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(iff $\liminf_{n \to \infty} I(x_n) \geq I(x)$ for any $x_n \to x$ in $\mathcal{X}$.)

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D3 I has compact level sets.

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D3 I has compact level sets.
(i.e. $I^{-1}([-\infty, c])$ is compact for every $c \in \mathbb{R}$)

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D3 I has compact level sets.
(i.e. $I^{-1}([-\infty, c])$ is compact for every $c \in \mathbb{R}$)

For any subset $S \subseteq \mathcal{X}$ we also define $I(S) = \inf\{I(s) : s \in S\}$.

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The Large Deviation Principle

Definition (LDP)

A sequence of probability measures $\{\mathbf{P}_n\}_{n\in\mathbb{N}}$ on the space $\mathcal{X}$ is said to satisfy the *Large Deviation Principle (LDP)* with rate n and good rate function I if

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L2 For any *closed* subset $C \subseteq \mathcal{X}$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbf{P}_n(C) \leq -I(C).$$

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L2 For any *closed* subset $C \subseteq \mathcal{X}$,

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L3 For any *open* subset $O \subseteq \mathcal{X}$,

$$\liminf_{n \to \infty} \frac{1}{n} \log \mathbf{P}_n(O) \geq -I(O).$$

Graph Limit Theory

We first define what is means for a graph sequences to have a limit.

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We first define what is means for a graph sequences to have a limit. *For a graph G, let $V(G)$ and $E(G)$ denote the vertex and edge set respectively.*

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We first define what is means for a graph sequences to have a limit. *For a graph G, let $V(G)$ and $E(G)$ denote the vertex and edge set respectively.*

- Let $\{G_n\}_{n\in\mathbb{N}}$ be a sequence of simple graphs whose number of vertices tend to infinity.

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- Let $\{G_n\}_{n \in \mathbb{N}}$ be a sequence of simple graphs whose number of vertices tend to infinity.

- For a fixed simple graph H, let $\text{hom}(H, G_n)$ denote the number of homomorphisms from H into G_n.

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- Let $\{G_n\}_{n\in\mathbb{N}}$ be a sequence of simple graphs whose number of vertices tend to infinity.

- For a fixed simple graph H, let $\mathrm{hom}(H, G_n)$ denote the number of homomorphisms from H into G_n.

- The probability that a random map $\phi : V(H) \to V(G_n)$ is a homomorphism is

$$t(H, G_n) = \frac{\mathrm{hom}(H, G_n)}{|V(G_n)|^{|V(H)|}}.$$

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Lovász and Szegedy proved the following:

Theorem

Suppose $t(H, G_n)$ tends to a limit $t(H)$ for every H.
Then there exists a symmetric measurable function

$$f : [0,1] \times [0,1] \to [0,1]$$

such that

$$\lim_{n \to \infty} t(H, G_n) = t(H, f) \stackrel{def.}{=} \int_{[0,1]^k} \prod_{\{i,j\} \in E(H)} f(x_i, x_j) dx_1 \cdots dx_k.$$

Conversely, every symmetric measurable function of the form
$f : [0,1] \times [0,1] \to [0,1]$ corresponds to the limit of an
appropriate graph sequence.

Graph Limit Theory

- The function f is the 'natural limit object' of $\{G_n\}_{n \in \mathbb{N}}$.

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Graph Limit Theory

- The function f is the 'natural limit object' of $\{G_n\}_{n\in\mathbb{N}}$.
- This gives us a 'working definition' for a 'graph limit'.

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Definition

A sequence of simple graphs $\{G_n\}_{n\in\mathbb{N}}$ whose number of vertices tend to infinity is said to converge to a symmetric measurable function $f : [0,1] \times [0,1] \to [0,1]$ if for every finite simple graph H,

$$\lim_{n\to\infty} t(H, G_n) = t(H, f).$$

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Erdős-Rényi Random Graphs

- The Erdős-Rényi random graph $G_{n,p}$ is a graph vertex set $\{1, \ldots, n\}$, and the law $\mathbf{P}(\{i,j\}$ is an edge $) = p$ independently for all $1 \leq i < j \leq n$.

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$$t(H, G_{n,p}) \xrightarrow{a.s.} p^{|E(H)|}, \text{ as } n \longrightarrow \infty.$$

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$$t(H, f) = \int_{[0,1]^k} \prod_{\{i,j\} \in E(H)} f(x_i, x_j) \mathrm{d}x_1 \cdots \mathrm{d}x_k = p^{|E(H)|}.$$

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- In other words, the graph sequence $\{G_{n,p}\}_{n \in \mathbb{N}}$ converges to $f \equiv p$ with probability 1.

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Large Deviation Principle for
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General information about this field of research:

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General information about this field of research:

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- However, less has been established with respect to *sparse
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Large Deviation Principle for
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Some interesting work regarding the Erdős-Rényi random
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Large Deviation Principle for Random Graphs: Research

Some interesting work regarding the Erdős-Rényi random graph:

- For an Erdős-Rényi random graph $G_{n,p}$, let $T_{n,p}$ denote the number of *triangles*.

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- For an Erdős-Rényi random graph $G_{n,p}$, let $T_{n,p}$ denote
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- It can be shown that $\mathbf{E}\, T_{n,p} = \binom{n}{3} p^3$.

- Consider the *rare* event $E_n = \{ T_{n,p} \geq (1+\delta)\mathbf{E}\, T_{n,p} \}$, for
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- Consider the *rare* event $E_n = \{T_{n,p} \geq (1+\delta)\mathbf{E}\, T_{n,p}\}$, for some $\delta > 0$.
- ? What is the asymptotic behaviour of $\mathbf{P}(E_n)$ as $n \to \infty$.

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- Consider the *rare* event $E_n = \{ T_{n,p} \geq (1 + \delta) \mathbf{E} T_{n,p} \}$, for some $\delta > 0$.
- ? What is the asymptotic behaviour of $\mathbf{P}(E_n)$ as $n \to \infty$.
- ? Can we characterize the *conditional distribution* of $G_{n,p}$ given that the rare event E_n has occurred.

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Goals for Honours Year Project

What I hope to fulfil by the end of this project:

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Goals for Honours Year Project

What I hope to fulfil by the end of this project:

- Better understand results concerning large deviations for:
  - Dense random graphs.
  - Sparse random graphs.

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- ? <u>Try</u> to answer some open questions in this field.

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Thank you very much for your attention.

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