**Question 1**

A person who has just graduated from high school can enter the workforce now (at time $t$) and earn the future income process

$$w_{t+j}^h = \delta_h^j w_t^h$$

for $j = 0 \dots T$, where $\delta_h > 1$. Alternatively, if the person goes to college and graduate school, they start working in period $t + k$. They earn nothing while in school for the $k$ periods, but have the following income process after they begin working

$$w_{t+j}^c = \delta_c^j w_t^c$$

for $j = k, \dots T$ and for $\delta_c > 1$. In either case, the agent retires at time $t + T + 1$. The discounts income at a rate $\beta \in (0, 1)$.

(a) Find a formula for the present discounted value of lifetime earnings <u>at time $t$</u> if they begin working in high school (i.e., $PV_t^h$) or if the go to college and begin working afterwords (i.e.i, $PV_t^c$). These formula should be in terms of $\beta, T, w_t^c, w_t^h, \delta_h, \delta_c$, and $k$.

(b) Assume that the consumer has period utility $u'(c) > 0, u''(c) < 0$, maximizes the present discounted value of consumption (as we did in class), and can borrow or save at an interest rate $R = 1/\beta$. Write an equation for starting college wages $w_t^c$ that makes the consumer indifferent between working now or going to college.[1]

(c) Would this indifference equation hold if the consumer could not borrow?

[1] Don't get caught up in reducing and simplifying this expression if it is difficult. I want to make sure you have set it up correctly as an implicit equation of model parameters. Another hint: you can only compare present discounted values if they reflect discounting from the same starting point (e.g. both at time $t$).

**Question 2**

There are two consumers ($i = 1, 2$) with potentially different consumption and income processes ($c_t^i$ and $y_t^i$), initial financial wealth $F_0^i = 0$, and identical preferences subject to an intertemporal budget constraint,

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t^i) \tag{1}$$

$$\text{s.t.} \sum_{t=0}^{\infty} \beta^t c_t^i = \sum_{t=0}^{\infty} \beta^t y_t^i \tag{2}$$

where $u'(c) > 0, u''(c) < 0, \beta \in (0, 1)$, and $\beta R = 1$. Assume that the two income processes are

$$y_t^1 = \{0, 1, 0, 1, \ldots\} \tag{3}$$

$$= \begin{cases} 0 & \text{if } t \text{ even} \\ 1 & \text{if } t \text{ odd} \end{cases} \tag{4}$$

$$y_t^2 = \{1, 0, 1, 0, \ldots\} \tag{5}$$

$$= \begin{cases} 1 & \text{if } t \text{ even} \\ 0 & \text{if } t \text{ odd} \end{cases} \tag{6}$$

(a) Apply the permanent income result to find $c_t^i$ for both agents.[2]

(b) For every $t$, compare $c_t^1 + c_t^2$ vs. $y_t^1 + y_t^2$. Would this comparison change if $\beta R \neq 1$? (no need to solve for the exact $c_t^i$ in that case)

(c) Assuming that both agents start with no financial wealth, i.e. $F_0^1 = F_0^2 = 0$, compute the asset trades between consumer 1 and 2 to support the $c_t^i$ where the period-by-period budget constraint for $i = 1, 2$ is

$$F_{t+1}^i = R(F_t^i + y_t^i - c_t^i)$$

[2]Hints: Note that if $a_t = \{1, 0, 1, 0 \ldots\}$ then $\sum_{t=0}^{\infty} \beta^t a_t = 1 + \beta^2 + \beta^4 + \ldots = \sum_{t=0}^{\infty} \left(\beta^2\right)^t$.

**Question 3**

Let $y_t \in \mathbb{R}$ be a sequence of dividends such that

$$y_{t+1} = y_t + w_{t+1}$$

where $w_{t+1} \sim N(\gamma, \sigma^2)$ for some $\sigma > 0$ and $\gamma \in \mathbb{R}$. i.e. $\mathbb{E}_t [w_{t+1}] = \gamma$ and $\mathbb{E}_t [(w_{t+1} - \mathbb{E}_t [w_{t+1}])^2] = \sigma^2$. An agent prices an asset as the expected PDV of dividends,

$$p_t = \mathbb{E}_t \left[ \sum_{j=0}^{\infty} \beta^j y_{t+j} \right]$$

Define the forecast error as the deviation in the expected value calculated at time $t$ from the actual value the next period. e.g. $FE^y_{t+1|t} \equiv y_{t+1} - \mathbb{E}_t [y_{t+1}]$.

(a) Convert the dividend process to one with a normalized Gaussian term, i.e. replace $w_{t+1}$ with a $\epsilon_{t+1} \sim N(0, 1)$

(b) Setup in our canonical Linear Gaussian State Space model, defining the appropriate state as $x_t$.

(c) Solve for $p_t$ in terms of $x_t$ and model intrinsics.[3]

(d) Find the expected forecast error of $x_{t+1}$: $\mathbb{E}_t \left[ FE^x_{t+1|t} \right] = \mathbb{E}_t [x_{t+1} - \mathbb{E}_t [x_{t+1}]]$

(e) Find the expected forecast error of $y_{t+1}$:[4] $\mathbb{E}_t \left[ FE^y_{t+1|t} \right] = \mathbb{E}_t [y_{t+1} - \mathbb{E}_t [y_{t+1}]]$

(f) Find the variance of forecast errors:

$$\begin{aligned}\mathbb{V}_t \left( FE^y_{t+1|t} \right) &\equiv \mathbb{E}_t \left[ \left( FE^y_{t+1|t} \right)^2 \right] - \left( \mathbb{E}_t \left[ FE^y_{t+1|t} \right] \right)^2 \\ &= \mathbb{E}_t \left[ (y_{t+1} - \mathbb{E}_t [y_{t+1}])^2 \right] - (y_{t+1} - \mathbb{E}_t [y_{t+1}])^2\end{aligned}$$

Interpret any dependence of the forecast error on the drift parameter, $\gamma$.

(g) Find the expected forecast error of $p_{t+1}$:[5] $\mathbb{E}_t \left[ FE^p_{t+1|t} \right] = \mathbb{E}_t [p_{t+1} - \mathbb{E}_t [p_{t+1}]]$

(h) Setup the problem recursively as $p_t$ define in terms of $p_{t+1}$. Solve the recursive problem with guess-and-verify, using your previous solution as a guide, and exploiting your Linear Gaussian state space setup. Feel free to leave things as matrices where appropriate.

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[3] Hint: You can use the appropriate formulas and leave it in terms of matrices if you have correctly put it into the state space.

[4] Hint: Very similar to the previous one, but you will need to use the $G$ matrix. I expect you to do the (simple) matrix algebra here.

[5] Hint: leave this in matrix form until the end, and then simplify.

**Question 4**

A government wants to minimize the following measure of tax distortions in an economy,

$$\sum_{t=0}^{\infty} \beta^t D(T_t)$$

where $\beta \in (0,1)$ and $D(T_t)$ is a measure of the costs of the distortion from $T_t$ total tax revenue at time $t$. Assume $D'(T_t) > 0$ and $D''(T_t) > 0$.

The government faces an exogenous stream of expenditures $\{G_t\}_{t=0}^{\infty}$ and faces the sequence of government budget constraints:

$$B_{t+1} = R(B_t + G_t - T_t)$$

where $B_{t+1}$ is the government debt issued at time $t$ due to be repaid at time $t+1$. Assume $B_0 = 0$. Finally, assume the government can borrow or lend at the gross interest rate $R = 1/\beta$ and that $\lim_{s\to\infty} \beta^s D'(T_s) B_s = 0$ (i.e., a no-ponzi condition).

(a) Does this remind you of any other model? If it is isomorphic to something we have done, describe the relationship in detail.

(b) Consider the expenditure process $G_t = \begin{cases} 1 & \text{if } t \text{ even} \\ 0 & \text{if } t \text{ odd} \end{cases}$

Find the optimal choice of taxes $\{T_t\}_{t=0}^{\infty}$

**Question 5**

A consumer receives an exogenous income $\{y_t\}_{t=0}^{\infty}$ that evolves according to the law of motion

$$y_{t+1} = y_t + \sigma_t \epsilon_{t+1}$$

where $\epsilon_{t+1} \sim N(0,1)$ are iid shocks, and

$$\sigma_t = \begin{cases} \bar{\sigma} > 0 & \text{for } t = 0, 1 \\ 0 & \text{for } t \geq 2 \end{cases}$$

Note the time variation of $\sigma_t$ compared to our baseline model. As usual, at time $t$ the consumer knows the full history $\{y_0, \ldots y_{t-1}, y_t\}$, but not the future values.

The consumer values consumption $c_t$ according to period utility, $u(c_t) = \alpha_0 - \frac{\alpha_1}{2} c_t^2$, where $\alpha_0, \alpha_1 > 0$. As in our standard model, the discount at rate $\beta$, save or borrow at interest rate $R = 1/\beta$, and choose consumption $c_t$ and financial wealth $F_{t+1}$ to maximize the expected present discounted value of consumption given $F_0$,

$$\max_{\{c_t, F_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) \right] \tag{7}$$

$$\text{s.t. } F_{t+1} = R\left(F_t + y_t - c_t\right), \qquad \text{for all } t \geq 0 \tag{8}$$

$$\lim_{T \to \infty} \mathbb{E}_0 \left[ \beta^T u'(c_T) F_T \right] = 0, \quad \text{Transversality Condition} \tag{9}$$

The consumer's optimality condition for this problem is the Euler equation,

$$u'(c_t) = \mathbb{E}_t \left[ u'(c_{t+1}) \right]$$

(a) Using the stochastic process above, roughly draw 5 "sample paths" for $\{y_t\}_{t=0}^{6}$ to get a sense for its dynamics.

(b) Is the stochastic process for $y_t$ first-order Markov? (i.e., only need $y_t$ to forecast $y_{t+1}$ rather than the whole history)

(c) From the Euler equation, find an expression relating consumption today in terms of expected consumption tomorrow.
In class we showed that the optimal choice of consumption (when $\beta R = 1$) in this case is,

$$c_t = (1 - \beta) \left( F_t + \mathbb{E}_t \left[ \sum_{j=0}^{\infty} \beta^j y_{t+j} \right] \right) \tag{10}$$

(d) From **??**, find a consumption function of the form:

$$c_t = \delta_0 + \delta_1 F_t + \phi_0 y_t + \phi_1 y_{t-1} \tag{11}$$

in terms of model parameters.

(e) Roughly draw the "sample paths" for $\{c_t\}_{t=0}^{6}$ from your previous sample paths of $y_t$.