

# Theory of Stochastic Interest Rates

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## 1 Simple 2-period Stochastic model

### 1.1 Basic Setup

- $t = \{0, 1\}$ , states =  $\{A, B\}$ , where state realized at  $t = 1$

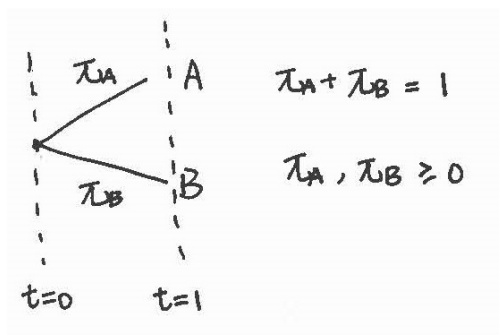


Figure 1: 2-period stochastic model

- Preferences (expected utility):

$$\mathbb{E} [u(c_0) + \beta u(c_1(s))] \tag{1}$$

$$= u(c_0) + \beta u(c_1(A)) \pi_A + \beta u(c_1(B)) \pi_B \tag{2}$$

- Endowment of Consumer (stochastic):

– At  $t = 0, y_0$

– At  $t = 1, \begin{cases} y_1(A) \text{ in state A} \\ y_1(B) \text{ in state B} \end{cases}$

- Commodities:

–  $c_0$ : consumption at  $t = 0$

- $c_1(A)$ : consumption in state  $A$  at  $t = 1$
- $c_1(B)$ : consumption in state  $B$  at  $t = 1$
- Prices:
  - $q_0$ : price of one unit of consumption at  $t = 0$  (No risk or uncertainty to resolve)
  - $q_1(A)$ : price of one unit of consumption at  $t = 1$  in state  $A$
  - $q_1(B)$ : price of one unit of consumption at  $t = 1$  in state  $B$

## 1.2 Consumer's problem

- For consumer, solve:

$$\max_{\{c_0, c_1(A), c_1(B)\}} u(c_0) + \beta u(c_1(A))\pi_A + \beta u(c_1(B))\pi_B \quad (3)$$

$$\text{s.t. } q_0 c_0 + q_1(A) c_1(A) + q_1(B) c_1(B) \leq q_0 y_0 + q_1(A) y_1(A) + q_1(B) y_1(B) \equiv w_0 \quad (4)$$

- Form Lagrangian:

$$\mathcal{L} = u(c_0) + \beta u(c_1(A))\pi_A + \beta u(c_1(B))\pi_B + \underbrace{\lambda}_{L.M.} \underbrace{[w_0 - q_0 c_0 - q_1(A) c_1(A) - q_1(B) c_1(B)]}_{\text{Present value of consumption}} \quad (5)$$

- F.O.N.C:

$$[c_0] : u'(c_0) - \lambda q_0 = 0 \quad (6)$$

$$[c_1(A)] : \beta u'(c_1(A))\pi_A - \lambda q_1(A) = 0 \quad (7)$$

$$[c_1(B)] : \beta u'(c_1(B))\pi_B - \lambda q_1(B) = 0 \quad (8)$$

And budget constraint:

$$w_0 - q_0 c_0 - q_1(A) c_1(A) - q_1(B) c_1(B) = 0 \quad (9)$$

By using (6), (7) and (8), we have:

$$\lambda = \frac{u'(c_0)}{q_0} \quad (10)$$

$$\beta u'(c_1(A))\pi_A = u'(c_0) \cdot \frac{q_1(A)}{q_0} \quad (11)$$

$$\beta u'(c_1(B))\pi_B = u'(c_0) \cdot \frac{q_1(B)}{q_0} \quad (12)$$

where  $q_0$  is an initial price level, just rescales all. Normalize  $q_0 = 1$  using (10). Consumer's take prices  $q_1(A)$ ,  $q_1(B)$  as given

$$q_1(A) = \beta \underbrace{\frac{u'(c_1(A))}{u'(c_0)}}_{\substack{\text{Marginal} \\ \text{utility} \\ \text{of allocation}}} \pi_A \quad (13)$$

$$q_1(B) = \beta \frac{u'(c_1(B))}{u'(c_0)} \underbrace{\pi_B}_{\text{probabilities}} \quad (14)$$

This is the Lucas '78 formulas for 2-period. If there is a representative consumer, then  $c_1(A) = y_1(A)$ , eats full endowment.

- Risk free:

$$q_1^{\text{RF}} = q_1(A) + q_1(B) = \frac{\beta}{u'(c_0)} [u'(c_1(A))\pi_A + u'(c_1(B))\pi_B] \quad (15)$$

$$= \underbrace{\beta \frac{\mathbb{E}[u'(c_1(s))]}{u'(c_0)}}_{\substack{\text{Ratio of average:} \\ \text{Expected marginal utility} \\ \text{to marginal utility today.}}} \quad (16)$$

where the risk free gross interest rate is:  $R^{\text{RF}} = \frac{1}{q_1^{\text{RF}}}$

**Does this Extend to Infinite Horizon?** Without proof, yes. Assume that the state process  $\{s_t\}$  is Markov, with transition probabilities  $\pi(s_{t+1}|s_t)$  and  $\pi(A|s_t) + \pi(B|s_t) = 1$ , then (14) and (16) become

$$q_t(s_{t+1}) = \beta \frac{u'(c_{t+1}(s_{t+1}))}{u'(c_t(s_t))} \underbrace{\pi(s_{t+1}|s_t)}_{\substack{\text{Conditional} \\ \text{Probability}}} \quad (17)$$

$$q_t^{\text{RF}} = \beta \frac{\mathbb{E}_t[u'(c_{t+1}(s_{t+1}))]}{u'(c_t(s_t))} \quad (18)$$

Section 2 and beyond will prove this in more generality, for claims on both period-by-period, and time-0 contingent claims.

## 2 Asset Trees

### 2.1 Introduction

- Consider a 2-state Markov chain with states  $S = \{\bar{s}_1, \bar{s}_2\}$  and transition matrix:

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \text{ where } p_{ij} = \text{Prob}(s_{t+1} = \bar{s}_j \mid s_t = \bar{s}_i) \quad (19)$$

$$\text{And } \pi_0 = \begin{pmatrix} \pi_{01} \\ \pi_{02} \end{pmatrix} = \begin{bmatrix} \text{Prob}(s_0 = \bar{s}_1) \\ \text{Prob}(s_0 = \bar{s}_2) \end{bmatrix} \text{ as initial probability distribution over states} \quad (20)$$

where  $\pi_{0,i} \geq 0$  and  $\sum_{i=1}^2 \pi_{0,i} = 1$ ,  $p_{i,j} \geq 0$  and  $\sum_{j=1}^2 p_{i,j} = 1$  (i.e. probabilities)

- Denote  $s_0^t$  as sequence of states from 0 to t. For example:

$$s_0^t = \{s_0, s_1, \dots, s_t\} \text{ for some } s_\tau \in S, \forall \tau = 0, \dots, t \quad (21)$$

This is an information set, a history.

- Example:

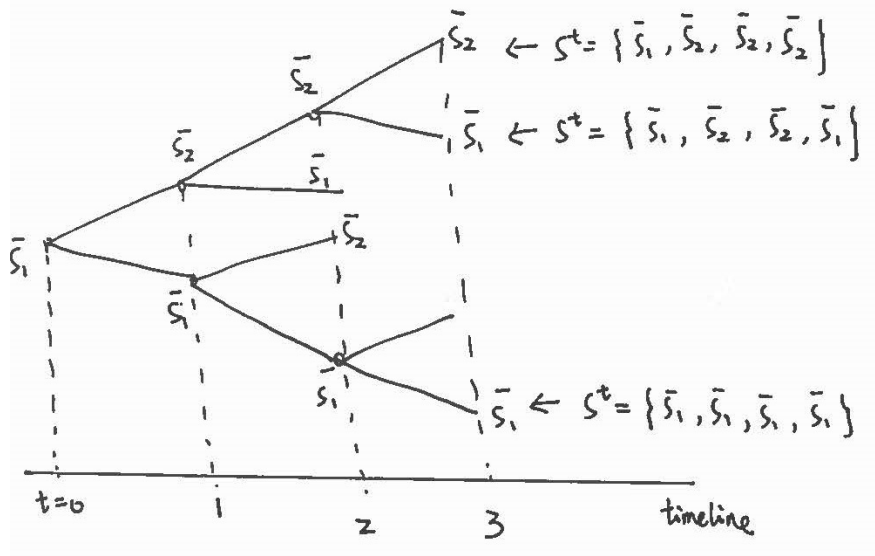


Figure 2: Asset Tree Example given  $\bar{s}_1$  at  $t = 0$

### 2.2 Probabilities of those jumps:

- In this way, build probabilities over histories  $s_0$ , which is denoted for simplicity as  $s^t$ .

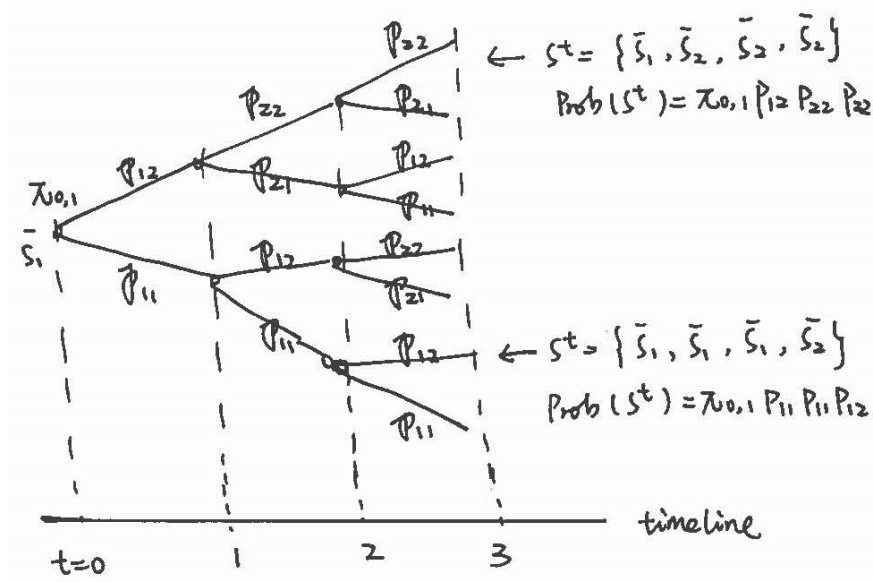


Figure 3: Jump Probabilities in Example

- Denote the probability of a particular history as

$$\text{Prob}(s_t = \bar{s}^t) \equiv \pi_t(s^t) \text{ (e.g. with Markov chain)} \quad (22)$$

$$\pi_t(\{\bar{s}_i, \bar{s}_j, \bar{s}_k, \bar{s}_\ell, \dots\}) = \underbrace{\pi_{0,i}}_{\text{Probability start in } i} \underbrace{p_{ij}}_{\text{Jump } i \text{ to } j} \underbrace{p_{jk}}_{\text{Jump } j \text{ to } k} p_{kl} \dots \quad (23)$$

- Note that:

$$\underbrace{\sum_{s^t}}_{\text{sum of all possible histories to time } t} \underbrace{\pi_t(s^t)}_{\text{Probability of history } s^t} = 1, \forall t, \text{ and } \pi_t(s^t) \geq 0 \quad (24)$$

## 2.3 Modelling risk

- At time 0, only knows  $s_0$ , does not know  $s_1, s_2, \dots$  but does know that at time  $t$  the history  $s^t$  might happen, and assigns probability  $\pi_t(s^t)$  that it will.
- Consumer's utility for a particular sequence of  $\{s_t\}$ :

$$\sum_{t=0} \beta^t u(c_t(s^t)), \text{ where } c_t(s^t) \text{ is consumption at } \underline{t} \text{ for } \underline{\text{history}} \ s^t \quad (25)$$

Hence, consumption can depend on a particular history of the state.

- But at time 0, the agent only has probabilities over possible histories. Then consumer's

expected utility at time 0 is:

$$\sum_t^T \sum_{s^t} \underbrace{\beta^t u(c_t(s^t))}_{\substack{\text{PDV given} \\ s^t \text{ happens}}} \underbrace{\pi_t(s^t)}_{\substack{\text{Probability} \\ s^t \text{ happens}}} \quad (26)$$

i.e. average discounted utilities:  $\beta^t u(c_t(s^t))$  across histories  $s^t$  at  $t$ , using probabilities  $\pi_t(s^t)$  as weights.

- Generalize the notion of the risk-free bond maturing at time  $t$  to be:

History-Date Contingent Claim on Consumption

- Claim on consumption:

$$\underbrace{c_t}_{\text{time } t} \underbrace{(s^t)}_{\text{history } s^t} \quad (27)$$

for every  $t$  every  $s^t$  (Huge number of assets)

### 3 Complete Markets (all assets exist)

#### 3.1 Basic setup

- At time 0, the consumer can buy or sell  $c_t(s^t)$  at price  $q_t^0(s^t)$ , which is the claim to 1 unit of consumption at  $t$  given history  $s^t$  and there is no return in other states.
  - We could have assets which pay in several states, but they would be *spanned* by this simple set of assets.
- Consumer faces  $\{q_t^0(s^t)\}$  as a price taker.

#### 3.2 Consumer Problem

- Consumers solve:

$$\max_{\{c_t(s^t)\}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t(s^t)) \pi_t(s^t) \quad (28)$$

$$\text{s.t. } \sum_{t=0}^{\infty} \underbrace{q_t^0(s^t)}_{\text{price}} \underbrace{c_t(s^t)}_{\text{quantity}} \leq \underbrace{w_0}_{\text{time 0 wealth}} \quad (29)$$

- One budget constraint in a Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t(s^t)) \pi_t(s^t) + \lambda \left( w_0 - \sum_{t=0}^{\infty} \sum_{s^t} q_t^0 c_t(s^t) \right) \quad (30)$$

where  $\lambda$  is Lagrangian Multiplier, will be the "marginal utility of wealth".

- Take the FONC into  $c_t(s^t)$

$$\beta^t u'(c_t(s^t)) \pi_t(s^t) = \lambda \underbrace{q_t^0(s^t)}_{\substack{\text{given as} \\ \text{price taker}}} , \text{ where } \lambda \text{ is to be determined from budget constraint} \quad (31)$$

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t q_t^0(s^t) c_t(s^t) = w_0 \quad (32)$$

### 3.3 Example with constant aggregate endowments

- Guess that  $q_t^0(s^t) = \beta^t \pi_t(s^t)$
- Then by (31):

$$\beta^t u'(c_t(s^t)) \pi_t(s^t) = \lambda \beta^t \pi_t(s^t) \quad (33)$$

$$\Rightarrow \lambda = u'(c_t(s^t)) \quad (34)$$

$$\Rightarrow q_0^0 = 1 \quad (35)$$

So if  $c_t(s^t) = \bar{c}$ , constant  $\forall t, \forall s^t$

- And from (32):

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) \bar{c} = w_0 \quad (36)$$

$$= \bar{c} \sum_{t=0}^{\infty} \beta^t \underbrace{\left[ \sum_{s^t} \pi_t(s^t) \right]}_{\substack{\text{must sum to 1} \\ \text{since a probability}}} \quad (37)$$

$$= \bar{c} \sum_{t=0}^{\infty} \beta^t \quad (38)$$

$$\Rightarrow \boxed{\bar{c} = w_0(1 - \beta)} \quad (39)$$

which means complete consumption smoothing across time and across history.

- What is price of 1 unit of consumption with certainty at time  $t$ ?

$$\sum_{s^t} q_t^0(s^t) \equiv \bar{q}_t^0 \quad (40)$$

$$= \sum_{s^t} \beta^t \pi_t(s^t) = \beta^t = \bar{q}_t^0 \quad (41)$$

We can compare this to the risk-free interest rate with constant endowment.

## 4 Lucas 1978 Model

### 4.1 Basic setup

- Pure endowment representative agent economy:

$$c_t(s^t) = y_t(s^t), \text{ which is exogenous stochastic} \quad (42)$$

i.e. In equilibrium, the representative consumer will consume the entire endowment as a price taker with  $q_t^0(s^t)$  prices. (42) is the feasible condition.

- Substitute (42) into FONC (32):

$$\beta^t u'(y_t(s^t)) \pi_t(s^t) = \lambda q_t^0(s^t) \quad (43)$$

$$q_t^0(s^t) = \frac{1}{\lambda} \beta^t u'(y_t(s^t)) \pi_t(s^t) \quad (44)$$

So if (44), then FONC hold.

- Budget constraint:

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t(s^t) \quad (45)$$

this holds immediately for any  $\lambda$  (i.e. price taker)

### 4.2 Competitive Equilibrium in Lucas 1978

- Given identical agents with exogenous stochastic endowment  $y_t(s^t)$  and complete markets.
- A feasible allocation  $\{c_t(s^t)\}$ :

$$c_t(s^t) \leq y_t(s^t), \forall t, s^t \quad (46)$$



and a price system  $\{q_t^0(s^t)\}$  is a competitive equilibrium if given  $\{q_t^0(s^t)\}$ ,  $\{c_t^0(s^t)\}$  solves the consumer's problem.

### 4.3 Example

- Two consumers ( $i = 1, 2$ ) with identical preferences:

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t^i(s^t)) \pi_t(s^t) \quad (47)$$

Note: no  $i$  on  $\beta^t$ ,  $u$  and  $\pi_t(s^t)$ . Same probabilities for all consumers.

– State:  $S = \{0, 1\}$

– Endowments:

$$y_t^1(s_t) = s_t \text{ and } y_t^2(s_t) = 1 - s_t \quad (48)$$

$$y_t^1(s_t) + y_t^2(s_t) = 1, \forall t, s^t \text{ (i.e. No aggregate risk)} \quad (49)$$

– Feasible allocation  $\{c_t^i(s^t)\}$ :

$$c_t^1(s^t) + c_t^2(s^t) \leq y_t^1(s^t) + y_t^2(s^t), \forall t, s^t \quad (50)$$

– Price system  $\{q_t^0(s^t)\}$  is same for all  $i$ .

- A competitive equilibrium is a price system and feasible allocation such that given the price system, the allocations solve each households problem for each  $i$ :

$$\max_{\{c_t^i(s^t)\}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t^i(s^t)) \pi_t(s^t) \quad (51)$$

$$\text{s.t. } \sum_{t=0}^{\infty} \sum_{s^t} s^t q_t^0(s^t) (c_t^i(s^t) - y_t^i(s^t)) \quad (52)$$

- How to solve?

– Difficult in general. Here, guess and verify:

$$q_t^0(s^t) = \beta^t \pi_t(s^t) \text{ (i.e. Same guess as a representative agent)} \quad (53)$$

– At guess, FONC for  $i$ :

$$\beta^t u'(c_t^i(s^t)) \pi_t(s^t) = \lambda_i \beta^t \pi_t(s^t) \quad (54)$$

$$\Rightarrow u'(c_t^i(s^t)) = \lambda_i, \forall i = 1, 2 \quad (55)$$

$$\Rightarrow c_t^i(s^t) = c^i \text{ (Perfect smoothing. LM is } i \text{ independent)} \quad (56)$$

– Feasibility:

$$c^1 + c^2 = 1 \quad (57)$$

– From budget constraint:

$$c^i \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi_t(s^t) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t y_t^i(s^t) \pi_t(s^t) \equiv w_0^i \quad (58)$$

$$\Rightarrow c^i = (1 - \beta) w_0^i \quad (59)$$

– Note:

$$w_0^1 + w_0^2 = \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi(s^t) \underbrace{\left[ y_t^1(s^t) + y_t^2(s^t) \right]}_I \quad (60)$$

$$= \sum_{t=0}^{\infty} \beta^t = \frac{1}{1 - \beta} \quad (61)$$

This is as far as we can get without specifying a particular  $\pi_t(s^t)$  process.

## 5 Complete vs Incomplete Market

### 5.1 Compete market

- Assume there exist assets for every possible history, i.e.  $q_t(s^t)$ , this is complete markets
- Consumer:

$$\max_{\{c_t(s^t)\}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t(s^t)) \cdot \pi_t(s^t) \quad (62)$$

$$\text{s.t. } \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) \left[ c_t(s^t) - y_t(s^t) \right] = 0 \quad (63)$$

where  $y_t$  is endowment in history  $s^t$

- FONC:

$$\beta^t u'(c_t(s^t)) \pi_t(s^t) = \lambda q_t^0(s^t) \quad (64)$$

Divide for histories at  $t, t + 1$ :

$$\frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)} = \beta \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} \cdot \frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)} \quad (65)$$

Note:  $\frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)} \equiv \pi_{t+1}(s^{t+1} \mid s^t)$ , which is the conditional probability of  $s^{t+1}$  given  $s^t$ .

- Note:

- Conditional probabilities are very easy to calculate here if markov, since it will only depend on last state. i.e.  $\pi_{t+1}(s^{t+1} \mid s^t) = \pi(s_{t+1} \mid s_t)$ , with markov chain  $[\pi_{ij}] = p$ , there are just transition probabilities  $p_{ij}$

- For example:  $S = \{A, B\}$ ,  $P = \begin{bmatrix} \pi_{AA} & \pi_{AB} \\ \pi_{BA} & \pi_{BB} \end{bmatrix}$ , then:

$$\pi_{t+1} \left( \left\{ s^{t+1} = s_2, s^t = s_1, \dots \right\} \mid \left\{ s^t = s_1, \dots \right\} \right) = \pi_{s_1 s_2} \quad (66)$$

- Define:

$$\frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)} \equiv q_{t+1}^t(s^{t+1} \mid s^t) \quad (67)$$

as the one-step ahead "pricing kernel". i.e. price at time  $t$  of  $t + 1$  consumption in state  $s^{t+1}$  given state  $s^t$  happened. So write pricing equation as:

$$\boxed{q_{t+1}^t(s^{t+1} \mid s^t) \equiv \beta \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} \pi_{t+1}(s^{t+1} \mid s^t)} \quad (68)$$

- Price, in history  $s^t$ , of a unit of consumption at time  $t + 1$  with certainty?

- Buy assets for every possible state

$$\underbrace{\sum_{s^{t+1} \mid s^t} q_{t+1}^t(s^{t+1} \mid s^t)}_{\substack{\text{price at node } s^t \\ \text{of a risk-free} \\ \text{claim to consumption} \\ \text{at } t + 1}} \equiv \underbrace{(R_t(s^t))^{-1}}_{\substack{\text{reciprocal of gross} \\ \text{one period} \\ \text{interest rate} \\ \text{of node } s^t}} \quad (69)$$

With (68):

$$1 = \beta R_t(s^t) \sum_{s^{t+1}|s^t} \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} \cdot \pi_t(s^{t+1} | s^t) \quad (70)$$

A little looser notation:

$$1 = \beta R_t \mathbb{E}_t \left[ \frac{u'(c_{t+1})}{u'(c_t)} \right] \quad (71)$$

Compare to permanent income hypothesis with exogenous  $R$ . Shows  $\beta R = 1$  is the natural solution if the consumer is able to achieve perfect income smoothing.

- Calculating risk free interest rates from the data:
  - Assume representative consumer with aggregate  $y_t(s^t)$
  - Determine probabilities for process you believe  $y_t(s^t)$  takes, then:

$$\frac{1}{R_t} = \beta \mathbb{E}_t \left[ \frac{u'(y_{t+1})}{u'(y_t)} \right] \quad (72)$$

using the process, for  $y_{t+1}$  from  $y_t$  for expectations.

- Note that for risk neutral consumers:

$$u(c) = c \cdot A \Rightarrow u'(c) = A \quad (73)$$

$$\Rightarrow \boxed{R\beta = 1}, \text{ for any stochastic process} \quad (74)$$

- Note that the complete markets pricing kernel have all the above but not vice versa. The above only holds "on average".

## 5.2 Incomplete Markets

- Motivation:
  - What if not all of the markets exist for all  $s^t$ ? Then cannot smooth completely at time 0
  - Extreme version, can only buy a 1-period risk free bond paying interest rate  $R_t(s^t)$ , holdings  $A_t(s^{t-1})$

- For consumer, given assets  $\underbrace{A_t}_{\text{Asset today}} \cdot \underbrace{(s^{t-1})}_{\text{previous history } s^{t-1}}$

$$\max_{\{c_t(s^t), A_{t+1}(s^t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} u(c_t(s^t)) \pi_t(s^t) \quad (75)$$

$$\text{s.t. } \underbrace{A_{t+1}(s^t)}_{\text{bond holdings tomorrow}} = \underbrace{R_t(s^t)}_{\substack{\text{interest paid on} \\ \text{holding} \\ \text{risk-free}}} \left[ \underbrace{y_t(s^t) - c_t(s^t)}_{\text{saving}} + \underbrace{A_t(\underbrace{s^{t-1}}_{\substack{\text{set from } s^{t-1} \text{ history} \\ \text{previous assets}}})}_{\text{previous assets}} \right] \quad (76)$$

Note: instead of 1 budget constraint, we have  $\lambda_t(s^t)$  possible multipliers.

- Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t(s^t)) \pi_t(s^t) \quad (77)$$

$$+ \sum_{t=0}^{\infty} \sum_{s^t} \lambda_t(s^t) \left[ A_t(s^{t-1}) + y_t(s^t) - c_t(s^t) - R_t^{-1}(s^t) A_{t+1}(s^t) \right] \quad (78)$$

- FONC:

$$[C_t(s^t)] : \beta^t u'(c_t(s^t)) \pi_t(s^t) = \lambda_t(s^t) \quad (79)$$

$$[A_{t+1}(s^t)] : -\lambda_t(s^t) R_t^{-1}(s^t) + \underbrace{\sum_{s^{t+1}|s^t} \lambda_{t+1}(s^{t+1})}_{\text{any might show up}} \quad (80)$$

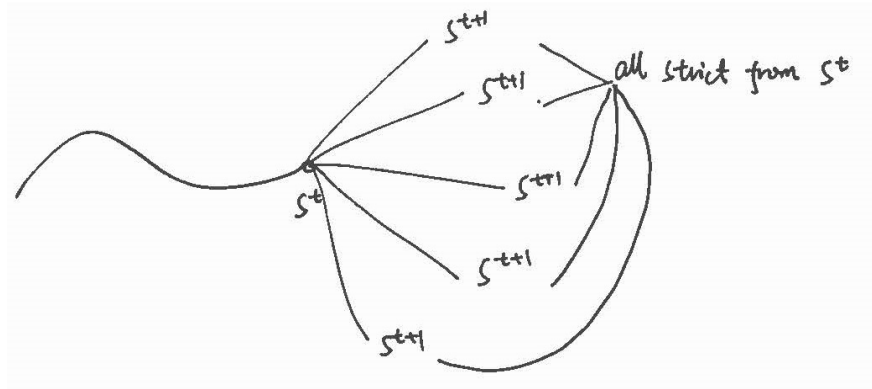


Figure 4: Example Path Conditioned on History  $s^t$

- Substitute (80) to (79)

$$R_t^{-1}(s^t)\beta^t u'(c_t(s^t)) \pi_t(s^t) = \beta^{t+1} \sum_{s^{t+1}|s^t} u'(c_{t+1}(s^{t+1})) \pi_{t+1}(s^{t+1}) \quad (81)$$

$$\Rightarrow 1 = \beta R_t(s^t) \sum_{s^{t+1}|s^t} \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} \cdot \pi_{t+1}(s^{t+1} | s^t) \quad (82)$$

$$\Rightarrow 1 = \beta \mathbb{E}_t \left[ R_t \frac{u'(c_{t+1})}{u'(c_t)} \right], \text{ same as the risk free calculated under complete market} \quad (83)$$

### 5.3 Punchlines:

- Under complete markets, intertemporal marginal rates of substitution:

$$\beta \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} \pi_t(s^{t+1} | s^t) \quad (84)$$

are equated for all consumers able to trade at relative prices  $q_{t+1}^t(s^{t+1} | s^t)$

- Under incomplete markets with only a risk-free security with gross returns  $R_t(s^t)$ , only the average intertemporal rates of substitutes.

$$\beta \sum_{s^{t+1}|s^t} \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} \pi_t(s^{t+1} | s_1^t) \quad (85)$$

are equated across consumers

- Permanent Income Hypothesis in Incomplete markets

- Stochastic  $y_t(s^t)$  with incomplete markets:

$$\underbrace{\frac{1}{R_t}}_{\text{interest rate}} = \beta \mathbb{E}_t \left[ \underbrace{\frac{u'(y_{t+1}(s^{t+1}))}{u'(y_t(s^t))}}_{\text{can use aggregate endowment}} \right] \quad (86)$$

- If markets were complete, consumers would eat a constant share of aggregate output.