# DMP model in continuous time
# Extension (I): Endogenous job-destruction rate

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## 1 Model

This model is originated from Mortensen and Pissarides (1994), where the firm decides whether to fire the worker based on the idiosyncratic productivity shocks.

### 1.1 Assumptions

(1). Household lives **infinitely** with **risk neutral** utility. Both household and firm discount the future at the rate $\rho$.

(2). March-specific productivity: $p + \varepsilon,\ \varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon}] \sim F(\varepsilon)$

- $p$ is constant and exogenous;
- Productivity shock rate: $\lambda$. $\varepsilon$ follows the Poisson process;
- **New-filled job** always produce with the highest productivity $\bar{\varepsilon}$.

(3). Match process: **CRS** matching technology, $m(u, v)$

(4). Search process: **Undirected search**.

- Match arrival rate: $f(\theta) = \frac{m(u,v)}{u}, q(\theta) = \frac{m(u,v)}{v}$
- **Endogenous** job destruction rate: $s = \lambda \cdot F(\varepsilon^*)$[1]

### 1.2 Labor demand (Firm)

(1) Production function:
Assume that the labor is the only input in production[2], and for realized match-specific productivity $\varepsilon$, the production per match is given by:

$$y(\varepsilon) = p + \varepsilon$$

The marginal cost of filling a job is the equilibrium wage $w(\varepsilon)$, which will be specified in Section 3.1.5. The marginal cost of posting a vacancy $\xi$ is constant and exogenous.

(2) Value function for filled job:

$$\rho J^F(\varepsilon) = p + \varepsilon - w(\varepsilon) + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} \left[J^F(\tilde{\varepsilon}) - J^V\right] dF(\tilde{\varepsilon}) - \lambda \left[J^F(\varepsilon) - J^V\right] \tag{1}$$

See the proof in the appendix.

(3) Value function for vacancy:

$$\rho J^V = -\xi + q(\theta)\left[J^F(\bar{\varepsilon}) - J^V\right] \tag{2}$$

[1] The $\varepsilon^*$ is the threshold of productivity shock for firing the worker, see details in Section 3.1.6.
[2] The linear productivity function is a common assumption in the search literature. For example, Postel–Vinay and Robin (2002) and Cahuc, Postel-Vinay, and Robin (2006) use a similar functional form for the flow productivity

The firm's value function for vacant job is derived same as in the baseline model with the additional asumption that the newly filled job always produced at $p+\bar{\varepsilon}$. Therefore, the value function $J^V$ is invariant to the match-specific productivity shock.

(4) Firm's decision:
The firm has three different decisions to make in this model: create vacancy, hire the worker and fire the worker. The first two decisions are the same as in baseline model, such that the first decision is characterized by the free entry condition, and the second decision is implicitly solved by imposing the free entry condition (see analysis in the baseline model). The third decision is the new element in this setup, and the firm needs to decide whether to (1) keep the job ($J^F(\varepsilon)$) or (2) destruct the job and turn it back to vacancy ($J^V$) if it receives negative productivity shocks. The productivity shock arriving at rate $\lambda$ will change the match specific productivity $p+\varepsilon$: if the productivity is too low, the firm has incentive to destruct it, which generates the endogenous job destruction rate (see detailed discussion in Section 3.1.6).

## 1.3 Labor supply (Household)

(1) Value function for employed worker:

$$\rho V^e(\varepsilon) = w(\varepsilon) + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} \left[V^e(\tilde{\varepsilon}) - V^u\right] dF\left(\tilde{\varepsilon}\right) - \lambda \left[V^e(\varepsilon) - V^u\right] \tag{3}$$

See the proof in the appendix.

(2) Value function for unemployed worker:

$$\rho V^u = b + f(\theta)\left[V^e(\bar{\varepsilon}) - V^u\right] \tag{4}$$

Same as in labor demand, the value function $V^u$ is invariant to the match-specific productivity shock, and it is derived in the same way as in baseline model.

(3) Worker's decision:
The household has only one decision to make in this model: accept or reject the job offer. Same as in baseline model, the household will always accept the wage offer because of free entry condition and Nash bargaining with linear utility, and one can abstract from the optimal stopping problem in the value functions of household.

## 1.4 Free entry

The firms will continue to create vacancies until the expected profit equals 0:

$$J^V = 0 \Rightarrow J^F(\bar{\varepsilon}) \overset{(1)}{=} \frac{\xi}{q(\theta)} \overset{(2)}{=} \frac{p + \bar{\varepsilon} - w(\bar{\varepsilon}) + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} J^F(\tilde{\varepsilon}) dF(\tilde{\varepsilon})}{\rho + \lambda} \tag{5}$$

The free entry condition pins down the equilibrium labor market tightness.

## 1.5 Wage determination

The risk neutrality of worker implies that the bargained wage satisfies the following condition

$$(1-\beta)\left[V^e(\varepsilon) - V^u\right] = \beta \cdot \left[J^F(\varepsilon) - J^V\right]$$

Solving the equation above with the value functions derived before and the free entry condition (see details in the appendix), we have:

$$w(\varepsilon) = (1-\beta)b + \beta \cdot \left[p + \varepsilon + \theta\xi\right] \tag{6}$$

Therefore, the workers receive their reservation wage $b$ and a share $\beta$ of the net surplus of the current match, which is the total productivity $p + \varepsilon + \theta\xi$ minus what workers give up $b$.

### 1.6 Endogenous destruction rate: $\varepsilon^*$

The endogenous destruction rate implies that the firm's problem is indeed an optimal stopping problem, and the firm decides whether to destruct the job according to the match-specific productivity $p+\varepsilon$. At the reservation productivity $\varepsilon^*$, the firm is indifferent between keeping and destructing the job:

$$J^F(\varepsilon^*) = J^v = 0$$

Therefore, the reservation productivity $\varepsilon^*$ satisfies the following condition:

$$p + \varepsilon^* - b - \frac{\beta}{1-\beta}\theta\xi + \frac{\lambda}{\rho+\lambda}\int_{\varepsilon^*}^{\bar{\varepsilon}} [\tilde{\varepsilon} - \varepsilon^*]\, dF(\tilde{\varepsilon}) = 0 \tag{7}$$

And it implies that the free entry condition could be written as:

$$J^F(\bar{\varepsilon}) = \frac{\xi}{q(\theta)} \overset{(3)}{=} \frac{1-\beta}{\rho+\lambda}[\bar{\varepsilon} - \varepsilon^*] \tag{8}$$

## 2 Stationary Equilibrium (Recursive form)

Denote $(z, \varepsilon)$ the state variable for household where $z \in \{e, u\}$ indicates the employment status of the agent, and $\varepsilon$ indicates the match-specific productivity if the agent is employed[3]. Similarly, denote $(\omega, \varepsilon)$ the state variable for firm, where $\omega \in \{F, V\}$ indicates whether the vacant job is filed or not.

A stationary equilibrium consists of:

- a set of policy functions $\{c(z, \varepsilon)\}$;
- a set of value functions $\left\{V^e(\varepsilon), V_u, J^F(\varepsilon), J^v\right\}$;
- a stationary distribution over unemployment and vacancy: $\{u, v\}$;
- a set of prices $\{w(\varepsilon)\}$;
- a set of endogenous labor market parameters $\{\varepsilon^*, \theta\}$

such that:

(1) UMP: the policy function $\{c(z, \varepsilon)\}$ solve the household's utility maximization problem. Since there is no saving decision, the households are hand-to-mouth

$$c(z, \varepsilon) = \begin{cases} w(\varepsilon), & \text{if } z = e \\ b, & \text{if } z = u \end{cases} \tag{9}$$

with the corresponding value functions:

$$\begin{aligned} \rho V^e(\varepsilon) &= w(\varepsilon) + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} [V^e(\tilde{\varepsilon}) - V^u]\, dF(\tilde{\varepsilon}) - \lambda [V^e(\varepsilon) - V^u] \\ \rho V^u &= b + f(\theta)[V^e(\bar{\varepsilon}) - V^u] \end{aligned}$$

(2) PMP: firms decide to destruct the job optimally when $\varepsilon < \varepsilon^*$ such that

$$p + \varepsilon^* - b - \frac{\beta}{1-\beta}\theta\xi + \frac{\lambda}{\rho+\lambda}\int_{\varepsilon^*}^{\bar{\varepsilon}} [\tilde{\varepsilon} - \varepsilon^*]\, dF(\tilde{\varepsilon}) = 0 \tag{10}$$

with corresponding value functions:

$$\begin{aligned} \rho J^F(\varepsilon) &= p + \varepsilon - w(\varepsilon) + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} \left[J^F(\tilde{\varepsilon}) - J^V\right] dF(\tilde{\varepsilon}) - \lambda \left[J^F(\varepsilon) - J^V\right] \\ \rho J^V &= -\xi + q(\theta)\left[J^F(\bar{\varepsilon}) - J^V\right] \end{aligned}$$

[3]Check whether $\varepsilon$ is the state variable.

(3) Nash bargaining: the equilibrium wage is determined by bargaining between the firm and the worker

$$w(\varepsilon) = (1-\beta)b + \beta \cdot [p + \varepsilon + \theta\xi] \tag{11}$$

(4) Stationary distribution:

$$\dot{u} = 0 \Rightarrow \lambda F(\varepsilon^*)(1-u) - f(\theta)\cdot u = 0 \Rightarrow u = \frac{\lambda F(\varepsilon^*)}{\lambda F(\varepsilon^*) + f(\theta)} \tag{12}$$

$$\dot{v} = 0 \Rightarrow \lambda F(\varepsilon^*)(1-u) - q(\theta)\cdot v = 0 \Rightarrow v = \theta u \tag{13}$$

(5) Free Entry: firms create vacancies at zero-profits

$$J^V = 0 \Rightarrow J^F(\bar{\varepsilon}) \overset{(1)}{=} \frac{\xi}{q(\theta)} \overset{(2)}{=} \frac{p + \bar{\varepsilon} - w(\bar{\varepsilon}) + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} J^F(\tilde{\varepsilon}) dF(\tilde{\varepsilon})}{\rho + \lambda} \overset{(3)}{=} \frac{1-\beta}{\rho+\lambda}[\bar{\varepsilon} - \varepsilon^*] \tag{14}$$

There are 6 unknown equilibrium object $\{c(z,\varepsilon), w(\varepsilon), u, v, \theta, \varepsilon^*\}$, which can be solved from the system of equations above.

# 3 Numerical method

Different from the baseline DMP model, the state variable for both household and firm are two-dimensional object, including the employment ($z_j$) or vacancy ($\omega_j$) status, as well as the match specific productivity ($\varepsilon$). However, the transition rates of the first state variable are endogenous and depend on the transition rate of the second state variable[4]. Beside, the endogenous job destruction rate also introduce an optimal stopping problem in worker and firm's value function.

Note that the Kolmogorov forward equation and the evaluation of free entry condition are the same as in the baseline model, which will not be listed here. The only difference is in the HJB equations of worker and firm.

## 3.1 Worker and firm's HJB

To be added

## 3.2 Summary of Algorithm

0. Create sparse grid on the dimension of $\varepsilon$.
1. Guess $\theta^0$, obtain labor market condition $(f, q, eps^*)$ from matching function and FE(3) in (14).
2. Calculate wage schedule on the grids $G(\varepsilon)$ from Nash bargaining (11).
3. Guess initial value function $\{V^0_{i,j}, J^0_f\}$.
4. Solve the HJB for worker and firm (to be added).
5. Evaluate the free entry condition, update $\theta^{n+1}$.
6. Update wage $w_j^{n+1}$ and labor market conditions $f(\theta^{n+1}), q(\theta^{n+1}), \varepsilon^*(\theta^{n+1})$ as in step 1 and 2.
7. Repeat until convergence from step 4.

[4] Note that in the model section, the second state variable is continuous in cross-sectional dimension, but it follows a Poisson distribution in the time dimension

# Appendix

**1. Value function for filled job:**

For the short period $\Delta \to 0$, the firm's value function of a filled job with realized match-specific productivity $\varepsilon$ is

$$J^F(\varepsilon) = \int_0^\Delta [p + \varepsilon - w(\varepsilon)]\, e^{-\rho t} dt + e^{-\rho\Delta} \cdot \left\{ \underbrace{e^{-\lambda\Delta}}_{\text{No shock}} J^F(\varepsilon) + \underbrace{\lambda\Delta e^{-\lambda\Delta}}_{\text{1 shock}} \mathbb{E}max\left\{J^F(\tilde{\varepsilon}), J^V\right\} + \underbrace{O(\Delta^2)}_{\text{More than 1 shock}} \right\}$$
$$= [p + \varepsilon - w(\varepsilon)]\,\Delta + e^{-(\rho+\lambda)\Delta} J^F(\varepsilon) \quad + \lambda\Delta \cdot e^{-(\rho+\lambda)\Delta} J^V + \lambda\Delta \cdot e^{-(\rho+\lambda)\Delta} \cdot \int_{\varepsilon^*}^{\bar{\varepsilon}} \left[J^F(\tilde{\varepsilon}) - J^V\right] dF(\tilde{\varepsilon})$$

$$\Rightarrow \underbrace{\frac{1 - e^{-(\rho+\lambda)\Delta}}{\Delta}}_{\lim_{\Delta\to 0}:\rho+\lambda} J^F(\varepsilon) = p + \varepsilon - w(\varepsilon) + \underbrace{\lambda \cdot e^{-(\rho+\lambda)\Delta}}_{\lim_{\Delta\to 0}:\lambda} \cdot \int_{\varepsilon^*}^{\bar{\varepsilon}} \left[J^F(\tilde{\varepsilon}) - J^V\right] dF(\tilde{\varepsilon}) + \underbrace{\lambda \cdot e^{-(\rho+\lambda)\Delta}}_{\lim_{\Delta\to 0}:\lambda} \cdot J^V$$

After rearranging the terms, one can obtain Equation (1).

**2. Value function for employed:**

For the short period $\Delta \to 0$, the worker's value function with realized match-specific productivity $\varepsilon$ is

$$V^e(\varepsilon) = \int_0^\Delta w(\varepsilon) \cdot e^{-\rho t} dt + e^{-\rho\Delta} \cdot \left\{ \underbrace{e^{-\lambda\Delta}}_{\text{No shock}} V^e(\varepsilon) + \underbrace{\lambda\Delta e^{-\lambda\Delta}}_{\text{1 shock}} \left[ F(\varepsilon^*) V^u + \int_{\varepsilon^*}^{\bar{\varepsilon}} V^e(\tilde{\varepsilon}) dF(\tilde{\varepsilon}) \right] + \underbrace{O(\Delta^2)}_{\text{More than 1 shock}} \right\}$$

$$\Rightarrow \underbrace{\frac{1 - e^{-(\rho+\lambda)\Delta}}{\Delta}}_{\lim_{\Delta\to 0}:\rho+\lambda} V^e(\varepsilon) = w(\varepsilon) + \underbrace{\lambda \cdot e^{-(\rho+\lambda)\Delta}}_{\lim_{\Delta\to 0}:\lambda} \cdot \int_{\varepsilon^*}^{\bar{\varepsilon}} V^e(\tilde{\varepsilon})\, dF(\tilde{\varepsilon}) + \underbrace{\lambda \cdot e^{-(\rho+\lambda)\Delta}}_{\lim_{\Delta\to 0}:\lambda} \cdot F(\varepsilon^*) V^u$$

After rearranging the terms, one can obtain Equation (3).

**3. Wage determination:**

From the worker's value function, we know:

$$V^e(\varepsilon) - V^u = \frac{1}{\rho + \lambda} \left\{ w(\varepsilon) + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} \left[V^e(\tilde{\varepsilon}) - V^u\right] dF(\tilde{\varepsilon}) - b - f(\theta) \left[V^e(\bar{\varepsilon}) - V^u\right] \right\} \tag{15}$$

where

$$V^e(\bar{\varepsilon}) - V^u \overset{NB}{=} \frac{\beta}{1-\beta} \cdot J^F(\bar{\varepsilon}) \overset{FE(1)}{=} \frac{\beta}{1-\beta} \cdot \frac{\xi}{q(\theta)}$$

Besides, from the value function of firm, we have:

$$J^F(\varepsilon) = \frac{1}{\rho + \lambda} \left\{ p + \varepsilon - w(\varepsilon) + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} J^F(w(\tilde{\varepsilon})) dF(\tilde{\varepsilon}) \right\} \tag{16}$$

Substituting the (15) and (16) into the Nash bargaining result, we have:

$$w(\varepsilon) - b + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} \left[ V^e(\tilde{\varepsilon}) - V^u - \frac{\beta}{1-\beta} J^F(\omega(\tilde{\varepsilon})) \right] dF(\tilde{\varepsilon}) = \frac{\beta}{1-\beta} \left[p + \varepsilon - w(\varepsilon) + \theta\xi\right]$$

where $V^e(\tilde{\varepsilon}) - V^u - \frac{\beta}{1-\beta} J^F(\omega(\tilde{\varepsilon})) = 0$ by Nash bargaining. After rearranging the terms, we obtain the wage equation (6).

# References


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