DMP model in continuous time
Extension (I): Endogenous job-destruction rate

Wenxuan WANG

MRes Toulouse School of Economics

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1 Model

This model is originated from Mortensen and Pissarides (1994), where the firm decides whether to fire the worker based on the idiosyncratic productivity shocks.

1.1 Assumptions

(1). Household lives **infinitely** with **risk neutral** utility. Both household and firm discount the future at the rate ρ.

(2). March-specific productivity: $p + \varepsilon, \ \varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon}] \sim F(\varepsilon)$

- p is constant and exogenous;
- Productivity shock rate: λ. If the shock happens, the match will have a new stochastic productivity component ε' independently drawn from the identical distribution $F(\varepsilon)$;
- **New-filled job** always produce with the highest productivity $\bar{\varepsilon}$.

(3). Match process: **CRS** matching technology, $m(u, v)$

(4). Search process: **Undirected search**.

- Match arrival rate: $f(\theta) = \frac{m(u,v)}{u}, q(\theta) = \frac{m(u,v)}{v}$
- **Endogenous** job destruction rate: $s = \lambda \cdot F(\varepsilon^*)$[1]

1.2 Labor demand (Firm)

(1) Production function:

Assume that the labor is the only input in production[2], and for realized match-specific productivity ε, the production per match is given by:

$$y(\varepsilon) = p + \varepsilon$$

[1] The ε^* is the threshold of productivity shock for firing the worker, see details in Section 3.1.6.

[2] The linear productivity function is a common assumption in the search literature. For example, Postel–Vinay and Robin (2002) and Cahuc, Postel-Vinay, and Robin (2006) use a similar functional form for the flow productivity

The marginal cost of filling a job is the equilibrium wage $w(\varepsilon)$, which will be specified in Section 3.1.5. The marginal cost of posting a vacancy ξ is constant and exogenous.

(2) Value function for filled job:

$$\rho J^F(\varepsilon) = p + \varepsilon - w(\varepsilon) + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} \left[J^F(\tilde{\varepsilon}) - J^V\right] dF(\tilde{\varepsilon}) - \lambda \left[J^F(\varepsilon) - J^V\right] \tag{1}$$

See the proof in the appendix.

(3) Value function for vacancy:

$$\rho J^V = -\xi + q(\theta)\left[J^F(\bar{\varepsilon}) - J^V\right] \tag{2}$$

The firm's value function for vacant job is derived same as in the baseline model with the additional asumption that the newly filled job always produced at $p + \bar{\varepsilon}$. Therefore, the value function J^V is invariant to the match-specific productivity shock.

(4) Firm's decision:

The firm has three different decisions to make in this model: create vacancy, hire the worker and fire the worker. The first two decisions are the same as in baseline model, such that the first decision is characterized by the free entry condition, and the second decision is implicitly solved by imposing the free entry condition (see analysis in the baseline model). The third decision is the new element in this setup, and the firm needs to decide whether to (1) keep the job ($J^F(\varepsilon)$) or (2) destruct the job and turn it back to vacancy (J^V) if it receives negative productivity shocks. The productivity shock arriving at rate λ will change the match specific productivity $p + \varepsilon$: if the productivity is too low, the firm has incentive to destruct it, which generates the endogenous job destruction rate (see detailed discussion in Section 3.1.6).

1.3 Labor supply (Household)

(1) Value function for employed worker:

$$\rho V^e(\varepsilon) = w(\varepsilon) + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} \left[V^e(\tilde{\varepsilon}) - V^u\right] dF(\tilde{\varepsilon}) - \lambda \left[V^e(\varepsilon) - V^u\right] \tag{3}$$

See the proof in the appendix.

(2) Value function for unemployed worker:

$$\rho V^u = b + f(\theta)\left[V^e(\bar{\varepsilon}) - V^u\right] \tag{4}$$

Same as in labor demand, the value function V^u is invariant to the match-specific productivity shock, and it is derived in the same way as in baseline model.

(3) Worker's decision:

The household has only one decision to make in this model: accept or reject the job offer. Same as in baseline model, the household will always accept the wage offer because of free entry condition and Nash bargaining with linear utility, and one can abstract from the optimal stopping problem in the value functions of household.

1.4 Free entry

The firms will continue to create vacancies until the expected profit equals 0:

$$J^V = 0 \Rightarrow J^F(\bar{\varepsilon}) \overset{(1)}{=} \frac{\xi}{q(\theta)} \overset{(2)}{=} \frac{p + \bar{\varepsilon} - w(\bar{\varepsilon}) + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} J^F(\tilde{\varepsilon}) dF(\tilde{\varepsilon})}{\rho + \lambda} \tag{5}$$

The free entry condition pins down the equilibrium labor market tightness.

1.5 Wage determination

The risk neutrality of worker implies that the bargained wage satisfies the following condition

$$(1 - \beta) \left[V^e(\varepsilon) - V^u\right] = \beta \cdot \left[J^F(\varepsilon) - J^V\right]$$

Solving the equation above with the value functions derived before and the free entry condition (see details in the appendix), we have:

$$w(\varepsilon) = (1 - \beta)b + \beta \cdot [p + \varepsilon + \theta\xi] \tag{6}$$

Therefore, the workers receive their reservation wage b and a share β of the net surplus of the current match, which is the total productivity $p + \varepsilon + \theta\xi$ minus what workers give up b.

1.6 Endogenous destruction rate: ε^*

The endogenous destruction rate implies that the firm's problem is indeed an optimal stopping problem, and the firm decides whether to destruct the job according to the match-specific productivity $p + \varepsilon$. At the reservation productivity ε^*, the firm is indifferent between keeping and destructing the job:

$$J^F(\varepsilon^*) = J^v = 0$$

Therefore, the reservation productivity ε^* satisfies the following condition:

$$p + \varepsilon^* - b - \frac{\beta}{1 - \beta}\theta\xi + \frac{\lambda}{\rho + \lambda} \int_{\varepsilon^*}^{\bar{\varepsilon}} \left[\tilde{\varepsilon} - \varepsilon^*\right] dF(\tilde{\varepsilon}) = 0 \tag{7}$$

And it implies that the free entry condition could be written as:

$$J^F(\bar{\varepsilon}) = \frac{\xi}{q(\theta)} \overset{(3)}{=} \frac{1 - \beta}{\rho + \lambda} \left[\bar{\varepsilon} - \varepsilon^*\right] \tag{8}$$

2 Stationary Equilibrium (Recursive form)

Denote (z, ε) the state variable for household where $z \in \{e, u\}$ indicates the employment status of the agent, and ε indicates the match-specific productivity if the agent is employed[3]. Similarly, denote (ω, ε) the state variable for firm, where $\omega \in \{F, V\}$ indicates whether the vacant job is filed or not.

A stationary equilibrium consists of:

[3]Check whether ε is the state variable.

- a set of policy functions $\{c(z,\varepsilon)\}$;
- a set of value functions $\left\{V^e(\varepsilon), V_u, J^F(\varepsilon), J^v\right\}$;
- a stationary distribution over unemployment and vacancy: $\{u, v\}$;
- a set of prices $\{w(\varepsilon)\}$;
- a set of endogenous labor market parameters $\{\varepsilon^*, \theta\}$

such that:

(1) UMP: the policy function $\{c(z,\varepsilon)\}$ solve the household's utility maximization problem. Since there is no saving decision, the households are hand-to-mouth

$$c(z,\varepsilon) = \begin{cases} w(\varepsilon), & \text{if } z = e \\ b, & \text{if } z = u \end{cases} \tag{9}$$

with the corresponding value functions:

$$\begin{aligned} \rho V^e(\varepsilon) &= w(\varepsilon) + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} \left[V^e(\tilde{\varepsilon}) - V^u\right] dF\left(\tilde{\varepsilon}\right) - \lambda \left[V^e(\varepsilon) - V^u\right] \\ \rho V^u &= b + f(\theta)\left[V^e(\bar{\varepsilon}) - V^u\right] \end{aligned}$$

(2) PMP: firms decide to destruct the job optimally when $\varepsilon < \varepsilon^*$ such that

$$p + \varepsilon^* - b - \frac{\beta}{1-\beta}\theta\xi + \frac{\lambda}{\rho+\lambda}\int_{\varepsilon^*}^{\bar{\varepsilon}} \left[\tilde{\varepsilon} - \varepsilon^*\right] dF(\tilde{\varepsilon}) = 0 \tag{10}$$

with corresponding value functions:

$$\begin{aligned} \rho J^F\left(\varepsilon\right) &= p + \varepsilon - w(\varepsilon) + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} \left[J^F\left(\tilde{\varepsilon}\right) - J^V\right] dF\left(\tilde{\varepsilon}\right) - \lambda\left[J^F\left(\varepsilon\right) - J^V\right] \\ \rho J^V &= -\xi + q(\theta)\left[J^F(\bar{\varepsilon}) - J^V\right] \end{aligned}$$

(3) Nash bargaining: the equilibrium wage is determined by bargaining between the firm and the worker

$$w(\varepsilon) = (1-\beta)b + \beta \cdot \left[p + \varepsilon + \theta\xi\right] \tag{11}$$

(4) Stationary distribution:

$$\dot{u} = 0 \Rightarrow \lambda F\left(\varepsilon^*\right)(1-u) - f(\theta)\cdot u = 0 \Rightarrow u = \frac{\lambda F\left(\varepsilon^*\right)}{\lambda F\left(\varepsilon^*\right) + f(\theta)} \tag{12}$$

$$\dot{v} = 0 \Rightarrow \lambda F\left(\varepsilon^*\right)(1-u) - q(\theta)\cdot v = 0 \Rightarrow v = \theta u \tag{13}$$

(5) Free Entry: firms create vacancies at zero-profits

$$J^V = 0 \Rightarrow J^F(\bar{\varepsilon}) \overset{(1)}{=} \frac{\xi}{q(\theta)} \overset{(2)}{=} \frac{p + \bar{\varepsilon} - w(\bar{\varepsilon}) + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} J^F(\tilde{\varepsilon}) dF(\tilde{\varepsilon})}{\rho+\lambda} \overset{(3)}{=} \frac{1-\beta}{\rho+\lambda}\left[\bar{\varepsilon} - \varepsilon^*\right] \tag{14}$$

There are 6 unknown equilibrium object $\{c(z,\varepsilon), w(\varepsilon), u, v, \theta, \varepsilon^*\}$, which can be solved from the system of equations above.

3 Numerical method

Different from the baseline DMP model, the state variable for both household and firm are two-dimensional object, including the employment ($z \in \{e, u\}$) or vacancy ($\omega \in \{f, v\}$) status, as well as the match specific productivity (ε). However, the transition rates of the first state variable are endogenous and depend on the transition rate of the second state variable[4]. Beside, the endogenous job destruction rate also introduce an optimal stopping problem in worker and firm's value function.

Note that the Kolmogorov forward equation and the evaluation of free entry condition are the similar to the baseline model. The only major difference is in the HJB equations of worker and firm.

Let's denote $J(\omega, \varepsilon)$ the value function of firm, $V(z, \varepsilon)$ the value function of household.

3.1 Firm's HJBVI

Recall that the optimal stopping problem arises from the third decision of the firm: whether to destruct the filled job, and the outside option is holding a vacancy with value $J(v) = 0$ by free entry condition. The optimal stopping problem writes as[5]:

$$J(f, \varepsilon) = \max_{\tau} \mathbb{E}_0 \int_0^{\tau} e^{-\rho t} \pi(\varepsilon_t) dt + e^{-\rho \tau} \cdot J(v)$$

where $\pi(\varepsilon_t) = p + \varepsilon_t - w(\varepsilon_t)$, the state variable ε_t follows Poisson process with arrival rate λ.

Denote the reservation productivity ε^* such that $J(f, \varepsilon^*) = J^V$. The one-dimensional optimal stopping problem satisfies the reservation value property: the firm will stop and destruct the job if $\varepsilon \leq \varepsilon^*$, keep the job if $\varepsilon > \varepsilon^*$.

For $\varepsilon > \varepsilon^*$ such that the firm does not destruct the job, the following HJB equation holds[6]:

$$\rho J(f, \varepsilon) = \pi(\varepsilon) + \lambda \mathbb{E}_{\tilde{\varepsilon}} \left[J(f, \tilde{\varepsilon}) - J(f, \varepsilon) \right] \tag{15}$$

Hence, for ε above the threshold ε^* we have $J(f, \varepsilon) \geq J(v)$, and for ε below the threshold ε^*, we have $J(f, \varepsilon) = J(v)$. The firm's problem can be solved as Linear Complementarity Problem (LCP)[7] inspired by Huang and Pang (1998).

3.1.1 HJBVI

Denote the set of ε for which there is no destruction by X. Then

$$\begin{aligned} \varepsilon \in X: \quad & J(f, \varepsilon) \geq J(v), \quad \rho J(f, \varepsilon) = \pi(\varepsilon) + \lambda \mathbb{E}_{\tilde{\varepsilon}} \left[J(f, \tilde{\varepsilon}) - J(f, \varepsilon) \right] \\ \varepsilon \notin X: \quad & J(f, \varepsilon) = J(v), \quad \rho J(f, \varepsilon) \geq \pi(\varepsilon) + \lambda \mathbb{E}_{\tilde{\varepsilon}} \left[J(f, \tilde{\varepsilon}) - J(f, \varepsilon) \right] \end{aligned}$$

This can also be written compactly as following HJB variational inequality (HJBVI):

$$\min \left\{ \rho J(f, \varepsilon) - \pi(\varepsilon) - \lambda \mathbb{E}_{\tilde{\varepsilon}} \left[J(f, \tilde{\varepsilon}) - J(f, \varepsilon) \right], J(f, \varepsilon) - J(v) \right\} = 0 \tag{16}$$

[4] Note that in the model section, the second state variable is continuous in cross-sectional dimension, but it follows a Poisson distribution in the time dimension

[5] If ε_t does not have continuous path, e.g. Poisson process, can I write it in this way (i.e. integral over time)

[6] One can obtain the analytical result (1) by expanding (15)

[7] See details in Benjamin Moll's website

3.1.2 Solving as LCP

With the finite difference method, we can approximate the function $J(f)$ at I discrete points in the space dimension, $\varepsilon_i, i = 1, ..., I$, and use the short-hand notation $J_{f,i} \equiv J(f, \varepsilon_i)$ and so on. Without the stopping option, we would have HJB (15) which is discretized as

$$\rho J_{f,i}^n = \pi_i^n + \lambda \left[\sum_{i'=1}^{I} g_{i'} J_{f,i'}^n - J_{f,i}^n \right]$$

where $\{g_i\}_{i=1}^I$ is the discretized probability mass functions[8] of the state variable ε. The matrix form is:

$$\rho J_f = \pi + \mathbf{A} J_f, \text{ where } \mathbf{A} = \lambda (\mathbf{G} - \mathbf{I})$$

$$\mathbf{G} = \begin{pmatrix} g_1 & g_2 & \cdots & g_I \\ \vdots & \vdots & \ddots & \vdots \\ g_1 & g_2 & \cdots & g_I \end{pmatrix}_{I \times I}$$

Instead, now J_f solves the variational inequality (16). The discretized analogue is:

$$\min\{\rho J_f - \pi - \mathbf{A} J_f, J_f - J_v\} = 0, \text{ with } J_v = \mathbf{0}_{I \times 1}$$

Equivalently

$$\begin{aligned} (J_f - J_v)'(\rho J_f - \pi - \mathbf{A} J_f) &= 0 \\ J_f &\geq J_v \\ \rho J_f - \pi - \mathbf{A} J_f &\geq 0 \end{aligned}$$

Note that the second and third equations imply that the first equation actually has to hold element-wise. Let's denote the "excess value" $z = J_f - J_v$ and $\mathbf{B} = \rho \mathbf{I} - \mathbf{A}$. Then the second equation is $z \geq 0$ and the third equation is

$$\mathbf{B} z + q \geq 0$$

where $q = -\pi + \mathbf{B} J_v$. Summarizing

$$\begin{aligned} z'(\mathbf{B} z + q) &= 0 \\ z &\geq 0 \\ \mathbf{B} z + q &\geq 0 \end{aligned} \tag{17}$$

This is the standard form for LCPs problem, so it can solve it with an LCP solver.

3.2 Worker's HJBVI

Because of linear utility, the Nash bargaining result leads to:

$$(1 - \beta) [V^e(\varepsilon) - V^u] = \beta \left[J^F(\varepsilon) - J^V \right]$$

[8] How to obtain? Exogenous or through stationary distribution?

and both the left-hand side and right-hand side of the equation above is increasing in ε, and the reservation productivity ε^* equates both the value functions for firm, i.e. $J^F(\varepsilon^*) = J^V$, and the value functions for worker, i.e. $V^e(\varepsilon^*) = V^u$.

Therefore, the forced job separation for workers can be interpreted as an optimal stopping problem such that the worker decides whether to leave the current job and become an unemployed job searcher. And worker's reservation productivity coincides with the one of firm's problem, which is equal to ε^*.

The worker will stop the current job and search as unemployed if $\varepsilon \leq \varepsilon^*$, keep the job if $\varepsilon > \varepsilon^*$. For $\varepsilon > \varepsilon^*$ such that the worker does not destruct the job, the following HJB equation holds[9]:

$$\rho V(e, \varepsilon) = w(\varepsilon) + \lambda \mathbb{E}_{\tilde{\varepsilon}} \left[V(e, \tilde{\varepsilon}) - V(e, \varepsilon) \right] \tag{18}$$

3.2.1 HJBVI

Since the reservation productivity is same for worker and firm, I will keep using the notation X for the set of ε for which there is no destruction. Then the following HJB variational inequality holds:

$$\min \left\{ \rho V(e, \varepsilon) - w(\varepsilon) - \lambda \mathbb{E}_{\tilde{\varepsilon}} \left[V(e, \tilde{\varepsilon}) - V(e, \varepsilon) \right], V(e, \varepsilon) - V(u) \right\} = 0 \tag{19}$$

with

$$\rho V(u) = b + f(\theta) \left[V(e, \bar{\varepsilon}) - V(u) \right] \tag{20}$$

3.2.2 Solving as LCP

The discretized analogue of the variational inequality (19) in matrix form is[10]:

$$\min\{\rho V_e - w - \mathbf{A} V_e, V_e - V_u\} = 0, \text{ with } V_u = c + \mathbf{C} \cdot V_e$$

where V_e (V_u) is the vector of discretized value function of being employed (unemployed), the matrix $\mathbf{A}$ is the same as in the previous section, and the vector $c = \frac{b}{\rho + f(\theta)} \cdot \mathbf{1}_{I \times 1}$, the matrix $\mathbf{C}$ is defined as:

$$\mathbf{C} = \frac{f(\theta)}{\rho + f(\theta)} \cdot \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}_{I \times I}$$

Equivalently

$$\begin{aligned} \left((\mathbf{I} - \mathbf{C}) V_e - c \right)' \left(\rho V_e - w - \mathbf{A} V_e \right) &= 0 \\ (\mathbf{I} - C) V_e &\geq c \\ \rho V_e - w - \mathbf{A} V_e &\geq 0 \end{aligned}$$

[9] One can obtain the analytical result (3) by expanding (18)

[10] Note that the outside option V_u is now a linear function of the objective value function V_e. In order to use LCP solver, one need to first decompose the $V_e - V_u$ into two part: one is linear in V_e, and another is independent from V_e.

Denote $\tilde{V}_e = (\mathbf{I}-\mathbf{C})V_e$, accordingly denote $\tilde{z} = \tilde{V}_e - c$ and $\tilde{\mathbf{B}} = (\rho\mathbf{I}-\mathbf{A})\cdot(\mathbf{I}-\mathbf{C})^{-1}$, $\tilde{q} = -w+\tilde{\mathbf{B}}c$. We have:

$$\begin{aligned} \tilde{z}'(\mathbf{B}\tilde{z}+\tilde{q}) &= 0 \\ \tilde{z} &\geq 0 \\ \mathbf{B}\tilde{z}+\tilde{q} &\geq 0 \end{aligned} \tag{21}$$

The system (21) may be able to be solved with a LCP solver.

3.3 Free entry condition

Define the free entry condition as following:

$$FE = -\xi + q(\theta)\cdot \iota_I J_f \tag{22}$$

If FE is positive, firms are too profitable and the guess for θ will have to be increased. Update at the end of the loop according to:

$$\theta^{n+1} = \theta^n + \Delta_\theta FE^n \tag{23}$$

where $\Delta_\theta > 0$ is a small number.

3.4 Summary of Algorithm

0: Choose an index $\boldsymbol{l}^0$ and construct associated regular sparse grid $\mathbf{G}^0$ in the space dimension of ε.
1: Initialize guess for θ^0:
2: Obtain labor market condition $(f(\theta^0), q(\theta^0))$;
3: Calculate wage schedule w^0 on $\mathbf{G}^0$ from Equation (11).
4: **for** $k \geq 0$ **do**
5: **for** $n \geq 0$ **do**
6: Solve for (V_e^n, J_f^n) according to HJBVI (17) and (21);
7: Update θ^{n+1} according to Equation (23);
8: Update wage schedule $\{w^{n+1}\}$ and labor market conditions $f(\theta^{n+1}), q(\theta^{n+1})$ as in step 1;
9: Continue if $|\theta^{n+1}-\theta^n| > \varepsilon^\theta$
end
10: Compute hierarchical surplus $\left(\boldsymbol{\alpha}_e^{H,k}, \boldsymbol{\alpha}_f^{H,k}\right)$, adapt grid $\mathbf{G}^{k+1}$ if adaptation criterion met
end

Remarks:

1. Note that only J_f^n is necessary to evaluate the free entry condition.

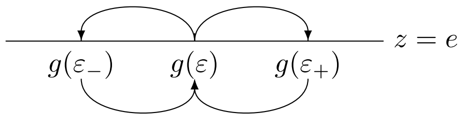
Question 1 : need to also check the convergence in $\{Ve, Jf\}$?

- If not: no need to solve for Ve.
- If so: Ve^{n+1} is solved in the algorithm only if θ is updated to $n+1$.

Question 2 : how to determine distribution of discretized ε_i?

- Exogenously given? How to discretize?

- Through stationary distribution? By LLN, the stationary cross-sectional distribution should converge to the underlying DGP $F(\varepsilon)$. May be infeasible .



The transition rate is same and equal to λ, we my have the following low of motion:

$$-\lambda g(\varepsilon) - \lambda g(\varepsilon) = \lambda g(\varepsilon_-) + \lambda g(\varepsilon_+) \Rightarrow g(\varepsilon_-) + g(\varepsilon_+) + 2g(\varepsilon) = 0$$

This condition implies that $g(\varepsilon) = 0, \forall \varepsilon$, since $g(\varepsilon) \geq 0$.

Question 3: is it possible to solve for ε^* numerically? Yes

One need to find the exact kink in J_f such that $J_f = 0$. Since LCP solves for the value functions on grids, no functional form is available.

Possible solution: recall the HJBVI of firm:

$$\min \left\{ \rho J(f,\varepsilon) - \pi(\varepsilon) - \lambda \mathbb{E}_{\tilde{\varepsilon}} \left[J(f,\tilde{\varepsilon}) - J(f,\varepsilon) \right], J(f,\varepsilon) - J(v) \right\} = 0 \tag{24}$$

The reservation productivity ε^* makes firm indifferent between the two options, and $J(f,\varepsilon^*) = 0$. Therefore, we have $\pi(\varepsilon^*) = -\lambda \mathbb{E}_{\tilde{\varepsilon}} J(f,\tilde{\varepsilon})$, the RHS is constant and its disretized analogue can be obtained after the whole algorithm.

Question 4: is it possible to solve for θ analytically? No

Recall that the three eqm. object $w(), \varepsilon^*, \theta$ is jointly pinned down by three equations: Wage bargaining(25), Firm's optimal stopping problem(26), and Free entry condition (27)

$$w(\varepsilon) = (1-\beta)b + \beta \cdot [p + \varepsilon + \theta\xi] \tag{25}$$

$$J^F(\bar{\varepsilon}) = \frac{\xi}{q(\theta)} \overset{(3)}{=} \frac{1-\beta}{\rho+\lambda} [\bar{\varepsilon} - \varepsilon^*] \tag{26}$$

$$J^F(\bar{\varepsilon}) = \frac{\xi}{q(\theta)} \overset{(2)}{=} \frac{p + \bar{\varepsilon} - w(\bar{\varepsilon}) + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} J^F(\tilde{\varepsilon}) dF(\tilde{\varepsilon})}{\rho + \lambda} \tag{27}$$

One can write $w(\varepsilon)$ and ε^* as functions of θ from equation (25) and (26), and substitute into equation (27) to solve for θ. However, $J^F(\cdot)$ is an unknown functional form, so θ cannot be solved analytically.

Appendix

1. Value function for filled job:

For the short period $\Delta \to 0$, the firm's value function of a filled job with realized match-specific productivity ε is

$$J^F(\varepsilon) = \int_0^{\Delta} [p+\varepsilon-w(\varepsilon)]\, e^{-\rho t} dt + e^{-\rho\Delta} \cdot \left\{ \underbrace{e^{-\lambda\Delta}}_{\text{No shock}} J^F(\varepsilon) + \underbrace{\lambda\Delta e^{-\lambda\Delta}}_{\text{1 shock}} \mathbb{E}max\left\{J^F(\tilde{\varepsilon}), J^V\right\} + \underbrace{O(\Delta^2)}_{\text{More than 1 shock}} \right\}$$

$$= [p+\varepsilon-w(\varepsilon)]\,\Delta + e^{-(\rho+\lambda)\Delta} J^F(\varepsilon) \quad + \lambda\Delta \cdot e^{-(\rho+\lambda)\Delta} J^V + \lambda\Delta \cdot e^{-(\rho+\lambda)\Delta} \cdot \int_{\varepsilon^*}^{\bar{\varepsilon}} \left[J^F(\tilde{\varepsilon}) - J^V\right] dF(\tilde{\varepsilon})$$

$$\Rightarrow \underbrace{\frac{1-e^{-(\rho+\lambda)\Delta}}{\Delta}}_{\lim_{\Delta\to 0}:\rho+\lambda} J^F(\varepsilon) = p+\varepsilon-w(\varepsilon) + \underbrace{\lambda \cdot e^{-(\rho+\lambda)\Delta}}_{\lim_{\Delta\to 0}:\lambda} \cdot \int_{\varepsilon^*}^{\bar{\varepsilon}} \left[J^F(\tilde{\varepsilon}) - J^V\right] dF(\tilde{\varepsilon}) + \underbrace{\lambda \cdot e^{-(\rho+\lambda)\Delta}}_{\lim_{\Delta\to 0}:\lambda} \cdot J^V$$

After rearranging the terms, one can obtain Equation (1).

2. Value function for employed:

For the short period $\Delta \to 0$, the worker's value function with realized match-specific productivity ε is

$$V^e(\varepsilon) = \int_0^{\Delta} w(\varepsilon) \cdot e^{-\rho t} dt + e^{-\rho\Delta} \cdot \left\{ \underbrace{e^{-\lambda\Delta}}_{\text{No shock}} V^e(\varepsilon) + \underbrace{\lambda\Delta e^{-\lambda\Delta}}_{\text{1 shock}} \left[F(\varepsilon^*)V^u + \int_{\varepsilon^*}^{\bar{\varepsilon}} V^e(\tilde{\varepsilon}) dF(\tilde{\varepsilon})\right] + \underbrace{O(\Delta^2)}_{\text{More than 1 shock}} \right\}$$

$$\Rightarrow \underbrace{\frac{1-e^{-(\rho+\lambda)\Delta}}{\Delta}}_{\lim_{\Delta\to 0}:\rho+\lambda} V^e(\varepsilon) = w(\varepsilon) + \underbrace{\lambda \cdot e^{-(\rho+\lambda)\Delta}}_{\lim_{\Delta\to 0}:\lambda} \cdot \int_{\varepsilon^*}^{\bar{\varepsilon}} V^e(\tilde{\varepsilon})\, dF(\tilde{\varepsilon}) + \underbrace{\lambda \cdot e^{-(\rho+\lambda)\Delta}}_{\lim_{\Delta\to 0}:\lambda} \cdot F(\varepsilon^*) V^u$$

After rearranging the terms, one can obtain Equation (3).

3. Wage determination:

From the worker's value function, we know:

$$V^e(\varepsilon) - V^u = \frac{1}{\rho+\lambda} \left\{ w(\varepsilon) + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} \left[V^e(\tilde{\varepsilon}) - V^u\right] dF(\tilde{\varepsilon}) - b - f(\theta)\left[V^e(\bar{\varepsilon}) - V^u\right] \right\} \tag{28}$$

where

$$V^e(\bar{\varepsilon}) - V^u \overset{NB}{=} \frac{\beta}{1-\beta} \cdot J^F(\bar{\varepsilon}) \overset{FE(1)}{=} \frac{\beta}{1-\beta} \cdot \frac{\xi}{q(\theta)}$$

Besides, from the value function of firm, we have:

$$J^F(\varepsilon) = \frac{1}{\rho+\lambda} \left\{ p+\varepsilon-w(\varepsilon) + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} J^F(w(\tilde{\varepsilon})) dF(\tilde{\varepsilon}) \right\} \tag{29}$$

Substituting the (28) and (29) into the Nash bargaining result, we have:

$$w(\varepsilon) - b + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} \left[V^e(\tilde{\varepsilon}) - V^u - \frac{\beta}{1-\beta} J^F(\omega(\tilde{\varepsilon})) \right] dF(\tilde{\varepsilon}) = \frac{\beta}{1-\beta} \left[p + \varepsilon - w(\varepsilon) + \theta\xi \right]$$

where $V^e(\tilde{\varepsilon}) - V^u - \frac{\beta}{1-\beta} J^F(\omega(\tilde{\varepsilon})) = 0$ by Nash bargaining. After rearranging the terms, we obtain the wage equation (6).

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