DMP model in continuous time

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1 Introduction

Job finding process is characterized by the investment in information made by both the worker and firm in the hope of locating a fruitful long-term relationship. Therefore, searching and matching frictions are important to understand the labor market dynamics. As the future is never known with certainty, the evaluation of the prospective benefits requires the formation of expectations. An acceptable job, then, is one that offers an expected stream of future benefits that has a value in excess of the option to continue to search for an even better alternative.

In the standard models of markets available prior to 1970, all of these complications were ignored. The best known model is of perfect competition. However, this approach assumes exchange in a centralized market in which information about the services traded as well as the price are perfect. Besides, there is no dynamics in this approach such that only the current value of employment opportunity matters, not the future streams of wages and profit associated with the employment opportunity. Therefore, the unemployment is involuntary and can not be explained in the equilibrium concept.

Because of the search and matching costs associated with the heterogeneity of jobs, individuals, and homes, this form of centralized trading is simply not possible in labor. The need to gather information about the properties of the job as well as the ask and bid price is still present. In response to these questions, Peter Diamond, Dale Mortensen and Chris Pissarides jointly developed the theory of equilibrium unemployment (Nobel prize 2010). Referred to DMP model, this framework is a general equilibrium model with two-sided search and match between worker and firm. The advantage of DMP model is its tractability, but it relies on the reduced-form matching function to introduce the friction in the economy.

Related literature DMP framework starts from the one-sided search theory proposed which was fully formalized in D. Mortensen (1970). McCall (1970) provided a similar formulation at about the same time based on the mathematical analysis of the optimal stopping problem model borrowed from stochastic decision theory. The essential assumptions of the formal optimal stopping model as applied here are that the worker cares only about the expected discounted stream of future wages offered by a job, and that an offer is a random draw from the distribution of possible wage offers known to the worker. Given these assumptions, the decision to accept or not is analogous to the problem of exercising a stock option. And the agent will accept the job offer if the wage exceeds his reservation wage.

But, the one-sided search model does not provide a complete theory of employment and wage determination useful for dynamic and policy analysis. The demand side of the market was not explicitly modeled in most papers published in the 1970s. The model of wage setting adopted in the macro literature is based on bilateral bargaining theory. In that setting, neither worker nor employer has the power to set the wage. Instead, the wage must be mutually agreed to as the outcome of bargaining between worker and employer. The "pie" to be divided in the bargain is equal to the wedge between the marginal value of a worker to the employer and the worker's reservation wage. This wedge is positive precisely because time and resources are required to find an alternative match partner. The early contributions in this area include D. Mortensen (1978), D. Mortensen (1982a), D. Mortensen (1982b), Diamond and Maskin (1979),Diamond (1982a), Diamond (1982b), Pissarides (1979) and Pissarides (1985). These works were among the first to formulate two-sided search models and to deal explicitly with the dual issues of existence and efficiency of search equilibrium. And the first generation of the DMP model was fully articulated in Pissarides (1990), and D. T. Mortensen (2011) provides a comprehensive review of the DMP model.

2 Baseline DMP model

2.1 Model

2.1.1 Assumptions

(1). Household lives **infinitely** with **risk neutral** utility. Both household and firm discount the future at the rate ρ.

(2). March-specific productivity: p, constant and exogenous $\Rightarrow$ homogeneous firm and worker

(3). Match process: **CRS** matching technology, $m(u, v)$

(4). Search process: **Undirected search**.

- Match arrival rate: $f(\theta) = \frac{m(u,v)}{u}$ for worker, $q(\theta) = \frac{m(u,v)}{v}$ for firm
- Exogenous job destruction rate: s

2.1.2 Labor demand (Firm)

(1) Production function:
Assume that the labor is the only input in production, the production per match is given by:

$$y = p$$

The marginal cost of filling a job is the equilibrium wage w, which will be specified in Section 2.1.5. The marginal cost of posting a vacancy ξ is constant and exogenous.

(2) Value function for filled job:

$$\rho J^F = p - w + s\left[J^V - J^F\right] \tag{1}$$

See the proof in the appendix.

(3) Value function for vacancy:

$$\rho J^V = -\xi + q(\theta)\left[J^F - J^V\right] \tag{2}$$

See the proof in the appendix.

(4) Firm's decision:
The firm has two different decisions to make in this model: create vacancy, hire the worker. The first decision consists of comparing the value of "creating" a vacancy (J^V) and "not creating" a vacancy, and the value of the latter is equal to 0. This decision is characterized by the free entry condition in the Section 2.1.4, see details later. The second decision consists of comparing the value of filling the vacancy (J^F) and keeping the vacancy (J^V). But as explained later, the free entry condition secures that $J^F > J^V = 0$, so the firm will always hire the worker once it meets with a worker. Therefore, both decisions are implicitly pinned down by the free entry condition, and we abstract from the optimal stopping problem when deriving the firm's value function.

2.1.3 Labor supply (Household)

(1) Value function for employed worker:

$$\rho V^e = w + s\left[V^u - V^e\right] \tag{3}$$

See the proof in the appendix.

(2) Value function for unemployed worker:

$$\rho V^u = b + f(\theta)\left[V^e - V^u\right] \tag{4}$$

See the proof in the appendix.

(3) Worker's decision:

The household has only one decision to make in this model: accept or reject the job offer. As discussed in McCall (1970), this optimal stopping problem satisfies the reservation wage property: the agent will accept the offer if the offered wage is higher than the reservation wage, i.e. $w \geq w^*$, and the agent is indifferent between accepting and rejecting the offer at the reservation wage, i.e. $V^e(w^*) = V^u$. But the free entry condition ($J^F > J^V = 0$) and Nash bargaining with linear utility ($(1-\beta)(V^e(w) - V^u) = \beta J^F > 0$)[1] secures that all wage offer w exceeds the reservation wage w^*, so the household will always accept the offer once he meets with the firm. Therefore, similar to the firm's problem, we abstract from the optimal stopping problem in the value functions of household.

2.1.4 Free entry

Note that the value function J^v is defined as the value function of "creating a vacancy", and the outside option for firm is "not creating" whose value is 0[2]. Therefore, the decision of vacancy creation is based on the comparison between J^V and 0. If $J^v > 0$, the firms will create infinite vacancy $v \to \infty$; if $J^v < 0$, no firm will create vacancy $v = 0$, both cases is not consistent with the definition of an equilibrium. There are two possible solutions to deal with this inconsistency: (1) impose free entry condition $J^v = 0$ with constant marginal cost of posting vacancy, (2) assume the marginal cost of posting vacancy ($\xi(v)$) is increasing in the vacancy rate. I follow the existing literature on DMP model[3] and assume the free entry condition.

Under free entry condition, the firms will continue to create vacancies until the expected profit equals 0:

$$J^V = 0 \Rightarrow J^F(\bar{\varepsilon}) \overset{(1)}{=} \frac{\xi}{q(\theta)} \overset{(2)}{=} \frac{p-w}{\rho+s} \geq 0 \tag{5}$$

Because both cost of posting vacancy (ξ) and the job-filling rate $q(\theta)$ are non-negative, the value function of filled job is always positive and the bargained wage is lower than the productivity ($w < p$). So the second decision for firm (hiring) is also implicitly pinned down by the free entry condition. Beside, this condition also determines the equilibrium labor market tightness θ.

2.1.5 Wage determination

The DMP model also assumes that the wage is determined by the pair-specific Nash bargaining between firm and worker. Note that the Nash bargaining requires no commitment, so the current bargained wage will not affect the future continuation value, i.e. J^V and V^u. Denote β the bargaining power of the worker, and the bargained wage solves the following maximization problem:

$$w = \max_{\tilde{w}} \left[J^F(\tilde{w}) - J^V\right]^{1-\beta} \left[V^e(\tilde{w}) - V^u\right]^{\beta}$$

The risk neutrality of worker implies that $-\frac{\partial J^F(\tilde{w})}{\partial \tilde{w}} = \frac{\partial V^e(\tilde{w})}{\partial \tilde{w}} = 1$, so the first-order condition writes:

$$(1-\beta)\left[V^e(w) - V^u\right] = \beta \cdot \left[J^F(w) - J^V\right]$$

Substituting the results implied by the free entry condition and the value functions derived before (see details in the appendix), we have:

$$w = (1-\beta)b + \beta \cdot [p + \theta\xi] \tag{6}$$

Therefore, the workers receive their reservation value b and a share β of the net surplus of the current match, which is the total productivity ($p + \theta\xi$) minus what workers give up (b).

2.2 Stationary Equilibrium (Recursive form)

Denote $z \in \{e, u\}$ the state variable for household, which indicates the employment status of the household. Similarly, denote $\omega \in \{F, V\}$ the state variable for firm, which indicates whether the vacant job is filed or not. Note that there is no heterogeneity in this economy.

[1]See details in the following two sections

[2]If interpret J^v as the value function of "keeping" the vacancy, the outside option becomes "filling" the vacancy whose value is represented by J^F, which corresponds to the second decision made by the firm.

[3]For example, see D. T. Mortensen and Pissarides (1994), Flinn and Heckman (1982) and Bobba, Flabbi, and Levy (2022)

A stationary equilibrium consists of:

- a set of policy functions $\{c(z)\}$;
- a set of value functions $\left\{V^e, V_u, J^F(\varepsilon), J^v\right\}$;
- a stationary distribution over unemployment and vacancy normalized by the total number of labor force: $\{u, v\}$;
- a set of price $\{w\}$;
- a set of endogenous labor market parameter (tightness) $\{\theta\}$.

such that[4]:

(1) UMP: Given the price $\{w\}$ and the labor market tightness $\{\theta\}$, the policy function $\{c(z)\}$ solve the household's utility maximization problem, and the household decides the optimal extensive margin of labor supply (always accept wage offer). Since there is no saving decision, the households are hand-to-mouth

$$c(z) = \begin{cases} w, & \text{if } z = e \\ b, & \text{if } z = u \end{cases} \tag{7}$$

with the corresponding value functions:

$$\begin{aligned} \rho V^e &= w + s\left[V^u - V^e\right] \\ \rho V^u &= b + f(\theta)\left[V^e - V^u\right] \end{aligned}$$

(2) PMP: Given the price $\{w\}$ and the labor market tightness $\{\theta\}$, firm decides the optimal extensive margin of labor demand (always hire the matched worker) with corresponding value functions:

$$\begin{aligned} \rho J^F &= p - w + s\left[J^V - J^F\right] \\ \rho J^V &= -\xi + q(\theta)\left[J^F - J^V\right] \end{aligned}$$

(3) Nash bargaining: Given the labor market tightness $\{\theta\}$, the equilibrium wage is determined by bargaining between the firm and the worker

$$w = (1-\beta)b + \beta \cdot [p + \theta\xi] \tag{8}$$

(4) Stationary distribution: Given the allocations and the prices $\{\theta\}$,

$$\dot{u} = 0 \Rightarrow s(1-u) - f(\theta) \cdot u = 0 \Rightarrow u = \frac{s}{s + f(\theta)} \tag{9}$$

$$\dot{v} = 0 \Rightarrow s(1-u) - q(\theta) \cdot v = 0 \Rightarrow v = \theta u \tag{10}$$

(5) Free Entry: Given the allocations and the prices $\{\theta\}$, job creation leads to free entry condition

$$J^V = 0 \Rightarrow J^F(\bar{\varepsilon}) \overset{(1)}{=} \frac{\xi}{q(\theta)} \overset{(2)}{=} \frac{p-w}{\rho+s} \tag{11}$$

There are 5 unknown equilibrium object $\{c(z), w, u, v, \theta\}$, which can be solved from the system of equations above.

2.3 Numerical method

Since there is no saving decision in this model, and the firm only produce with labor input, the state variables for both household (z) and firm (ω) is an one-dimensional object, taking value between two possible values: $z \in \{e, u\}, \omega \in \{f, v\}$, indicating respectively whether the household is employed (e) or not (u), and whether the firm has filled vacancy (f) or not (v). Denote $\mathbf{z} = (e, u)', \omega = (f, v)'$ the vectors of state variable for household and firm.

[4] Note that the market clearing conditions are not specified in any DMP model, because the equilibrium wage is determined by the Nash bargaining instead of the conventional labor market clearing (i.e. $u = 0$).

2.3.1 Worker's HJB

Denote $V_j \equiv V(z_j), j \in \{1,2\}$ the value function of the household with employment status z_j, so $V_1 = V^e(w), V_2 = V^u$ in the section above. We solve numerically the HJB with **semi-implicit method** as in Achdou, Han, Lasry, Lions, and Moll (2021) where the method is referred to "implicit method". The value function $V_j^n, n = 1, ...$ is updated according to

$$\frac{V_j^{n+1} - V_j^n}{\Delta} + \rho V_j^{n+1} = c_j^n + \lambda_j^{HH}(V_{-j}^{n+1} - V_j^{n+1}), \quad j \in \{1,2\} \tag{12}$$

where Δ is the step size, which can be arbitrarily large in the semi-implicit scheme. c_j^n is the equilibrium consumption[5] defined in Equation (7), and the λ_j^{HH} is the state transition rate for household. In particular, $\mathbf{c}^n = (w^n, b)'$, and $\boldsymbol{\lambda}^{HH} = (s, f(\theta))'$, where w^n is the nth update equilibrium wage, b is the unemployment utility, s is the exogenous job destruction rate, $f(\theta)$ is the endogenous job finding rate depending on the equilibrium labor market tightness θ.

Note that because the type of employment status is discrete and there is no saving problem, we don't have the partial derivative of the value function in the HJB (12). Therefore, the first-difference method and upwind scheme is not necessary for numerical solution to DMP model.

Equation (12) is a system of 2×1 linear equations which can be written in matrix notation as:

$$\frac{1}{\Delta}\left(\mathbf{V}^{n+1} - \mathbf{V}^n\right) + \rho\mathbf{V}^{n+1} = \mathbf{V}^n + \mathbf{c}^n + \mathbf{A}^n \cdot \mathbf{V}^{n+1} \tag{13}$$

where

$$\mathbf{A}^n = \begin{pmatrix} -s & s \\ f(\theta^n) & -f(\theta^n) \end{pmatrix}, \quad \mathbf{c}^n = \begin{pmatrix} w^n \\ b \end{pmatrix} \tag{14}$$

Note that different from the Huggett model in Achdou et al. (2021), the matrix $\mathbf{A}$ is not sparse and constant to the iterations in the DMP model because of the lack of saving decision. This system can in turn be written as

$$\mathbf{B}^n\mathbf{V}^{n+1} = \mathbf{b}^n, \ \mathbf{B}^n = \left(\frac{1}{\Delta} + \rho\right)\mathbf{I}_2 - \mathbf{A}^n, \ \mathbf{b}^n = \mathbf{c}^n + \frac{1}{\Delta}\mathbf{V}^n \tag{15}$$

2.3.2 Firm's HJB

This is relatively easy since the value function of unfilled vacancy (J_v) is zero from free entry condition (see next subsection), and we only need to specify the value function of filled vacancy (J_f). The value function $J_f^n, n = 1, 2, ...$ can be updated according to:

$$\frac{J_f^{n+1} - J_f^n}{\Delta} + \rho J_f^{n+1} = \pi^n - s \cdot J_f^{n+1} \Rightarrow J_f^{n+1} = \left(\frac{1}{\Delta} + \rho + s\right)^{-1}\left(\pi^n + \frac{1}{\Delta}J_f^n\right) \tag{16}$$

where $\pi^n = p - w^n$ is the flow profit for the firm with filled vacancy.

2.3.3 Free entry condition

Define the free entry condition as following:

$$FE = -\xi + q(\theta)J^f \tag{17}$$

If FE is positive, firms are too profitable and the guess for θ will have to be increased. Update at the end of the loop according to:

$$\theta^{n+1} = \theta^n + \Delta_\theta FE^n \tag{18}$$

where $\Delta_\theta > 0$ is a small number.

[5] Because I assume the household is risk neutral, the consumption level c_j^n also represents the utility $u(c_j^n)$. In the code, it is equivalent to set the CRRA parameter to 0.

2.3.4 Kolmogorov forward equation

Denote $g_j, j \in \{1,2\}$ the distribution of household in the space of employment status z_j, so $g_1 \equiv 1-u, g_2 \equiv u$. The stationary distribution of worker flow satisfies:

$$0 = -\lambda_j g_j + \lambda_{-j} g_{-j}, j \in \{1,2\} \tag{19}$$

Similarly, the first-difference method and upwind scheme is not required because of no saving and discrete employment status. And equation (19) is a system of 2×1 linear equations which can be written in matrix notation as:

$$\mathbf{A}^T \mathbf{g} = 0 \tag{20}$$

where $\mathbf{A}$ is defined as Equation (14) in Section 2.2.1, and $\mathbf{g} = (g_1, g_2)' = (1-u, u)$.

Therefore, the stationary distribution $\mathbf{g}^l, l = 1, 2, ...$ can be updated according to:

$$\frac{\mathbf{g}^{l+1} - \mathbf{g}^l}{\Delta_{KF}} = \mathbf{A}^T \mathbf{g}^{l+1} \Rightarrow \left(\frac{1}{\Delta_{KF}}\mathbf{I}_2 - \mathbf{A}^T\right)\mathbf{g}^{l+1} = \frac{1}{\Delta_{KF}}\mathbf{g}^l \tag{21}$$

, where Δ_{KF} is the step size.

Note that we only need to solve the stationary distribution of worker flow, since the stationarity of the vacancy distribution is directly satisfied by imposing the matching technology: $v = \theta u$.

2.3.5 Parametric assumptions

Assumes that the matching technology takes form of Cobb-Douglas function:

$$m(u, v) = u^\eta v^{1-\eta}$$

Following Bardóczy (2017), let the elasticity of matching with respect to unemployment be $\eta = 0.72$, and impose the Hosios condition such that the bargaining power of worker is equal to the matching elasticity $\beta = \eta$. In the standard DMP model with linear utility, this condition guarantees efficiency. Let the labor-output ratio $p = 1$, and the unemployment utility $b = 0.5$. The rest of the parameters also follow the calibration in Bardóczy (2017): $\rho = 0.4$, $s = 0.034$, $\xi = 0.4$, where the first is the discount rate for both household and firm, the second is the job destruction rate, the third is the cost of posting vacancy.

2.3.6 Summary of Algorithm

This algorithm is based on Achdou et al. (2021) and Bardóczy (2017). Achdou et al. (2021) provides solutions for the HJB as well as the Kolmogorov forward equation, and Bardóczy (2017) provides the global algorithm for the DMP model with saving decision, namely KMS model. Therefore, the household and firm's problem are solved as described in Achdou et al. (2021), the free entry condition is solved as described in Bardóczy (2017). And the global algorithm is similar to Bardóczy (2017), where finding the stationary equilibrium requires a fixed-point algorithm over the labor market tightness θ.

1. Guess labor market tightness, θ_0.

This guess will be updated based on the free entry condition (**??**), taken the cost of posting vacancy (ξ) as given. Given θ^n (n = 0,1,...), the labor market variables are[6]

$$f(\theta^n) = (\theta^n)^{1-\eta}, \quad q(\theta^n) = (\theta^n)^{-\eta} \tag{22}$$

2. Calculate the initial wage schedule, w^0.

According to the Nash bargaining result (28), calculate the equilibrium wage schedule $w(\theta^0)$ taken the bargaining power (β) as given.

3. Guess the initial value function $\mathbf{V}^0$ and $\mathbf{J}^0$.

A natural initial guess for the latter is the value function of "staying put":

[6]Note that in Bardóczy (2017), the equilibrium $u = \frac{s}{s+f(\theta)}$ and $v = \theta u$ is obtained directly from the Kolmogorov forward equation. The Kolmogorov forward equation is not required in the following algorithm since the worker distribution is summarized in the labor market tightness (θ) and the matching functions ($f(\theta), q(\theta)$). Therefore, I provide a separate code, *"KF.m"*, to solve Kolmogorov forward equation numerically as in (21), which does not appear in the main function, *"VFI.m"* ,solving for the stationary equilibrium labor market tightness and wage.

$$V_j^0 = \frac{c_j^0}{\rho}, \quad J_f^0 = \frac{\pi^0}{\rho}, \quad \mathbf{c}^0 = (w^0, b)', \quad \pi^0 = p - w^0$$

4. Solve the worker's and firm's problem.
For $n = 0, 1, 2, ...$, calculate $\mathbf{V}^{n+1}$ and $\mathbf{J}^{n+1}$ from (15) and (16).

5. Evaluate the free entry condition.
For $n = 0, 1, 2, ...$, update θ^{n+1} according to (17) and (18).

6. Update the wage schedule and labor market conditions.
For $n = 0, 1, 2, ...$, update w^{n+1} according to the Nash bargaining result (28), and update the labor market conditions according to (22).

7. Repeat until convergence from step 4[7].

[7]The convergence is different from aiyagari model, where we could have a outer-loop for equilibrium prices, and two inner-loops for HJB and KF, i.e. for a given equilibrium prices (r, w), compute the stationary equilibrium, then check the market clearing condition. However, in DMP, we need to update the labor market tightness (as well as the wage schedule) accordingly with the value functions (HJBs).

3 Extension (I): Endogenous job-destruction rate

3.1 Model

This model is originated from D. T. Mortensen and Pissarides (1994), where the firm decides whether to fire the worker based on the idiosyncratic productivity shocks.

3.1.1 Assumptions

(1). Household lives **infinitely** with **risk neutral** utility. Both household and firm discount the future at the rate ρ.

(2). March-specific productivity: $p+\varepsilon,\ \varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon}] \sim F(\varepsilon)$

- p is constant and exogenous;
- Productivity shock rate: λ. ε follows the Poisson process;
- **New-filled job** always produce with the highest productivity $\bar{\varepsilon}$.

(3). Match process: **CRS** matching technology, $m(u, v)$

(4). Search process: **Undirected search**.

- Match arrival rate: $f(\theta)=\frac{m(u,v)}{u}, q(\theta)=\frac{m(u,v)}{v}$
- **Endogenous** job destruction rate: $s=\lambda \cdot F(\varepsilon^*)$[8]

3.1.2 Labor demand (Firm)

(1) Production function:
Assume that the labor is the only input in production[9], and for realized match-specific productivity ε, the production per match is given by:

$$y(\varepsilon)=p+\varepsilon$$

The marginal cost of filling a job is the equilibrium wage $w(\varepsilon)$, which will be specified in Section 3.1.5. The marginal cost of posting a vacancy ξ is constant and exogenous.

(2) Value function for filled job:

$$\rho J^F(\varepsilon)=p+\varepsilon-w(\varepsilon)+\lambda \int_{\varepsilon^*}^{\bar{\varepsilon}}\left[J^F(\tilde{\varepsilon})-J^V\right] dF(\tilde{\varepsilon})-\lambda\left[J^F(\varepsilon)-J^V\right] \tag{23}$$

See the proof in the appendix.

(3) Value function for vacancy:

$$\rho J^V=-\xi+q(\theta)\left[J^F(\bar{\varepsilon})-J^V\right] \tag{24}$$

The firm's value function for vacant job is derived same as in the baseline model with the additional asumption that the newly filled job always produced at $p+\bar{\varepsilon}$. Therefore, the value function J^V is invariant to the match-specific productivity shock.

(4) Firm's decision:
The firm has three different decisions to make in this model: create vacancy, hire the worker and fire the worker. The first two decisions are the same as in baseline model, such that the first decision is characterized by the free entry condition, and the second decision is implicitly solved by imposing the free entry condition (see analysis in the baseline model). The third decision is the new element in this setup, and the firm needs to decide whether to (1) keep the job ($J^F(\varepsilon)$) or (2) destruct the job and turn it back to vacancy (J^V) if it receives negative productivity shocks. The productivity shock arriving at rate λ will change the match specific productivity $p+\varepsilon$: if the productivity is too low, the firm has incentive to destruct it, which generates the endogenous job destruction rate (see detailed discussion in Section 3.1.6).

[8]The ε^* is the threshold of productivity shock for firing the worker, see details in Section 3.1.6.
[9]The linear productivity function is a common assumption in the search literature. For example, Postel–Vinay and Robin (2002) and Cahuc, Postel-Vinay, and Robin (2006) use a similar functional form for the flow productivity

3.1.3 Labor supply (Household)

(1) Value function for employed worker:

$$\rho V^e(\varepsilon) = w(\varepsilon) + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} \left[V^e(\tilde{\varepsilon}) - V^u\right] dF\left(\tilde{\varepsilon}\right) - \lambda \left[V^e(\varepsilon) - V^u\right] \tag{25}$$

See the proof in the appendix.

(2) Value function for unemployed worker:

$$\rho V^u = b + f(\theta) \left[V^e(\bar{\varepsilon}) - V^u\right] \tag{26}$$

Same as in labor demand, the value function V^u is invariant to the match-specific productivity shock, and it is derived in the same way as in baseline model.

(3) Worker's decision:
The household has only one decision to make in this model: accept or reject the job offer. Same as in baseline model, the household will always accept the wage offer because of free entry condition and Nash bargaining with linear utility, and one can abstract from the optimal stopping problem in the value functions of household.

3.1.4 Free entry

The firms will continue to create vacancies until the expected profit equals 0:

$$J^V = 0 \Rightarrow J^F(\bar{\varepsilon}) \overset{(1)}{=} \frac{\xi}{q(\theta)} \overset{(2)}{=} \frac{p + \bar{\varepsilon} - w(\bar{\varepsilon}) + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} J^F(\tilde{\varepsilon}) dF(\tilde{\varepsilon})}{\rho + \lambda} \tag{27}$$

The free entry condition pins down the equilibrium labor market tightness.

3.1.5 Wage determination

The risk neutrality of worker implies that the bargained wage satisfies the following condition

$$(1-\beta)\left[V^e(\varepsilon) - V^u\right] = \beta \cdot \left[J^F(\varepsilon) - J^V\right]$$

Solving the equation above with the value functions derived before and the free entry condition (see details in the appendix), we have:

$$w(\varepsilon) = (1-\beta)b + \beta \cdot \left[p + \varepsilon + \theta\xi\right] \tag{28}$$

Therefore, the workers receive their reservation wage b and a share β of the net surplus of the current match, which is the total productivity $p + \varepsilon + \theta\xi$ minus what workers give up b.

3.1.6 Endogenous destruction rate: ε^*

The endogenous destruction rate implies that the firm's problem is indeed an optimal stopping problem, and the firm decides whether to destruct the job according to the match-specific productivity $p + \varepsilon$. At the reservation productivity ε^*, the firm is indifferent between keeping and destructing the job:

$$J^F\left(\varepsilon^*\right) = J^v = 0$$

Therefore, the reservation productivity ε^* satisfies the following condition:

$$p + \varepsilon^* - b - \frac{\beta}{1-\beta}\theta\xi + \frac{\lambda}{\rho + \lambda} \int_{\varepsilon^*}^{\bar{\varepsilon}} \left[\tilde{\varepsilon} - \varepsilon^*\right] dF(\tilde{\varepsilon}) = 0 \tag{29}$$

And it implies that the free entry condition could be written as:

$$J^F(\bar{\varepsilon}) = \frac{\xi}{q(\theta)} \overset{(3)}{=} \frac{1-\beta}{\rho + \lambda}\left[\bar{\varepsilon} - \varepsilon^*\right] \tag{30}$$

3.2 Stationary Equilibrium (Recursive form)

Denote (z, ε) the state variable for household where $z \in \{e, u\}$ indicates the employment status of the agent, and ε indicates the match-specific productivity if the agent is employed[10]. Similarly, denote (ω, ε) the state variable for firm, where $\omega \in \{F, V\}$ indicates whether the vacant job is filed or not.

A stationary equilibrium consists of:

- a set of policy functions $\{c(z, \varepsilon)\}$;
- a set of value functions $\left\{V^e(\varepsilon), V_u, J^F(\varepsilon), J^v\right\}$;
- a stationary distribution over unemployment and vacancy: $\{u, v\}$;
- a set of prices $\{w(\varepsilon)\}$;
- a set of endogenous labor market parameters $\{\varepsilon^*, \theta\}$

such that:

(1) UMP: the policy function $\{c(z, \varepsilon)\}$ solve the household's utility maximization problem. Since there is no saving decision, the households are hand-to-mouth

$$c(z, \varepsilon) = \begin{cases} w(\varepsilon), & \text{if } z = e \\ b, & \text{if } z = u \end{cases} \tag{31}$$

with the corresponding value functions:

$$\begin{aligned} \rho V^e(\varepsilon) &= w(\varepsilon) + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} \left[V^e(\tilde{\varepsilon}) - V^u\right] dF\left(\tilde{\varepsilon}\right) - \lambda \left[V^e(\varepsilon) - V^u\right] \\ \rho V^u &= b + f(\theta) \left[V^e(\bar{\varepsilon}) - V^u\right] \end{aligned}$$

(2) PMP: firms decide to destruct the job optimally when $\varepsilon < \varepsilon^*$ such that

$$p + \varepsilon^* - b - \frac{\beta}{1-\beta}\theta\xi + \frac{\lambda}{\rho+\lambda} \int_{\varepsilon^*}^{\bar{\varepsilon}} \left[\tilde{\varepsilon} - \varepsilon^*\right] dF(\tilde{\varepsilon}) = 0 \tag{32}$$

with corresponding value functions:

$$\begin{aligned} \rho J^F\left(\varepsilon\right) &= p + \varepsilon - w(\varepsilon) + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} \left[J^F\left(\tilde{\varepsilon}\right) - J^V\right] dF\left(\tilde{\varepsilon}\right) - \lambda \left[J^F\left(\varepsilon\right) - J^V\right] \\ \rho J^V &= -\xi + q(\theta) \left[J^F(\bar{\varepsilon}) - J^V\right] \end{aligned}$$

(3) Nash bargaining: the equilibrium wage is determined by bargaining between the firm and the worker

$$w(\varepsilon) = (1-\beta)b + \beta \cdot \left[p + \varepsilon + \theta\xi\right] \tag{33}$$

(4) Stationary distribution:

$$\dot{u} = 0 \Rightarrow \lambda F\left(\varepsilon^*\right)(1-u) - f(\theta) \cdot u = 0 \Rightarrow u = \frac{\lambda F\left(\varepsilon^*\right)}{\lambda F\left(\varepsilon^*\right) + f(\theta)} \tag{34}$$

$$\dot{v} = 0 \Rightarrow \lambda F\left(\varepsilon^*\right)(1-u) - q(\theta) \cdot v = 0 \Rightarrow v = \theta u \tag{35}$$

(5) Free Entry: firms create vacancies at zero-profits

$$J^V = 0 \Rightarrow J^F(\bar{\varepsilon}) \overset{(1)}{=} \frac{\xi}{q(\theta)} \overset{(2)}{=} \frac{p + \bar{\varepsilon} - w(\bar{\varepsilon}) + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} J^F(\tilde{\varepsilon}) dF(\tilde{\varepsilon})}{\rho+\lambda} \overset{(3)}{=} \frac{1-\beta}{\rho+\lambda}\left[\bar{\varepsilon} - \varepsilon^*\right] \tag{36}$$

There are 6 unknown equilibrium object $\{c(z, \varepsilon), w(\varepsilon), u, v, \theta, \varepsilon^*\}$, which can be solved from the system of equations above.

[10] Check whether ε is the state variable.

3.3 Numerical method

Different from the baseline DMP model, the state variable for both household and firm are two-dimensional object, including the employment (z_j) or vacancy (ω_j) status, as well as the match specific productivity (ε). However, the transition rates of the first state variable are endogenous and depend on the transition rate of the second state variable[11]. Beside, the endogenous job destruction rate also introduce an optimal stopping problem in worker and firm's value function.

Note that the Kolmogorov forward equation and the evaluation of free entry condition are the same as in the baseline model, which will not be listed here. The only difference is in the HJB equations of worker and firm.

3.3.1 Worker and firm's HJB

To be added

3.3.2 Summary of Algorithm

0. Create sparse grid on the dimension of ε.
1. Guess θ^0, obtain labor market condition (f, q, eps^*) from matching function and FE(3) in (36).
2. Calculate wage schedule on the grids $G(\varepsilon)$ from Nash bargaining (33).
3. Guess initial value function $\{V^0_{i,j}, J^0_f\}$.
4. Solve the HJB for worker and firm (to be added).
5. Evaluate the free entry condition, update θ^{n+1}.
6. Update wage w^{n+1}_j and labor market conditions $f(\theta^{n+1}), q(\theta^{n+1}), \varepsilon^*(\theta^{n+1})$ as in step 1 and 2.
7. Repeat until convergence from step 4.

[11]Note that in the model section, the second state variable is continuous in cross-sectional dimension, but it follows a Poisson distribution in the time dimension

Appendix

Baseline model

1. Value function for filled job:

For the short period $\Delta \to 0$, the firm's value function of a filled job is

$$
\begin{aligned}
J^F &= \int_0^\Delta [p-w]\, e^{-\rho t} dt + e^{-\rho\Delta} \cdot \left\{ \underbrace{e^{-s\Delta}}_{\text{No shock}} J^F + \underbrace{s\Delta e^{-s\Delta}}_{\text{1 shock}} J^V + \underbrace{O(\Delta^2)}_{\text{More than 1 shock}} \right\} \\
&= [p-w]\,\Delta + e^{-(\rho+s)\Delta} J^F \quad + s\Delta \cdot e^{-(\rho+s)\Delta} J^V
\end{aligned}
$$

$$
\Rightarrow \underbrace{\frac{1-e^{-(\rho+s)\Delta}}{\Delta}}_{\lim_{\Delta\to 0}:\rho+s} J^F = p - w + \underbrace{s \cdot e^{-(\rho+s)\Delta}}_{\lim_{\Delta\to 0}:s} \cdot J^V
$$

After rearranging the terms, one can obtain Equation (1).

2. Value function for vacancy:

For the short period $\Delta \to 0$, the firm's value function of a vacant job is

$$
\begin{aligned}
J^V &= \int_0^\Delta -\xi \cdot e^{-\rho t} dt + e^{-\rho\Delta} \cdot \left\{ \underbrace{e^{-q(\theta)\Delta}}_{\text{No match}} J^V + \underbrace{q(\theta)\Delta e^{-q(\theta)\Delta}}_{\text{1 match}} J^F + \underbrace{O(\Delta^2)}_{\text{More than 1 match}} \right\} \\
&= -\xi \cdot \Delta + e^{-(\rho+q(\theta))\Delta} J^V \quad + q(\theta)\Delta \cdot e^{-(\rho+q(\theta))\Delta} J^F
\end{aligned}
$$

$$
\Rightarrow \underbrace{\frac{1-e^{-(\rho+q(\theta))\Delta}}{\Delta}}_{\lim_{\Delta\to 0}:\rho+q(\theta)} J^V = -\xi + \underbrace{q(\theta) \cdot e^{-(\rho+q(\theta))\Delta}}_{\lim_{\Delta\to 0}:q(\theta)} \cdot J^F
$$

After rearranging the terms, one can obtain Equation (2).

3. Value function for employed:

For the short period $\Delta \to 0$, the worker's value function is

$$
V^e = \int_0^\Delta w \cdot e^{-\rho t} dt + e^{-\rho\Delta} \cdot \left\{ \underbrace{e^{-s\Delta}}_{\text{No shock}} V^e + \underbrace{s\Delta e^{-s\Delta}}_{\text{1 shock}} V^u + \underbrace{O(\Delta^2)}_{\text{More than 1 shock}} \right\}
$$

$$
\Rightarrow \underbrace{\frac{1-e^{-(\rho+s)\Delta}}{\Delta}}_{\lim_{\Delta\to 0}:\rho+s} V^e = w + \underbrace{s \cdot e^{-(\rho+s)\Delta}}_{\lim_{\Delta\to 0}:s} \cdot V^u
$$

After rearranging the terms, one can obtain Equation (3).

4. Value function for unemployed:

For the short period $\Delta \to 0$, the value function of an unemployed job seeker is

$$
V^u = \int_0^\Delta b \cdot e^{-\rho t} dt + e^{-\rho\Delta} \cdot \left\{ \underbrace{e^{-f(\theta)\Delta}}_{\text{No match}} V^u + \underbrace{f(\theta)\Delta e^{-f(\theta)\Delta}}_{\text{1 match}} V^e + \underbrace{O(\Delta^2)}_{\text{More than 1 match}} \right\}
$$

$$\Rightarrow \underbrace{\frac{1-e^{-(\rho+f(\theta))\Delta}}{\Delta}}_{\lim_{\Delta\to 0}:\rho+f(\theta)} V^u = b + \underbrace{f(\theta)\cdot e^{-(\rho+f(\theta))\Delta}}_{\lim_{\Delta\to 0}:f(\theta)} \cdot V^e$$

After rearranging the terms, one can obtain Equation (4).

5. Wage determination:
Recall the maximization problem

$$w = \max_{\tilde{w}} \left[J^F(\tilde{w}) - J^V\right]^{1-\beta} \left[V^e(\tilde{w}) - V^u\right]^{\beta}$$

Take the first-order condition, we have:

$$\begin{aligned} 0 = (1-\beta)\cdot \left[J^F(\tilde{w}) - J^V\right]^{-\beta} \left[V^e(\tilde{w}) - V^u\right]^{\beta} \cdot \underbrace{\frac{\partial J^F(\tilde{w})}{\partial \tilde{w}}}_{=-1} \\ + \beta\cdot \left[J^F(\tilde{w}) - J^V\right]^{1-\beta} \left[V^e(\tilde{w}) - V^u\right]^{1-\beta} \cdot \underbrace{\frac{\partial V^e(\tilde{w})}{\partial \tilde{w}}}_{=1} \end{aligned}$$

Rearranging the terms, we have:

$$\begin{aligned} (1-\beta)\underbrace{\left[V^e(\tilde{w}) - V^u\right]}_{\frac{w-b}{\rho+s+f(\theta)}} &= \beta \underbrace{\left[J^F(\tilde{w}) - J^V\right]}_{\text{FE(1)}=\ \frac{\theta\xi}{f(\theta)}} \\ \Rightarrow w &= \frac{\beta}{1-\beta}\left[\rho + s + f(\theta)\right]\frac{\theta\xi}{f(\theta)} + b \\ &= \frac{1}{1-\beta}\left\{\beta\left[p - w - \theta\xi\right] + (1-\beta)b\right\} \\ &= (1-\beta)b + \beta(p + \theta\xi) \end{aligned}$$

Extension (I): Endogenous job-destruction rate

1. Value function for filled job:

For the short period $\Delta \to 0$, the firm's value function of a filled job with realized match-specific productivity ε is

$$\begin{aligned} J^F(\varepsilon) &= \int_0^{\Delta} \left[p + \varepsilon - w(\varepsilon)\right] e^{-\rho t} dt + e^{-\rho\Delta}\cdot \left\{ \underbrace{e^{-\lambda\Delta}}_{\text{No shock}} J^F(\varepsilon) + \underbrace{\lambda\Delta e^{-\lambda\Delta}}_{\text{1 shock}} \mathbb{E}max\left\{J^F(\tilde{\varepsilon}), J^V\right\} + \underbrace{O(\Delta^2)}_{\text{More than 1 shock}} \right\} \\ &= \left[p + \varepsilon - w(\varepsilon)\right]\Delta + e^{-(\rho+\lambda)\Delta} J^F(\varepsilon) \quad + \lambda\Delta\cdot e^{-(\rho+\lambda)\Delta} J^V + \lambda\Delta\cdot e^{-(\rho+\lambda)\Delta}\cdot \int_{\varepsilon^*}^{\bar{\varepsilon}} \left[J^F(\tilde{\varepsilon}) - J^V\right] dF(\tilde{\varepsilon}) \end{aligned}$$

$$\Rightarrow \underbrace{\frac{1-e^{-(\rho+\lambda)\Delta}}{\Delta}}_{\lim_{\Delta\to 0}:\rho+\lambda} J^F(\varepsilon) = p + \varepsilon - w(\varepsilon) + \underbrace{\lambda\cdot e^{-(\rho+\lambda)\Delta}}_{\lim_{\Delta\to 0}:\lambda} \cdot \int_{\varepsilon^*}^{\bar{\varepsilon}} \left[J^F(\tilde{\varepsilon}) - J^V\right] dF(\tilde{\varepsilon}) + \underbrace{\lambda\cdot e^{-(\rho+\lambda)\Delta}}_{\lim_{\Delta\to 0}:\lambda} \cdot J^V$$

After rearranging the terms, one can obtain Equation (23).

2. Value function for employed:

For the short period $\Delta \to 0$, the worker's value function with realized match-specific productivity ε is

$$V^e(\varepsilon) = \int_0^\Delta w(\varepsilon) \cdot e^{-\rho t} dt + e^{-\rho\Delta} \cdot \left\{ \underbrace{e^{-\lambda\Delta}}_{\text{No shock}} V^e(\varepsilon) + \underbrace{\lambda\Delta e^{-\lambda\Delta}}_{\text{1 shock}} \left[F(\varepsilon^*) V^u + \int_{\varepsilon^*}^{\bar{\varepsilon}} V^e(\tilde{\varepsilon}) dF(\tilde{\varepsilon}) \right] + \underbrace{O(\Delta^2)}_{\text{More than 1 shock}} \right\}$$

$$\Rightarrow \underbrace{\frac{1 - e^{-(\rho+\lambda)\Delta}}{\Delta}}_{\lim_{\Delta\to 0}:\rho+\lambda} V^e(\varepsilon) = w(\varepsilon) + \underbrace{\lambda \cdot e^{-(\rho+\lambda)\Delta}}_{\lim_{\Delta\to 0}:\lambda} \cdot \int_{\varepsilon^*}^{\bar{\varepsilon}} V^e(\tilde{\varepsilon})\, dF(\tilde{\varepsilon}) + \underbrace{\lambda \cdot e^{-(\rho+\lambda)\Delta}}_{\lim_{\Delta\to 0}:\lambda} \cdot F(\varepsilon^*) V^u$$

After rearranging the terms, one can obtain Equation (25).

3. Wage determination:

From the worker's value function, we know:

$$V^e(\varepsilon) - V^u = \frac{1}{\rho+\lambda} \left\{ w(\varepsilon) + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} \left[V^e(\tilde{\varepsilon}) - V^u \right] dF(\tilde{\varepsilon}) - b - f(\theta) \left[V^e(\bar{\varepsilon}) - V^u \right] \right\} \tag{37}$$

where

$$V^e(\bar{\varepsilon}) - V^u \overset{NB}{=} \frac{\beta}{1-\beta} \cdot J^F(\bar{\varepsilon}) \overset{FE(1)}{=} \frac{\beta}{1-\beta} \cdot \frac{\xi}{q(\theta)}$$

Besides, from the value function of firm, we have:

$$J^F(\varepsilon) = \frac{1}{\rho+\lambda} \left\{ p + \varepsilon - w(\varepsilon) + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} J^F(w(\tilde{\varepsilon})) dF(\tilde{\varepsilon}) \right\} \tag{38}$$

Substituting the (37) and (38) into the Nash bargaining result, we have:

$$w(\varepsilon) - b + \lambda \int_{\varepsilon^*}^{\bar{\varepsilon}} \left[V^e(\tilde{\varepsilon}) - V^u - \frac{\beta}{1-\beta} J^F(\omega(\tilde{\varepsilon})) \right] dF(\tilde{\varepsilon}) = \frac{\beta}{1-\beta} \left[p + \varepsilon - w(\varepsilon) + \theta\xi \right]$$

where $V^e(\tilde{\varepsilon}) - V^u - \frac{\beta}{1-\beta} J^F(\omega(\tilde{\varepsilon})) = 0$ by Nash bargaining. After rearranging the terms, we obtain the wage equation (28).

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