Optimal stopping problem

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1 Baseline stopping problem

This note is based on the material from Benjamin Moll's website.

1.1 Problem

Consider an optimal stopping problem with state variable (x) which is a diffusion process[1].

$$v(x) = \max_{\tau} \mathbb{E}_0 \int_0^{\tau} e^{-\rho t} u(x_t)\, dt + e^{-\rho \tau} S(x_\tau), \quad dx_t = \mu(x_t)\, dt + \sigma(x_t)\, dW_t \tag{1}$$

This is the simplest problem from Stokey's book. One interpretations that v is the value of a firm who may live forever. u are profits and x is the **currently realized** demand which evolves exogenously. One can close the firm and sell it at a scrap value $S(x)$. The problem is to choose an optimal shut-down time (stopping time) τ. Assume $\mu(x) < 0$ for all x to make sure that the firm shuts down eventually.

This problem can be rewritten as:

$$v(x) = \lim_{\Delta \to 0} \int_0^{\Delta} u(x) \cdot e^{-\rho t} dt + e^{-\rho \Delta} \mathbb{E}_0 \max\{v(\tilde{x}), S(\tilde{x})\} \tag{2}$$

which implies that the firm is making the discrete choice between "stopping" and "not stopping" at any point in the continous time.

1.1.1 HJB Equation

This one-dimensional optimal stopping problem, same as in McCall (1970), satisfies the "reservation value" property. To solve it, we need to find the "reservation demand" x^* such that the firm is indifferent between staying and exiting when the **currently realized** demand level is x^*. Since $u(\cdot)$ is an increasing function, $v(x)$ is increasing in the **currently realized** demand level x, the firm will stop when the demand level x falls below the reservation demand, $x \leq x^*$[2].

1. Value function for firm that not stops

[1] Note that the diffusion is a continuous time Markov process with **continuous sample paths**. But the Poisson arrival process, N_t = number of arrivals up to time t, is not a diffusion because its sample paths are not continuous: N_t is discontinuous at any arrival time

[2] $x \leq x^* \Rightarrow v(x) \leq v(x^*) = S(x^*)$

For $x > x^*$ such that the firm does not exit, the following HJB equation[3] holds (see proof in the appendix)

$$\rho v(x) = u(x) + \mu(x)v'(x) + \frac{\sigma^2(x)}{2}v''(x) \tag{3}$$

Hence, for x above the exit threshold x^* we have $v(x) \geq S(x)$, and for x below the exit threshold x^*, we have $v(x) = S(x)$. Traditional methods impose "value matching" and "smooth pasting" conditions: $v(x^*) = S(x^*)$ and $v'(x^*) = S'(x^*)$. We here pursue a different strategy.

2. HJBVI

To this end let's denote the set of x for which there is no exit by X. Then

$$\begin{aligned} x \in X: &\quad v(x) \geq S(x), \quad \rho v(x) = u(x) + \mu(x)v'(x) + \tfrac{\sigma^2(x)}{2}v''(x) \\ x \notin X: &\quad v(x) = S(x), \quad \rho v(x) \geq u(x) + \mu(x)v'(x) + \tfrac{\sigma^2(x)}{2}v''(x) \end{aligned}$$

This can also be written compactly as follows[4]

$$\min\left\{\rho v(x) - u(x) - \mu(x)v'(x) - \frac{\sigma^2(x)}{2}v''(x), v(x) - S(x)\right\} = 0 \tag{4}$$

In mathematics, (4) is called a "HJB variational inequality." See e.g. Barles, Daher, and Romano (1995) and Tourin (2013).

An alternative, and perhaps economically more intuitive way of writing this equation is

$$\rho v(x) = \max\left\{u(x) + \mu(x)v'(x) + \frac{\sigma^2(x)}{2}v''(x), \rho S(x)\right\} \tag{5}$$

However, the HJBVI formulation (4) will be useful in the numerical solution below.

Note that rather than imposing the smooth pasting condition as is usually done in economics, this is now a result. That is, one can prove that the HJBVI (4) implies the smooth pasting condition $v'(x^*) = S'(x^*)$. See e.g. Oksendal's book who calls "smooth pasting" the "high contact (or smooth fit) principle."

1.2 Finite Difference Method

1.2.1 Solving as Linear Complementarity Problem (LCP)

This is inspired by Huang and Pang (1998) who realize the connection between the HJB for American options and LCP.

Without the exit option, we would have HJB (3) which is discretized as

$$\rho v_i^n = u_i + \mu_i \left(v_i^n\right)' + \frac{\sigma_i^2}{2}\left(v_i^n\right)''$$

or in matrix form

$$\rho v = u + \mathbf{A}v$$

[3] By including $\mu(x)$ and $\sigma(x)$ in the value function, this condition states that if the currently realized demand level satisfies this condition, the x_t will never be too low for firm to stay.

[4] To see this take any two functions $f(x)$ and $g(x)$ and consider the following statement: for all x either $f(x) \geq 0, g(x) = 0$ or $f(x) = 0, g(x) \geq 0$. This statement can be written compactly as $\min\{f(x), g(x)\} = 0$ for all x.

Instead, now v solves the variational inequality (4). The discretized analogue is:

$$\min\{\rho v - u - \mathbf{A}v, v - S\} = 0$$

Equivalently

$$\begin{aligned}(v - S)'(\rho v - u - \mathbf{A}v) &= 0 \\ v &\geq S \\ \rho v - u - \mathbf{A}v &\geq 0\end{aligned}$$

Note that the second and third equations imply that the first equation actually has to hold element-wise.

Let's denote the ··excess value" $z = v - S$ and $\mathbf{B} = \rho\mathbf{I} - \mathbf{A}$. Then the second equation is $z \geq 0$ and the third equation is

$$\mathbf{B}z + q \geq 0$$

where $q = -u + \mathbf{B}S$. Summarizing

$$\begin{aligned}z'(\mathbf{B}z + q) &= 0 \\ z &\geq 0 \\ \mathbf{B}z + q &\geq 0\end{aligned}$$

This is the standard form for LCPs problem, see details in Wikipedia, so it can solve it with an LCP solver.

References

Barles, G., Daher, C., & Romano, M. (1995). Convergence of numerical schemes for parabolic equations arising in finance theory. *Mathematical Models and Methods in Applied Sciences*, *05*(01), 125-143.

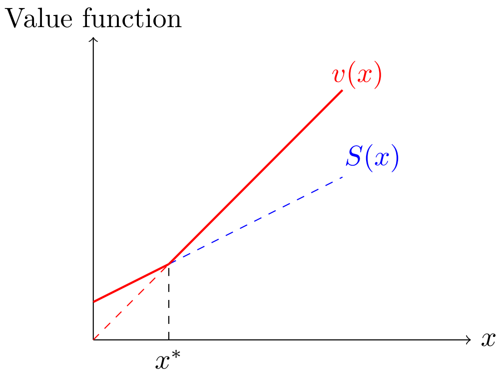
Huang, J., & Pang, J. S. (1998). Option pricing and linear complementarity. *Journal of Computational Finance*, *2*, 31-60.

McCall, J. J. (1970). Economics of information and job search. *The Quarterly Journal of Economics*, *84*(1), 113-126.

Appendix

HJB for firm that never stops:

The "reservation value" property implies that $v(x) = S(x)$ for $x \leq x^*$. A simple illustration would be:



Therefore, for a very short period Δ such that $x_t = x > x^*, t \in [0, \Delta]$, the value function (2) can be rewritten as

$$\begin{aligned}
v(x) &= \int_0^{\Delta} u(x) \cdot e^{-\rho t} dt + e^{-\rho \Delta} \cdot \mathbb{E}_0 \left[v(x_{\Delta}) \right] \\
&= u(x) \cdot \Delta + \frac{\mathbb{E}_0 \left[v(x_{\Delta}) \right]}{1 + \rho \Delta} \\
\rho v(x) &= u(x) + \frac{\mathbb{E}_0 \left[v(x_{\Delta}) - v(x) \right]}{\Delta} \\
&= u(x) + \mathbb{E}_0 \left(\frac{dv(x)}{dt} \right)
\end{aligned}$$

The second equality is obtained from $e^{-x} \approx \frac{1}{1+x}$ for $x \to 0$. By *Ito's Lemma*[5], we know that

$$\begin{aligned}
dv(x_t) &= v_x \cdot d(x_t) + \frac{1}{2} v_{xx} d(x_t)^2 \\
&= v'(x_t) \left(\mu(x_t) dt + \sigma(x_t) dW_t \right) + \frac{1}{2} v''(x_t) \left(\mu(x_t) dt + \sigma(x_t) dW_t \right)^2
\end{aligned}$$

Because $dW_t^2 = dt$, $\mathbb{E}\left(W_t\right) = dt^2 = 0$, the HJB wirtes

$$\rho v(x) = u(x) + v'(x) \mu(x) + \frac{\sigma^2(x)}{2} v''(x), x > x^*$$

[5]Since $v(\cdot)$ is time invariant, $\frac{dv}{dt} = 0$