清 华 大 学

综 合 论 文 训 练

题目：Vlasov-Poisson 系统的适定性

系　　别：

专　　业：

姓　　名：

指导教师：

2020 年 4 月 16 日

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中文摘要

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关键词：TeX；LaTeX；CJK；模板；论文

ABSTRACT

Recent researches on the Vlasov-Poisson system have been concluded in this literature review, and relevant results concerning Vlasov-Maxwell system are also presented as extension. The well-known Landau damping, a varitey of long-term time asymptotic behaviour of Vlasov-Poisson system, is introduced concisely before the main body.

The literature mainly centres on the topic of well-posedness in the Vlasov-Poisson system, discussing about the existence and uniqueness problem in the framework of Lions et al. (1991) in Sect. 2. Some ”small” cases are known to have global existence and a case of blow up with ”large” enough spherically symmetric initial data. Nonlinear stability proof for the stable solutions in Vlasov-Poisson system are introduced in Sect. 3, mainly based on the energy-Casimir methoud which is an elegant method to settle stability issue in conservative systems. As the byproducts in the above proof, regularites about some concrete quantities, *e.g.* $\mathbf{E}$ and ρ, would be exhibited in Sect. 4.

Keywords: TeX; LaTeX; CJK; template; thesis

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Contents

Chapter 1 Introduction

Terms marked with *color* mean they are not yet verified.

1.1 Background

The Vlasov type equation is studied as a governing equation describing the multi-body motion of the particle swarm, in which the anistropic velocity distribution contributes a significant influence to the dynamics of the system. Phase space distribution $f(t, \mathbf{x}, \mathbf{v}) \geqslant 0$ ($x \in \mathbb{R}_x^3,\ v \in \mathbb{R}_v^3,\ t \geqslant 0$) with initial data $f_0(\mathbf{x}, \mathbf{v}) = f(0, \mathbf{x}, \mathbf{v})$ determines the particle density at $(t, \mathbf{x}, \mathbf{v})$, *i.e.*, the number of particles per unit volume in phase space. When coupled with Maxwell equations as the electromagnetic field governing rule, Vlasov equation is capable to decide the dynamic scenario for particle-field interaction, named Vlasov-Maxwell system (VM).

$$\text{(VM \& RVM)} \begin{cases} f_t + \mathbf{a}(\mathbf{v}) \cdot \boldsymbol{\nabla}_x f + (\mathbf{E} + \mathbf{a}(\mathbf{v}) \times \mathbf{B}) \cdot \boldsymbol{\nabla}_v f = 0 \\ \mathbf{E}_t = \boldsymbol{\nabla} \times \mathbf{B} - \mathbf{j} \\ \mathbf{B}_t = -\boldsymbol{\nabla} \times \mathbf{E} \\ \boldsymbol{\nabla} \cdot \mathbf{E} = \rho, \quad \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \end{cases} \tag{1-1}$$

in which $\mathbf{a}(\mathbf{v}) \in \{\mathbf{v}, \hat{\mathbf{v}}\},\ \hat{\mathbf{v}} := \mathbf{v}/\sqrt{1 + |\mathbf{v}|^2}$ and

$$\rho(t, \mathbf{x}) = \int_{\mathbb{R}_v^3} f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}, \quad \mathbf{j}(t, \mathbf{x}) = \int_{\mathbb{R}_v^3} \mathbf{a}(\mathbf{v}) f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}. \tag{1-2}$$

Here $\mathbf{E}, \mathbf{B}, \rho$ and $\mathbf{j}$ expressed electric field, magnetic field, spatial density and current density respectively. The more dedicated Einstein theory considers relativistic effect when $\mathbf{a}(\mathbf{v}) = \hat{\mathbf{v}}$ and limit the maximum velocity the particles can reach, transforming the Vlasov-Poisson to relativistic Vlasov-Poisson (RVP) and the Vlasov-Maxwell to relativistic Vlasov-Maxwell (RVM).

One could easily capture the physical essential of the Vlasov type equaiton by observing the $\mu(\mathbf{E} + \mathbf{a}(\mathbf{v}) \times \mathbf{B})$ term which is indeed the acceleration of particles located at $(\mathbf{x}, \mathbf{v})$ in the phase space.

Furthermore, under the assumption that the electrostatic force dominates the interaction, *i.e.* the Lorentz force could be treated as zero, magnetic force is omitted and then comes the Vlasov-Poisson system (VP). $\mathbf{E}$ could be expressed as the gradient of the electrostatic potential under the assumption.

$$\mathbf{E}(t, \mathbf{x}) = \nabla_x \phi(t, \mathbf{x}), \quad \phi = \frac{1}{4\pi} \frac{1}{|\mathbf{x}|} * \rho \tag{1-3}$$

$$\text{(VP \& RVP)} \begin{cases} \partial_t f + \mathbf{a}(\mathbf{v}) \cdot \nabla_x f + \mu \nabla_x \phi \cdot \nabla_v f = 0 \\ \Delta\phi = \rho(f) := \int_{\mathbb{R}^3_v} f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v} \end{cases} \tag{1-4}$$

where, $\mu \in \{+, -\}, \mathbf{a}(\mathbf{v}) \in \{\mathbf{v}, \hat{\mathbf{v}}\}$, and $\hat{\mathbf{v}} := \mathbf{v}/\sqrt{1 + |\mathbf{v}|^2}$. The sign of μ indicates different physical scenario, while "+" for the plasma physics case and "-" for the stellar dynamics case. (VP) in fact indicates the rules of swarn motion governed by the scalar potential field-particle interaction, in which that the scalar potential field is generated by the particles themselves and that the stellar dynamics (plasma physics) case shows the absorbing gravitation (the repulsive Coulomb force) respectively.

Multi-species extended equation of (VP) and (RVP) could be easily established by adding q_i and m_i in the original ones. Note that μ must be "+" at the present because the stellar dynamics case only allow single species situation.

$$\text{(Multi-species VP \& RVP)} \begin{cases} \partial_t f_i + \mathbf{a}(\mathbf{v}) \cdot \nabla_x f_i + q_i/m_i \nabla_x \phi \cdot \nabla_v f_i = 0, \\ \Delta\phi = \rho(f) := \int_{\mathbb{R}^3_v} \sum_i q_i f_i d\mathbf{v} \end{cases} \tag{1-5}$$

Though (VM) results give many heuristic clues in the research of (VP) and much more is known about (VP). In this paper we mainly investigate the characteristics of (VP) and (RVP).

1.2 Basic Properties of Vlasov-Poisson Equation

1.2.1 Characteristics in Various Coordinate Systems

Frequently used in the context of (VP) system research are the characteristics: $s \mapsto \mathbf{X}(s, t, \mathbf{x}, \mathbf{v})$, $s \mapsto \mathbf{V}(s, t, \mathbf{x}, \mathbf{v})$ defined as the solutions of the following corresponding system of ordinary differential equations:

$$\begin{aligned}\frac{d\mathbf{X}}{ds} &= a(\mathbf{V}) = \{\mathbf{V}, \mathbf{V}/\sqrt{1+|\mathbf{V}|^2}\} \\ \frac{d\mathbf{V}}{ds} &= \gamma \mathbf{E}(\mathbf{X}, s)\end{aligned} \tag{1-6}$$

indicating the trace of a particle who arrives at $(\mathbf{x}, \mathbf{v})$ at the time of $s = t$, *i.e.*, $\mathbf{X}(t, t, \mathbf{x}, \mathbf{v}) = \mathbf{x}$, $\mathbf{V}(t, t, \mathbf{x}, \mathbf{v}) = \mathbf{v}$. Hence $||f(t, \cdot, \cdot)||_\infty = ||f_0||_\infty < \infty$ by initial data boundedness assumption.

The characteristics with *spherical symmetric* initial data, *i.e.*, that $f_0(U\mathbf{x}, U\mathbf{v}) = f_0(\mathbf{x}, \mathbf{v})$ for any rotation matrix U on $\mathbb{R}^3$, would be further specified. The initial data would induce the spherically symmetric solution with simplified argument $r = |\mathbf{x}|$, $u = |\mathbf{v}|$, $\alpha = \angle(\mathbf{x}, \mathbf{v})$ and t (cf. [3,8]①). The density ρ depends then only on r and t:

$$\rho(t, r) = 2\pi \int_0^\infty \int_0^\pi f(t, r, u, \alpha) u^2 \sin\alpha d\alpha du, \tag{1-7}$$

the same as the electrostatic potential $\phi(r, t)$:

$$\phi(t, r) = -\frac{1}{r} \int_0^r \lambda^2 \rho(t, \lambda) d\lambda - \int_r^\infty \lambda \rho(t, \lambda) d\lambda \tag{1-8}$$

The electric field $\mathbf{E}$ as the gradient of the potential and new notation $M(t, r)$ introduced as below:

$$\mathbf{E}(t, \mathbf{x}) = \nabla_x u = \frac{\mathbf{x}}{r^3} \int_0^r \lambda^2 \rho(t, \lambda) d\lambda = \frac{\mathbf{x}}{r^3} M(t, r) \tag{1-9}$$

Notice that $M(t, r)$ is essentially the integral of ρ on the volume the sphere with radius r except a factor of 4π, showing $\lim_{r\to\infty} M(t, r) = \mathscr{M}/4\pi$. Moreover, $|\mathbf{E}| = r^{-2} M(t, r)$.

Spherical symmetry brings in the below simplification of the characteristics' ordinary differential equations and note that dA/dt is decided by the $d(\mathbf{X} \cdot \mathbf{V})/ds =$

① Not yet found, mentioned in *Glassey, Schaeffer - 1985 - On symmetric solutions of the relativistic Vlasov-Poisson system*

$d(RU \cos A)/ds$. Curly brackets $\{..., ...\}$ includes the terms for (VP) in the left and for (RVP) in the right.

$$\begin{cases} \dfrac{dR}{ds} = |\mathbf{a}(\mathbf{V})| \cos A = \left\{ U \cos A, \dfrac{U \cos A}{\sqrt{1+U^2}} \right\} \\ \dfrac{dU}{ds} = \left|\dfrac{d\mathbf{V}}{ds}\right| \cos < \dfrac{d\mathbf{V}}{ds}, \mathbf{a}(\mathbf{V}) >= \gamma \dfrac{\cos A}{R^2} M(s,R) \\ \dfrac{dA}{ds} = -\left(\gamma \dfrac{M(s,R)}{R^2 U} + \left\{ \dfrac{U}{R}, \dfrac{U}{R\sqrt{1+U^2}} \right\} \right) \sin A \end{cases} \tag{1-10}$$

1.2.2 Constant Physical Quantities

Proposition 1.1: *(Constant Physical Quantities)* Let f be a classical solution of (RVP) on some time interval $(0,T)$ with nonnegative initial data $f_0 \in C_0^1(\mathbb{R}^6)$. Then the following properties hold:

1. If f_0 vanished for $|x| > k$, then $f(t, \mathbf{x}, \mathbf{v})$ vanished for $|\mathbf{x}| > t + k$ (casuality).
2. The total mass is conserved, *i.e.*, $\iint_{\mathbb{R}^6} f \, d\mathbf{v} d\mathbf{x} =$ constant $= \mathscr{M}$.
3. The total energy is conserved, *i.e.*,

$$\text{(RVP)} \int_{\mathbb{R}^3_x} \left\langle \int_{\mathbb{R}^3_v} \sqrt{1+|\mathbf{v}|^2} f \, d\mathbf{v} + \frac{1}{2}\gamma |\mathbf{E}|^2 \right\rangle d\mathbf{x} = \text{ constant } = \mathscr{E}_0 \tag{1-11}$$

$$\text{(VP)} \int_{\mathbb{R}^3_x} \left\langle \int_{\mathbb{R}^3_v} \frac{|\mathbf{v}|^2}{2} f \, d\mathbf{v} + \frac{1}{2}\gamma |\mathbf{E}|^2 \right\rangle d\mathbf{x} = \text{ constant } = \mathscr{E}_0 \tag{1-12}$$

4. If $\gamma = +1$, then there exists a constant C (depending on $\|f_0\|_\infty$ and $\mathscr{E}_0$) such that $||\rho(t)||_{4/3} \leqslant C$ for $0 \leqslant t < T$.

proof TODO: The proof of (RVP) to be reorganized from Glassey et al. (1985). □

1.3 Well-posedness Results

1.3.1 With Compact Support in "v"

For classical solutions, it is well known that the existence and uniqueness result of the Vlasov-Poisson system solution have been presented by Iordanskii ① in dimension

① The paper [16] listed in the reference of Lions et al. (1991) is missing, refered as Iordanskii, S.V.: The Cauehy problem for the kinetic equation of plasma. Transl., II. Ser., Am. Math. Soc. 35, 351-363 (1964)

1, Ukai et al. (1978) in dimension 2, Bardos et al. (1985) in dimensions 3 for small data. The case of (nearly) classical symmetric data has been treated by Batt (1977), Wollman (1980), Horst et al. (1981), Schaeffer (1987), among whom Schaeffer (1987) treated the relativistic case of symmetric data in one paper.

In particular, in three space dimensions, global weak solutions exist (Arsen'ev (1975); Abdallah (1994)), and global classical solutions exist if the Cauchy data are small enough given by Bardos et al. (1985).

When we turn to the relativistic case, at first sight, (RVP) seems "better" than its classical version, since $|\hat{\mathbf{v}}| < 1$. Thus "higher moment difficulties" well-known in the classical case, will not occur. Moreover, for the same reason we have the casuality along characteristics due to the limited velocity. These favorable circumstances are diminished somewhat by examination of the total energy integral.① One may hope then that (RVM) is better behaved than (VM), but only when $\gamma = +1$.

In the plasma physics case, $\rho \in L^{4/3}(\mathbb{R}^3_x)$ is worse than the result for (VP) itself, where $\rho \in L^{5/3}(\mathbb{R}^3_x)$. However, Batt's and Wollman's methods [*cf.* Batt (1977), Wollman (1980)] can still be adapted and used to show the existence of global spherically symmetric solutions when $\gamma = +1$. In contrast to the plasma case, Glassey et al. (1985) stated the existence of solution of stellar dynamics is weakened in the sense that a restrction on the size of initial data is necessary. Only "small" radial solutions with $40M^{2/3}\|f_0\|_\infty^{1/3}$ are confirmed to exist in the large for (RVP) with $\gamma = -1$. Indeed, if $\gamma = -1$ and the initial energy $\mathscr{E}_0$ is negative, it is shown that the life-span of such a radial, classical solution is finite.

The outstanding result for the full 3D Maxwell-Vlasov system remains that of Glassey et al. (1986), who were able to prove a global existence result under the hypothesis of compactly supported (for all time !) particle density.

Theorem 1.1: *(Glassey-Strauss)* For (RVM) system, assume the standard regularity $f_0 \in C^1_0$ and $\mathbf{E}_0, \mathbf{B}_0 \in C^2$ for the initial data and assume there exists a continuous function $\beta(t)$ such that for all $\mathbf{x} \in \mathbb{R}^3$

$$f(t, \mathbf{x}, \mathbf{v}) = 0 \quad \text{for} \quad |\mathbf{v}| > \beta(t) \tag{1-13}$$

Then there exists a unique C^1 solution of the system for all t.

① Why? I don't knwo why it is diminished.

The Glassey-Strauss proof relies on showing uniform bounds in time for the $||\mathbf{E}||_\infty, ||\mathbf{B}||_\infty$, $||f||_\infty$ as well as of all their first spatial derivatives. They started by rewriting (RVM) as follows:

$$\begin{cases} \partial_t f + \hat{\mathbf{v}} \cdot \nabla_x f + (\mathbf{E} + \hat{\mathbf{v}} \times \mathbf{B}) \cdot \nabla_v f = 0, \quad (\mathbf{x}, \mathbf{v}) \in \mathbb{R}^3 \times \mathbb{R}^3 \\ \mathbf{E}_{tt} - \Delta \mathbf{E} = -\left(\partial_t \mathbf{j} + \nabla_x \rho\right) = -\int_{\mathbb{R}^3} \left(\nabla_x f + \hat{\mathbf{v}} \partial_t f\right) d\mathbf{v} \\ \mathbf{B}_{tt} - \Delta \mathbf{B} = \nabla_x \times \mathbf{j} = \int_{\mathbb{R}^3} \left(\hat{\mathbf{v}} \times \nabla_x\right) f d\mathbf{v} \\ f(0, \mathbf{x}, \mathbf{v}) = f_0(x, v) > 0, \mathbf{E}(0, \mathbf{x}) = \mathbf{E}_0(\mathbf{x}), \mathbf{B}(0, \mathbf{x}) = \mathbf{B}_0(\mathbf{x}) \end{cases} \tag{1-14}$$

Then they represent the fields $\mathbf{E}$ and $\mathbf{B}$ using the explicit form of the fundamental solution of $\square = \partial_t^2 - \Delta$ in physical space. For example for E they write

$$\mathbf{E}(t, \mathbf{x}) = \mathbf{E}_0(t, \mathbf{x}) + \frac{1}{2\pi} \int_0^t ds \int_{C_{t,s}} \frac{d\mathbf{y}}{|\mathbf{y} - \mathbf{x}|} \left(\int \left(\nabla + \hat{\mathbf{v}} \partial_t\right) f(s - |\mathbf{y} - \mathbf{x}|, \mathbf{y}, \mathbf{v}) d\mathbf{v} \right) \tag{1-15}$$

where $\mathbf{E}_0$ is a solution of the homogeneous equation $\square \mathbf{E}_0 = 0$, $C_{t,s}$ is the cone $C_{t,s} = \{|\mathbf{y} - \mathbf{x}| \leqslant t - s\}$ and $\nabla = (\partial_1, \partial_2, \partial_3)$. The main trouble is that the existence of $\nabla + \hat{\mathbf{v}}\partial_t$ requires the estimates for the second order derivatives of f if $||\nabla \mathbf{E}||_\infty$ estimate requested. Glassey-Strauss eases the requirement by decomposing $\nabla + \hat{\mathbf{v}}\partial_t$ into fields $T_i = \partial_{y_i} - \omega_i \partial_t$, with $\omega_i = \frac{(\mathbf{y}-\mathbf{x})_i}{|\mathbf{y}-\mathbf{x}|}$, and $S = \partial_t + \hat{\mathbf{v}} \cdot \nabla = -(\mathbf{E} + \hat{\mathbf{v}} \times \mathbf{B}) \cdot \nabla_v$, in terms of which the ∂_t, ∂_x can be reexpressed, *e.g.*,

$$\partial_{y_i} = \frac{\omega_i S}{1 + \hat{v} \cdot \omega} + \left(\delta_{i,j} - \frac{\omega_i \hat{v}_j}{1 + \hat{v} \cdot \omega}\right) T_j \tag{1-16}$$

Note the v derivative of S operator coorporates with the v-integration in equation (1-15). Then the estimate of $||\mathbf{E}||_\infty$ and $||\mathbf{B}||_\infty$ by $||f||_\infty$ is acquired, as long as the denominator $1 + \hat{\mathbf{v}} \cdot \omega$ is bounded away from zero guaranted by the assumption (1-13).

Furthermore, to estimate the norm of the derivatives of $\mathbf{E}$ and $\mathbf{B}$, Glassey-Strauss restricted $\nabla\mathbf{E}$ and $\nabla\mathbf{B}$ by $\sup_{\tau \leqslant t} \|Df(\tau)\|$, with $D = (\nabla x, \nabla v)$, according to the following inequality that for any t in a fixed interval of time $[0, T]$,

$$\left\|\nabla_x \mathbf{E}(t)\right\|_\infty + \left\|\nabla_x \mathbf{B}(t)\right\|_\infty \leqslant C_T \left(1 + \log^+ \left(\sup_{\tau \leqslant t} \|Df(\tau)\|_\infty\right)\right). \tag{1-17}$$

On the other hand, $||Df(t)||_\infty$ estimate is gained by using the transport equation of (RVM) directly.

$$\|Df(t)\|_\infty \leqslant C_T \int_0^t \left(1 + \left\|\nabla_x \mathbf{E}(\tau)\right\|_\infty + \left\|\nabla_x \mathbf{B}(\tau)\right\|_\infty\right) \|Df(\tau)\|_\infty d\tau \tag{1-18}$$

The above two inequalities combined with the Gronwall's inequality gives a bound for all the quantities involved. Continue the proof by a recursive method and then the proof finished.

Staffilani et al. (2001) reorganized the (RVM) to fit their method based on Fourier Transform and proved the essentially same theorem with Glassey et al. (1985) .

$$\begin{cases} \partial_t f + \hat{\mathbf{v}} \cdot \nabla_x f + \boldsymbol{\alpha}(v)\boldsymbol{\Phi} \cdot \nabla_v f = 0, \quad (\mathbf{x}, \mathbf{v}) \in \mathbb{R}^3 \times \mathbb{R}^3 \\ \boldsymbol{\Phi}_{tt} - \Delta\boldsymbol{\Phi} = \mathbf{J}(t, \mathbf{x}) \\ f(0, \mathbf{x}, \mathbf{v}) = f_0(\mathbf{x}, \mathbf{v}) > 0, \quad \boldsymbol{\Phi}(0, \mathbf{x}) = \boldsymbol{\Phi}_0(\mathbf{x}) \end{cases} \tag{1-19}$$

$$\begin{aligned} &\boldsymbol{\Phi}(t, \mathbf{x}) = (\mathbf{E}(t, \mathbf{x}), \mathbf{B}(t, \mathbf{x})) \\ &\boldsymbol{\alpha}(\mathbf{v})\boldsymbol{\Phi} = \mathbf{E} + \hat{\mathbf{v}} \times \mathbf{B} \\ &\mathbf{J}(t, \mathbf{x}) = \int_{\mathbb{R}^3} M(\mathbf{v})\nabla_x f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v} + \int_{\mathbb{R}^3} N(\mathbf{v})\boldsymbol{\Phi}(t, \mathbf{x}) \cdot \nabla_v f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v} \end{aligned} \tag{1-20}$$

with $M(\mathbf{v})$ and $N(\mathbf{v})$ matrices depending only on $\mathbf{v}$.

Theorem 1.2: (Klainerman-Staffilani) Consider the IVP (1-19) with $f_0 \in C_0^1\left(\mathbb{R}^3 \times \mathbb{R}^3\right)$ and $\Phi_0(x) \in C^1\left(\mathbb{R}^3\right)$ Assume that, on any fixed interval of time $[0, T]$

$$\|\Phi\|_{L^\infty_{x,t}([0,T]\times\mathbb{R}^3)} < C$$

Then the system (1-19) admits a unique $C^1\left([0, T] \times \mathbb{R}^3\right)$ solution.

1.3.2 Without Compact Support in "v"

The question of global well-posedness for (VP) system without the compact assumpyion in "v" has been considered by many authors. The Vlasov-Poisson system has been tackled successfully, for large data, by Pfaffelmoser (1992), Lions et al. (1991), Schaeffer (1991).

Lions et al. (1991), based on the representation formula built by the characteristic method considering the source term, proves the propagation of moments in v higher than 3. More precisely, if $|\mathbf{v}|^m f_0 \in L^1(\mathbb{R}^6)$ for all $m < m_0$, with $m_0 > 3$, then we build a solution of Vlasov-Poisson equations satisfying $|\mathbf{v}|^m f(t, \mathbf{x}, \mathbf{v}) \in L^1(\mathbb{R}^6)$ for all $m < m_0$, with $m_0 > 3$ for any $t > 0$. Moreover, for $m_0 > 6$, *Sobolev injections* deduces that $\mathbf{E} \in L^\infty([0, T] \times \mathbb{R}^3)$ for any $t > 0$, and, following Horst ①, f is a smooth function if $f_0(x, v)$ is smooth.

For the Vlasov-Maxwell system, Glassey et al. (1986) deduced the result that the classical solution can be globally extended with the strong assumption that f has compact support in $\mathbf{v}$ for all the time. Later Glassey et al. (1990) were able to remove the additional support hypothesis for the 2+1/2 dimensional system whose $\mathbf{x} \in \mathbb{R}^2, \mathbf{v} \in \mathbb{R}^3$.

Continuation criterion for the global existence of the Vlasov-Maxwell system may be a more relaxed condition than assuming the compact support in "$\mathbf{v}$". Robert et al. (1989) proved that, if the initial data decay at rate $|\mathbf{v}|^{-7}$ as $|\mathbf{v}| \to \infty$ and the imposed assumption that bound "$\iint_{\mathbb{R}^3} |\mathbf{v}| f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v} \leq \text{Constant}$" holds, then the solution extends globall. Luk et al. (2016), Kunze (2015), Pallard (2015) and Patel (2018) have developed recent improvements on the continuation criterion. To release the limitation of the compact support and get rid of the assumptions on the solution itself, Wang (2018) studied the propagation of regularity and the long time behavior of the 3D RVM system for suitably small initial data.

1.4 Asymptotic Behaviour

The electrostatic Vlasov-Poisson equation (VP) suffices to induce the well-known Landau Damping phenomenon in the plasma. Linearized Vlasov-Poisson equation is capable to describe the behaviour of the electron distribution function in a plasma: given a uniform steady distribution $f_0(\mathbf{v})$ and an initial perturbation $f_0(\mathbf{x}, \mathbf{v})$, the equation describes the evolution of this perturbation $f(t, \mathbf{x}, \mathbf{v})$ under the action of the electrostatic potential $\phi(t, \mathbf{x})$.

Landau damping arose when physicists solved equation (lineaized VP) by means

① The paper [14] listed in the reference of Lions et al. (1991) is missing, refered as Horst, E.: Global strong solutions of Vlasov's Equation. Necessary and sufficient conditions for their existence. Partial differential equations. Banach Cent. Publ. 19, 143 153 (1987) of a Fourier-Laplace transform (cf. Krall et al. (1973)) and wondered whether the $\phi(t, \mathbf{x})$ behaved like plane waves when t is large. They found that there is no way to exhibit damped plane waves except by providing an analytical extension of the Laplace transform of f, which is only possible by assuming that f_0 and g are analytic with respect to $\mathbf{v}$. Although very strong, these hypotheses are verified in numerous physical situations for instance in the case of a Maxwellian distribution: $f_0(v) = \exp\left(-v^2\right)$. (TODO: This paragraph needs to be reorganized)

Wenyin is unfamiliar with the topic of operator theory employed in the relevant papers, so he might just give a short introduction about this topic. However, this topic is tightly related to the Landau damping, Wenyin would try to figure it out though he has little idea about *the resolvent, Dunfold formula and the spectrum of an operator.*

$$\text{(linearized VP)} \quad \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f_0(\mathbf{v}) = 0 \tag{1-21}$$

Degond (1986) studied the spectral theory of the linearized Vlasov-Poisson equation in order to prove that its solution behaves, for large times, like a sum of plane waves. (lineaized VP) is transformed to such a form $\dot{f} = T \cdot f; \quad f(0) = f_0$ that could be solved by semigroup theory: $f(t) = \exp(tT) \cdot f_0$. The Dunford formula (Hille, 1948) then relates it to the resolvent $R_\lambda = (\lambda - T)^{-1}$. A deformation of the path of integration in the Dunford formula allows to give an asymptotic expansion for f without damped wave.

To obtain the following distribution function expansion involving damped waves, an analytical extension of the resolvent of the equation is necessary.

$$f(x, v, t) = \sum_{s=1}^{S} \alpha_s(v) e^{\lambda_s t + i n_s x} + \mathcal{O}\left(e^{rt}\right), \quad r < \min_{1 \leqslant s \leqslant S} \left(\operatorname{Re} \lambda_s\right) \tag{1-22}$$

where r can be negative, and $\alpha_s(v)$ are well-defined functions of v. Then the poles λ_s of this extension are no longer eigenvalues of T and must be interpreted as eigenmodes, associated to "generalized eigenfunctions" which actually are linear functionals on a Banach space of analytic functions.

This work is an attempt at a mathematical explanation of Landau damping in terms of eigenmodes and scattering theory. In is shown that the potential $\phi(t, x)$ and the transform $f(t, x, \xi)$ of $f(t, x, v)$ actually admits the expansions:

$$\phi(x, t) = \sum_{s=1}^{S} c_s e^{\lambda_s t + i n_s x} + \mathcal{O}\left(e^{rt}\right), \quad r < \min_{1 \leqslant s \leqslant S} \left(\operatorname{Re} \lambda_s\right) \tag{1-23}$$

$$\check{f}(x, \xi, t) = \sum_{s=1}^{S} \check{\alpha}_s(\xi) e^{\lambda_s t + i n_s x} + \mathcal{O}\left(e^{rt}\right), \quad r < \min_{1 \leqslant s \leqslant S} \left(\operatorname{Re} \lambda_s\right) \tag{1-24}$$

Landau damping is examined by Krall et al. (1973). Beside this approach, another successful theory has been developed by Kampen (1955) and Case (1959), using a ”normal mode expansion”.

Chapter 2 Existence and Uniqueness

2.1 Existence

For the basic 3D (non-relativistic) Vlasov-Poisson system problem, we first revisited the proof of Lions and Perthame (1991) to confirm the existence and uniqueness of a weak solution for a standard initial data without the compact assumption in "v".

Theorem 2.1: *(Lions-Perthame)* Let $f_0 \geqslant 0, f_0 \in L^1 \bigcap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$. We assume that

$$\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} |\mathbf{v}|^m f_0(\mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v} < +\infty \quad \text{if } m < m_0, \tag{2-1}$$

where $3 < m_0$. Then, there exists a solution $f \in C(\mathbb{R}^+; L^p(\mathbb{R}^3 \times \mathbb{R}^3)) \bigcap L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$ (for all $1 \leqslant p < +\infty$) of Vlasov-Poisson system satisfying

$$\sup_{t\in[0,T]} \iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} |\mathbf{v}|^m f(t, \mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v} < +\infty \tag{2-2}$$

In the first step of the proof, we prove some general estimates on $\mathbf{E}$. Then, we conclude assuming that the time interval $(0, T)$ on which we solve the Vlasov-Poisson system is small enough and that $\mathbf{E}$ is bounded in $L^{3/2}(\mathbb{R}^3_x)$, in fact which is false in general but holds true for bounded domains of $\mathbb{R}^3_x$. We relax this assumption on $\mathbf{E}$ in a fourth and final step.

Due to the solution lying in general functional space $\mathscr{D}'$, $\mathbf{E} \cdot \nabla_v f = \operatorname{div}_v(\mathbf{E}f)$ ①

proof Simplify the proof by handling smooth solutions first, *i.e.*, C^∞ with compact support. The *standard approximation method* (see[5,6]) would be enough to say the estimates are uniform and expand the function space. Treat the $\mathbf{E} \cdot \nabla_v f$ on the RHS as a source term safely in Vlasov-Poisson equation (1-4), trace the characteristic and then integrate in v to acquire $\rho(t, x)$.

① There seems to be a typo in the paper where $\mathbf{E} \cdot \nabla_x f = \operatorname{div}_v(\mathbf{E}f)$

Definition 2.1: Denote the supreme on $[0,t]$ of k-order moment of the solution in $|\mathbf{v}|$ as follows:

$$M_k(t) = \sup_{0\leqslant s\leqslant t} \int |\mathbf{v}|^k f(s) d\mathbf{x} d\mathbf{v} \tag{2-3}$$

(i) *General estimates on* $\mathbf{E}(t)$ *and* $M_k(t)$.

Follow the characteristic line along which $\left(\mathrm{div}_v\, Ef\right)$ is treated as the source term,

$$\begin{aligned} &\dot{\mathbf{X}}(s) = -\mathbf{v}, \quad \mathbf{X}(0) = x \\ &\dot{\mathbf{V}}(s) = 0, \qquad \mathbf{V}(0) = \mathbf{v}, \quad 0 \leqq s \leqq T \end{aligned} \tag{2-4}$$

$$\begin{aligned} f(t,x,v) &= -\int_0^t \left(\mathrm{div}_v\, Ef\right)(t-s, x-vs, v) ds + f_0(x-vt, v) \\ &= -\int_0^t \mathrm{div}_v[Ef(t-s, x-vs, v)] ds \\ &\quad - \int_0^t s\, \mathrm{div}_x[Ef(t-s, x-vs, v)] ds + f_0(x-vt, v) \end{aligned} \tag{2-5}$$

The density comes from f integrated in $\mathbf{v}$ naturally,

$$\rho(t,\mathbf{x}) = -\,\mathrm{div}_x \int_0^t s \int_{\mathbb{R}_v^3} [\mathbf{E}f(t-s, \mathbf{x}-\mathbf{v}s, \mathbf{v})] d\mathbf{v} ds + \int_{\mathbb{R}_v^3} f_0(\mathbf{x}-\mathbf{v}t, \mathbf{v}) d\mathbf{v} \tag{2-6}$$

The above equation (2-6) is then transformed to the following inequality concerning $\mathbf{E}$ while the $\int_{\mathbb{R}_v^3} f_0(\mathbf{x}-\mathbf{v}t, \mathbf{v}) d\mathbf{v}$ term could be controlled by a constant while only relies on the initial data:

$$
\begin{aligned}
\|\mathbf{E}(t,\cdot)\|_{m+3} \leqslant & C + C\left\| \int_0^t s \int_{\mathbb{R}^3} \mathbf{E} f(t-s, \mathbf{x}-\mathbf{v}s, \mathbf{v}) d\mathbf{v} ds \right\|_{m+3} \\
\leqslant & C + C\left\| \int_0^{t_0} \dots \right\| + C\left\| \int_{t_0}^{t} \dots \right\|
\end{aligned} \tag{2-7}
$$

and *deduces*① the following inequality between the derivative of $M_k(t)$ and its power.

$$
\frac{d}{dt} M_k(t) \leqslant |\frac{d}{dt} \int |v|^k f(t) dx dv| \leqslant C \|\mathbf{E}(t)\|_{3+k} M_k(t)^{\frac{k+2}{k+3}}, \tag{2-8}
$$

with which one can imply by Gronwall lemma that,

$$
M_k(t) \leqq C \left\{ M_k(0) + \left(t \sup_{s \in (0,t)} \|\mathbf{E}(s)\|_{3+k} \right)^{k+3} \right\} \tag{2-9}
$$

confirming the bound of $M_k(t)$, which is restricted by the supreme $\mathbf{E}$ during the time $(0, t)$

(ii) *Small time estimates.*

Based on the above $\|\mathbf{E}(t,\cdot)\|_{m+3}$ estimate (2-7) and *its integration on the time interval* $(0, t_0)$ for any $r > 3/2$ and $t_0 \leqslant t \leqslant T$. ②

$$
\left\| \int_0^{t_0} s ds \sup_{\tau \in (0,T)} \left(\int_{R^3} |\mathbf{E}|^r(\tau, y) \frac{dy}{s^3} \right)^{1/r} \left(\int_{\mathbb{R}^3} f(t-s, x-vs, v) dv \right)^{1/r'} \right\|_{m+3} \|f\|_\infty^{(r'-1)/r'} \leqslant C t_0^\gamma \left(1 + M_m(t)^\delta\right) \tag{2-10}
$$

where r' is the conjugate exponent of r, $1/r + 1/r' = 1$, $1 \leqslant r' < 3$ and C only depends upon the initial data, $\gamma = 2 - 3/r > 0$ and $\delta = 3(k+1)/(m+3)^2 > 0$, $m_0 > k > m$. Here comes the limitation for $m_0 > 3$.

By choosing proper $\delta, \gamma > 0; \gamma = 2 - 3/r$ and $\delta = 3(k+3)/(m+3)^2, m_0 > k > m$, it follows that there exists a bound for the above inequality, (2-7),

① Wenyin has trouble understanding $\left| \frac{d}{dt} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^k f(t) dx dv \right| \leqq \int_{\mathbb{R}^3 \times \mathbb{R}^3} k |E(t)| f(t) |v|^{k-1} dx dv$

② Though Wenyin knows Young's inequality but this formula is too complicated to know what happens in it. Where does the s^3 in the denominator of $|\mathbf{E}|$ interal come from?

$$\|\mathbf{E}(t)\|_{m+3} \leqq C\left(1 + t^{\gamma} M_m(t)\right)^{\delta}.$$

While in the first step, we introduced (2-8), the derivative of $M_k(t)$ is bounded by the product of $\|\mathbf{E}(t,\cdot)\|_{3+k}$ and $M_k(t)^{\frac{k+2}{k+3}}$. For $k = m$, the above two equations jointly give

$$\frac{d}{dt} M_m(t) \leqq C\left(M_m(t)^{\frac{m+2}{m+3}} + t^{\gamma} M_m(t)^{\delta + \frac{m+2}{m+3}}\right),$$

by which we acquire a bound imposed by $M_k(t)$ itself on a small time interval $[0, t_0]$ exclusive of the appearance $\|\mathbf{E}(t,\cdot)\|_{3+k}$.

In the following proof, step (iii) firstly proves the boundedness of the $M_m(t)$ on any time interval $(0, T)$ in the case $\mathbf{E}(t,\cdot) \in L^{3/2}$, and then step (iv) expanded the range to the general case.

(iii) *The case when* $\mathbf{E} \in L^{3/2}$.

Here is an strong limitation on the functional space that the $\mathbf{E} \in L^{3/2}(\mathbb{R}^3)$, which will be eased in the next step to be $\mathbf{E} \in L^{3/2,\infty}(\mathbb{R}^3)$. ①, relaxed to the standard condition in the next step.

The estimates of $\| \int_{t_0}^{t} \|$ of (2-7) can be acquired by a similar process in step iiwith $r = 3/2$.

$$\|E(t)\|_{m+3} \leqq C + C t_0^{\gamma} M_m(t)^{\delta} + C\left|\log t_0\right| \sup_{\tau \in (0,T)} \|E(\tau)\|_{3/2} M_m(t)^{\frac{1}{m+3}} \tag{2-11}$$

The estimate $\|\mathbf{E}(t,\cdot)\|_{3+m}$ in (2-7) combined with the bound acquired from (2-10), choosing proper t_0, can give a bound of the derivative of $M_m(t)$ restricted by $M_m(t)$ itself on any time interval $(0, T)$.

$$\frac{d}{dt} M_m(t) \leqq C\left(1 + M_m(t)\left|\log M_m(t)\right|\right) \tag{2-12}$$

makes it possible to say $M_m(t)$ is bounded on any interval $(0, T)$.

(iv) *The general case.*

① I don't really understand the definition of the weaker $L^{3/2,\infty}$ lebesgue space, does that mean either $L^{3/2}$ or L^{∞}?

When $\mathbf{E} \notin L^{3/2}(\mathbb{R}^3)$, similar to the step (i) treating $\mathbf{E} \cdot \nabla_v f$ as a source term, the $\mathbf{E}$ is decomposed into two parts: $\mathbf{E} = \mathbf{E}_1 + \mathbf{F}$

$$E_1 = \frac{\alpha}{4\pi}\left(\chi_R(x)\nabla\frac{1}{|x|}\right) * \rho, \tag{2-13}$$

where $0 \leqslant \chi_R \leqslant 1$ is smooth such that $\chi_R(x) = 1$ if $|x| \leqslant R$ and $\chi_R(x) = 0$ if $|x| \geqslant 2R$. ① The separation of field indeed let the field generated by inner particles and outer particles be clearly divided. Since $\chi_R(x)\nabla\frac{1}{|\mathbf{x}|}$ benefits from boundedness in $L^1 \bigcap L^{3/2}$ now, it follows that

$$||E_1(t)||_{L^{3/2}\cap L^{15/4}} \leqq C(R), \tag{2-14}$$

and the aforementioned step two and three can also be applied. While for the $\mathbf{F}$, we have

$$\|F\|_{L^\infty}, \|\nabla F\|_{L^\infty}, \left\|D^2 F\right\|_{L^\infty} \leqq C/R \tag{2-15}$$

$$\begin{aligned} &\dot{\mathbf{X}}(s) = -\mathbf{V}(s), && \mathbf{X}(0) = x \\ &\dot{\mathbf{V}}(s) = -\mathbf{F}(t-s, \mathbf{X}(s)), && \mathbf{V}(0) = \mathbf{v}, \quad 0 \leqq s \leqq T \end{aligned} \tag{2-16}$$

$$f(t, \mathbf{x}, \mathbf{v}) = \int_0^t \nabla_V \left(\mathbf{E}_1 f\right)(t-s, \mathbf{X}(s), \mathbf{V}(s))ds + f_0(\mathbf{X}(t), \mathbf{V}(t)) \tag{2-17}$$

I think there is typo in the original proof, that the integral should have a minus symbol just as the case in the first step.

Treat the $\mathbf{E}_1 \cdot \nabla_v f$ as source term on the RHS of the Vlasov-Poisson equation and trace the characteristic again like step (i), then it yields similar equation as (2-7).

① But I am a little confused by the numerator why it is α *rather than 1*. I think it is a typo.

$$\|\mathbf{E}(t)\|_{m+3} \leqq C + C \left\| \int_0^t \int_{\mathbb{R}^3} \left(\frac{\partial x}{\partial V} \right) E_1 f(t-s, \mathbf{X}(s), \mathbf{V}(s)) ds d\mathbf{v} \right\|_{m+3} \tag{2-18}$$

$\square$

According to the *standard approximation method*, it is said that it can keep the desired estimate, but I have not yet understood it.

2.2 Uniqueness

2.3 Blow-up Situations

Glassey et al. (1985) discusses about the existence problem of spherically symmertric solutions to the Cauchy problem for the 3D relativistic Vlasov-Poisson (RVP) system.

The paper mainly talks about spherically symmertric solutions, *i.e.*, the radial ones.

2.3.1 Velocity Bound in both cases

Lemma 2.1: There exists a constant C_1 such that for $r \geqslant 0$ and $0 \leqslant t < T$

$$|\mathbf{E}(x,t)| = \frac{M(r,t)}{r^2} \leqq \begin{cases} \min\left(Mr^{-2}, 100M^{1/3}\|\hat{f}\|_\infty^{2/3} P^2(t)\right) & \text{if } \gamma = -1 \\ \min\left(Mr^{-2}, C_1 P^{5/3}(t)\right) & \text{if } \gamma = +1 \end{cases}$$

Definition 2.2: The highest speed the solution f has on the time interval $[0,t]$.

$$P(t) = \sup\{U(s,0,r,u,\alpha) : 0 \leqslant s \leqslant t, (r,u,\alpha) \in \text{ support } f\}$$

Lemma 2.2: For all $z \geqslant 1$,

$$\xi^{-1}(z) \leqslant \left[\left(z + A^{-1}\mathbf{B}^{-1}\right)^2 - 1\right]^{1/2}$$

Lemma 2.3: For all $t \in [0, T_0]$

$$U(t) \leqslant U(0) + \xi^{-1}(\sqrt{1+U^2(0)}), \text{ when } \gamma = -1$$

2.3.1.1 The stellar dynamics case

Theorem 2.2: Let f be a classical solution of (RVP) on some time interval $[0, T)$ with $\gamma = -1$ and smooth, nonnegative, spherically symmetric data fwhich has compact support and vanishes for $(r, u, \alpha) \notin (0, \infty) \times (0, \pi)$. If $40 M^{2/3} \|f^{\circ}\|_{\infty}^{1/3} < 1$, then $P(t)$ is uniformly bounded on $[0, T)$, and hence (RVP) possesses a global classical solution.

2.3.1.2 The plasma physics case

Definition 2.3: To control the derivative of characteristic $R(s)$, here comes a function:

$$G(r,t) = -\int_r^{\infty} \min\left(M\lambda^{-2}, C_1 P^{5/3}(t)\right) d\lambda, r \geqslant 0 \text{ and } t \geqslant 0 \tag{2-19}$$

Lemma 2.4: Assume either $\dot{R} \geqq 0$ on $\left[t_1, t_2\right]$ or $\dot{R} \leqq 0$ on $\left[t_1, t_2\right]$. Assume $\gamma = +1$. Then

$$|\sqrt{1+U^2\left(t_2\right)} - \sqrt{1+U^2\left(t_1\right)}| \leqslant \left|G\left(R\left(t_2\right), t_2\right) - G\left(R\left(t_1\right), t_2\right)\right| \tag{2-20}$$

Remark 2.1: There exists a positive constant C_2

$$\left|G\left(r_1, t\right) - G\left(r_2, t\right)\right| \leqq C_2 P^{5/6}(t) \text{ for all } r_1 \geqq 0, r_2 \geqq 0, \text{ and } t \geqq 0 \tag{2-21}$$

Lemma 2.5: Assume $\gamma = +1 . \dot{R}$ can be zero for at most one value of s. If $\dot{R}\left(t_1\right) = 0$, then R has an absolute minimum at t_1.

Theorem 2.3: Let f be a classical solution of (RVP) on some time interval $[0, T)$ with $\gamma = +1$ and smooth, nonnegative, spherically symmetric data f_0 which has compact support and vanishes for $(r, u, \alpha) \notin (0, \infty) \times (0, \infty) \times (0, \pi)$. Then $P(t)$ is uniformly bounded on $[0, T)$, and hence (RVP) possesses a global classical solution.

proof According to Lemma 1.6 and its following remark,

$$\begin{aligned} \sqrt{1+U^2\left(t_2\right)} & \leqslant \sqrt{1+U^2\left(t_1\right)} + |G\left(R\left(t_2\right), t_2\right) - G\left(R\left(t_1\right), t_2\right) \\ & \leqslant \sqrt{1+U^2\left(t_1\right)} + C_2 P^{5/6}\left(t_2\right) \end{aligned} \tag{2-22}$$

Either $\dot{R}$ never vanishes or vanishes at one value,

$$\sqrt{1+U^2(t)} \leqslant \sqrt{1+U^2(0)} + 2 C_2 P^{5/6}(t) \tag{2-23}$$

holds for $t \in [0, T)$ as long as we apply Eq. 2-22 at most twice. Naturally,

$$P(t) \leqslant \sqrt{1 + P^2(t)} \leqslant \sqrt{1 + P^2(0)} + 2C_2 P^{5/6}(t) \tag{2-24}$$

induces that $P(t)$ has upper bound. □

2.3.2 Blow-up of Radial Solutions in the case of stellar dynamics system

By utility of the negative $\mathscr{E}_0$,

Theorem 2.4: Let f_0 be smooth, nonnegative, radial and of compact support on $\mathbb{R}^6$. Let $f(t, \mathbf{x}, \mathbf{v})$ be a classical solution of (RVP)on an interval $0 < t < T$ for which $-\infty < \mathscr{E}_0 < 0$. Then $T < \infty$.

proof The "dilation identity" is used below, and here is its derivation.

$$\begin{aligned}
\frac{d}{dt} \iint\limits_{\mathbb{R}^6} \mathbf{x} \cdot \mathbf{v} f \, d\mathbf{v} d\mathbf{x} =& \iint\limits_{\mathbb{R}^6} \mathbf{x} \cdot \mathbf{v} \left[-\hat{\mathbf{v}} \cdot \nabla_x f + \mathbf{E} \cdot \nabla_v f \right] d\mathbf{v} d\mathbf{x} = \cdots \text{(a lot omitted)} \\
=& \iint\limits_{R^6} \frac{|\mathbf{v}|^2 f}{\sqrt{1 + |\mathbf{v}|^2}} d\mathbf{v} d\mathbf{x} - \int\limits_{R^3} \rho \mathbf{x} \cdot \mathbf{E} d\mathbf{x} = \cdots \text{(a lot omitted)} \\
=& \iint\limits_{R^6} \frac{|\mathbf{v}|^2 f}{\sqrt{1 + |\mathbf{v}|^2}} d\mathbf{v} d\mathbf{x} - \frac{1}{2} \int\limits_{\mathbb{R}^3} |\nabla u|^2 d\mathbf{x} \\
=& \mathscr{E}_0 - \iint\limits_{\mathbb{R}^6} \frac{1}{\sqrt{1 + |\mathbf{v}|^2}} f \, d\mathbf{v} d\mathbf{x}
\end{aligned} \tag{2-25}$$

Clearly, we have a coarse upper bound estimation of $\iint_{\mathbb{R}^6} \mathbf{x} \cdot \mathbf{v} f \, d\mathbf{v} d\mathbf{x}$, which is a part of $\frac{d}{dt} \iint_{\mathbb{R}^6} r^2 \sqrt{1 + |\mathbf{v}|^2} f \, d\mathbf{v} d\mathbf{x}$.

$$\iint\limits_{\mathbb{R}^6} \mathbf{x} \cdot \mathbf{v} f \, d\mathbf{v} d\mathbf{x} \leqslant \iint\limits_{\mathbb{R}^6} \mathbf{x} \cdot \mathbf{v} f_0 d\mathbf{v} d\mathbf{x} + \mathscr{E}_0 t$$

$$
\begin{aligned}
\frac{d}{dt}\iint\limits_{\mathbb{R}^6} r^2\sqrt{1+|\mathbf{v}|^2} f\, d\mathbf{v} d\mathbf{x} &= 2\iint\limits_{\mathbf{R}^6} \mathbf{x}\cdot\mathbf{v} f\, d\mathbf{v} d\mathbf{x} - \int\limits_{\mathbb{R}^3} r^2 \mathbf{E}\cdot\mathbf{j} d\mathbf{x} \\
&\text{(by } \mathbf{j} = \int\limits_{\mathbb{R}^3} \hat{\mathbf{v}} f\, d\mathbf{v} \text{ and } \left|\int\limits_{\mathbb{R}^3} r^2\mathbf{E}\cdot\mathbf{j} d\mathbf{x}\right| \leqslant M^2) \\
&\leqslant 2\left(\iint\limits_{\mathbb{R}^6} \mathbf{x}\cdot\mathbf{v} f_0 d\mathbf{v} d\mathbf{x} + \mathscr{E}_0 t\right) + M^2 \\
&\leqslant \text{Constant} + 2\mathscr{E}_0 t \\
\Rightarrow 0 \leqslant \iint\limits_{\mathbb{R}^6} r^2\sqrt{1+|\mathbf{v}|^2} f\, d\mathbf{v} d\mathbf{x} &\leqslant C + Ct + \mathscr{E}_0 t^2
\end{aligned}
\tag{2-26}
$$

However, since we assume f_0 is a "large" enough initial data, *i.e.*, satisfying the hypothesis that $\mathscr{E}_0 < 0$. There must be some time the solution blows up. □

Chapter 3 Stability

Large amount of attention has been put on the problem of stability of stationary solutions of the Vlasov-Poisson system, both in the stellar dynamics and the plasma physics cases. The energy-Casimir method has been used to prove non-linear stability for various conservative systems, and Rein (1994) employed the method to prove non-linear stability of the Vlasov-Poisson system in three grometrically different setting. The three settings are the situations where the ion density is replaced by a given fixed ion background, the plasma can be spatially periodic, or can be restricted to a bounded domain.

With the exception of the first case, stationary solutions exist in these settings and also in the stellar dynamic case. In the physics literature there are numerous investigations of the problem of stability of these stationary solutions, both linear and non-linear, and we refer to [1,2]① and the monographs [4,5]② for references. In contrast, very little rigorous mathematics exists on this problem. In [2,9]③ non-linear stability of stationary solutions in a spatially periodic, plasma physics setting is established for the Vlasov-Poisson system and the relativistic Vlasov-Maxwell system, respectively, see also [lo, 111④. The problem of linear stability is investigated in [l]⑤, both for the plasma physics and the stellar dynamics cases. The phenomenon of Landau damping is established mathematically in [6]⑥ for the one-dimensional, linearized Vlasov-Poisson system. The starting point of the present investigation is a general method to assert non-linear stability of stationary solutions for (infinite-dimensional, degenerate) Hamiltonian systems, which is presented in [S]⑦. We briefly review this method. Let the system under consideration be described by the equation of motion

① Not yet found
② Not yet found
③ Not yet found
④ Not yet found
⑤ Not yet found
⑥ Not yet found
⑦ Not yet found

$$\dot{u} = A(u)$$

on some state space $X, A : D(A) \to X$ a (non-linear) operator, and let u_0 be the stationary solution whose stability we want to investigate. The following steps lead to a stability result for u_0:

1. Find the energy (Hamiltonian) $H : X + \mathbb{R}$ of the system; $dH(u(t))/dt = 0$ along solutions.
2. Relate u_0 to another conserved quantity Casimir functional $C : X + \mathbb{R}$ such that u_0 is a critical point of $H_C := H + C$, i.e. $DH_C(u_0) = 0$.
3. Show that the quadratic part in the expansion of H_c at u_0

   $$H_C(u) = H_C(u_00) + DH_C(u_0)(u - u_0) + D^2 H_C(u_0)(u - u_0, u - u_0) + ...$$

   is positive definite, more precisely, find a norm $|| \cdot ||_a$ on X such that

   $$H_C(u) - H_C(u_0) - DH_C(u_0)(U - u_0) \geqslant C||u - u_0||_a^2 \in X,$$

   for some $C > 0$.
4. Find a norm $|| \cdot ||_b$ on X with respect to which H_c is continuous at u_0.

If Steps (1)-(3) can be carried through, then for any solution

and with Step (4) we conclude that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $||u(0) - u_0||_b < \delta$ implies

$$\left\|u(t) - u_0\right\|_a^2 \leqslant \frac{1}{C} \left|H_C(u(0)) - H_C\left(u_0\right)\right|,$$

i.e. u_0 is non-linearly stable.

Though the energy-Casimir method is elegant and appealing, in [S] it is pointed out that the appearance of large velocities could cause the method to run into trouble.

Chapter 4　Regularities of Some Quantities

List of Equations

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