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题目：Vlasov-Poisson 系统的适定性

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中文摘要

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关键词：TeX；LaTeX；CJK；模板；论文

ABSTRACT

Recent researches on the Vlasov-Poisson system have been concluded in this literature review. The well-posedness problem, when the Vlasov-Poisson system has a global and unique solution has been studied for a long time. The main part of the literature review contains the approximation method used to converge to a local-in-time solution, proving the local well-posedness. The well-known Landau damping, a varitey of long-term time asymptotic behaviour of Vlasov-Poisson system, is introduced concisely before the main body.

The literature mainly centres on the topic of well-posedness in the Vlasov-Poisson system, discussing about the existence and uniqueness problem. After the method we used in Chap 2 to solve the local well-posedness problem, global solutions existence problem are presented in Chap 3 and Chap 4 respectively for non-relativistic and relativistic cases. For the relativistic Vlasov-Poisson problem, it is shown with compact veclocity support, that $\mu = 1$ situations have global existence, while for $\mu = -1$ ”small” enough cases are known to have global existence and a case of blow up with ”large” enough spherically symmetric initial data.

Keywords: TeX; LaTeX; CJK; template; thesis

Contents

Chapter 1 Introduction

1.1 Background

The Vlasov type equation is studied as a governing equation describing the multi-body motion of the particle swarm, in which the anistropic velocity distribution contributes a significant influence to the dynamics of the system. Phase space distribution $f(t, \mathbf{x}, \mathbf{v}) \geqslant 0$ ($x \in \mathbb{R}_x^3,\ v \in \mathbb{R}_v^3,\ t \geqslant 0$) with initial data $f_0(\mathbf{x}, \mathbf{v}) = f(0, \mathbf{x}, \mathbf{v})$ determines the particle density at $(t, \mathbf{x}, \mathbf{v})$, *i.e.*, the number of particles per unit volume in phase space. When coupled with Maxwell equations as the electromagnetic field governing rule, Vlasov equation is capable to decide the dynamic scenario for particle-field interaction, named Vlasov-Maxwell system (VM) .

$$\text{(VM \& RVM)} \begin{cases} f_t + \mathbf{a}(\mathbf{v}) \cdot \boldsymbol{\nabla}_x f + (\mathbf{E} + \mathbf{a}(\mathbf{v}) \times \mathbf{B}) \cdot \boldsymbol{\nabla}_v f = 0 \\ \mathbf{E}_t = \boldsymbol{\nabla} \times \mathbf{B} - \mathbf{j} \\ \mathbf{B}_t = -\boldsymbol{\nabla} \times \mathbf{E} \\ \boldsymbol{\nabla} \cdot \mathbf{E} = \rho, \quad \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \end{cases} \tag{1-1}$$

in which $\mathbf{a}(\mathbf{v}) \in \{\mathbf{v}, \hat{\mathbf{v}}\},\ \hat{\mathbf{v}} := \mathbf{v}/\sqrt{1 + |\mathbf{v}|^2}$ and

$$\rho(t, \mathbf{x}) = \int\limits_{\mathbb{R}_v^3} f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}, \quad \mathbf{j}(t, \mathbf{x}) = \int\limits_{\mathbb{R}_v^3} \mathbf{a}(\mathbf{v}) f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}. \tag{1-2}$$

Here $\mathbf{E}, \mathbf{B}, \rho$ and $\mathbf{j}$ expressed electric field, magnetic field, spatial density and current density respectively. The more dedicated Einstein theory considers relativistic effect when $\mathbf{a}(\mathbf{v}) = \hat{\mathbf{v}}$ and limit the maximum velocity the particles can reach, transforming the Vlasov-Poisson to relativistic Vlasov-Poisson (RVP) and the Vlasov-Maxwell to relativistic Vlasov-Maxwell (RVM) .

One could easily capture the physical essential of the Vlasov type equaiton by observing the $\mu(\mathbf{E} + \mathbf{a}(\mathbf{v}) \times \mathbf{B})$ term which is indeed the acceleration of particles located at $(\mathbf{x}, \mathbf{v})$ in the phase space.

Furthermore, under the assumption that the electrostatic force dominates the interaction, *i.e.* the Lorentz force could be treated as zero, magnetic force is omitted and then comes the Vlasov-Poisson system (VP) . $\mathbf{E}$ could be expressed as the gradient of the electrostatic potential under the assumption.

$$\mathbf{E}(t,\mathbf{x}) = -\nabla_x \phi(t,\mathbf{x}), \quad \phi = \frac{1}{|\mathbf{x}|} * \rho \tag{1-3}$$

$$\text{(VP \& RVP)} \begin{cases} \partial_t f + \mathbf{a}(\mathbf{v}) \cdot \nabla_x f + \mu \nabla_x \phi \cdot \nabla_v f = 0 \\ \Delta\phi = \rho(f) := \int_{\mathbb{R}^3_v} f(t,\mathbf{x},\mathbf{v}) d\mathbf{v} \end{cases} \tag{1-4}$$

where, $\mu \in \{+,-\}, \mathbf{a}(\mathbf{v}) \in \{\mathbf{v}, \hat{\mathbf{v}}\}$, and $\hat{\mathbf{v}} := \mathbf{v}/\sqrt{1+|\mathbf{v}|^2}$. The sign of μ indicates different physical scenario, while "+" for the plasma physics case and "-" for the stellar dynamics case. (VP) in fact indicates the rules of swarn motion governed by the scalar potential field-particle interaction, in which that the scalar potential field is generated by the particles themselves and that the stellar dynamics (plasma physics) case shows the absorbing gravitation (the repulsive Coulomb force) respectively.

Multi-species extended equation of (VP) and (RVP) could be easily established by adding q_i and m_i in the original ones. Note that μ must be "+" at the present because the stellar dynamics case only allow single species situation.

1.2 Characteristics

Frequently used in the context of Vlasov-problem research are the characteristics: $s \mapsto \mathbf{X}(s,t,\mathbf{x},\mathbf{v})$, $s \mapsto \mathbf{V}(s,t,\mathbf{x},\mathbf{v})$ defined as the solutions of the following corresponding system of ordinary differential equations:

$$\begin{aligned} \frac{d\mathbf{X}}{ds} &= a(\mathbf{V}) = \{\mathbf{V}, \mathbf{V}/\sqrt{1+|\mathbf{V}|^2}\} \\ \frac{d\mathbf{V}}{ds} &= \gamma \mathbf{E}(\mathbf{X}, s) \end{aligned} \tag{1-5}$$

indicating the trace of a particle who arrives at $(\mathbf{x},\mathbf{v})$ at the time of $s = t$, *i.e.*, $\mathbf{X}(t,t,\mathbf{x},\mathbf{v}) = \mathbf{x}$, $\mathbf{V}(t,t,\mathbf{x},\mathbf{v}) = \mathbf{v}$. Hence $||f(t,\cdot,\cdot)||_\infty = ||f_0||_\infty < \infty$ by initial data boundedness assumption. Sometimes we use $\mathbf{X}(s)$ directly, when $(t,\mathbf{x},\mathbf{v})$ are known, to simplify the notation, especially when we are studying the traces of characteristics.

1.3 Well-posedness Results

We will study only classical solutions of (VP), i.e., solutions for which the characteristic system of (1.1) has unique classical solutions. In this case the existence of local solutions is known for every $n \in \mathbf{N}$ by Horst et al. (1981a). Global existence is known in the following cases:

Non-relativistic Situation:

(i) $n = 2$, Illner et al. (1979)

(ii) $n = 3$ and f_0 spherically symmetric, Horst et al. (1982)

(iii) $n = 3$ and f_0 cylindrically symmetric Hellwig (1964)

(iv) $n = 3$, Lions et al. (1991)

(v) $n = 4$ and f_0 spherically symmetric and small, Hellwig (1964)

Relativistic Situation:

(i) $n = 3$, $\mu = 1$ and f_0 spherically symmetric; $\mu = -1$, f_0 spherically symmetric and small Glassey et al. (1985). Both with compact support assumption.

(ii) $n = 3$, $\mu = 1$, localized sphercial symmetric data, Wang (2003).

Further, it is known that for $n \geqslant 4$ there are f_0 (even in $C_0^\infty\left(\mathbf{R}^n \times \mathbf{R}^n\right)\big)$, so that the corresponding solutions exist only on a finite time interval [9].

1.3.1 With Compact Support in ”v”

For classical solutions, it is well known that the existence and uniqueness result of the Vlasov-Poisson system solution have been presented by Iordanskii ① in dimension 1, Ukai et al. (1978) in dimension 2, Bardos et al. (1985) in dimensions 3 for small data. The case of (nearly) classical symmetric data has been treated by Batt (1977), Wollman (1980), Horst et al. (1981b), Schaeffer (1987), among whom Schaeffer (1987) treated the relativistic case of symmetric data in one paper.

In particular, in three space dimensions, global weak solutions exist (Arsen'ev (1975); Abdallah (1994)), and global classical solutions exist if the Cauchy data are small enough given by Bardos et al. (1985).

When we turn to the relativistic case, at first sight, (RVP) seems ”better” than its classical version, since $|\hat{\mathbf{v}}| < 1$. Thus ”higher moment difficulties” well-known in

① The paper [16] listed in the reference of Lions et al. (1991) is missing, refered as Iordanskii, S.V.: The Cauehy problem for the kinetic equation of plasma. Transl., II. Ser., Am. Math. Soc. 35, 351-363 (1964) the classical case, will not occur. Moreover, for the same reason we have the casuality along characteristics due to the limited velocity. These favorable circumstances are diminished somewhat by examination of the total energy integral.① One may hope then that (RVM) is better behaved than (VM) , but only when $\gamma = +1$.

In the plasma physics case, $\rho \in L^{4/3}(\mathbb{R}_x^3)$ is worse than the result for (VP) itself, where $\rho \in L^{5/3}(\mathbb{R}_x^3)$. However, Batt's and Wollman's methods [*cf.* Batt (1977), Wollman (1980)] can still be adapted and used to show the existence of global spherically symmetric solutions when $\gamma = +1$. In contrast to the plasma case, Glassey et al. (1985) stated the existence of solution of stellar dynamics is weakened in the sense that a restrction on the size of initial data is necessary. Only "small" radial solutions with $40M^{2/3}\|f_0\|_\infty^{1/3}$ are confirmed to exist in the large for (RVP) with $\gamma = -1$. Indeed, if $\gamma = -1$ and the initial energy $\mathscr{E}_0$ is negative, it is shown that the life-span of such a radial, classical solution is finite.

1.3.2 Without Compact Support in "v"

The question of global well-posedness for (VP) system without the compact assumpyion in "v" has been considered by many authors. The Vlasov-Poisson system has been tackled successfully, for large data, by Pfaffelmoser (1992), Lions et al. (1991), Schaeffer (1991).

Lions et al. (1991), based on the representation formula built by the characteristic method considering the source term, proves the propagation of moments in v higher than 3. More precisely, if $|\mathbf{v}|^m f_0 \in L^1(\mathbb{R}^6)$ for all $m < m_0$, with $m_0 > 3$, then we build a solution of Vlasov-Poisson equations satisfying $|\mathbf{v}|^m f(t, \mathbf{x}, \mathbf{v}) \in L^1(\mathbb{R}^6)$ for all $m < m_0$, with $m_0 > 3$ for any $t > 0$. Moreover, for $m_0 > 6$, Sobolev injections deduces that $\mathbf{E} \in L^\infty([0, T] \times \mathbb{R}^3)$ for any $t > 0$, and, following Horst ②, f is a smooth function if $f_0(x, v)$ is smooth.

Luk et al. (2016), Kunze (2015), Pallard (2015) and Patel (2018) have developed recent improvements on the continuation criterion. To release the limitation of the compact support and get rid of the assumptions on the solution itself, Wang (2018)

① Why? I don't knwo why it is diminished.

② The paper [14] listed in the reference of Lions et al. (1991) is missing, refered as Horst, E.: Global strong solutions of Vlasov's Equation. Necessary and sufficient conditions for their existence. Partial differential equations. Banach Cent. Publ. 19, 143 153 (1987) studied the propagation of regularity and the long time behavior of the 3D RVM system for suitably small initial data.

1.4 Landau Damping

The electrostatic Vlasov-Poisson equation (VP) demonstrates the well-known long-time asymptotic phenomenon, named Landau damping in plasma physics. Linearized Vlasov-Poisson equation, the div operator working on f_0 rather than f, is capable to depict the asymptotic behaviour of the distribution function and linear Landau Damping theory firstly developed.

Physicists solved the lineaized (VP) by means of a Fourier-Laplace transform (cf. Krall et al. (1973)) and wondered whether the $\phi(t, \mathbf{x})$ behaved like plane waves after t being large enough. They found that there is no way to exhibit damped plane waves except by providing an analytical extension of the Laplace transform of f, which is only possible with strong analytic assumptions on involved functions. However, these hypotheses are verified in numerous physical situations with these assumptions.

Beside this approach, another successful theory which has been developed long time ago is established by Kampen (1955) and Case (1959), using a "normal mode expansion". Degond (1986) studied the spectral theory of the linearized (VP) in order to prove that its solution behaves, for large times, like a sum of plane waves. It is shown that to obtain the distribution function expansion expressed by damped waves, an analytical extension of the resolvent of the (VP) is necessary.

Beyond the linearized study, which has been studied for a long period in the theory of Landau damping. Mouhot et al. (2011) established the theory of exponential Landau damping in analytic regularity where the damping phenomenon is reinterpreted in terms of transfer of regularity between kinetic and spatial variables, rather than exchanges of energy. The study revealed that the phase mixing is the driving mechanism of damping.

1.5 Denotation

Reference Horst et al. (1981a) contains a sufficient condition for global existence with quite general assumptions. Because we use this condition (somewhat modified), most of the following notation and definitions are taken from there. We also use constants C which do not depend on f_0 and K which do. The first index i always denotes the number of the lemma or theorem, where it is defined.

For any two numbers A and B, we use $A \lesssim B$ and $B \gtrsim A$ to denote $A \lesssim CB$, where C is an absolute constant.

$\omega_N := 2 \cdot \pi^{N/2}/\Gamma(N/2)$ is the surface area of the $(N-1)$ -dimensional sphere. If $I \subset [0, \infty)$ we let $C_+(I) := \{f : I \to [0, \infty) | f$ is continuous and non-decreasing on $I\}$

Integration and measurability are always meant with respect to the Lebesgue measure. $\mathrm{L}_\infty\left(\mathbb{R}^M, \mathbb{R}^L\right)$ and $\|\cdot\|_\infty$ are also defined as usual. $\mathrm{L}_p\left(\mathbb{R}^M\right) := L_p\left(\mathbb{R}^M, \mathbb{R}\right)$

For all $f : \mathbb{R}^M \to \mathbb{R}^L$ we let $\operatorname{lip}(f) := \sup_{z \neq w} |z-w|^{-1} |f(z) - f(w)|$

$$\operatorname{Lip}\left(\mathbb{R}^M, \mathbb{R}^L\right) := \left\{f : \mathbb{R}^M \to \mathbb{R}^L | \operatorname{lip}(f) < \infty\right\}, \quad \operatorname{Lip}\left(\mathbb{R}^M\right) := \operatorname{Lip}\left(\mathbb{R}^M, \mathbb{R}\right)$$

Chapter 2 Local Solutions of Vlasov-Poisson System

Here we present the standard approximation methods adopted by Horst to investigate the well-posedness of the generalized (VP) system problem in N dimenstion space.

2.1 Definition Preparation

$$\text{(VP \& RVP)} \begin{cases} \partial_t f + \mathbf{a}(\mathbf{v}) \cdot \nabla_x f + \mathbf{E} \cdot \nabla_v f = 0 \\ \mathbf{E}(t, \mathbf{x}) = \mu \int \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^N} \rho(t, \mathbf{x}) \mathrm{d}\mathbf{x} \end{cases} \tag{2-1}$$

We always use $f_0 \in \mathrm{L}_1\left(\mathbb{R}^{2N}\right)$ to denote the initial data for the problem to be considered. Let $m := \|f_0\|_1$ denote the initial mass, *i.e.* L_1 norm of initial phase function. The singularity of the field integral kernel is modified with a parameter ε as follows.

$$\mathbf{e}^{\varepsilon}(\mathbf{z}) = \gamma \cdot \frac{\mathbf{z}}{\left(|\mathbf{z}|^2 + \varepsilon\right)^{N/2}}, \qquad \mathbf{z} \in \mathbb{R}^N, \varepsilon \geqslant 0$$

Since we have weakened the singularity by the ε, the solutions to the $\mathrm{VP}^{\varepsilon}$ are redefined as follow.

Definition 2.1: *(The solution of VP^{ε})*

Assume that $I \subset [0, \infty)$ is an interval with $0 \in I$. We say that $f^{\varepsilon} : I \times \mathbb{R}^{2N} \to \mathbb{R}$ is a solution of problem $\mathrm{VP}^{\varepsilon}$($\varepsilon \geqslant 0$ fixed) on I, if

(1) $f(0, \cdots) = f_0$

(2) $\left((\mathbf{y}, \mathbf{u}) \mapsto \mathbf{e}^{\varepsilon}(\mathbf{x} - \mathbf{y}) \cdot f^{\varepsilon}(t, \mathbf{y}, \mathbf{u})\right) \in \mathrm{L}_1\left(\mathbb{R}^{2N}, \mathbb{R}^N\right)$ for all $t \in I, \mathbf{x} \in \mathbb{R}^N$

(3) $\mathbf{E}^{\varepsilon}(t, \mathbf{x}) := \iint_{\mathbb{R}^6} \mathbf{e}^{\varepsilon}(\mathbf{x} - \mathbf{y}) \cdot f^{\varepsilon}(t, \mathbf{y}, \mathbf{u}) \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{u}$ is defined to eliminate the orignal singularity of integral kernel. It is required that $\mathbf{E}^{\varepsilon}$ is continuous on $I \times \mathbb{R}^N$, $\mathbf{E}^{\varepsilon}(t, \cdot) \in \mathrm{C}_b^0\left(\mathbb{R}^N, \mathbb{R}^N\right) \cap \mathrm{Lip}\left(\mathbb{R}^N, \mathbb{R}^N\right)$ for all $t \in I$ and there exist $C_{\rho}^{\varepsilon}, C_{lip(E)}^{\varepsilon} \in C_+(I)$ such that for all $t \in I$

$$\|\mathbf{E}^{\varepsilon}(t, \cdot)\|_{\infty} \leqslant C_E^{\varepsilon}(t), \quad \mathrm{lip}\left(\mathbf{E}^{\varepsilon}(t, \cdot)\right) \leqslant C_{\mathrm{lip}(E)}^{\varepsilon}(t)$$

(4) The solution f^ε is defined according to the characteristic trajectories of the VP$^\varepsilon$ problem, along which the value f^ε should not change. The characteristic under the modified electrical field $\mathbf{E}^\varepsilon$ is

$$\dot{\mathbf{X}}^\varepsilon = \mathbf{V}^\varepsilon, \dot{\mathbf{V}}^\varepsilon = \mathbf{E}^\varepsilon\left(t, \mathbf{X}^\varepsilon\right), \quad t \in I, \tag{2-2}$$

which has a unique solution on I for any initial value thanks to the regularities given in (3). $f(t, \mathbf{x}, \mathbf{v})$ is given by the characteristic with initial condition $\mathbf{X}^\varepsilon(t) = \mathbf{x}, \mathbf{V}^\varepsilon(t) = \mathbf{v}$, *i.e.*

$$f^\varepsilon(t, \mathbf{X}^\varepsilon(t), \mathbf{V}^\varepsilon(t)) = f^\varepsilon(0, \mathbf{X}^\varepsilon(0), \mathbf{V}^\varepsilon(0)) = f_0(\mathbf{X}^\varepsilon(0), \mathbf{V}^\varepsilon(0)), \quad t \in I$$

if $I = [0, \infty)$, f^ε is named a global solution, otherwise a local solution.

Theorem 2.1: *(Well-posedness of VP$^\varepsilon$ when $\varepsilon > 0$)*

If $\varepsilon > 0$, there exists a unique global solution f^ε of P^ε. If I is a subinterval of $[0, \infty)$ with $0 \in I$ then $f^\varepsilon|_{I\times\mathbb{R}^{2N}}$ is a solution on I and this solution is unique.

proof The result can be acquired with the methods of [11] TODO. □

Definition 2.2: Assume f^ε is a solution of VP$^\varepsilon$($\varepsilon \geqslant 0$ fixed) on I. For any $t_0 \in I, (\mathbf{x}, \mathbf{v}) \in \mathbb{R}^{2N}$, let $\mathbf{X}(s, t_0, \mathbf{x}, \mathbf{v}), \mathbf{V}(t, t_0, \mathbf{x}, \mathbf{v}) : I \times I \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^n$ denote the solution of the characteristic ordinary differential system that satisfies the initial condition

$$\mathbf{X}^\varepsilon\left(t_0, t_0, \mathbf{x}, \mathbf{v}\right) = \mathbf{x}, \quad \mathbf{V}^\varepsilon\left(t_0, t_0, \mathbf{x}, \mathbf{v}\right) = \mathbf{v}$$

Lemma 2.1: *(Features of characteristcs)* Then the following statements are valid

(i) $\mathbf{X}^\varepsilon, \mathbf{V}^\varepsilon$ are continuous on $I \times I \times \mathbb{R}^{2N}$

(ii) For any $t_1, t_2 \in I$ the function $\mathbf{X}^\varepsilon\left(t_1, t_2, \cdot\right)$ is a (Lebesgue) measure preserving homeomorphism of $\mathbb{R}^{2N}$ onto $\mathbb{R}^{2N}$.

(iii) For any $t_1, t_2, t_3 \in I$

$$(\mathbf{X}^\varepsilon, \mathbf{V}^\varepsilon)\left(t_1, t_2, \cdot\right) \circ (\mathbf{X}^\varepsilon, \mathbf{V}^\varepsilon)\left(t_2, t_3, \cdot\right) = (\mathbf{X}^\varepsilon, \mathbf{V}^\varepsilon)\left(t_1, t_3, \cdot\right)$$

especially the $t_3 = t_1$ case tells that the inverse of $(\mathbf{X}^\varepsilon, \mathbf{V}^\varepsilon)\left(t_1, t_2, \cdot\right)$ is $(\mathbf{X}^\varepsilon, \mathbf{V}^\varepsilon)\left(t_2, t_1, \cdot\right)$.

(iv) If $\frac{\partial}{\partial x_3} E^t(t, \mathbf{x})$ exists and is continuous on $I \times \mathbb{R}^N$, then $\mathbf{X}^e$ is continuously differentiable with respect to all arguments and satisfies the system of first order partial differential equations

$$\frac{\partial}{\partial t_1}\mathbf{X}^t\left(t, t_1, x\right) + x_v \cdot \boldsymbol{\nabla}_x \mathbf{X}^t\left(t, t_1, x\right) + E^{\varepsilon}\left(t_1, \mathbf{x}\right) \cdot \boldsymbol{\nabla}_v X_i^t\left(t, t_1, x\right) = 0$$

(v) As f^{ε} is an integral of (1.2.1), we have for all $t, t_1 \in I, x \in \mathbb{R}^{2N}$

$$f^{\varepsilon}\left(t, \mathbf{X}^{\varepsilon}\left(t, t_1, \mathbf{x}, \mathbf{v}\right), \mathbf{V}^{\varepsilon}\left(t, t_1, \mathbf{x}, \mathbf{v}\right)\right) = f_0\left(\mathbf{X}^{\varepsilon}(0, t, x)\right), \text{ especially for } t = t_1$$

$f^{\varepsilon}(t, \mathbf{x}, \mathbf{v}) = f_0\left(\mathbf{X}^{\varepsilon}(0, t, \mathbf{x}, \mathbf{v}), \mathbf{V}^{\varepsilon}(0, t, \mathbf{x}, \mathbf{v})\right)$ Therefore $f^{\varepsilon}(t, \cdot)$ has the same range as f_0 for all $t \in I$, hence $f^{\varepsilon} \geqslant 0$, if and only if $f_0 \geqslant 0$ and $\sup|f^{\varepsilon}(t, x)| = \sup|f_0(x)|$.$f^{\varepsilon}$ is continuous on $I \times \mathbb{R}^{2N}$, if and only if f_0 is continuous on $\mathbb{R}^{2N}$. If $\boldsymbol{\nabla}_x E^{\varepsilon}(t, \mathbf{x})$ exists and is continuous on $I \times \mathbb{R}^N$, then (iv) and (1.4.4) imply that Φ^c is differentiable (continuously differentiable), if and only if f_0 is differentiable (continuously differentiable) and in this case f^{ε} satisfies the partial differential equation

$$\frac{\partial}{\partial t} f^{\varepsilon}(t, x) + \mathbf{v} \cdot \boldsymbol{\nabla}_x f^{\varepsilon}(t, \mathbf{x}, \mathbf{v}) + \mathbf{E}^{\varepsilon}(t, \mathbf{x}) \cdot \boldsymbol{\nabla}_v f^{\varepsilon}(t, \mathbf{x}, \mathbf{v}) = 0$$

(vi) (ii) and (1.4.4) imply that for any $t, t_1 \in I$ and any measurable function $\sigma : \mathbb{R}^{2N} \to \mathbb{R}^M$ we have $\sigma \in \mathrm{L}_1\left(\mathbb{R}^{2N}, \mathbb{R}^M\right)$, if and only if $\sigma \circ \mathbf{X}^{\varepsilon}\left(t_1, t, \cdot\right) \in \mathrm{L}_1\left(\mathbb{R}^{2N}, \mathbb{R}^M\right)$. In this case

$$(1.4.6) \int \sigma(x) \mathrm{d}x^{2N} = \int \sigma\left(\mathbf{X}^{\varepsilon}\left(t_1, t, x\right)\right) \mathrm{d}x^{2N}$$

Especially $\left(t_1 = 0, \sigma = |f_0|^p\right)$ we have for all $t \in I$ that $f^{\varepsilon}(t, \cdot) \in \mathrm{L}_p\left(\mathbb{R}^{2N}\right)$, if and only if $f_0 \in \mathrm{L}_p\left(\mathbb{R}^{2N}\right)$ and in this case (1.4.7) $\| f^{\varepsilon}(t, \cdot)|_p = \| f_0\big|_p, 1 \leqslant p < \infty$

(vii) We define $\rho^{\varepsilon}(t, \mathbf{x}, \mathbf{v}) := \int f^{\varepsilon}(t, \mathbf{x}, \mathbf{v}) \mathrm{d}\mathbf{v}, \rho^{\varepsilon}_{abs}(t, \mathbf{x}) := \int |f^{\varepsilon}(t, \mathbf{x}, \mathbf{v})| \, \mathrm{d}\mathbf{v}$. For fixed $t \in I$ these functions are defined for almost all $x_s \in \mathbb{R}^N$ because of (vi) and (1.1). *By* Fubini's theorem $\rho^{\varepsilon}(t, \cdot), \rho^{\varepsilon}_{abs}(t, \cdot) \in \mathrm{L}_1\left(\mathbb{R}^N\right)$ for all $t \in I$ and $\|\rho^{\varepsilon}(t, \cdot)\|_1 \leqslant$

$\left\| |\rho^{\varepsilon}_{abs}|_e (t,\cdot) \right\|_1 = \|f_0\|_1 = m$ and

$$\mathbf{E}^{\varepsilon}(t,\mathbf{x}) = \int \mathbf{e}^{\varepsilon}(\mathbf{x}-\mathbf{y})\,\rho^{\epsilon}(t,\mathbf{y})\,\mathrm{d}\mathbf{y}$$

proof All this is standard theory of first order partial differential equations, cf. [9], p. 131 ff. The proof that $\mathbf{X}^{e}\left(t,t_1,\cdot\right)$ is measure preserving (even if no differentiability of $\mathbf{E}^{\varepsilon}$ is assumed can be found in [3], p. 62. The special case that E^{c} is continuously differentiable with respect to x_s is discussed in [16], ch. III and V □

2.2 Bounds for $\mathbf{E}^{\varepsilon}$ and ρ^{ε}

Traditional analysis works well to solve the problem VP$^{\varepsilon}(\varepsilon > 0)$ due to the removed singularity of integral kernel. However, we need to illustrate that f^{ε} uniformly converges to f^0 as $\varepsilon \to 0$ on $I \times \mathbb{R}^N \times \mathbb{R}^N$ and confirm the uniqueness. Therefore we need conditions on f which assure that the characteristcs ordinary equations $\mathbf{X}(s,t_0,\mathbf{x},\mathbf{v}), \mathbf{V}(s,t_0,\mathbf{x},\mathbf{v})$ have a unique solution for given parameter $t_0 \in I, \mathbf{x},\mathbf{v} \in \mathbb{R}^N$. In this section we take the first steps into this direction. An obvious estimate yields that $\left|E^{\varepsilon}\left(t,x_s\right)\right| \leqslant \int \left|x_s - y_s\right|^{1-N} \cdot |f^{\varepsilon}|_Q\left(t,y_s\right)\mathrm{d}x_s^N$ Here the time t plays only the role of a parameter, whereas integration is with respect to y_s.

Therefore we can use the following lemma to find a bound for $\|E^{\varepsilon}(t,\cdot)\|_{\infty}$

Lemma 2.2: Assume $0 < \alpha < N, p \in (1,\infty], q \in [1,\infty)$ $p > N/(N-\alpha) > q$. Assume further $\sigma \in \mathrm{L}_p\left(\mathbb{R}^N\right) \cap \mathrm{L}_q\left(\mathbb{R}^N\right)$. Then for all $\mathbf{z} \in \mathbb{R}^N$

$$\int |\mathbf{z}-\mathbf{w}|^{-\alpha} \cdot |\sigma(\mathbf{w})|\mathrm{d}\mathbf{w} \leqslant \bar{C}(N,\alpha,p,q) \cdot \|\sigma\|_p^{\lambda} \cdot \|\,\sigma|_q^{\mu} \tag{2-3}$$

with constants $\tilde{C}(N,\alpha,p,q)$, $\lambda := (\alpha/N - 1 + 1/q)/(1/q - 1/p)$ and $\mu := 1 - \lambda$.

Let $\tilde{C}_{min}(N,\alpha,p,q)$ denote the smallest constant such that (2-3) remains true for all $\sigma \in \mathrm{L}_p\left(\mathbb{R}^N\right) \cap \mathrm{L}_q\left(\mathbb{R}^N\right)$

proof Fix $R > 0$ and divide the integral of (2-3) into two parts, one for intergral I_1 inside a sphere (radius R) while the other I_2 outside.

By Hölder's inequality,

$$I_1 \leqslant \left(\int\limits_{|\mathbf{z}-\mathbf{x}|\leqslant R} |\mathbf{z}-\mathbf{x}|^{-\alpha p'} \mathrm{d}\mathbf{x} \right)^{1/p'} \|\sigma\|_p = \text{ const. } R^{(N/p')-\alpha} \|\sigma\|_p$$

$$I_2 \leqslant \left(\int\limits_{|\mathbf{z}-\mathbf{x}|> R} |\mathbf{z}-\mathbf{x}|^{-\alpha q'} \mathrm{d}\mathbf{x} \right)^{1/q'} \|\sigma\|_q = \text{ const. } R^{(N/q')-\alpha} \|\sigma\|_q$$

p' and q' being the numbers with $p^{-1} + p'^{-1} = q^{-1} + q'^{-1} = 1$.

Thus $I_1 + I_2 \leqslant$ const. $\left(R^{(N/p')-\alpha} \cdot \|\sigma\|_p + R^{(N/q')-\alpha} \cdot \|\sigma\|_q \right)$ The minimum of the right-hand side as a function of R is equal to const. $\|\sigma\|_p^{\lambda} \cdot \|\sigma\|_q^{\mu}$. , *i.e.* $\tilde{C}(N, \alpha, p, q)\|\sigma\|_p^{\lambda} \cdot \|\sigma\|_q^{\mu}$. □

This lemma shows us that it makes sense to investigate the properties of $\left\| f^{\varepsilon} \rho_{\theta}(t, \cdot) \right\|_p$. We already know that always $\left\| f^{\varepsilon}|_{\rho}(t, \cdot) \right\|_1 = m$. Now we consider the case $p = \infty$, $1 < p < \infty$ will become important in part II.

Assumption 2.1: Assume that f^{ε} is a solution of $\mathrm{VP}^{\varepsilon}$($\varepsilon \geqslant 0$ fixed) on I.

1. We say that f^{ε} satisfies (ρ_{abs} control condition) on I, if there exists an $C^{\varepsilon}_{\rho,abs}(t) \in C_+(I)$ such that for all $t \in I$

$$\|\rho^{\varepsilon}_{abs}(t, \cdot)\|_{\infty} \leqslant C^{\varepsilon}_{\rho,abs}(t) \tag{2-4}$$

2. We say that $f_0 := f^{\varepsilon}(0, \cdot, \cdot)$ satisfies (sup f_0 control condition) , if

$$\int^{*} \underbrace{\sup\left\{ |f_0(\mathbf{x}, \mathbf{u})| \,|\, \mathbf{x}, \mathbf{u} \in \mathbb{R}^N, |\mathbf{u} - \mathbf{v}| \leqslant a \right\}}_{\hat{f}_v(\mathbf{v})} \mathrm{d}\mathbf{v} \leqslant K_1 \cdot \left(K_2 + a\right)^N \tag{2-5}$$

   for suitable constants K_1, K_2 and all $a \geqslant 0$ ($\int^{*} \ldots \mathrm{d}x_0^N$ denotes the upper Lebesgue integral)

(sup f_0 control condition) demonstrates that f_0 is bounded, *i.e.* $f_0 \in \mathrm{L}_{\infty}\left(\mathrm{R}^{2N}\right)$. As $f_0 \in \mathrm{L}_1\left(\mathbb{R}^{2N}\right)$, (sup f_0 control condition) indeed shows strong control on f_0 that $f_0 \in \mathrm{L}_p\left(\mathrm{R}^{2N}\right)$ for all $1 \leqslant p \leqslant \infty$.

Lemma 2.3: f_0 satisfies (sup f_0 control condition) , if there exists constants $\alpha > N$ and $K \geqslant 0$ such that $|f_0(\mathbf{x},\mathbf{v})| \leqslant K \cdot (1+|\mathbf{v}|)^{-\alpha}$ for all $\mathbf{x},\mathbf{v} \in \mathbb{R}^N$

proof This is easily verified when $N = 1$ by showing that the $K \cdot (1+|\mathbf{v}|)^{-\alpha}$ satisfies (sup f_0 control condition) . While for the $N > 1$ cases, establish the relation between $|\mathbf{v}|$ and its components first, $(1+|\mathbf{v}|)^{-\alpha} \leqslant \left(1+\left|v_1\right|\right)^{-\alpha/N} \cdots \cdot \left(1+\left|v_N\right|\right)^{-\alpha/N}$ and note that,

$$\begin{aligned}&\sup\left\{|f_0(\mathbf{x},\mathbf{u})| \,\middle|\, \mathbf{x},\mathbf{u} \in \mathbb{R}^{2N}, \left|\mathbf{u}-\mathbf{v}\right| \leqslant a\right\}\\ &\leqslant K \cdot \sup\left\{\left(1+\left|u_1\right|\right)^{-\alpha/N} \,\middle|\, |u_1 - v_1| \leqslant a\right\} \cdot \cdots \cdot \sup\left\{\left(1+\left|u_N\right|\right)^{-\alpha/N} \,\middle|\, |u_N - v_N| \leqslant a\right\}\end{aligned}$$

□

Do upper Lebesgue integral in v on both sides and note the right-hand side is measurable and its integral could be acquired using Fubini's theorem.

Lemma 2.4: Assume that f^{ε} is a solution of $\mathrm{VP}^{\varepsilon}(\varepsilon \geqslant 0$ fixed) on I. Assume further that f satisfies (sup f_0 control condition) (with constants K_1 and K_2). Then

(i) f^{ε} satisfies (ρ_{abs} control condition) on I.

(ii) For each fixed $t_0 \in I$ the function $f^{\varepsilon}\left(t_0,\cdot\right)$ also satisfies (sup f_0 control condition) with the same constants K_1 and $K_2 + f_v^{e}\left(t_0\right)$

proof (i) For all $t \in I$, let

$$\begin{aligned}h_v^{\varepsilon}(t) :&= \sup\left\{|\mathbf{V}^{\varepsilon}(0,\tau,\mathbf{x},\mathbf{v}) - \mathbf{v}| \,|\mathbf{x},\mathbf{v} \in \mathrm{R}^N, 0 \leqslant \tau \leqslant t\right\}\\ &= \sup\left\{|\mathbf{V}^{\varepsilon}(\tau,0,\mathbf{x},\mathbf{v}) - \mathbf{v}| \,|\mathbf{x},\mathbf{v} \in \mathbb{R}^N, 0 \leqslant \tau \leqslant t\right\}\end{aligned}$$

The maximum possible velocity change should be controlled by the supreme field intensity.

$$\leqslant \int_0^t \sup\left\{\|\mathbf{E}^{\varepsilon}(\tau,\cdot)\|_{\infty} 0 \leqslant \tau \leqslant r\right\} \mathrm{d}r \leqslant \int_0^t h_E^{\varepsilon}(r)\mathrm{d}r < \infty$$

It follows that only the particles with a 'neighboring' velocity can arrive at $(\mathbf{x},\mathbf{v})$ at time t, because the maximal change of velocity is controlled by the $C_v^{\varepsilon}(t)$, for all $x \in \mathrm{R}^{2N}$

$$\begin{aligned}|f^{\varepsilon}(t,\mathbf{x},\mathbf{v})| &= \left|f_0\left(\mathbf{X}^{\varepsilon}(0,t,\mathbf{x},\mathbf{v}),\mathbf{V}^{\varepsilon}(0,t,\mathbf{x},\mathbf{v})\right)\right|\\ &\leqslant \sup\left\{|f_0(\mathbf{y},\mathbf{u})| \,\middle|\, \mathbf{y},\mathbf{u} \in \mathrm{R}^N, \left|\mathbf{u}-\mathbf{v}\right| \leqslant C_v^{\varepsilon}(t)\right\}\end{aligned}$$

Taking $\int \ldots d\mathbf{v}$ of the left-hand side and $\int_{\ldots}^{*} \ldots d\mathbf{v}$ of the right-hand side proves $\rho_{abs}^{\varepsilon}(t,\mathbf{x}) \leqslant K_1 \cdot \left(K_2 + C_v^{\varepsilon}(t)\right)^N$.

(ii) Fix $t_0 \in I$,

$$\begin{aligned} &\sup\left\{\left|f^{\varepsilon}\left(t_0,\mathbf{y},\mathbf{u}\right)\right| \left|\mathbf{y},\mathbf{u} \in \mathbb{R}^{2N}, \left|\mathbf{u}-\mathbf{v}\right| \leqslant a\right.\right\} \\ =&\sup\left\{\left|f_0\left(\mathbf{X}^{\varepsilon}\left(0,t_0,\mathbf{y},\mathbf{u}\right),\mathbf{V}^{\varepsilon}\left(0,t_0,\mathbf{y},\mathbf{u}\right)\right)\right| \left|\mathbf{y},\mathbf{u} \in \mathbb{R}^{N}, |\mathbf{u}-\mathbf{v}| \leqslant a\right.\right\} \\ \leqslant&\sup\left\{|f_0(\mathbf{y},\mathbf{u})| \left|\mathbf{y},\mathbf{u} \in \mathrm{R}^{N}, \left|\mathbf{u}-\mathbf{v}\right| \leqslant a + C_v^{\varepsilon}\left(t_0\right)\right.\right\} \end{aligned}$$

It follows that an integral would suffice to show that $f(t_0,\cdot,\cdot)$ satisfy the (sup f_0 control condition) . ☐

Remark. Condition $(f1)$ is much weaker than the conditions used by other authors in the same context, cf. [5], [2], [19], [21], [15]. (2.6) Lemma Assume that f^{ε} is a solution of $\mathrm{VP}^{\varepsilon}(\varepsilon \geqslant 0$ fixed) on I. Assume further that f is continuous on $\mathbb{R}^{2N}$ and satisfies $(f1)$. Then f^{ε} is continuous on $I \times \mathbb{R}^N$

Theorem 2.2: *(Boundness Conditions Equivalence)*

Assume $I \subset [0,\infty)$ is an interval with $0 \in I$. Assume further that f_0 satisfies $(f_0 1)$ Then the following statements are equivalent:

(i) There exists an $H_\rho(t) \in C_+(I)$, such that

$$|\rho^{\varepsilon}(t,\mathbf{x})| \leqslant H_\rho(t) \text{ for all } \varepsilon > 0, t \in I, \mathbf{x} \in \mathbb{R}^{\mathrm{N}}$$

(ii) There exists an $H_E(t) \in C_+(I)$, such that

$$|\mathbf{E}^{\varepsilon}(t,\mathbf{x})| \leqslant H_E(t) \text{ for all } \varepsilon > 0, t \in I, \mathbf{x} \in \mathbb{R}^{\mathrm{N}}$$

(iii) There exists an $H_v(t) \in C_+(I)$, such that

$$|\mathbf{V}^{\varepsilon}(t,0,\mathbf{x},\mathbf{v}) - \mathbf{v}| \leqslant C_v^{\varepsilon}(t) \leqslant h_v(t) \text{ for all } \varepsilon > 0, t \in I, \mathbf{x},\mathbf{v} \in \mathbb{R}^N$$

Definition 2.3: *(Boundness Condition)*

If the above conditions are satisfied, we say that "I satisfies the boundedness condition" If $N = 1,2$, then $[0,\infty)$ satisfies the boundedness condition. If $N \geqslant 3$, there exists a $T \in (0,\infty]$, which depends only on $N, \mathscr{M} = \|f_0\|_1$ and K_1, K_2 (the constants from definition (2.3)), such that $[0,T)$ satisfies the boundedness condition.

2.3 Lipschitz Continuity of $\mathbf{E}^\varepsilon$ and ρ^ε

In this section we show that $\mathbf{E}^\varepsilon(t,\cdot)$ and $\rho^\varepsilon(t,\cdot)$ are Lipschitz continuous, if f_0 is sufficiently nice. If I satisfies the boundedness condition, the Lipschitz constants do not depend on ε

Lemma 2.5: Let $\varepsilon \geqslant 0, \gamma = \pm 1$. Assume that $\sigma : I \times \mathbb{R}^N \to \mathbb{R}$ satisfies $\sigma(t,\cdot) \in \mathrm{L}_1\left(\mathbb{R}^N\right) \cap \mathrm{L}_\infty\left(\mathbb{R}^N\right) \cap \mathrm{Lip}\left(\mathbb{R}^N\right)$ for all $t \in I$. Let $\mathbf{E}^\varepsilon(t,\mathbf{x}) := \int \mathbf{e}^\varepsilon(\mathbf{x}-\mathbf{y}) \cdot \sigma(t,\mathbf{y})\, \mathrm{d}\mathbf{y}, t \in I, \mathbf{x}, \mathbf{y} \in \mathbb{R}^N$. Then

(i) $\mathbf{E}^\varepsilon(t,\cdot) \in C_b^1\left(\mathbb{R}^N, \mathbb{R}^N\right)$ for all $t \in I$ and

$$\left|\frac{\partial E_i}{\partial x_j}(t,\mathbf{x})\right| \leqslant \omega_N(\delta_{ij}/N + N) + N\omega_N \ln(1+\mathrm{lip}(\sigma(t,\cdot)))\|\sigma(t,\cdot)\|_\infty + N\|\sigma(t,\cdot)\|_1 \quad \text{for all } 1 \leqslant i,j \leqslant N, t \in I, \mathbf{x} = \left(x_1,\ldots,x_N\right) \in \mathbb{R}^N$$

and therefore $\mathbf{E}^\varepsilon(t,\cdot) \in \mathrm{Lip}\left(\mathbb{R}^N, \mathbb{R}^N\right)$ and

$$\mathrm{lip}(\mathbf{E}(t,\cdot)) \leqslant \omega_N\left(N^{-1} + N^2 \cdot \log(1+\mathrm{lip}(\sigma(t,\cdot)))\right) \cdot \|\sigma(t,\cdot)\|_\infty + N^2\left(\omega_N + \|\sigma(t,\cdot)\|_1\right)$$

(ii) If σ is continuous on $I \times \mathbb{R}^N$ and if $\mathrm{lip}(\sigma(t,\cdot))$ and $\|\sigma(t,\cdot)\|_1$ are bounded uniformly in t on every compact subinterval of I, then the partial derivatives $\partial E_i/\partial x_j$ are continuous on $I \times \mathbb{R}^N$

proof (i) The calculation of $\mathbf{E}_i^\varepsilon(t,\mathbf{x})$ derivative component can be divided into three parts, for which a spherical region decomposition is necessary. Let $\mathbf{x}$ be in $B(\mathbf{z}, d_1)$, an open sphere with center $\mathbf{z}$ and radius d_1.

$$\frac{\partial}{\partial x_j} E_i^\varepsilon(\mathbf{x}) = \int\limits_{d_1<|\mathbf{x}-\mathbf{y}|\leqslant d_2} + \int\limits_{d_2<|\mathbf{x}-\mathbf{y}|} + \int\limits_{|\mathbf{x}-\mathbf{y}|\leqslant d_1} \frac{\partial}{\partial x_j} e_i^\varepsilon(\mathbf{x}-\mathbf{y})\sigma(t,\mathbf{y})\mathrm{d}\mathbf{y} =: I_{1,1} + I_{1,2} + I_2 \tag{2-6}$$

Definition of $\mathbf{e}^\varepsilon$ gives $\left|\frac{\partial}{\partial x_j} e_i^\varepsilon(\mathbf{x}-\mathbf{y})\right| \leqslant N\left|\mathbf{x}-\mathbf{y}\right|^{-N}$ to control the estimate, and *cf.* [10], section 4.4.1, theorem 3 helps settle the singularity calculation problem of I_2. TODO! The details would not be present here because the result in fact is not optimal due to the casual value of d_1, d_2.

(ii) TODO ☐

Assumption 2.2: Assume that f^ε is a solution of $\mathrm{VP}^\varepsilon(\varepsilon \geqslant 0$ fixed) on I.

1. We say that f^ε satisfies $\underline{\mathrm{lip}_x(\rho)\ \textit{Control Condition}\ \text{on}\ I}$ if there exists a $C^\varepsilon_{lip_x(\rho)}(t) \in C_+(I)$ such that for $t \in I$

$$\mathrm{lip}\left(\rho^\varepsilon(t,\cdot)\right) \leqslant C^\varepsilon_{lip_x(\rho)}(t) \tag{2-7}$$

2. We say that f_0 satisfies $\underline{\mathrm{lip}(f_0)\ \textit{in v Sphere Control Condition}}$ if there exists an $h \in C_+([0,\infty))$ such that for all $a \geqslant 0$

$$\begin{aligned}\int^* \sup\{ \frac{|f_0(\mathbf{y},\mathbf{u}) - f_0(\mathbf{z},\mathbf{w})|}{|(\mathbf{y},\mathbf{u}) - (\mathbf{z},\mathbf{w})|} \Big| (\mathbf{y},\mathbf{u}),(\mathbf{z},\mathbf{w}) \in \mathbb{R}^N \times \mathbb{R}^N, \\ (\mathbf{y},\mathbf{u}) \neq (\mathbf{z},\mathbf{w}), |\mathbf{u}-\mathbf{v}|, |\mathbf{w}-\mathbf{v}| \leqslant a\} \mathrm{d}\mathbf{v} \leqslant h(a)\end{aligned} \tag{2-8}$$

Remark. It is easily shown that ($\mathrm{lip}(f_0)$ in v sphere control condition) implies $f_0 \in \mathrm{Lip}\left(\mathbb{R}^{2N}\right)$

Lemma 2.6: Assume f_0 satisfies ($\mathrm{lip}(f_0)$ in v sphere control condition) , if f_0 is differentiable and there exists constants $\alpha > N$ and $K \geqslant 0$ such that $\left|\nabla_{x,v} f_0(\mathbf{x},\mathbf{v})\right| \leqslant K \cdot (1+|\mathbf{v}|)^{-\alpha}$ for all $\mathbf{x},\mathbf{v} \in \mathbb{R}^N$

proof By the mean value theorem the integrand of (3.3.1) is bounded by $K \cdot (\max\{1, 1+|\mathbf{v}|-a\})^{-\alpha}$. Let $h(a) := K \cdot \omega_N \cdot \int_0^\infty r^{N-1} \cdot (\max\{1, 1+r-a\})^{-\alpha} \mathrm{d}r$ □

Lemma 2.7: Assume that f_0 satisfies $(f_0 1)$ and $\mathrm{lip}(f_0)$ *in v Sphere Control Condition*,

1. If f^ε is a solution of VP^ε on $I(\varepsilon \geqslant 0$ fixed), then f^ε satisfies $(f2)$
2. If I satisfies the boundedness condition, there exist functions $C_\rho, C_E \in C_+(I)$ such that for all $\varepsilon > 0, t \in I$

$$\mathrm{lip}\left(\mathbf{E}^\varepsilon(t,\cdot)\right) \leqslant H_E(t), \quad \mathrm{lip}\left(\rho^\varepsilon(t,\cdot)\right) \leqslant C_\rho(t)$$

proof We have shown in lemma (2.5) that $\sup\left\{|\mathbf{v}^\varepsilon(0,\tau,x) - \mathbf{v}| \cdot |\mathbf{x},\mathbf{v} \in \mathbb{R}^{2N}\ 0 \leqslant \tau \leqslant t\right\} = f^\varepsilon_v(t) < \infty$ for all $t \in I$. Let $g^\varepsilon(t) := \sup\left\{\mathrm{lip}\left(\mathbf{E}^\varepsilon(r,\cdot)\right) | 0 \leqslant r \leqslant t\right\}$ □

2.4 $\varepsilon \to 0$ Approximation

In this section we prove that a solution of P^0 exists on I, if and only if I satisfies the boundedness condition. We start with three technical lemmas.

Lemma 2.8: For all $\mathbf{z} \in \mathbb{R}^N, \varepsilon, \eta \geqslant 0$ we have that

$$|\mathbf{e}^\varepsilon(\mathbf{z}) - \mathbf{e}^\eta(\mathbf{z})| \leqslant 2 \cdot N \cdot |\mathbf{z}|^{-N+1/2} \cdot \left|\varepsilon^{1/4} - \eta^{1/4}\right|$$

proof

$$\begin{aligned} |\mathbf{e}^\varepsilon(\mathbf{z}) - \mathbf{e}^\eta(\mathbf{z})| =& \left| \int_\eta^\varepsilon \frac{\partial}{\partial \lambda} \mathbf{e}^\lambda(\mathbf{z}) \mathrm{d}\lambda \right| = \left| -(N/2) \cdot \int_\eta^\varepsilon \left(\mathbf{z}^2 + \lambda\right)^{-N/2-1} \mathbf{z} \mathrm{d}\lambda \right| \\ \leqslant& (N/2) \cdot \left| \int_\eta^\varepsilon \left(\mathbf{z}^2 + \lambda\right)^{-(N+1)/2} \mathrm{d}\lambda \right| \leqslant (N/2) \cdot |\mathbf{z}|^{-N+1/2} \cdot \left| \int_\eta^\varepsilon \lambda^{-3/4} \mathrm{d}\lambda \right| \\ =& 2N \cdot |\mathbf{z}|^{-N+1/2} \cdot \left|\varepsilon^{1/4} - \eta^{1/4}\right| \end{aligned} \tag{2-9}$$

□

To accurately depict the singularity of $\mathbf{e}^0$, $\mathbf{e}^\varepsilon$ is divided into two parts to investigate.

Definition 2.4: For all $\varepsilon \geqslant 0$ and $\mathbf{z} \in \mathbb{R}^N$ let $\mathbf{e}^\varepsilon =: \mathbf{e}^{\varepsilon,1} + \mathbf{e}^{\varepsilon,2}$ in which

(i) $\mathbf{e}^{\varepsilon,1}(\mathbf{z}) := \mu \cdot \left(\varepsilon + \max\left\{1, \mathbf{z}^2\right\}\right)^{-N/2} \cdot \mathbf{z}$,

(ii) $\mathbf{e}^{\varepsilon,2}(\mathbf{z}) := \mathbf{e}^\varepsilon(\mathbf{z}) - \mathbf{e}^{\varepsilon,1}(\mathbf{z})$.

Lemma 2.9: Then

(i) $\mathbf{e}^{\varepsilon,1} \in \mathrm{L}_\infty\left(\mathbb{R}^N, \mathbb{R}^N\right) \cap \mathrm{Lip}\left(\mathbb{R}^N, \mathbb{R}^N\right), \left\|\mathbf{e}^{\varepsilon,1}\right\|_\infty \leqslant 1, \mathrm{lip}\left(\mathbf{e}^{\varepsilon,1}\right) \leqslant N^2$

(ii) $\mathbf{e}^{\varepsilon,2} \in \mathrm{L}_1(\mathbb{R}^N, \mathbb{R}^N), \left\|\mathbf{e}^{\varepsilon,2}\right\|_1 \leqslant \omega_n \cdot N/(N+1), \mathrm{supp}\left(\mathbf{e}^{\varepsilon,2}\right) = \left\{\mathbf{z} \in \mathbb{R}^N \middle| |\mathbf{z}| \leqslant 1\right\}$

proof $\mathrm{lip}\left(\mathbf{e}^{\varepsilon,1}\right) \leqslant \max\left\{(1+\varepsilon)^{-N/2}, \sup_{|z|\geqslant 1} \max_{1\leqslant i \leqslant N j=1} \sum_{0=1}^N \left|\frac{\partial}{\partial z_j} e_i^\varepsilon(z)\right|\right\}$ $\leqslant \max\left\{1, \sup_{|z|\geqslant 1} N^2 \cdot |z|^{-N}\right\} = N^2$ $|\mathbf{e}^{\varepsilon,2}\|_1 = \omega_N \cdot \int_0^1 r^N \left(\left(r^2+\varepsilon\right)^{-N/2} - (1+\varepsilon)^{-N/2}\right) \mathrm{d}r \leqslant \omega_N \cdot \int_0^1 r^N \left(r^{-N} - 1\right) \mathrm{d}r = \omega_N \cdot N/(N+1)$, as the integrand is for $0 \leqslant r \leqslant 1$ non-increasing in ε (its derivative with respect to ε is non-positive). □

In the proof of the next theorem we need the Bivariant Gronwall lemma.

Theorem 2.3: Assume that $f_0 \in \mathrm{L}_1\left(\mathbb{R}^{2N}\right)$ satisfies (sup f_0 control condition) and (lip(f_0) in v sphere control condition) . Assume that $I \subset [0, \infty)$ is an interval with $0 \in I$. Then a solution f^0 of VP^0 exists on I, if and only if I satisfies the boundedness condition. In this case f^0 is unique and $f^0 = \lim_{\varepsilon \to 0} f^{\varepsilon}$, uniformly on $I_1 \times \mathbb{R}^{2N}$ for all compact subsets I_1 of I $\varepsilon \to 0$ If f_0 is continuously differentiable on $\mathbb{R}^{2N}$, then f^0 is continuously differentiable on $I \times \mathbb{R}^{2N}$ and satisfies Vlasov's equation. Remark. In view of theorem (2.8) this proves global existence of a solution of P^0 for $N = 1, 2$ and at least local existence for $N \geqslant 3$

Proof of (4.4). It is sufficient to prove the theorem for compact I. (The theorem is true, if and only if it is true for all compact subintervals of I.) Thus assume $I = [0, T], T > 0$

"$\Rightarrow$" of the well-posedness theorem: If I satisfies the boundedness condition, then a unique solution of P^0 exists. We have shown in \$2 and \$3 that there exist constants ρ G_E, F_v, ρ, G_Q such that for all $\varepsilon > 0, t \in I$

$$
\begin{aligned}
&|\mathbf{E}^{\varepsilon}(t, \cdot)|_{\infty} \leqslant \rho, \quad \operatorname{lip}\left(\mathbf{E}^{\varepsilon}(t, \cdot)\right) \leqslant G_E \\
&\sup\left\{|\mathbf{v}^{\varepsilon}(0, t, x) - \mathbf{v}|\ |\mathbf{x}, \mathbf{v} \in \mathbb{R}^{2N}\right\} \leqslant F_v \\
&\|\rho^{\varepsilon}(t, \cdot)\|_{\infty} \leqslant \left.||f^{\varepsilon}|_e(t, \cdot)\right|_{\infty} \leqslant \rho, \quad \operatorname{lip}\left(\rho^{\varepsilon}(t, \cdot)\right) \leqslant G_e
\end{aligned}
$$

Definition 2.5: Now let $\varepsilon > 0$ and $\eta > 0$ for the time being be fixed. Define for $t, \tau \in I$

$$
f^{\varepsilon,\eta}(t, \tau) := \sup\left\{ |(\mathbf{X}^{\varepsilon}, \mathbf{V}^{\varepsilon})(t, \tau, \mathbf{x}, \mathbf{v}) - (\mathbf{X}^{\eta}, \mathbf{V}^{\eta})(t, \tau, \mathbf{x}, \mathbf{v})| \,\middle|\, \mathbf{x}, \mathbf{v} \in \mathbb{R}^{2N} \right\},
$$

i.e. the supremum of the norms of the final state change $|(\mathbf{X}^{\varepsilon} - \mathbf{X}^{\eta}, \mathbf{V}^{\varepsilon} - \mathbf{V}^{\eta})|$, caused by the variantional parameter ε modifying the field singularity, among all the particles moving from time τ to t.

To prove that we could acquire $\mathbf{X}^0$ by the limit of $\mathbf{X}^{\varepsilon}$ ($\varepsilon > 0$), we need to prove that $\mathbf{X}$ uniformly converge as $\varepsilon \to 0$ on $I \times I \times \mathbb{R}^N \times \mathbb{R}^N$, *i.e.* $\lim_{\varepsilon \to 0} \mathbf{X}^{\varepsilon}$ exists and the convergence rate does not rely on $(s, t, \mathbf{x}, \mathbf{v})$ but the field modification parameter ε. We will prove that by the Cauchy method, that is, proving $f^{\varepsilon,\eta}(t, \tau)$ smaller than $K|\varepsilon^k - \eta^k|$ for a certain $k > 0$, where K are constants independent on f_0.

Because of (1.2) and (1.3)f is bounded on $I \times I$. As $(\mathbf{X}^{\varepsilon}, \mathbf{V}^{\varepsilon})$ are known as continuous functions at least when $\varepsilon > 0$, $f(t, \tau) = \sup\left\{\left|\mathbf{X}^{\varepsilon}\left(t, \tau, x_n\right) - \mathbf{X}^{\eta}\left(t, \tau, x_n\right)\right|\ |n \in \mathrm{N}\right\}$ for any sequence $\left(x_n\right)_{\mathrm{neN}}$ dense in $\mathbb{R}^{2N}$. This shows that for each $t \in I$ $f(t, \cdot)$ and $f(\cdot, t)$ are measurable.

Lemma 2.10:

$$f^{\varepsilon,\eta}(t,\tau) \leqslant K|\varepsilon^{1/4} - \eta^{1/4}| \text{ for all } t,\tau \in I, \varepsilon, \eta > 0 \tag{2-10}$$

where K is a constant independent on f_0. If the lemma holds, it shows that $(\mathbf{X}^\varepsilon)$ is uniformly convergent on $I \times I \times \mathbb{R}^n \times \mathbb{R}^n$.

proof To control $f^{\varepsilon,\eta}$, some estimates concerning $\|\mathbf{E}^\varepsilon(t,\cdot) - \mathbf{E}^\eta(t,\cdot)\|_\infty$, $\|\rho^\varepsilon - \rho^\eta\|_\infty$ will be introduced and used sequently.

$$\begin{aligned}
&|(\mathbf{X}^\varepsilon, \mathbf{V}^\varepsilon)(t,\tau,\mathbf{x},\mathbf{v}) - (\mathbf{X}^\eta, \mathbf{V}^\eta)(t,\tau,\mathbf{x},\mathbf{v})| \\
=&\left| \int_\tau^t \left(\mathbf{V}^\varepsilon(r,\tau,\mathbf{x},\mathbf{v}) - \mathbf{V}^\eta(r,\tau,\mathbf{x},\mathbf{v}), \mathbf{E}^\varepsilon\left(r, \mathbf{X}^\varepsilon(r,\tau,\mathbf{x},\mathbf{v})\right) - \mathbf{E}^\eta\left(r, \mathbf{X}^\eta(r,\tau,\mathbf{x},\mathbf{v})\right)\right) \mathrm{d}r \right| \\
\leqslant &\left| \int_\tau^t f(r,\tau)\mathrm{d}r \right| + \int_\tau^t |\mathbf{E}^\varepsilon(r,\mathbf{X}^\varepsilon(r)) - \mathbf{E}^\varepsilon\left(r,\mathbf{X}^\eta(r)\right| \mathrm{d}r + \int_\tau^t |\mathbf{E}^\varepsilon(r,\mathbf{X}^\eta(r)) - \mathbf{E}^\eta\left(r,\mathbf{X}^\eta(r)\right| \mathrm{d}r
\end{aligned} \tag{2-11}$$

To control the $\mathbf{E}^\varepsilon(t,\mathbf{x}) - \mathbf{E}^\eta(t,\mathbf{x})$ integral, we need to estimate $\|\mathbf{E}^\varepsilon(t,\cdot) - \mathbf{E}^\eta(t,\cdot)\|_\infty$.

$$\begin{aligned}
\mathbf{E}^\varepsilon(t,\mathbf{x}) - \mathbf{E}^\eta(t,\mathbf{x}) &=: I_1 + I_2 + I_3 \text{ in which} \\
I_1 &= \int \left(\mathbf{e}^\varepsilon(\mathbf{x}-\mathbf{y}) - \mathbf{e}^\eta(\mathbf{x}-\mathbf{y})\right) \cdot \rho^\varepsilon(t,\mathbf{y})\,\mathrm{d}\mathbf{y} \\
I_2 &= \int \mathbf{e}^{\eta,1}(\mathbf{x}-\mathbf{y}) \cdot \left(\rho^\varepsilon(t,\mathbf{y}) - \rho^\eta(t,\mathbf{y})\right) \mathrm{d}\mathbf{y} \\
I_3 &= \int \mathbf{e}^{\eta,2}(\mathbf{x}-\mathbf{y}) \cdot \left(\rho^\varepsilon(t,\mathbf{y}) - \rho^\eta(t,\mathbf{y})\right) \mathrm{d}\mathbf{y}
\end{aligned}$$

Because of lemma (2.1) and (4.2) TODO

$$\begin{aligned}
|I_1| &\leqslant \int 2N \cdot |\mathbf{x}-\mathbf{y}|^{-N+1/2} \cdot \left|\varepsilon^{1/4} - \eta^{1/4}\right| \cdot |\rho^\varepsilon(t,\mathbf{y})|\,\mathrm{d}\mathbf{y} \\
&\leqslant 2N\mathscr{M}^{1/2N} C(N, N-1/2, \infty, 1) \cdot C_\rho^{(N-1/2)/N} \cdot \left|\varepsilon^{1/4} - \eta^{1/4}\right| \\
&=: K_5 \cdot \left|\varepsilon^{1/4} - \eta^{1/4}\right| \\
|I_2| &= \left| \int \left(\mathbf{e}^{\eta,1}\left(\mathbf{x} - \mathbf{X}^\varepsilon(t,0,\mathbf{y},\mathbf{u})\right) - \mathbf{e}^{\eta,1}\left(\mathbf{x} - \mathbf{X}^\eta(t,0,\mathbf{y},\mathbf{u})\right)\right) \cdot f_0(\mathbf{y},\mathbf{u})\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{u} \right| \\
&\leqslant \mathscr{M} \cdot \operatorname{lip}\left(\mathbf{e}^{\eta,1}\right) \cdot f(t,0) \leqslant \mathscr{M} \cdot N^2 \cdot f(t,0) =: K_4 \cdot f(t,0)
\end{aligned}$$

Proposition 2.1: To control $|I_3|$, we need to estimate $\|\rho^\varepsilon - \rho^\eta\|_\infty$ in advance,

$$
\begin{aligned}
|\rho^\varepsilon(t,\mathbf{x}) - \rho^\eta(t,\mathbf{x})| &= \left| \int f_0\left((\mathbf{X}^\varepsilon, \mathbf{V}^\varepsilon)(0,t,\mathbf{x},\mathbf{v})\right) - f_0\left((\mathbf{X}^\eta, \mathbf{V}^\eta)(0,t,\mathbf{x},\mathbf{v})\right) \mathrm{d}\mathbf{v} \right| \text{ for all } \mathbf{x} \in \mathbb{R}^N \\
&\leqslant \int^{*} \sup \left\{ \frac{|f_0(\mathbf{y},\mathbf{u}) - f_0(\mathbf{z},\mathbf{w})|}{|(\mathbf{y},\mathbf{u}) - (\mathbf{z},\mathbf{w})|} \middle| y \neq w, |\mathbf{u}-\mathbf{v}|, |\mathbf{w}-\mathbf{v}| \leqslant F_0 \right\} \\
&\quad \cdot |\mathbf{X}^\varepsilon(0,t,\mathbf{x},\mathbf{v}) - \mathbf{X}^\eta(0,t,\mathbf{x},\mathbf{v})| \,\mathrm{d}\mathbf{v} \leqslant h\left(F_v\right) \cdot f(0,t)
\end{aligned}
\tag{2-12}
$$

where the function h was introduced in definition (3.3) TODO

$$
\left|I_3\right| \leqslant \left\|\mathbf{e}^{\eta,2}\right\|_1 \cdot \|\rho^\varepsilon(t,\cdot) - \rho^\eta(t,\cdot)\|_\infty \leqslant \omega_N(N/(N+1)) \cdot h\left(F_v\right) \cdot f(0,t) =: K_3 \cdot f(0,t)
$$

Hence we conclude that there exist constants K_3, K_4 and K_5, depending on f_0, but not on ε and η, such that for all $t \in I$,

$$
\|\mathbf{E}^\varepsilon(t,\cdot) - \mathbf{E}^\eta(t,\cdot)\|_\infty \leqslant K_3 \cdot f(0,t) + K_4 \cdot f(t,0) + K_5 \left|\varepsilon^{1/4} - \eta^{1/4}\right|, \tag{2-13}
$$

We can now turn back to the (2-11) to push the inequality forward

$$
\begin{aligned}
&|(\mathbf{X}^\varepsilon, \mathbf{V}^\varepsilon)(t,\tau,\mathbf{x},\mathbf{v}) - (\mathbf{X}^\eta, \mathbf{V}^\eta)(t,\tau,\mathbf{x},\mathbf{v})| \\
&\leqslant \left| \int_\tau^t \left(1 + G_E\right) \cdot f(r,\tau) + K_3 \cdot f(0,r) + K_4 \cdot f(r,0) + K_5 \cdot \left|\varepsilon^{1/4} - \eta^{1/4}\right| \mathrm{d}r \right| \\
\Rightarrow f(t,\tau) &\leqslant \left| \int_\tau^t \left(\left(1 + G_E\right) \cdot f(r,\tau) + \max\left\{K_3, K_4\right\} \cdot (f(0,r) + f(r,0)) \right.\right. \\
&\quad \left.\left. + K_5 \cdot \left|\varepsilon^{1/4} - \eta^{1/4}\right|\right) \mathrm{d}r \right| \text{ for any } t, \tau \in I
\end{aligned}
\tag{2-14}
$$

The equations looks complicated but we can utilize the bivariant Gronwall lemma in appdenx A.1 to make it clear. We deduce that there exists a constant K, which does not depend on ε and η, such that for all $t, \tau \in I$,

$$
f(t,\tau) = \sup\left\{|\mathbf{X}^\varepsilon(t,\tau,\mathbf{x},\mathbf{v}) - \mathbf{X}^\eta(t,\tau,\mathbf{x},\mathbf{v})| \,|\mathbf{x},\mathbf{v} \in \mathbb{R}^{2N}\right\} \leqslant K \cdot \left|\varepsilon^{1/4} - \eta^{1/4}\right| \tag{2-15}
$$

□

This shows that $\left(X^{\varepsilon}\right)_{\varepsilon>0}$ is uniformly convergent as $\varepsilon \to 0$ on $I \times I \times \mathbb{R}^{2N}$ We interrupt the argument to note the following: If a solution f^0 of P^0 exists on I and I satisfies the boundedness condition, then we can prove (4.4.1) for $\eta = 0$ (maybe with different constants $\rho, G_E, F_v, \rho, G_e$. As a matter of fact the same constants will work, but we do not know this beforehand.) It follows that $\mathbf{X}^0(t,\tau,\mathbf{x},\mathbf{v}) = \lim_{\epsilon\to 0}\mathbf{X}^{\varepsilon}(t,\tau,\mathbf{x},\mathbf{v})$. Therefore $\mathbf{X}^0$ and also $f^0(t,\mathbf{x},\mathbf{v}) = f_0\left(\mathbf{X}^0(0,t,x)\right)$ are uniquely determined. This proves the uniqueness part of the theorem.

We continue the main argument: Define for $t,\tau \in I, \mathbf{x},\mathbf{v} \in \mathbb{R}^{2N}$ $\mathbf{X}^0(t,\tau,\mathbf{x},\mathbf{v}) := \lim_{\epsilon\to 0}\mathbf{X}^{\varepsilon}(t,\tau,\mathbf{x},\mathbf{v}), f^0(t,x) := f_0\left(\mathbf{X}^0(0,t,x)\right)$. Then (i) $\mathbf{X}^0$ is continuous on $I \times \mathbb{R}^N$ (ii) $\sup\left\{\left|\mathbf{v}^0(0,t,x) - x_0\right| |\mathbf{x},\mathbf{v} \in \mathbb{R}^{2N}, t \in I\right\} \leqslant F_v$ Because of inequality 2°$\mathbf{E}^{\varepsilon}$ converges uniformly. Define for $t \in I, \mathbf{x} \in \mathbb{R}^N$ $E^0(t,\mathbf{x}) := \lim_{\epsilon\to 0}\mathbf{E}^{\varepsilon}(t,\mathbf{x})$. Then (iii) E^0 is continuous on $I \times \mathbb{R}^N$ (iv) $E^0(t,\cdot) \in \mathrm{C}_b^0\left(\mathbb{R}^N,\mathbb{R}^N\right) \cap \mathrm{Lip}\left(\mathbb{R}^N,\mathbb{R}^N\right)$ for all $t \in I$ and $\left|E^0(t,\cdot)\right|_{\infty} \leqslant \rho$ $\mathrm{lip}\left(E^0(t,\cdot)\right) \leqslant G_E$ For all $\varepsilon > 0$ we have

$$\mathbf{X}^{\varepsilon}(t,\tau,\mathbf{x},\mathbf{v}) = x + \int_{\tau}^{t}\left(\mathbf{x}^{\epsilon}(r,\tau,x), \mathbf{E}^{\varepsilon}\left(r,\mathbf{x}^{\epsilon}(r,\tau,x)\right)\mathrm{d}r\right.$$

The uniform convergence of $\mathbf{X}^{\epsilon}$ and $\mathbf{E}^{\varepsilon}$ implies that this equation remains true for $\varepsilon = 0$. This proves

(v) $\mathbf{X}^0$ satisfies the differential equation (1.2.1) with initial condition (1.4.1) $\mathbf{X}^0(t,\tau,\cdot)$ is therefore a measure preserving homeomorphism for all $t,\tau \in I$. Hence (vi) $f^0(t,\cdot) \in \mathrm{L}_1\left(\mathbb{R}^{2N}\right)$ for all $t \in I$ and $\left|f^0(t,\cdot)\right|_1 = m$ This shows that $f_{\rho}^0(t,\cdot)$ is well-defined and $\left\|f_{\rho}^0(t,\cdot)\right\|_1 \leqslant m$ for all $t \in I$ Inequality 1° can now be proved analogously for $\eta = 0$. This implies that f_{ρ}^{ϵ} converges uniformly on $I \times \mathbb{R}^N$ to ρ^0. Thus (vii) $f_{\rho}^0(t,\cdot) \in \mathrm{L}_{\infty}\left(\mathbb{R}^N\right) \cap \mathrm{Lip}\left(\mathbb{R}^N\right)$ for all $t \in I$ and $\left|\rho^0(t,\cdot)\right|_{\infty} \leqslant \rho, \mathrm{lip}\left(\rho^0(t,\cdot)\right)$ $\leqslant G_e$ Inequality 2° can now be proved analogously for $\eta = 0$. This implies that $\mathbf{E}^{\varepsilon}(t,\mathbf{x})$ converges uniformly on $I \times \mathbb{R}^N$ to $\int \mathbf{e}^0(\mathbf{x}-\mathbf{y}) \cdot f^0(t,y)\mathrm{d}y^{2N}$. This expression must therefore be equal to $E^0(t,\mathbf{x})$ All in all we have shown that f^0 is the unique solution of P^0 on I. As ($f_0$2) implies $f_0 \in \mathrm{Lip}\left(\mathbb{R}^{2N}\right)$, we have that

$$\begin{aligned}
&\left|f^{\epsilon}(t,x) - f^0(t,x)\right| = \left|f_0\left(\mathbf{X}^{\epsilon}(0,t,x)\right) - f_0\left(\mathbf{X}^0(0,t,x)\right)\right| \\
&\leqslant \mathrm{lip}(f_0) \cdot \sup\left\{\left|\mathbf{X}^{\epsilon}(0,t,x) - \mathbf{X}^0(0,t,x)\right| |\mathbf{x},\mathbf{v} \in \mathbb{R}^{2N}, t \in I\right\} \to 0, \text{ if } \varepsilon \to 0
\end{aligned}$$

Therefore f^e converges uniformly on $I \times \mathbb{R}^{2N}$ to f^0 Lemma (2.6) and lemma (3.1) imply (viii) $E^0(t, \mathbf{x})$ is continuously differentiable with respect to $\mathbf{x}$ If f_0 is continuously differentiable, then f^0 is continuously differentiable on $I \times \mathbb{R}^{2N}$ and satisfies Vlasov's equation (cf. lemma (1.4)).

"$\Leftarrow$" of the well-posedness theorem: If a solution f^0 of P^0 exists on I, then I satisfies the boundedness condition.

If $N = 1, 2$, there is nothing to be proved (cf. theorem (2.8)). Now let $N \geqslant 3$ and assume that a solution f^0 of P^0 exists on $I = [0, T]$ By theorem (2.8) there exists a $T_1 \in \left]0, T\right]$, such that $\left[0, T_1\right[$ satisfies the boundedness condition. Therefore there exists a largest interval $I_2 \subset I$ with left endpoint 0 that satisfies the boundedness condition. We have either $I_2 = \left[0, T_2\right]$ or $I_2 = \left[0, T_2\right[$ for some $T_2 \in \left]0, T\right]$. If $I_2 = I$, the proof is finished. Thus assume I_2 I As I is compact, it follows with lemma (2.5) that

$$\sup\left\{\left|\mathbf{v}^0(0, t, x) - \mathbf{v}\right| |\mathbf{x}, \mathbf{v} \in \mathbb{R}^{2N}, t \in I\right\} =: F_0^0 < \infty$$

f_0 satisfies ($f_0 1$) with constants K_1 and K_2, as defined in definition (2.3). Now let $K_1^* := K_1, K_2^* := K_2 + F_v^0 + 1$ In the proof of theorem (2.8) we have shown the following: Assume that $\psi \in \mathrm{L}_1\left(\mathbb{R}^{2N}\right)$ satisfies ($f_0 1$) with constants K_1^* and K_2^* and that $\|\psi\|_1 \leqslant m$. Then there exists a $\vartheta > 0$ and a $B \geqslant 0$, both numbers depending only on K_1^*, K_2^* and m, such that for all $\varepsilon > 0$ the solution Ψ^ε of P^ε on $[0, \infty)$ with initial datum ψ satisfies $\left\|\Psi_{f_0}^\varepsilon(t, \cdot)\right\|_\infty \leqslant B$ for all $t \in [0, \vartheta]$ Let $T_0 := \max\left\{0, T_2 - \vartheta/2\right\}$. As $T_0 < T_2$ we know that $I_0 := \left[0, T_0\right]$ satisfies the boundedness condition. We have shown in the first part of this proof that $\mathbf{X}^\varepsilon$ converges uniformly on $I_0 \times I_0 \times \mathbb{R}^{2N}$ to $\mathbf{X}^0$. Hence there exists an $\varepsilon_0 > 0$, such that $\sup\left\{\left|\mathbf{v}^\epsilon(0, t, x) - x_0\right| |\mathbf{x}, \mathbf{v} \in \mathbb{R}^{2N}, t \in I_0\right\} \leqslant F_v^0 + 1$ for all $\varepsilon \in \left]0, \varepsilon_0\right]$. Lemma (2.5) implies that $f^\varepsilon\left(T_0, \cdot\right)$ satisfies ($f_0 1$) with constants K_1^* and K_2^* for all $\varepsilon \in \left]0, \varepsilon_0\right]$.

We now claim that $\left[0, T_0 + \theta\right]$ of satisfies the boundedness condition:

$$\begin{aligned}
&\sup\left\{|\rho^\varepsilon(t, \cdot)|_\infty \,|\varepsilon > 0, t \in \left[0, T_0 + \vartheta\right]\right\} = \max\left\{B_1, B_2, B_3\right\} \\
&\text{with } B_1 = \sup\left\{|\rho^\varepsilon(t, \cdot)|_\infty \,|\varepsilon > 0, t \in I_0\right\} \\
&B_2 = \sup\left\{|\rho^\epsilon(t, \cdot)|_\infty \,|\varepsilon > \varepsilon_0, t \in \left[0, T_0 + \vartheta\right]\right\} \\
&B_3 = \sup\left\{|\rho^\varepsilon(t, \cdot)|_\infty \,|0 < \varepsilon \leqslant \varepsilon_0, t \in \left[T_0, T_0 + \vartheta\text{f}\right.\right\}
\end{aligned}$$

We know that $B_1 < \infty$, as I_0 satisfies the boundedness condition. Furthermore $B_2 < \infty$ because of lemma (2.5) and the remark after theorem (1.3) We now show that $B_3 \leqslant B < \infty$: Take any $\varepsilon \in \left] 0, \varepsilon_0 \right]$ and let $\psi = f^{\varepsilon}\left(T_0, \cdot\right) .\psi$ satisfies $(f_0 1)$ with constants K_1^* and K_2^* and $\|\psi\|_1 = m$. For all $t \in \left[T_0, T_0 + \vartheta\right]$ we have $f^{\epsilon}(t, \cdot) = \Psi^{\epsilon}\left(t - T_0, \cdot\right)$ and therefore $|\rho^{e}(t, \cdot)|_{0\infty} = |\Psi_e^e\left(t - T_0, \cdot\right)\|_{\infty} \leqslant B$ Thus we have proved that $\left[0, T_0 + \theta\right]$ of satisfies the boundedness condition. This is a contradiction as $\left[0, T_0 + \theta\right] \cap I \geqslant I_2$ and I_2 was the largest subinterval of $.I$ with left endpoint 0 that satisfies the boundedness condition.

In short, we finished the well-posedness theorem of local solutions for Vlasove equations in this chapter.

Assume that $f_0 \in \mathrm{L}_1\left(\mathbb{R}^{2N}\right)$ and is continuously differentiable and that there exist an $\alpha > N$ and $aK \geqslant 0$ such that

$$|f_0(x)| \leqslant K \cdot (1 + |\mathbf{v}|)^{-a}, \left|\frac{\mathrm{d}}{\mathrm{d}x} f_0(\dot{x})\right| \leqslant K \cdot (1 + |\mathbf{v}|)^{-\alpha} \text{ for all } \mathbf{x}, \mathbf{v} \in \mathbb{R}^{2N}$$

Assume I $\subset$ $[0, \infty)$ is an interval with $0 \in I$ Then a solution f^0 of the initial value problem P^0 on I exists, if and only if I satisfies the boundedness condition. In this case the solution is unique and it satisfies Vlasov's equation. Moreover $f^0 = \lim_{\varepsilon \to 0} f^{\varepsilon}$, uniformly on $[0, T] \times \mathbb{R}^{2N}$ for all $T \in I$ If $N = 1, 2$, then $[0, \infty)$ satisfies the boundedness condition. If $N \geqslant 3$ there exists a $T \in]0, \infty]$ (which may depend on f_0) such that $[0, T[$ satisfies the boundedness condition.

Chapter 3 Global Solutions of non-relativistic Vlasov-Poisson System

We restrict the Vlasov-Poisson problem in the 3D (non-relativistic) case in this chapter, and revisited the proof of to confirm the existence and uniqueness of classical solutions, by Pfaffelmoser (1992) and Lions and Perthame (1991) respectively, for the initial data without the compact assumption in "**v**".

3.1 Global Existence

Pfaffelmoser (1992) proved the global existence of $\gamma = \pm 1$ cases, with almost the same assumptions given in last chapter. The existence result is shown by indicating that the supremum of velocity change of a local solution can not be controlled by a function $H_v \in C_+\left(\mathbf{R}_0^+\right)$, however, it did. The control aim is reached by pointing out how much acceleration of a particle can be given by the "neighboring" particles near its characteristic trajectory.

In this section we consider the trajectories passing through a point $\mathbf{x}$ of the configuration-space at a time t and subsets of the velocity-space at $\mathbf{x}$ and at time t, that are defined by certain properties of the trajectories. We study, how the "largeness" of the subsets depends on these properties.

Assumption 3.1: (i) $f_0 \in L_1\left(\mathbb{R}^n \times \mathbb{R}^n\right), f_0 \geqslant 0$ satisfying f_0 satisfies (sup f_0 control condition) with constants K_1, K_2 and (lip(f_0) in v sphere control condition) with $H \in C_+([0, \infty))$

(ii) $\int_{\mathbb{R}^n} \mathbf{v}^2 f_0(\mathbf{x}, \mathbf{v}) d\mathbf{x}, \mathbf{v} < \infty$

Let f be the corresponding maximal solution of (VP) on $[0, T)$.

Definition 3.1: We define $h_{\mathbf{v}}, h_E, h_\rho : [0, T) \to [0, +\infty)$ by

$$
\begin{aligned}
h_v(t) &:= \sup\left\{|\mathbf{V}(0, \tau, \mathbf{x}, \mathbf{v}) - \mathbf{v}| \,|\, \mathbf{x}, \mathbf{v} \in \mathbb{R}^n, 0 \leqslant \tau \leqslant t\right\} \\
h_E(t) &:= \sup\left\{|\mathbf{E}(\tau, \mathbf{x})| \,|\, \mathbf{x} \in \mathbb{R}^3, 0 \leqslant \tau \leqslant t\right\} \\
h_\rho(t) &:= \sup\left\{|\rho(\tau, \mathbf{x})| \,|\, \mathbf{x} \in \mathbb{R}^3, 0 \leqslant \tau \leqslant t\right\}
\end{aligned}
$$

These functions are nondecreasing due to the supremum range. Their inter-control relation is presented in the following lemma.

Lemma 3.1: There exist $K_{10,E}, K_{10,\rho} > 0$, so that for all $t \in [0, T)$ we have

$$h_E(t) \leqslant K_{10,E} h_\rho^{4/9}(t)$$

$$h_v(t) \leqslant \int_0^t h_E(s) ds$$

$$h_\rho(t) \leqslant K_{10,\rho} \left(1 + h_{\mathbf{v}}(t)\right)^3$$

proof See Pfaffelmoser (1992), lemma 10. $\square$

Proposition 3.1: If there exists a $H_v \in C_+ \left(\mathbf{R}_0^+\right)$, so that for all $t \in [0, T[$ we have

$$h_v(t) \leqslant H_v(t)$$

then $T = \infty$, i.e., (VP) has a global solution.

proof The proposition is a clear result with means of Horst et al. (1981a). For solutions f on an bounded interval $[0, t)$, they can be continued onto an interval $[0, t + \varepsilon)$, where $\varepsilon = \varepsilon(h_v(t))$ depends monotonically decreasing on $h_{\mathbf{v}}(t)$. For bounded $T < \infty$ we can take $t > T - \varepsilon\left(H_v(T)\right)$, which is sufficient to continue f onto $[0, t+\varepsilon) \subsetneq [0, T)$, which is a contradiction to the maximality of $[0, T)$. The exact argument can be found in [12]

Definition 3.2: (i) For $t \in [0, T)$ $\mathbf{x} \in \mathbb{R}^3, 0 < \Delta_1 \leqslant t$ and $d > 0$, define

$$\Psi_1(t, \mathbf{x}) := \left\{\mathbf{v} \in \mathbb{R}^3 \,\middle|\, \exists s \in \left[t - \Delta_1, t\right] : \left|\mathbf{V}(s, t, \mathbf{x}, \mathbf{v}) - \mathbf{v}\right| > d\right\}$$

(ii) Let $\Omega \subset \mathbb{R}^3, t \in \left[0, T\right[, 0 < \Delta_2 \leqslant \Delta_1 \leqslant t, R > 0$ and $\mathbf{x} \in \mathbb{R}^3$. Further let $(\mathbf{X}^*, \mathbf{V}^*)$ be a solution of the characteristic system and define

$$\Psi_2(t, \mathbf{x}) := \left\{\mathbf{v} \in \Omega \,\middle|\, \exists s \in \left[t - \Delta_1, t - \Delta_2\right] : \left|\mathbf{X}(s, t, \mathbf{x}, \mathbf{v}) - \mathbf{X}^*(s)\right| \leqslant R\right\}$$

by which in section 6 we will estimate the influence of "high" densities near a given trajectory on its acceleration. This will turn out to be the crucial part of the proof.

Assumption 3.2:

$$\lim_{t \to T} h_v(t) = \lim_{t \to T} h_\rho(t) = \infty$$

Definition 3.3: Define for $0 \leqslant \alpha \leqslant \beta, R > 0$ and $s \in [0, T)$, $(\mathbf{X}^*, \mathbf{V}^*)$ of the characteristic system the following

$$\begin{aligned} G_\alpha^\beta(s) &:= \left\{ \mathbf{x} \in B_R(\hat{X}(s)) | \alpha \leqslant \rho(s, \mathbf{x}) \leqslant \beta \right\} \\ N_\alpha^\beta(s) &:= \| \rho(s) \cdot 1_{G_1^{(s)}} \|_3 \end{aligned}$$

Theorem 3.1: Let f_0 satisfy the assumption made in this chapter. Then (VP) has a global solution, and for all $\varepsilon > 0$ there exists a $K_{19,8} > 0$, so that for all $t \geqslant 0$ we have

$$h_v(t) \leqslant K_{19,6}(1 + t)^{51/11 + \varepsilon}$$

3.2 Uniqueness

The uniqueness proof is based on the Lions-Perthame theorem and require the moment of $m \leqslant 6$ exist.

Theorem 3.2: *(Lions-Perthame)* Let $f_0 \geqslant 0, f_0 \in L^1 \bigcap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$. We assume that

$$\int\limits_{\mathbb{R}^3 \times \mathbb{R}^3} |\mathbf{v}|^m f_0(\mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v} < +\infty \quad \text{if } m < m_0, \tag{3-1}$$

where $3 < m_0$. Then, there exists a solution $f \in C(\mathbb{R}^+; L^p(\mathbb{R}^3 \times \mathbb{R}^3)) \bigcap L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$ (for all $1 \leqslant p < +\infty$) of Vlasov-Poisson system satisfying

$$\sup_{t \in [0,T]} \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} |\mathbf{v}|^m f(t, \mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v} < +\infty \tag{3-2}$$

Theorem 3.3: We make the assumptions of Theorem 1, (10), (42), then the solution of Vlasov-Poisson Equation such that $\rho \in L^\infty\left((0, T) \times \mathbb{R}^3_{\mathbf{x}}\right)$ is unique.

Remarks. 1. The boundedness of ρ in L^∞ implies, by Corollary 5 , that the solutions of Vlasov-Poisson Equation are smooth. Thus Theorem 6 applies to classical solutions. 2. Notice that (42) and $\rho \in L^{\infty}\left((0, T) \times \mathbb{R}_{\mathbf{x}}^{3}\right)$ implies (10) and the assumptions of Theorem 6 could be slightly improved. 3. Of course the difficulty in Lemma 3 is that $\mathbf{E}$ is not lipschitz continuous in $\mathbf{x}$ We now turn to the proof of these results.

proof Proof of Corollary 5. Thanks to the representation formula of f with the characteristic curves we find, for $\left|\mathbf{x}_{1}-\mathbf{x}_{2}\right| \leqq 1/2$

$$\left|\rho\left(\mathbf{x}_{1}, t\right)-\rho\left(\mathbf{x}_{2}, t\right)\right| \leqq \int_{\mathbb{R}_{2}}\left|f_{0}\left(X_{1}(0), V_{1}(0)\right)-f_{0}\left(X_{2}(0), V_{2}(0)\right)\right| d \mathbf{v}$$

where $\left(X_{i}, V_{i}\right)$ satisfies (43) with $v_{1}=v_{2}=\mathbf{v}$. since $\mathbf{E}$ is bounded in L^{∞}, we have, following (19)

$$\left|X_{i}(0)-\mathbf{v} t-\mathbf{x}_{i}\right| \leqq R, \quad\left|V_{i}(0)-\mathbf{v}\right| \leqq R$$

and thus we may estimate the above terms by

$$\begin{aligned}&\int_{\mathbf{R}_{\mathrm{p}}} \sup \left\{\left|\nabla f_{\mathrm{o}}\right|(y+\mathbf{v} t, w) ;\left|y-\mathbf{x}_{1}\right| \leqq R,|w-\mathbf{v}| \leqq R\right\} d \mathbf{v} \\ &\quad \cdot \sup_{\mathbf{v} \in \mathbb{R}^{3}}\left\{\left|X_{1}(0)-X_{2}(0)\right|+\left|V_{1}(0)-V_{2}(0)\right|\right\} \leqq C(R, T)\left|\mathbf{x}_{1}-\mathbf{x}_{2}\right|^{2}\end{aligned}$$

thanks to (42) and (44). This shows that $\rho(\cdot, t)$ is Hölder continuous and the $C^{1, \alpha}$ regularity of $\mathbf{E}(\cdot, t)$ follows from Schauder estimates. Then, we obtain that the characteristic curves are Lipschitz continuous (we may take $\alpha=1$ in (45)) and thus $\mathbf{E}(\cdot, t)$ belongs to $C^{1 . \beta}\left(\mathbb{R}^{3}\right)$ for all $\beta<1$ We may now conclude the uniqueness proof. Proof of Theorem 6. First, let us notice that an clementary modification of the proof of Corollary 5 gives, thanks to the L^{2} bound in (42) (48)

$$\nabla_{\mathbf{v}} f(t, \mathbf{x}, \mathbf{v}) \in L^{\infty}\left((0, T) \times \mathbb{R}_{\mathbf{x}}^{3} ; L^{2}\left(\mathbb{R}_{\mathbf{v}}^{3}\right)\right)$$

for any solution f. Secondly, we set

$$D(t)=\sup_{0 \leqslant s \leqslant t}\left\|\left(f_{1}-f_{2}\right)\right\|_{L^{2}\left(\mathbf{R}^{\prime}\right)}$$

for two possible solutions f_{1}, f_{2} of $(1)-(3)$ and we claim that (49)

$$\frac{d}{d t} D(t) \leqq C(T)\left\|\left(E_{1}-E_{2}\right)(t)\right\|_{L^{2}\left(R^{3}\right)}$$

Indeed Vlasov equations give

$$\frac{\partial}{\partial t}\left(f_1-f_2\right)^2+\mathbf{v}\cdot\boldsymbol{\nabla}_{\mathbf{x}}\left(f_1-f_2\right)^2+E_1\cdot\boldsymbol{\nabla}_{\mathbf{v}}\left(f_1-f_2\right)^2 \leqq 2\left|E_2-E_1\right|\cdot\left|f_1-f_2\right|\cdot\left|\boldsymbol{\nabla}_e f_2\right|$$

and thus, using (48)

$$\begin{aligned}\frac{d}{dt}D(t)^2 &\leqq 2\int\limits_{\substack{\mathbb{R}_{\mathbf{x}}\\1}} \left|\left(E_2-E_1\right)(t)\right| \mathbb{P}\left(f_1-f_1\right)(t,\mathbf{x},\mathbf{v})\left\|_{L^2(\mathbb{R}_j)}\right\| \boldsymbol{\nabla}_{\mathbf{v}} f_2(t,\mathbf{x},\mathbf{v})\|_{L^2(\mathbf{R}\})}d\mathbf{x}\\ &\leqq C(T)\left\|\left(E_1-E_2\right)(t)\right\|_{L^2\left(\mathbb{R}^3_{\mathbf{x}}\right)} D(t)\end{aligned}$$

□

which clearly proves (49). Finally, we use formula (28) which gives

$$\begin{aligned}&\left\|\left(E_1-E_2\right)(t)\right\|_{L^2(\mathbf{R}^3_{\mathbf{x}})}\\ &\leqq C\left\|\int_0^t\int_{\mathbb{R}^3}\left(E_1-E_2\right)(\mathbf{x}-\mathbf{v}s,t-s)f_1(t-s,\mathbf{x}-\mathbf{v}s,\mathbf{v})s\,ds\,ds\,d\mathbf{v}\right\|_{L^2(\mathbf{R}^3_{\mathbf{x}})}\\ &\quad+C\left\|\int_0^t\int_{\mathbf{R}^3}E_2(\mathbf{x}-\mathbf{v}s,t-s)\left(f_2-f_1\right)(t-s,\mathbf{x}-\mathbf{v}s,\mathbf{v})s\,ds\,ds\,d\mathbf{v}\right\|_{L^2(\mathbb{R}_2)}\end{aligned}$$

The first term in the r.h.s. of this inequality may be estimated by

$$\begin{aligned}&\left\|\int_0^t \tfrac{s\,ds}{s^{3/2}}\mathbb{I}\left(E_1-E_2\right)(y,t-s)\right\|_{L^2(\mathbf{R},y)}\left(\int_{\mathbb{R}\})}f_1^2(t-s,\mathbf{x}-\mathbf{v}s,\mathbf{v})d\mathbf{v}\right)^{1/2}\|_{L^2\left(\mathbb{R}^3_{\mathbf{x}}\right)}\\ &\leqq \int_0^t \tfrac{ds}{s^{1/2}}\left\|f_1(t-s)\right|_{L^2(\mathbf{R}^6)}\left\|\left(E_1-E_2\right)(t-s)\right\|_{L^2(\mathbb{R}^3)}\\ &\leqq Ct^{1/2}\sup_{s\leqslant t}\left[\left(E_1-E_2\right)(s)\right\|_{L^2(R^3)}\end{aligned}$$

Processing the other term in the same way yields

$$\left\|\left(E_2-E_1\right)(t)\right\|_{L^2(\mathbf{R})} \leqq \frac{1}{2}\sup_{0\leqslant s\leqslant t}\left|\left(E_2-E_1\right)(s)\right|_{L^2(\mathbf{R}^3)}+C(T)D(t)$$

for $t \leqq t_0$ small enough. From this one easily deduces that

$$\left\|\left(E_2-E_1\right)(t)\right\|_{L^2(\mathbb{R}^3)} \leqq C(T)D(t)$$

and combining this inequality with (49) just shows that $D(t)=0$ by a Gronwall argument. Therefore $f_1=f_2$ for $t \leqq t_0$ and Theorem 6 is proved.

Chapter 4 Global Solutions of relativistic Vlasov-Poisson System

In this chapter, we present some useful global existence resutls with spherical symmetric data for relativistic Vlasov-Poisson System.

4.1 Spherical Symmetry

The characteristics with *spherical symmetric* initial data, *i.e.*, that $f_0(U\mathbf{x}, U\mathbf{v}) = f_0(\mathbf{x}, \mathbf{v})$ for any rotation matrix U on $\mathbb{R}^3$, would be further specified. Now that the distribution f is radial, as a result, we know that the density function $\rho(t, x)$ is radial at least for $t = 0$,

$$\forall x \in \mathbb{R}^3, \quad \rho(t, Ux) = \int_{\mathbb{R}^3} f(t, Ux, v)dv = \int_{\mathbb{R}^3} f(t, Ux, U\omega)|\det(U)|d\omega = \int_{\mathbb{R}^3} f(t, x, \omega)d\omega = \rho(t, x)$$

which indicates the $\mathbf{E}$ is also radial.

Let $\mathbf{X}_1, \mathbf{V}_1$ be the solution of characteristic system with initial data $\mathbf{X}_1(0) = \mathbf{x}, \mathbf{V}_1(0) = \mathbf{v}$ while $\mathbf{X}_2, \mathbf{V}_2$ with $(U\mathbf{x}, U\mathbf{v})$ as initial data.

$$\begin{aligned}\mathbf{X}_1(t) =& \int_0^t \int_0^s \mathbf{E}(\tau, \mathbf{X}_1(\tau))\mathrm{d}\tau + \mathbf{V}_1(0)\mathrm{d}s + \mathbf{X}_1(0)\\ =& U^{-1} \int_0^t \int_0^s \mathbf{E}(\tau, \mathbf{X}_2(\tau))\mathrm{d}\tau + \mathbf{V}_2(0)\mathrm{d}s + U^{-1}\mathbf{X}_2(0) = U^{-1}\mathbf{X}_2,\end{aligned}$$

then we deduce that $f(t, \mathbf{X}_1(t), \mathbf{V}_1(t)) = f(t, \mathbf{X}_2(t), \mathbf{V}_2(t)) = f(t, U\mathbf{X}_1(t), U\mathbf{V}_1(t))$ for all t, *i.e.* the radial property of intial data propagates.

Therefore, The initial data spherical symmetry would induce the spherically symmetric solution with simplified argument $r := |\mathbf{x}|$, $u := |\mathbf{v}|, \alpha := \angle(\mathbf{x}, \mathbf{v})$ and t. f, ρ and other quantities could be redefined in a more essential way, thanks to the spherical geometry, *i.e.* $f(t, r, u, \alpha) := f(t, r\hat{\mathbf{x}}, \mathbf{v})$, where $\hat{\mathbf{x}}$ denotes one of the standard unit vector coordinate basis, $|\mathbf{v}| = u$ and $\angle(\mathbf{v}, \hat{\mathbf{x}}) = \alpha$. The density ρ depends then only on r and

t:

$$\rho(t,r) = 2\pi \int_0^\infty \int_0^\pi f(t,r,u,\alpha)u^2 \sin\alpha d\alpha du, \tag{4-1}$$

the same as the electrostatic potential $\phi(r,t)$:

$$\phi(t,r) = -\frac{1}{r}\int_0^r \lambda^2 \rho(t,\lambda)d\lambda - \int_r^\infty \lambda\rho(t,\lambda)d\lambda \tag{4-2}$$

The electric field $\mathbf{E}$ as the negative of the gradient of the potential and new notation $M(t,r)$ introduced as below:

$$\mathbf{E}(t,\mathbf{x}) = \nabla_x \phi = \frac{\mathbf{x}}{r^3}\int_0^r \lambda^2 \rho(t,\lambda)d\lambda = \frac{\mathbf{x}}{r^3}M(t,r) \tag{4-3}$$

Notice that $M(t,r)$ is essentially the integral of ρ on the volume the sphere with radius r except a factor of 4π, showing $\lim_{r\to\infty} M(t,r) = m/4\pi$. Moreover, $|\mathbf{E}| = r^{-2}M(t,r)$.

Spherical symmetry brings in the below simplification of the characteristics' ordinary differential equations and note that dA/dt could be decided by the $d(\mathbf{X}\cdot\mathbf{V})/ds = d(RU\cos A)/ds$. Curly brackets $\{...,...\}$ includes the terms for (VP) in the left and for (RVP) in the right.

$$\begin{cases} \dfrac{dR}{ds} = |\mathbf{a}(\mathbf{V})|\cos A = \left\{ U\cos A, \dfrac{U\cos A}{\sqrt{1+U^2}} \right\} \\ \dfrac{dU}{ds} = \left|\dfrac{d\mathbf{V}}{ds}\right| \cos < \dfrac{d\mathbf{V}}{ds}, \mathbf{a}(\mathbf{V}) >= \gamma \dfrac{\cos A}{R^2} M(s,R) \\ \dfrac{dA}{ds} = -\left(\gamma \dfrac{M(s,R)}{R^2 U} + \left\{ \dfrac{U}{R}, \dfrac{U}{R\sqrt{1+U^2}} \right\} \right) \sin A \end{cases} \tag{4-4}$$

The simplified results of (RVP) will be helpful in the following proof.

4.2 Existence

Global existence results of (RVP) in 3D has been studied by Glassey et al. (1985) with compact distribution function support and by Wang (2003). Glassey et al. (1985)

restrict the $|\mathbf{E}|$ to prove that supremum of the velocity can be controlled by a function $H_v \in C_+(R_0^+)$. While Wang (2003) controlled the norm of electric field $\|\mathbf{E}(t,\cdot)\|_\infty$ to achieve the global existence . Their approaches are concisely introduced as follow.

To control the L_x^∞ -norm of the acceleration term $\nabla_x \bar{\phi}$, now it's a standard argument to show that it is controlled by a high moment of the distribution function, see also Lemma 2.2 2.3. Propagation of moments. We define

$$M_n(t,x) := \int_{\mathbb{R}^3} (1+|v|)^n f(t,x,v)dv, \quad M_n(t) := \int_{\mathbb{R}^3} M_n(t,x)dx, \quad n := \lceil N_0/10 \rceil$$

Local theory and the reduction of the proof. Because our assumption on the initial data is stronger than the assumption imposed on the distribution function in [7], by using the same argument used by Luk-Strain [ ] for the relativistic Vlasov-Maxwell system, which is more difficult, we can reduce the proof of global existence to the L_x^∞ -estimate of the scalar field $\nabla_x \phi$, which corresponds to the acceleration of the speed of particles.

Lemma 4.1: There exists a constant C such that for $r \geqslant 0$ and $0 \leqslant t < T$

$$|\mathbf{E}(\mathbf{x},t)| = \frac{M(r,t)}{r^2} \leqq \begin{cases} \min\left(Mr^{-2}, 100M^{1/3}\|\hat{f}\|_\infty^{2/3}P^2(t)\right) & \text{if } \gamma = -1 \\ \min\left(Mr^{-2}, CP^{5/3}(t)\right) & \text{if } \gamma = +1 \end{cases}$$

Definition 4.1: The highest speed the solution f has on the time interval $[0,t]$.

$$P(t) = \sup\{U(s,0,r,u,\alpha) : 0 \leqslant s \leqslant t, (r,u,\alpha) \in \text{ support } f\}$$

The paper mainly talks about spherically symmertric solutions, *i.e.*, the radial ones.

Theorem 4.1: Let f be a classical solution of (RVP) on some time interval $[0,T)$ with $\gamma = -1$ and smooth, nonnegative, spherically symmetric data fwhich has compact support and vanishes for $(r,u,\alpha) \notin (0,\infty) \times (0,\pi)$. If $40M^{2/3}\ \|f^\circ\|_\infty^{1/3} < 1$, then $P(t)$ is uniformly bounded on $[0,T)$, and hence (RVP) possesses a global classical solution.

Theorem 4.2: Let f be a classical solution of (RVP) on some time interval $[0,T)$ with $\gamma = +1$ and smooth, nonnegative, spherically symmetric data f_0 which has compact support and vanishes for $(r,u,\alpha) \notin (0,\infty) \times (0,\infty) \times (0,\pi)$. Then $P(t)$ is uniformly bounded on $[0,T)$, and hence (RVP) possesses a global classical solution.

From the conservation laws (1.2), we know that $M_1(t)$ is always bounded from the above. Moreover we define

$$\tilde{M}_n(t) := (1+t)^{2n} + \sup M_n(s)$$

We have two basic estimates for the L^∞_x -norm of the acceleration term $\nabla_x \phi$, which will be elaborated in the next two Lemmas. The first estimate (2.3) is available mainly because of the radial symmetry and the conservation law. The second estimate (2.5) is standard

List of Equations

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声　明

本人郑重声明：所呈交的学位论文，是本人在导师指导下，独立进行研究工作所取得的成果。尽我所知，除文中已经注明引用的内容外，本学位论文的研究成果不包含任何他人享有著作权的内容。对本论文所涉及的研究工作做出贡献的其他个人和集体，均已在文中以明确方式标明。

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Appendix A Useful Inequalities

A.1 Gronwall Inequalities

Lemma A.1: *(Nonlinear Gronwall Lemma)*

Assume that $t_0, t_1 \in \mathbb{R}, t_0 < t_1$, that $F : \left[t_0, t_1\right) \times [0, \infty) \to [0, \infty)$ is continuous and that $F(t, r) \geqslant F\left(t, r'\right)$ for all $r \geqslant r'$(*i.e.* non-decreasing in r). Assume further that $f_0 \in [0, \infty), T \in (t_0, t_1]$ and that $g : I \to [0, \infty)$, in the interval $I := [t_0, T)$, is the maximal upper solution of the initial value problem $g' = F(t, g), g\left(t_0\right) = f_0$. If $f : I \to [0, \infty)$ is measurable and locally bounded and for all $t \in I$

$$f(t) \leqslant f_0 + \int_{t_0}^{t} F(r, f(r)) \mathrm{d}r$$

then it follows that $f(t) \leqslant g(t)$ for all $t \in I$

proof Please *cf.* [7], theorem 4.1 to handle continuous f, otherwise let $h(t) := f_0 + \int_{t_0}^{t} F(r, f(r)) \mathrm{d}r$. h is continuous and still satisfy $h(t) \leqslant f_0 + \int_{t_0}^{t} F(r, h(r)) \mathrm{d}r$, $f(t) \leqslant h(t) \leqslant g(t)$. $\square$

Lemma A.2: *(Bivariant Gronwall Lemma)*

Let $T > 0, I := [0, T]$. Assume that $g : I \times I \to [0, \infty)$ is bounded and that for each $t \in I$ the functions $g(t, \cdot)$ and $g(\cdot, t)$ are measurable. Assume further that there exist constants D_1, D_2, D_3 such that for all $t, \tau \in I$ (4.3.1) $g(t, \tau) \leqslant \left|\int_{\tau}^{t} \left(D_1 \cdot g(r, \tau) + D_2 \cdot (g(0, r) + g(r, 0)) + D_3\right) \mathrm{d}r\right|$. Then there exists a constant D, which depends only on D_1, D_2 and T, such that for all $t, \tau \in I$ we have $g(t, \tau) \leqslant D \cdot D_3$

Proof. Take any $g^* \in C_+(I)$ such that $g^*(t) \geqslant \sup\{g(0, u) + g(u, 0) | 0 \leqslant u \leqslant t\}$ for all $t \in I$. Let $\tau \in I$ be fixed. Then for all $t \in [\tau, T]$ we have

$$g(t, \tau) \leqslant \int_{\tau}^{t} \left(D_1 \cdot g(r, \tau) + D_2 \cdot g(r) + D_3\right) \mathrm{d}r$$

We conclude with lemma (2.7) that

$$g(t,\tau) \leqslant \int_{\tau}^{t} \exp\left(D_1 \cdot (t-r)\right) \cdot \left(D_2 \cdot g^*(r) + D_3\right) \mathrm{d}r \leqslant \exp\left(D_1 \cdot T\right) \cdot \left| \int_{\tau}^{t} \left(D_2 \cdot g^*(r) + D_3\right) \mathrm{d}r \right|$$

and this can analogously be shown for all $t \in [0,\tau]$. Thus for all $t \in I$

$$g(0,t) + g(t,0) \leqslant 2 \cdot \exp\left(D_1 \cdot T\right) \cdot \int_0^t \left(D_2 \cdot g^*(r) + D_3\right) \mathrm{d}r$$

As the right-hand side is non-decreasing on I, this implies

$$\sup\{g(0,u) + g(u,0) | 0 \leqslant u \leqslant t\} \leqslant 2 \cdot \exp\left(D_1 \cdot T\right) \cdot \int_0^t \left(D_2 \cdot g^*(r) + D_3\right) \mathrm{d}r$$

There exists a non-increasing sequence in $C_+(I)$, which converges almost everywhere to $\sup\{g(0,u) + g(u,0) | 0 \leqslant u \leqslant t\}$. Thus we have shown

$$\begin{aligned} &\sup\{g(0,u) + g(u,0) | 0 \leqslant u \leqslant t\} \\ &\leqslant 2 \cdot \exp\left(D_1 \cdot T\right) \cdot \int_0^t \left(D_2 \cdot \sup\{g(0,u) + g(u,0) | 0 \leqslant u \leqslant r\} + D_3\right) \mathrm{d}r \end{aligned}$$

Another application of (2.7) yields

$$\sup\{g(0,u) + g(u,0) | 0 \leqslant u \leqslant t\}$$

$$\begin{aligned} &\leqslant \int_0^t \left(\exp\left(2 \cdot \exp\left(D_1 \cdot T\right) \cdot D_2 \cdot (t-r)\right) \cdot 2 \cdot \exp\left(D_1 \cdot T\right) \cdot D_3\right) \mathrm{d}r \\ &\leqslant T \cdot \exp\left(2 \cdot \exp\left(T \cdot D_1\right) \cdot D_2 \cdot T\right) \cdot 2 \cdot \exp\left(D_1 \cdot T\right) \cdot D_3 =: D_4 \cdot D_3 \end{aligned}$$

We insert this into (4.3.1) and get

$$g(t,\tau) \leqslant \left| \int_{\tau}^{t} \left(D_1 \cdot g(r,\tau) + \left(1 + D_4\right) \cdot D_3\right) \mathrm{d}r \right|$$

Lemma (2.7), applied for the third time, now yields

$$
\begin{aligned}
g(t,\tau) &\leqslant \left| \int_{\tau}^{t} \exp\left(D_1 \cdot |t-r|\right)\left(1+D_4\right) \cdot D_3 \mathrm{d}r \right| \\
&\leqslant T \cdot \exp\left(D_1 \cdot T\right) \cdot \left(1+D_4\right) \cdot D_3 =: D \cdot D_3
\end{aligned}
$$

Appendix B Conservation Laws

Proposition B.1: If f_0 vanished for $|x| > k$, then $f(t, \mathbf{x}, \mathbf{v})$ vanished for $|\mathbf{x}| > t + k$ (casuality).

Lemma B.1: *(Conservation Laws)* Let f be a classical solution of (RVP) on some time interval $[0, T)$ with nonnegative initial data $f_0 \in C^1(\mathbb{R}^6)$. Then the following properties hold:

(a) The total mass is conserved, *i.e.*, $\iint_{\mathbb{R}^6} f\, d\mathbf{v} d\mathbf{x} =$ constant $= m$.

(b) The total energy is conserved, *i.e.*,

$$\text{(RVP)} \int_{\mathbb{R}^3_x} \left\langle \int_{\mathbb{R}^3_v} \sqrt{1 + |\mathbf{v}|^2} f\, d\mathbf{v} + \frac{1}{2}\gamma |\mathbf{E}|^2 \right\rangle d\mathbf{x} = \text{ constant } =: \mathscr{E}_0 \tag{B-1}$$

$$\text{(VP)} \int_{\mathbb{R}^3_x} \left\langle \int_{\mathbb{R}^3_v} \frac{|\mathbf{v}|^2}{2} f\, d\mathbf{v} + \frac{1}{2}\gamma |\mathbf{E}|^2 \right\rangle d\mathbf{x} = \text{ constant } =: \mathscr{E}_0 \tag{B-2}$$

proof (RVP) to be reorganized from Glassey et al. (1985).

(a) We apply Eq. (2) to note that $|\mathbf{X}(s, t, \mathbf{x}, \mathbf{v}) - \mathbf{x}| = \left| \int_t^s \hat{\mathbf{V}}(\xi, t, \mathbf{x}, \mathbf{v}) d\xi \right| \leqq |t - s|$. In particular, $|X(0, t, \mathbf{x}, \mathbf{v}) - \mathbf{x}| \leqq t$. Thus whenever $|\mathbf{x}| > k + t$, we have $|\mathbf{X}(0, t, \mathbf{x}, \mathbf{v})| \geqq |\mathbf{x}| - |\mathbf{X}(0, t, \mathbf{x}, \mathbf{v}) - \mathbf{x}| > k$, and so by hypothesis and (3), $f(t, \mathbf{x}, \mathbf{v}) = f_0(\mathbf{X}(0, t, \mathbf{x}, \mathbf{v}), \mathbf{V}(0, t, \mathbf{x}, \mathbf{v})) = 0$

(b) follows by simply integrating (RVP) in v and x.

(c) Multiplying (RVP) by $\sqrt{1 + |\mathbf{v}|^2}$ and integrating in v, we obtain

$$\frac{\partial}{\partial t} \int \sqrt{1 + |\mathbf{v}|^2} f\, d\mathbf{v} + \int \mathbf{v} \cdot \nabla_x f\, d\mathbf{v} - \gamma \mathbf{j} \cdot \mathbf{E} = 0, \quad \mathbf{j} = \int \hat{\mathbf{v}} f\, d\mathbf{v} \tag{B-3}$$

For non-relativistic (VP) , multiply it by $|\mathbf{v}|^2$ and we acquire similarly

$$\frac{\partial}{\partial t} \int |\mathbf{v}|^2 f\, d\mathbf{v} + \int \mathbf{v} \cdot \nabla_x |\mathbf{v}|^2 f\, d\mathbf{v} - 2\gamma \mathbf{j} \cdot \mathbf{E} = 0, \quad \mathbf{j} = \int \mathbf{v} f\, d\mathbf{v} \tag{B-4}$$

We have defined $\mathbf{E} = -\nabla\phi$, where $\Delta\phi = \rho$. Multiplying by ϕ, we have

$$\int_{\Omega^3} |\mathbf{E}|^2 dx = -\int_{R^3} \rho\phi d\mathbf{x}$$

and hence

$$\begin{aligned}\frac{d}{dt}\int_{R^3} |\mathbf{E}|^2 dx &= -\int_{R^3} \rho_t\phi dx - \int \rho u_t dx = -\int \rho_t\phi dx - \int_{R^3} u_t \Delta\phi dx \\ &= -\int_{\mathbb{R}^3} \rho_t\phi dx + \frac{1}{2}\frac{d}{dt}\int_{R^3} |E|^2 d\mathbf{x} \text{ (integrate by parts)}\end{aligned}$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\int_{\mathrm{R}^3} |\mathbf{E}|^2 d\mathbf{x} = -\int_{\mathrm{R}^3} \rho_t\phi d\mathbf{x}$$

Therefore

Next, integrating (VP) and (RVP) in $\mathbf{v}$, we get the conservation law for both cases

$$\rho_t + \nabla_x \cdot \mathbf{j} = 0, \quad \mathbf{j} = \left\{\int \mathbf{v} f d\mathbf{v}, \int \hat{\mathbf{v}} f d\mathbf{v}\right\}$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega^3} |\mathbf{E}|^2 d\mathbf{x} = -\int_{R^3} \rho_t\phi d\mathbf{x} = \int_{\Omega^3} \phi\nabla_x \cdot j d\mathbf{x} = -\int_{\mathbb{R}^3} \mathbf{j}\cdot\nabla_x\phi d\mathbf{x} = -\int_{n^3} \mathbf{j}\cdot\mathbf{E} d\mathbf{x}$$

Now using this and (B-3), (B-4) we have

$$\begin{aligned}&\text{(VP) } \frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\mathbf{v}|^2}{2} f d\mathbf{v} + \frac{1}{2}\gamma|\mathbf{E}|^2 d\mathbf{x} \\ &= -\frac{1}{2}\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3} \mathbf{v}\cdot\nabla_x f|\mathbf{v}|^2 d\mathbf{v} - 2\gamma\mathbf{j}\cdot\mathbf{E}\right) d\mathbf{x} - \gamma\int_{R^3} \mathbf{j}\cdot\mathbf{E} d\mathbf{x} \\ &= -\iint_{\mathbb{R}^3\times\mathbb{R}^3} \nabla_x \cdot (f\mathbf{v}^3) d\mathbf{v} d\mathbf{x} = 0\end{aligned}$$

$$
\begin{aligned}
\text{(RVP)}\ \frac{d}{dt}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \sqrt{1+|v|^2} f\,d\mathbf{v} + \frac{1}{2}\gamma|\mathbf{E}|^2 d\mathbf{x} \\
= -\int\limits_{\mathbb{R}^3}\left(\int\limits_{\mathbb{R}^3} v\cdot\boldsymbol{\nabla}_x f\,d\mathbf{v} - \gamma j\cdot\mathbf{E}\right)d\mathbf{x} - \gamma\int\limits_{R^3}\mathbf{j}\cdot\mathbf{E}d\mathbf{x} \\
= -\int\limits_{\mathbb{R}^3 R^3} \boldsymbol{\nabla}_x\cdot(f\mathbf{v})d\mathbf{v}d\mathbf{x} = 0
\end{aligned}
$$

□

which proves the total energy $\mathscr{E}_0$ does not change with respect to time in both non-relativistic adn relativistic cases.

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