Evaluate the integrals and prove they converge: Wenzhi Tseng

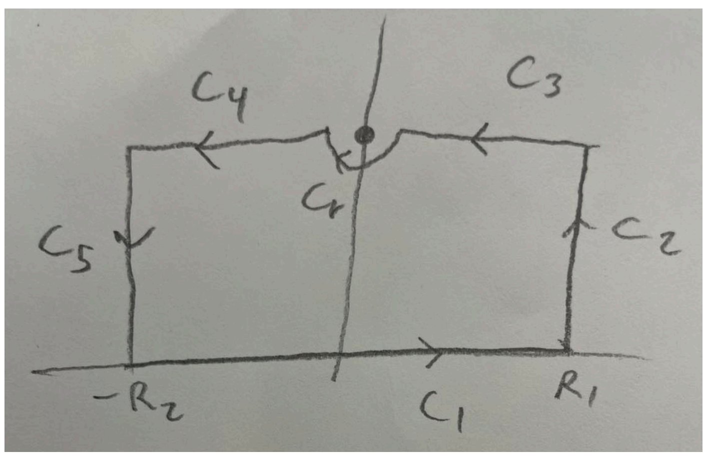
$$K = \int_0^1 \frac{\ln x}{x^2 - 1}\,dx = \int_0^\infty \frac{x}{2\sinh x}\,dx \qquad J = \int_0^1 \frac{\ln^2 x}{x^2 + 1}\,dx = \int_0^\infty \frac{x^2}{2\cosh x}\,dx$$

The integral equalities follow from the substitution $u = \ln x$ and subsequently noting that $e^u/(e^{2u} \pm 1) = 1/(e^u \pm e^{-u})$. Recall that $\cosh z = (e^z + e^{-z})/2$ is an even function and $\sinh z = (e^z - e^{-z})/2$ is an odd function. To see J converges, note the $x^2/\cosh x$ is zero at $x = 0$ and positive for $x > 0$, so

$$0 \le \int_0^\infty \frac{x^2}{2\cosh x}\,dx = \int_0^\infty \frac{x^2}{e^x + e^{-x}}\,dx \le \int_0^\infty \frac{x^2}{e^x}\,dx = 2$$

To see K converges, note $\lim_{x\to 0} \frac{x}{2\sinh x} = \lim_{x\to 0} \frac{1}{2\cosh x} = \frac{1}{2}$ so there are no convergence issues at $x = 0$. Further note $x/\sinh x$ is positive for $x > 0$ and decays much faster than $1/x^2$ as $x \to \infty$. Hence K converges.

We first find K. We integrate $f(z) = z/(2\sinh z) = ze^z/(e^{2z} + 1)$ over the indented rectangle with vertices $R_1, R_1 + i\pi/2, -R_2 + i\pi/2$, and $-R_2$ (with $R_1, R_2 \to \infty$) with the indent (of radius $r \to 0$) at the isolated singularity $i\pi$.



By solving a quadratic equation in e^z, the solutions to $2\sinh z = 0$ are $e^{i\pi n}$, $n \in \mathbb{Z}$. To prove that $i\pi n$, $n \in \mathbb{Z}$ is a simple pole of f, we find the Taylor series of $2\sinh z$ expanded around $i\pi n$ using $a_k = f^{(k)}(i\pi n)/k!$ to compute the coefficients. We notice the first nonzero term in the Taylor series is the z term, so $z = i\pi n$ is a simple zero of f. If $n \neq 0$, then z is analytic and nonzero at $i\pi n$, so $f(z) = z/(2\sinh z)$ has a simple pole at $i\pi n$. If $n = 0$, then $i\pi n = 0$ and 0 is clearly a simple zero of z, so $f(z) = z/(2\sinh z)$ has a removable singularity at 0 (so we do not need an indent at zero in our contour). There are no other poles of f on or inside the contour.

On C_1, we clearly have $\int_{C_1} f \to 2K$ as $R_1, R_2 \to \infty$ (note the integrand is even). On C_2 we parameterize $z = R_1 + it, t \in (0,1)$ so for $R_1 > 1$,

$$\left|\int_{C_2} f\right| \le \int_{C_2} |f||dz| = \int_0^1 \left|\frac{(R_1+it)e^{R_1+it}}{e^{2R_1+i2t}-1}\right| |idt| \le \int_0^1 \frac{\sqrt{R_1^2+t^2}e^{R_1}}{e^{2R_1}-1} dt$$

$$\le \int_0^1 \frac{\sqrt{R_1^2+R_1^2}e^{R_1}}{e^{2R_1}-1} dt = \frac{\sqrt{2}R_1 e^{R_1}}{e^{2R_1}-1}$$

which clearly goes to 0 as $R_1 \to \infty$ (use L'Hopital's for a proof). So $\int_{C_2} f \to 0$ as $R_1 \to \infty$.
A similar calculation shows $\int_{C_5} f \to 0$ as $R_2 \to \infty$.
Since $i\pi$ is a simple pole, by the fractional residue theorem we have

$$\lim_{r\to 0}\int_{C_r} f = -\pi i \operatorname{Res}\left(\frac{z}{2\sinh z}, i\pi\right) = -\pi i \left.\frac{z}{(2\sinh z)'}\right|_{z=i\pi} = -\pi i \frac{i\pi}{2\cosh(i\pi)} = -\frac{\pi^2}{2}$$

We have $z = t + i\pi$ on C_3 and C_4, so

$$\int_{C_3} f + \int_{C_4} f = \int_{R_1}^{r} \frac{(t+i\pi)e^{t+i\pi}}{e^{2t+2\pi i}-1} dt + \int_{-r}^{-R_2} \frac{(t+i\pi)e^{t+i\pi}}{e^{2t+2\pi i}-1} dt$$

$$= \left(\int_{R_1}^{r} + \int_{-r}^{-R_2}\right) \frac{-(t+i\pi)e^t}{e^{2t}-1} dt = \left(\int_{-R_2}^{-r} + \int_r^{R_1}\right)\left(\frac{te^t}{e^{2t}-1} + \frac{i\pi e^t}{e^{2t}-1}\right) dt$$

Apply the substitution $u = e^t$ for the second term,

$$= \left(\int_{-R_2}^{-r} + \int_r^{R_1}\right)\frac{t}{2\sinh t} dt + i\pi\left(\int_{e^{-R_2}}^{e^{-r}} + \int_{e^r}^{e^{R_1}}\right)\frac{du}{u^2-1}$$

$$= \left(\int_{-R_2}^{-r} + \int_r^{R_1}\right)\frac{t}{2\sinh t} dt + \frac{i\pi}{2}\left(\left[\ln\left|\frac{u-1}{u+1}\right|\right]_{e^{-R_2}}^{e^{-r}} + \left[\ln\left|\frac{u-1}{u+1}\right|\right]_{e^r}^{e^{R_1}}\right)$$

After a good deal of simplifying using the laws of logs, we get

$$= \left(\int_{-R_2}^{-r} + \int_r^{R_1}\right)\frac{t}{2\sinh t} dt + \frac{i\pi}{2}\left(\ln\left|\frac{e^{R_1}-1}{e^{R_1}+1}\right| - \ln\left|\frac{e^{-R_2}-1}{e^{-R_2}+1}\right|\right)$$

Since K converges, we can take $r \to 0$ and $R_1, R_2 \to \infty$,

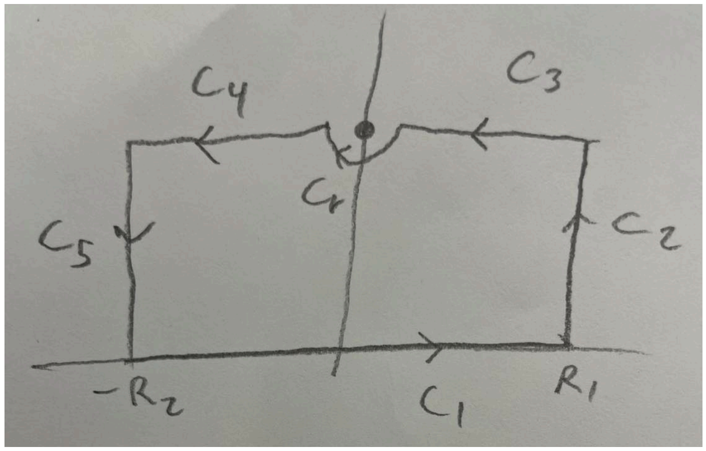
$$\to \int_{-\infty}^{\infty} \frac{t}{2\sinh t} dt = 2K$$

Then, by Cauchy's theorem,

$$2K - \frac{\pi^2}{2} + 2K = 0$$

$$\Rightarrow K = \frac{\pi^2}{8}$$

Now we compute J. We will integrate $f(z) = z^2/(2\cosh z) = z^2e^z/(e^{2z}+1)$ over the indented rectangle with vertices $R_1, R_1 + i\pi/2$, $-R_2 + i\pi/2$, and $-R_2$ (with $R_1, R_2 \to \infty$) with the indent (of radius $r \to 0$) at the simple pole $i\pi/2$.



To prove that $i\pi/2$ is a simple pole of f, we find the Taylor series of $2\cosh z$ expanded around $i\pi/2$ using $a_k = f^{(k)}(i\pi/2)/k!$ to find the coefficients. We notice the first nonzero term in the Taylor series is the z term, so $z = i\pi/2$ is a simple zero of f. Then, since z^2 is analytic and nonzero at $i\pi/2$, it follows that $f(z) = z^2/(2\cosh z)$ has a simple pole at $i\pi/2$. By solving a quadratic equation in e^z, the solutions to $2\cosh z = 0$ are $e^{i(\pi k + \pi/2)}$, $k \in \mathbb{Z}$, so there are no poles of f on or inside the contour.

On C_1, we clearly have $\int_{C_1} f \to 2J$ as $R_1, R_2 \to \infty$ (note the integrand is even). On C_2 we parameterize $z = R_1 + it$, so for $R_1 > 1$,

$$\left|\int_{C_2} f\right| \le \int_{C_2} |f||dz| = \int_0^{\frac{1}{2}} \left|\frac{(R_1+it)^2e^{R_1+it}}{e^{2R_1+i2t}+1}\right| |idt| \le \int_0^{\frac{1}{2}} \frac{(R_1^2+t^2)e^{R_1}}{e^{2R_1}-1} dt$$

$$\le \int_0^{\frac{1}{2}} \frac{(R_1^2+R_1^2)e^{R_1}}{e^{2R_1}-1} dt = \frac{R_1^2 e^{R_1}}{e^{2R_1}-1}$$

which clearly goes to 0 as $R_1 \to \infty$ (use L'Hopital's for a proof). So $\int_{C_2} f \to 0$ as $R_1 \to \infty$.
A similar calculation shows $\int_{C_5} f \to 0$ as $R_2 \to \infty$.
Since $i\pi/2$ is a simple pole, by the fractional residue theorem we have

$$\lim_{r\to 0}\int_{C_r} f = -\pi i \operatorname{Res}\left(\frac{z^2}{2\cosh z}, \frac{i\pi}{2}\right) = -\pi i \left.\frac{z^2}{(2\cosh z)'}\right|_{z=i\pi/2} = -\pi i \frac{-\pi^2/4}{2\sinh(i\pi/2)} = \frac{\pi^3}{8}$$

We have $z = t + i\pi/2$ on C_3 and C_4, so

$$\int_{C_3} f + \int_{C_4} f = \int_{R_1}^{r} \frac{(t+i\pi/2)^2 e^{t+i\pi/2}}{e^{2t+i\pi}+1} dt + \int_{-r}^{-R_2} \frac{(t+i\pi/2)^2 e^{t+i\pi/2}}{e^{2t+i\pi}+1} dt$$

$$= -\left(\int_{-R_2}^{-r} + \int_{r}^{R_1}\right) \frac{(t^2 + ti\pi - \pi^2/4)ie^t}{-e^{2t} + 1} dt$$

$$= \left(\int_{-R_2}^{-r} + \int_{r}^{R_1}\right)\left(\frac{it^2e^t}{e^{2t} - 1} - \frac{\pi te^t}{e^{2t} - 1} - \frac{i(\pi^2/4)e^t}{e^{2t} - 1}\right) dt$$

$$= \left(\int_{-R_2}^{-r} + \int_{r}^{R_1}\right)\left(\frac{it^2}{2\sinh t} - \frac{\pi t}{2\sinh t} - \frac{i(\pi^2/4)}{2\sinh t}\right) dt$$

Note that $x^2/\sinh x$ and $1/\sinh x$ are both positive for $x > 0$ and decay much faster than $1/x^2$ as $x \to \pm\infty$, so their integrals have no convergence issues at $\pm\infty$. Since we already know K converges as well, we can take $R_1, R_2 \to \infty$ at this stage,

$$\to \left(\int_{-\infty}^{-r} + \int_{r}^{\infty}\right)\left(\frac{it^2}{2\sinh t} - \frac{\pi t}{2\sinh t} - \frac{i(\pi^2/4)}{2\sinh t}\right) dt$$

Note that the first and third terms are odd functions,

$$\left(\int_{-\infty}^{-r} + \int_{r}^{\infty}\right)\left(-\frac{\pi t}{2\sinh t}\right) dt$$

Since K converges, we take $r \to 0$ and use the previous result $K = \pi^2/8$,

$$= -\pi\left(\frac{\pi^2}{8} + \frac{\pi^2}{8}\right) = -\frac{\pi^3}{4}$$

Then, by Cauchy's theorem,

$$2J + \frac{\pi^3}{8} - \frac{\pi^3}{4} = 0$$

$$\Rightarrow J = \frac{\pi^3}{16}$$