Complex Analysis: Functions with Branch Cuts and Integration

This document is meant to serve as an educational resource for students who wish to learn more about complex functions with multiple branches, as it is a topic that I feel is often underemphasized or underexplained. In particular, when I tried to learn complex analysis from textbooks and websites, I felt that authors did not fully take the time to explain things carefully and often failed to provide adequate definitions of the terminology they were using. I hope this document will provide a more patient, detailed overview of the topic at the introductory level. This is the resource I wish I had when first learning about functions with branch cuts and how to integrate them.

There are three sections. The first focuses on defining and understanding functions with multiple branches and assumes only the first few weeks of an undergraduate complex analysis course. It defines all the relevant terminology and then goes into constructing branches of $\sqrt{1-z^2}$ by multiplying branches of $\sqrt{1-z}$ and $\sqrt{1+z}$ together. This is not a common exercise but I think it goes a long way in demonstrating why we must be careful when manipulating square roots in $\mathbb{C}$, and it shows how continuity can be gained or lost when multiplying together different branches. It also explicitly demonstrates how the branch of a function such as $\sqrt{z}$ depends on where $\arg z$ comes from, which is important when a particular branch of a function must be selected to solve certain contour integrals. Overall I strongly believe it is a very useful exercise for students who are unfamiliar with branches and branch cuts. The second section contains worked examples of integrating functions with branch cuts and presumes that the student has already been exposed to residues. We notably avoid using Laurent series to calculate residues in this section, as I believe it is more instructive to find the equivalent residue of $-1/z^2 f(1/z)$ at zero by constructing an appropriate branch of $1/z^2 f(1/z)$. However, the third section does discuss how to find Taylor and Laurent series of functions with branch cuts and how different branches will have different series, and we conclude with the traditional method of finding residues at infinity by computing the appropriate Laurent series.

If you find errors or have suggestions, do let me know at wtseng9@gmail.com, I highly appreciate them!

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03/02/24

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Branches and Branch Cuts

Definition 1.1
Let $f(z)$ be a function of $\arg z$ (such as $\log z = \log|z| + i\arg z$) such that f is continuous and single-valued when we specify $\arg z$ to only take on values from a certain set E. Then $f: D \to \mathbb{C}$ where $D := \{z \in \mathbb{C} : \arg z \in E\}$ is a *branch* of $f(z)$. That is, for our purposes, a branch of f is a *function* specified by two things: the set of values E that $\arg z$ is allowed to take on, and the formula for f itself.

Example 1.2
Let $D_1 = \{z \in \mathbb{C} : \arg z \in (-\pi, \pi)\}$ and $D_2 = \{z \in \mathbb{C} : \arg z \in (\pi, 3\pi)\}$ and define functions $g_1: D_1 \to \mathbb{C}, g_1(z) = \log z$ and $g_2: D_2 \to \mathbb{C}, g_2(z) = \log z$. Note $D_1 = D_2 = \mathbb{C}\backslash(-\infty, 0]$, so both g_1 and g_2 have the same domain. However, they are different functions because, for example, $g_1(1) = \log|1| + i\arg(1) = 0$ but $g_2(1) = \log|1| + i\arg(1) = 2\pi$. It is precisely because $\arg z$ took on different values in D_1 and D_2 that we have $g_1(1) \neq g_2(1)$, and in fact, $g_1(z) \neq g_2(z)$ for all $z \in \mathbb{C}\backslash(-\infty, 0]$. So we say that g_1 and g_2 are different branches of $\log z$ with domain $\mathbb{C}\backslash(-\infty, 0]$. We emphasize that the set of values that $\arg z$ takes on has uniquely specified the branch: the fact that $\arg z \in (-\pi, \pi)$ implies the domain of g_1 is $\mathbb{C}\backslash(-\infty, 0]$ and determines the output of g_1 at every element in that domain.

In fact, there are infinitely many branches of $\log z$ with domain $\mathbb{C}\backslash(-\infty, 0]$, each corresponding to $\arg z$ lying in the interval $(-\pi + 2\pi k, \pi + 2\pi k)$ for some $k \in \mathbb{Z}$. There are also infinitely many branches of $\log z$ with domain $\mathbb{C}\backslash[0, \infty)$, each corresponding to $\arg z$ lying in the interval $(2\pi k, 2\pi + 2\pi k)$ for some $k \in \mathbb{Z}$.

Definition 1.3
A branch cut of a function $f(z)$ is a portion of domain that is removed from $\mathbb{C}$ to ensure that $f(z)$ is continuous and single-valued.

Example 1.4
Returning to g_1, the domain was $D_1 = \{z \in \mathbb{C} : \arg z \in (-\pi, \pi)\}$ was equal to $\mathbb{C}\backslash(-\infty, 0]$. That is, the domain is $\mathbb{C}$ but excluding the ray $(-\infty, 0]$, so we say that g_1 has branch cut $(-\infty, 0]$.

Definition 1.5 (informal)[1]
A branch point z_0 of a function $f(z)$ is a point at which f is discontinuous on the circle $\{|z - z_0| = \epsilon\}$ for all ϵ small enough. That is, z_0 is a branch point of f if there exists $r > 0$ such that f is discontinouous on $\{|z - z_0| = \epsilon\}$ for all $\epsilon \in (0, r)$.

Example 1.6
Consider $z_0 = 0$ and $f(z) = \arg z$. Let z traverse the circle $\{|z - z_0| = \epsilon\}$ counter-clockwise with arbitrary starting point a. Then $\arg z$ will have increased by 2π by the time z makes a full revolution and returns to a, meaning $\arg z$ will approach two different values (separated by 2π) at a, making $\arg z$ discontinuous at a. Hence $z_0 = 0$ is a

[1] This definition is far from satisfactory, but it will do for an introduction.

branch point of $\arg z$. It follows that certain functions of $\arg z$ such as $\log z$ and $\sqrt{z}$ also have 0 as a branch point.

Definition 1.7
We say $z = \infty$ is a branch point of $f(z)$ if $z = 0$ is a branch point of $f(1/z)$.

Example 1.8
Consider $f(z) = \arg z$. Then $f(1/z) = \arg(1/z) = \arg(\bar{z}/|z|^2) = -\arg z$. Since $z = 0$ is a branch point of $\arg z$, it is also a branch point of $-\arg z$. So $z = \infty$ is a branch point of $\arg z$. It follows that $z = \infty$ is also a branch point of functions like $\log z$ and $\sqrt{z}$.

Theorem 1.9 (informal)
A branch cut must connect all branch points of a function in order for the function to be continuous.

Proof: Omitted, but there is an intuition. Essentially, we must remove enough of the domain so that there will be no circle around a branch point on which f is discontinuous. For the case of $\arg z$, if a branch cut did not connect 0 and ∞, i.e. if there was a "gap" in the branch cut, then we could have a circle around the branch point that runs through the gap and f would be discontinuous on this circle for the same reason described in Example 1.6. Therefore, the branch cut must remove a curve that connects 0 and ∞. We will see that, for more complicated functions with, let's say, two branch points, the function may be discontinuous on a circle containing one of the two branch points but continuous on a circle containing *both* of them.

Example 1.10
Hence any branch cut of $\arg z$, $\log z$, and $\sqrt{z}$ must connect $z = 0$ and $z = \infty$. Common examples include $[0, \infty)$ and $(-\infty, 0]$, but the branch cuts $[0, i\infty)$ and $[0, -i\infty)$ along the imaginary axis are also sometimes useful.

Definition 1.12
The principal branch of a function is the branch corresponding to $\arg z \in (-\pi, \pi)$. For example, the principal branch of $\arg z$ is denoted $\mathrm{Arg}(z)$ and has image $(-\pi, \pi)$. The principal branch of $\log z$ is denoted $\mathrm{Log}(z)$ and is defined as $\log|z| + i\, Arg(z)$.

Example 1.11
The definition of the complex logarithm $\log z = \log|z| + i \arg z$ implies that the set of values that $\arg z$ takes on *uniquely* specifies the branch of log. However, this is not always the case. One example is the complex square root, defined as $\sqrt{z} = e^{\frac{1}{2}\log|z|} e^{\frac{1}{2} i \arg z} = \left|\sqrt{z}\right| e^{\frac{1}{2} i \arg z}$. We can let $\arg z$ take on values from $(-\pi + 2\pi k, \pi + 2\pi k)$ for any $k \in \mathbb{Z}$ and the branch of $\sqrt{z}$ will still be the same since $e^{\frac{1}{2} i \arg z} = e^{\frac{1}{2} i(\arg z + 2\pi k)}$ for all $z \in \mathbb{C}$.

There are only two branches of $\sqrt{z}$ with branch cut $(-\infty, 0]$, corresponding to the positive and negative root. To see this, we first define a specific branch of $\sqrt{z}$. Define

$g(z)$ to be the principal branch of $\sqrt{z}$. So $g(z) = e^{\frac{1}{2}\log|z|}e^{i\frac{1}{2}\operatorname{Arg} z}$ and has domain $\mathbb{C}\backslash(-\infty, 0]$. Now, for any z in $\mathbb{C}\backslash(-\infty, 0]$, $\arg z$ must equal $\operatorname{Arg} z + 2k\pi$ for some $k \in \mathbb{Z}$. Then,

$$\sqrt{z} = e^{\frac{1}{2}\log z} = e^{\frac{1}{2}\log|z|}e^{i\frac{1}{2}\arg z} = e^{\frac{1}{2}\log|z|}e^{i\frac{1}{2}(\operatorname{Arg} z + 2k\pi)} = \left(e^{\frac{1}{2}\log|z|}e^{i\frac{1}{2}\operatorname{Arg} z}\right)e^{ik\pi} = \pm g(z)$$

That is, any branch of $\sqrt{z}$ with branch cut $(-\infty, 0]$ is either the positive or the negative of $g(z)$, hence these are the only two branches: choosing $\arg z \in (-\pi + 2k\pi, \pi + 2k\pi)$ will lead to $\sqrt{z}$ being equal to $g(z)$ and $-g(z)$ when k is even and odd respectively. This reasoning can be easily extended to show that any $\sqrt{z}$ whose branch cut is a straight line segment from 0 to ∞ only has two branches that are negatives of each other, and there are generalizations for z^α for any $\alpha \in \mathbb{C}$ as well.

Exercise 1.14
Show there are only three branches of $z^{1/3}$ with branch cut $(-\infty, 0]$, and that if we denote one of the branches to be $g(z)$, then the other two branches are $e^{i2\pi/3}g(z)$ and $e^{i4\pi/3}g(z)$.

Solution:
Let $g(z)$ be the principal branch of $z^{1/3}$, so $g(z)$ has branch cut $(-\infty, 0]$. Again, we have $\arg z = \operatorname{Arg} z + 2k\pi$ for some $k \in \mathbb{Z}$. Then,

$$z^{1/3} = e^{\frac{1}{3}\log z} = e^{\frac{1}{3}\log|z|}e^{i\frac{1}{3}\arg z} = e^{\frac{1}{3}\log|z|}e^{i\frac{1}{3}(\operatorname{Arg} z + 2k\pi)} = \left(e^{\frac{1}{3}\log|z|}e^{i\frac{1}{3}\operatorname{Arg} z}\right)e^{i\frac{2\pi k}{3}} = g(z)e^{i\frac{2\pi k}{3}}$$

Hence the three branches are $g(z)$, $g(z)e^{i2\pi/3}$, and $g(z)e^{i4\pi/3}$ (any value of $k \in \mathbb{Z}$ will lead to one of these three branches).

Example 1.15
We can construct some branches of $\arg(1-z)$ by starting with a given branch of $\arg z$ and applying transformations to obtain $\arg(1-z)$. For example, we plot some values of $\operatorname{Arg}(z)$, $\operatorname{Arg}(-z)$, and $\operatorname{Arg}(1-z)$ below in succession:

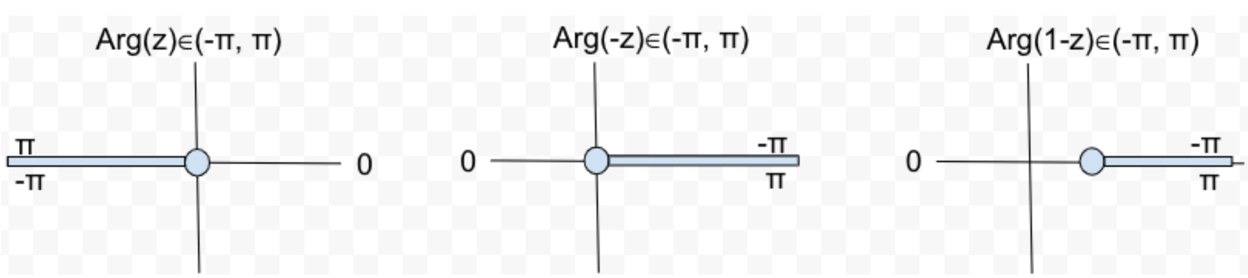

Fig 1.1

where branch cuts are colored in blue (so $\operatorname{Arg}(1-z)$ has branch cut $[1, \infty)$ for instance). To construct another branch of $\arg(1-z)$, we may start with the branch of $\arg z$ satisfying $\arg z \in (0, 2\pi)$, and plot $\arg(-z)$ and $\arg(1-z)$ just like before:

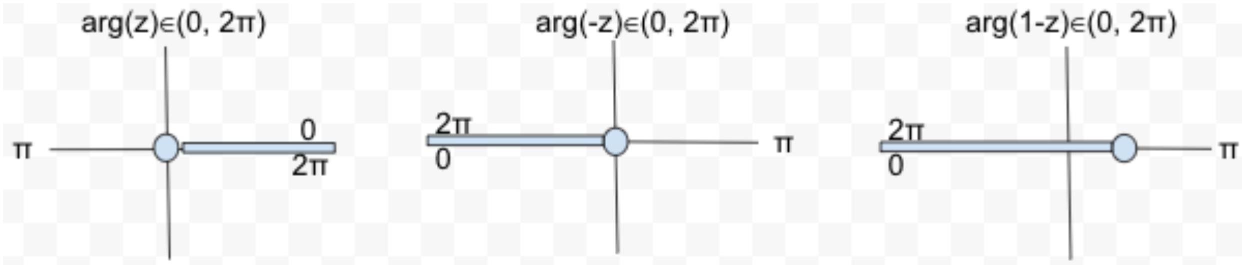

Fig 1.2

where $\arg(1-z)$ has branch cut $(-\infty, 1]$.

In general, for whatever branch of $\arg z$ we start with, we can obtain a corresponding branch of $\arg(1-z)$ with just a few sketches. We can also obtain a branch of $\arg(z-1)$ (which can be obtained from $\arg z$ with just a single translation), etc.

Example 1.16

We now illustrate a more complicated example. There are many branches of $\sqrt{1-z^2}$ depending on what branch and branch cuts we choose, but we can construct such branches of $\sqrt{1-z^2}$ as follows: take a branch of $\sqrt{1-z}$ and a branch of $\sqrt{1+z}$ and multiply them together. We have:

$$\sqrt{1-z}\sqrt{1+z} = e^{i\log\sqrt{1-z}}e^{i\arg\sqrt{1-z}}e^{i\log\sqrt{1+z}}e^{i\arg\sqrt{1+z}}$$

$$= \left|\sqrt{1-z}\right|\left|\sqrt{1+z}\right|e^{i\frac{1}{2}(\arg(1-z)+\arg(1+z))} = \left|\sqrt{1-z^2}\right|e^{i\arg\sqrt{1-z^2}}$$

This is the polar form for $\sqrt{1-z^2}$, so this is precisely a branch of $\sqrt{1-z^2}$ that we constructed from our $\sqrt{1-z}$ and $\sqrt{1+z}$, where at each input z, our $\sqrt{1-z^2}$ has argument equal to $(1/2)(\arg(1-z)+\arg(1+z))$ and magnitude equal to $\left|\sqrt{1-z^2}\right|$. Since our $\sqrt{1-z}$ and $\sqrt{1+z}$ were both single-valued, our $\sqrt{1-z^2}$ is also single-valued. However, the region over which our $\sqrt{1-z^2}$ is continuous depends on the branches chosen for $\sqrt{1-z}$ and $\sqrt{1+z}$. We will do some examples next to illustrate this.[2] At this point, it is important to clarify that our $\arg(1-z)$ and $\arg(1+z)$ are different functions and are not related to each other: our branch of $\sqrt{1-z}$ is a function of $\arg(1-z)$ and our branch of $\sqrt{1+z}$ is a function of $\arg(1+z)$.

Suppose we take the branch of $\sqrt{1+z}$ with $\arg(1+z) \in (0, 2\pi)$, so the branch cut is $[-1, \infty)$. Suppose we take the branch of $\sqrt{1-z}$ from Fig 1.2 with $\arg(1-z) \in (0, 2\pi)$, so the branch cut is $(-\infty, 1]$. We sketch the branch cuts and label the arguments on the diagrams below:

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[2] Our derivation also shows that decomposing a preexisting branch of $\sqrt{1-z^2}$ into $\sqrt{1-z}\sqrt{1+z}$ is not easy because you would have to carefully work out which branches of $\sqrt{1-z}$ and $\sqrt{1+z}$ were multiplied together to get the $\sqrt{1-z^2}$ you have. There are often multiple possibilities.

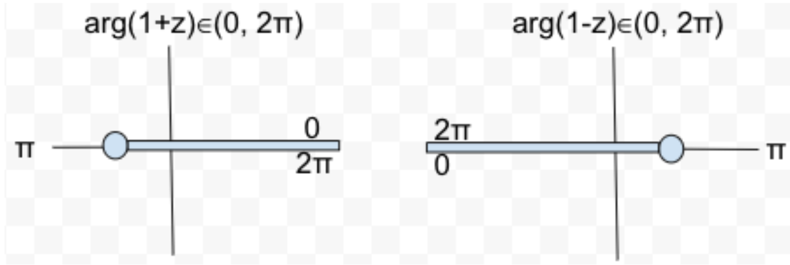


Fig. 1.3

We are multiplying $\sqrt{1+z}$ and $\sqrt{1-z}$ together at each point in the complex plane to find $\sqrt{1-z^2}$. Since $\arg\sqrt{1-z^2} = (1/2)(\arg(1-z) + \arg(1+z))$, we just need to add the arguments of $1-z$ and $1+z$ and then divide by two at each point in the complex plane to find $\arg\sqrt{1-z^2}$. We use this process to plot select values of $\arg\sqrt{1-z^2}$ below.

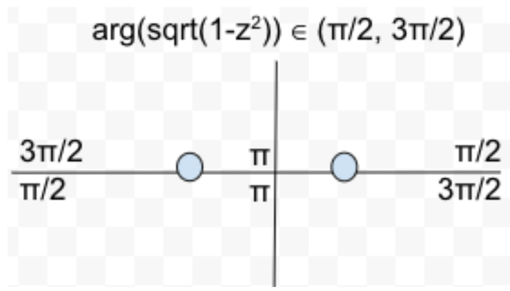


Fig. 1.4

We notice that $\arg\sqrt{1-z^2} \in (\pi/2, 3\pi/2)$. We also notice that $\arg\sqrt{1-z^2}$ approaches $3\pi/2$ from below $[1,\infty)$ but approaches $\pi/2$ from above $[1,\infty)$, so $\sqrt{1-z^2}$ will approach $\left|\sqrt{1-z^2}\right|e^{i3\pi/2}$ from below $[1,\infty)$ but approach $\left|\sqrt{1-z^2}\right|e^{i\pi/2}$ from above $[1,\infty)$. A similar situation occurs with $(-\infty,-1]$. Therefore our $\sqrt{1-z^2}$ is discontinuous only across the rays $(-\infty,-1]$ and $[1,\infty)$, so we must remove these rays to ensure our $\sqrt{1-z^2}$ is continuous. That is, our choices of branch for $\sqrt{1+z}$ and $\sqrt{1-z}$ resulted in a branch of $\sqrt{1-z^2}$ with branch cut $(-\infty,-1] \cup [1,\infty)$ as shown below:

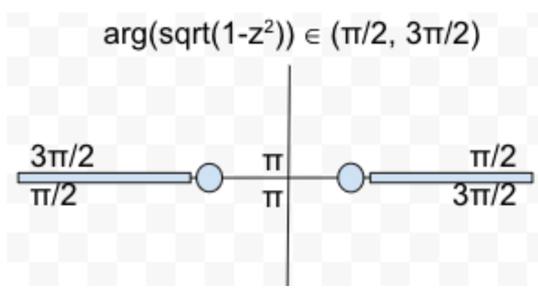


Fig. 1.5

Notice that our $\sqrt{1+z}$ and $\sqrt{1-z}$ are themselves discontinuous across the interval $(-1,1)$, but this continuity is regained for $\sqrt{1-z^2}$. What is going on is that the product $\sqrt{1+z}\sqrt{1-z} = \sqrt{1-z^2}$ happens to approach the same value if you approach the interval $(-1,1)$ from above and below the real axis (in this case, $\arg\sqrt{1-z^2}$ approaches

π from both above and below). Therefore, we can continuously extend $\sqrt{1-z^2}$ to the interval $(-1,1)$ even though this interval was not in the domains of $\sqrt{1+z}$ and $\sqrt{1-z}$.

It also seems like the this branch cut does not connect the branch points -1 and 1. However, $z=\infty$ is considered to be a "single point" in the extended complex plane (you can imagine the Riemann sphere to see this) so in fact, the rays $(-\infty,-1]$ and $[1,\infty)$ can be said to meet at $z=\infty$.

The last thing to point out is that $\arg\sqrt{1-z^2}$ being continuous and single-valued is a *sufficient* condition for $\sqrt{1-z^2}$ being continuous and single-valued, but not a *necessary* one. As we will see in Example 1.18, it is possible for $\sqrt{1-z^2}$ to be continuous across a line even if $\arg\sqrt{1-z^2}$ is discontinuous across that line. But in this example, both $\sqrt{1-z^2}$ and $\arg\sqrt{1-z^2}$ have the same branch cuts and are continuous outside $(-\infty,-1]\cup[1,\infty)$.

Exercise 1.17
Consider a new set of branch cuts for $\sqrt{1+z}$ and $\sqrt{1-z}$ as shown below:

Fig. 1.6

By picking branches of $\sqrt{1+z}$ and $\sqrt{1-z}$ that are consistent with the above branch cuts, show that the product $\sqrt{1+z}\sqrt{1-z}$ has the exact same branch cut as before, that is, the branch cut is still $(-\infty,-1]\cup[1,\infty)$.

Solution:
There are multiple possibilities, but one option is to choose $\arg(1-z)$ from Fig. 1.1 and the principal branch of $\arg(1+z)$. Proceeding along the same lines as Example 1.16, we compute $\arg\sqrt{1-z^2}=(1/2)(\arg(1+z)+\arg(1-z))$ and plot some values:

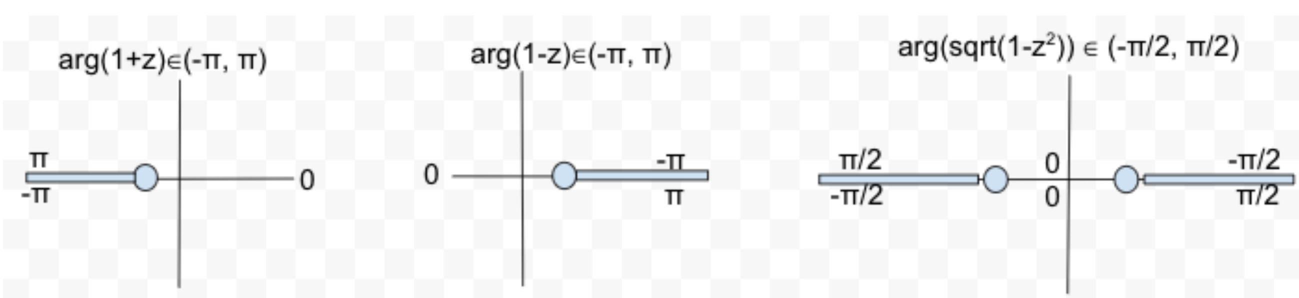


Fig. 1.7

We see that this branch of $\sqrt{1-z^2}$ also has the branch cut $(-\infty,-1] \cup [1,\infty)$ but is ultimately a different branch from the one in Example 1.16 since, for example, this branch of $\sqrt{1-z^2}$ will approach $\left|\sqrt{1-z^2}\right|e^{i0} = 1$ as $z \to 0$ but the branch in Example 1.16 will approach $\left|\sqrt{1-z^2}\right|e^{i\pi} = -1$ as $z \to 0$. In fact, you may notice that this branch is just the negative of the branch in Example 1.16.

Example 1.18

Consider the branch cuts for $\arg(1+z)$ and $\arg(1-z)$ shown below:

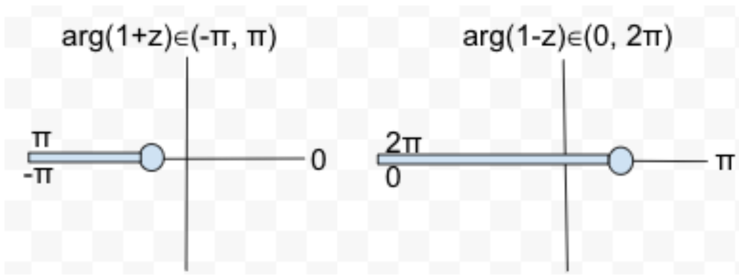


Fig. 1.8

We have chosen the same $\arg(1+z)$ as in Exercise 1.17 and $\arg(1-z)$ from Fig. 1.2. Just like before, we compute $\arg\sqrt{1-z^2} = (1/2)(\arg(1+z) + \arg(1-z))$ and plot some values:

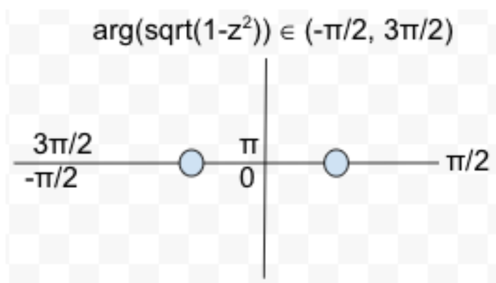


Fig. 1.9

However, note $\sqrt{1-z^2} = \left|\sqrt{1-z^2}\right|e^{i\arg\sqrt{1-z^2}}$ has the same value if $\arg\sqrt{1-z^2}$ equals $3\pi/2$ or $-\pi/2$. That is, although $\arg\sqrt{1-z^2}$ is discontinuous across the ray $(-\infty,-1)$, the function $\sqrt{1-z^2}$ itself is not! Therefore we can extend our $\sqrt{1-z^2}$ continuously over $(-\infty,-1)$, leading to the following branch cut for $\sqrt{1-z^2}$ below:

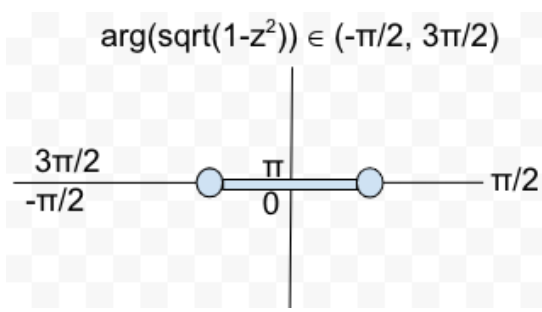


Fig. 1.10

So in this case, we end up with a branch of $\sqrt{1-z^2}$ with branch cut $[-1,1]$. **Note that $[-1,1]$ is the branch cut for $\sqrt{1-z^2}$, but $\arg\sqrt{1-z^2}$ would need the branch cut $(-\infty,1]$ in order to be continuous. Do not get the branch cuts confused! In this document, all plots will display the values of $\arg f(z)$ but the branch cuts shown will be the branch cuts for $f(z)$ itself.**

Exercise 1.19
Suppose we had chosen the branch cut $[-1,\infty)$ for $\sqrt{1+z}$ and the branch cut $[1,\infty)$ for $\sqrt{1-z}$. Show that the product $\sqrt{1+z}\sqrt{1-z}=\sqrt{1-z^2}$ also has the branch cut $[-1,1]$.

Solution:
Take the branch of $\arg(1+z)$ from Fig. 1.3 and the branch of $\arg(1-z)$ from Fig. 1.1. You should have found:

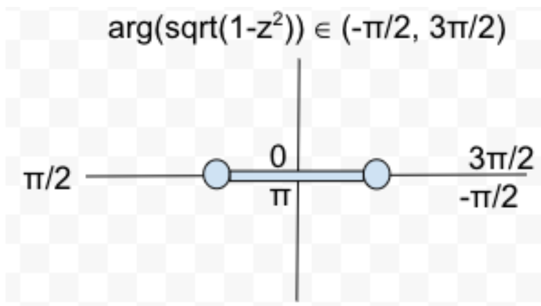


Fig. 1.11

So we once again have found two different set of branch cuts for $\sqrt{1+z}$ and $\sqrt{1-z}$ that lead to the same branch cut $[-1,1]$ for $\sqrt{1-z^2}$, but the branch itself is still different (again, notice this branch and the one in Example 1.18 are negatives of each other).

Example 1.20
So far, we have been multiplying together various branches of $\sqrt{1-z}$ and $\sqrt{1+z}$ to create different branches of $\sqrt{1-z^2}$. However, we could also consider the product $\sqrt{-1}\sqrt{z-1}\sqrt{1+z}$. Now, there are three things we have to pick: a branch for $\sqrt{1+z}$, a branch for $\sqrt{z-1}$, and we can pick $\sqrt{-1}$ to be either i or $-i$. For example, consider the following (principal) branches of $\sqrt{z-1}$ and $\sqrt{1+z}$. Pick $\sqrt{-1}=i$.

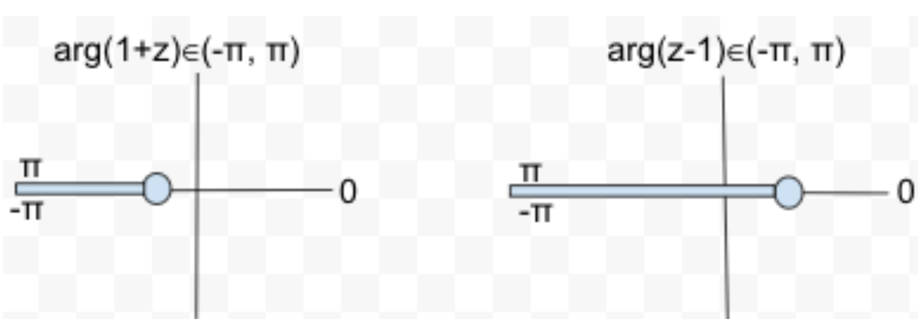


Fig. 1.12

We now define $\sqrt{1-z^2} = \sqrt{-1}\sqrt{z-1}\sqrt{1+z} = i\sqrt{z-1}\sqrt{1+z}$. We choose $\arg(i) = \pi/2$, although we just as well could have chosen $\arg(i) = 5\pi/2$, $-3\pi/2$, etc. Then,

$$\arg\sqrt{1-z^2} = \arg(i) + \arg\sqrt{z-1}\sqrt{1+z} = \pi/2 + (1/2)(\arg(z-1) + \arg(1+z))$$

That is, we still add together $\arg(z-1)$ and $\arg(1+z)$ and then divide by two at each point in the complex plane, but now we also add on $\pi/2$ afterwards. Hence,

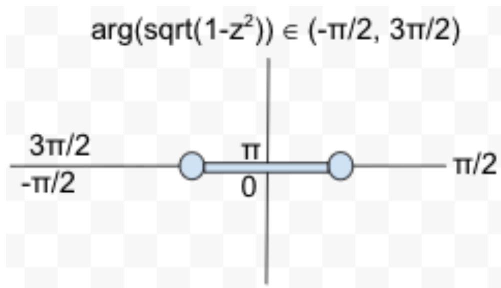


Fig. 1.13

This is precisely the same figure as Fig. 1.10, obtained from a (slightly) different computation. Had we chosen $\arg(i) = 5\pi/2$, $-3\pi/2$, or any other valid value, we would still have the same branch of $\sqrt{1-z^2}$ since all these values of $\arg(i)$ differ by an integer multiple of 2π.

Given the many possibilities for constructing a branch of $\sqrt{1-z^2}$, the number of distinct branches of $\sqrt{1-z^2}$ seems daunting. However, as it turns out, if we restrict ourselves to the branch cuts $(-\infty,-1], (-\infty,1], [-1,\infty)$, and $[1,\infty)$ for $\sqrt{1-z}$ and $\sqrt{1+z}$, then there are only four branches of $\sqrt{1-z^2}$ that we can produce: two branches of $\sqrt{1-z^2}$ with branch cut $(-\infty,-1] \cup [1,\infty)$ that are negatives of each other, and two branches of $\sqrt{1-z^2}$ with branch cut $[-1,1]$ that are negatives of each other. Multiple constructions can lead to the same branch. We encourage you to find out why, but the reason comes down to the fact that a square root function with a given branch cut only has two branches, differing only by sign. As a result, regardless of how many square root functions you multiply together to produce your branch of $\sqrt{1-z^2}$, the end result will only ever differ by sign. If we were working with a cube root or something more complicated, there would be more possibilities.
It is natural to wonder how to compute Laurent series of different branches of the same function. This topic will be deferred to Chapter 3.

Coming up next are three examples of contour integration. They will be explained in great detail, but subsequent explanations will be less detailed for the sake of concision. If something confuses you, we encourage you to refer back to the examples. Also, **if you need a refresher on computing residues, the residue at infinity, or other related theorems, see Appendix 1. We will not explain these things much while we solve our integrals.**

Although I have said it before, I will reiterate again here: **all plots will display the values of $\arg f(z)$ but the branch cuts shown will be the branch cuts for $f(z)$ itself. As we have seen, sometimes $f(z)$ and $\arg f(z)$ will have the same branch cuts, but sometimes they will not.** However, when it comes to integration, we usually do not care about the branch cuts of $\arg f(z)$ since it is $f(z)$ that we are trying to integrate, not its argument.

Lastly, there is something to be said about the convergence of our integrals. If we are working with an integral that does not converge, the residue theorem could still give us a deceptive answer (usually the principal value of the integral, if it exists). The point is, you have to be sure that your integrals converge. However, determining the convergence of integrals is not our focus, so from now on**, you may assume all integrals converge unless otherwise stated**.

(Example 1) Using contour integration, evaluate:

$$J = \int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}\,dx$$

We define $f(z) = \sqrt{1-z^2}/(1+z^2)$. We are integrating from -1 to 1, so it makes sense to choose a branch of $\sqrt{1-z^2}$ with branch cut $[-1,1]$ as then we can integrate around this branch cut using a dogbone contour (with an extra circle of radius R) to extract J. We sketch the contour to integrate over below. C_3 and C_5 are circles of radii $r \to 0$ and C_R is a circle of radius $R \to \infty$. The points $\pm i$ are simple poles of $f(z)$.

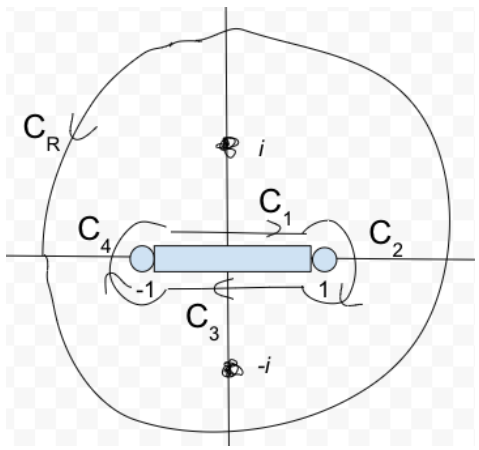


Fig. 2.1

We will use the branch of $\sqrt{1-z^2}$ from Fig 1.10 and we include the plot of its argument again below for convenience.

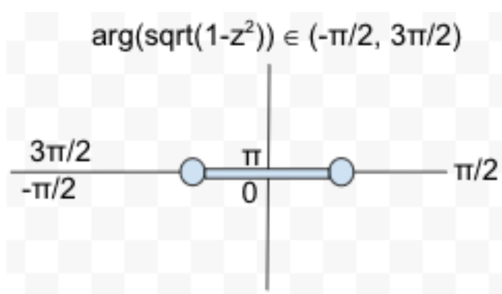


Fig. 2.2

By the residue theorem,

$$\int_{C_R+C_1+C_2+C_3+C_4} f = 2\pi i\big(\mathrm{Res}(f,i) + \mathrm{Res}(f,-i)\big)$$

We first show $\int_{C_2} f$ and $\int_{C_4} f$ vanish as $r \to 0$. We have $z = -1 + re^{i\theta}$ on C_4. Hence, by triangle inequalities, we have:

$$|f(z)| = \frac{|\sqrt{1-z^2}|}{|1+z^2|} = \frac{|-i\sqrt{z-1}\sqrt{1+z}|}{|1+z^2|} \le \frac{\sqrt{2+r}\sqrt{r}}{2r-r^2} = \frac{\sqrt{2+r}}{\sqrt{r}(2-r)}$$

for r small enough.[3] Then by ML lemma, we have:

$$0 \le \left|\int_{C_4} f(z)dz\right| \le 2\pi r \frac{\sqrt{2+r}}{\sqrt{r}(2-r)} = \frac{2\pi\sqrt{r}\sqrt{2+r}}{2-r}$$

Since $\frac{2\pi\sqrt{r}\sqrt{2+r}}{2-r} \to 0$ as $r \to 0$, we have $\int_{C_4} f \to 0$ as $r \to 0$. A similar calculation shows $\int_{C_2} f \to 0$. We now handle $\int_{C_1} f$ and $\int_{C_3} f$. Looking at Fig. 2.2, as $r \to 0$ we have $\arg\sqrt{1-z^2} \to \pi$ on C_1 and $\arg\sqrt{1-z^2} \to 0$ on C_3. We parameterize $z = t + i\epsilon$ on C_1 and $z = t - i\epsilon$ on C_3. Then,

$$\int_{C_1} f = \int_{C_1} \frac{\sqrt{1-z^2}}{1+z^2} dz = \int_{C_1} \frac{|\sqrt{1-z^2}|e^{i\arg\sqrt{1-z^2}}}{1+z^2} dz$$

$$= \lim_{\epsilon\to 0} \int_{-1+r}^{1-r} \frac{\left|\sqrt{1-(t+i\epsilon)^2}\right| e^{i\arg\sqrt{1-(t+i\epsilon)^2}}}{1+(t+i\epsilon)^2} dt \to \int_{-1}^{1} \frac{|\sqrt{1-t^2}|e^{i\pi}}{1+t^2} dt = -J$$

as $r \to 0$.[4] Similarly, for C_3, we have:

$$\int_{C_3} f = \lim_{\epsilon\to 0} \int_{1-r}^{-1+r} \frac{\left|\sqrt{1-(t-i\epsilon)^2}\right| e^{i\arg\sqrt{1-(t-i\epsilon)^2}}}{1+(t-i\epsilon)^2} dt \to \int_{1}^{-1} \frac{|\sqrt{1-t^2}|e^{i0}}{1+t^2} dt = -J$$

as $r \to 0$. It remains to handle $\int_{C_R} f$. We have:

$$\int_{C_R} f(z)\,dz = -2\pi i \operatorname{Res}(f,\infty) = 2\pi i \operatorname{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right) = 2\pi i \operatorname{Res}\left(\frac{\sqrt{1-1/z^2}}{1+z^2}, 0\right)$$

Laurent series could also be used to find the residue at infinity (see Example 5) but it is interesting to compute this equivalent residue at zero. Note the term $\sqrt{1-1/z^2}$ came from substituting $1/z$ in place of z in our branch of $\sqrt{1-z^2}$, so the branch of $\sqrt{1-1/z^2}$ is specified, let it be denoted $g(z)$. Although we have $|g(z)| = |\sqrt{z^2-1}/z|$, in order to write $g(z) = \sqrt{z^2-1}/z$ we must ensure the arguments also agree, that is, we require $\arg g(z) = \arg(\sqrt{z^2-1}/z)$, but since $\arg(\sqrt{z^2-1}/z) = \arg\sqrt{z^2-1} - \arg z$ by the laws of $\arg$, what we need is to choose appropriate branches of $\arg\sqrt{z^2-1}$ and $\arg z$ such that our $\arg g(z)$ equals $\arg\sqrt{z^2-1} - \arg z$.

We start by plotting our branch of $\arg g(z)$ so we know what to work towards. We have our plot of $\arg\sqrt{1-z^2}$ in Fig. 2.2, but the plot of $g(z) = \arg\sqrt{1-1/z^2}$ will be different due to the inversion $z \mapsto 1/z$. By noting $1/z = \bar{z}/|z|^2$ and $\bar{z}$ is the reflection of z

---

[3] Precisely, $|1+z^2| \ge |1-|z|^2| = \left|1-\left|-1+re^{i\theta}\right|^2\right| \ge |1-|-1+r|^2| = 2r - r^2$ for $r \ll 1$.

[4] If you are wondering why we are able to pass the limit $\lim_{\epsilon\to 0}$ under the integral, see Appendix 2.

across the real axis, we can use Fig. 2.2 to plot some values of $\arg g(z)$ below.[5]

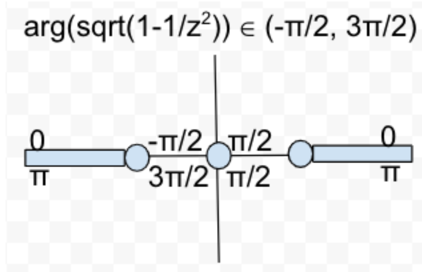


Fig. 2.3

We see that the branch cut of $g(z)$ is $(-\infty,-1] \cup [1,\infty)$ and note $z = 0$ is an isolated singularity of $g(z)$ and so must also be removed from the domain.

We now must choose branches of $\arg\sqrt{z^2-1}$ and $\arg z$ such that $\arg g(z) = \arg\sqrt{z^2-1} - \arg z$. It may take some trial and error to find appropriate branches, but one possibility is to take $\arg(1+z) \in (-\pi,\pi)$ and $\arg(z-1) \in (0,2\pi)$ and define $\sqrt{z^2-1} = \sqrt{1+z}\sqrt{z-1}$, with some values plotted below,

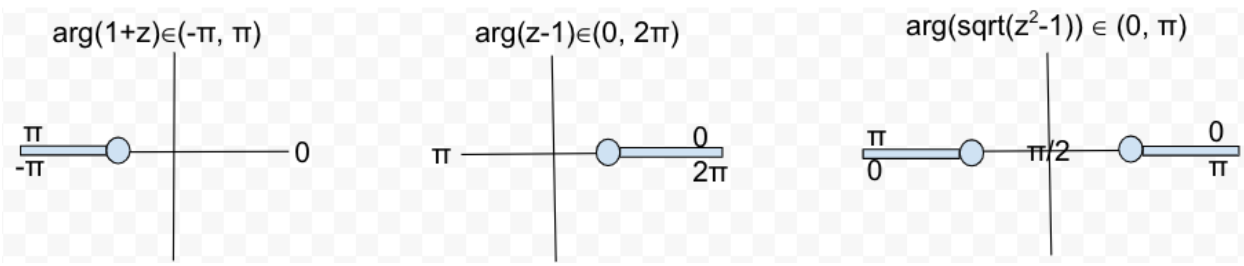


Fig. 2.4

and to take the principal branch of $\arg z$, denoted $\mathrm{Arg}(z)$. Now we plot some values of $\arg\sqrt{z^2-1} - \mathrm{Arg}(z)$ below,

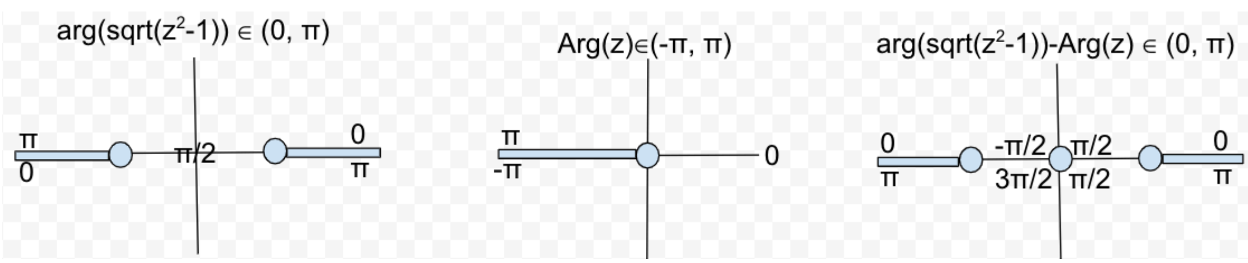


Fig. 2.5

But this plot of $\arg\sqrt{z^2-1} - \mathrm{Arg}(z)$ is identical to our plot of $\arg g(z)$ in Fig. 2.3, so for our choice of branch of $\arg\sqrt{z^2-1}$ and $\mathrm{Arg}(z)$, we indeed have $\arg g(z) = \arg\sqrt{z^2-1} - \mathrm{Arg}(z)$ as desired. To clarify everything, this means that if we define

[5] As an example, take z just above $(-1,0)$ and very close to 0. Then $\bar{z}$ is just below $(-1,0)$ and very close to 0. But since z is close to 0, $|z|$ is very small so $\bar{z}/|z|^2 = 1/z$ will be very far from the origin but still below the negative real axis. But from Fig. 2.2, this is where $\arg\sqrt{1-z^2} \approx -3\pi/2$. So $\arg\sqrt{1-1/z^2} \approx -3\pi/2$ for z just above $(-1,0)$ and very close to 0, as shown in Fig. 2.3.

$h(z) = \left|\sqrt{z^2-1}\right| e^{i \arg \sqrt{z^2-1}}$ where $\arg \sqrt{z^2-1}$ is the branch from Fig. 2.4, then we have $g(z) = h(z)/z$, or in other words, $g(z)$ and $h(z)$ are branches of $\sqrt{1-1/z^2}$ and $\sqrt{z^2-1}$ respectively satisfying $\sqrt{1-1/z^2} = \sqrt{z^2-1}/z$.

So now we can write:

$$2\pi i \operatorname{Res}\left(\frac{\sqrt{1-1/z^2}}{1+z^2}, 0\right) = 2\pi i \operatorname{Res}\left(\frac{g(z)}{1+z^2}, 0\right) = 2\pi i \operatorname{Res}\left(\frac{h(z)}{z(1+z^2)}, 0\right)$$

$$= 2\pi i \lim_{z \to 0} \frac{h(z)}{1+z^2} = 2\pi i \, h(0)$$

Note $h(0) = \sqrt{-1}$ which can equal i or $-i$. But note $\arg h(z) \in (0, \pi)$ (see Fig. 2.4) and $\arg(-i)$ can never lie in $(0, \pi)$, so it must be that $h(0) = i$,

$$= 2\pi i(i) = -2\pi$$

So $\int_{C_R} f \to -2\pi$ as $R \to \infty$. We now find the residues of f at the simple poles $\pm i$. Let $t(z)$ denote our branch of $\sqrt{1-z^2}$ from Fig. 2.2, or equivalently, $t(z) = (1+z^2)f(z)$.

$$\operatorname{Res}(f, i) = \lim_{z \to i} \frac{t(z)}{z+i} = \frac{t(i)}{2i} \qquad \operatorname{Res}(f, -i) = \lim_{z \to -i} \frac{t(z)}{z-i} = \frac{t(-i)}{-2i}$$

So $t(i) = \sqrt{2}$, which could be either $\left|\sqrt{2}\right|$ or $-\left|\sqrt{2}\right|$. One way to deduce which root is to note $\arg t(z) \in (-\pi/2, 3\pi/2)$ (see Fig. 2.2), so either $\arg t(i)$ equals 0 (upon which $t(i) = \left|\sqrt{2}\right| e^{i0} = \left|\sqrt{2}\right|$) or $\arg t(i)$ equals π (upon which $t(i) = \left|\sqrt{2}\right| e^{i\pi} = -\left|\sqrt{2}\right|$). But looking carefully at Fig. 2.2 we see $\arg t(z) \in (\pi/2, 3\pi/2)$ in the upper-half plane, and $z = i$ is in the upper-half plane, therefore $\arg t(i)$ must be π, implying $t(i) = -\sqrt{2}$. Similarly, we note $\arg t(z) \in (-\pi/2, \pi/2)$ in the lower-half plane, and $z = -i$ is in the lower-half plane, so we must have $\arg t(-i) = 0$, implying $t(-i) = \left|\sqrt{2}\right|$.

$$\Rightarrow \operatorname{Res}(f, i) = \frac{-\left|\sqrt{2}\right|}{2i} \qquad \operatorname{Res}(f, -i) = \frac{\left|\sqrt{2}\right|}{-2i}$$

Then, returning to residue theorem,

$$-2J - 2\pi = 2\pi i\left(\frac{-\left|\sqrt{2}\right|}{2i} + \frac{\left|\sqrt{2}\right|}{-2i}\right)$$

$$\Rightarrow J = \pi\left(\sqrt{2} - 1\right)$$

(Example 2) Using contour integration, evaluate:

$$J = \int_1^\infty \frac{dx}{x\sqrt{x^2-1}}$$

Let $f(z) = \frac{1}{z\sqrt{z^2-1}}$. It makes sense to choose a branch of $\sqrt{z^2-1}$ with branch cut $(-\infty,-1] \cup [1,\infty)$ as then we can integrate around it to extract J.

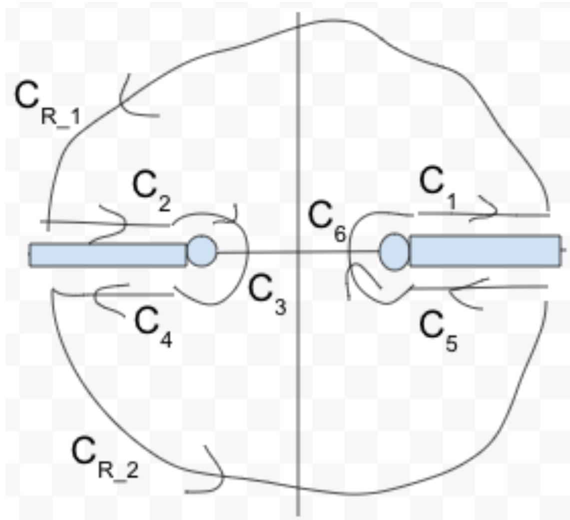


Fig. 2.6

Let C_{R_1}, C_{R_2} be semicircles of radius $R \to \infty$ and ;et C_3, C_6 be circles of radius $r \to 0$. Pick the branch of $\sqrt{z^2-1}$ from Fig. 2.4, and we include the plot of its argument again below for convenience.

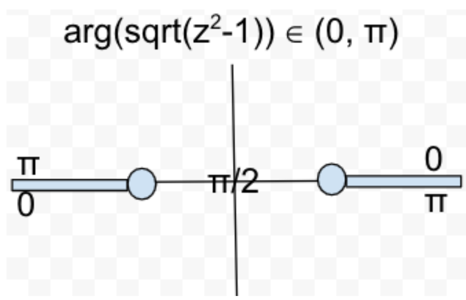


Fig. 2.7

By residue theorem,

$$\int_{C_{R_1}+C_{R_2}+C_1+C_2+C_3+C_4+C_5+C_6} f = 2\pi i\,\mathrm{Res}(f,0)$$

We show $\int_{C_{R_1}} f$ vanishes as $R \to \infty$. On C_{R_1}, we have $z = Re^{i\theta}$, so

$$|f(z)| = \left|\frac{1}{z\sqrt{z^2-1}}\right| \le \frac{1}{R\sqrt{R^2-1}}$$

Hence,

$$0 \le \left|\int_{C_{R_1}} f(z)dz\right| \le \pi R \frac{1}{R\sqrt{R^2-1}} = \frac{2\pi}{\sqrt{R^2-1}}$$

Since $\frac{2\pi}{\sqrt{R^2-1}} \to 0$ as $R \to \infty$, we have $\int_{C_{R_1}} f \to 0$. Similarly, $\int_{C_{R_2}} f \to 0$ as $R \to \infty$.
We now show $\int_{C_3} f$ vanishes as $r \to 0$. We have $z = -1 + re^{i\theta}$ on C_3, so:

$$|f(z)| = \left|\frac{1}{z\sqrt{z^2-1}}\right| = \frac{1}{|-1+re^{i\theta}|\left|\sqrt{re^{i\theta}(-2+re^{i\theta})}\right|} \leq \frac{1}{(1-r)\sqrt{r}\sqrt{2-r}}$$

for r small enough. Hence,

$$0 \leq \left|\int_{C_3} f(z)dz\right| \leq 2\pi r \frac{1}{(1-r)\sqrt{r}\sqrt{2-r}} = \frac{2\pi\sqrt{r}}{(1-r)\sqrt{2-r}}$$

Since $\frac{2\pi\sqrt{r}}{(1-r)\sqrt{2-r}} \to 0$ as $r \to 0$, we have $\int_{C_3} f \to 0$. A similar calculation shows $\int_{C_6} f \to 0$.
We now compute the integrals over C_1, C_2, C_4, C_5. We can use Fig. 2.7 to help us find $\arg\sqrt{z^2-1}$ on these contours. We have $z = t + i\epsilon$ on C_1,

$$\int_{C_1} f = \int_{C_1} \frac{dz}{z\left|\sqrt{z^2-1}\right|e^{i\arg\sqrt{z^2-1}}} = \lim_{\epsilon\to 0}\int_{1+r}^{R} \frac{dt}{(t+i\epsilon)\left|\sqrt{(t+i\epsilon)^2-1}\right|e^{i\arg\sqrt{(t+i\epsilon)^2-1}}}$$

$$\to \int_1^\infty \frac{dt}{t\left|\sqrt{t^2-1}\right|e^{i0}} = J$$

as $r \to 0$, $R \to \infty$. We have $z = -t + i\epsilon$ on C_2,

$$\int_{C_2} f = \lim_{\epsilon\to 0}\int_R^{1+r} \frac{-dt}{(-t+i\epsilon)\left|\sqrt{(-t+i\epsilon)^2-1}\right|e^{i\arg\sqrt{(-t+i\epsilon)^2-1}}}$$

$$\to \int_\infty^1 \frac{-dt}{(-t)\left|\sqrt{t^2-1}\right|e^{i\pi}} = J$$

as $r \to 0$, $R \to \infty$. Similarly, we have $\int_{C_4} f \to J$ and $\int_{C_5} f \to J$.
We now calculate the residue at zero. Let $h(z)$ denote our branch of $\sqrt{z^2-1}$. Note that $h(0) = \sqrt{-1}$ which must equal i and not $-i$ since the argument of $-i$ can never lie in the set of values $(0,\pi)$ that $\arg h(z)$ takes on. So,

$$\mathrm{Res}(f,0) = \lim_{z\to 0}\frac{1}{\sqrt{z^2-1}} = \frac{1}{i}$$

Hence, returning to residue theorem,

$$4J = 2\pi i\left(\frac{1}{i}\right) \Rightarrow J = \frac{\pi}{2}$$

Incidentally, this integral is solved fastest using the substitution $x = \sec\theta$.

(Example 3) Using contour integration, evaluate:

$$J = \int_{-1}^{1} \left(\frac{1-x}{1+x}\right)^{\frac{1}{6}} dx$$

Define $f(z) = \left(\frac{1-z}{1+z}\right)^{1/6}$. We use the exact same contour as in Fig. 2.1, that is, a dogbone contour with a circle of radius $R \gg 1$. We choose the principal branches for $\sqrt{1+z}$ and $\sqrt{z-1}$, that is, $\arg(1+z) \in (-\pi, \pi)$ implying branch cut $(-\infty, -1]$ and $\arg(z-1) \in (-\pi, \pi)$ implying branch cut $(-\infty, 1]$. Pick $\sqrt[6]{-1} = e^{-i\pi/6}$ and $\arg\left(e^{-i\pi/6}\right) = -\pi/6$. Then, we define:

$$\left(\frac{1-z}{1+z}\right)^{\frac{1}{6}} = \sqrt[6]{-1}\,\frac{\sqrt[6]{z-1}}{\sqrt[6]{1+z}} = e^{-i\frac{\pi}{6}}\frac{\sqrt[6]{z-1}}{\sqrt[6]{z+1}}$$

which implies:

$$\arg\left(\frac{1-z}{1+z}\right)^{\frac{1}{6}} = -\frac{\pi}{6} + \frac{1}{6}\left(\arg(z-1) - \arg(1+z)\right)$$

So at each point in the complex plane, we subtract $\arg(1+z)$ from $\arg(z-1)$, then divide by six, then subtract $\pi/6$ in order to find the argument of $f(z) = \left(\frac{1-z}{1+z}\right)^{1/6}$.

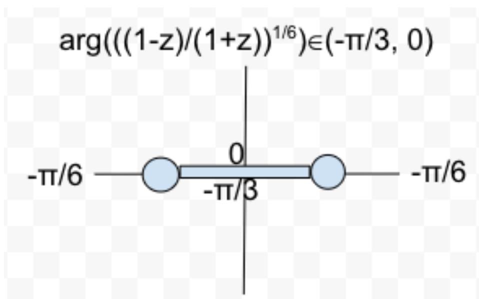


Fig. 2.8

So our $\left(\frac{1-z}{1+z}\right)^{1/6}$ has branch cut $[-1, 1]$ and $\arg\left(\frac{1-z}{1+z}\right)^{1/6} \in (-\pi/3, 0)$.
On C_4, we have $z = -1 + re^{i\theta}$, so

$$|f(z)| = \frac{\left|\sqrt[6]{z-1}\right|}{\left|\sqrt[6]{z+1}\right|} \leq \frac{\sqrt[6]{2+r}}{\sqrt[6]{r}}$$

Hence,

$$0 \leq \left|\int_{C_4} f(z)dz\right| \leq 2\pi r \frac{\sqrt[6]{2+r}}{\sqrt[6]{r}} = 2\pi i\, r^{\frac{5}{6}} \sqrt[6]{2+r}$$

which tends to 0 as $r \to 0$. Hence $\int_{C_4} f \to 0$. Similarly, $\int_{C_2} f \to 0$.
On C_1, we have $\arg f \to 0$, so:

$$\int_{C_1} f(z)dz \to \int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{\frac{1}{6}} e^{i0} dt = J$$

as $r \to 0$.

On C_3, we have $\arg f \to -\pi/3$, so:

$$\int_{C_3} f(z)dz \to \int_1^{-1} \left(\frac{1-t}{1+t}\right)^{\frac{1}{6}} e^{-i\frac{\pi}{3}} dt = -\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) J$$

as $r \to 0$. For C_R, we have:

$$\int_{C_R} f(z)dz = -2\pi i \operatorname{Res}(f, \infty) = 2\pi i \operatorname{Res}\left(\frac{1}{z^2}\left(\frac{1-1/z}{1+1/z}\right)^{\frac{1}{6}}, 0\right) = 2\pi i \operatorname{Res}\left(\frac{1}{z^2}\left(\frac{z-1}{z+1}\right)^{\frac{1}{6}}, 0\right)$$

Our above branch of $\left(\frac{z-1}{z+1}\right)^{1/6}$ is just $f(1/z)$, but for convenience, let $g(z) = f(1/z)$. Once again, since we have only replaced z with $1/z$ in our branch of f to obtain g, we still have $\arg g(z) \in (-\pi/3, 0)$. We notice $z = 0$ is an order two pole,

$$\Rightarrow 2\pi i \operatorname{Res}\left(\frac{1}{z^2}\left(\frac{z-1}{z+1}\right)^{\frac{1}{6}}, 0\right) = 2\pi i \lim_{z\to 0} \frac{d}{dz}[g(z)] = 2\pi i \lim_{z\to 0} \frac{1}{3} g(z) \frac{1}{z^2-1}$$

where we used chain rule to calculate the derivative.[6] Note $g(0) = (-1)^{1/6}$ and that the only sixth root of -1 consistent with $\arg g(z) \in (-\pi/3, 0)$ is $e^{-i\pi/6}$,

$$= 2\pi i \frac{1}{3}(-1)^{\frac{1}{6}} \frac{1}{-1} = -\frac{2\pi i}{3} e^{-i\frac{\pi}{6}} = -\frac{2\pi i}{3}\left(\frac{\sqrt{3}}{2} - \frac{1}{2} i\right)$$

Note there are no residues inside the contour. Then, by Cauchy's theorem,

$$\int_{C_R + C_1 + C_2 + C_3 + C_4} f(z)dz = 0$$

$$\Rightarrow J - \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) J - \frac{2\pi i}{3}\left(\frac{\sqrt{3}}{2} - \frac{1}{2} i\right) = 0$$

$$\Rightarrow \frac{J}{2} - \frac{\pi}{3} + \frac{i\sqrt{3}}{2} J - \frac{\pi i\sqrt{3}}{3} = 0$$

Equating either real or imaginary parts yields:

$$J = \frac{2\pi}{3}$$

[6] We are technically not using power rule here to take the derivative. For details, see Appendix 3.

(Question 1) Now you try. We start with an easier one. Solutions are on the next page. Using contour integration, evaluate:

$$J = \int_0^1 \frac{\sqrt{x - x^2}}{1 + x^3}\,dx$$

1. Construct $\sqrt{z - z^2}$ with branch cut $[0, 1]$. What function and contour will you pick? After you answer, check with the solutions before proceeding.

2. Show that $\int_{C_2} f \to 0$ and $\int_{C_4} f \to 0$ as $r \to 0$.

3. Show $\int_{C_1} f \to J$ and $\int_{C_3} f \to J$ as $r \to 0$.

4. Show that $\int_{C_R} f \to 0$ as $R \to \infty$.

5. What are the poles of f inside the contour? Find their residues. Remember, you can check Appendix 1 if you need help computing residues.

6. Hence find the value of J using the residue theorem.

(Solutions to Question 1)

1.

Pick $f(z) = \frac{\sqrt{z-z^2}}{1+z^3}$ and define $\sqrt{z-z^2} = -i\sqrt{z}\sqrt{z-1}$ where $\arg(-i) = -\pi/2$, $\sqrt{z}$ has branch cut $[0,\infty)$ with $\arg z \in (0, 2\pi)$, and $\sqrt{z-1}$ has branch cut $[1,\infty)$ with $\arg(z-1) \in (0,2\pi)$. Recall, we add $\arg z$ and $\arg(z-1)$ together, divide by two, and subtract $\pi/2$ to find $\arg\sqrt{z-z^2}$ as per laws of arg. Choose a dogbone contour with circle radius $R \to \infty$.

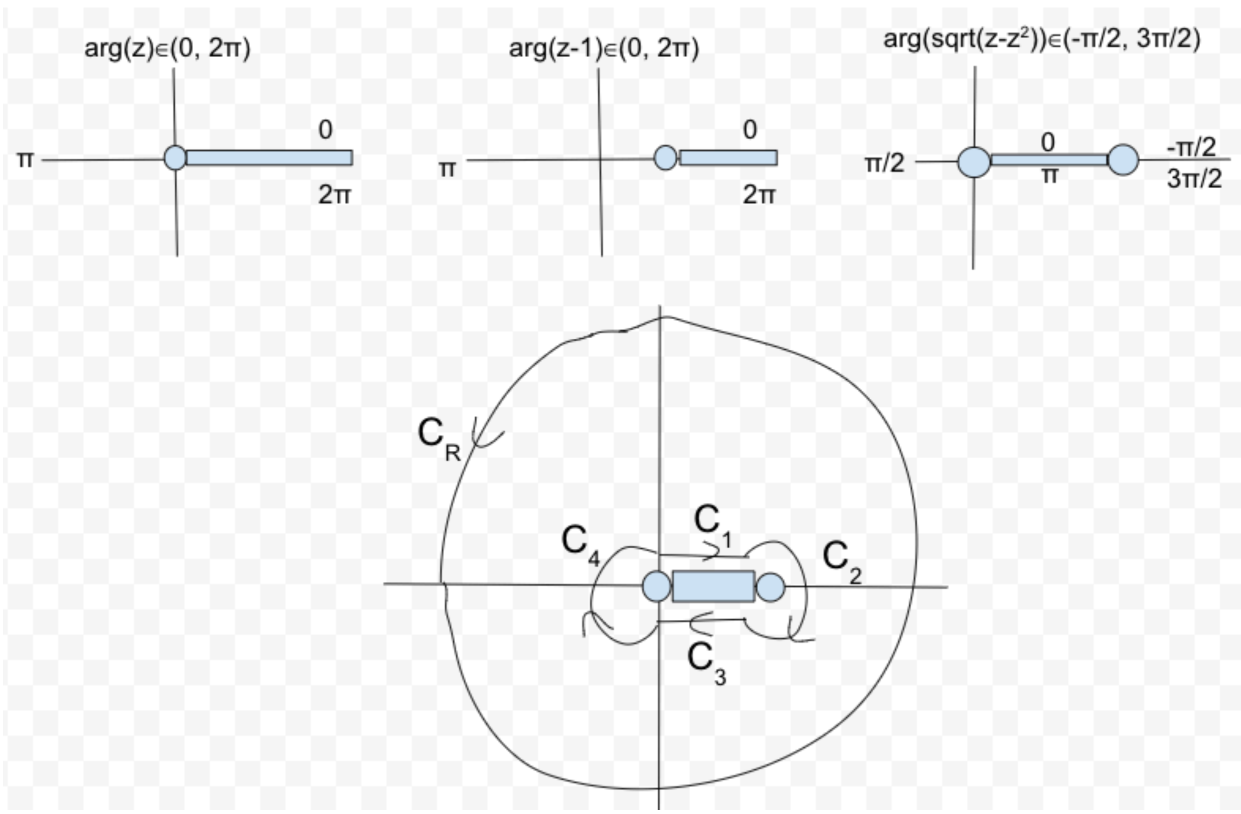


Fig. 2.9

2.

On C_2, we have $z = 1 + re^{i\theta}$. Note that for $r \ll 1$, we have:

$$|1+z^3| \geq 1 - |z^3| \geq 1 - \left|1 + 3re^{i\theta} + 3r^2e^{i2\theta} + r^3e^{i3\theta}\right| \geq 3r - 3r^2 - r^3$$

So $|f(z)| = \frac{|\sqrt{z}||\sqrt{z-1}|}{|1+z^3|} \leq \frac{\sqrt{1+r}\sqrt{r}}{3r-3r^2-r^3}$ hence $\left|\int_{C_2} f\right| \leq 2\pi r \frac{\sqrt{1+r}\sqrt{r}}{3r-3r^2-r^3}$ which goes to 0 as $r \to 0$, hence $\int_{C_2} f \to 0$. On C_4, we have $z = re^{i\theta}$, so $|f(z)| \leq \frac{\sqrt{r}\sqrt{1+r}}{1-r^3}$ hence $\left|\int_{C_4} f\right| \leq 2\pi r \frac{\sqrt{r}\sqrt{1+r}}{1-r^3}$ which also goes to 0 as $r \to 0$, hence $\int_{C_4} f \to 0$.

3.
Parameterize $z = t + i\epsilon$ and $z = t - i\epsilon$ on C_1 and C_3 respectively. Then we find that $\int_{C_1} f \to \int_0^1 \frac{\sqrt{t-t^2}}{1+t^3} e^{i0} dt = J$ and $\int_{C_3} f \to \int_1^0 \frac{\sqrt{t-t^2}}{1+t^3} e^{i\pi} dt = J$ as $r \to 0$.

4.
We parameterize $z = Re^{i\theta}$ on C_R. So $|f(z)| \le \frac{\sqrt{R}\sqrt{1+R}}{R^3-1}$ hence $\left|\int_{C_R} f\right| \le 2\pi R \frac{\sqrt{R}\sqrt{1+R}}{R^3-1}$ which tends to 0 as $R \to \infty$ (this can be proved by L'Hopital's rule). Hence $\int_{C_R} f \to 0$.

5.
The poles are $z = -1$ and $z = e^{\pm i\pi/3}$, all are simple poles and inside the contour. Then, by referencing Fig. 2.9 and noting that $\arg\sqrt{z - z^2} \in (-\pi/2, 3\pi/2)$, we have $\operatorname{Res}(f, -1) = \lim_{z\to -1} \frac{\sqrt{z-z^2}}{3z^2} = \frac{\sqrt{-2}}{3} = \frac{|\sqrt{2}|e^{i\pi/2}}{3}$ and $\operatorname{Res}\left(f, e^{i\pi/3}\right) = \lim_{z\to e^{i\pi/3}} \frac{\sqrt{z-z^2}}{3z^2} = \frac{\sqrt{1}}{3e^{i2\pi/3}} = \frac{|1|e^{i0}}{3e^{i2\pi/3}}$ and $\operatorname{Res}\left(f, e^{-i\pi/3}\right) = \lim_{z\to e^{-i\pi/3}} \frac{\sqrt{z-z^2}}{3z^2} = \frac{\sqrt{1}}{3e^{-i2\pi/3}} = \frac{|1|e^{i\pi}}{3e^{-i2\pi/3}}$. We explain the computation of the residue at $e^{-i\pi/3}$. Let $g(z)$ denote our branch of $\sqrt{z - z^2}$. Then $g\left(e^{-i\pi/3}\right) = \sqrt{1}$ equals either 1 or -1. But from Fig. 2.9, we see that $\arg g(z) \in (-\pi/2, 3\pi/2)$, so $\arg g\left(e^{-i\pi/3}\right)$ equals either 0 or π. But noting $\arg g(z) \in (\pi/2, 3\pi/2)$ in the lower-half plane, which is where $e^{-i\pi/3}$ is, we must have $\arg g\left(e^{-i\pi/3}\right) = \pi$, so $g\left(e^{-i\pi/3}\right) = |1|e^{i\pi}$. The other two residues are computed with similar reasoning.

6.
The residue theorem says:

$$\int_{C_1+\cdots+C_4+C_R} f = 2\pi i\left(\operatorname{Res}(f,-1) + \operatorname{Res}\left(f, e^{i\frac{\pi}{3}}\right) + \operatorname{Res}\left(f, e^{-i\frac{\pi}{3}}\right)\right)$$

$$\Rightarrow 2J = 2\pi i\left(\frac{i\sqrt{2}}{3} + \frac{e^{i0}}{3e^{i2\pi/3}} + \frac{e^{i\pi}}{3e^{-i2\pi/3}}\right) = \frac{2\pi i}{3}\left(i\sqrt{2} + e^{-\frac{2\pi i}{3}} + e^{\frac{5\pi i}{3}}\right)$$

$$\Rightarrow J = \frac{\pi}{3}\left(\sqrt{3} - \sqrt{2}\right)$$

(Question 2) Using contour integration, evaluate:

$$J = \int_0^1 \frac{\sqrt{x - x^2}}{x^2 + 1}\, dx$$

This is the same integral as Question 1 but the denominator is $x^2 + 1$ instead of $x^3 + 1$. If you have already solved Question 1, hopefully you can see that we can reuse or modify much of that previous work to help us solve this integral. In fact, you should use the exact same branch of $\sqrt{z - z^2}$ and the same contour. The trouble comes from the integral over C_R which, unlike in Question 1, will not tend to zero as $R \to \infty$ (do you see why?)

1. Use the same branch of $\sqrt{z - z^2}$ and the same contour as in Question 1. Show that $\int_{C_R} f = 2\pi i \operatorname{Res}\left(\frac{\sqrt{1/z - 1/z^2}}{1+z^2}, 0\right)$. Draw a plot of $\arg\sqrt{1/z - 1/z^2}$.

2. Find a branch of $\sqrt{z - 1}$ satisfying $\sqrt{1/z - 1/z^2} = \sqrt{z - 1}/z$.

3. Hence find $\int_{C_R} f$ as $R \to \infty$.

4. Hence find the value of J.

(Solutions to Question 2)

1.
We have $\int_{C_R} f = -2\pi i \operatorname{Res}(f, \infty) = 2\pi i \operatorname{Res}\left(\frac{1}{z^2}\frac{\sqrt{1/z-1/z^2}}{1+1/z^2}, 0\right) = 2\pi i \operatorname{Res}\left(\frac{\sqrt{1/z-1/z^2}}{1+z^2}, 0\right)$. Note that we simply replaced z with $1/z$ in our branch of $\sqrt{z-z^2}$ to get $\sqrt{1/z - 1/z^2}$. Since $1/z = \bar{z}/|z|^2$ and $\bar{z}$ is the reflection of z across the real axis, we can use the plot of $\arg\sqrt{z-z^2}$ in Fig. 2.9 to create the plot of $\arg\sqrt{1/z-1/z^2}$ shown below (see the explanations surrounding Fig. 2.3 and Fig. 2.4 for a previous example of this).

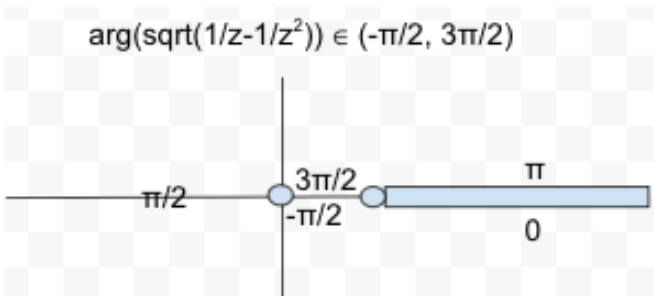


Fig. 2.10

So our $\sqrt{1/z-1/z^2}$ has branch cut $[1, \infty)$ and a singularity at zero.

2.
We have $\arg\sqrt{1/z-1/z^2} = \arg\sqrt{z-1} - \arg z = (1/2)\arg(z-1) - \arg z$ for appropriate branches of $\arg(z-1)$ and $\arg z$. If we choose $\arg z \in (0, 2\pi)$ and $\arg(z-1) \in (2\pi, 4\pi)$ then we can plot $(1/2)\arg(z-1) - \arg z$ below:

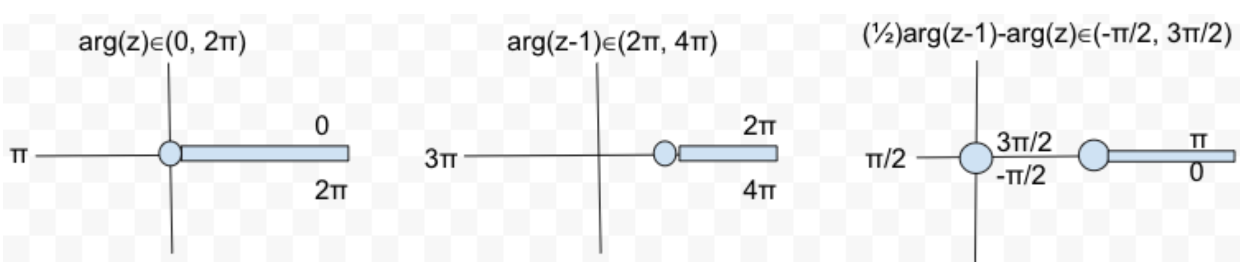


Fig. 2.11

The plot of $\arg\sqrt{z-1} - \arg z$ is identical to the plot of $\arg\sqrt{1/z-1/z^2}$ in Fig. 2.10, so our $\arg\sqrt{1/z-1/z^2}$ indeed equals our $\arg\sqrt{z-1} - \arg z$.

3.
So we have:

$$\int_{C_R} f = 2\pi i \operatorname{Res}\left(\frac{\sqrt{1/z - 1/z^2}}{1+z^2}, 0\right) = 2\pi i \operatorname{Res}\left(\frac{\sqrt{z-1}}{z(1+z^2)}, 0\right) = 2\pi i \lim_{z\to 0} \frac{\sqrt{z-1}}{1+z^2}$$

$$= 2\pi i\sqrt{-1} = 2\pi i(-i) = 2\pi$$

where $\sqrt{-1} = -i$ and not i because $\arg i$ can never lie in the set of values $(\pi, 2\pi)$ that $\arg\sqrt{z-1}$ takes on (note we chose $\arg(z-1) \in (2\pi, 4\pi)$ so $\arg\sqrt{z-1} \in (\pi, 2\pi)$).

4.
The only poles in the contour are $z = \pm i$, both are simple poles. We have $\operatorname{Res}(f, \pm i) = \lim_{z\to\pm i} \frac{\sqrt{z-z^2}}{z\pm i} = \frac{\sqrt{1\pm i}}{\pm 2i}$. Let $g(z)$ denote our branch of $\sqrt{z-z^2}$. Recall $\arg g(z) \in (-\pi/2, 3\pi/2)$. Then the possible values for $\arg g(i) = \arg\sqrt{1+i}$ are $\pi/8$ and $9\pi/8$. Returning to our plot of $\arg g(z)$ in Fig. 2.9, since i lies in the upper-half plane where $\arg g(z)$ takes on values between $-\pi/2$ and $\pi/2$, we must have $\arg g(i) = \pi/8$. Then $\operatorname{Res}(f, i) = \left|\sqrt[4]{2}\right| e^{i\pi/8}/2i$. Similarly, the values for $\arg g(-i) = \arg\sqrt{1-i}$ that are consistent with $\arg g(z) \in (-\pi/2, 3\pi/2)$ are $-\pi/8$ and $7\pi/8$. Since $-i$ is in the lower-half plane where $\arg g(z)$ takes on values between $\pi/2$ and $3\pi/2$, we must have $\arg g(-i) = 7\pi/8$. So $\operatorname{Res}(f, -i) = \left|\sqrt[4]{2}\right| e^{i7\pi/8}/(-2i)$.

The justification that $\int_{C_3} f \to 0$, $\int_{C_5} f \to 0$, $\int_{C_2} f \to J$, and $\int_{C_4} f \to J$ as $r \to 0$ is similar to that used in Question 1. Then by residue theorem,

$$2J + 2\pi = 2\pi i\left(\frac{\sqrt[4]{2}e^{i\frac{\pi}{8}}}{2i} - \frac{\sqrt[4]{2}e^{i\frac{7\pi}{8}}}{2i}\right)$$

$$\Rightarrow 2J + 2\pi = \pi\sqrt[4]{2}\left(2\cos\frac{\pi}{8}\right) = \pi\sqrt[4]{2}\sqrt{2+\sqrt{2}}$$

$$\Rightarrow J = \frac{\pi}{2}\left(-2 + \sqrt{2+2\sqrt{2}}\right)$$

(Question 3) We aim to solve the integral:

$$I = \int_1^\infty \frac{x^3\sqrt{x^2-1}}{x^6-1}dx$$

We could try and integrate this function directly, but the $x^6 - 1$ in the denominator means that the function $\frac{z^3\sqrt{z^2-1}}{z^6+1}$ has four poles (note $z = \pm 1$ are not poles since they are branch points). This means we would have to compute four residues, which is rather tedious. We notice $x^6 - 1$ can be factored as $(x^3+1)(x^3-1)$. Perhaps there is something we can do here.

1. Show that:

$$\frac{x^3\sqrt{x^2-1}}{x^6-1} = \frac{1}{2}\left(\frac{\sqrt{x^2-1}}{x^3+1} + \frac{\sqrt{x^2-1}}{x^3-1}\right)$$

Now define the following two integrals:

$$J = \int_1^\infty \frac{\sqrt{x^2-1}}{x^3+1}dx \qquad K = \int_1^\infty \frac{\sqrt{x^2-1}}{x^3-1}dx$$

2. Construct an appropriate branch of $\sqrt{z^2-1}$. What contour will you pick? After you answer, check with the solutions before proceeding.

3. Integrate $f(z) = \frac{\sqrt{z^2-1}}{z^3+1}$ around the contour chosen above. What do you find?

(Solutions to Question 3)

1.
Just add the fractions together.

2.
Take the branch of $\sqrt{z^2-1}$ from Fig. 2.4, which is copied below. We use the exact same contour as in Fig. 2.6, a double keyhole contour.

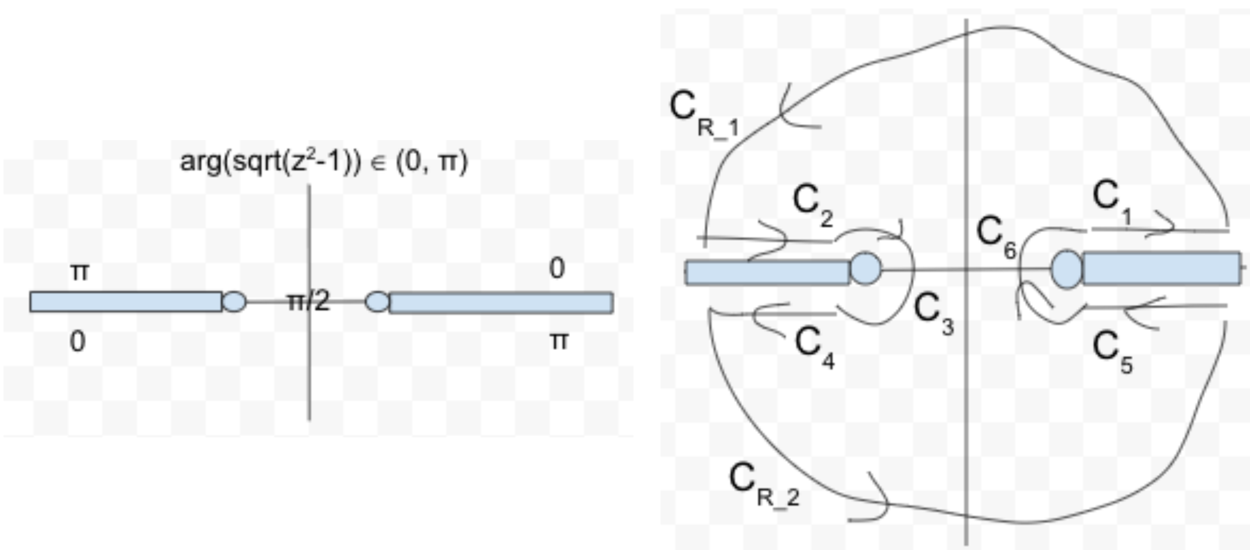


Fig. 2.12

3.
We will integrate f over the contour.
On C_3, we have $|f(z)| \le \frac{\sqrt{2+r}\sqrt{r}}{3r-3r^2-r^3}$ so $\left|\int_{C_3} f\right| \le \frac{2\pi\sqrt{2+r}\sqrt{r}}{3-3r-r^2}$, which vanishes as $r \to 0$, hence $\int_{C_3} f \to 0$. Similarly, $\int_{C_6} f \to 0$. On C_{R_1}, we have $|f(z)| \le \frac{\sqrt{1+R}\sqrt{1+R}}{R^3-1}$ so $\left|\int_{C_{R_1}} f\right| \le \frac{\pi R(1+R)}{R^3-1}$ which vanishes as $R \to \infty$, so $\int_{C_{R_1}} f \to 0$. Similarly, $\int_{C_{R_2}} f \to 0$ as $R \to \infty$. Now, as $r \to 0$ and $R \to \infty$, we have the following:
On C_1 we have $z = t + i\epsilon$ so $\int_{C_1} f \to \int_1^\infty \frac{\sqrt{t^2-1}}{t^3+1} e^{i0} dt = J$
On C_2 we have $z = -t + i\epsilon$ so $\int_{C_2} f \to \int_\infty^1 \frac{\sqrt{t^2-1}}{1+(-t)^3} e^{i\pi}(-dt) = \int_1^\infty \frac{\sqrt{t^2-1}}{t^3-1} dt = K$
On C_4 we have $z = -t - i\epsilon$ so $\int_{C_4} f \to \int_1^\infty \frac{\sqrt{t^2-1}}{1+(-t)^3} e^{i0}(-dt) = \int_1^\infty \frac{\sqrt{t^2-1}}{t^3-1} dt = K$
On C_5 we have $z = t - i\epsilon$ so $\int_{C_5} f \to \int_\infty^1 \frac{\sqrt{t^2-1}}{t^3+1} e^{i\pi} dt = J$
The only poles of f are $z = e^{\pm i\pi/3}$, both simple poles and inside the contour. Note that $z = -1$ is a branch point of f and so is not a pole of f. Keeping in mind that $\arg\sqrt{z^2-1} \in (0,\pi)$,

$$\operatorname{Res}\left(f, e^{i\pi/3}\right) = \lim_{z \to e^{i\pi/3}} \frac{\sqrt{z^2 - 1}}{3z^2} = \frac{\left|\sqrt[4]{3}\right| e^{i5\pi/12}}{3e^{i2\pi/3}} = \left|\frac{1}{\sqrt[4]{27}}\right| e^{-i\frac{\pi}{4}}$$

$$\operatorname{Res}\left(f, e^{-i\pi/3}\right) = \lim_{z \to e^{-i\pi/3}} \frac{\sqrt{z^2 - 1}}{3z^2} = \frac{\left|\sqrt[4]{3}\right| e^{i7\pi/12}}{3e^{-i2\pi/3}} = \left|\frac{1}{\sqrt[4]{27}}\right| e^{i\frac{5\pi}{4}}$$

Then by residue theorem,

$$2J + 2K = 2\pi i \left(\frac{1}{\sqrt[4]{27}} e^{-i\frac{\pi}{4}} + \frac{1}{\sqrt[4]{27}} e^{i\frac{5\pi}{4}}\right) = \frac{2\pi i}{\sqrt[4]{27}}\left(-i\sqrt{2}\right)$$

$$\Rightarrow J + K = \frac{\pi\sqrt{2}}{\sqrt[4]{27}}$$

But $J + K$ is precisely twice the value of I. Therefore,

$$I = \frac{\pi}{\sqrt[4]{27}\sqrt{2}}$$

However, note that we were not able to find the values of J or K individually. When you integrated f over the contour, you probably expected to find J, but because of the parameterizations along the negative real axis, things worked out in such a way that we found I instead.

(Question 4)

We will evaluate the following two integrals:

$$J = \int_0^\infty \frac{\log x}{x^2+1}\,dx \qquad\qquad K = \int_0^\infty \frac{\log x}{x^2-1}\,dx$$

where $\log x$ is the natural logarithm.

1. Integrate $g(z) = \log z/(z^2+1)$ over a keyhole contour. Focus on the integrals over the non-circular portions of the contour. What do you notice? If you already know what is going to happen, you can skip to the next part.

2. Integrate $f(z) = \log^2 z/(z^2+1)$ over the same contour above. Hence find J. You may use the fact that $\lim_{r\to 0} r(\ln r)^n = 0$ and $\lim_{R\to\infty} R^{-1}(\ln R)^n = 0$ for all $n \in \mathbb{N}$ (these two limits can be proved using repeated application of L'Hopital's rule).

3. We now evaluate K. Integrate $f(z) = \log z/(z^2-1)$ over the contour below:

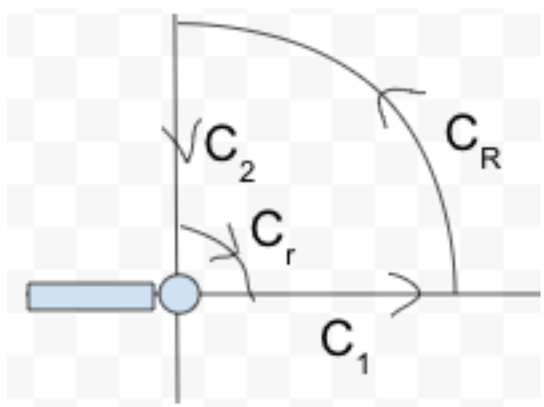


Fig. 2.13

where C_r and C_R are quarter-circles of radii r and R respectively and $\log z$ is chosen with branch cut $(-\infty, 0]$.

(Solutions to Question 4)

1.
For $g(z) = \log z\,/(z^2+1)$, we will use the branch of $\log z$ that has $\arg z \in (0, 2\pi)$, so the branch cut is $[0, \infty)$. Accordingly, we pick the keyhole with $[0, \infty)$ as the slit. However, you could also pick the principal branch of $\log z$ and use the keyhole with slit $(-\infty, 0]$.

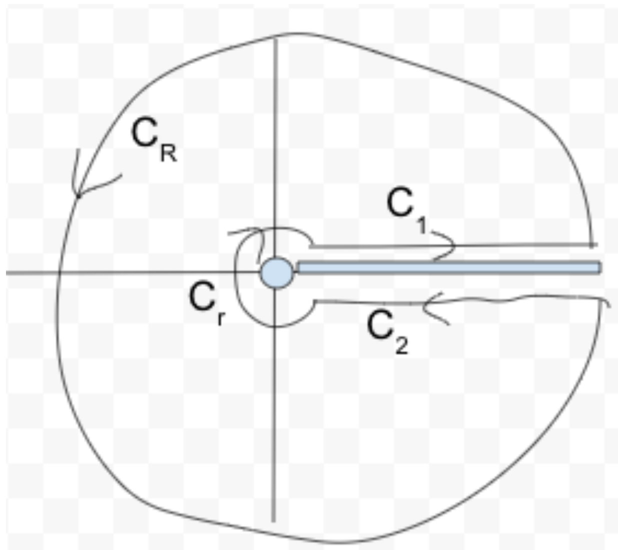


Fig. 2.14

On C_1, we have $z = t + i\epsilon$, hence

$$\int_{C_1} g = \lim_{\epsilon\to 0}\int_r^R \frac{\log|t+i\epsilon| + i\arg(t+i\epsilon)}{(t+i\epsilon)^2+1}\,dt \to \int_0^\infty \frac{\log|t| + i0}{t^2+1}\,dt = J$$

as $r \to 0$, $R \to \infty$. On C_2, we have $z = t - i\epsilon$, hence

$$\int_{C_2} g = \lim_{\epsilon\to 0}\int_R^r \frac{\log|t-i\epsilon| + i\arg(t-i\epsilon)}{(t-i\epsilon)^2+1}\,dt \to \int_\infty^0 \frac{\log|t| + i2\pi}{t^2+1}\,dt$$

$$= -\int_0^\infty \frac{\log|t|}{t^2+1}\,dt - \int_0^\infty \frac{i2\pi}{t^2+1}\,dt = -J - 2\pi i[\arctan t]_0^\infty = -J - 2\pi i\frac{\pi}{2}$$

as $r \to 0$, $R \to \infty$. But in that case, when we use the residue theorem, we will have $\int_{C_1} g + \int_{C_2} g = J + \left(-J - 2\pi i\frac{\pi}{2}\right) = -2\pi i\frac{\pi}{2}$ and notice that the J terms have cancelled each other out! Clearly the integrals over C_R and C_3 will not produce a copy of J, so we will not be able to find the value of J with this setup. We will have to either *change our contour* or *change the function* we integrate.

2.
On C_R, we parameterize $z = Re^{i\theta}$, so for R large enough, we have:

$$|f(z)| = \frac{\log^2|z| + (\arg z)^2}{|z^2+1|} \le \frac{\log^2 R + (2\pi)^2}{R^2-1} \le \frac{\log^2 R + \log^2 R}{R^2 - R^2/2} = \frac{2\log^2 R}{R^2/2}$$

hence $\left|\int_{C_R} f\right| \le 2\pi R \frac{2\log^2 R}{R^2/2} = 8\pi R^{-1}\log^2 R$ which goes to 0 as $R \to \infty$, hence $\int_{C_R} f \to 0$. Similarly, on C_r, we have $z = re^{i\theta}$, so for r small enough, we have:

$$\left|\int_{C_r} f\right| \le 2\pi r \frac{\log^2 r + (2\pi)^2}{1-r^2} \le 2\pi r \frac{\log^2 r + \log^2 r}{1-r^2} = \frac{4\pi r \log^2 r}{1-r^2}$$

which goes to 0 as $r \to 0$. Hence $\int_{C_r} f \to 0$. On C_1, we have $z = t + i\epsilon$, hence

$$\int_{C_1} f = \lim_{\epsilon\to 0}\int_r^R \frac{(\log|t+i\epsilon| + i\arg(t+i\epsilon))^2}{(t+i\epsilon)^2+1}dt = \int_r^R \frac{(\log|t| + i0)^2}{t^2+1}dt \to \int_0^\infty \frac{\log^2|t|}{t^2+1}dt$$

as $r \to 0$, $R \to \infty$. On C_2, we have $z = t - i\epsilon$, hence

$$\int_{C_1} f = \lim_{\epsilon\to 0}\int_R^r \frac{(\log|t-i\epsilon| + i\arg(t-i\epsilon))^2}{(t-i\epsilon)^2+1}dt = -\int_r^R \frac{(\log|t| + i2\pi)^2}{t^2+1}dt$$

$$= -\int_r^R \frac{\log^2|t|}{t^2+1}dt - \int_r^R \frac{4\pi i\log|t|}{t^2+1}dt + \int_r^R \frac{4\pi^2}{t^2+1}dt$$

The poles of f are $z = \pm i$, and both are simple poles that are inside the contour. So $\mathrm{Res}(f, \pm i) = \lim_{z\to\pm i} \frac{\log^2 z}{z \pm i} = \frac{(\log|\pm i| + i\arg(\pm i))^2}{\pm 2i}$. Since $\arg z \in (0, 2\pi)$, we have $\arg i = \pi/2$ and $\arg(-i) = 3\pi/2$. Therefore, we have $\mathrm{Res}(f, i) = -\pi^2/8i$ and $\mathrm{Res}(f, -i) = 9\pi^2/8i$. Hence $2\pi i\big(\mathrm{Res}(f,i) + \mathrm{Res}(f,-i)\big) = 2\pi^3$. Then by residue theorem,

$$\int_r^R \frac{\log^2|t|}{t^2+1}dt + \left(-\int_r^R \frac{\log^2|t|}{t^2+1}dt - \int_r^R \frac{4\pi i\log|t|}{t^2+1}dt + \int_r^R \frac{4\pi^2}{t^2+1}dt\right) + \int_{C_r + C_R} f = 2\pi^3$$

$$\Rightarrow -\int_r^R \frac{4\pi i\log|t|}{t^2+1}dt + \int_r^R \frac{4\pi^2}{t^2+1}dt + \int_{C_r+C_R} f = 2\pi^3$$

Now we take $r \to 0$ and $R \to \infty$,[7]

$$\Rightarrow -4\pi i J + 4\pi^2\frac{\pi}{2} = 2\pi^3$$

$$\Rightarrow J = 0$$

Incidentally, the integral is solved fastest using the substitution $x = 1/u$ and noticing that the resulting integral is just the negative of the original.

---

[7] By delaying taking these limits until *after* the terms $\int_r^R \frac{\log^2|t|}{t^2+1}dt$ cancelled out, we never had to worry about whether $\int_0^\infty \frac{\log^2|t|}{t^2+1}dt$ converges. As it turns out, $\int_0^\infty \frac{\log^2|t|}{t^2+1}dt$ does converge, so it would have been okay to let $r \to 0, R \to \infty$ at an earlier stage, but this will not always be the case.

3.
We have $z = \rho e^{i\theta}, \theta \in (0, \pi/2)$ on both C_R and C_r, so for ρ large or small enough, $\left|\int_{C_R} f\right|, \left|\int_{C_2} f\right| \le \frac{\pi}{2}\rho\frac{\sqrt{\log^2\rho + (\pi/2)^2}}{|\rho^2-1|} \le \frac{\pi\sqrt{2}}{2}\frac{\rho|\log\rho|}{|\rho^2-1|}$. Taking $\rho \to \infty$ and $\rho \to 0$ implies $\int_{C_R} f \to 0$ and $\int_{C_r} f \to 0$ respectively. Note $z = 1$ is a removable singularity of f, hence the function extends continuously over $z = 1$, so we do not need an indent around $z = 1$, hence we have $\int_{C_3} f = \int_{1+r}^{R} \frac{\log|t|}{t^2-1} e^{i0} dt$ which goes to K as $r \to 0, R \to \infty$. Lastly, $z = it$ on C_1 so $\int_{C_1} f = \int_{R}^{1+r} \frac{\log(it)}{(it)^2-1} i dt = \int_{1+r}^{R} \frac{\log|t| + i\pi/2}{t^2+1} i dt \to i\int_0^\infty \frac{\log|t|}{t^2+1} dt - \int_0^\infty \frac{\pi/2}{t^2+1} dt = 0 - \pi^2/4$ as $r \to 0$ and $R \to \infty$ (note the first integral is precisely J, which we proved is zero). There are no singularities in the contour, so by Cauchy's theorem, we have $K - \pi^2/4 = 0$, or

$$K = \frac{\pi^2}{4}$$

Notice that integrating along the imaginary axis was productive because the function becomes real-valued on that axis due to the $z^2 - 1$ in the denominator. In general, we see that **any polynomial $\boldsymbol{z^n + c, c \in \mathbb{R}}$ is real-valued on the lines $\boldsymbol{\{z \in \mathbb{C} : \arg z = \pi/n\}}$ and $\boldsymbol{\{z \in \mathbb{C} : \arg z = 2\pi/n\}}$.** Consequently, when integrating functions with these polynomials in the denominator, **it is useful to consider (indented) circular sector contours of angle $\boldsymbol{\pi/n}$ or $\boldsymbol{2\pi/n}$.** In fact, this is a good rule of thumb when n is any positive real number.

(Question 5)

Based on our discussion on the previous page, evaluate the following two integrals where $b \in \mathbb{R}$, $b > 1$ (the integrals will not converge otherwise).

$$J_b = \int_0^\infty \frac{1}{x^b + 1} dx \qquad\qquad K_b = \int_0^\infty \frac{\log x}{x^b + 1} dx$$

Both integrals can actually be evaluated at once using a single application of the residue theorem.

1. What function and contour will you pick? After you answer, check with the solutions before proceeding.

2. For the contour and function chosen above, how many singularities of f are there inside the contour? Then, use the residue theorem to show that:

$$\begin{cases} K_b - K_b \cos\left(\frac{2\pi}{b}\right) + \frac{2\pi}{b} J_b \sin\left(\frac{2\pi}{b}\right) = \frac{2\pi^2}{b^2} \cos\left(\frac{\pi}{b}\right) \\ K_b \sin\left(\frac{2\pi}{b}\right) + \frac{2\pi}{b} J_b \cos\left(\frac{2\pi}{b}\right) = -\frac{2\pi^2}{b^2} \sin\left(\frac{\pi}{b}\right) \end{cases}$$

   Hence evaluate J_b and K_b.

3. Now, for $b \in \mathbb{R}$, $b > 1$, evaluate the integral

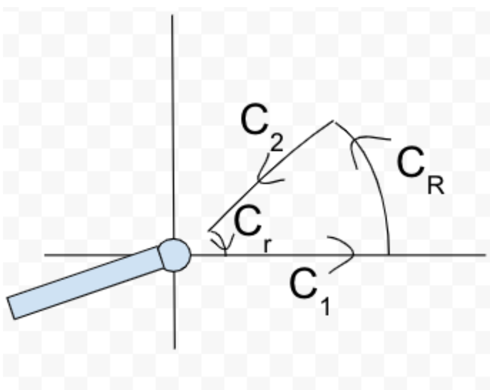
$$L_b = \int_0^\infty \frac{\log x}{x^b - 1} dx$$

   Which contour and function will you pick? Keep in mind that we have already solved J_b and K_b. After you answer, check with the solutions before proceeding.

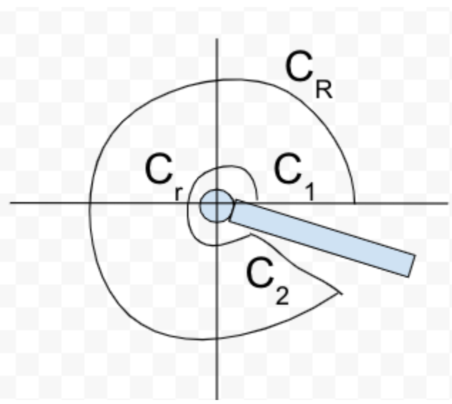
4. Using the function and contour chosen above, evaluate L_b.

(Solutions to Question 5)

1.
If we use an indented circular sector of angle π/b, we notice that the parameterization $z = te^{i\pi/b}$ will cause $z^b + 1$ to become $-z^b + 1$, and this will eventually lead to a big mess, and you will end up having to do more work (potentially using the residue theorem twice) to get out of it. It is faster and easier to use the indented circular sector of angle $2\pi/b$. As for which function to pick, we pick $f(z) = \log z/(z^b + 1)$. Note that, unlike with keyhole contours, we don't have to worry about our desired integrals cancelling out anywhere since we are not integrating around the same branch cut twice. We simply need to choose a branch cut for $\log z$ that does not run into our contour. Since $b > 1$, the ray $\{z \in \mathbb{C} : \arg z = -\pi(1 - 1/b)\} \cup \{0\}$ is a suitable branch cut, and we sketch some examples below (note the ray is positioned "halfway" between C_1 and C_2). We pick the branch of $\log z$ with this branch cut and $\arg z \in \big(-\pi(1 - 1/b), \pi(1 + 1/b)\big)$ and we define $z^b = e^{b \log z}$ with the same $\log z$ (so z^b has the same branch cut). We integrate $f(z) = \log z/(z^b + 1)$ over the contour below where C_r and C_R are circular arcs of angle $2\pi/b$ of radius r and R respectively.



$b = 8$
Aperture of angle $\pi/4$
Branch cut is $\{\arg z = -7\pi/8\}$



$b = 8/7$
Aperture of angle $7\pi/4$
Branch cut is $\{\arg z = -\pi/8\}$

Fig. 2.15

2.
Let z be an isolated singularity of f. Then $e^{b \log z} = -1$,

$$b \log z = i\pi(1 + 2n),\ n \in \mathbb{Z}$$

$$\log|z| + i \arg z = i\pi(1 + 2n)/b$$

$$\arg z = \pi(1 + 2n)/b$$

But since we are only looking for singularities inside the contour for the purposes of residue calculus, we also have $0 < \pi(1 + 2n)/b < 2\pi/b$, which simplifies to $|2n| < 1$. This forces $n = 0$, hence the only singularity of f in the contour is $e^{i\pi/b}$. To see that $e^{i\pi/b}$ is a simple pole, it suffices to show $\lim_{z \to e^{i\pi/b}} \big(z - e^{i\pi/b}\big) f(z)$ exists. Since our branch of $\log z$ is analytic at $e^{i\pi/b}$, it is enough to show $\lim_{z\to e^{i\pi/b}} \frac{z-e^{i\pi/b}}{z^b+1}$ exists, but a single application of L'Hopital's rule shows the limit to be $-e^{i\pi/b}/b$.

Using ML-lemma, it is not hard to show $\int_{C_R} f \to 0$ and $\int_{C_r} f \to 0$ as $R \to \infty$ and $r \to 0$ respectively. We also have $\int_{C_1} f = \int_r^R \frac{\log|t|}{t^b+1} dt \to K_n$ as $r \to 0, R \to \infty$. However, on C_2, we have $z = te^{i2\pi/b}$, so:

$$\int_{C_2} f = \int_R^r \frac{\log\left(te^{i2\pi/b}\right)}{(te^{i2\pi/b})^b + 1} e^{i2\pi/b} dt = -\int_r^R \frac{\log|t| + i2\pi/n}{t^b + 1} e^{i2\pi/b} dt$$

$$\to -e^{i2\pi/b} K_b - \frac{2\pi}{b} i e^{i2\pi/b} J_b$$

as $r \to 0, R \to \infty$. Since the only pole in the contour is the simple pole $e^{i\pi/b}$, we have:

$$\operatorname{Res}\left(f, e^{i\pi/b}\right) = \lim_{z\to e^{i\pi/b}} \left(z - e^{i\pi/b}\right) \frac{\log z}{z^b + 1}$$

The limit $\lim_{z\to e^{i\pi/b}} \frac{z-e^{i\pi/b}}{z^b+1}$ was previously shown to equal $-e^{i\pi/b}/b$,

$$= -\frac{e^{i\pi/b}}{b} \log\left(e^{i\pi/b}\right) = -\frac{i\pi}{b^2} e^{i\pi/b}$$

Then by residue theorem,

$$K_b - e^{i2\pi/b} K_b - \frac{2\pi}{b} i e^{i2\pi/b} J_b = \frac{2\pi^2}{b^2} e^{i\pi/b}$$

Writing $e^{i2\pi/b} = \cos(2\pi/b) + i\sin(2\pi/b)$ and equating real and imaginary parts yields:

$$\Rightarrow \begin{cases} K_b - K_b \cos\left(\frac{2\pi}{b}\right) + \frac{2\pi}{b} J_b \sin\left(\frac{2\pi}{b}\right) = \frac{2\pi^2}{b^2} \cos\left(\frac{\pi}{b}\right) \\ K_b \sin\left(\frac{2\pi}{b}\right) + \frac{2\pi}{b} J_b \cos\left(\frac{2\pi}{b}\right) = -\frac{2\pi^2}{b^2} \sin\left(\frac{\pi}{b}\right) \end{cases}$$

How you solve this system is up to you, but one way goes as follows: divide the first equation by $\sin(2\pi/b)$, divide the second equation by $\cos(2\pi/b)$, then subtract the two equations from each other to get:[8]

$$K_b \csc\left(\frac{2\pi}{b}\right) - K_b \left(\frac{\cos 2\pi/b}{\sin 2\pi/b} + \frac{\sin 2\pi/b}{\cos 2\pi/b}\right) = \frac{2\pi^2}{b^2} \left(\frac{\cos \pi/b}{\sin 2\pi/b} + \frac{\sin \pi/b}{\cos 2\pi/b}\right)$$

After using trigonometric sum identities and simplifying,

$$\Rightarrow K_b = \frac{2\pi^2}{b^2} \frac{\cos\left(\frac{\pi}{b}\right)}{\cos\left(\frac{2\pi}{b}\right) - 1} = -\frac{\pi^2}{b^2} \csc^2\left(\frac{\pi}{b}\right) \cos\left(\frac{\pi}{b}\right)$$

[8] The values of b for which either $\sin(2\pi/b)$ or $\cos(2\pi/b)$ are zero are isolated, so we may continuously extend our formulas for J_b and K_b to these b (but do not forget we are only considering $b > 1$).

Substituting back into the second equation and simplifying using double-angle identities yields:

$$J_b = \frac{\pi}{b}\csc\left(\frac{\pi}{b}\right)$$

3.

If you used a sector of angle $2\pi/b$, the contour would run into the pole $z = e^{i2\pi/b}$, and although we could integrate around the pole, that is more work. As it turns out, a sector of angle π/b is now advantageous since it will cause K_b and J_b to appear. Therefore we use the same contour as Fig. 2.14 but now C_r and C_R are sectors of angle π/b. We pick $f(z) = \log z/(z^b - 1)$ with $\log z$ and z^b the same as before. Note $z = 1$ is a removable singularity of f (proved in Appendix 1), so the function extends continuously over $z = 1$, so we do not need an indent around $z = 1$.

4.

Once again, by using ML-lemma, we have $\int_{C_R} f \to 0$ and $\int_{C_r} f \to 0$ as $R \to \infty$ and $r \to 0$ respectively. Since $z = 1$ is a removable singularity of f, the function extends continuously over $z = 1$ so $\int_{C_1} f \to L_b$. On C_2 we have $z = te^{i\pi/b}$ hence:

$$\int_{C_2} f = \int_R^r \frac{\log\left(te^{i\pi/b}\right)}{(te^{i\pi/b})^b - 1} e^{i\pi/b}\,dt = \int_r^R \frac{\log|t| + i\pi/b}{t^b + 1} e^{i\pi/b}\,dt \to e^{\frac{i\pi}{b}}\left(K_b + \frac{i\pi}{b}J_b\right)$$

as $r \to 0, R \to \infty$. Lastly, the zeroes of $z^b - 1$ are e^{i2bn}, $n \in \mathbb{Z}$, none of which lie inside our contour (and $z = 1$ was handled earlier) so there are no singularities inside our contour. Then by Cauchy's theorem,

$$L_b + e^{\frac{i\pi}{b}}\left(K_b + \frac{i\pi}{b}J_b\right) = 0$$

$$\Rightarrow L_b = -\left(\cos\frac{\pi}{b} + i\sin\frac{\pi}{b}\right)\left(-\frac{\pi^2}{b^2}\csc^2\left(\frac{\pi}{b}\right)\cos\left(\frac{\pi}{b}\right) + i\frac{\pi^2}{b^2}\csc\left(\frac{\pi}{b}\right)\right)$$

Equating real parts and simplifying,

$$\Rightarrow L_b = \frac{\pi^2}{b^2}\csc^2\left(\frac{\pi}{b}\right)$$

which incidentally leads to the interesting equation $L_b = (J_b)^2$.

(Question 6)

For $a, b \in \mathbb{R}$ and $0 \leq a < 1$ and $b > 1$, evaluate the integrals:

$$J = \mathrm{PV}\int_0^\infty \frac{1}{x^a(x^b-1)}dx \qquad K = \int_0^\infty \frac{\log x}{x^a(x^b-1)}dx$$

where J is the principal value of $\int_0^\infty \frac{1}{x^a(x^b-1)}dx$. You may assume the integral of $\frac{1}{x^a(x^b-1)}$ diverges on any interval containing $x = 1$ and that $\frac{1}{x^a(x^b-1)}$ is absolutely integrable on the intervals $(0, 1-r)$ and $(1+r, \infty)$ for all $r > 0$. And as usual, you may assume the integral K converges. If you need to, check Appendix 1 for a refresher on principal value integrals and the "fractional residue theorem," which will also be used in this question.

1. What function and contour will you pick? After you answer, check with the solutions before proceeding. It may be helpful to note that $z = 1$ is a removable singularity of $\log z/(z^b - 1)$ for $b > 1$.

2. Using the function and contour chosen above, find J and K. We offer guided solutions below.

   a. Show $\int_{C_R} f \to 0$ and $\int_{C_r} f \to 0$ as $R \to \infty$ and $r \to 0$ respetively.

   b. Using the fractional residue theorem, show:

   $$\lim_{r\to 0}\int_{C_3} f(z)dz = \frac{2\pi^2}{b^2}e^{i2\pi(1-a)/b}$$

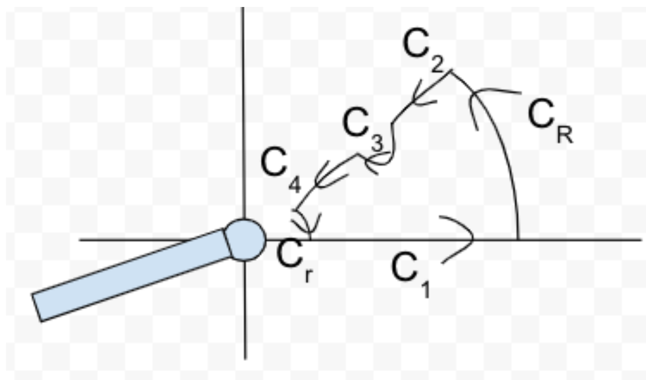
   c. Let $\gamma = 2\pi(1-a)/b$. Show that:

   $$\begin{cases} (1-\cos\gamma)K + \dfrac{2\pi}{b}J\sin\gamma + \dfrac{2\pi^2}{b^2}\cos\gamma = 0 \\ -K\sin\gamma - \dfrac{2\pi}{b}J\cos\gamma + \dfrac{2\pi^2}{b^2}\sin\gamma = 0 \end{cases}$$

   d. Hence find J and K.

(Solutions to Question 6)

1.
Pick $f(z) = \frac{\log z}{z^a(z^b-1)}$ with $\log z$, z^a, and z^b defined exactly as in the previous question: with branch cut $\{\arg z = -\pi(1-1/b)\} \cup \{0\}$ and $\arg z \in \left(-\pi(1-1/b), \pi(1+1/b)\right)$ for all three functions. We integrate f over the indented circular sector of angle $2\pi/b$ with an indent C_3 around the pole at $z = e^{i2\pi/b}$. Since $z = 1$ is a removable singularity of f (see Appendix 1), the function f extends continuously over $z = 1$, so we do not need an indent around $z = 1$.



$b = 8$
Aperture of angle $\pi/4$
Branch cut is $\{\arg z = -7\pi/8\}$

Fig. 26

2.
Keep in mind $b > 1$ and $0 \le a < 1$. For $z = \rho e^{i\theta}, \theta \in (0, 2\pi/b)$ and ρ small or large enough, we have $|f(z)| = \frac{|\log|z| + i\arg z|}{|z|^a|z^b-1|} \le \frac{\sqrt{\log^2\rho + 4\pi^2/b^2}}{\rho^a|\rho^b-1|} \le \frac{\sqrt{2}|\log\rho|}{\rho^a|\rho^b-1|}$ and therefore $\left|\int_{C_r} f\right|, \left|\int_{C_R} f\right| \le \frac{2\pi}{b}\frac{\rho^{1-a}\sqrt{2}|\log\rho|}{|\rho^b-1|} = \frac{2\sqrt{2}\pi}{b}\frac{\rho^{1-a}\log\rho}{\rho^b-1}$ (note the signs work out in such a way that we can remove the absolute value signs regardless of whether ρ is very small or very large). Since $0 < 1-a \le 1$, the numerator $\rho^{1-a}\log\rho$ vanishes as $\rho \to 0$ (use L'Hopital's) and the whole expression goes to 0 as $\rho \to \infty$ since $\frac{\rho^{1-a}\log\rho}{\rho^b-1} \le \frac{\rho^{1-a}\log\rho}{\rho^b - \rho^b/2} = \frac{2\log\rho}{\rho^{b+a-1}}$ which goes to 0 as $\rho \to \infty$ by L'Hopital's (note $b + a - 1 > 0$). So $\int_{C_R} f \to 0$ and $\int_{C_r} f \to 0$. It is easy to see $\int_{C_1} f = \int_r^R \frac{\log|t|}{|t|^a(|t|^b-1)}dt$. On C_2, we parameterize $z = te^{i2\pi/b}$, hence:

$$\int_{C_2} f = \int_R^{1+r} \frac{\log\left(te^{i2\pi/b}\right)}{e^{a\log\left(te^{i2\pi/b}\right)}\left(e^{b\log\left(te^{i2\pi/b}\right)} - 1\right)} e^{i2\pi/b}dt = -\int_{1+r}^R \frac{\log|t| + i2\pi/b}{|t|^a e^{i2\pi a/b}(|t|^b - 1)} e^{i2\pi/b}dt$$

$$= -e^{i2\pi(1-a)/b}\int_{1+r}^R \frac{\log|t|}{|t|^a(|t|^b-1)}dt - \frac{2\pi i}{b}e^{i2\pi(1-a)/b}\int_{1+r}^R \frac{dt}{|t|^a(|t|^b-1)}$$

Similarly, on C_4, we have:

$$\int_{C_4} f = -e^{i2\pi(1-a)/b} \int_r^{1-r} \frac{\log|t|}{|t|^a(|t|^b-1)} dt - \frac{2\pi i}{b} e^{i2\pi(1-a)/b} \int_r^{1-r} \frac{dt}{|t|^a(|t|^b-1)}$$

Note we cannot take limits yet since $\int_0^\infty 1/(x^a(x^b-1))dx$ diverges.
We notice that C_3 is the arc of a circle of angle π around the simple pole $e^{i2\pi/b}$ traversed clockwise, hence by fractional residue theorem, $\int_{C_3} f \to -\pi i \operatorname{Res}(f, e^{i2\pi/b})$ as $r \to 0$. We have:

$$\operatorname{Res}(f, e^{i2\pi/b}) = \lim_{z\to e^{i2\pi/b}} \frac{\log z / z^a}{(z^b-1)'} = \lim_{z\to e^{i2\pi/b}} \frac{\log z / z^a}{z^b(b/z)} = \frac{2\pi i}{b^2} e^{i2\pi(1-a)/b}$$

Hence $\int_{C_3} f \to \frac{2\pi^2}{b^2} e^{i2\pi(1-a)/b}$ as $r \to 0$.
There are no singularities inside the contour, hence by Cauchy's theorem,

$$\int_{C_r+C_R} f + \int_r^R \frac{\log|t|}{|t|^a(|t|^b-1)} dt - e^{\frac{i2\pi(1-a)}{b}} \left(\int_r^{1-r} + \int_{1+r}^R\right) \frac{\log|t|}{|t|^a(|t|^b-1)} dt$$
$$- \frac{2\pi i}{b} e^{\frac{i2\pi(1-a)}{b}} \left(\int_r^{1-r} + \int_{1+r}^R\right) \frac{dt}{|t|^a(|t|^b-1)} + \int_{C_3} f = 0$$

We write $\int_r^R \frac{\log|t|}{|t|^a(|t|^b-1)} dt = \left(\int_r^{1-r} + \int_{1-r}^{1+r} + \int_{1+r}^R\right) \frac{\log|t|}{|t|^a(|t|^b-1)} dt$ and simplify,

$$\int_{C_r+C_R} f + \int_{1-r}^{1+r} \frac{\log|t|}{|t|^a(|t|^b-1)} dt + \left(1 - e^{\frac{i2\pi(1-a)}{b}}\right)\left(\int_r^{1-r} + \int_{1+r}^R\right) \frac{\log|t|}{|t|^a(|t|^b-1)} dt$$
$$- \frac{2\pi i}{b} e^{\frac{i2\pi(1-a)}{b}} \left(\int_r^{1-r} + \int_{1+r}^R\right) \frac{dt}{|t|^a(|t|^b-1)} + \int_{C_3} f = 0$$

Now we take $r \to 0$ and $R \to \infty$. We denote $\gamma = 2\pi(1-a)/b$ for concision.

$$(1 - e^{i\gamma})K - \frac{2\pi i}{b} e^{i\gamma} J + \frac{2\pi^2}{b^2} e^{i\gamma} = 0$$

Equating real and imaginary parts,

$$\begin{cases} (1-\cos\gamma)K + \dfrac{2\pi}{b} J \sin\gamma + \dfrac{2\pi^2}{b^2}\cos\gamma = 0 \\ -K\sin\gamma - \dfrac{2\pi}{b} J\cos\gamma + \dfrac{2\pi^2}{b^2}\sin\gamma = 0 \end{cases}$$

Solving the system (see Question 5 for a similar example) yields:

$$J = -\frac{\pi}{b}\cot\left(\frac{\pi(1-a)}{b}\right) \qquad K = \frac{\pi^2}{b^2}\csc^2\left(\frac{\pi(1-a)}{b}\right)$$

Remember, $-\frac{\pi}{b}\cot\left(\frac{\pi(1-a)}{b}\right)$ is the principal value of the integral, but the integral itself diverges. Note that our result for K is consistent with our calculation of L_n in Question 5.

(Question 7)

For $0 < b < 4$, evaluate the integral:

$$J = \int_0^\infty \frac{\log(1 + x^4)}{x^{b+1}} dx$$

1. We first construct a branch of $\log(1 + z^4)$. The laws of $\log$ motivate us to write $\log(1 + z^4) = \log\left(z - e^{i\pi/4}\right) + \log\left(z - e^{i3\pi/4}\right) + \log\left(z - e^{i5\pi/4}\right) + \log\left(z - e^{i7\pi/4}\right)$. We choose the branches as follows:

Function	Branch cut	Domain of $\arg\left(f_k(z)\right)$
$f_1(z) = \log\left(z - e^{i\pi/4}\right)$	$\left\{z : z = re^{i\pi/4}, r \geq 1\right\}$	$(-7\pi/4, \pi/4)$
$f_2(z) = \log\left(z - e^{i3\pi/4}\right)$	$\left\{z : z = re^{i3\pi/4}, r \geq 1\right\}$	$(-5\pi/4, 3\pi/4)$
$f_3(z) = \log\left(z - e^{i5\pi/4}\right)$	$\left\{z : z = re^{i5\pi/4}, r \geq 1\right\}$	$(-3\pi/4, 5\pi/4)$
$f_4(z) = \log\left(z - e^{i7\pi/4}\right)$	$\left\{z : z = re^{i7\pi/4}, r \geq 1\right\}$	$(-\pi/4, 7\pi/4)$

   Using the above table, draw plots of $\arg f_k(z)$ for each $k = 1, 2, 3, 4$.

2. Define $g(z) = f_1(z) + \cdots + f_4(z)$ be our branch of $\log(1 + z^4)$. Sketch a plot of $\arg g(z)$. After you answer, check with the solutions before proceeding.

3. We integrate $f(z) = \log(1 + z^4)/z^{b+1}$ where $\log(1 + z^4)$ is the branch as defined above and we choose z^{b+1} with branch cut $[0, -i\infty)$ and $\arg z \in (-\pi/2, 3\pi/2)$. Which contour should we integrate f over? After you answer, check with the solutions before proceeding.

4. Hence find J. We offer guided solutions below.

   a. Show $\int_{C_R} f \to 0$ as $R \to \infty$. Then show $\int_{C_2} f \to 0$ as $r \to 0$.

   b. Show $\int_{C_5} f \to 0$ and $\int_{C_8} f \to 0$ as $r \to 0$.

   c. Show that:

   $$\lim_{r \to 0} \int_{C_4 + C_6 + C_7 + C_9} f = -\frac{2\pi i}{b}\left(e^{-ib\pi/4} + e^{-ib3\pi/4}\right)$$

   d. Hence find J.

(Solutions to Question 7)

1.

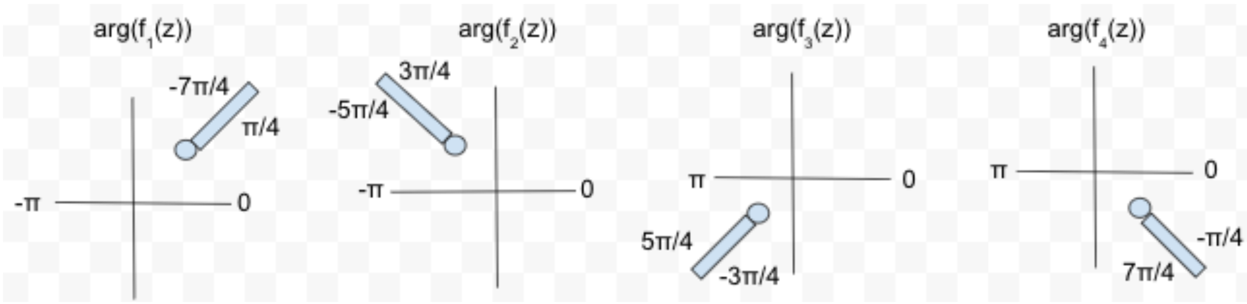


Fig. 2.17

2.
We have $\arg g(z) = \arg f_1(z) + \cdots + \arg f_4(z)$. We see that $\arg g(z) \in (-\pi, \pi)$.

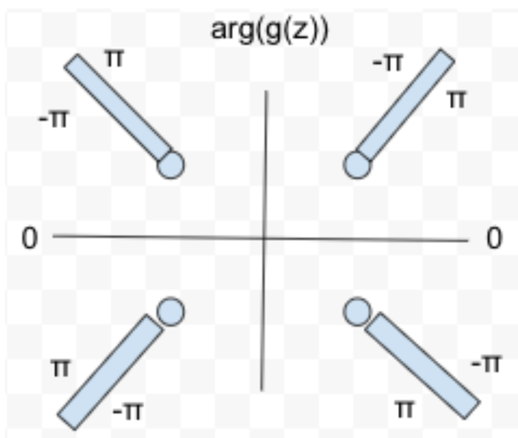


Fig. 2.18

3.
We use the following contour:

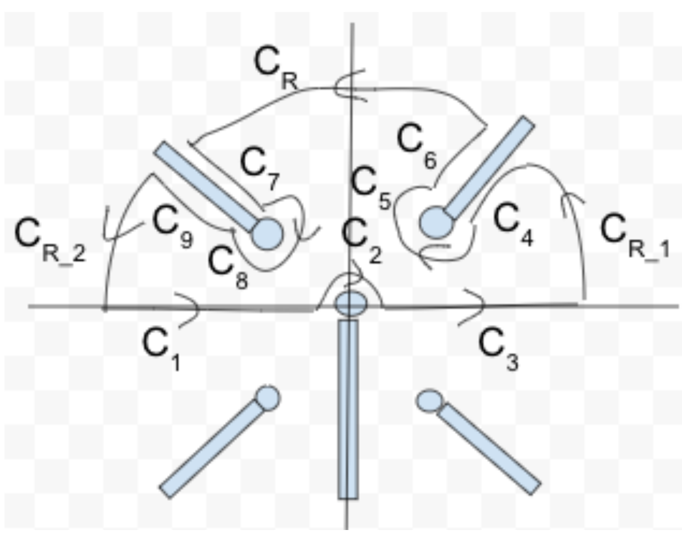


Fig. 2.19

Note that $e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}, 0,$ and ∞ are the branch points of f (the $\log(1+z^4)$ term contributes $e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4,}, e^{i7\pi/4}, \infty$ while the z^{b+1} term contributes 0 and ∞). However, we must keep the branch cuts for $\log(1+z^4)$ and z^{b+1} separate in our minds: $\arg(\log(1+z^4))$ is zero on both the negative and positive real axis, but $\arg z$ (from z^{b+1}) is zero on the positive real axis and π on the negative real axis. As usual, C_2, C_5, C_9 have radius $r \to 0$ and C_R, C_{R_1}, C_{R_2} have radius $R \to \infty$.

4.

For $z = Re^{i\theta}, \theta \in (0,\pi)$ and R large enough, we have $|f(z)| = \frac{\sqrt{\log^2|1+z^4|+(\arg(1+z^4))^2}}{|z|^{b+1}} \leq \frac{\sqrt{\log^2(1+R^4)+\pi^2}}{R^{b+1}} \leq \frac{\sqrt{\log^2(1+R^4)+\log^2(1+R^4)}}{R^{b+1}} = \frac{\sqrt{2}\log(1+R^4)}{R^{b+1}}$ hence $\left|\int_{C_R} f\right| \leq \frac{\pi}{2}\frac{\sqrt{2}\log(1+R^4)}{R^b}$ which goes to 0 as $R \to \infty$ by L'Hopital's. Hence $\int_{C_R} f$ vanishes as $R \to \infty$. Similarly, the integrals over C_{R_1} and C_{R_2} vanish as $R \to \infty$. We have $z = re^{i\theta}, \theta \in (0,\pi)$ on C_2 and note $\arg(1 + r^4e^{i4\theta}) \in [-\theta_0, \theta_0]$ for some small $\theta_0 > 0$ when r is very small,

$$|f(z)| \leq \frac{\sqrt{\log^2(1+r^4) + (\arg(1+r^4e^{i4\theta}))^2}}{r^{b+1}} = \frac{\sqrt{\log^2(1+r^4) + \arctan^2\left(\frac{r^4\sin(4\theta)}{1+r^4\cos(4\theta)}\right)}}{r^{b+1}}$$

$$\leq \frac{\sqrt{\log^2(1+r^4) + \arctan^2\left(\frac{r^4}{1-r^4}\right)}}{r^{b+1}}$$

As $r \to 0$, this becomes a $0/0$ limit, but before L'Hopital's, we try to simplify a little. We aim to show $\log(1+r^4) \leq \arctan(r^4/(1-r^4))$ for all $r > 0$ small enough. Both are zero when $r = 0$, so it suffices to show the derivative of $\arctan(r^4/(1-r^4))$ is greater than the derivative of $\log(1+r^4)$ for all $r > 0$ small enough, but this is readily apparent: we have $\left(\arctan(r^4/(1-r^4))\right)' = 4r^3/(2r^8 - 2r^4 + 1)$ which is certainly greater than $(\log(1+r^4))' = 4r^3/(1+r^4)$ for $r > 0$ small enough. Hence,

$$\leq \frac{\sqrt{\arctan^2\left(\frac{r^4}{1-r^4}\right) + \arctan^2\left(\frac{r^4}{1-r^4}\right)}}{r^{b+1}} = \frac{\sqrt{2}\arctan\left(\frac{r^4}{1-r^4}\right)}{r^{b+1}}$$

Hence $\left|\int_{C_2} f\right| \leq \pi\sqrt{2}\arctan\left(\frac{r^4}{1-r^4}\right)/r^b$ which goes to 0 as $r \to 0$ by L'Hopital's. Hence $\int_{C_2} f \to 0$ as $r \to 0$.

On C_5, we have $z = e^{i\pi/4} + re^{i\theta}$. Writing $\log(1+z^4)$ as $f_1 + \cdots + f_4$, we have:

$$|f(z)| \leq \left|\frac{\log(re^{i\theta}) + \log(\sqrt{2} + re^{i\theta}) + \log(\sqrt{2}(1+i) + re^{i\theta}) + \log(i\sqrt{2} + re^{i\theta})}{(e^{i\pi/4} + re^{i\theta})^{b+1}}\right|$$

Note $\left|e^{\frac{i\pi}{4}} + re^{i\theta}\right|^{b+1} = \left(\left(\frac{1}{\sqrt{2}} + r\cos\theta\right)^2 + \left(\frac{1}{\sqrt{2}} + r\sin\theta\right)^2\right)^{\frac{b+1}{2}} \geq \left(\left(\frac{1}{\sqrt{2}} - r\right)^2 + \left(\frac{1}{\sqrt{2}} - r\right)^2\right)^{\frac{b+1}{2}}$ for r small enough and $\theta \in [0,\pi]$,

$$\leq \frac{|\log(r)| + \log(\sqrt{2}+r) + \log(2+r) + \log(\sqrt{2}+r)}{\sqrt{2}^{b+1}\left((1/\sqrt{2}) - r\right)^{b+1}}$$

$$\Rightarrow \left|\int_{C_5} f\right| \leq 2\pi r\frac{|\log(r)| + \log(\sqrt{2}+r) + \log(2+r) + \log(\sqrt{2}+r)}{\sqrt{2}^{b+1}\left((1/\sqrt{2}) - r\right)^{b+1}}$$

which goes to 0 as $r \to 0$ (note $r \log r \to 0$ as $r \to 0$). Hence $\int_{C_5} f \to 0$ as $r \to 0$. A similar calculation shows $\int_{C_8} f \to 0$ as $r \to 0$. Next, we work on C_4, C_6, C_7, C_9. Recall that $\arg(\log(1+z^4)) \in (-\pi, \pi)$ (see Fig. 2.18) and $\arg(z^{b+1}) \in (-\pi/2, 3\pi/2)$. Then we have:

$z = te^{i\pi/4} - i\epsilon$ on C_4 hence $\int_{C_4} f = \int_R^{1+r} \frac{\log|1-t^4|+i\pi}{|t|^{b+1}e^{i(b+1)\pi/4}} e^{i\pi/4} dt$

$z = te^{i\pi/4} + i\epsilon$ on C_6 hence $\int_{C_6} f = \int_{1+r}^{R} \frac{\log|1-t^4|-i\pi}{|t|^{b+1}e^{i(b+1)\pi/4}} e^{i\pi/4} dt$

$z = te^{i3\pi/4} + i\epsilon$ on C_7 hence $\int_{C_7} f = \int_R^{1+r} \frac{\log|1-t^4|+i\pi}{|t|^{b+1}e^{i(b+1)3\pi/4}} e^{i3\pi/4} dt$

$z = te^{i3\pi/4} - i\epsilon$ on C_9 hence $\int_{C_9} f = \int_{1+r}^{R} \frac{\log|1-t^4|-i\pi}{|t|^{b+1}e^{i(b+1)3\pi/4}} e^{i3\pi/4} dt$

It can be verified that $\int_1^\infty dt/t^{b+1} = 1/b$ for $b > 0$. Then, adding it all together,

$$\int_{C_4+C_6+C_7+C_9} f = -2\pi i\left(e^{-ib\pi/4} + e^{-ib3\pi/4}\right) \int_{1+r}^{R} \frac{dt}{t^{b+1}}$$

$$\to -\frac{2\pi i}{b}\left(e^{-ib\pi/4} + e^{-ib3\pi/4}\right)$$

as $r \to 0, R \to \infty$. Lastly, $z = -t$ on C_1 and $z = t$ on C_2. Recall $\log(1+z^4)$ has argument 0 on the real axis but $\arg z$ (for z^{b+1}) has argument 0 and π on the positive and negative real axis respectively. Then as $r \to 0, R \to \infty$, we have $\int_{C_2} f \to \int_0^\infty \frac{\log|1+t^4|+i0}{|t|^{b+1}e^{i(b+1)0}} dt = J$ and $\int_{C_1} f \to -\int_\infty^0 \frac{\log|1+t^4|+i0}{|t|^{b+1}e^{i(b+1)\pi}} dt = -e^{-ib\pi} J$. Then by Cauchy's theorem,

$$\left(1 - e^{-ib\pi}\right) J - \frac{2\pi i}{b}\left(e^{-ib\pi/4} + e^{-ib3\pi/4}\right) = 0$$

Write $1 - e^{-ib\pi} = \left(e^{ib\pi/2} - e^{-ib\pi/2}\right)/e^{ib\pi/2} = 2ie^{-ib\pi/2}\sin(b\pi/2)$ and simplify,

$$\Rightarrow J = \frac{\pi}{b}\left(\frac{e^{ib\pi/4} + e^{-ib\pi/4}}{\sin(b\pi/2)}\right) = \frac{\pi}{b}\left(\frac{2\cos(b\pi/4)}{\sin(b\pi/2)}\right)$$

$$= \frac{\pi}{b}\left(\frac{2\cos(b\pi/4)}{2\sin(b\pi/4)\cos(b\pi/4)}\right)$$

$$= \frac{\pi}{b}\csc\left(\frac{b\pi}{4}\right)$$

(Question 8)

Evaluate:

$$J = \int_0^\infty \frac{\log(1+x^2)}{x^2(x^4+1)}\,dx$$

1. Construct an appropriate branch of $\log(1+z^2)$. Draw a plot of $\arg(\log(1+z^2))$. Which contour and function will you use? After you answer, check with the solutions before proceeding. It may be helpful to show that $\log(1+z^2)/z^2$ has a removable singularity at zero.

2. Hence find J. You may assume the following result, which is obtained using partial fractions decomposition:

$$\int_1^\infty \frac{1}{t^2(t^4+1)}\,dt = 1 - \frac{\pi + \log\left(3+2\sqrt{2}\right)}{4\sqrt{2}}$$

We offer guided solutions below.

a. Show $\int_{C_R} f \to 0$ as $R \to \infty$ and $\int_{C_5} f \to 0$ as $r \to 0$.

b. Hence find J.

(Solutions to Question 8)

1.

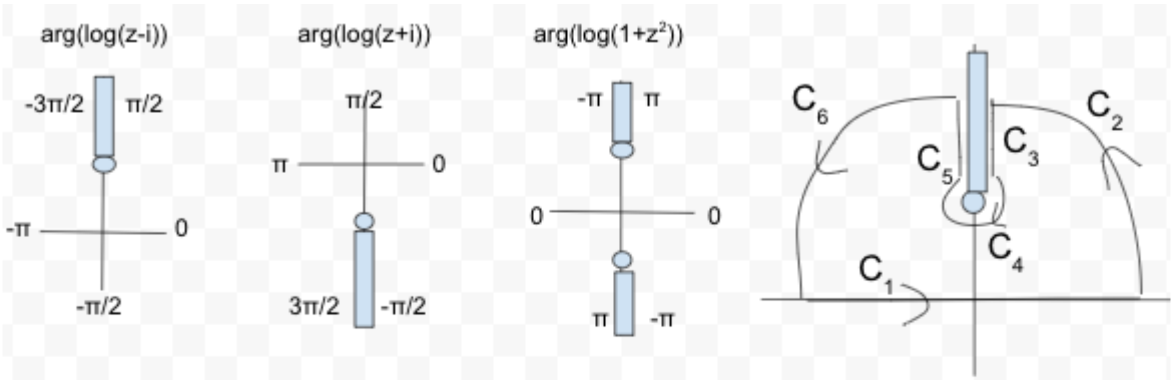


Fig. 2.20

Choose $\log(z-i)$ with branch cut $[i,\infty)$ and $\arg(z-i)\in(-3\pi/2,\pi/2)$ and choose $\log(z+i)$ with branch cut $[-i,\infty)$ and $\arg(z+i)\in(-\pi/2,3\pi/2)$. Then we define $\log(1+z^2)=\log(z-i)+\log(z+i)$ to be our branch. We integrate $f(z)=\frac{\log(1+z^2)}{z^2(z^4+1)}$ over the above semicircular contour. By manually computing the Taylor series of $\log(1+z^2)$ around 0, we see the first nonzero term is the z^2 term (you don't even have to bother finding the coefficient), hence $z=0$ is a removable singularity of f, so we do not need an indent around zero.

2.

Using $|\log(1+z^2)|\leq|\log(z+i)|+|\log(z-i)|\leq 2\log(1+R)$ when $z=Re^{i\theta}$, it is easy to see $\int_{C_2}f,\int_{C_6}f\to 0$ as $R\to\infty$ using ML-lemma. On C_4 we have $z=i+re^{i\theta}$, therefore $|f(z)|\leq\frac{\log(2+r)+\log(r)}{\left|(i+re^{i\theta})^2\right|\left|\left|(i+re^{i\theta})^4\right|-1\right|}\leq\frac{\log(2+r)+\log(r)}{(1-2r-r^2)(4r-6r^2-4r^3-r^4)}$ for r small enough, hence $\left|\int_{C_4}f\right|\leq\frac{2\pi(\log(2+r)+\log(r))}{(1-2r-r^2)(4-6r-4r^2-r^3)}$ which goes to 0 as $r\to 0$. Hence $\int_{C_4}f\to 0$ as $r\to 0$. Since $\log(1+z^2)$ has argument 0 on the real axis and the integrand is even, we see $\int_{C_1}f\to 2J$ as $r\to 0, R\to\infty$. Next we have $z=it+\epsilon$ and $z=it-\epsilon$ on C_3 and C_5 respectively, so $\int_{C_3}f=\int_R^{1+r}\frac{\log|1+(it)^2|+i\pi}{(it)^2((it)^4+1)}idt$ and $\int_{C_5}f=\int_{1+r}^R\frac{\log|1+(it)^2|-i\pi}{(it)^2((it)^4+1)}idt$. Then,

$$\int_{C_4+C_6}f=\int_{1+r}^R\frac{-2\pi}{t^2(t^4+1)}dt\to-2\pi\left(1-\frac{\pi+\log\left(3+2\sqrt{2}\right)}{4\sqrt{2}}\right)$$

as $r\to 0, R\to\infty$. The poles of f in the contour are $e^{i\pi/4}$ and $e^{i3\pi/4}$, both simple poles. Then since $\arg(\log(1+z^2))\in(-\pi,\pi)$, we have:

$$\operatorname{Res}\left(f,e^{i\pi/4}\right)=\lim_{z\to e^{i\pi/4}}\frac{\log(1+z^2)/z^2}{4z^3}=\frac{\log(1+i)}{4e^{i5\pi/4}}=\frac{\log\sqrt{2}+i\pi/4}{4e^{i5\pi/4}}$$

$$\operatorname{Res}\left(f,e^{i3\pi/4}\right)=\lim_{z\to e^{i3\pi/4}}\frac{\log(1+z^2)/z^2}{4z^3}=\frac{\log(1-i)}{4e^{i5\pi/4}}=\frac{\log\sqrt{2}-i\pi/4}{4e^{i15\pi/4}}$$

Then by residue theorem,

$$2J - 2\pi\left(1 - \frac{\pi + \log\left(3 + 2\sqrt{2}\right)}{4\sqrt{2}}\right) = 2\pi i\left(\frac{\log\sqrt{2} + i\pi/4}{4e^{i5\pi/4}} + \frac{\log\sqrt{2} - i\pi/4}{4e^{i15\pi/4}}\right)$$

$$J = \pi\left(1 - \frac{\pi + 2\log\left(6 + 4\sqrt{2}\right)}{8\sqrt{2}}\right)$$

(Question 9)

For $a > 0$, $b \in \mathbb{R}\backslash[0, a]$, and $n \in \mathbb{N}$, evaluate:

$$J = \int_0^a \frac{x^{\frac{n-1}{n}}(a-x)^{\frac{1}{n}}}{b-x}\,dx$$

Note that this integral could be solved using Laurent series, but we have not developed the necessary tools yet. If $n = 1$, it can be shown that $J = a + (b-a)\ln|-1 + a/b|$ using basic integration techniques, so from now on we will assume $n \geq 2$.

1. What contour and function will you choose? After you answer, check with the solutions before proceeding. Keep in mind you will have to construct the branch of your function carefully. Check Example 1 or Example 3 for a refresher.

2. Hence find J. We offer guided solutions below.

   a. Show $\int_{C_2} f \to 0$ and $\int_{C_4} f \to 0$ and $r \to 0$.

   b. Using the fractional residue theorem or otherwise, show that:

$$\lim_{r\to 0}\int_{C_5} f = \begin{cases} 2\pi i|b^{n-1}(a-b)|^{1/n}e^{-i\pi/n} & b < 0 \\ -2\pi i|b^{n-1}(a-b)|^{1/n}e^{-i\pi/n} & b > a \end{cases}$$

   c. Find a branch of $(az-1)^{1/n}$ satisfying:

$$\left(\frac{1}{z}\right)^{\frac{n-1}{n}}\left(a - \frac{1}{z}\right)^{\frac{1}{n}} = \frac{(az-1)^{\frac{1}{n}}}{z}$$

   d. Hence show $\int_{C_R} f \to 2\pi i e^{-i\pi/n}(-b + a/n)$ as $R \to \infty$.

   e. Hence find J.

(Solutions to Question 9)

1.

We wish to integrate $f(z) = \frac{z^{(n-1)/n}(a-z)^{1/n}}{b-z}$ over a dogbone contour. We therefore construct our branch of $z^{(n-1)/n}(a-z)^{1/n}$ by writing:

$$z^{\frac{n-1}{n}}(a-z)^{\frac{1}{n}} = z^{\frac{n-1}{n}}(z-a)^{\frac{1}{n}}(-1)^{\frac{1}{n}}$$

where we pick $z^{(n-1)/n}$ with branch cut $(-\infty, 0]$ and $\arg z \in (-\pi, \pi)$, we pick $(z-a)^{1/n}$ with branch cut $(-\infty, a]$ and $\arg(z-a) \in (-\pi, \pi)$, and we pick $(-1)^{1/n} = e^{-i\pi/n}$ with $\arg e^{-i\pi/n} = -\pi/n$. Then $\arg\left(z^{(n-1)/n}(a-z)^{1/n}\right) = \frac{n-1}{n}\arg z + \frac{1}{n}\arg(z-a) - \frac{\pi}{n}$ hence we can plot some values of $\arg\left(z^{(n-1)/n}(a-z)^{1/n}\right)$ below:

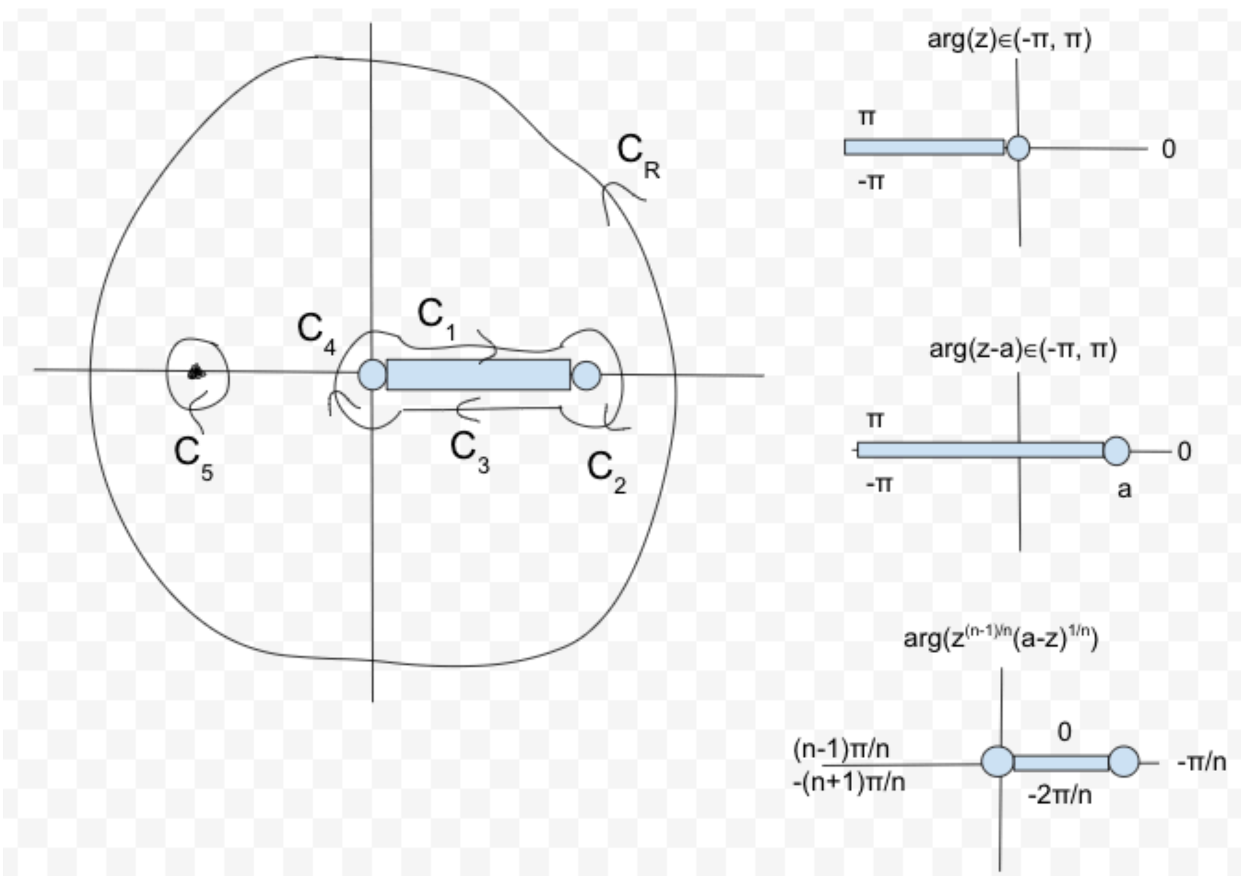


Fig. 2.21

Note $e^{i(n-1)\pi/n} = e^{-i(n+1)\pi/n}$, so our branch of $z^{(n-1)/n}(a-z)^{1/n}$ has branch cut $[0, a]$ and $\arg\left(z^{(n-1)/n}(a-z)^{1/n}\right) \in (-(n+1)\pi/n, 0)$. We have drawn b to be negative, but b could be anywhere in $\mathbb{R}\backslash[0, a]$, and we would just draw a circle around it small enough that it does not intersect the dogbone. The outer circle is large enough to fully contain the circle around b and the dogbone. As usual, C_2, C_4, C_5 have radius $r \to 0$ and C_R has radius $R \to \infty$.

2.
We have $z = re^{i\theta}$ on C_4 so for r small enough, $|f(z)| \leq \frac{r(a+r)}{b-r}$ hence $\left|\int_{C_4} f\right| \leq 2\pi \frac{r^2(a+r)}{b-r}$ which goes to 0 as $r \to 0$, so $\int_{C_4} f \to 0$. We have $z = a + re^{i\theta}$ on C_2 so for r small enough, $|f(z)| \leq \frac{(a+r)r}{|b-|a \pm r||}$ (which sign we take depends on how close b is to a) hence $\left|\int_{C_2} f\right| \leq 2\pi \frac{r^2(a+r)}{|b-|a\pm r||}$ which goes to 0 as $r \to 0$, so $\int_{C_2} f \to 0$. As $r \to 0$, we also find $\int_{C_1} f \to \int_0^a \frac{|t^{(n-1)/n}(a-t)^{1/n}|e^{i0}}{b-t} dt = J$ and $\int_{C_3} f \to \int_a^0 \frac{|t^{(n-1)/n}(a-t)^{1/n}|e^{-i2\pi/n}}{b-t} dt = -e^{-i2\pi/n}J$.
Since $z = b$ is a simple pole and C_5 is a circle arc, the fractional residue theorem says:

$$\int_{C_5} f \to -2\pi i \operatorname{Res}(f, b) = -2\pi i \lim_{z \to b} z^{\frac{n-1}{n}}(a-z)^{\frac{1}{n}} = -2\pi i \left|b^{\frac{n-1}{n}}(a-b)^{\frac{1}{n}}\right| e^{i \arg\left(b^{\frac{n-1}{n}}(a-b)^{\frac{1}{n}}\right)}$$

as $r \to 0$. Looking at Fig. 2.21, if $b > a$ then $\arg\left(b^{(n-1)/n}(a-b)^{1/n}\right) = -\pi/n$. If $b < 0$, then either $\arg\left(b^{(n-1)/n}(a-b)^{1/n}\right)$ is $(n-1)\pi/n$ or $-(n+1)\pi/n$, but both result in $e^{i\arg\left(b^{(n-1)/n}(a-b)^{1/n}\right)}$ having the same value of $-e^{-i\pi/n}$. Hence,

$$\lim_{r \to 0} \int_{C_5} f = \begin{cases} 2\pi i |b^{n-1}(a-b)|^{1/n} e^{-i\pi/n} & b < 0 \\ -2\pi i |b^{n-1}(a-b)|^{1/n} e^{-i\pi/n} & b > a \end{cases}$$

It remains to tackle $\int_{C_R} f$. We have:

$$\int_{C_R} f = -2\pi i \operatorname{Res}(f, \infty) = 2\pi i \operatorname{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right) = 2\pi i \operatorname{Res}\left(\frac{(1/z)^{\frac{n-1}{n}}(a-1/z)^{\frac{1}{n}}}{z(bz-1)}, 0\right)$$

The following steps are similar to what we did in Example 1. We wish to write $(1/z)^{(n-1)/n}(a-1/z)^{1/n} = (az-1)^{1/n}/z$, but we must choose branches for $(az-1)^{1/n}$ and $\arg z$ such that this is true. The argument of $(1/z)^{(n-1)/n}(a-1/z)^{1/n}$ still lies in the interval $(-(n+1)\pi/n, 0)$ since we have only applied inversion. By noting $1/z = \bar{z}/|z|^2$, we can plot $\arg\left((1/z)^{(n-1)/n}(a-1/z)^{1/n}\right)$ below:

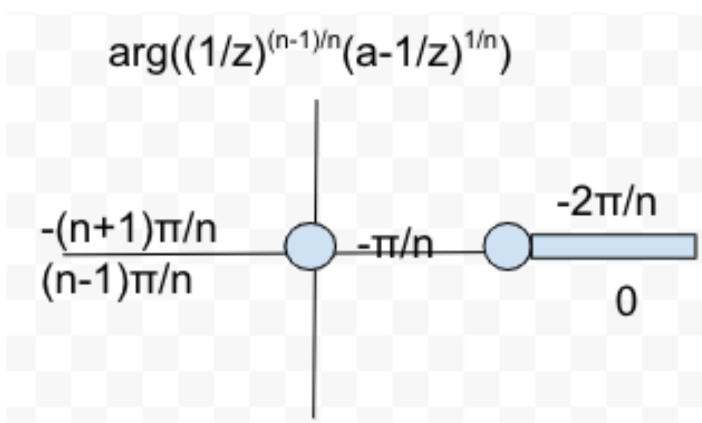


Fig. 2.22

We choose the principal branch of $\arg z$ and the branch of $\arg(az-1)$ with branch cut $[1/a,\infty)$ and $\arg(az-1)\in(-2\pi,0)$, so $\frac{1}{n}\arg(az-1)=\arg(az-1)^{1/n}\in(-2\pi/n,0)$.

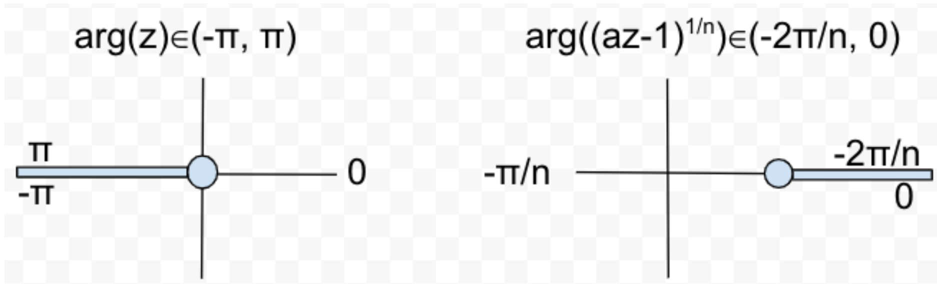


Fig. 2.23

It can be verified that our $\arg(az-1)^{1/n}-\arg z$ will indeed result in the exact same plot as that in Fig. 2.21, that is, our $\arg(az-1)^{1/n}-\arg z$ indeed equals our branch of $\arg\left((1/z)^{(n-1)/n}(a-1/z)^{1/n}\right)$. Then,

$$2\pi i \operatorname{Res}\left(\frac{(1/z)^{\frac{n-1}{n}}(a-1/z)^{\frac{1}{n}}}{z(bz-1)},0\right)=2\pi i\operatorname{Res}\left(\frac{(az-1)^{\frac{1}{n}}}{z^2(bz-1)},0\right)=2\pi i\lim_{z\to 0}\frac{d}{dz}\left[\frac{(az-1)^{\frac{1}{n}}}{bz-1}\right]$$

$$=2\pi i\lim_{z\to 0}\frac{d}{dz}\left[\frac{g(z)}{bz-1}\right]=2\pi i\lim_{z\to 0}\frac{(bz-1)g(z)\frac{1}{n}\left(\frac{a}{az-1}\right)-bg(z)}{(bz-1)^2}$$

where we used chain rule to differentiate $g(z)$. Since $\arg g(z)\in(-2\pi/n,0)$, we must have $\arg g(0)=-\pi/n$, therefore $g(0)=|g(0)|e^{i\arg g(0)}=e^{-i\pi/n}$, so our residue becomes:

$$=2\pi i e^{-i\pi/n}\left(\frac{a}{n}-b\right)$$

There are no poles in the contour $C_R+C_1+\cdots+C_5$, so by Cauchy's theorem, if $b<0$,

$$J-e^{-i2\pi/n}J+2\pi i|b^{n-1}(a-b)|^{1/n}e^{-i\pi/n}+2\pi i e^{-i\pi/n}\left(\frac{a}{n}-b\right)=0$$

We write $1-e^{-i2\pi/n}=\left(e^{i\pi/n}-e^{-i\pi/n}\right)/e^{i\pi/n}=2ie^{-i\pi/n}\sin(\pi/n)$ and simplify,

$$\Rightarrow J=\pi\left(-|b^{n-1}(a-b)|^{1/n}-\frac{a}{n}+b\right)\csc\left(\frac{\pi}{n}\right)$$

We do a similar calculation for $b>a$. Then,

$$J=\begin{cases}\pi\left(-|b^{n-1}(a-b)|^{1/n}-\frac{a}{n}+b\right)\csc\left(\frac{\pi}{n}\right) & b<0\\ \pi\left(|b^{n-1}(a-b)|^{1/n}-\frac{a}{n}+b\right)\csc\left(\frac{\pi}{n}\right) & b>a\end{cases}$$

Taylor and Laurent Series of Functions with Branch Cuts

Finding the Laurent series of simpler functions is sometimes not so hard. The important thing to recognize is that different branches of a function will have different Laurent series. Although it is usually easier to calculate residues using tricks and theorems, it is sometimes easier to calculate the residue at infinity by finding the Laurent series.

Example 2.1
As a simple example, let $f(z)$ and $g(z)$ both be $\log(z)$ with branch cut $(-\infty, 0]$ but f has $\arg z \in (-\pi, \pi)$ and g has $\arg z \in (\pi, 3\pi)$. We find the Laurent series of f and g centered at $z = 1$ on the disk $D = \{|z-1| < 1\}$. Since f and g are analytic on D, the Laurent series is just a Taylor series $\sum_{k=0}^{\infty} a_k (z-1)^k$ where $a_k = f^{(k)}(1)/k!$. Since the derivatives of $\log(z)$ are rational functions $1/z$, $-1/z^2$, etc. that do not depend on branch, the only term that will differ in the Taylor series of f and g are the constant terms. Indeed, we find that:

$$f(z) = 0 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(z-1)^k \qquad g(z) = 2\pi i + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(z-1)^k$$

Example 2.2
For another example, we find all Laurent series of $\sqrt{z}$ with branch cut $(-\infty, 0]$ centered at $z = 2$. Recall that there are only two such branches, which are $f_1(z) \coloneqq +g(z)$ and $f_2(z) \coloneqq -g(z)$ where $g(z)$ is the principal branch of $\sqrt{z}$. Each branch only has one Laurent series around $z = 2$ which must converge on $D \coloneqq \{|z-2| < 2\}$. Note the Laurent series is once again a Taylor series since $\sqrt{z}$ is analytic on D. Note that f_1 is positive on the positive real axis but f_2 is negative on the positive real axis. Then to find our Taylor series, we again compute the coefficients manually using $a_k = f_1^{(k)}(2)/k!$ to find $f_1'(z) = f_1(z)/2z$, $f_1''(z) = -f_1(z)/4z^2$, … and $f_1(2) = \sqrt{2}$. Then,

$$f_1(z) = \sqrt{2} + \frac{\sqrt{2}}{4}(z-2) - \frac{\sqrt{2}}{2!\,16}(z-2)^2 + \frac{3\sqrt{2}}{3!\,64}(z-2)^3 + \cdots$$

Then the Taylor series of f_2 is just the negative of the Taylor series of f_1,

$$f_2(z) = -\sqrt{2} - \frac{\sqrt{2}}{4}(z-2) + \frac{\sqrt{2}}{2!\,16}(z-2)^2 - \frac{3\sqrt{2}}{3!\,64}(z-2)^3 - \cdots$$

Especially for functions of the form $z^\alpha, \alpha \in \mathbb{C}$, finding Taylor series by computing derivatives is tedious. There is sometimes a better way using binomial coefficients.

Definition 2.3
For $a \in \mathbb{C}$ and $n \in \mathbb{N}_{\geq 0}$ we define the binomial coefficient “a choose n” by

$$\binom{a}{0} = 1 \qquad \binom{a}{n} = \frac{a(a-1)\dots(a-n+1)}{n!}, n \geq 1$$

Theorem 2.4
Binomial Theorem: Let $a \in \mathbb{C}$ and let $g(z)$ be an analytic branch of $(1+z)^a$ defined on the open disk $D \coloneqq \{|z|<1\}$. Then for all $z \in D$, we have:

$$g(z) = g(0)\sum_{k=0}^{\infty}\binom{a}{k}z^k$$

Proof: First we determine the convergence of the series using the ratio test.

$$\lim_{k\to\infty}\left|\frac{\binom{a}{k+1}z^{k+1}}{\binom{a}{k}z^k}\right| = \lim_{k\to\infty}\frac{|a-k|}{k+1}|z| = \lim_{k\to\infty}\frac{\sqrt{(\mathrm{Re}(a)-k)^2+\left(\mathrm{Im}(a)\right)^2}}{k+1}|z| = |z|$$

Hence the series converges for $|z|<1$, that is, on D. To prove $g(z) = g(0)\sum_{k=0}^{\infty}\binom{a}{k}z^k$ on D, let $f(z) \coloneqq \sum_{k=0}^{\infty}\binom{a}{k}z^k$. Since f is a convergent power series on D, f is analytic on D and we can use term-by-term differentiation to find f',

$$f'(z) = \underbrace{\sum_{k=1}^{\infty}\binom{a}{k}kz^{k-1}}_{(1)} = \sum_{k=0}^{\infty}\binom{a}{k+1}(k+1)z^k = \underbrace{\sum_{k=0}^{\infty}\binom{a}{k}(a-k)z^k}_{(2)}$$

where the above series also converge on D. Then,

$$(1+z)f'(z) = f'(z) + z'f(z) = \underbrace{\sum_{k=0}^{\infty}\binom{a}{k}(a-k)z^k}_{(2)} + z\underbrace{\sum_{k=1}^{\infty}\binom{a}{k}kz^{k-1}}_{(1)}$$

$$= \sum_{k=0}^{\infty}\binom{a}{k}(a-k)z^k + \sum_{k=0}^{\infty}\binom{a}{k}kz^k = af(z)$$

Hence $(1+z)f'(z) - af(z) = 0$ on D. Now consider $f(z)/g(z)$. Note $g(z) \neq 0$ on D. Then by writing $g(z) = e^{a\log(1+z)}$, we have:

$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z)f'(z) - f(z)g(z)\frac{a}{1+z}}{\left(g(z)\right)^2}$$

$$= \frac{1}{g(z)(1+z)}\left((1+z)f'(z) - af(z)\right) = 0$$

for all $z \in D$. Then $f(z)/g(z)$ is constant on D, that is, $f(z)/g(z) = c$ on D for some fixed $c \in \mathbb{C}$. Plugging in $z=0$ and noting $f(0) = \binom{a}{0} = 1$, we find $1/g(0) = c$. Therefore, we have $g(z) = g(0)f(z)$ on D as desired.

Example 2.5
For an example of using the binomial theorem, we return to the calculation of the residue at infinity from Example 1. Back then, we had calculated the residue of $f(z) = \sqrt{1-z^2}/(1+z^2)$ at infinity using $\mathrm{Res}(f,\infty) = -\mathrm{Res}\big((1/z^2)f(1/z),0\big)$ since this residue at zero was more accessible. However, we are now able to find the Laurent series of f at infinity, allowing us to find $\mathrm{Res}(f(z),\infty)$ directly. We need the Laurent series of f that converges on $\{|z| \geq R\}$ for some $R > 0$. We will find the Laurent series of our branch of $\sqrt{1-z^2}$ and $1/(1+z^2)$ separately and then multiply them together. We can write any branch of $\sqrt{1-z^2}$ in terms of some branch of $\sqrt{1-1/z^2}$ as follows:

$$\sqrt{1-z^2} = \sqrt{-z^2\left(1-\frac{1}{z^2}\right)} = \sqrt{-1}\, z\sqrt{1-\frac{1}{z^2}} = \pm iz\sqrt{1-\frac{1}{z^2}}$$

Previously, we would draw a plot of $\arg\sqrt{1-1/z^2}$ and carefully work out which branch of $\sqrt{1-1/z^2}$ and which $\pm$ sign would lead to the particular branch of $\sqrt{1-z^2}$ that we decided to use. However, we will now find all possible Laurent series for $\sqrt{1-z^2}$ on $\{|z| \geq R\}$ and pick the one that corresponds to our particular branch. Let $w = -1/z^2$. Then for $|w| < 1$, we can write $\sqrt{1+w}$ as a Laurent series using binomial coefficients. At $w = 0$, we have $\sqrt{1+w} = \sqrt{1} = \pm 1$. Then by the binomial theorem,

$$\sqrt{1-\frac{1}{z^2}} = \sqrt{1+w} = \pm\sum_{k=0}^{\infty}\binom{1/2}{k}w^k = \pm\sum_{k=0}^{\infty}\binom{1/2}{k}\left(-\frac{1}{z^2}\right)^k$$

Therefore,

$$\sqrt{1-z^2} = \pm i\sum_{k=0}^{\infty}\binom{1/2}{k}\frac{(-1)^k}{z^{2k-1}} = \pm i\left(z - \binom{1/2}{1}\frac{1}{z} + \binom{1/2}{2}\frac{1}{z^3} - \cdots\right)$$

So the above series converges for $|w| < 1$ which is if and only if $|z| > 1$. So we have found the possible Laurent series for $\sqrt{1-z^2}$ that converge on $\{|z| > 1\}$. Looking at the plot of $\arg\sqrt{1-z^2}$ in Fig. 2.2 (shown again below), we see that the branch of $\sqrt{1-z^2}$ we used in Example 1 is positive imaginary on the positive real axis (note $e^{i\pi/2} = i$).

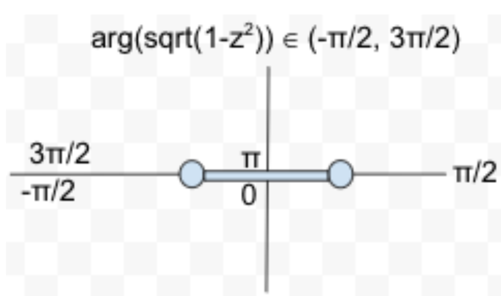


Fig. 3.1

Then, by taking $z \to \infty$ along the positive real axis in our Laurent series (so all terms will vanish except the first term), it is clear that our particular branch of $\sqrt{1-z^2}$ must have the Laurent series:

$$i\left(z-\binom{1/2}{1}\frac{1}{z}+\binom{1/2}{2}\frac{1}{z^3}-\cdots\right)$$

which converges for $|z|>1$.

We now must find the Laurent series of $1/(1+z^2)$ that converges for $|z|>1$. Luckily, this is much more straightforward. For $|z|>1$, we have:

$$\frac{1}{1+z^2}=\frac{1/z^2}{1+1/z^2}=\frac{1}{z^2}\sum_{k=0}^{\infty}\left(-\frac{1}{z^2}\right)^k=\frac{1}{z^2}-\frac{1}{z^4}+\frac{1}{z^6}-\cdots$$

Multiplying the two Laurent series together,

$$f(z)=\frac{\sqrt{1-z^2}}{1+z^2}=i\left(z-\binom{1/2}{1}\frac{1}{z}+\binom{1/2}{2}\frac{1}{z^3}-\cdots\right)\left(\frac{1}{z^2}-\frac{1}{z^4}+\frac{1}{z^6}-\cdots\right)$$

The residue of $f(z)$ at infinity is defined as the *negative* of the coefficient of z^{-1} in this expansion, so $\mathrm{Res}(f,\infty)=-i$, which is consistent with our previous work in Example 1.

Was it faster to use Laurent series to find the residue of f at infinity, or was it faster to calculate the residue of $\frac{1}{z^2}f\left(\frac{1}{z}\right)$ at zero like we did in Example 1? It really does depend on the integrand. If the integrand is the product of functions with relatively straightforward Laurent series, then perhaps Laurent series are the way to go. On the other hand, it is not immediately obvious how to find the Laurent series of $\left(\frac{1-z}{1+z}\right)^{1/6}$.

(Question 10)

Let $a < b$ be real numbers. Using contour integration, evaluate:

$$J = \int_a^b \sqrt{(x-a)(b-x)}\, dx$$

1. Let $c = b - a$. Show that:

$$J = \int_0^c \sqrt{x(c-x)} dx$$

2. Which function and contour will you choose? After you answer, check with the solutions before proceeding.

3. Show $\int_{C_1+C_2+C_3+C_4} f \to 2J$ as $r \to 0$.

4. By finding the appropriate Laurent series of f, calculate $\mathrm{Res}(f, \infty)$. Hence find $\lim\limits_{R\to\infty} \int_{C_R} f$.

5. Hence find J.

(Solutions to Question 10)

1.
Use the substitution $x - a \mapsto x$.

2.
We will integrate a branch of $\sqrt{z(c-z)}$ over a dogbone contour traversed clockwise with an extra outer circle of radius $R \to \infty$ traversed counterclockwise. We construct our branch as follows: pick $\sqrt{z}$ with branch cut $(-\infty, 0]$ and $\arg z \in (-\pi, \pi)$, pick $\sqrt{z-c}$ with branch cut $(-\infty, c]$ and $\arg(z-c) \in (-\pi, \pi)$, and pick $\sqrt{-1} = -i$ with $\arg(-i) = -\pi/2$. Then we will define $f(z) = \sqrt{z}\sqrt{z-c}\sqrt{-1} = -i\sqrt{z}\sqrt{z-c}$ to be our branch of $\sqrt{z(c-z)}$, so $\arg\sqrt{z(c-z)} = \frac{1}{2}\arg z + \frac{1}{2}\arg(z-c) - \frac{\pi}{2}$. Then we plot $\arg\sqrt{z(c-z)}$ below:

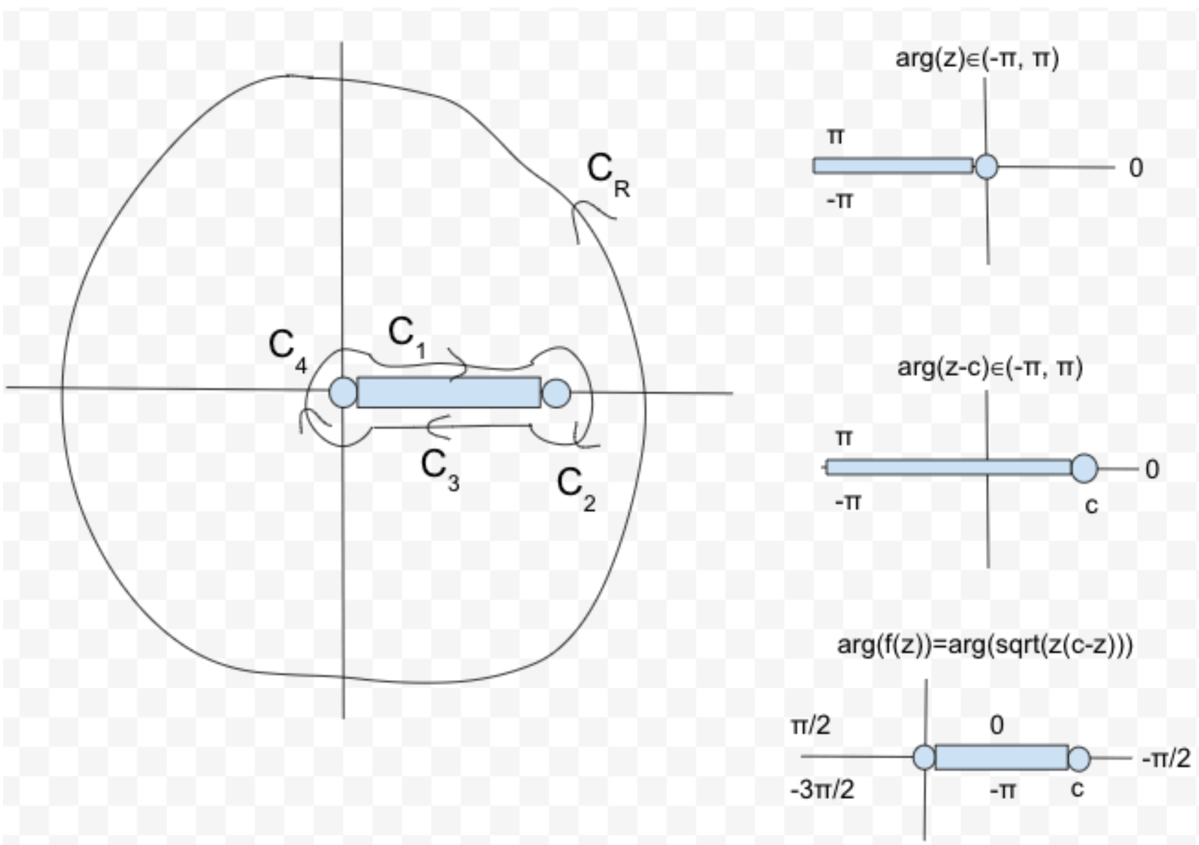


Fig. 3.2

So our branch of $\sqrt{z(c-z)}$ has branch cut $[0, c]$ and $\arg\sqrt{z(c-z)} \in (-3\pi/2, \pi/2)$.

3.
On C_2, we have $z = c + re^{i\theta}$, so $|f(z)| = \left|\sqrt{(c + re^{i\theta})re^{i\theta}}\right| \leq \sqrt{(c + r)r}$ therefore $\left|\int_{C_2} f\right| \leq 2\pi r\sqrt{(c + r)r}$ which clearly goes to 0 as $r \to 0$, so $\int_{C_2} f \to 0$ as $r \to 0$. Similarly, $\int_{C_4} f \to 0$ as $r \to 0$. On C_1, we have $z = t + i\epsilon$ so $\int_{C_1} f \to \int_{-1}^{1} \sqrt{t(c - t)}\, e^{i0} = J$ as $r \to 0$. On C_3, we have $z = t - i\epsilon$ so $\int_{C_3} f \to \int_{1}^{-1} \sqrt{t(c - t)}\, e^{-i\pi} dt = J$ as $r \to 0$.

4.
We need the Laurent series of f that converges on $\{z \geq \rho\}$ for some $\rho > 0$. For some branch of $\sqrt{1 - c/z}$, we have:

$$f(z) = \sqrt{z(c - z)} = \sqrt{-z^2\left(1 - \frac{c}{z}\right)} = \sqrt{-1}\, z\sqrt{1 - \frac{c}{z}} = \pm iz\sqrt{1 - \frac{c}{z}}$$

For $|-c/z| < 1$, that is, for $|z| > c$, we have $\sqrt{1 - \frac{c}{z}} = \pm\sum_{k=0}^{\infty}\binom{1/2}{k}\left(-\frac{c}{z}\right)^k$, therefore

$$f(z) = \pm i\sum_{k=0}^{\infty}\binom{1/2}{k}\frac{(-c)^k}{z^{k-1}} = \pm i\left(z - c\binom{1/2}{1} + c^2\binom{1/2}{2}\frac{1}{z} - c^3\binom{1/2}{3}\frac{1}{z^2} + \cdots\right)$$

Looking at Fig. 3.2, our f is negative imaginary on the positive real axis. Taking $z \to \infty$ along the positive real axis in our Laurent series (so all terms will vanish except the first two terms), we see that our particular branch has Laurent series

$$f(z) = -i\left(z - c\binom{1/2}{1} + c^2\binom{1/2}{2}\frac{1}{z} - c^3\binom{1/2}{3}\frac{1}{z^2} + \cdots\right)$$

Then $\mathrm{Res}(f, \infty)$ is the negative of the coefficient of z^{-1},

$$\mathrm{Res}(f, \infty) = i\binom{1/2}{2}c^2 = i\frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!}c^2 = -\frac{1}{8}ic^2$$

Hence $\lim_{R\to\infty}\int_{C_R} f = -2\pi i\,\mathrm{Res}(f, \infty) = -\pi c^2/4 = -\pi(b - a)^2/4$.

5.
There are no poles in the contour $C_1 + C_2 + C_3 + C_4 + C_R$, so by Cauchy's theorem,

$$2J - \frac{\pi}{4}(b - a)^2 = 0$$

$$J = \frac{\pi}{8}(b - a)^2$$

More Practice Questions

1. For $0 < a < b$, evaluate:
$$\int_0^\infty \frac{x^{a-1}}{x^b+1}\,dx$$
2. Let $b > 1$. By integrating a branch of $\frac{1}{z^b-1}$ around an appropriate contour, evaluate:
$$\mathrm{PV}\int_0^\infty \frac{1}{x^b-1}\,dx$$
3. For $a > 0$, evaluate:
$$\int_0^\infty \frac{\log x}{\sqrt{x}(a+x)^2}\,dx$$
4. For $b > 0$, evaluate:
$$\int_0^\infty \frac{\log^2 x}{x^2+b^2}\,dx$$
5. By integrating a branch of $\frac{\log^2 z}{z^2-1}$ around an appropriate contour and using the result $\int_0^\infty \frac{\log x}{x^2+1}\,dx = 0$, evaluate:
$$\int_0^\infty \frac{\log^2 x}{x^2-1}\,dx$$
6. For $a < b$ and $n \in \mathbb{Z}^+$, prove that:
$$\int_a^b \frac{x^n}{\sqrt{(x-a)(b-x)}}\,dx = \pi \sum_{k=0}^{n} (-1)^k \binom{-1/2}{k}\binom{n}{k}(b-a)^k a^{n-k}$$
7. Let $0 < a < 1$ and $a = m/n$ where $m, n \in \mathbb{Z}^+$, n is odd, and $\gcd(m,n) = 1$. Also let $b > 0$ and $c \in \mathbb{R}\backslash[-b, b]$. Prove that:[9]
$$\int_{-b}^{b} \frac{1}{x-c}\left(\frac{x-b}{x+b}\right)^a dx = \pi\left(\left(\frac{c-b}{c+b}\right)^a - 1\right)\csc(\pi a)$$
8. Let $b > 0$. By integrating a branch of $\frac{\log(1+z^3)}{z^2+b^2}$ over an appropriate contour, evaluate,
$$\int_1^\infty \frac{x^2+b^2}{x^4-b^2x^2+b^4}\,dx$$

[9] Essentially we want a to be a rational number in $(0, 1)$ in lowest terms with odd denominator. The odd denominator ensures $(c-b)^a$ and $(c+b)^a$ are well-defined for all $c \in \mathbb{R}\backslash[-b, b]$.

9. Let $n \in \mathbb{N}$, $n \geq 2$ and define $f(z) = z^n \log z$ where $\log z$ is chosen with branch cut $[0, \infty)$ and $\arg z \in (0, 2\pi)$. Using induction, prove that for $m \in \mathbb{N}$, $1 \leq m \leq n$,

$$f^{(m)}(z) = z^{n-m} \frac{n!}{(n-m)!} \left(\log(z) + \sum_{k=0}^{m-1} \frac{1}{n-k} \right)$$

where $f^{(m)}(z)$ is the m^{th} derivative of $f(z)$. Hence find an expression for $f^{(m)}(z)$ when $m \geq n+1$ and subsequently use residue theory to evaluate:

$$\int_0^\infty \frac{x^n}{(1+x)^{2n}} dx$$

Appendix 1: Miscellaneous Review

Let z_0 be an isolated singularity of $f(z)$ and $f(z)$ has the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

which converges for $0 < |z - z_0| < \rho$ for some $\rho > 0$. The principal part of the series is defined to be:

$$\sum_{n=-\infty}^{-1} a_n (z - z_0)^n = \frac{a_{-1}}{z - z_0} + \frac{a_{-2}}{(z - z_0)^2} + \cdots$$

1. If there are no terms with negative powers of $z - z_0$, that is, if the principal part is zero, then we say z_0 is a removable singularity of f.

2. If $a_{-k} \neq 0$ for some $k \in \mathbb{N}$ and $a_{-n} = 0$ for all $n > k$, then we say that z_0 is a pole of f of order k. Poles of order one are called simple poles.

3. If the principal part is infinite, so there are infinitely many terms with negative powers of $z - z_0$, then z_0 is an essential singularity of f.

Example: Define $g(z) = \log z / (z^b - 1)$ for $b \in \mathbb{R}, b > 1$ where $\log z$ and z^b are both defined on the positive real axis, $\log(1) = 0$, and $1^b = 1$. Prove that $z = 1$ is a removable singularity of g.

So our $\log z$ and $z^b - 1$ are analytic at $z = 1$, so they have Taylor expansions around $z = 1$ that converge on a neighborhood around $z = 1$. Recall that for a function f, we can calculate its Taylor coefficients using $f^{(k)}(z_0)/k!$ and therefore:

$$\log z = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \frac{(z-1)^4}{4} + \cdots$$

$$z^b - 1 = b(z-1) + \frac{b(b-1)}{2}(z-1)^2 + \frac{b(b-1)(b-2)}{3}(z-1)^3 + \cdots$$

Then,

$$g(z) = \frac{\log z}{z^b - 1} = \frac{1 - \frac{1}{2}(z-1) + \cdots}{b + \frac{b(b-1)}{2}(z-1) + \cdots}$$

By the uniqueness of the Laurent expansion, this must equal the Laurent series $\sum_{n=-\infty}^{\infty} a_n (z-1)^n$ of $g(z)$ centered at $z = 1$. But plugging in $z = 1$ yields $1/b$, so the Laurent series is in fact defined at $z = 1$ (with $a_0 = 1/b$) and therefore has zero principal part. So $z = 1$ is a removable singularity of g. Alternatively, one could simply use L'Hopital's rule to show $\lim_{z \to 1} \frac{\log z}{z^b - 1} = 1/b$.

Calculating Residues

Let z_0 be an isolated singularity of $f(z)$ and $f(z)$ has the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

which converges for $0 < |z - z_0| < \rho$ for some $\rho > 0$. Then the residue of $f(z)$ at z_0, denoted $\mathrm{Res}(f, z_0)$, is defined to be the coefficient a_{-1} of $1/(z - z_0)$ in this expansion. Importantly, the residue at z_0 is obtained from the Laurent series that converges *immediately around* z_0.

To calculate residues, we list some useful results below.

1. If f has a removable singularity at z_0, then $\mathrm{Res}(f, z_0) = 0$.

   Proof: Since z_0 is a removable singularity, the Laurent series of f that converges immediately around z_0 has no principal part, so the residue is zero.

2. If f has a simple pole or removable singularity at z_0, then

   $$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

   Proof: Follows immediately from considering the appropriate Laurent series of f.

3. If $f(z)$ has a pole of order k at z_0, then

   $$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} [(z - z_0)^k f(z)]$$

   Proof: Consider the appropriate Laurent series of f and use term-by-term differentiation. The case $k = 2$ becomes $\mathrm{Res}(f, z_0) = \lim_{z \to z_0} \frac{d}{dz} [(z - z_0)^2 f(z)]$.

4. If f and g are analytic at z_0 and g has a simple zero at z_0, then

   $$\mathrm{Res}\left(\frac{f}{g}, z_0\right) = \frac{f(z_0)}{g'(z_0)}$$

   Proof: So f/g either has a simple pole or a removable singularity at z_0. Noting $g(z_0) = 0$, we can use the second result to conclude:

   $$\mathrm{Res}\left(\frac{f}{g}, z_0\right) = \lim_{z \to z_0} (z - z_0) \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f(z)}{\frac{g(z) - g(z_0)}{z - z_0}} = \frac{f(z_0)}{g'(z_0)}$$

   A useful corollary is the case $f(z) = 1$, that is, $\mathrm{Res}(1/g, z_0) = 1/g'(z_0)$ when g has a simple zero at z_0.

The Residue at Infinity

Let $f(z)$ be analytic for $|z| \geq R$ for some $R > 0$ with Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ that converges for $|z| \geq R$. Then we define $\mathrm{Res}(f, \infty) = -a_{-1}$. Note the additional negative sign.

In the case that there is a piecewise smooth closed curve C traversed counterclockwise such that $f(z)$ is meromorphic inside C but analytic outside C, then we additionally have:

$$\mathrm{Res}(f, \infty) = -\frac{1}{2\pi i}\int_C f(z)dz = -\mathrm{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right)$$

We emphasize again that C is being traversed counterclockwise. The proof of the first equality goes as follows: let C_R be a circle (traversed counterclockwise) of radius R large enough such that C is entirely contained inside C_R. Since f is analytic outside C, we have $\int_{C_R - C} f = 0$ by Cauchy's theorem, or $\int_C f = \int_{C_R} f$. Since $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ for $|z| \geq R$, we have

$$-\frac{1}{2\pi i}\int_C f(z)dz = -\frac{1}{2\pi i}\int_{C_R} f(z)dz = -\frac{1}{2\pi i}\int_{C_R} \sum_{n=-\infty}^{\infty} a_n z^n \, dz$$

Laurent series converge uniformly so we can interchange limits as needed. We parameterize $z = Re^{i\theta}$,

$$= -\frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_0^{2\pi} a_n R^n e^{in\theta}\, Rie^{i\theta} d\theta = -\frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_0^{2\pi} a_n R^n e^{in\theta}\, Rie^{i\theta} d\theta$$

Note that $\int_0^{2\pi} e^{i(n+1)\theta}\, d\theta$ is zero for all integers $n \neq -1$ but equals 2π when $n = -1$,

$$= -\frac{1}{2\pi i} 2\pi i a_{-1} = -a_{-1} = \mathrm{Res}(f, \infty)$$

To prove the second equality, we recommend an online article about the residue at infinity, see the references section.[10] But if you wish to try the proof for yourself, apply the inversion $z \mapsto 1/z$ to the integral $\int_{C_R} f(z)dz$. What will this inversion do to C_R and the isolated singularities of f inside C_R? After sorting out the integral, apply the residue theorem.

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[10] Orloff, Jeremy. *9.6: Residue at ∞*. LibreTexts, https://math.libretexts.org/Bookshelves/Analysis/Complex_Variables_with_Applications_(Orloff)/09%3A_Residue_Theorem/9.06%3A_Residue_at.

The Residue Theorem

Let C be a piecewise smooth closed curve traversed counterclockwise and suppose $f(z)$ is analytic on a region containing C except at a finite number of isolated singularities $z_1, \dots, z_n$ that are inside C. Then,

$$\int_C f(z)dz = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(f, z_j)$$

Since we have discussed the residue at infinity, we present a slightly more interesting view of the residue theorem by considering the Riemann sphere. Suppose we map the extended complex plane $\mathbb{C} \cup \{\infty\}$ to a sphere using a stereographic projection. Note that $z = \infty$ is mapped to the north pole of the sphere. If f were analytic on $\mathbb{C}$ except at the isolated singularities $z_1, \dots, z_n$, then f is analytic on the Riemann sphere except at $z_1, \dots, z_n$ and possibly ∞.

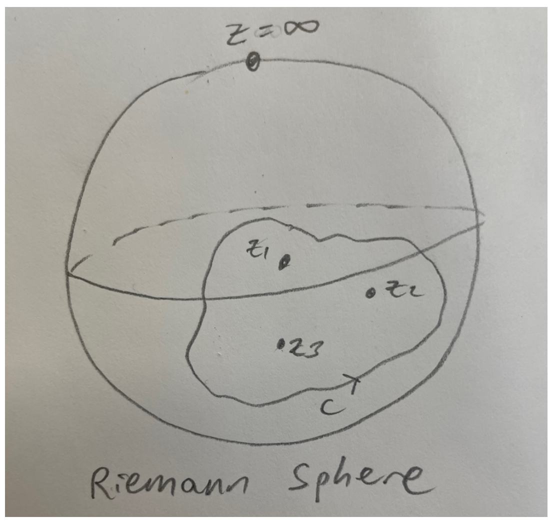

Fig. A1

If the curve C is traversed counterclockwise, then the interior of C is the region containing z_1, z_2, z_3 and the residue theorem tells us $\int_C f(z)dz = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(f, z_i)$. However, if we now traversed C *clockwise*, then the interior is now the region *not* containing z_1, z_2, z_3 and the only possibly nonzero residue inside C is the one at infinity. Then the residue theorem immediately gives us the result from the previous page:

$$\mathrm{Res}(f, \infty) = \frac{1}{2\pi i} \int_{-C} f(z)dz = -\frac{1}{2\pi i} \int_C f(z)dz$$

where C is traversed counterclockwise. Note $\mathrm{Res}(f, \infty) = 0$ if f is analytic at ∞.

We will not prove the residue theorem, there is plenty of that elsewhere. However, if you need a reminder, the proof comes from integrating the appropriate Laurent series of f around a small circle around each of the isolated singularities $z_1, \dots, z_n$.

Fractional Residue Theorem

Next is a useful result about integrating over circular arcs around simple poles. One textbook[11] calls it the fractional residue theorem, but it seems like no one else does.

(Fractional residue theorem)
Let z_0 be a simple pole of $f(z)$ and let C_r be the arc of the circle $\{|z - z_0| = r\}$ of angle α traversed counterclockwise. Then,

$$\lim_{r\to 0} \int_{C_r} f(z)\, dz = \alpha i \operatorname{Res}(f, z_0)$$

So α would be negative if the curve was traversed clockwise.

Proof: Since z_0 is a simple pole of f, we can write f as a Laurent series that converges on $\{0 < |z - z_0| < \rho\}$ for some $\rho > 0$.

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$$

Parameterizing the arc C_r with $z = z_0 + re^{i\theta}$, $\theta_0 < \theta < \theta_0 + \alpha$ where θ_0 is the starting angle,

$$\lim_{r\to 0} \int_{C_r} f(z)\, dz = \lim_{r\to 0} \int_{\theta_0}^{\theta_0+\alpha} \left(\frac{a_{-1}}{re^{i\theta}} + a_0 + a_1 re^{i\theta} + \cdots\right) rie^{i\theta} d\theta$$

$$= \lim_{r\to 0} \int_{\theta_0}^{\theta_0+\alpha} \left(a_{-1} + a_0 re^{i\theta} + a_1 r^2 e^{i2\theta} + \cdots\right) i d\theta$$

Whether you integrate term-by-term or pass the limit into the integral (note Laurent series converge uniformly, allowing these manipulations), the desired result follows.

[11] Gamelin, Theodore W. *Complex Analysis*. Springer, 2001

Cauchy Principal Value

Consider the integral $\int_{-\infty}^{\infty}(1/x)dx$. We know this integral diverges, but if we graph $1/x$, it looks like the integral *should* equal zero since the area to the right of the line $x = 0$ is perfectly cancelled out by the (negatively signed) area to the left. The principal value of an integral allows us to assign such a value to otherwise divergent integrals.

Suppose $f(x)$ is continuous on $[a, x_0)$ and $(x_0, b]$. We define the *principal value* of the integral $\int_a^b f(x)dx$ to be

$$\mathrm{PV}\int_a^b f(x)dx = \lim_{\epsilon\to 0}\left(\int_a^{x_0-\epsilon} f(x)dx + \int_{x_0+\epsilon}^b f(x)dx\right)$$

when the limit exists. If f is not continuous at a or b, or $a = -\infty$ or $b = \infty$, then it is required for f to be absolutely integrable on $(a, x_0 - r)$ and $(x_0 + r, b)$ for all $r > 0$ in order for the principal value to exist.

The principal value is similarly defined if f has multiple isolated discontinuities in $[a, b]$. If $f(x)$ is continuous on $[a, x_0), (x_0, x_1), \ldots, (x_{n-1}, x_n)$, and $(x_n, b]$, then

$$\mathrm{PV}\int_a^b f(x)dx = \lim_{\epsilon_0,\ldots,\epsilon_n\to 0}\left(\int_a^{x_0-\epsilon_0} + \int_{x_0+\epsilon_0}^{x_1-\epsilon_1} + \cdots + \int_{x_n+\epsilon_n}^{b}\right) f(x)dx$$

when the limits exist.

Referring to the earlier example of $1/x$, it is not hard to verify that $\mathrm{PV}\int_{-\infty}^{\infty}(1/x)dx = 0$.

The principal value of an integral equals the value of that integral when the integral converges.

Appendix 2

Passing the limit $\epsilon \to 0$ under the integral as follows:

$$\lim_{\epsilon\to 0}\int_{-1+r}^{1-r} \underbrace{\frac{\left|\sqrt{1-(t+i\epsilon)^2}\right| e^{i\arg\sqrt{1-(t+i\epsilon)^2}}}{1+(t+i\epsilon)^2}}_{h(t+i\epsilon)} dt = \int_{-1+r}^{1-r} \lim_{\epsilon\to 0}\frac{\left|\sqrt{1-(t+i\epsilon)^2}\right| e^{i\arg\sqrt{1-(t+i\epsilon)^2}}}{1+(t+i\epsilon)^2}$$

$$= \int_{-1+r}^{1-r} \frac{\left|\sqrt{1-t^2}\right| e^{i\arg\sqrt{1-t^2}}}{1+t^2}$$

needs some justification. The reason boils down to the fact that we are integrating over a compact set, in this case $[-1+r, 1-r]$. Let our integrand be denoted $h(t+i\epsilon)$.

Lemma 1: A continuous function over a compact set is uniformly continuous on that set.

This result is proven in most real analysis courses, and can also be shown to hold in $\mathbb{C}$. In this case, our integrand $h(t+i\epsilon)$ is indeed continuous on $[-1+r, 1-r]$ for any fixed $r > 0$ and for all $\epsilon > 0$, and is therefore uniformly continuous on $[-1+r, 1-r]$.

Lemma 2: Suppose $f(z)$ is uniformly continuous on K and $f(z + c/n)$ is defined on K for all $n \in \mathbb{N}$ and fixed $c \in \mathbb{C}$. Then $f(z + c/n) \to f$ uniformly as $n \to \infty$.

Since $f(z)$ is uniformly continuous on K, for all $\epsilon > 0$ there is $\delta > 0$ such that for all $w, w + c/n \in K$, $|w - (w + c/n)| = |c|/n < \delta$ implies $|f(w) - f(w + c/n)| < \epsilon$. In that case, given any $\epsilon > 0$, pick $N > |c|/\delta$. Then $|c|/n < |c|/N < \delta$, so we immediately have $|f(w) - f(w + c/n)| < \epsilon$ from before, hence $f(z + c/n) \to f$ uniformly as $n \to \infty$.

Then, by writing $\epsilon = 1/n$ (so $n \to \infty \Leftrightarrow \epsilon \to 0$) and by taking $c = i$ in Lemma 2, we see that $h(t+i\epsilon) \to h(t)$ uniformly on $[-1+r, 1-r]$ as $\epsilon \to 0$. Therefore, we are justified in passing the limit $\epsilon \to 0$ under the integral.

Appendix 3

Differentiating a branch of a function often requires use of the chain rule.

Let $g(z)$ be a branch of $\left(\frac{z-1}{z+1}\right)^{\frac{1}{6}}$. We cannot use power rule since the exponent is not a natural number (recall that the proof of the power rule required the exponent to be a natural number). Instead, we use chain rule as follows:

$$\frac{d}{dz}\Bigg[\underbrace{e^{\frac{1}{6}\log\left(\frac{z-1}{z+1}\right)}}_{g(z)}\Bigg] = \underbrace{e^{\frac{1}{6}\log\left(\frac{z-1}{z+1}\right)}}_{g(z)}\frac{d}{dz}\left[\frac{1}{6}\log\left(\frac{z-1}{z+1}\right)\right] = \frac{1}{6}g(z)\frac{\left(\frac{z-1}{z+1}\right)'}{\frac{z-1}{z+1}}$$

$$\frac{1}{6}g(z)\frac{\frac{z+1-(z-1)}{(z+1)^2}}{\frac{z-1}{z+1}} = \frac{1}{3}g(z)\frac{1}{z^2-1}$$

When we use chain rule, the branch of the differentiated function is clear: in this case, the $e^{\frac{1}{6}\log\left(\frac{z-1}{z+1}\right)}$ term in the differentiated function is the same branch of $e^{\frac{1}{6}\log\left(\frac{z-1}{z+1}\right)}$ as in the original function.

In fact, for $a \in \mathbb{C}$, we have the general result:

$$\frac{d}{dz}[z^a] = \frac{d}{dz}\left[e^{a\log z}\right] = e^{a\log z}\frac{a}{z} = z^a\frac{a}{z}$$

which looks consistent with the power rule, but we could not use the power rule to derive it. Note again that the original and differentiated function are using the same branch of z^a.

Solutions to More Practice Questions

1.
Choose the branch cut $[0, \infty)$ and $\arg z \in (0, 2\pi)$ for both z^{a-1} and z^b. We integrate $f(z) = \frac{z^{a-1}}{z^b+1}$ over the indented circular sector of angle $2\pi/b$. Using ML-lemma, the integral over the two circular portions vanish. The only pole is $e^{i\pi/b}$, a simple pole, and the residue is $\lim_{z \to e^{i\pi/b}} \frac{z^{a-1}}{(z^b b/z)} = -e^{i\pi a/b}/b$. Let $J = \int_0^\infty \frac{x^{a-1}}{x^b+1} dx$. Then by residue theorem,

$$\left(1 - e^{i2\pi a/b}\right) J = -2\pi i e^{i\pi a/b}/b$$

$$\Rightarrow J = \frac{\pi}{b} \csc\left(\frac{\pi a}{b}\right)$$

2.
We integrate $f(z) = \frac{1}{z^b-1}$ over the below contour with $\arg z \in (-\pi/2, 3\pi/2)$ for z^b, but principal branch would work too. Other functions and contours will work too.

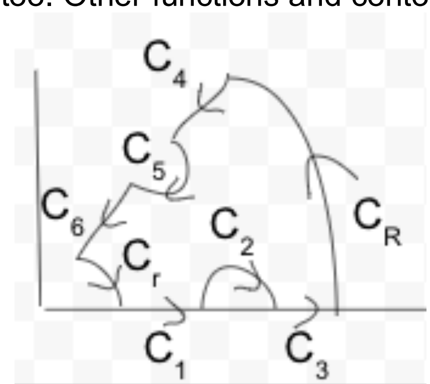

Fig. A2

The integrals over C_R and C_r vanish by ML-lemma. By fractional residue theorem, we have $\int_{C_2} f = -\pi i \operatorname{Res}(f, 1) = -\pi i \lim_{z \to 1} \frac{1}{bz^b/z} = -\pi i/b$ and $\int_{C_5} f = -\pi i \operatorname{Res}\left(f, e^{i2\pi/b}\right) = -\pi i \lim_{z \to e^{i2\pi/b}} \frac{1}{bz^b/z} = -(\pi i/b) e^{i2\pi/b}$. Let $J = \operatorname{PV} \int_0^\infty \frac{1}{x^b-1} dx$. Parameterize $z = te^{i2\pi/b}$ on C_4 and C_6. Then by Cauchy's theorem,

$$\int_{C_r + C_R + C_2 + C_5} f + \left(\int_0^{1-r} + \int_{1+r}^R\right) \frac{dt}{t^b - 1} + \left(\int_R^{1+r} + \int_{1-r}^0\right) \frac{e^{i2\pi/b} dt}{t^b - 1} = 0$$

Take $r \to 0$ and $R \to \infty$,

$$-\frac{\pi i}{b} - \frac{\pi i}{b} e^{i2\pi/b} + \left(1 - e^{i2\pi/b}\right) J = 0$$

$$\Rightarrow J = -\frac{\pi}{b} \cot\left(\frac{\pi}{b}\right)$$

Note that this is consistent with our results from Question 6.

3.
Choose $\sqrt{z}$ with $\arg z \in (0, 2\pi)$ and $\log z$ with $\arg z \in (0, 2\pi)$, so both have branch cut $[0, \infty)$. We integrate the function $f(z) = \log z / \left(\sqrt{z}(a+z)^2\right)$ over the corresponding keyhole contour. Let $J = \int_0^\infty \frac{\log x}{\sqrt{x}(a+x)^2} dx$. The integral over the two circular portions vanish by ML-lemma. The only pole is at $z = -a$ which is an order two pole. So $\operatorname{Res}(f, -a) = \lim_{z \to -a} \frac{d}{dz}\left[\frac{\log z}{\sqrt{z}}\right] = \lim_{z \to -a} \frac{\sqrt{z}/z - \log z/(2\sqrt{z})}{\sqrt{z}^2} = \frac{-i|\sqrt{a}|/a - (\log|a| + i\pi)/(2i|\sqrt{a}|)}{-a} = \frac{-2 + \log|a| + i\pi}{2ia\sqrt{a}}$.
Hence by residue theorem,

$$J + \int_\infty^0 \frac{\log|t| + i2\pi}{|\sqrt{t}|e^{i\pi}(a+t)^2} dt = 2\pi i \left(\frac{-2 + \log|a| + i\pi}{2ia\sqrt{a}}\right)$$

$$2J + \int_0^\infty \frac{2\pi i}{|\sqrt{t}|(a+t)^2} dt = \frac{\pi}{a\sqrt{a}}(-2 + \log|a| + i\pi)$$

Equating real and imaginary parts,

$$\int_0^\infty \frac{\log x}{\sqrt{x}(a+x)^2} dx = \frac{\pi}{2a\sqrt{a}}(-2 + \log a) \qquad \int_0^\infty \frac{dx}{\sqrt{x}(a+x)^2} = \frac{\pi}{2a\sqrt{a}}$$

4.
There are multiple options but we integrate $f(z) = \log^2 z / (z^2 + b^2)$ over a semicircle in the upper-half plane with indent at $z = 0$, and we choose the branch cut $[0, -i\infty)$ and $\arg z \in (-\pi/2, 3\pi/2)$ for $\log z$. Let $J = \int_0^\infty \frac{\log^2 x}{x^2 + b^2} dx$. The integral over the two circular portions vanish by ML-lemma. The only pole inside the contour is $z = ib$, and we have $\operatorname{Res}(f, ib) = \lim_{z \to ib} \frac{\log^2 z}{z + ib} = \frac{(\log|b| + i\pi/2)^2}{2ib}$. Then by residue theorem,

$$J + \int_\infty^0 \frac{(\log|-t| + i\pi)^2}{(-t)^2 + b^2}(-dt) = 2\pi i \frac{(\log|b| + i\pi/2)^2}{2ib}$$

$$J + \int_0^\infty \frac{\log^2 t + 2\pi i \log t - \pi^2}{t^2 + b^2} dt = \frac{\pi}{b}(\log|b| + i\pi/2)^2$$

$$2J + 2\pi i \int_0^\infty \frac{\log t}{t^2 + b^2} dt - \frac{\pi^3}{2b} = \frac{\pi}{b}(\log|b| + i\pi/2)^2$$

Equating real and imaginary parts,

$$\int_0^\infty \frac{\log^2 x}{x^2 + b^2} dx = \frac{\pi}{2b}\left(\frac{\pi^2}{4} + \log^2 b\right) \qquad \int_0^\infty \frac{\log x}{x^2 + b^2} dx = \frac{\pi}{2b}\log b$$

5.
Again there are multiple options but we integrate $f(z) = \log^2 z/(z^2-1)$ over the same contour as in Fig. 2.13, where we have chosen the principal branch of $\log z$ (the branch with $\arg z \in (-\pi/2, 3\pi/2)$ is also possible). Let $J = \int_0^\infty \frac{\log^2 x}{x^2-1}dx$. The integrals over C_2 and C_R vanish by ML-lemma. Since $z=1$ is a removable singularity of $f(z)$, we do not need an indent around $z=1$. Then by Cauchy's theorem,

$$0 = J + \int_\infty^0 \frac{(\log|it| + i\pi/2)^2}{(it)^2-1} i dt = J + i\int_0^\infty \frac{\log^2 t + \pi i \log t - \pi^2/4}{t^2+1}dt$$

Equating real parts and noting that we showed $\int_0^\infty \frac{\log t}{t^2+1}dt = 0$ in Question 4,

$$\int_0^\infty \frac{\log^2 x}{x^2-1}dx = 0$$

Alternatively, the integral is quickly solved using the substitution $x \mapsto 1/x$.

6.
Using the substitution $x - a \mapsto x$, the integral equals $J \coloneqq \int_0^c \frac{(x+a)^n}{\sqrt{x(c-x)}}dx$ where $c = b-a$. We use the same branch of $\sqrt{z(c-z)} \coloneqq g(z)$ as constructed in Fig. 3.2, and we integrate $f(z) = (z+a)^n/g(z)$ over the same dogbone contour. The integrals over the circular portions vanish by ML-lemma. Then by residue theorem, $2J = 2\pi i \operatorname{Res}(f,\infty)$. Note that $1/g(z)$ is a branch of $(z(c-z))^{-1/2}$ that is positive-imaginary for z on the positive real axis. Again, there are only two branches of $(z(c-z))^{-1/2}$ with the branch cut $[0,c]$, which are negatives of each other. We have:

$$\frac{1}{g(z)} = \left(z(c-z)\right)^{-1/2} = \left(-z^2\left(1-\frac{c}{z}\right)\right)^{-1/2} = \pm i\frac{1}{z}\left(1-\frac{c}{z}\right)^{-1/2} = \pm\frac{i}{z}\sum_{k=0}^{\infty}\binom{-1/2}{k}\left(-\frac{c}{z}\right)^k$$
$$= \pm i\left(\frac{1}{z} - \binom{-1/2}{1}\frac{c}{z^2} + \binom{-1/2}{2}\frac{c^2}{z^3} + \cdots\right)$$

and the series converges for $|z| > c$. Note that $\binom{-1/2}{k}$ is negative and positive when k is odd and even respectively, so all coefficients in the series of $(1-c/z)^{-1/2}$ are in fact positive. Then our branch of $(z(c-z))^{-1/2}$ uses the positive sign in the above series, that is,

$$\frac{(z+a)^n}{g(z)} = i\left(\frac{1}{z} - \binom{-1/2}{1}\frac{c}{z^2} + \binom{-1/2}{2}\frac{c^2}{z^3} + \cdots\right)\sum_{k=0}^{n}\binom{n}{k}z^k a^{n-k}$$

Collecting terms, the coefficient of z^{-1} is found to be $i\sum_{k=0}^n (-1)^k \binom{-1/2}{k}\binom{n}{k}c^k a^{n-k}$. Then, since $2J = 2\pi i \operatorname{Res}(f,\infty)$,

$$J = \pi\sum_{k=0}^{n}(-1)^k\binom{-1/2}{k}\binom{n}{k}(b-a)^k a^{n-k}$$

7.
There are many constructions, but we choose the principal branch for both $(z-b)^a$ and $(z+b)^a$ and define $g(z)=\frac{(z-b)^a}{(z+b)^a}$ to be our branch of $\left(\frac{z-b}{z+b}\right)^a$. Let $f(z)=\frac{1}{z-c}g(z)$. Since $\arg(z-b),\arg(z+b)\in(-\pi,\pi)$ and $\arg g(z)=a\arg(z-b)-a\arg(z+b)$, it can be shown that $f(z)$ has branch cut $[-b,b]$ and is positive just above $(-b,b)$. We will integrate f over the dogbone contour from $-b$ to b clockwise, leaving only the residues at $z=c$ and $z=\infty$ to worry about.

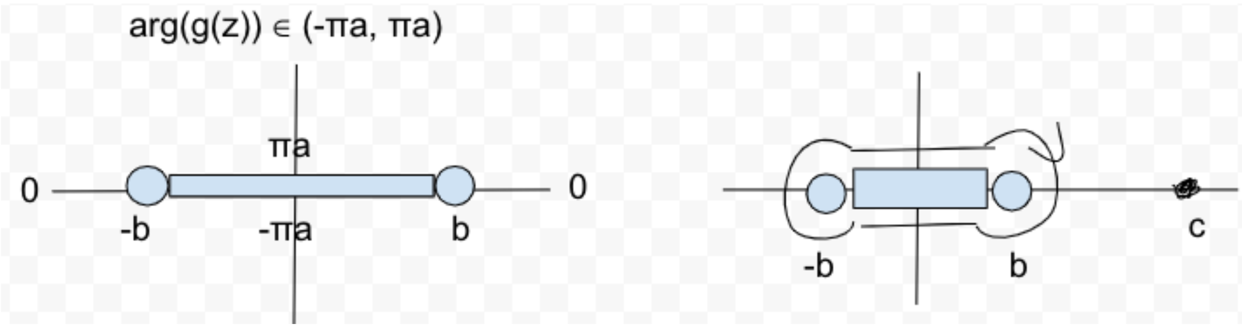

Fig. A3

Let $J=\int_{-b}^{b}\frac{1}{x-c}\left(\frac{x-b}{x+b}\right)^a dx$. The integral over the two circular portions vanish by ML-lemma. Note that:

$$\operatorname{Res}(f,\infty)=-\operatorname{Res}\left(\frac{1}{z^2}f\left(\frac{1}{z}\right),0\right)=-\operatorname{Res}\left(\frac{1}{z(1-cz)}\left(\frac{1-bz}{1+bz}\right)^a,0\right)$$

where $\left(\frac{1-bz}{1+bz}\right)^a=g(1/z)$. Noting that our branch of g implies $\lim_{z\to0}(\arg g(1/z))=0$ so[12] $\lim_{z\to0}g(1/z)=1$,

$$-\operatorname{Res}\left(\frac{1}{z(1-cz)}\left(\frac{1-bz}{1+bz}\right)^a,0\right)=-\lim_{z\to0}\frac{1}{1-cz}\left(\frac{1-bz}{1+bz}\right)^a=-1$$

Lastly, $\operatorname{Res}(f,c)=\lim_{z\to c}g(z)=\left|\left(\frac{c-b}{c+b}\right)^a\right|e^{i0}$. Then by residue theorem,

$$\int_{-b}^{b}\frac{1}{t-c}\left|\left(\frac{t-b}{t+b}\right)^a\right|e^{i\pi a}dt+\int_{b}^{-b}\frac{1}{t-c}\left|\left(\frac{t-b}{t+b}\right)^a\right|e^{-i\pi a}dt=2\pi i\left(\left|\left(\frac{c-b}{c+b}\right)^a\right|-1\right)$$

$$J\left(e^{i\pi a}-e^{-i\pi a}\right)=2\pi i\left(\left|\left(\frac{c-b}{c+b}\right)^a\right|-1\right)$$

$$J=\pi\left(\left(\frac{c-b}{c+b}\right)^a-1\right)\csc(\pi a)$$

Note the absolute value sign can be removed since $\frac{c-b}{c+b}>0$ for all $c\in\mathbb{R}/[-b,b]$.

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[12] We have $\lim_{z\to0}g(1/z)=\left|\lim_{z\to0}g(1/z)\right|e^{i\arg\left(\lim_{z\to0}g(1/z)\right)}=1e^{i0}=1.$

8.
We define f_1, f_2, f_3 as follows:

Function	Branch cut	Domain of $\arg(f_k(z))$
$f_1(z) = \log(z+1)$	$(-\infty, -1]$	$(-\pi, \pi)$
$f_2(z) = \log\left(z - e^{i\pi/3}\right)$	$\{z : z = re^{i\pi/3}, r \geq 1\}$	$(-5\pi/3, \pi/3)$
$f_3(z) = \log\left(z - e^{i5\pi/3}\right)$	$\{z : z = re^{i5\pi/3}, r \geq 1\}$	$(-\pi/3, 5\pi/3)$

Define $\log(1+z^3) = f_1(z) + f_2(z) + f_3(z)$. Further define z^a with branch cut $[0, \infty)$ and $\arg z \in (0, 2\pi)$. We integrate $f(z) = \log(1+z^3)/(z^2+b^2)$ over the below contour:

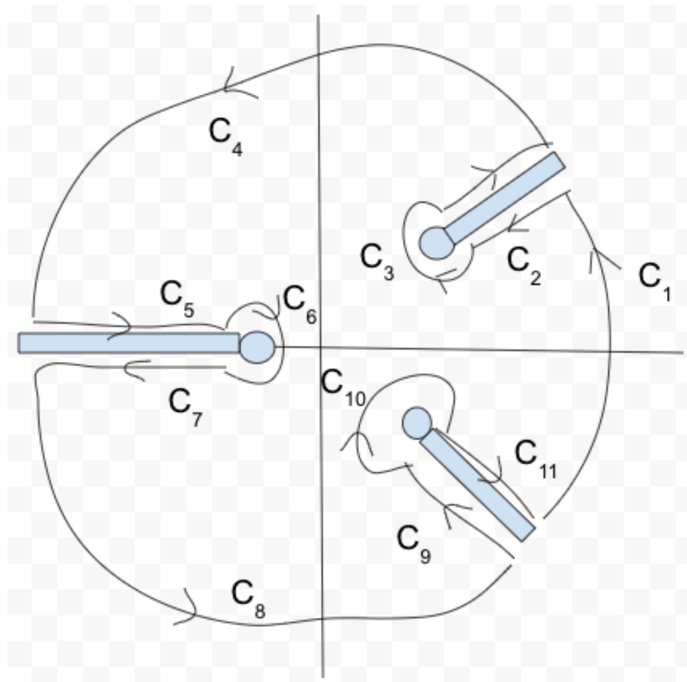

Fig. A4

The integrals over $C_1, C_3, C_5, C_7, C_9, C_{11}$ will vanish by ML-lemma, see Question 7 for a similar situation. Next, we have $\int_{C_2} f \to \int_\infty^1 \frac{\log|1+t^3|+i\pi}{t^2e^{i2\pi/3}+b^2} e^{i\pi/3} dt$, $\int_{C_4} f \to \int_1^\infty \frac{\log|1+t^3|-i\pi}{t^2e^{i2\pi/3}+b^2} e^{i\pi/3} dt$, $\int_{C_6} f \to \int_\infty^1 \frac{\log|1+t^3|+i\pi}{t^2+b^2}(-dt)$, $\int_{C_8} f \to \int_1^\infty \frac{\log|1+t^3|-i\pi}{t^2+b^2}(-dt)$, $\int_{C_{10}} f \to \int_\infty^1 \frac{\log|1+t^3|+i\pi}{t^2e^{i10\pi/3}+b^2} e^{i5\pi/3} dt$, and $\int_{C_{12}} f \to \int_1^\infty \frac{\log|1+t^3|-i\pi}{t^2e^{i10\pi/3}+b^2} e^{i5\pi/3} dt$. Adding all these integrals together, we find

$$\int_{C_2+C_4+C_6+\cdots+C_{12}} f = -2\pi i \int_1^\infty \frac{e^{i\pi/3}}{t^2e^{i2\pi/3}+b^2} - \frac{1}{t^2+b^2} + \frac{e^{i5\pi/3}}{t^2e^{i10\pi/3}+b^2} dt$$

$$-2\pi i \int_1^\infty \frac{1+\sqrt{3}i}{t^2\left(-1+\sqrt{3}i\right)+2b^2} + \frac{1-\sqrt{3}i}{t^2\left(-1-\sqrt{3}i\right)+2b^2} dt + \int_1^\infty \frac{2\pi i}{t^2+b^2} dt$$

$$= -2\pi i \int_1^\infty \frac{t^2+b^2}{t^4-b^2t^2+b^4} dt + 2\pi i \left[\frac{1}{b}\arctan\left(\frac{t}{b}\right)\right]_1^\infty$$

$$= -2\pi iJ + 2\pi i\left(\frac{\pi}{2b} - \frac{1}{b}\arctan\frac{1}{b}\right)$$

We now compute the residues of f inside the contour, keeping in mind that we have $\mathrm{Res}(f, ib) = \lim_{z\to ib} \frac{\log(1+z^3)}{z+ib} = \frac{\log(1-ib^3)}{2ib} = \frac{\log|1-ib^3| + i\arg(1-ib^3)}{2ib} = \frac{\log|\sqrt{1+b^6}| - i\arctan b^3}{2ib}$ and similarly, $\mathrm{Res}(f, -ib) = \lim_{z\to -ib} \frac{\log(1+z^3)}{z-ib} = \frac{\log|\sqrt{1+b^6}| + i\arctan b^3}{-2ib}$. Then by residue theorem,

$$-2\pi iJ + 2\pi i\left(\frac{\pi}{2b} - \frac{1}{b}\arctan\frac{1}{b}\right) = 2\pi i\big(\mathrm{Res}(f, ib) + \mathrm{Res}(f, -ib)\big)$$

$$-J + \frac{\pi}{2b} - \frac{1}{b}\arctan\frac{1}{b} = \frac{-2i\arctan b^3}{2bi}$$

Since $\arctan x + \arctan 1/x = \pi/2$ for $x > 0$,

$$J = \frac{\arctan(b) + \arctan(b^3)}{b}$$

9.
Base case $n = 1$: We have $f'(z) = z^n(1/z) + nz^{n-1}\log z = z^{n-1}(1 + n\log z)$, which is indeed equal to $f^{(1)}(z) = z^{n-1}\frac{n!}{(n-1)!}\left(\log(z) + \sum_{k=0}^{0}\frac{1}{n-k}\right) = z^{n-1}n(\log z + 1/n)$. Assuming true for given m, $1 \le m < n$,

$$f^{(m+1)}(z) = \left(f^{(m)}\right)'(z) = \frac{d}{dz}\left[z^{n-m}\frac{n!}{(n-m)!}\left(\log(z) + \sum_{k=0}^{m-1}\frac{1}{n-k}\right)\right]$$

$$= \frac{n!}{(n-m)!}\left(z^{n-m}\left(\frac{1}{z}\right) + (n-m)z^{n-m-1}\left(\log(z) + \sum_{k=0}^{m-1}\frac{1}{n-k}\right)\right)$$

$$= z^{n-m-1}\frac{n!}{(n-m)!}\left(1 + (n-m)\left(\log(z) + \sum_{k=0}^{m-1}\frac{1}{n-k}\right)\right)$$

$$= z^{n-m-1}\frac{n!}{(n-m-1)!}\left(\frac{1}{n-m} + \left(\log(z) + \sum_{k=0}^{m-1}\frac{1}{n-k}\right)\right)$$

$$= z^{n-m-1}\frac{n!}{(n-m-1)!}\left(\log(z) + \sum_{k=0}^{m}\frac{1}{n-k}\right)$$

which proves the induction. Note the formula only holds when $1 \le m \le n$ since the $(n-m)!$ term will no longer make sense if $m \ge n+1$. However, observing $f^{(n)}(z) = n!\left(\log(z) + \sum_{k=0}^{n-1}\frac{1}{n-k}\right)$, we see that $f^{(n+1)}(z) = n!/z$, upon which we deduce (by induction or otherwise) that $f^{(n+m)}(z) = n!\,(m-1)!\,(-1)^{m-1}/z^m$ for $m \in \mathbb{Z}^+$, or equivalently, $f^{(m)}(z) = n!\,(m-n-1)!\,(-1)^{m-n-1}/z^{m-n}$ for $m \ge n+1$. Now to solve the integral, we integrate $g(z) = z^n\log z/(1+z)^{2n}$ over the keyhole contour in Fig. 2.14

with keyhole along $[0,\infty)$. Parameterizing $z = Re^{i\theta}$ and noting $\log|z| \le |z|$, we have

$$|g(z)| \le \frac{|z|^n|\log|z| + i\arg z|}{|1+z|^{2n}} \le \frac{|z|^n\sqrt{\log^2|z| + \arg^2 z}}{\left|1-|z|\right|^{2n}} \le \frac{R^n\sqrt{R^2+4\pi^2}}{|1-R|^{2n}}$$

Then by ML-lemma,

$$\left|\int_{C_R} g(z)dz\right| \le 2\pi\frac{R^{n+1}\sqrt{R^2+4\pi^2}}{|1-R|^{2n}}$$

which goes to 0 as $R \to \infty$ and as $R \to 0$, so the contributions from the contour's circular arcs vanish. Then by residue theorem, noting $z = -1$ is a pole of order $2n$,

$$\int_0^\infty \frac{t^n \log|t|}{(1+t)^{2n}}dt + \int_\infty^0 \frac{t^n(\log|t| + 2\pi i)}{(1+t)^{2n}}dt = 2\pi i \operatorname{Res}(f,-1)$$

$$\int_0^\infty \frac{t^n}{(1+t)^{2n}}dt = -\operatorname{Res}(f,-1) = -\frac{1}{(2n-1)!}\lim_{z\to-1}\frac{d^{2n-1}}{dz^{2n-1}}[(1+z)^{2n}g(z)]$$

$$= -\frac{1}{(2n-1)!}\lim_{z\to-1} f^{(2n-1)}(z)$$

Using the formula found for $f^{(m)}(z)$ when $m \ge n+1$,

$$= -\frac{1}{(2n-1)!}\lim_{z\to-1}\frac{n!\,(n-2)!\,(-1)^n}{z^{n-1}}$$

$$= \frac{n!\,(n-2)!}{(2n-1)!}$$

It should be noted that our result can be extended using the gamma function, and the integral $\int_0^\infty \frac{x^b}{(1+x)^{2b}}dx$ in fact converges for all $b > 1$, so we may conjecture for $b > 1$ that

$$\int_0^\infty \frac{x^b}{(1+x)^{2b}}dx = \frac{\Gamma(b+1)\Gamma(b-1)}{\Gamma(2b)} = \mathrm{B}(b-1, b+1)$$

This is indeed true and can be proven by substituting $x = 1-u$ and then $v = 1/u$,

$$\int_0^\infty \frac{x^b}{(1+x)^{2b}}dx = \int_1^\infty (u-1)^b u^{-2b}du = \int_0^1 (1-v)^b v^{b-2}dv = \mathrm{B}(b-1, b+1)$$

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