Evaluate:

$$I=\int_0^{\frac{\pi}{2}}\frac{x}{\sin x}\,dx \qquad J=\int_0^1\frac{(\arcsin x)(\arccos x)}{x}\,dx \qquad K=\int_0^1\frac{\arcsin^2 x}{x}\,dx$$

We treat I as a complex line integral and write

$$I=\int_0^{\frac{\pi}{2}}\frac{2iz}{e^{iz}-e^{-iz}}\,dz=\int_0^{\frac{\pi}{2}}\frac{2ize^{iz}}{e^{i2z}-1}\,dz=\int_0^{\frac{\pi}{2}}\frac{2ize^{iz}}{(e^{iz}-1)(e^{iz}+1)}\,dz$$

$$=\int_0^{\frac{\pi}{2}}\frac{iz}{e^{iz}-1}+\frac{iz}{e^{iz}+1}\,dz=\int_0^{\frac{\pi}{2}}\frac{ize^{-iz}}{1-e^{-iz}}+\frac{ize^{-iz}}{1+e^{-iz}}\,dz$$

$$=\left[z\operatorname{Log}\left(1-e^{-iz}\right)\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\operatorname{Log}\left(1-e^{-iz}\right)dz-\left[z\operatorname{Log}\left(1+e^{-iz}\right)\right]_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\operatorname{Log}\left(1+e^{-iz}\right)dz$$

and we take the principal branches $\operatorname{Log}\left(1-e^{-iz}\right)$ and $\operatorname{Log}\left(1+e^{-iz}\right)$ to be our antiderivatives since they are defined on $(a,\pi/2)$ as $a\to 0^+$ (when $z\in(a,\pi/2)$, $1-e^{-iz}$ represents the circular arc from $a\approx 0$ to $1+i$ and $1+e^{-iz}$ represents the circular arc from $1+e^{-ia}\approx 2$ to $1-i$, both of which are contained in the domain of $\operatorname{Log} z$). Using L'Hopital's rule, $\lim\limits_{z\to 0} z\log\left(1-e^{-iz}\right)=\lim\limits_{z\to 0}\frac{\log\left(1-e^{-iz}\right)}{1/z}=\lim\limits_{z\to 0}\frac{iz^2e^{-iz}}{1-e^{-iz}}$, but since $\lim\limits_{z\to 0}\frac{z^2}{1-e^{-iz}}=\lim\limits_{z\to 0}\frac{2z}{ie^{-iz}}=0$, it follows that the original limit is also 0. Substitute $u=e^{-iz}$ and $u=-e^{-iz}$ in the first and second integrals respectively.

$$=\frac{\pi}{2}\operatorname{Log}(1+i)-i\lim_{b\to 1}\int_b^{-i}\frac{\operatorname{Log}(1-u)}{u}\,du-\frac{\pi}{2}\operatorname{Log}(1-i)+i\int_{-1}^{i}\frac{\operatorname{Log}(1-u)}{u}\,du$$

where the analyticity of $\operatorname{Log}(1-u)/u$ on $\mathbb{C}\backslash[1,\infty)$ ensures the two above integrals are path independent (note $u=0$ is a removable singularity, and to see why $[1,\infty)$ is the branch cut, recall $\operatorname{Log}(u-1)$ has branch cut $(-\infty,-1]$ with argument $\pm\pi$ above and below the branch cut respectively, so $\operatorname{Log}(1-u)$ will have branch cut $[1,\infty)$ with argument ∓ 1 above and below the branch cut respectively).

$$=\frac{\pi}{2}\left(\frac{1}{2}\ln 2+\frac{i\pi}{4}\right)+i\lim_{b\to 1}\left(\operatorname{Li}_2(-i)-\operatorname{Li}_2(b)\right)-\frac{\pi}{2}\left(\frac{1}{2}\ln 2-\frac{i\pi}{4}\right)-i\left(\operatorname{Li}_2(i)-\operatorname{Li}_2(-1)\right)$$

Recalling $\operatorname{Li}_2(z)=\sum_{k=1}^{\infty}z^k/k^2$ for $|z|<1$, for $|z_0|=1$ we treat $\operatorname{Li}_2(z_0)$ as the limit of the infinite series as $z\to z_0$ from within the unit circle,

$$=i\frac{\pi^2}{8}+\left(i\sum_{k=1}^{\infty}\frac{(-i)^k}{k^2}\right)-i\frac{\pi^2}{6}+i\frac{\pi^2}{8}-\left(i\sum_{k=1}^{\infty}\frac{i^k}{k^2}\right)-i\frac{\pi^2}{12}$$

The two series converge absolutely so we may combine them,

$$i(2i)\left(-1+\frac{1}{3^2}-\frac{1}{5^2}+\frac{1}{7^2}-\cdots\right)=2\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\cdots\right)$$

$$=\mathbf{2G}$$

where G denotes Catalan's constant.

To solve J, we write $\arccos x = -\arcsin x + \pi/2$ (easily verified by taking $\cos$ of both sides) and substitute $x = \sin u$,

$$J = \int_0^1 \frac{(\arcsin x)\left(-\arcsin x + \frac{\pi}{2}\right)}{x} dx = \int_0^{\frac{\pi}{2}} \frac{u\left(\frac{\pi}{2} - u\right)}{\tan u} du$$

Substituting $u \mapsto -u + \pi/2$,

$$= \int_0^{\frac{\pi}{2}} \frac{u\left(\frac{\pi}{2} - u\right)}{\cot u} du$$

Hence,

$$2J = \int_0^{\frac{\pi}{2}} \frac{u\left(\frac{\pi}{2} - u\right)}{\tan u} + \frac{u\left(\frac{\pi}{2} - u\right)}{\cot u} du = 2\int_0^{\frac{\pi}{2}} \frac{u\left(\frac{\pi}{2} - u\right)}{\sin 2u} du = \frac{1}{4}\int_0^{\pi} \frac{u(\pi - u)}{\sin u} du$$

We proceed similarly as before using complex sine,

$$\Rightarrow 8J = \int_0^{\pi} \frac{2iu(\pi - u)}{e^{iu} - e^{-iu}} du = \int_0^{\pi} \frac{u(\pi - u)ie^{-iz}}{1 - e^{-iz}} + \frac{u(\pi - u)ie^{-iz}}{1 + e^{-iz}} du$$

$$= \left[u(\pi - u)\operatorname{Log}\left(1 - e^{-iu}\right)\right]_0^{\pi} - \int_0^{\pi} (\pi - 2u)\operatorname{Log}\left(1 - e^{-iu}\right) du - \left[u(\pi - u)\operatorname{Log}\left(1 + e^{-iu}\right)\right]_0^{\pi} + \int_0^{\pi} (\pi - 2u)\operatorname{Log}\left(1 + e^{-iu}\right) du$$

$$= -\pi\int_0^{\pi} \operatorname{Log}\left(1 - e^{-iu}\right) du + 2\int_0^{\pi} u\operatorname{Log}\left(1 - e^{-iu}\right) du + \pi\int_0^{\pi} \operatorname{Log}\left(1 + e^{-iu}\right) du - 2\int_0^{\pi} u\operatorname{Log}\left(1 + e^{-iu}\right) du$$

Substituting $w = \pm e^{-iu}$, we see $\int \operatorname{Log}\left(1 \mp e^{-iu}\right) du = i\int \operatorname{Log}(1 - w)/w\, dw = -i\mathrm{Li}_2(w) = -i\mathrm{Li}_2\left(\pm e^{-iu}\right)$, hence

$$= \pi i\left(\mathrm{Li}_2(e^{-i\pi}) - \mathrm{Li}_2(e^{-i0})\right) + 2\left([-iu\mathrm{Li}_2(e^{-iu})]_0^{\pi} - \int_0^{\pi} -i\mathrm{Li}_2(e^{-iu}) du\right) - \pi i\left(\mathrm{Li}_2(-e^{-i\pi}) - \mathrm{Li}_2(-e^{-i0})\right) - 2\left([-iu\mathrm{Li}_2(-e^{-iu})]_0^{\pi} - \int_0^{\pi} -i\mathrm{Li}_2(-e^{-iu}) du\right)$$

Substitute $w = \pm e^{-iu}$ in the first and second integrals respectively,

$$= \pi i\left(-\frac{\pi^2}{12} - \frac{\pi^2}{6}\right) + 2\left(-\pi i\left(-\frac{\pi^2}{12}\right) - \int_1^{-1} \frac{\mathrm{Li}_2(w)}{w} dw\right) - \pi i\left(\frac{\pi^2}{6} + \frac{\pi^2}{12}\right) - 2\left(-\pi i\left(\frac{\pi^2}{6}\right) - \int_{-1}^{1} \frac{\mathrm{Li}_2(w)}{w} dw\right)$$

Simplifying and using definition of polylogarithm,

$$= 4\int_{-1}^{1} \frac{\mathrm{Li}_2(w)}{w} dw = 4\left(\mathrm{Li}_3(1) - \mathrm{Li}_3(-1)\right)$$

Again we may treat $\mathrm{Li}_3(\pm 1)$ as the limit of the series $\sum_{k=1}^{\infty} z^k/k^3$ as $z \to \pm 1$ from within the unit circle, and note the series converge absolutely at $z = \pm 1$ so we may combine them,

$$= 4\left(\sum_{k=1}^{\infty} \frac{1}{k^3} - \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}\right) = 4\left(\frac{2}{1^3} + \frac{2}{3^3} + \frac{2}{5^3} + \cdots\right) = 8\sum_{k=1}^{\infty} \frac{1}{(2k+1)^3}$$

$$= 8\left(\sum_{k=1}^{\infty} \frac{1}{k^3} - \sum_{k=1}^{\infty} \frac{1}{(2k)^3}\right) = 8\left(\zeta(3) - \frac{1}{8}\zeta(3)\right) = 7\zeta(3)$$

Recall this was $8J$. Hence,

$$J = \frac{7}{8}\zeta(3)$$

To solve K, start by writing $\arcsin x = -\arccos x + \pi/2$,

$$K = \int_0^1 \frac{\arcsin x\left(\frac{\pi}{2} - \arccos x\right)}{x} dx = \int_0^1 \frac{\pi}{2}\left(\frac{\arcsin x}{x}\right) - \frac{(\arcsin x)(\arccos x)}{x} dx$$

We have just found $\int_0^1 (\arcsin x)(\arccos x)/x \, dx$ and the integral $\int_0^1 \arcsin x \,/x \, dx$ was proved in the document "Differentiating Under Integral Examples" to be $\pi \ln 2\,/2$, hence

$$K = \frac{\pi^2}{4}\ln 2 - \frac{7}{8}\zeta(3)$$