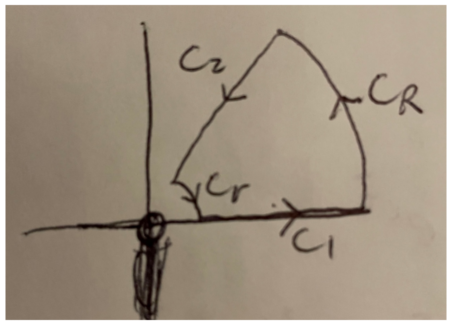
We aim to evaluate: Wenzhi Tseng

$$J = \int_0^\infty \cos(x^a)\,dx \qquad K = \int_0^\infty \sin(x^a)\,dx$$

$$L = \int_0^\infty \frac{\sin(x^a)}{x^a}\,dx$$

given that the above integrals all converge for $a > 1$.

We integrate $f(z) = e^{iz^a}$ where z^a has branch cut $[0, -i\infty)$ and $\arg z \in (-\pi/2, 3\pi/2)$ over the indented circular sector of angle $\pi/2a$.



(The first attempt used the circular sector of angle $2\pi/a$, which looks more natural, but the integration eventually led to a $0 = 0$ situation, so although the manipulations were legal, they were ultimately unhelpful)

We have $z = \rho e^{i\theta}, \theta \in (0, 2\pi/a)$ on C_r and C_R. Writing $z^a = |z^a| i \arg(z^a)$, we have

$$|f(z)| = \left|e^{i|z^a| i \arg(z^a)}\right| = e^{-a|z^a| \arg z} \le e^{-a\rho^a}$$

Then $\left|\int_{C_r} f\right| \le \frac{\pi}{2a}\rho e^{-a\rho^a}$ which goes to 0 as $\rho \to 0$ and $\left|\int_{C_R} f\right| \le \frac{\pi}{2a}\rho e^{-a\rho^a}$ which also goes to 0 as $\rho \to \infty$ (use L'Hopital's, the limit goes to zero since $a > 1$). Therefore the integrals over C_r and C_R both vanish as $r \to 0$ and $R \to \infty$ respectively.

Parameterizing $z = t$ on C_1, we quickly find $\int_{C_1} f \to J + iK$ as $r \to 0$ and $R \to \infty$. On C_2, we have $z = te^{i\pi/2a}$, so

$$\int_{C_2} f = \int_\infty^0 e^{i\left(te^{i\frac{\pi}{2a}}\right)^a} e^{i\frac{\pi}{2a}} dt = -e^{i\frac{\pi}{2a}} \int_0^\infty e^{-t^a}\,dt$$

Substitute $u = t^a$, so $du = at^{a-1}dt$,

$$= -e^{i\frac{\pi}{2a}} \frac{1}{a} \int_0^\infty e^{-u} u^{\frac{1}{a}-1}\,du = -e^{i\frac{\pi}{2a}} \frac{1}{a} \Gamma\left(\frac{1}{a}\right)$$

The function f is analytic in and around the contour, so by Cauchy's theorem,

$$J + iK - e^{i\frac{\pi}{2a}} \frac{1}{a} \Gamma\left(\frac{1}{a}\right) = 0$$

Equating real and imaginary parts, we have:

$$J = \frac{1}{a}\Gamma\left(\frac{1}{a}\right)\cos\left(\frac{\pi}{2a}\right) \qquad K = \frac{1}{a}\Gamma\left(\frac{1}{a}\right)\sin\left(\frac{\pi}{2a}\right)$$

The above expressions are also the analytic continuations of the complex functions $f_1(a) = \int_0^\infty \cos(x^a)\,dx$ and $f_2(a) = \int_0^\infty \sin(x^a)\,dx$ respectively to the whole complex plane excepting the singularities of $\Gamma(1/a)$.

We now evaluate L. Using integration by parts,

$$L = \int_0^\infty x^{-a}\sin(x^a)\ dx = \left[\frac{x^{-a+1}}{-a+1}\sin(x^a)\right]_0^\infty - \int_0^\infty \frac{x^{-a+1}}{-a+1}\cos(x^a)\,ax^{a-1}dx$$

Since $a > 1$, we have $\lim_{x\to\infty}\frac{x^{-a+1}}{-a+1}\sin(x^a) = \lim_{x\to\infty}\frac{\sin(x^a)}{(1-a)x^{a-1}} = 0$,

$$= 0 + \frac{a}{a-1}\int_0^\infty \cos(x^a)\,dx$$

$$= \frac{1}{a-1}\Gamma\left(\frac{1}{a}\right)\cos\left(\frac{\pi}{2a}\right)$$

It should be noted that the integral

$$\int_0^\infty \frac{\cos(x^a)}{x^a}dx$$

diverges for all real numbers a. This can be seen by noting that the first term in the Taylor series of $\cos(x^a)$ centered at $x = 0$ is 1, and the integral $\int_0^\infty 1/x^a\,dx$ diverges for all $a \in \mathbb{R}$.