We will solve the famous Dirichlet integral:

$$\int_0^\infty \frac{\sin\lambda}{\lambda}\,d\lambda$$

You may assume the Dirichlet integral converges. Proving it converges is a whole other matter.

1. Verify that:

   $$\int_0^\infty \lambda e^{-\lambda x}\,dx = 1$$

   for all $\lambda > 0$. The function $\lambda e^{-\lambda x}$ is the exponential distribution and appears a lot in probability.

2. Observe that we have:

   $$\int_0^\infty \frac{\sin\lambda}{\lambda}\,d\lambda = \int_0^\infty \frac{\sin\lambda}{\lambda}\left(\int_0^\infty \lambda e^{-\lambda x}\,dx\right)d\lambda = \int_0^\infty \int_0^\infty e^{-\lambda x}\sin\lambda\,d\lambda\,dx$$

   Using integration by parts twice, evaluate:

   $$\int_0^\infty e^{-\lambda x}\sin\lambda\,d\lambda$$

   Your answer will be in terms of x.

3. Hence find the value of the Dirichlet integral.

4. Go online and find the Laplace transform of $\sin x$. What do you notice?

5. We know that $e^{i\lambda} = \cos\lambda + i\sin\lambda$ for $\lambda \in \mathbb{R}$. It may be tempting to consider the following argument, similar to before:

   $$\int_0^\infty \frac{e^{i\lambda}}{\lambda}\,d\lambda = \int_0^\infty \int_0^\infty \frac{e^{i\lambda}}{\lambda}\lambda e^{-\lambda x}\,d\lambda\,dx = \int_0^\infty \int_0^\infty e^{i\lambda}e^{-\lambda x}\,d\lambda\,dx$$

   But in fact, all this is rather illegal since you do not know if $\int_0^\infty \frac{e^{i\lambda}}{\lambda}d\lambda$ converges (it does not) and you cannot just treat i like any other real number here. It is true that considering $e^{i\lambda}$ would make our life easier and it would avoid integration by parts and all that stuff. But what we *would* need to do is to integrate in the complex plane in order for our manipulations to be legal.