The Complex Exponential, Sine, and Cosine From First Principles

We present one avenue by which we may rigorously derive the well-known properties of the complex exponential, sine, and cosine functions. Our approach will be to define e^z as a power series and prove everything from there. The equivalency of various definitions will be discussed afterwards. Footnotes contain more advanced details.

Definition 1
We define the complex exponential function e^z by

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

for all $z \in \mathbb{C}$. The series converges absolutely for all $z \in \mathbb{C}$ by ratio test, so e^z is well-defined. We can extend e^z to the extended complex plane $\mathbb{C} \cup \{\infty\}$ by defining $e^\infty = \infty$.

Theorem 2
For all $z, w \in \mathbb{C}$ we have $e^{z+w} = e^z e^w$.

Proof:
By collecting terms of the same degree, we have[1]

$$e^z e^w = \left(\sum_{m=0}^{\infty} \frac{z^m}{m!}\right)\left(\sum_{m=0}^{\infty} \frac{w^m}{m!}\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{z^k w^{n-k}}{k!\,(n-k)!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} z^k w^{n-k}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} = \sum_{n=0}^{\infty} \frac{1}{n!} (z+w)^n = e^{z+w}$$

Definition 3
We define $\sin z$ and $\cos z$ by the following power series:

$$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} \qquad\qquad \cos z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}$$

for all $z \in \mathbb{C}$. Both series converge absolutely for all $z \in \mathbb{C}$ by ratio test, so both $\sin z$ and $\cos z$ are well-defined. From these definitions, it follows that $\cos(-z) = \cos(z)$ and $\sin(-z) = -\sin z$ for all $z \in \mathbb{C}$. That is, $\cos z$ is even and $\sin z$ is odd.

[1] The second equality is justified by Merten's theorem, since it is not obvious that collecting terms of the same degree actually leaves the limit unchanged. The key is that the series converge absolutely.

Theorem 4 (Euler's Identity):
For all $z \in \mathbb{C}$, we have $e^{iz} = \cos(z) + i\sin(z)$. Consequently, when $z \in \mathbb{R}$, we have $\cos z = \mathrm{Re}(e^z)$ and $\sin z = \mathrm{Im}(e^z)$.

Proof:
Follows from writing out the power series for e^{iz} and collecting odd and even powered terms.[2]

Theorem 5 (Complex definition of sine and cosine):
For all $z \in \mathbb{C}$, we have

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

Proof:
Using Euler's Identity and the even/oddness of $\cos z$ and $\sin z$, we have $e^{-iz} = \cos z - i\sin z$. Of course we also have $e^{iz} = \cos z + i\sin z$. Solving the two equations simultaneously completes the proof.

Corollary 6
For all $z \in \mathbb{C}$, we have $\sin^2 z + \cos^2 z = 1$.

Proof:
Firstly, it should be noted that this identity is not trivial and could not be deduced from elementary trigonometry or the Pythagorean Theorem. After all, z could be any complex number, upon which it is impossible to interpret z as an angle.

We have

$$\sin^2 z + \cos^2 z = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 = \frac{e^{i2z} - 2 + e^{-i2z}}{-4} + \frac{e^{i2z} + 2 + e^{-i2z}}{4} = 1$$

Corollary 7
The complex number $e^{i\theta}$ has length one for $\theta \in \mathbb{R}$.

Proof:
Since $\theta \in \mathbb{R}$, we are assured (by Definition 2) that $\sin\theta$ and $\cos\theta$ are both real numbers (since a convergent infinite sum of real numbers must converge to a real number). Then $|e^{i\theta}| = |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$. Note how this reasoning fails if θ had a nonzero imaginary part, since we would then no longer be assured that $\cos\theta$ and $\sin\theta$ are purely real.

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[2] The rearrangement of terms in the series is allowed since e^{iz} converges absolutely.

Corollary 8
We have $\sin(z+w) = \sin z \cos w + \sin w \cos z$ and $\cos(z+w) = \cos z \cos w - \sin z \sin w$ for all $z, w \in \mathbb{C}$. Consequently, the double angle identities are obtained by taking $z = w$.

Proof:
We have

$$\cos(z+w) + i\sin(z+w) = e^{i(z+w)} = e^{iz+iw} = e^{iz}e^{iw} = (\cos z + i\sin z)(\cos w + i\sin w)$$

Expanding the right-hand side and equating real and imaginary parts completes the proof.

Remark
It is NOT always true that $(e^z)^w = e^{zw}$. It is only easily seen to be true if w is a nonnegative integer, upon which $(e^z)^w = \underbrace{e^z e^z \dots e^z}_{w \text{ times}} = e^{z+z+\cdots+z} = e^{wz}$. However this argument does not hold for other values of w since, for example, it does not make sense to multiply something to itself "$4+5i$" times, reflecting the fact that the operation of raising to a complex power requires the complex logarithm (recall we define $z^\alpha = e^{\alpha \log z}$ for $\alpha \in \mathbb{C}$). Consequently, when writing something like $(e^z)^w$, it should be treated as a composition of functions $g \circ f$ where $g(z)$ is some branch of $z^w = e^{w \log z}$ and $f(z) = e^z$. See appendix for an explicit example of when $(e^z)^w = e^{zw}$ fails.

Next, we prove one of the most important properties of the complex exponential. This is where we start using calculus.

Theorem 9
The following two statements are equivalent.

1. $f(z) = e^z$ as defined in Definition 1.
2. $f(0) = 1$ and f is complex-differentiable on all of $\mathbb{C}$ with $f'(z) = f(z)$ for all $z \in \mathbb{C}$.

Proof:
Let $f(z) = e^z$. Using the series definition, it is easy enough to verify that $f(0) = 1$, and $f'(z) = f(z)$ can be deduced by differentiating the series term-by-term.[3] To prove the other implication, let g be any function satisfying $g(0) = 1$ and $g'(z) = g(z)$ on $\mathbb{C}$. Since g is differentiable on $\mathbb{C}$, g has a Taylor expansion $g(z) = \sum_{k=0}^{\infty} a_k z^k$ convergent on all of $\mathbb{C}$. Then $g(z) = g'(z)$ implies $a_0 + a_1 z + a_2 z^2 + \cdots = a_1 + 2a_2 z + 3a_3 z^2 + \cdots$ on $\mathbb{C}$. Before we equate coefficients, we first prove we are actually allowed to do so, since it is not a trivial result.

Lemma: Let $h_1(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k$ and $h_2(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k$ be two power series converging on the same set E. If $h_1 = h_2$ on E, then $b_k = c_k$ for all $k \geq 0$.

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[3] Differentiating term-by-term to recover the derivative is allowed as the power series converges uniformly on any $\{|z| \leq R\}$ by the M-test. So e^z is an entire function when defined as a power series.

Proof: Since $h_1 = h_2$ on E and power series are infinitely differentiable, we have $h_1^{(k)}(z_0) = h_2^{(k)}(z_0)$ for all $k \geq 0$. But $h_1^{(k)}(z_0)/k! = b_k$ and $h_2^{(k)}(z_0)/k! = c_k$ are precisely the formulae for Taylor coefficients, so $b_k = c_k$ for all $k \geq 0$.

Therefore we are justified in equating coefficients and we conclude $a_0 = a_1$, $a_1 = 2a_2$, $a_2 = 3a_3$, and so on, that is, $a_{k-1} = ka_k$ for all $k \geq 1$. Since $f(0) = 1$, we have $a_0 = 1$. Then either by observation or by induction, we quickly deduce $a_k = 1/k!$ which implies $g(z) = \sum_{k=0}^{\infty} z^k/k! = e^z$ as desired.

Theorem 10
The function $f(z) = e^z$ never attains 0.

Proof:
Let $f(z) = e^z$ and suppose towards a contradiction there exists $z_0 \in \mathbb{C}$ with $f(z_0) = 0$. Since we now know $f = f'$, we repeatedly differentiate both sides to conclude $f' = f''$, $f'' = f'''$, and so on. It follows that all derivatives of f are equal. Then in particular we have $0 = f(z_0) = f'(z_0) = f''(z_0) = f'''(z_0) = \cdots$ But this implies the Taylor series for f centered at z_0, given by $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$, is identically zero on all of $\mathbb{C}$ (and thus convergent on all of $\mathbb{C}$), implying $f(z) = e^z$ is just the constant zero function, which is clearly a contradiction (for instance, it contradicts $f(0) = 1$).

Now, most people would agree that the two most famous properties of the exponential function are $e^{z+w} = e^z e^w$ for all $z, w \in \mathbb{C}$ and $e^z = \frac{d}{dz}[e^z]$ for all $z \in \mathbb{C}$. These two properties are much closer than you may think.

Theorem 11
Let $g: \mathbb{C} \to \mathbb{C}$ be a function that is complex-differentiable on all of $\mathbb{C}$ and $g(0) = 1$. If $g(v + w) = g(v)g(w)$ for all $v, w \in \mathbb{C}$, then $g(z) = e^{az}$ for some $a \in \mathbb{C}$.

Proof:
Let $g(v + w) = g(v)g(w)$ for all $v, w \in \mathbb{C}$. Then,

$$g'(z) = \lim_{h \to 0} \frac{g(z + h) - g(z)}{h} = \lim_{h \to 0} \frac{g(z)g(h) - g(z)}{h} = g(z) \lim_{h \to 0} \frac{g(h) - 1}{h}$$

$$= g(z) \lim_{h \to 0} \frac{g(h) - g(0)}{h} = g(z)g'(0)$$

So we have shown $g'(z) = g'(0)g(z)$. Let $g(z)$ have the Taylor series $g(z) = \sum_{k=0}^{\infty} a_k z^k$, so $a_1 + 2a_2 z + 3a_3 z^2 + \cdots = a_1(a_0 + a_1 z + a_2 z^2 + \cdots)$. Equating coefficients and proceeding along a similar argument as in Theorem 9, we deduce $a_k = (a_1)^k/k!$ which implies $g(z) = \sum_{k=0}^{\infty} (a_1)^k z^k/k! = e^{a_1 z}$.

Summary

So what have we found after all this? By combining the results of Theorems 2, 9, and 11, we see that the following three statements are equivalent for a function $f: \mathbb{C} \to \mathbb{C}$.

(1) $f(z) = \sum_{k=0}^{\infty} z^k/k!$
(2) $f(0) = 1$ and $f(z)$ is differentiable on $\mathbb{C}$ with $f'(z) = f(z)$ on $\mathbb{C}$
(3) $f(z)$ is differentiable on $\mathbb{C}$, $f(0) = f'(0) = 1$, and $f(v + w) = f(v)f(w)$ for all $v, w \in \mathbb{C}$.

We have proven the implications (3) ⇔ (1) ⇔ (2). Therefore, *any* of the above three statements can in fact be used as the starting definition for e^z. The functions $\sin z$, $\cos z$, $\sinh z$, and $\cosh z$ also have multiple possible starting definitions from which all other properties may be derived. Most commonly they are defined via a power series or a differential equation with certain initial conditions, as shown below.

The following are equivalent on $\mathbb{C}$:

(1) $g(z) = \sum_{k=0}^{\infty} (-1)^k z^{2k+1}/(2k+1)!$
(2) $g(z) = \left(e^{iz} - e^{-iz}\right)/2i$
(3) $g''(z) = -g(z)$, $g'(0) = 1$, and $g(0) = 0$

and $g(z) = \sin z$.

The following are equivalent on $\mathbb{C}$:

(1) $g(z) = \sum_{k=0}^{\infty} (-1)^k z^{2k}/(2k)!$
(2) $g(z) = \left(e^{iz} + e^{-iz}\right)/2$
(3) $g''(z) = -g(z)$, $g'(0) = 0$, and $g(0) = 1$

and $g(z) = \cos z$.

The following are equivalent on $\mathbb{C}$:

(1) $g(z) = \sum_{k=0}^{\infty} z^{2k+1}/(2k+1)!$
(2) $g''(z) = g(z)$, $g'(0) = 1$, and $g(0) = 0$
(3) $g(z) = -i\sin(iz) = (e^z - e^{-z})/2$

and $g(z) = \sinh z$.

The following are equivalent on $\mathbb{C}$:

(1) $g(z) = \sum_{k=0}^{\infty} z^{2k}/(2k)!$
(2) $g''(z) = g(z)$, $g'(0) = 0$, and $g(0) = 1$
(3) $g(z) = \cos(iz) = (e^z + e^{-z})/2$

and $g(z) = \cosh z$.

Appendix

To see an example of when $(e^z)^w = e^{zw}$ fails, take $z = \ln(2) + 2\pi i$ and $w = i$. If we treat $(e^z)^w$ as $g(e^z)$ where $g(z)$ is the principal branch of z^w, then $(e^z)^w = e^{w\,\mathrm{Log}(e^z)} = e^{i\left(\ln\left|e^{\ln(2)+2\pi i}\right| + i\,\mathrm{Arg}\left(e^{\ln(2)+2\pi i}\right)\right)} = e^{i(\ln(2)+i\,\mathrm{Arg}(2))} = e^{i\ln(2)}$, which is clearly not equal to $e^{zw} = e^{-2\pi + i\ln(2)}$.