Evaluate:

$$I \coloneqq \int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}\,dx$$

Since the integrand is nonnegative and defined on $[-1,1]$ and $\sqrt{1-x^2} \leq 1$ on $[-1,1]$,

$$0 \leq \int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2} \leq \int_{-1}^{1} \frac{1}{1+x^2} = \frac{\pi}{2}$$

hence the integral converges.
We have:

$$I \coloneqq \int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2 x}{1+\sin^2 x}\,dx = 2\int_{0}^{\frac{\pi}{2}} \frac{\cos^2 x}{1+\sin^2 x}\,dx \qquad (1)$$

$$= 2\int_{0}^{\frac{\pi}{2}} \frac{1-\sin^2 x}{1+\sin^2 x}\,dx = 2\int_{0}^{\frac{\pi}{2}} \frac{1+\sin^2 x}{1+\sin^2 x} - \frac{2\sin^2 x}{1+\sin^2 x}\,dx = \pi - 4\int_{0}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+\sin^2 x}\,dx \qquad (2)$$

Equate the right-hand sides of (1) and (2),

$$\Rightarrow \int_{0}^{\frac{\pi}{2}} \frac{\cos^2 x}{1+\sin^2 x}\,dx = \frac{\pi}{2} - 2\int_{0}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+\sin^2 x}\,dx$$

$$\Rightarrow \left(\int_{0}^{\frac{\pi}{2}} \frac{\cos^2 x}{1+\sin^2 x}\,dx + \int_{0}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+\sin^2 x}\,dx\right) + \int_{0}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+\sin^2 x}\,dx = \frac{\pi}{2}$$

$$\Rightarrow \int_{0}^{\frac{\pi}{2}} \frac{1}{1+\sin^2 x}\,dx + \int_{0}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+\sin^2 x}\,dx = \frac{\pi}{2} \qquad (3)$$

Note:

$$\int_{0}^{\frac{\pi}{2}} \frac{1}{1+\sin^2 x}\,dx = \int_{0}^{\frac{\pi}{2}} \frac{\sec^2 x}{\sec^2 x + \tan^2 x}\,dx = \int_{0}^{\infty} \frac{dx}{2x^2+1} = \left[\frac{1}{\sqrt{2}}\arctan\left(x\sqrt{2}\right)\right]_{0}^{\infty} = \frac{\pi}{2\sqrt{2}}$$

Substituting into (3),

$$\Rightarrow \int_{0}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+\sin^2 x}\,dx = \frac{\pi}{2} - \frac{\pi}{2\sqrt{2}}$$

But returning to (1) and (2), we have:

$$I = \pi - 4\int_{0}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+\sin^2 x} = \pi - 4\left(\frac{\pi}{2} - \frac{\pi}{2\sqrt{2}}\right) = \pi\left(\sqrt{2}-1\right)$$

Details:
1st equality: Substitute $x \mapsto \sin x$.
2nd equality: Even integrand.
10th equality: Substitute $x \mapsto \arctan x$ and use $\sec^2 x = 1 + \tan^2 x$.

For alternate solution using complex analysis, see “complex_integrals” document.