We evaluate the following integrals:

$$I_k := \int_0^1 x^n (\ln x)^k dx \qquad\qquad \int_0^1 x^x dx$$

for $n \geq 0$ and $k \in \mathbb{N}_0$. The first integral will be used to solve the second one.
To solve the first integral, use integration by parts to create a recursion:

$$I_k = \int_0^1 x^n (\ln x)^k dx = \left[\frac{x^{n+1}}{n+1}(\ln x)^k\right]_0^1 - \int_0^1 \frac{x^{n+1}}{n+1} k(\ln x)^{k-1}\frac{1}{x}dx$$

$$= 0 - \lim_{x\to 0}\left(\frac{x^{n+1}(\ln x)^k}{n+1}\right) - \frac{k}{n+1}\int_0^1 x^n (\ln x)^{k-1} dx$$

We can evaluate the limit using L' Hopital's rule:

$$\lim_{x\to 0}\left(\frac{x^{n+1}(\ln x)^k}{n+1}\right) = \lim_{x\to 0}\left(\frac{(\ln x)^k}{(n+1)x^{-(n+1)}}\right) = \lim_{x\to 0}\left(\frac{k(\ln x)^{k-1}\frac{1}{x}}{-(n+1)^2 x^{-(n+2)}}\right)$$
$$= \lim_{x\to 0}\left(\frac{k(\ln x)^{k-1}}{-(n+1)^2 x^{-(n+1)}}\right)$$

Hence we see that repeated application of L' Hopital's rule will reduce the power of $\ln x$ but keep the denominator $x^{-(n+1)}$ the same. Then, after k applications of L' Hopital's rule, the limit will become $\lim_{x\to 0}\frac{c}{x^{-(n+1)}}$ for some constant c, but this limit is just 0.

$$\Rightarrow I_k = -\frac{k}{n+1}\int_0^1 x^n (\ln x)^{k-1} dx = -\frac{k}{n+1} I_{k-1}$$

Using this recursion, we have:

$$I_0 = \int_0^1 x^n dx = \frac{1}{n+1}$$

$$I_1 = -\frac{1}{n+1} I_0 = -\frac{1}{(n+1)^2}$$

$$I_2 = -\frac{2}{n+1} I_1 = \frac{2}{(n+1)^3}$$

Hence, we deduce that:

$$I_k = \frac{(-1)^k k!}{(n+1)^{k+1}}$$

(this can be proved by induction using the recursion)

We now evaluate the second integral.

$$\int_0^1 x^x dx = \int_0^1 e^{x \ln x} dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(x \ln x)^n}{n!} dx$$

To interchange the sum and integral, it suffices to show $\sum_{n=0}^{\infty} \frac{(x \ln x)^n}{n!}$ converges uniformly. We will use the M-test. Note,

$$\left|\frac{(x \ln x)^n}{n!}\right| = \frac{|x|^n |\ln x|^n}{n!} \leq \frac{|x|^n |x|^n}{n!} = \frac{|x|^{2n}}{n!} \leq \frac{1}{n!} := M_n$$

But $\sum_{n=0}^{\infty} M_n$ converges to e, hence the original sum converges uniformly. Then,

$$\int_0^1 \sum_{n=0}^{\infty} \frac{(x \ln x)^n}{n!} dx = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 (x \ln x)^n \, dx = \sum_{n=0}^{\infty} \frac{1}{n!} I_n = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{(-1)^n n!}{(n+1)^{n+1}} \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^{n+1}} = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \cdots \approx 0.783$$

In summary, for $n \geq 0$ and $k \in \mathbb{N}_0$,

$$\int_0^1 x^n (\ln x)^k dx = \frac{(-1)^k k!}{(n+1)^{k+1}}$$

$$\int_0^1 x^x dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^{n+1}}$$