Evaluate the integral and prove it converges: Wenzhi Tseng

$$J = -\int_0^1 \frac{\ln x}{x^2+1}\,dx = \int_0^\infty \frac{x}{2\cosh x}\,dx$$

The equality follows from the substitution $u = \ln x$ and subsequently noting that $e^u/(e^{2u}+1) = 1/(e^u + e^{-u})$. Recall that $\cosh z = (e^z + e^{-z})/2$. To see J converges, note $x/\cosh x$ is zero at $x = 0$ and is positive for $x > 0$, so

$$0 \le \int_0^\infty \frac{x}{2\cosh x}\,dx = \int_0^\infty \frac{x}{e^x + e^{-x}}\,dx \le \int_0^\infty \frac{x}{e^x}\,dx = 1$$

So J converges. Because $x/\cosh x$ is odd, contour integration is not so obvious here. Instead, we use integration by parts,

$$J = -\int_0^1 \frac{\ln x}{x^2+1}\,dx = -[(\arctan x)(\ln x)]_0^1 + \int_0^1 \frac{\arctan x}{x}\,dx$$

Using L'Hopital's, we have:

$$\lim_{x\to 0}(\arctan x)(\ln x) = \lim_{x\to 0}\frac{\arctan x}{1/\ln x} = \lim_{x\to 0}\frac{1/(1+x^2)}{\frac{-1}{(\ln x)^2 x}} = \lim_{x\to 0}\frac{-x(\ln x)^2}{1+x^2} = 0$$

where the last equality follows from $\lim_{x\to 0} x(\ln x)^2 = 0$, which is itself proved using two applications of L'Hopital's. Then,

$$J = \int_0^1 \frac{\arctan x}{x}\,dx = \int_0^1 \sum_{n=0}^\infty (-1)^n \frac{x^{2n}}{2n+1}\,dx$$

Note the power series converges on $[-1, 1]$ (can prove convergence at $x = \pm 1$ using alternating series test), hence it converges uniformly on $[0, 1]$ (Abel's theorem) so we can interchange integration and summation,

$$= \sum_{n=0}^\infty \int_0^1 (-1)^n \frac{x^{2n}}{2n+1}\,dx = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^2} = G$$

where G denotes Catalan's constant.