

Assuming they converge, evaluate the following integrals:

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$$I = \int_0^1 \frac{\ln^2 x}{x^2 - 1} dx \quad J = \int_0^{\frac{\pi}{2}} \theta \ln(\cos \theta) d\theta$$

We first solve  $I$ . Using partial fractions,

$$\int \frac{\ln^2 x}{x^2 - 1} dx = \frac{1}{2} \left( \int \frac{\ln^2 x}{x - 1} dx - \int \frac{\ln^2 x}{x + 1} dx \right)$$

Substitute  $u = x - 1$  and  $u = x + 1$  in the first and second integrals respectively,

$$= \frac{1}{2} \left( \int \frac{\ln^2(u + 1)}{u} du - \int \frac{\ln^2(u - 1)}{u} du \right)$$

Now use integration by parts, noting we can write  $\int u du = \ln(-u)$ ,

$$= \left[ \frac{\ln(-u) \ln^2(u + 1)}{2} - \int \frac{\ln(-u) \ln(u + 1)}{u + 1} du \right] - \left[ \frac{\ln u \ln^2(u - 1)}{2} - \int \frac{\ln u \ln(u - 1)}{u - 1} du \right]$$

Undoing the substitutions in the first and third terms and substituting  $u = v - 1$  and  $u = 1 - v$  in the integrals,

$$= \frac{1}{2} (\ln^2 x) \ln \left( \frac{1 - x}{1 + x} \right) + \int \frac{-\ln(1 - v) \ln(v)}{v} dv - \int \frac{-\ln(1 - v) \ln(-v)}{v} dv$$

From here on we make use of polylogarithms, see appendix for a brief review. We again use integration by parts,

$$\begin{aligned} &= \frac{1}{2} (\ln^2 x) \ln \left( \frac{1 - x}{1 + x} \right) + \left[ (\text{Li}_2(v)) \ln(v) - \int \frac{\text{Li}_2(v)}{v} dv \right] - \left[ (\text{Li}_2(v)) \ln(-v) - \int \frac{\text{Li}_2(v)}{v} dv \right] \\ &= \frac{1}{2} (\ln^2 x) \ln \left( \frac{1 - x}{1 + x} \right) + (\text{Li}_2(v)) \ln(v) - \text{Li}_3(v) - (\text{Li}_2(v)) \ln(-v) + \text{Li}_3(v) \end{aligned}$$

Undoing the substitutions,

$$= \frac{1}{2} (\ln^2 x) \ln \left( \frac{1 - x}{1 + x} \right) + (\ln x) (\text{Li}_2(x) - \text{Li}_2(-x)) + \text{Li}_3(-x) - \text{Li}_3(x) + c, c \in \mathbb{R}$$

Let the above antiderivative be  $F(x)$ . So we need to compute  $\lim_{x \rightarrow 1} F(x) - \lim_{x \rightarrow 0} F(x)$ . Note that  $\text{Li}_s(x)$  has the power series expansion  $\text{Li}_s(x) = \sum_{k=1}^{\infty} x^k / k^s$  for  $|x| \leq 1$ , and the sum converges absolutely by ratio test for  $s > 0$ . Then, we are justified in the following series manipulations:

$$\text{Li}_s(x) - \text{Li}_s(-x) = \left( x + \frac{x^2}{2^s} + \frac{x^3}{3^s} + \dots \right) - \left( -x + \frac{x^2}{2^s} + \frac{x^3}{3^s} - \dots \right) = 2 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)^s}$$

Then,

$$F(x) = \frac{1}{2} (\ln^2 x) \ln \left( \frac{1 - x}{1 + x} \right) + 2(\ln x) \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)^2} - 2 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)^3} + c$$

We now compute the limits required to find  $\lim_{x \rightarrow 0} F(x)$ . The most troublesome limit is

$$\lim_{x \rightarrow 0} (\ln x) \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)^2} = \lim_{x \rightarrow 0} \frac{\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)^2}}{1/\ln x}$$

The sum clearly converges at  $x = 1$ , so it converges absolutely on  $[-1, 1]$  and is thus infinitely differentiable and term-by-term differentiation is allowed and all derivatives will converge on  $(0, 1)$ . Then, by L'Hopital's,

$$= \lim_{x \rightarrow 0} \frac{\sum_{k=0}^{\infty} \frac{x^{2k}}{2k+1}}{-1/x \ln^2 x} = \lim_{x \rightarrow 0} \frac{\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}}{-1/\ln^2 x} = \lim_{x \rightarrow 0} \frac{\sum_{k=0}^{\infty} x^{2k}}{2/x \ln^3 x} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{x \ln^3 x}{1 - x^2} = 0$$

since  $\lim_{x \rightarrow 0} x \ln^3 x = 0$  (also by repeated L'Hopital's).

It can also be shown that  $\lim_{x \rightarrow 0} (\ln^2 x) \ln \left( \frac{1-x}{1+x} \right) = \lim_{x \rightarrow 1} (\ln^2 x) \ln \left( \frac{1-x}{1+x} \right) = 0$  using repeated L'Hopital's. Putting everything together, we have  $\lim_{x \rightarrow 0} F(x) = 0$  and thus

$$\int_0^1 \frac{\ln^2 x}{x^2 - 1} dx = \lim_{x \rightarrow 1} F(x) - \lim_{x \rightarrow 0} F(x) = -2 \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3}$$

But notice,

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} + \sum_{k=1}^{\infty} \frac{1}{(2k)^3} = \sum_{k=1}^{\infty} \frac{1}{k^3} = \zeta(3)$$

So the second sum is just  $\zeta(3)/8$ . Then  $\sum_{k=0}^{\infty} 1/(2k+1)^3 = 7\zeta(3)/8$ , so

$$\int_0^1 \frac{\ln^2 x}{x^2 - 1} dx = -\frac{7}{4} \zeta(3)$$

Now we solve  $J$ . We will freely use the following result, which is proved (using symmetry tricks) in my document "ln(sinx) and related integrals"

$$\int_0^{\pi/2} \ln(\sin \theta) d\theta = \int_0^{\pi/2} \ln(\cos \theta) d\theta = -\frac{\pi}{2} \ln 2 \quad (1)$$

Substituting  $\theta \mapsto \pi/2 - \theta$  and using law of logs,

$$J = \int_0^{\pi/2} \theta \ln(\cos \theta) d\theta = \int_0^{\pi/2} \left( \frac{\pi}{2} - \theta \right) \ln(\sin \theta) d\theta = \frac{\pi}{2} \int_0^{\pi/2} \ln(\sin \theta) d\theta - \int_0^{\pi/2} \theta \ln(\sin \theta) d\theta$$

Rearranging and using (1),

$$\Rightarrow \int_0^{\pi/2} \theta \ln(\cos \theta) d\theta + \int_0^{\pi/2} \theta \ln(\sin \theta) d\theta = -\frac{\pi^2}{4} \ln 2 \quad (2)$$

On the other hand, by laws of logs,

$$\int_0^{\frac{\pi}{2}} \theta \ln(\sin \theta) d\theta - \int_0^{\frac{\pi}{2}} \theta \ln(\cos \theta) d\theta = \int_0^{\frac{\pi}{2}} \theta \ln(\tan \theta) d\theta \quad (3)$$

Define the following function for  $t \geq 0$ ,

$$J(t) = \int_0^{\frac{\pi}{2}} (\arctan(t \tan \theta)) \ln(\tan \theta) d\theta$$

Now differentiate under the integral. We skip over verifying the convergence/uniform convergence of the relevant integrals.

$$\Rightarrow J'(t) = \int_0^{\frac{\pi}{2}} \frac{\tan \theta}{1 + t^2 \tan^2 \theta} \ln(\tan \theta) d\theta$$

Substitute  $x = \tan \theta$ ,

$$= \int_0^{\infty} \frac{x \ln x}{(1 + t^2 x^2)(1 + x^2)} dx$$

This integral suggests contour integration due to the polynomial in the denominator and the rather straightforward function  $x \ln x$  in the numerator. We choose  $\log z$  with branch cut  $[0, \infty)$  and  $\arg z \in (0, 2\pi)$  and integrate the function  $f(z) = \frac{z \log^2 z}{(1 + t^2 z^2)(1 + z^2)}$  over the corresponding keyhole contour. For  $t \neq 0$  and  $t \neq 1$ , note  $f$  has four simple poles  $\pm i, \pm i/t$  on the imaginary axis inside the contour. By residue theorem, we find

$$\begin{aligned} & \int_0^{\infty} \frac{x(\ln x)^2}{(1 + t^2 x^2)(1 + x^2)} dx - \int_0^{\infty} \frac{x(\ln x + 2\pi i)^2}{(1 + t^2 x^2)(1 + x^2)} dx = 2\pi i \sum \text{Res} \\ & -4\pi i \int_0^{\infty} \frac{x \ln x}{(1 + t^2 x^2)(1 + x^2)} dx + 4\pi^2 \int_0^{\infty} \frac{x}{(1 + t^2 x^2)(1 + x^2)} dx = 2\pi i \frac{\ln^2 t - 2\pi i \ln t}{t^2 - 1} \end{aligned}$$

Equating real and imaginary parts yields

$$\int_0^{\infty} \frac{x \ln x}{(1 + t^2 x^2)(1 + x^2)} dx = \frac{\ln^2 t}{2(1 - t^2)} \quad \int_0^{\infty} \frac{x}{(1 + t^2 x^2)(1 + x^2)} dx = \frac{\ln t}{t^2 - 1}$$

So  $J'(t) = \frac{\ln^2 t}{2(1 - t^2)}$  for  $t > 0, t \neq 1$ . But note  $\lim_{t \rightarrow 1} \frac{\ln^2 t}{2(1 - t^2)} = 0$  by L'Hopital's so we can extend  $J'(t) = \int_0^{\infty} \frac{x \ln x}{(1 + t^2 x^2)(1 + x^2)} dx$  to a continuous function on  $(0, \infty)$  where we define  $J'(1) = 0$ .

The antiderivative of  $\frac{\ln^2 t}{2(1 - t^2)}$  was computed previously,

$$\Rightarrow J(t) = -\frac{1}{2} \left( \frac{1}{2} (\ln^2 t) \ln \left| \frac{1 - t}{1 + t} \right| + (\ln t) (\text{Li}_2(t) - \text{Li}_2(-t)) - \text{Li}_3(t) + \text{Li}_3(-t) \right) + c$$

for some  $c \in \mathbb{R}$ . However, since  $J(0) = \int_0^{\frac{\pi}{2}} (\arctan(0 \tan \theta)) \ln(\tan \theta) d\theta = 0$ , taking  $t \rightarrow 0$  in the above expression (also done previously) and solving for  $c$  shows that  $c = 0$ .

Recall our original goal was to find the value of  $J(1)$  and substitute it into (3). The computation of  $J(1)$  was also done previously, so we have

$$\int_0^{\frac{\pi}{2}} \theta \ln(\tan \theta) d\theta = J(1) = \frac{7}{8}\zeta(3)$$

Look familiar? That's right. Our integrals  $I$  and  $J$  are related:

$$\int_0^1 \frac{\ln^2 x}{x^2 - 1} dx = -2 \int_0^{\frac{\pi}{2}} \theta \ln(\tan \theta) d\theta = -\frac{7}{4}\zeta(3)$$

How coincidental. But now we finish this question. Returning to (3), we now have

$$\int_0^{\frac{\pi}{2}} \theta \ln(\sin \theta) d\theta - \int_0^{\frac{\pi}{2}} \theta \ln(\cos \theta) d\theta = \frac{7}{8}\zeta(3)$$

And from (2), we had

$$\int_0^{\frac{\pi}{2}} \theta \ln(\cos \theta) d\theta + \int_0^{\frac{\pi}{2}} \theta \ln(\sin \theta) d\theta = -\frac{\pi^2}{4} \ln 2$$

Solving simultaneously yields:

$$\int_0^{\frac{\pi}{2}} \theta \ln(\sin \theta) d\theta = \frac{7}{16}\zeta(3) - \frac{\pi^2}{8} \ln 2 \quad \int_0^{\frac{\pi}{2}} \theta \ln(\cos \theta) d\theta = -\frac{7}{16}\zeta(3) - \frac{\pi^2}{8} \ln 2$$

## Appendix: The Polylogarithm

In its most general form, the polylogarithm  $\text{Li}_s(z)$  of order  $s$  is a complex-valued function defined as a power series and extended via analytic continuation, but for the purposes of our integration problem, we will only consider  $s = 2, 3, 4, \dots$  and  $z \in [-1, 1]$ .

For  $|x| \leq 1$  and  $s = 2, 3, 4, \dots$  the following power series converges absolutely by ratio test (and p-series test for  $x = 1$ ) and is defined to be  $\text{Li}_s(x)$ .

$$\text{Li}_s(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^s} = x + \frac{x^2}{2^s} + \frac{x^3}{3^s} + \dots$$

Consequently we can differentiate term-by-term,

$$\begin{aligned} \frac{d}{dx} \text{Li}_{s+1}(x) &= \frac{d}{dx} \left[ x + \frac{x^2}{2^{s+1}} + \frac{x^3}{3^{s+1}} + \dots \right] = 1 + \frac{x}{2^s} + \frac{x^2}{3^s} + \dots \\ &= \frac{1}{x} \left( x + \frac{x^2}{2^s} + \frac{x^3}{3^s} + \dots \right) = \frac{\text{Li}_s(x)}{x} \end{aligned}$$

So  $\text{Li}_{s+1}(x)$  is an antiderivative of  $\text{Li}_s(x)/x$ .

We now prove the following identity for  $|x| \leq 1$ .

$$\text{Li}_2(x) = - \int_0^x \frac{\ln(1-v)}{v} dv$$

We first prove the integral  $\int_0^x \ln(1-v)/v dv$  exists and is continuous for  $x \in [-1, 1]$ . Note  $\lim_{v \rightarrow 0} \ln(1-v)/v = -1$  by L'Hopital's, so we can extend the integrand to a continuous function at  $v = 0$  and thus the integral converges as  $v \rightarrow 0$ , hence  $\int_0^x \ln(1-v)/v dv$  converges for all  $x \in [-1, 1]$ . It remains to show  $\int_0^1 \ln(1-v)/v dv$  converges since  $v = 1$  is the only other point at which the integrand is undefined. Since we already know the integral converges as  $v \rightarrow 0$ , it suffices to show something like  $\int_{2/3}^1 \ln(1-v)/v dv$  converges. Substitute  $u = -\ln(1-v)$ , then

$$\begin{aligned} 0 &\leq - \int_{2/3}^1 \frac{\ln(1-v)}{v} dv = \int_{\ln 3}^{\infty} \frac{u}{e^u - 1} du \leq \int_1^{\infty} \frac{u}{e^u - 1} du \leq \int_1^{\infty} \frac{u}{e^u - \frac{1}{2}e^u} du \\ &= 2 \int_1^{\infty} u e^{-u} du \leq 2 \int_0^{\infty} u e^{-u} du = 2\Gamma(1) = 2 \end{aligned}$$

Hence  $\int_0^1 \ln(1-v)/v dv$  converges. Since we have now shown  $\int_0^x \ln(1-v)/v dv$  converges for all  $|x| \leq 1$  and clearly the integrand is continuous on  $(0, 1)$ , we have that  $\int_0^x \ln(1-v)/v dv$  is continuous for  $x \in [-1, 1]$ .

Now we prove the identity. Using Taylor series,

$$- \int_0^x \frac{\ln(1-v)}{v} dv = - \int_0^x \sum_{k=1}^{\infty} \frac{-v^{k-1}}{k} dv$$

Fix  $x \in (-1, 1)$ . Then when considering the integral, we have  $0 < |v| \leq |x|$ . Then the summand  $-v^{k-1}/k$  is bounded by  $|x|^{k-1}/k$  on the interval  $[-x, x]$  and  $\sum_{k=1}^{\infty} |x|^{k-1}/k$  converges by ratio test, so by M-test, the series converges uniformly on  $[-x, x]$ , allowing us to interchange integration and summation,

$$= \sum_{k=1}^{\infty} \int_0^x \frac{v^{k-1}}{k} dv = \sum_{k=1}^{\infty} \left[ \frac{v^k}{k^2} \right]_0^x = \sum_{k=1}^{\infty} \frac{x^k}{k^2} = \text{Li}_2(x)$$

It remains to show the identity holds at  $x = \pm 1$ . However this follows from the fact that  $\text{Li}_2(x) = \sum_{k=1}^{\infty} x^k/k^2$  is continuous on  $[-1, 1]$  (by term-by-term continuity) and  $\int_0^x \ln(1-v)/v dv$  is also continuous on  $[-1, 1]$  (shown previously), and since the two functions already coincide on  $(-1, 1)$ , continuity forces them to also coincide at the endpoints  $x = \pm 1$ .