Complex Analysis: Functions with Branch Cuts and Integration

This document is meant to serve as an educational resource for students who wish to learn more about complex functions with multiple branches, as it is a topic that I feel is often underemphasized or underexplained. In particular, when I tried to learn complex analysis from textbooks and websites, I felt that authors did not fully take the time to explain things carefully and often failed to provide adequate definitions of the terminology they were using. I hope this document will provide a more patient, detailed overview of the topic at the introductory level. This is the resource I wish I had when first learning about functions with branch cuts and how to integrate them.

There are three sections. The first focuses on defining and understanding functions with multiple branches and assumes only the first few weeks of an undergraduate complex analysis course. It defines all the relevant terminology and then goes into constructing branches of $\sqrt{1 - z^2}$ by multiplying branches of $\sqrt{1 - z}$ and $\sqrt{1 + z}$ together. This is not a common exercise but I think it goes a long way in demonstrating why we must be careful when manipulating square roots in $\mathbb{C}$, and it shows how continuity can be gained or lost when multiplying together different branches. It also explicitly demonstrates how the branch of a function such as $\sqrt{z}$ depends on where $\arg z$ comes from, which is important when a particular branch of a function must be selected to solve certain contour integrals. Overall I strongly believe it is a very useful exercise for students who are unfamiliar with branches and branch cuts. The second section contains worked examples of integrating functions with branch cuts and presumes that the student has already been exposed to residues. We notably avoid using Laurent series to calculate residues in this section, as I believe it is more instructive to find the equivalent residue of $-1/z^2 f(1/z)$ at zero by constructing an appropriate branch of $1/z^2 f(1/z)$. However, the third section does discuss how to find Taylor and Laurent series of functions with branch cuts and how different branches will have different series, and we conclude with the traditional method of finding residues at infinity by computing the appropriate Laurent series.

There are of course faster techniques for solving many of the integrals in this document (such as differentiation under the integral sign), and even within my solutions there are probably ways to streamline my working. If you find a faster method or notice something, do let me know at wtseng9@gmail.com.

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Branches and Branch Cuts

Definition 1.1
Let $f(z)$ be a function of $\arg z$ (such as $\log z = \log|z| + i \arg z$) such that f is continuous and single-valued when we specify $\arg z$ to only take on values from a certain set E. Then $f: D \to \mathbb{C}$ where $D := \{z \in \mathbb{C} : \arg z \in E\}$ is a *branch* of $f(z)$. That is, for our purposes, a branch of f is a function specified by two things: the set of values E that $\arg z$ is allowed to take on, and the formula for f itself.

Example 1.2
Let $D_1 = \{z \in \mathbb{C} : \arg z \in (-\pi, \pi)\}$ and $D_2 = \{z \in \mathbb{C} : \arg z \in (\pi, 3\pi)\}$ and define functions $g_1: D_1 \to \mathbb{C}, g_1(z) = \log z$ and $g_2: D_2 \to \mathbb{C}, g_2(z) = \log z$. Note $D_1 = D_2 = \mathbb{C} \backslash (-\infty, 0]$, so both g_1 and g_2 have the same domain. However, they are different functions because, for example, $g_1(1) = \log|1| + i \arg(1) = 0$ but $g_2(1) = \log|1| + i \arg(1) = 2\pi$. It is precisely because $\arg z$ took on different values in D_1 and D_2 that we have $g_1(1) \neq g_2(1)$, and in fact, $g_1(z) \neq g_2(z)$ for all $z \in \mathbb{C} \backslash (-\infty, 0]$. So we say that g_1 and g_2 are different branches of $\log z$ with domain $\mathbb{C} \backslash (-\infty, 0]$. We emphasize that the set of values that $\arg z$ takes on has uniquely specified the branch: the fact that $\arg z \in (-\pi, \pi)$ implies the domain of g_1 is $\mathbb{C} \backslash (-\infty, 0]$ and determines the output of g_1 at every element in that domain.

In fact, there are infinitely many branches of $\log z$ with domain $\mathbb{C} \backslash (-\infty, 0]$, each corresponding to $\arg z$ lying in the interval $(-\pi + 2\pi k, \pi + 2\pi k)$ for some $k \in \mathbb{Z}$. There are also infinitely many branches of $\log z$ with domain $\mathbb{C} \backslash [0, \infty)$, each corresponding to $\arg z$ lying in the interval $(2\pi k, 2\pi + 2\pi k)$ for some $k \in \mathbb{Z}$.

Definition 1.3
A branch cut of a function is a portion of domain that is removed from $\mathbb{C}$ to ensure the function is continuous and single-valued.

Example 1.4
Returning to g_1, the domain was $D_1 = \{z \in \mathbb{C} : \arg z \in (-\pi, \pi)\}$ was equal to $\mathbb{C} \backslash (-\infty, 0]$. That is, the domain is $\mathbb{C}$ but excluding the ray $(-\infty, 0]$, so we say that g_1 has branch cut $(-\infty, 0]$.

Definition 1.5 (informal)
A branch point z_0 of a function $f(z)$ is a point at which there exists a circle $\{|z - z_0| = \epsilon\}$ on which f is discontinuous.

Example 1.6
Consider $z_0 = 0$ and $f(z) = \arg z$. Let z traverse the circle $\{|z - z_0| = \epsilon\}$ counterclockwise with arbitrary starting point a. Then $\arg z$ will have increased by 2π by the time z traverses the circle and returns to a, meaning $\arg z$ will approach two different values (separated by 2π) at a, making $\arg z$ discontinuous at a. Hence $z_0 = 0$ is a branch point of $\arg z$. It follows that certain functions of $\arg z$ such as $\log z$ and $\sqrt{z}$ also have 0 as a branch point.

Definition 1.7
We say $z = \infty$ is a branch point of $f(z)$ if $z = 0$ is a branch point of $f(1/z)$.

Example 1.8
Consider $f(z) = \arg z$. Then $f(1/z) = \arg(1/z) = \arg(\bar{z}/|z|^2) = -\arg z$. Since $z = 0$ is a branch point of $\arg z$, it is also a branch point of $-\arg z$. So $z = \infty$ is a branch point of $\arg z$. It follows that $z = \infty$ is also a branch point of functions like $\log z$ and $\sqrt{z}$.

Theorem 1.9 (informal)
A branch cut must connect all branch points of a function in order for the function to be continuous.

Proof: Omitted, but there is an intuition. Essentially, we must remove enough of the domain so that there will be no circle around a branch point on which f is discontinuous. For the case of $\arg z$, if a branch cut did not connect 0 and ∞, i.e. if there was a "gap" in the branch cut, then we could have a circle around the branch point that runs through the gap and f would be discontinuous on this circle for the same reason described in Example 1.6. Therefore, the branch cut must remove a curve that connects 0 and ∞. The curve should be simply connected.

Example 1.10
Hence any branch cut of $\arg z$, $\log z$, and $\sqrt{z}$ must connect $z = 0$ and $z = \infty$. Common examples include $[0, \infty)$ and $(-\infty, 0]$, but the branch cuts $[0, i\infty)$ and $[0, -i\infty)$ along the imaginary axis are also sometimes useful.

Definition 1.12
The principal branch of a function is the branch corresponding to $\arg z \in (-\pi, \pi)$. For example, the principal branch of $\arg z$ is denoted $\mathrm{Arg}(z)$ and has image $(-\pi, \pi)$. The principal branch of $\log z$ is denoted $\mathrm{Log}(z)$ and is defined as $\log|z| + i\,\mathrm{Arg}(z)$.

Example 1.11
The definition of the complex logarithm $\log z = \log|z| + i \arg z$ implies that the set of values that $\arg z$ takes on *uniquely* specifies the branch of $\log$. However, this is not always the case. One example is the complex square root, defined as $\sqrt{z} = e^{\frac{1}{2}\log|z|}e^{\frac{1}{2}i\arg z} = |\sqrt{z}|e^{\frac{1}{2}i\arg z}$. We can let $\arg z$ take on values from $(-\pi + 2\pi k, \pi + 2\pi k)$ for any $k \in \mathbb{Z}$ and the branch of $\sqrt{z}$ will still be the same since $e^{\frac{1}{2}i\arg z} = e^{\frac{1}{2}i(\arg z + 2\pi k)}$ for all $z \in \mathbb{C}$.

There are only two branches of $\sqrt{z}$ with branch cut $(-\infty, 0]$, corresponding to the positive and negative root. To see this, we first define a specific branch of $\sqrt{z}$. Define $g(z)$ to be the principal branch of $\sqrt{z}$. So $g(z) = e^{\frac{1}{2}\log|z|}e^{i\frac{1}{2}\mathrm{Arg}\,z}$ and has domain $\mathbb{C}\backslash(-\infty, 0]$. Now, for any z in $\mathbb{C}\backslash(-\infty, 0]$, $\arg z$ must equal $\mathrm{Arg}\,z + 2k\pi$ for some $k \in \mathbb{Z}$. Then,

$$\sqrt{z} = e^{\frac{1}{2}\log z} = e^{\frac{1}{2}\log|z|}e^{i\frac{1}{2}\arg z} = e^{\frac{1}{2}\log|z|}e^{i\frac{1}{2}(\text{Arg }z + 2k\pi)} = \left(e^{\frac{1}{2}\log|z|}e^{i\frac{1}{2}\text{Arg }z}\right)e^{ik\pi} = \pm g(z)$$

That is, any branch of $\sqrt{z}$ with branch cut $(-\infty, 0]$ is either the positive or the negative of $g(z)$, hence these are the only two branches: choosing $\arg z \in (-\pi + 2k\pi, \pi + 2k\pi)$ will lead to $\sqrt{z}$ being equal to $g(z)$ and $-g(z)$ when k is even and odd respectively. This reasoning can be easily extended to show that any $\sqrt{z}$ whose branch cut is a straight line segment from 0 to ∞ only has two branches that are negatives of each other, and there are generalizations for z^{α} for any $\alpha \in \mathbb{C}$ as well.

Exercise 1.14
Show there are only three branches of $z^{1/3}$ with branch cut $(-\infty, 0]$, and that if we denote one of the branches to be $g(z)$, then the other two branches are $e^{i2\pi/3}g(z)$ and $e^{i4\pi/3}g(z)$.

Solution:
Let $g(z)$ be the principal branch of $z^{1/3}$, so $g(z)$ has branch cut $(-\infty, 0]$. Again, we have $\arg z = \text{Arg } z + 2k\pi$ for some $k \in \mathbb{Z}$. Then,

$$z^{1/3} = e^{\frac{1}{3}\log z} = e^{\frac{1}{3}\log|z|}e^{i\frac{1}{3}\arg z} = e^{\frac{1}{3}\log|z|}e^{i\frac{1}{3}(\text{Arg }z + 2k\pi)} = \left(e^{\frac{1}{3}\log|z|}e^{i\frac{1}{3}\text{Arg }z}\right)e^{i\frac{2\pi k}{3}} = g(z)e^{i\frac{2\pi k}{3}}$$

Hence the three branches are $g(z)$, $g(z)e^{i2\pi/3}$, and $g(z)e^{i4\pi/3}$ (any value of $k \in \mathbb{Z}$ will lead to one of these three branches).

We now illustrate a more complicated example. There are many branches of $\sqrt{1-z^2}$ depending on what branch and branch cuts we choose, but we can construct such branches of $\sqrt{1-z^2}$ as follows: take a branch of $\sqrt{1-z}$ and a branch of $\sqrt{1+z}$ and multiply them together. We have:

$$\sqrt{1-z}\sqrt{1+z} = e^{i\log\sqrt{1-z}}e^{i\arg\sqrt{1-z}}e^{i\log\sqrt{1+z}}e^{i\arg\sqrt{1+z}}$$

$$= |\sqrt{1-z}||\sqrt{1+z}|e^{i\frac{1}{2}(\arg(1-z)+\arg(1+z))} = \left|\sqrt{1-z^2}\right|e^{i\arg\sqrt{1-z^2}}$$

This is the polar form for $\sqrt{1-z^2}$, so this is precisely a branch of $\sqrt{1-z^2}$ that we constructed from our $\sqrt{1-z}$ and $\sqrt{1+z}$, where at each input z, our $\sqrt{1-z^2}$ has argument equal to $(1/2)(\arg(1-z)+\arg(1+z))$ and magnitude equal to $|\sqrt{1-z^2}|$. Since our $\sqrt{1-z}$ and $\sqrt{1+z}$ were both single-valued, our $\sqrt{1-z^2}$ is also single-valued. However, the region over which our $\sqrt{1-z^2}$ is continuous depends on the branches chosen for $\sqrt{1-z}$ and $\sqrt{1+z}$. We will do some examples next to illustrate this.[1] At this point, it is important to clarify that our $\arg(1-z)$ and $\arg(1+z)$ are different functions and are not related to each other: our branch of $\sqrt{1-z}$ is a function of $\arg(1-z)$ and our branch of $\sqrt{1+z}$ is a function of $\arg(1+z)$.

Suppose we take the branch of $\sqrt{1+z}$ with $\arg(1+z) \in (0, 2\pi)$, so the branch cut is $[0,\infty)$. Suppose we take the branch of $\sqrt{1-z}$ with $\arg(1-z) \in (-\pi, \pi)$, so the branch cut is $(-\infty, 0]$. We sketch the branch cuts and label the arguments on the diagrams below:
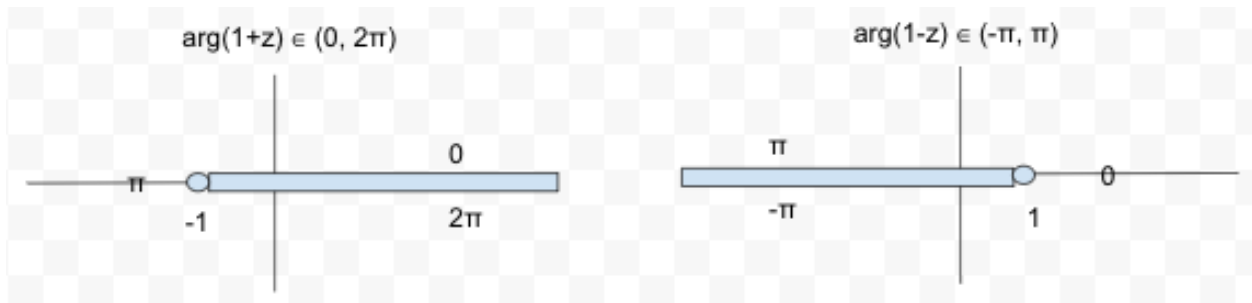

Fig. 1

We are multiplying $\sqrt{1+z}$ and $\sqrt{1-z}$ together at each point in the complex plane to find $\sqrt{1-z^2}$. Since $\arg\sqrt{1-z^2} = (1/2)(\arg(1-z)+\arg(1+z))$, we just need to add the arguments of $1-z$ and $1+z$ and then divide by two at each point in the complex plane to find $\arg\sqrt{1-z^2}$. We use this process to plot some values of $\arg\sqrt{1-z^2}$ below.

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[1] Our derivation also shows that decomposing a preexisting branch of $\sqrt{1-z^2}$ into $\sqrt{1-z}\sqrt{1+z}$ is not easy because you would have to carefully work out which branches of $\sqrt{1-z}$ and $\sqrt{1+z}$ were multiplied together to get the $\sqrt{1-z^2}$ you have. There are often multiple possibilities.
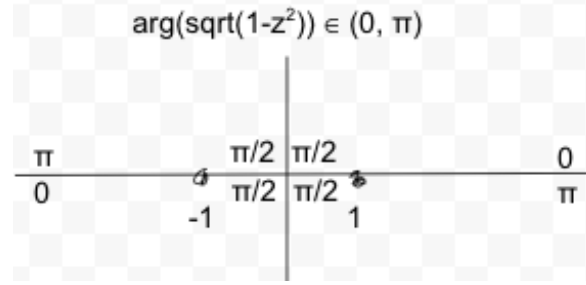
arg(sqrt(1-z²)) ∈ (0, π)

π/2 π/2
π/2 π/2

π   0
0   π
-1   1

Fig. 2

We notice that $\arg\sqrt{1-z^2} \in (0,\pi)$. We also notice that $\arg\sqrt{1-z^2}$ approaches π from below $[1,\infty)$ but approaches 0 from above $[1,\infty)$, so $\sqrt{1-z^2}$ will approach $\left|\sqrt{1-z^2}\right|e^{i\pi}$ from below $[1,\infty)$ but approach $\left|\sqrt{1-z^2}\right|e^{i0}$ from above $[1,\infty)$. A similar situation occurs with $(-\infty,-1]$. Therefore our $\sqrt{1-z^2}$ is discontinuous only across the rays $(-\infty,-1]$ and $[1,\infty)$, so we must remove these rays to ensure our $\sqrt{1-z^2}$ is continuous. That is, our choices of branch for $\sqrt{1+z}$ and $\sqrt{1-z}$ resulted in a branch of $\sqrt{1-z^2}$ with branch cut $(-\infty,-1] \cup [1,\infty)$ as shown below:
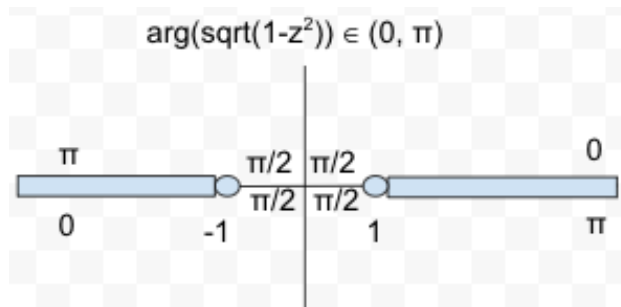
arg(sqrt(1-z²)) ∈ (0, π)

π/2 π/2
π/2 π/2

π   0
0   π
-1   1

Fig. 3

Notice that our $\sqrt{1+z}$ and $\sqrt{1-z}$ are themselves discontinuous across the interval $(-1,1)$, but this continuity is regained for $\sqrt{1-z^2}$. What is going on is that the product $\sqrt{1+z}\sqrt{1-z} = \sqrt{1-z^2}$ happens to approach the same value if you approach the interval $(-1,1)$ from above and below the real axis (in this case, $\arg\sqrt{1-z^2}$ approaches $\pi/2$ from both above and below). Therefore, we can continuously extend $\sqrt{1-z^2}$ to the interval $(-1,1)$ even though this interval was not in the domains of $\sqrt{1+z}$ and $\sqrt{1-z}$.

It also seems like the this branch cut does not connect the branch points -1 and 1. However, $z = \infty$ is considered to be a "single point" in the extended complex plane (you can imagine the Riemann sphere to see this) so in fact, the rays $(-\infty,-1]$ and $[1,\infty)$ can be said to meet at $z = \infty$.

The last thing to point out is that $\arg\sqrt{1-z^2}$ being continuous and single-valued is a *sufficient* condition for $\sqrt{1-z^2}$ being continuous and single-valued, but not a *necessary* one. As we will see later, it is possible for $\sqrt{1-z^2}$ to be continuous across a line even if $\arg\sqrt{1-z^2}$ is discontinuous across that line. But in this example, both $\sqrt{1-z^2}$ and $\arg\sqrt{1-z^2}$ have the same branch cuts.

Exercise 1.15

Keep the same branch cuts as before. If we had chosen $\arg(1+z) \in (2\pi, 4\pi)$ and $\arg(1-z) \in (-\pi, \pi)$, what branch of $\sqrt{1-z^2}$ would we obtain? How about if we had chosen $\arg(1+z) \in (2\pi, 4\pi)$ and $\arg(1-z) \in (\pi, 3\pi)$?

Solution:

The first construction would lead to a different branch of $\sqrt{1-z^2}$ than in Fig. 3 but with the same branch cut. The second construction would lead to the exact same branch as in Fig. 3 since $e^{iz} = e^{i(z+2\pi)}$.

Exercise 1.16

Consider a new set of branch cuts for $\sqrt{1+z}$ and $\sqrt{1-z}$ shown below:

Fig. 4

By picking branches for $\sqrt{1+z}$ and $\sqrt{1-z}$ that are consistent with the above branch cuts, show that the product $\sqrt{1+z}\sqrt{1-z} = \sqrt{1-z^2}$ has the exact same branch cut as before, that is, the branch cut is still $(-\infty, -1] \cup [1, \infty)$.

Therefore, looking at Fig. 1 and Fig. 4, we see that *two different sets of branch cuts for $\sqrt{1+z}$ and $\sqrt{1-z}$ lead to the same branch cut for $\sqrt{1-z^2}$*. Of course, which branch of $\sqrt{1-z^2}$ we get depends on which branches were picked for $\sqrt{1+z}$ and $\sqrt{1-z}$.

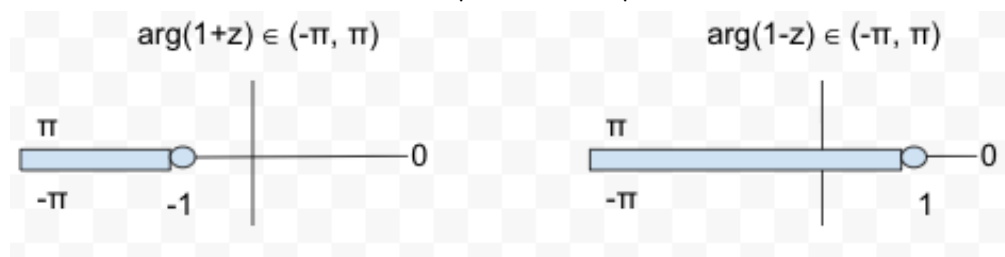Consider another set of branch cuts for $\sqrt{1+z}$ and $\sqrt{1-z}$ shown below:

Fig. 5

So we have the branch cut $(-\infty, -1]$ for both $\arg(1+z)$ and $\arg(1-z)$ and we have chosen $\arg(1+z), \arg(1-z) \in (-\pi, \pi)$. By adding the arguments and then dividing by two at each point in the complex plane just like before, we plot some values of $\arg\sqrt{1-z^2}$ below:
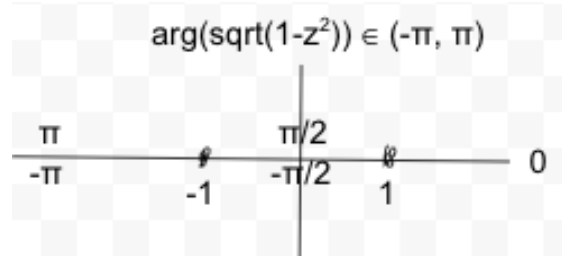
arg(sqrt(1-z²)) ∈ (-π, π)

Fig. 6

However, note $\sqrt{1-z^2} = \left|\sqrt{1-z^2}\right|e^{i\arg\sqrt{1-z^2}}$ has the same value if $\arg\sqrt{1-z^2} = \pm\pi$. That is, although $\arg\sqrt{1-z^2}$ is discontinuous across the ray $(-\infty, -1)$, the function $\sqrt{1-z^2}$ itself is not! Therefore we can extend our $\sqrt{1-z^2}$ continuously over $(-\infty, -1)$, leading to the following branch cut for $\sqrt{1-z^2}$ below:
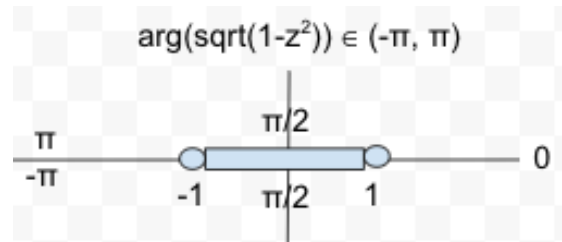

arg(sqrt(1-z²)) ∈ (-π, π)

Fig. 7

So in this case, we end up with a branch of $\sqrt{1-z^2}$ with branch cut $[-1, 1]$ and $\arg\sqrt{1-z^2} \in (-\pi, \pi)$. **Note that $[-1, 1]$ is the branch cut for $\sqrt{1-z^2}$, but $\arg\sqrt{1-z^2}$ would need the branch cut $(-\infty, 1]$ in order to be continuous. Do not get the branch cuts confused! In this document, all plots will display the values of $\arg f(z)$ but the branch cuts shown will be the branch cuts for $f(z)$ itself.**

Exercise 1.17
Suppose we had chosen the branch cut $[-1, \infty)$ for $\sqrt{1+z}$ and the branch cut $[1, \infty)$ for $\sqrt{1-z}$. Show that the product $\sqrt{1+z}\sqrt{1-z} = \sqrt{1-z^2}$ also has the branch cut $[-1, 1]$. So we once again have found two different set of branch cuts for $\sqrt{1+z}$ and $\sqrt{1-z}$ that lead to the same branch cut $[-1, 1]$ for $\sqrt{1-z^2}$, but of course the precise branch of $\sqrt{1-z^2}$ depends on which branches were picked for $\sqrt{1+z}$ and $\sqrt{1-z}$.

So far, we have been multiplying together various branches of $\sqrt{1-z}$ and $\sqrt{1+z}$ to create different branches of $\sqrt{1-z^2}$. However, there are other constructions. We could also consider the product $\sqrt{-1}\sqrt{z-1}\sqrt{1+z}$. Now, there are three things we have to pick: a branch for $\sqrt{1+z}$, a branch for $\sqrt{z-1}$, and we can pick $\sqrt{-1}$ to be either i or $-i$.

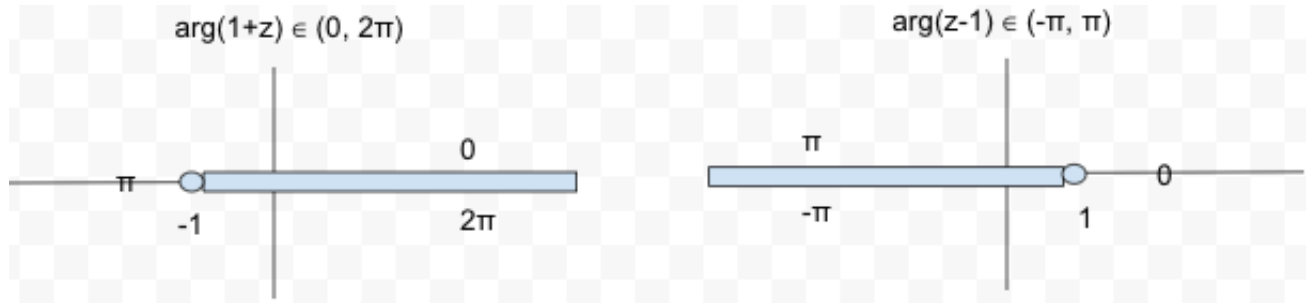For example, consider the following branch cuts for $\sqrt{z-1}$ and $\sqrt{1+z}$. Pick $\sqrt{-1} = i$.

Fig. 8

We now define $\sqrt{1-z^2} = \sqrt{-1}\sqrt{z-1}\sqrt{1+z} = i\sqrt{z-1}\sqrt{1+z}$. We choose $\arg(i) = \pi/2$, although we just as well could have chosen $\arg(i) = 5\pi/2, -3\pi/2$, etc. Then,

$$\arg\sqrt{1-z^2} = \arg(i) + \arg\sqrt{z-1}\sqrt{1+z} = \pi/2 + (1/2)(\arg(z-1) + \arg(1+z))$$

That is, we still add together $\arg(z-1)$ and $\arg(1+z)$ and then divide by two at each point in the complex plane, but now we also add on $\pi/2$ afterwards. Hence,
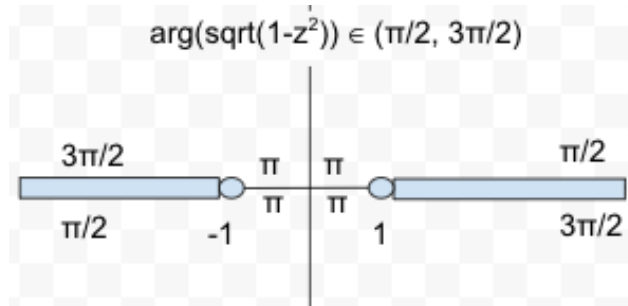

Fig. 9

So we have the branch cut $(-\infty, -1] \cup [1, \infty)$ for $\sqrt{1-z^2}$ and $\arg\sqrt{1-z^2} \in (\pi/2, 3\pi/2)$. Notice that this is the same diagram as Fig. 3 but all the arguments are increased by $\pi/2$. Had we chosen $\arg(i) = 5\pi/2, -3\pi/2$, or any other valid value, we would still have the same branch cut but all the arguments would instead have been increased by $5\pi/2$, $-3\pi/2$, etc.

Given the many possibilities for constructing a branch of $\sqrt{1-z^2}$, the number of distinct branches of $\sqrt{1-z^2}$ seems daunting. However, as it turns out, if we restrict ourselves to the branch cuts $(-\infty, -1], (-\infty, 1], [-1, \infty)$, and $[1, \infty)$ for $\sqrt{1-z}$ and $\sqrt{1+z}$, then there are only four branches of $\sqrt{1-z^2}$ that we can produce: two branches of $\sqrt{1-z^2}$ with branch cut $(-\infty, -1] \cup [1, \infty)$ and two branches of $\sqrt{1-z^2}$ with branch cut $[-1, 1]$. Many constructions will lead to the same branch. We encourage you to find out why, but the reason comes down to the fact that a square root function with a given branch cut only has two branches, differing only by sign. As a result, regardless of how many square root functions you multiply together to produce your branch of $\sqrt{1-z^2}$, the end result will only ever differ by sign. If we were working with a cube root or something more complicated, there would be more possibilities.

It is natural to wonder how to compute Laurent series of different branches of the same function. This topic will be deferred to Chapter 3.

Coming up next are three examples of contour integration. They will be explained in great detail, but subsequent explanations will be less detailed for the sake of concision. If something confuses you, we encourage you to refer back to the examples. Also, **if you need a refresher on computing residues, the residue at infinity, or other related theorems, see Appendix 1. We will not explain these things much while we solve our integrals.**

Although I have said it before, I will reiterate again here: **all plots will display the values of $\arg f(z)$ but the branch cuts shown will be the branch cuts for $f(z)$ itself. As we have seen, sometimes $f(z)$ and $\arg f(z)$ will have the same branch cuts, but sometimes they will not.** However, when it comes to integration, we usually do not care about the branch cuts of $\arg f(z)$ since it is $f(z)$ that we are trying to integrate, not its argument.

Lastly, there is something to be said about the convergence of our integrals. If we are working with an integral that does not converge, the residue theorem could still give us a deceptive answer (usually the principal value of the integral, if it exists). The point is, you have to be sure that your integrals converge. However, determining the convergence of integrals is not our focus, so from now on, **you may assume all integrals converge unless otherwise stated**.

(Example 1) Using contour integration, evaluate:

$$J = \int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}\,dx$$

We define $f(z) = \frac{\sqrt{1-z^2}}{1+z^2}$. We are integrating from -1 to 1, so it makes sense to choose a branch of $\sqrt{1-z^2}$ with branch cut $[-1,1]$ as then we can integrate around this branch cut using a dogbone contour (with an extra circle of radius R) to extract J. We sketch the contour to integrate over below. C_3 and C_5 are circles of radii $r \to 0$ and C_R is a circle of radius $R \to \infty$. The points $\pm i$ are simple poles of $f(z)$.
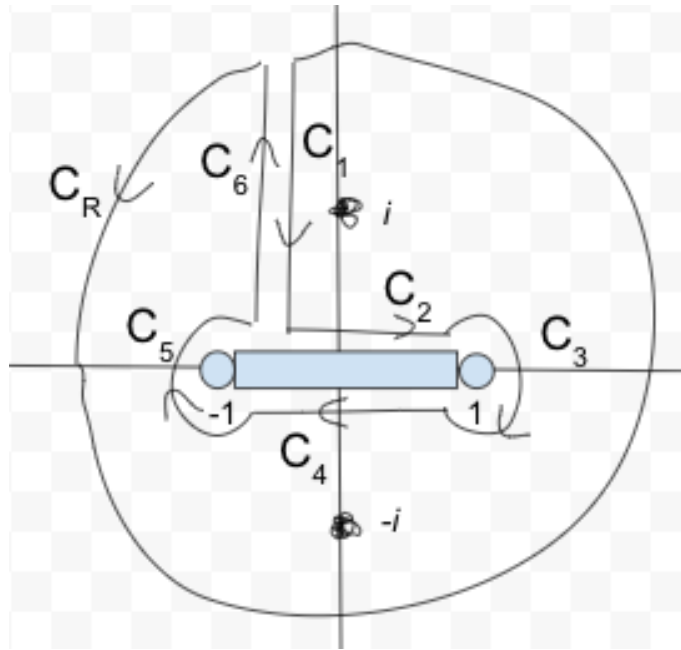

Fig. 10

However, since our integrand is real-valued, it is advisable to have $\sqrt{1-z^2}$ be *purely real* on at least one of C_2 and C_4. We can construct an appropriate $\sqrt{1-z^2}$ as follows: pick the branch cut $(-\infty, -1]$ for $\sqrt{1+z}$ with $\arg(1+z) \in (-\pi, \pi)$ and the branch cut $(-\infty, 1]$ for $\sqrt{z-1}$ with $\arg(z-1) \in (-\pi, \pi)$, and pick $\sqrt{-1} = -i$.
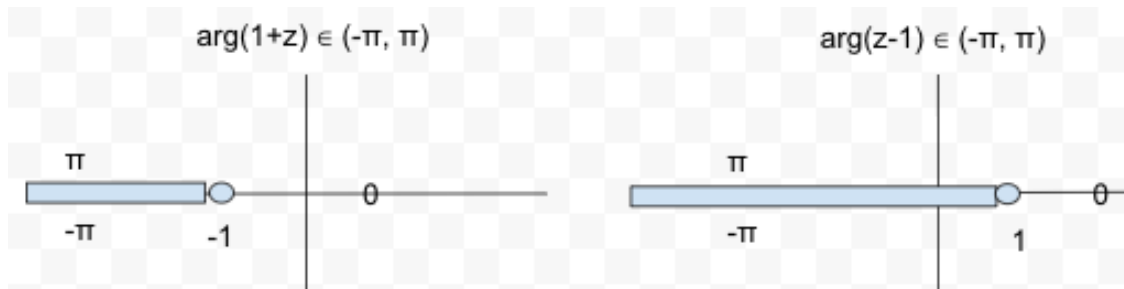

Fig. 11

We define $\sqrt{1-z^2} = \sqrt{-1}\sqrt{z-1}\sqrt{1+z} = -i\sqrt{z-1}\sqrt{1+z}$. Choose $\arg(-i) = -\pi/2$. It follows that $\arg\sqrt{1-z^2} = -\pi/2 + (1/2)(\arg(z-1) + \arg(1+z))$. We plot the values of $\arg\sqrt{1-z^2}$ below:
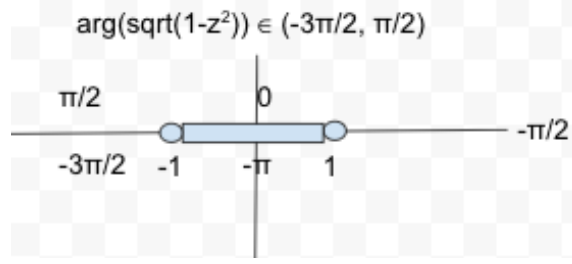

Fig. 12

Hence we have the branch cut $[-1,1]$ for $\sqrt{1-z^2}$ and $\arg\sqrt{1-z^2} \in (-3\pi/2, \pi/2)$. Note that $\sqrt{1-z^2}$ will indeed take on purely real values just above and below $[-1,1]$ since e^{i0} and $e^{-i\pi}$ are both purely real.

As $R \to \infty$ and $r \to 0$, the contributions from C_1 and C_6 will cancel and we can treat C_2 as an uninterrupted line segment. Then, by residue theorem,

$$\int_{C_R+C_2+C_3+C_4+C_5} f = 2\pi i\left(\text{Res}(f,i) + \text{Res}(f,-i)\right)$$

We first show $\int_{C_3} f$ and $\int_{C_5} f$ vanish as $r \to 0$. We have $z = -1 + re^{i\theta}$ on C_5. Hence, by triangle inequalities, we have:

$$|f(z)| = \frac{\left|\sqrt{1-z^2}\right|}{|1+z^2|} = \frac{\left|-i\sqrt{z-1}\sqrt{1+z}\right|}{|1+z^2|} \leq \frac{\sqrt{2+r}\sqrt{r}}{2r-r^2} = \frac{\sqrt{2+r}}{\sqrt{r}(2-r)}$$

for r small enough.[2] Then by ML lemma, we have:

$$0 \leq \left|\int_{C_5} f(z)dz\right| \leq 2\pi r\frac{\sqrt{2+r}}{\sqrt{r}(2-r)} = \frac{2\pi\sqrt{r}\sqrt{2+r}}{2-r}$$

Since $\frac{2\pi\sqrt{r}\sqrt{2+r}}{2-r} \to 0$ as $r \to 0$, we have $\int_{C_5} f \to 0$ as $r \to 0$. A similar calculation shows $\int_{C_3} f \to 0$.

We now handle $\int_{C_2} f$ and $\int_{C_4} f$. Looking at Fig. 12, as $r \to 0$ we have $\arg\sqrt{1-z^2} \to 0$ on C_2 and $\arg\sqrt{1-z^2} \to -\pi$ on C_4. We parameterize $z = t + i\epsilon$ on C_2 and $z = t - i\epsilon$ on C_4. Then,

$$\int_{C_2} f = \int_{C_2} \frac{\sqrt{1-z^2}}{1+z^2}dz = \int_{C_2} \frac{\left|\sqrt{1-z^2}\right|e^{i\arg\sqrt{1-z^2}}}{1+z^2}dz$$

---

[2] Note that $|1+z^2| \geq |1-|z^2|| = \left|1 - \left|(-1+re^{i\theta})^2\right|\right| = \left|1 - |1 - 2re^{i\theta} + r^2e^{i2\theta}|\right| \geq 2r - r^2$ for $r \ll 1$. There is a bit of visual reasoning used in the last inequality.

$$= \lim_{\epsilon \to 0} \int_{-1+r}^{1-r} \frac{\left|\sqrt{1-(t+i\epsilon)^2}\right| e^{i \arg \sqrt{1-(t+i\epsilon)^2}}}{1+(t+i\epsilon)^2}\, dt \to \int_{-1}^{1} \frac{\left|\sqrt{1-t^2}\right| e^{i0}}{1+t^2}\, dt = J$$

as $r \to 0$.[3] Similarly, for C_4, we have:

$$\int_{C_4} f = \lim_{\epsilon \to 0} \int_{1-r}^{-1+r} \frac{\left|\sqrt{1-(t-i\epsilon)^2}\right| e^{i \arg \sqrt{1-(t-i\epsilon)^2}}}{1+(t-i\epsilon)^2}\, dt \to \int_{1}^{-1} \frac{\left|\sqrt{1-t^2}\right| e^{-i\pi}}{1+t^2}\, dt = J$$

as $r \to 0$. It remains to handle $\int_{C_R} f$. We have:

$$\int_{C_R} f(z)\, dz = -2\pi i \operatorname{Res}(f, \infty) = 2\pi i \operatorname{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right) = 2\pi i \operatorname{Res}\left(\frac{\sqrt{1-1/z^2}}{1+z^2}, 0\right)$$

Note that $\left|\sqrt{1-1/z^2}\right| = \left|\sqrt{z^2-1}/z\right|$. However, if we are to say $\sqrt{1-1/z^2} = \sqrt{z^2-1}/z$, we must also be sure that $\arg \sqrt{1-1/z^2} = \arg\left(\sqrt{z^2-1}/z\right) = \arg \sqrt{z^2-1} - \arg z$, so we must choose appropriate branches for $\sqrt{1-1/z^2}$, $\sqrt{z^2-1}$, and $\arg z$. The term $\sqrt{1-1/z^2}$ simply came from substituting $1/z$ in place of z in our branch of $\sqrt{1-z^2}$, so the branch of $\sqrt{1-1/z^2}$ is already known. We must find a branch of $\sqrt{z^2-1}$ and a branch of $\arg z$ that will produce this specific branch of $\sqrt{1-1/z^2}$. The general idea is this: we have $\arg \sqrt{1-1/z^2} = \arg \sqrt{z^2-1} - \arg z$ by laws of $\arg$. Then we will quite literally pick a branch of $\arg z$ out of thin air and see if there is a branch of $\sqrt{z^2-1}$ that makes $\sqrt{1-1/z^2} = \sqrt{z^2-1}/z$ true for all z in the domain of our $\sqrt{1-1/z^2}$. Note, Laurent series could also be used to find the residue at infinity, see Example 5.

We have a plot of $\arg \sqrt{1-z^2}$ in Fig. 12, but the plot of $\arg \sqrt{1-1/z^2}$ will be different due to the inversion $z \mapsto 1/z$. By noting $1/z = \bar{z}/|z|^2$ and $\bar{z}$ is the reflection of z across the real axis, we can use Fig. 12 to plot some values of $\arg \sqrt{1-1/z^2}$ below.[4]
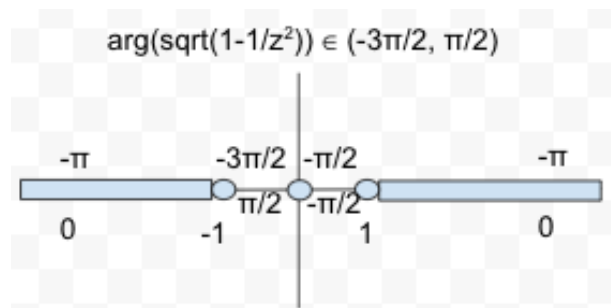

Fig. 13

[3] If you are wondering why we are able to pass the limit $\lim_{\epsilon \to 0}$ under the integral, see Appendix 2.
[4] As an example, take z just above $(-1, 0)$ and very close to 0. Then $\bar{z}$ is just below $(-1, 0)$ and very close to 0. But since z is close to 0, $|z|$ is very small so $\bar{z}/|z|^2 = 1/z$ will be very far from the origin but still below the negative real axis. But from Fig. 12, this is where $\arg \sqrt{1-z^2} \approx -3\pi/2$. So $\arg \sqrt{1-1/z^2} \approx -3\pi/2$ for z just above $(-1, 0)$ and very close to 0, as shown in Fig. 13.

Note that $z = 0$ is an isolated singularity of $\sqrt{1 - 1/z^2}$ and so must be removed from the domain. Using laws of arg, we have $\arg\sqrt{z^2 - 1} = \arg\sqrt{1 - 1/z^2} + \arg z$, so we will add together $\arg z$ and $\sqrt{1 - 1/z^2}$ at each point in $\mathbb{C}$ to find $\arg\sqrt{z^2 - 1}$, but we first must choose a branch of $\arg z$. One natural choice is the principal branch for $\arg z$, that is, we pick $\arg z \in (-\pi, \pi)$. Now we can plot some values of $\arg\sqrt{z^2 - 1}$ below:
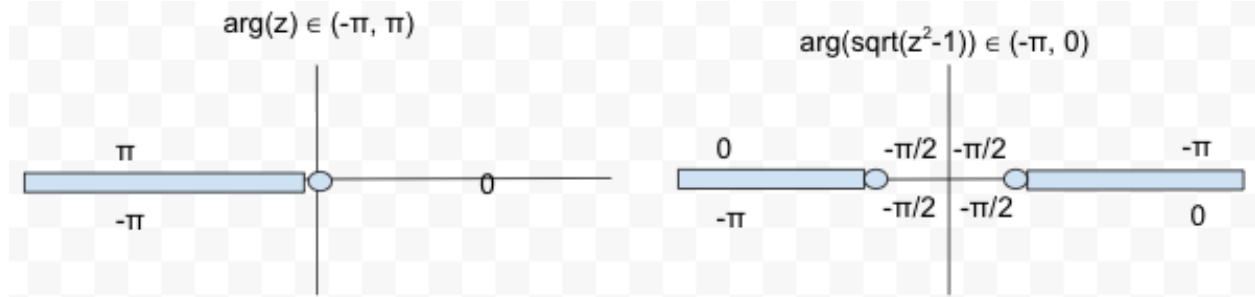

Fig. 14

So we have $\sqrt{z^2 - 1}$ with branch cut $(-\infty, -1] \cup [1, \infty)$ and $\arg\sqrt{z^2 - 1} \in (-\pi, 0)$. Note continuity has been regained at $z = 0$ since $\arg\sqrt{z^2 - 1} \to -\pi/2$ as $z \to 0$ from all directions.

Incidentally, if we had chosen $\arg z \in (0, 2\pi)$ instead of $\arg z \in (-\pi, \pi)$, you can verify that we would end up with the same branch of $\sqrt{z^2 - 1}$ but with $\arg\sqrt{z^2 - 1} \in (-\pi, 2\pi)$ instead of $\arg\sqrt{z^2 - 1} \in (-\pi, 0)$ (see the discussion around Fig. 7 and Fig. 8 for a similar instance where a function $\sqrt{1 - z^2}$ is continuously extended over an interval across which $\arg\sqrt{1 - z^2}$ is discontinuous).

So now we can write:

$$2\pi i \operatorname{Res}\left(\frac{\sqrt{1 - 1/z^2}}{1 + z^2}, 0\right) = 2\pi i \operatorname{Res}\left(\frac{\sqrt{z^2 - 1}}{z(1 + z^2)}, 0\right)$$

$$= 2\pi i \lim_{z \to 0}\frac{\sqrt{z^2 - 1}}{1 + z^2} = 2\pi i \sqrt{-1} = 2\pi i(-i) = 2\pi$$

where $\sqrt{-1} = -i$ and not i since the argument of i can never lie in the set of values $(-\pi, 0)$ that $\arg\sqrt{z^2 - 1}$ takes on. So we have $\int_{C_R} f \to 2\pi$ as $R \to \infty$.

We now find the residues of f at the simple poles $\pm i$. Let $g(z)$ denote our particular branch of $\sqrt{1 - z^2}$. So $g(i) = \sqrt{2}$, and since $\arg g(z) \in (-3\pi/2, \pi/2)$, $\arg g(i)$ could equal 0 or $-\pi$, so $g(i)$ could equal $\left|\sqrt{2}\right|e^{i0}$ or $\left|\sqrt{2}\right|e^{-i\pi}$. But looking at Fig. 12, we see that $\arg g(z) \in (-\pi/2, \pi/2)$ in the upper-half plane, which is where $z = i$ is. Therefore we must have $\arg g(i) = 0$ since 0 is in $(-\pi/2, \pi/2)$ but $-\pi$ is not, so $g(i) = \left|\sqrt{2}\right|e^{i0}$. Similarly, we still have $g(-i) = \sqrt{2}$, but since $\arg g(z) \in (-3\pi/2, -\pi/2)$ in the lower-half plane, which is where $z = -i$ is, we must have $\arg g(-i) = -\pi$, so $g(-i) = \left|\sqrt{2}\right|e^{-i\pi}$. Then,

$$\text{Res}(f, i) = \lim_{z \to i} \frac{\sqrt{1 - z^2}}{z + i} = \lim_{z \to i} \frac{\left|\sqrt{2}\right| e^{i0}}{z + i} = \frac{\sqrt{2}}{2i}$$

$$\text{Res}(f, -i) = \lim_{z \to -i} \frac{\sqrt{1 - z^2}}{z - i} = \lim_{z \to -i} \frac{\left|\sqrt{2}\right| e^{-i\pi}}{z - i} = \frac{\sqrt{2}}{2i}$$

Hence, returning to residue theorem,

$$2J + 2\pi = 2\pi i \left(\frac{\sqrt{2}}{2i} + \frac{\sqrt{2}}{2i} \right)$$

$$\Rightarrow J = \pi\left(\sqrt{2} - 1\right)$$

It is worth noting that this integral could have been solved with a dogbone traversed clockwise and the computations would end up the exact same (do you see why? See Appendix 1, Residue Theorem).

(Example 2) Using contour integration, evaluate:

$$J = \int_1^\infty \frac{dx}{x\sqrt{x^2 - 1}}$$

Let $f(z) = \frac{1}{z\sqrt{z^2-1}}$. It makes sense to choose $\sqrt{z^2 - 1}$ with branch cut $(-\infty, -1] \cup [1, \infty)$ as then we can integrate around it to extract J.
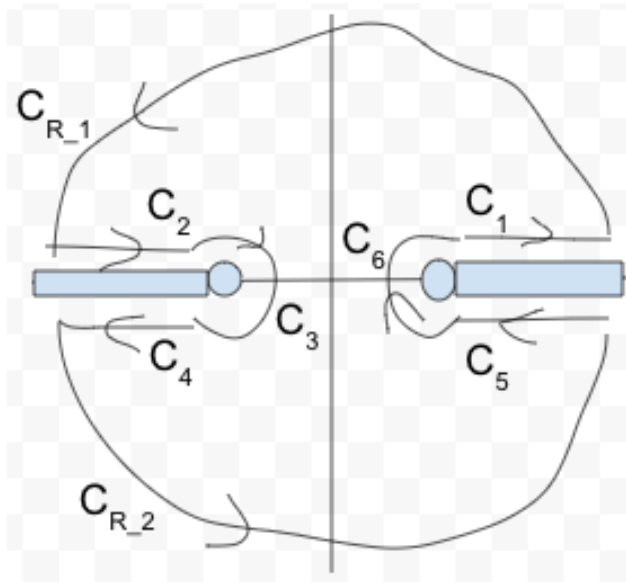

Fig. 15

C_{R_1}, C_{R_2} are semicircles of radius $R \to \infty$ and C_3, C_6 are circles of radius $r \to 0$. We also would like $\sqrt{z^2 - 1}$ to be real-valued on at least one of C_1, C_2, C_4, and C_5. We notice that the constructions in Fig. 1 and Fig. 3 appear most appropriate, so pick the branch cut $(-\infty, 1]$ for $\sqrt{z - 1}$ with $\arg(z - 1) \in (-\pi, \pi)$ and the branch cut $[-1, \infty)$ for $\sqrt{z + 1}$ with $\arg(z + 1) \in (0, 2\pi)$. Then $\sqrt{z - 1}\sqrt{z + 1} = \sqrt{z^2 - 1}$ with the following plot of $\arg \sqrt{z^2 - 1}$ below, which is identical to Fig. 3.
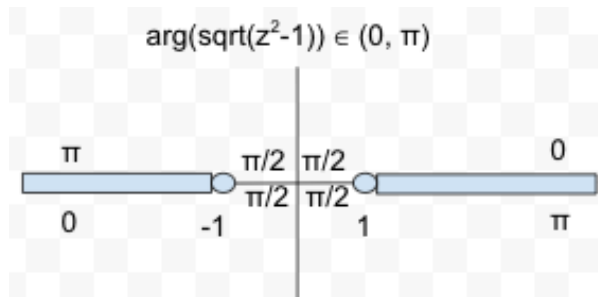

Fig. 16

By residue theorem,

$$\int_{C_{R_1}+C_{R_2}+C_1+C_2+C_3+C_4+C_5+C_6} f = 2\pi i \operatorname{Res}(f, 0)$$

We show $\int_{C_{R_1}} f$ vanishes as $R \to \infty$. On C_{R_1}, we have $z = Re^{i\theta}$, so

$$|f(z)| = \left|\frac{1}{z\sqrt{z^2-1}}\right| \le \frac{1}{R\sqrt{R^2-1}}$$

Hence,

$$0 \le \left|\int_{C_{R_1}} f(z)dz\right| \le \pi R \frac{1}{R\sqrt{R^2-1}} = \frac{2\pi}{\sqrt{R^2-1}}$$

Since $\frac{2\pi}{\sqrt{R^2-1}} \to 0$ as $R \to \infty$, we have $\int_{C_{R_1}} f \to 0$. Similarly, $\int_{C_{R_2}} f \to 0$ as $R \to \infty$.

We now show $\int_{C_3} f$ vanishes as $r \to 0$. We have $z = -1 + re^{i\theta}$ on C_3, so:

$$|f(z)| = \left|\frac{1}{z\sqrt{z^2-1}}\right| = \frac{1}{|-1+re^{i\theta}|\left|\sqrt{re^{i\theta}(-2+re^{i\theta})}\right|} \le \frac{1}{(1-r)\sqrt{r}\sqrt{2-r}}$$

for r small enough. Hence,

$$0 \le \left|\int_{C_3} f(z)dz\right| \le 2\pi r \frac{1}{(1-r)\sqrt{r}\sqrt{2-r}} = \frac{2\pi\sqrt{r}}{(1-r)\sqrt{2-r}}$$

Since $\frac{2\pi\sqrt{r}}{(1-r)\sqrt{2-r}} \to 0$ as $r \to 0$, we have $\int_{C_3} f \to 0$. A similar calculation shows $\int_{C_6} f \to 0$.

We now compute the integrals over C_1, C_2, C_4, C_5. We can use Fig. 16 to help us find $\arg\sqrt{z^2-1}$ on these contours. We have $z = t + i\epsilon$ on C_1,

$$\int_{C_1} f = \int_{C_1} \frac{dz}{z\left|\sqrt{z^2-1}\right|e^{i\arg\sqrt{z^2-1}}} = \lim_{\epsilon \to 0}\int_{1+r}^{R} \frac{dt}{(t+i\epsilon)\left|\sqrt{(t+i\epsilon)^2-1}\right|e^{i\arg\sqrt{(t+i\epsilon)^2-1}}}$$

$$\to \int_1^\infty \frac{dt}{t\left|\sqrt{t^2-1}\right|e^{i0}} = J$$

as $r \to 0, R \to \infty$. We have $z = -t + i\epsilon$ on C_2,

$$\int_{C_2} f = \lim_{\epsilon \to 0}\int_R^{1+r} \frac{-dt}{(-t+i\epsilon)\left|\sqrt{(-t+i\epsilon)^2-1}\right|e^{i\arg\sqrt{(-t+i\epsilon)^2-1}}}$$

$$\to \int_\infty^1 \frac{-dt}{(-t)\left|\sqrt{t^2-1}\right|e^{i\pi}} = J$$

as $r \to 0, R \to \infty$. Similarly, we have $\int_{C_4} f \to J$ and $\int_{C_5} f \to J$.

We now calculate the residue at zero. Let $g(z)$ denote our branch of $\sqrt{z^2-1}$. Note that $g(0) = \sqrt{-1}$ which must equal i and not $-i$ since the argument of $-i$ can never lie in the set of values $(0, \pi)$ that $\arg g(z)$ takes on. So,

$$\text{Res}(f, 0) = \lim_{z \to 0} \frac{1}{\sqrt{z^2-1}} = \frac{1}{i}$$

Hence, returning to residue theorem,

$$4J = 2\pi i\left(\frac{1}{i}\right) \Rightarrow J = \frac{\pi}{2}$$

Incidentally, this integral is solved fastest using the substitution $x = \sec\theta$.

(Example 3) Using contour integration, evaluate:

$$J = \int_{-1}^{1} \left(\frac{1-x}{1+x}\right)^{\frac{1}{6}} dx$$

Define $f(z) = \left(\frac{1-z}{1+z}\right)^{1/6}$. We use the exact same contour as in Fig. 10, that is, a dogbone contour with a circle of radius $R \to \infty$. We once again choose f such that f is real valued on at least one of C_2 and C_4. To accomplish this, we choose the branch cut $(-\infty, -1]$ for $\sqrt{1+z}$ with $\arg(1+z) \in (-\pi, \pi)$ and the branch cut $(-\infty, 1]$ for $\sqrt{z-1}$ with $\arg(z-1) \in (-\pi, \pi)$ (identical to Fig. 11). Pick $\sqrt[6]{-1} = e^{-i\pi/6}$ and $\arg(e^{-i\pi/6}) = -\pi/6$. Then, we define:

$$\left(\frac{1-z}{1+z}\right)^{\frac{1}{6}} = \sqrt[6]{-1}\frac{\sqrt[6]{z-1}}{\sqrt[6]{1+z}} = e^{-i\frac{\pi}{6}}\frac{\sqrt[6]{z-1}}{\sqrt[6]{z+1}}$$

which implies:

$$\arg\left(\frac{1-z}{1+z}\right)^{\frac{1}{6}} = -\frac{\pi}{6} + \frac{1}{6}(\arg(z-1) - \arg(1+z))$$

So at each point in the complex plane, we add together $\arg(z-1)$ and $\arg(1+z)$, then divide by six, then subtract $\pi/6$ in order to find the argument of $\left(\frac{1-z}{1+z}\right)^{1/6}$.
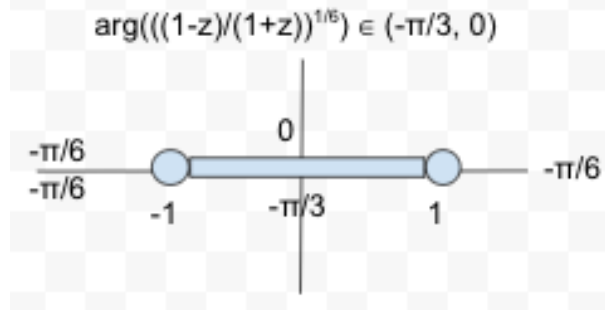
Fig. 17

So our $\left(\frac{1-z}{1+z}\right)^{1/6}$ has branch cut $[-1, 1]$ and $\arg\left(\frac{1-z}{1+z}\right)^{1/6} \in (-\pi/3, 0)$.

On C_5, we have $z = -1 + re^{i\theta}$, so

$$|f(z)| = \frac{|\sqrt[6]{z-1}|}{|\sqrt[6]{z+1}|} \le \frac{\sqrt[6]{2+r}}{\sqrt[6]{r}}$$

Hence,

$$0 \le \left|\int_{C_5} f(z)dz\right| \le 2\pi r \frac{\sqrt[6]{2+r}}{\sqrt[6]{r}} = 2\pi i \, r^{\frac{5}{6}} \sqrt[6]{2+r}$$

which tends to 0 as $r \to 0$. Hence $\int_{C_5} f \to 0$. Similarly, $\int_{C_3} f \to 0$.
On C_2, we have $\arg f \to 0$, so:

$$\int_{C_2} f(z)dz \to \int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{\frac{1}{6}} e^{i0}dt = J$$

as $r \to 0$.

On C_4, we have $\arg f \to -\pi/3$, so:

$$\int_{C_4} f(z)dz \to \int_1^{-1} \left(\frac{1-t}{1+t}\right)^{\frac{1}{6}} e^{-i\frac{\pi}{3}}dt = -\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)J$$

as $r \to 0$. For C_R, we have:

$$\int_{C_R} f(z)dz = -2\pi i \operatorname{Res}(f,\infty) = 2\pi i \operatorname{Res}\left(\frac{1}{z^2}\left(\frac{1-1/z}{1+1/z}\right)^{\frac{1}{6}},0\right) = 2\pi i \operatorname{Res}\left(\frac{1}{z^2}\left(\frac{z-1}{z+1}\right)^{\frac{1}{6}},0\right)$$

Let the above $\left(\frac{z-1}{z+1}\right)^{1/6}$ be denoted $g(z)$. Once again, since we have only replaced z with $1/z$ in our branch of f to obtain g, we still have $\arg g(z) \in (-\pi/3,0)$. We notice $z = 0$ is an order two pole,

$$\Rightarrow 2\pi i \operatorname{Res}\left(\frac{1}{z^2}\left(\frac{z-1}{z+1}\right)^{\frac{1}{6}},0\right) = 2\pi i \lim_{z\to 0} \frac{d}{dz}[g(z)] = 2\pi i \lim_{z\to 0} \frac{1}{3} g(z) \frac{1}{z^2-1}$$

where we used chain rule to calculate the derivative.[5] Note $g(0) = (-1)^{1/6}$ and that the only sixth root of -1 consistent with $\arg g(z) \in (-\pi/3,0)$ is $e^{-i\pi/6}$,

$$= 2\pi i \frac{1}{3}(-1)^{\frac{1}{6}}\frac{1}{-1} = -\frac{2\pi i}{3}e^{-i\frac{\pi}{6}} = -\frac{2\pi i}{3}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

Note there are no residues inside the contour. Then, by Cauchy's theorem,

$$\int_{C_1+C_2+\cdots+C_6+C_R} f(z)dz = 0$$

$$\Rightarrow J - \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)J - \frac{2\pi i}{3}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = 0$$

$$\Rightarrow \frac{J}{2} - \frac{\pi}{3} + \frac{i\sqrt{3}}{2}J - \frac{\pi i\sqrt{3}}{3} = 0$$

Equating either real or imaginary parts yields:

$$J = \frac{2\pi}{3}$$

---

[5] We cannot use power rule here to take the derivative. For details, see Appendix 3.

(Question 1) Now you try. We start with an easier one. Solutions are on the next page. Using contour integration, evaluate:

$$J = \int_0^1 \frac{\sqrt{x - x^2}}{1 + x^3} \, dx$$

1. Construct $\sqrt{z - z^2}$ with branch cut $[0, 1]$ and $\arg \sqrt{z - z^2} \in (-\pi/2, 3\pi/2)$ and such that it is real-valued just above the branch cut $[0, 1]$. Which function and contour will you pick? After you answer, check with the solutions before proceeding.

2. Show that $\int_{C_2} f \to 0$ and $\int_{C_4} f \to 0$ as $r \to 0$.

3. Show $\int_{C_1} f \to J$ and $\int_{C_3} f \to J$ as $r \to 0$.

4. Show that $\int_{C_R} f \to 0$ as $R \to \infty$.

5. What are the poles of f inside the contour? Find their residues. Remember, you can check Appendix 1 if you need help computing residues.

6. Hence find the value of J using the residue theorem.

(Solutions to Question 1)

1.

Pick $f(z) = \frac{\sqrt{z-z^2}}{1+z^3}$ and define $\sqrt{z-z^2} = -i\sqrt{z}\sqrt{z-1}$ where $\arg(-i) = -\pi/2$, $\sqrt{z}$ has branch cut $[0, \infty)$ with $\arg z \in (0, 2\pi)$, and $\sqrt{z-1}$ has branch cut $[1, \infty)$ with $\arg(z-1) \in (0, 2\pi)$. Recall, we add $\arg z$ and $\arg(z-1)$ together, divide by two, and subtract $\pi/2$ to find $\arg \sqrt{z-z^2}$ as per laws of $\arg$. Choose a dogbone contour with circle radius $R \to \infty$.
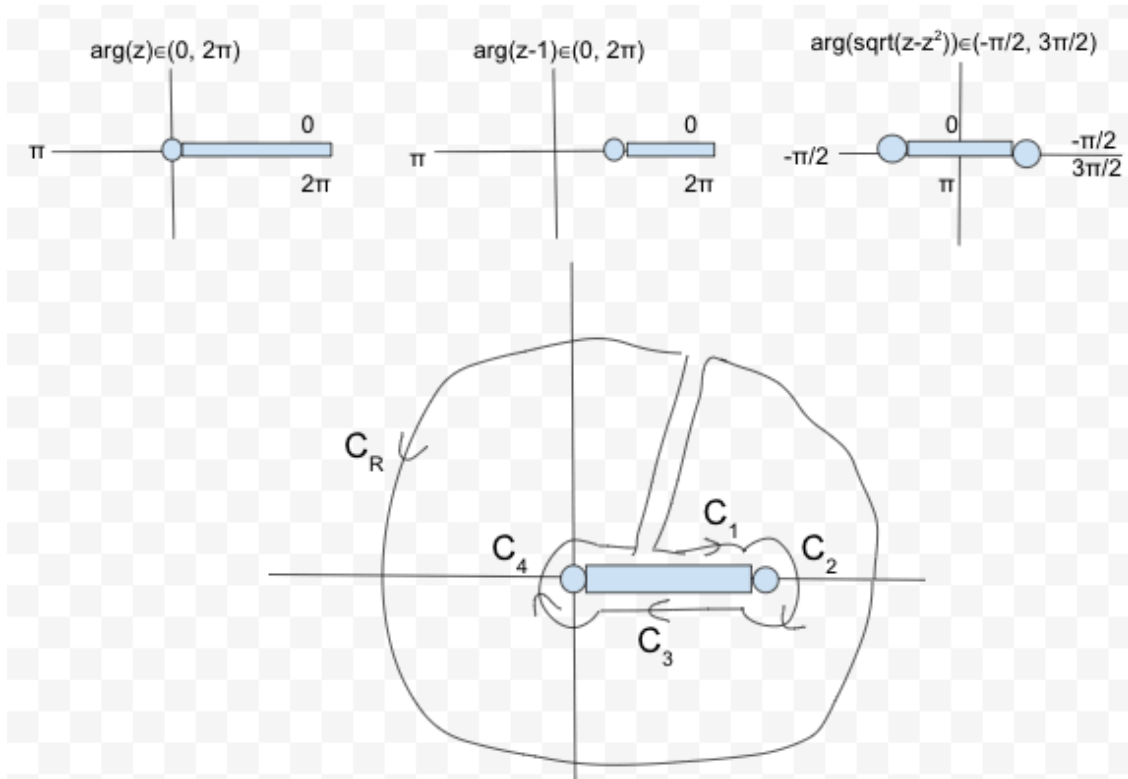
Fig. 18

2.
On C_2, we have $z = 1 + re^{i\theta}$. Note that for $r \ll 1$, we have:

$$|1 + z^3| \geq 1 - |z^3| \geq 1 - |1 + 3re^{i\theta} + 3r^2e^{i2\theta} + r^3e^{i3\theta}| \geq 3r - 3r^2 - r^3$$

So $|f(z)| = \frac{|\sqrt{z}||\sqrt{z-1}|}{|1+z^3|} \leq \frac{\sqrt{1+r}\sqrt{r}}{3r-3r^2-r^3}$ hence $\left|\int_{C_2} f\right| \leq 2\pi r \frac{\sqrt{1+r}\sqrt{r}}{3r-3r^2-r^3}$ which goes to 0 as $r \to 0$, hence $\int_{C_2} f \to 0$. On C_4, we have $z = re^{i\theta}$, so $|f(z)| \leq \frac{\sqrt{r}\sqrt{1+r}}{1-r^3}$ hence $\left|\int_{C_4} f\right| \leq 2\pi r \frac{\sqrt{r}\sqrt{1+r}}{1-r^3}$ which also goes to 0 as $r \to 0$, hence $\int_{C_4} f \to 0$.

3.
Parameterize $z = t + i\epsilon$ and $z = t - i\epsilon$ on C_1 and C_3 respectively. Then we find that $\int_{C_1} f \to \int_0^1 \frac{\sqrt{t-t^2}}{1+t^3} e^{i0} dt = J$ and $\int_{C_3} f \to \int_1^0 \frac{\sqrt{t-t^2}}{1+t^3} e^{i\pi} dt = J$ as $r \to 0$.

4.

There are two methods. The first is to parameterize $z = Re^{i\theta}$ on C_R. So $|f(z)| \leq \frac{\sqrt{R}\sqrt{1+R}}{R^3-1}$ hence $\left|\int_{C_R} f\right| \leq 2\pi R \frac{\sqrt{R}\sqrt{1+R}}{R^3-1}$ which tends to 0 as $R \to \infty$ (this can be proved by L'Hopital's rule). Hence $\int_{C_R} f \to 0$. The other method is to use residue at infinity. We have:

$$\int_{C_R} f = -2\pi i\, \text{Res}(f, \infty) = 2\pi i\, \text{Res}\left(\frac{1}{z^2}\frac{\sqrt{1/z-1/z^2}}{1+1/z^3}, 0\right) = 2\pi i\, \text{Res}\left(\frac{z\sqrt{1/z-1/z^2}}{z^3+1}, 0\right)$$

We notice that the numerator $z\sqrt{1/z-1/z^2}$ is equal to some branch of $\sqrt{z-1}$, but we do not even have to bother figuring out which one because the function would become $\frac{\sqrt{z-1}}{z^3+1}$ which is analytic at $z = 0$ regardless of branch. Then $z = 0$ is a removable singularity, so the principal part around $z = 0$ is zero, so the residue at $z = 0$ must be 0. So $\int_{C_R} f = 0$.

5.

The poles are $z = -1$ and $z = e^{\pm i\pi/3}$, all are simple poles and inside the contour (we are assuming that the two paths connecting the dogbone to the outer circle are close enough to the imaginary axis so that $z = e^{i\pi/3}$ is safely inside the contour). Then, by referencing Fig. 18 and noting that $\arg\sqrt{z-z^2} \in (-\pi/2, 3\pi/2)$, we have $\text{Res}(f, -1) = \lim_{z \to -1} \frac{\sqrt{z-z^2}}{3z^2} = \frac{\sqrt{-2}}{3} = \frac{|\sqrt{2}|e^{i\pi/2}}{3}$ and $\text{Res}(f, e^{i\pi/3}) = \lim_{z \to e^{i\pi/3}} \frac{\sqrt{z-z^2}}{3z^2} = \frac{\sqrt{1}}{3e^{i2\pi/3}} = \frac{|1|e^{i0}}{3e^{i2\pi/3}}$ and $\text{Res}(f, e^{-i\pi/3}) = \lim_{z \to e^{-i\pi/3}} \frac{\sqrt{z-z^2}}{3z^2} = \frac{\sqrt{1}}{3e^{-i2\pi/3}} = \frac{|1|e^{i\pi}}{3e^{-i2\pi/3}}$. We explain the computation of the residue at $e^{-i\pi/3}$. Let $g(z)$ denote our branch of $\sqrt{z-z^2}$. So $g(e^{-i\pi/3}) = \sqrt{1}$, and since $\arg g(z) \in (-\pi/2, 3\pi/2)$, $\arg g(e^{-i\pi/3})$ could equal 0 or π, so $g(e^{-i\pi/3})$ could equal $|1|e^{i0}$ or $|1|e^{i\pi}$. However, from Fig. 18, we see that $\arg g(z) \in (\pi/2, 3\pi/2)$ in the lower-half plane, which is where $e^{-i\pi/3}$ is. Therefore, we must have $\arg g(e^{-i\pi/3}) = \pi$ since π is in $(\pi/2, 3\pi/2)$ but 0 is not, so $g(e^{-i\pi/3}) = |1|e^{i\pi}$. The other two residues are computed with similar reasoning.

6.

The residue theorem says:

$$\int_{C_1+\cdots+C_4+C_R} f = 2\pi i \left(\text{Res}(f, -1) + \text{Res}\left(f, e^{i\frac{\pi}{3}}\right) + \text{Res}\left(f, e^{-i\frac{\pi}{3}}\right)\right)$$

$$\Rightarrow 2J = 2\pi i \left(\frac{i\sqrt{2}}{3} + \frac{e^{i0}}{3e^{i2\pi/3}} + \frac{e^{i\pi}}{3e^{-i2\pi/3}}\right) = \frac{2\pi i}{3}\left(i\sqrt{2} + e^{-\frac{2\pi i}{3}} + e^{\frac{5\pi i}{3}}\right)$$

$$\Rightarrow J = \frac{\pi}{3}\left(\sqrt{3} - \sqrt{2}\right)$$

(Question 2) Using contour integration, evaluate:

$$J = \int_0^1 \frac{\sqrt{x - x^2}}{x^2 + 1} dx$$

This is the same integral as Question 1 but the denominator is $x^2 + 1$ instead of $x^3 + 1$. If you have already solved Question 1, hopefully you can see that we can reuse or modify much of that previous work to help us solve this integral. In fact, you should use the exact same branch of $\sqrt{z - z^2}$ and the same contour. The trouble comes from the integral over C_R which, unlike in Question 1, will not tend to zero as $R \to \infty$ (do you see why?)

1. Use the same branch of $\sqrt{z - z^2}$ and the same contour as in Question 1. Show that $\int_{C_R} f = 2\pi i \operatorname{Res}\left(\frac{\sqrt{1/z - 1/z^2}}{1 + z^2}, 0\right)$. Draw a plot of $\arg\sqrt{1/z - 1/z^2}$.

2. Find a branch of $\sqrt{z - 1}$ satisfying $\sqrt{1/z - 1/z^2} = \sqrt{z - 1}/z$.

3. Hence find $\int_{C_R} f$ as $R \to \infty$.

4. Hence find the value of J.

(Solutions to Question 2)

1.

We have $\int_{C_R} f = -2\pi i \operatorname{Res}(f, \infty) = 2\pi i \operatorname{Res}\left(\frac{1}{z^2} \frac{\sqrt{1/z - 1/z^2}}{1 + 1/z^2}, 0\right) = 2\pi i \operatorname{Res}\left(\frac{\sqrt{1/z - 1/z^2}}{1 + z^2}, 0\right)$. Note that we simply replaced z with $1/z$ in our branch of $\sqrt{z - z^2}$ to get $\sqrt{1/z - 1/z^2}$. Since $1/z = \bar{z}/|z|^2$ and $\bar{z}$ is the reflection of z across the real axis, we can use the plot of $\arg\sqrt{z - z^2}$ in Fig. 18 to create the plot of $\arg\sqrt{1/z - 1/z^2}$ shown below (see the explanations surrounding Fig. 13 and Fig. 14 for a previous example of this).

arg(sqrt(1/z-1/z²)) ∈ (-π/2, 3π/2)

3π/2
π/2   -π/2

π
0
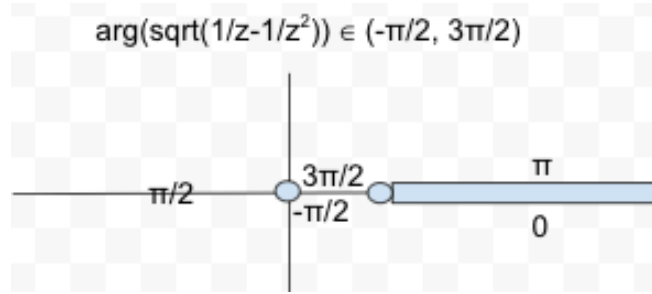
Fig. 19

2.

We have $\arg\sqrt{z - 1} = \arg\sqrt{1/z - 1/z^2} + \arg z$. If we choose the branch of $\arg z$ satisfying $\arg z \in (0, 2\pi)$, then we get the following plot of $\arg\sqrt{z - 1}$ below:

arg(sqrt(z-1)) ∈ (π, 2π)

3π/2
3π/2   3π/2

π
2π
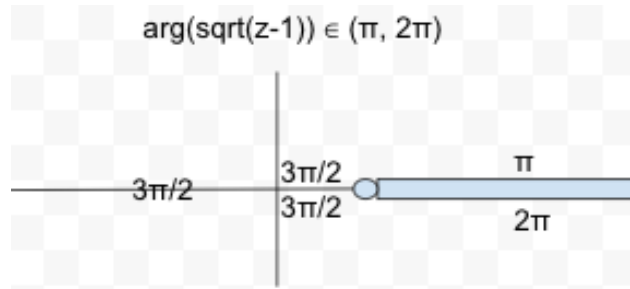
Fig. 20

So we have the branch cut $[1, \infty)$ for $\sqrt{z - 1}$ with $\arg\sqrt{z - 1} \in (\pi, 2\pi)$, and we have regained continuity at zero. Choosing $\arg z \in (-\pi, \pi)$ would also have worked.

3.

So we have:

$$\int_{C_R} f = 2\pi i \operatorname{Res}\left(\frac{\sqrt{1/z - 1/z^2}}{1 + z^2}, 0\right) = 2\pi i \operatorname{Res}\left(\frac{\sqrt{z - 1}}{z(1 + z^2)}, 0\right) = 2\pi i \lim_{z \to 0} \frac{\sqrt{z - 1}}{1 + z^2}$$

$$= 2\pi i \sqrt{-1} = 2\pi i(-i) = 2\pi$$

where $\sqrt{-1} = -i$ and not i because $\arg i$ can never lie in the set of values $(\pi, 2\pi)$ that $\arg\sqrt{z - 1}$ takes on.

4.
The only poles in the contour are $z = \pm i$, and both are simple poles. We have

$\text{Res}(f, \pm i) = \lim\limits_{z \to \pm i} \frac{\sqrt{z - z^2}}{z \pm i}$. Let $g(z)$ denote our branch of $\sqrt{z - z^2}$. Recall that $\arg g(z) \in$
$(-\pi/2, 3\pi/2)$. Then the possible values for $\arg g(i) = \arg \sqrt{1 + i}$ are $\pi/8$ and $9\pi/8$.
Returning to our plot of $\arg g(z)$ in Fig. 18, since i lies in the upper-half plane where
$\arg g(z)$ takes on values between $-\pi/2$ and $\pi/2$, we must have $\arg g(i) = \pi/8$. Then
$\text{Res}(f, i) = |\sqrt[4]{2}|e^{i\pi/8}/2i$. Similarly, the values for $\arg g(-i) = \arg \sqrt{1 - i}$ that are
consistent with $\arg g(z) \in (-\pi/2, 3\pi/2)$ are $-\pi/8$ and $7\pi/8$, and since $-i$ is in the
lower-half plane where $\arg g(z)$ takes on values between $\pi/2$ and $3\pi/2$, we must have
$\arg g(-i) = 7\pi/8$. So $\text{Res}(f, -i) = |\sqrt[4]{2}|e^{i7\pi/8}/(-2i)$.

It is not hard to show $\int_{C_3} f \to 0$, $\int_{C_5} f \to 0$, $\int_{C_2} f \to J$, and $\int_{C_4} f \to J$ as $r \to 0$, the
justification is similar to that used in Question 1. Then by residue theorem,

$$2J + 2\pi = 2\pi i \left(\frac{\sqrt[4]{2}e^{i\frac{\pi}{8}}}{2i} - \frac{\sqrt[4]{2}e^{i\frac{7\pi}{8}}}{2i} \right)$$

$$\Rightarrow 2J + 2\pi = \pi\sqrt[4]{2}\left(2\cos\frac{\pi}{8} \right) = \pi\sqrt[4]{2}\sqrt{2 + \sqrt{2}}$$

$$\Rightarrow J = \frac{\pi}{2}\left(-2 + \sqrt{2\sqrt{2} + 2} \right)$$

(Question 3) We aim to solve the integral:

$$I = \int_1^\infty \frac{x^3\sqrt{x^2-1}}{x^6-1}\,dx$$

We could try and integrate this function directly, but the x^6-1 in the denominator means that the function $z^3\sqrt{z^2-1}/(z^6+1)$ has four poles (note $z=\pm1$ are not poles since they are branch points). This means we would have to compute four residues, which is rather tedious. We notice x^6-1 can be factored as $(x^3+1)(x^3-1)$. Perhaps there is something we can do here.

1. Show that:
$$\frac{x^3\sqrt{x^2-1}}{x^6-1} = \frac{1}{2}\left(\frac{\sqrt{x^2-1}}{x^3+1} + \frac{\sqrt{x^2-1}}{x^3-1}\right)$$

Now define the following two integrals:

$$J = \int_1^\infty \frac{\sqrt{x^2-1}}{x^3+1}\,dx \qquad K = \int_1^\infty \frac{\sqrt{x^2-1}}{x^3-1}\,dx$$

2. Construct an appropriate branch of $\sqrt{z^2-1}$. What contour will you pick? After you answer, check with the solutions before proceeding.

3. Integrate $f(z) = \sqrt{z^2-1}/(z^3+1)$ around the contour chosen above. What do you find?

(Solutions to Question 3)

2.
We are integrating from 1 to ∞, so it seems natural to use the construction given by either Fig. 1 or Fig. 4, as they both result in the same function $\sqrt{z^2 - 1} = \sqrt{z - 1}\sqrt{z + 1}$ with branch cut $(-\infty, -1] \cup [1, \infty)$ and $\arg \sqrt{z^2 - 1} \in (0, \pi)$. To be explicit, one possibility is to take the branch cut $(-\infty, -1]$ for $\sqrt{z + 1}$ with $\arg(z + 1) \in (-\pi, \pi)$ and the branch cut $[1, \infty)$ for $\sqrt{z - 1}$ with $\arg(z - 1) \in (0, 2\pi)$. The plot of $\arg \sqrt{z^2 - 1}$ is:
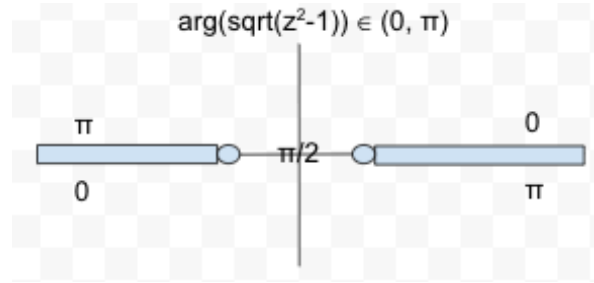

$$\arg(\mathrm{sqrt}(z^2\text{-}1)) \in (0,\ \pi)$$

Fig. 21

We use the exact same contour as in Fig. 15, a double keyhole contour.

3.
We will integrate f over the contour.

On C_3, we have $|f(z)| \le \frac{\sqrt{2+r}\sqrt{r}}{3r - 3r^2 - r^3}$ so $\left|\int_{C_3} f\right| \le \frac{2\pi\sqrt{2+r}\sqrt{r}}{3 - 3r - r^2}$, which vanishes as $r \to 0$, hence $\int_{C_3} f \to 0$. Similarly, $\int_{C_6} f \to 0$. On C_R, we have $|f(z)| \le \frac{\sqrt{1+R}\sqrt{1+R}}{R^3 - 1}$ so $\left|\int_{C_R} f\right| \le \frac{2\pi R(1+R)}{R^3 - 1}$ which vanishes as $R \to \infty$, so $\int_{C_R} f \to 0$. Now, as $r \to 0$, $R \to \infty$, we have the following:

On C_1 we have $z = t + i\epsilon$ so $\int_{C_1} f \to \int_1^\infty \frac{\sqrt{t^2 - 1}}{t^3 + 1} e^{i0} dt = J$

On C_2 we have $z = -t + i\epsilon$ so $\int_{C_2} f \to \int_\infty^1 \frac{\sqrt{t^2 - 1}}{1 + (-t)^3} e^{i\pi}(-dt) = \int_1^\infty \frac{\sqrt{t^2 - 1}}{t^3 - 1} dt = K$

On C_4 we have $z = -t - i\epsilon$ so $\int_{C_4} f \to \int_1^\infty \frac{\sqrt{t^2 - 1}}{1 + (-t)^3} e^{i0}(-dt) = \int_1^\infty \frac{\sqrt{t^2 - 1}}{t^3 - 1} dt = K$

On C_5 we have $z = t - i\epsilon$ so $\int_{C_5} f \to \int_\infty^1 \frac{\sqrt{t^2 - 1}}{t^3 + 1} e^{i\pi} dt = J$

The only poles of f are $z = e^{\pm i\pi/3}$, both simple poles and inside the contour. Note that $z = -1$ is a branch point of f and so is not a pole of f. Keeping in mind that $\arg \sqrt{z^2 - 1} \in (0, \pi)$,

$$\mathrm{Res}\left(f, e^{i\pi/3}\right) = \lim_{z \to e^{i\pi/3}} \frac{\sqrt{z^2 - 1}}{3z^2} = \frac{\left|\sqrt[4]{3}\right| e^{i5\pi/12}}{3e^{i2\pi/3}} = \left|\frac{1}{\sqrt[4]{27}}\right| e^{-i\frac{\pi}{4}}$$

$$\mathrm{Res}\left(f, e^{-i\pi/3}\right) = \lim_{z \to e^{-i\pi/3}} \frac{\sqrt{z^2 - 1}}{3z^2} = \frac{\left|\sqrt[4]{3}\right| e^{i7\pi/12}}{3e^{-i2\pi/3}} = \left|\frac{1}{\sqrt[4]{27}}\right| e^{i\frac{5\pi}{4}}$$

Then by residue theorem,

$$2J + 2K = 2\pi i \left(\frac{1}{\sqrt[4]{27}} e^{-i\frac{\pi}{4}} + \frac{1}{\sqrt[4]{27}} e^{i\frac{5\pi}{4}} \right) = \frac{2\pi i}{\sqrt[4]{27}} \left(-i\sqrt{2} \right)$$

$$\Rightarrow J + K = \frac{\pi\sqrt{2}}{\sqrt[4]{27}}$$

Integrating g over the contour would lead to this same result.
But, looking at part (1), we see $J + K$ is precisely two times the value of I. Therefore,

$$I = \frac{\pi}{\sqrt[4]{27}\sqrt{2}}$$

However, note that we were not able to find the values of J or K individually. When you integrated f over the contour, you probably expected to find J, but because of the parameterizations along the negative real axis, things worked out in such a way that we found I instead. It is natural to wonder then, how would we evaluate J and K? As it turns out, there is no good way. It can be shown that J and K are related to the roots of some rather unpleasant polynomials, in the sense that the roots of the polynomials have a very messy form in radicals. Whether you use residues or partial fractions, there is no way around dealing with these roots. Do not try it now, it is not worth it. In fact, if the denominator of the integrand was a polynomial of high enough degree, sometimes the integral will not even have a solution in radicals. The moral of the story is, sometimes you will not be able to find a closed form solution for an integral, and it is not your fault.

(Question 4)

We will evaluate the following two integrals:

$$J = \int_0^\infty \frac{\log x}{x^2 + 1} dx \qquad\qquad K = \int_0^\infty \frac{\log x}{x^2 - 1} dx$$

where $\log x$ is the natural logarithm.

1. Integrate $g(z) = \log z / (z^2 + 1)$ over a keyhole contour. Focus on the integrals over the non-circular portions of the contour. What do you notice? If you already know what is going to happen, then you can skip to the next part.

2. Integrate $f(z) = \log^2 z / (z^2 + 1)$ over the same contour above. Hence find J. You may use the fact that $\lim_{r \to 0} r (\ln r)^n = 0$ and $\lim_{R \to \infty} R^{-1} (\ln R)^n = 0$ for all $n \in \mathbb{N}$ (these two limits can be proved using repeated application of L'Hopital's rule).

3. We now evaluate K. Integrate $f(z) = \log^2 z / (z^2 - 1)$ over a modified keyhole contour (what is the modification?) Which branch and branch cut of $\log z$ will you pick? After you answer, check with the solutions before proceeding.

4. Using the function and contour chosen above, find the value of K. You may assume the following result:

(Corollary of dominated convergence theorem)
Let $\{f_n(z)\}$ be a sequence of continuous complex-valued functions defined on a compact set K that converges pointwise to some $f(z)$ on K. If there exists $g(z)$ such that $|f_n(z)| \leq g(z)$ on K for all $n \in \mathbb{N}$ and the integral $\int_K g(z) dz$ converges, then:

$$\lim_{n \to \infty} \int_K f_n(z) dz = \int_K f(z) dz$$

Note that taking $r \to 0$ is equivalent to taking $n \to \infty$ if we set $r = 1/n$.

(Solutions to Question 4)

1.
For $g(z) = \log z / (z^2 + 1)$, we will use the branch of $\log z$ that has $\arg z \in (0, 2\pi)$, so the branch cut is $[0, \infty)$. Accordingly, we pick the keyhole with $[0, \infty)$ as the slit. However, you could also pick the principal branch of $\log z$ and use the keyhole with slit $(-\infty, 0]$.
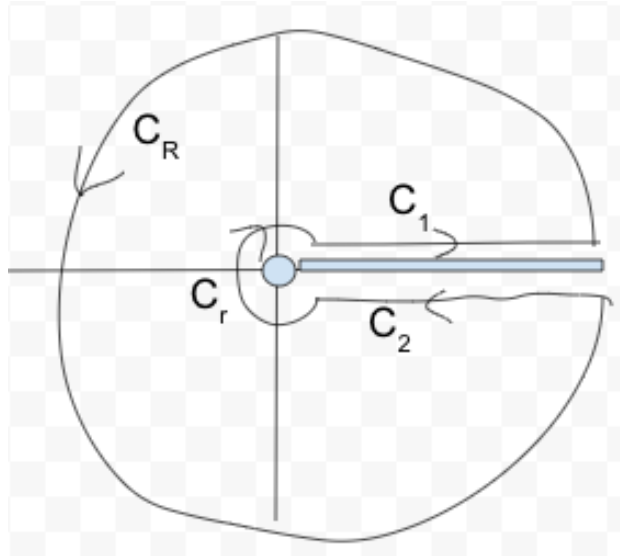

Fig. 22

On C_1, we have $z = t + i\epsilon$, hence

$$\int_{C_1} g = \lim_{\epsilon \to 0} \int_r^R \frac{\log|t + i\epsilon| + i \arg(t + i\epsilon)}{(t + i\epsilon)^2 + 1} dt \to \int_0^\infty \frac{\log|t| + i0}{t^2 + 1} dt = J$$

as $r \to 0$, $R \to \infty$. On C_2, we have $z = t - i\epsilon$, hence

$$\int_{C_2} g = \lim_{\epsilon \to 0} \int_R^r \frac{\log|t - i\epsilon| + i \arg(t - i\epsilon)}{(t - i\epsilon)^2 + 1} dt \to \int_\infty^0 \frac{\log|t| + i2\pi}{t^2 + 1} dt$$

$$= -\int_0^\infty \frac{\log|t|}{t^2 + 1} dt - \int_0^\infty \frac{i2\pi}{t^2 + 1} dt = -J - 2\pi i [\arctan t]_0^\infty = -J - 2\pi i \frac{\pi}{2}$$

as $r \to 0$, $R \to \infty$. But in that case, when we use the residue theorem, we will have $\int_{C_1} g + \int_{C_2} g = J + \left(-J - 2\pi i \frac{\pi}{2}\right) = -2\pi i \frac{\pi}{2}$ and notice that the J terms have cancelled each other out! Clearly the integrals over C_R and C_3 will not produce a copy of J, so we will not be able to find the value of J with this setup. We will have to either *change our contour* or *change the function* we integrate.

2.

On C_R, we parameterize $z = Re^{i\theta}$, so for R large enough, we have:

$$|f(z)| = \frac{\log^2|z| + (\arg z)^2}{|z^2 + 1|} \leq \frac{\log^2 R + (2\pi)^2}{R^2 - 1} \leq \frac{\log^2 R + \log^2 R}{R^2 - R^2/2} = \frac{2\log^2 R}{R^2/2}$$

hence $\left|\int_{C_R} f\right| \leq 2\pi R \frac{2\log^2 R}{R^2/2} = 8\pi R^{-1} \log^2 R$ which goes to 0 as $R \to \infty$, hence $\int_{C_R} f \to 0$.

Similarly, on C_3, we have $z = re^{i\theta}$, so for r small enough, we have:

$$\left|\int_{C_3} f\right| \leq 2\pi r \frac{\log^2 r + (2\pi)^2}{1 - r^2} \leq 2\pi r \frac{\log^2 r + \log^2 r}{1 - r^2} = \frac{4\pi r \log^2 r}{1 - r^2}$$

which goes to 0 as $r \to 0$. Hence $\int_{C_3} f \to 0$. On C_1, we have $z = t + i\epsilon$, hence

$$\int_{C_1} f = \lim_{\epsilon \to 0} \int_r^R \frac{(\log|t + i\epsilon| + i\arg(t + i\epsilon))^2}{(t + i\epsilon)^2 + 1} dt = \int_r^R \frac{(\log|t| + i0)^2}{t^2 + 1} dt \to \int_0^\infty \frac{\log^2|t|}{t^2 + 1} dt$$

as $r \to 0$, $R \to \infty$. On C_2, we have $z = t - i\epsilon$, hence

$$\int_{C_1} f = \lim_{\epsilon \to 0} \int_R^r \frac{(\log|t - i\epsilon| + i\arg(t - i\epsilon))^2}{(t - i\epsilon)^2 + 1} dt = -\int_r^R \frac{(\log|t| + i2\pi)^2}{t^2 + 1} dt$$

$$= -\int_r^R \frac{\log^2|t|}{t^2 + 1} dt - \int_r^R \frac{4\pi i \log|t|}{t^2 + 1} dt + \int_r^R \frac{4\pi^2}{t^2 + 1} dt$$

The poles of f are $z = \pm i$, and both are simple poles that are inside the contour. So $\text{Res}(f, \pm i) = \lim_{z \to \pm i} \frac{\log^2 z}{z \pm i} = \frac{(\log|\pm i| + i\arg(\pm i))^2}{\pm 2i}$. Since $\arg z \in (0, 2\pi)$, we have $\arg i = \pi/2$ and $\arg(-i) = 3\pi/2$. Therefore, we have $\text{Res}(f, i) = -\pi^2/8i$ and $\text{Res}(f, -i) = 9\pi^2/8i$. Hence $2\pi i (\text{Res}(f, i) + \text{Res}(f, -i)) = 2\pi^3$. Then by residue theorem,

$$\int_r^R \frac{\log^2|t|}{t^2 + 1} dt + \left(-\int_r^R \frac{\log^2|t|}{t^2 + 1} dt - \int_r^R \frac{4\pi i \log|t|}{t^2 + 1} dt + \int_r^R \frac{4\pi^2}{t^2 + 1} dt\right) + \int_{C_3 + C_R} f = 2\pi^3$$

$$\Rightarrow -\int_r^R \frac{4\pi i \log|t|}{t^2 + 1} dt + \int_r^R \frac{4\pi^2}{t^2 + 1} dt + \int_{C_3 + C_R} f = 2\pi^3$$

Now we take $r \to 0$ and $R \to \infty$,[6]

$$\Rightarrow -4\pi i J + 4\pi^2 \frac{\pi}{2} = 2\pi^3$$

$$\Rightarrow J = 0$$

Incidentally, the integral is solved fastest using the substitution $x = 1/u$ and noticing that the resulting integral is just the negative of the original.

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[6] By delaying taking these limits until *after* the terms $\int_r^R \frac{\log^2|t|}{t^2+1} dt$ cancelled out, we never had to worry about whether $\int_0^\infty \frac{\log^2|t|}{t^2+1} dt$ converges. As it turns out, $\int_0^\infty \frac{\log^2|t|}{t^2+1} dt$ does converge, so it would have been okay to let $r \to 0, R \to \infty$ at an earlier stage, but this will not always be the case.

3.
There is a clever choice of function and contour that leads to a relatively quick solution, but it is not so obvious and we will look at it later in Example 4. For now, we will integrate $f(z) = \log^2 z /(z^2 - 1)$ with $\arg z \in (0, 2\pi)$ for $\log z$ and use keyhole contour with an indent around $z = 1$ as shown below:
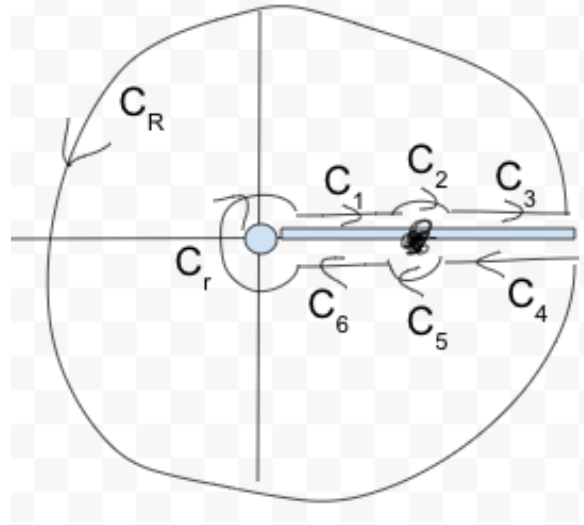

Fig. 23

Notice we must integrate around the singularity $z = 1$ in a semicircle when we get close to it, on C_2 and C_5 in this case. This is because, even though $z = 1$ is removed from the domain of f, the function f still tends to ∞ in all directions as z approaches 1, so when we integrate close to $z = 1$, our function is still behaving as if there is a true isolated singularity nearby, so we must act accordingly and integrate around it.

4.
The proof of $\int_{C_R} f \to 0$ and $\int_{C_r} f \to 0$ is essentially the same as in the previous part. Let C_2 and C_5 be semicircles of radius $r \to 0$ as well. Then we have $\int_{C_1} f = \int_r^{1-r} \frac{\log^2|t|}{t^2-1} dt$, $\int_{C_3} f = \int_{1+r}^R \frac{\log^2|t|}{t^2-1} dt$, $\int_{C_4} f = \int_R^{1+r} \frac{(\log|t|+i2\pi)^2}{t^2-1} dt$, and $\int_{C_6} f = \int_{1-r}^r \frac{(\log|t|+i2\pi)^2}{t^2-1} dt$. Then,

$$\int_{C_1+C_3+C_4+C_5} f = \int_r^{1-r} \frac{4\pi^2}{t^2-1} dt + \int_{1+r}^R \frac{4\pi^2}{t^2-1} dt - \int_r^{1-r} \frac{4\pi i \log|t|}{t^2-1} dt - \int_{1+r}^R \frac{4\pi i \log|t|}{t^2-1} dt$$

A quick computation shows $\int_0^1 \frac{dt}{t^2-1}$ and $\int_1^R \frac{dt}{t^2-1}$ both diverge, so we cannot take $r \to 0$ yet (but $R \to \infty$ would be fine). We simplify further hoping that some terms will cancel.

$$= 2\pi^2 \left(\left[\ln\left|\frac{x-1}{x+1}\right|\right]_r^{1-r} + \left[\ln\left|\frac{x-1}{x+1}\right|\right]_{1+r}^R \right) - 4\pi i \left(\int_r^{1-r} \frac{\log|t|}{t^2-1} dt + \int_{1+r}^\infty \frac{\log|t|}{t^2-1} dt \right)$$

$$= 2\pi^2 \left(\ln\left|\frac{(r+1)(r+2)}{(r-1)(r-2)}\right| + \ln\left|\frac{R-1}{R+1}\right| \right) - 4\pi i \left(\int_r^{1-r} \frac{\log|t|}{t^2-1} dt + \int_{1+r}^R \frac{\log|t|}{t^2-1} dt \right)$$

where we used laws of log to simplify. You should have found that two $\ln r$ terms cancelled each other out. Now we can take $r \to 0$ and $R \to \infty$ to obtain:

$$\lim_{r \to 0, R \to \infty} \int_{C_1 + C_3 + C_4 + C_5} f = -4\pi i \int_0^\infty \frac{\log|t|}{t^2 - 1} \, dt = -4\pi i K$$

Now we work on $\int_{C_2} f$ and $\int_{C_5} f$. As it turns out, the ML-lemma is not as helpful here (the $\arg z$ term will be problematic on C_5) so we will solve the integrals directly. We parameterize $z = 1 + re^{i\theta}$ on C_2 and C_5, hence:

$$\lim_{r \to 0} \int_{C_2} f = \lim_{r \to 0} \int_\pi^0 \frac{\left(\log|1 + re^{i\theta}| + i \arg(1 + re^{i\theta})\right)^2}{(1 + re^{i\theta})^2 - 1} \, rie^{i\theta} \, d\theta$$

$$= \lim_{r \to 0} \int_0^\pi - \frac{\left(\log|1 + re^{i\theta}| + i \arg(1 + re^{i\theta})\right)^2}{2 + re^{i\theta}} \, id\theta$$

Note that:

$$\left| -\frac{\left(\log|1 + re^{i\theta}| + i \arg(1 + re^{i\theta})\right)^2}{2 + re^{i\theta}} \, i \right| \leq \frac{\log^2 |1 + r| + (2\pi)^2}{2 - r} := g(\theta)$$

and $g(\theta)$ is integrable on $[0, \pi]$, so we can pass the limit $r \to 0$ under the integral by dominated convergence. Observing that $\lim_{r \to 0} \arg(1 + re^{i\theta}) = 0$ since we are on C_2, our integral becomes:

$$\int_0^\pi -\frac{(0 + i0)^2}{2 + 0} \, id\theta = 0$$

The proof for C_5 is essentially the same but now we have $\lim_{r \to 0} \arg(1 + re^{i\theta}) = 2\pi$, hence:

$$\lim_{r \to 0} \int_{C_5} f = \lim_{r \to 0} \int_0^{-\pi} \frac{\left(\log|1 + re^{i\theta}| + i \arg(1 + re^{i\theta})\right)^2}{2 + re^{i\theta}} \, id\theta = \int_{-\pi}^0 -\frac{(0 + i2\pi)^2}{2 + 0} \, id\theta = 2\pi^3 i$$

Since $z = 1$ lies on the branch cut, the only pole inside the contour is $z = -1$, a simple pole. Then $\text{Res}(f, -1) = \lim_{z \to -1} \frac{\log^2 z}{z - 1} = \frac{(i\pi)^2}{-2} = \pi^2/2$. Then by residue theorem,[7]

$$-4\pi i K + 2\pi^3 i = 2\pi i \left(\frac{\pi^2}{2}\right)$$

$$\Rightarrow K = \frac{\pi^2}{4}$$

---

[7] In general, you want to avoid evaluating integrals directly like what we have done here. In fact, direct integration is sometimes not even possible. However, the reality is you might not always be clever enough to find the function and contour that makes computation easy. So I give you an example of brute force integration here, so you have seen it at least once.

(Example 4) We will now solve the previous integral $K = \int_0^\infty \frac{\log x}{x^2-1} dx$ using a faster way. We will integrate $f(z) = \log z /(z^2 - 1)$ over the quarter-circle contour below, where we choose the principal branch for $\log z$.
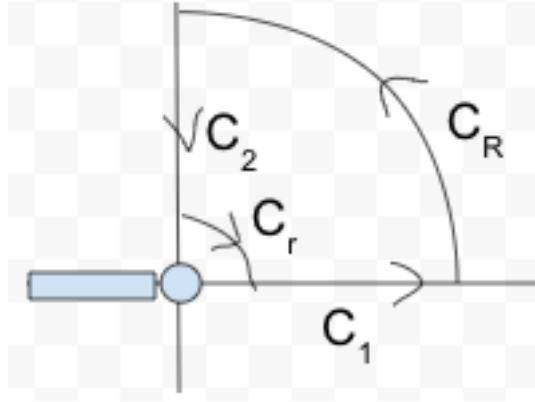

Fig. 24

Other fast methods use a semicircle contour in the upper-half plane, but since we have already proved $\int_0^\infty \frac{\log x}{x^2+1} dx = 0$ in the previous question, the quarter circle will prove even faster. We have $z = \rho e^{i\theta}, \theta \in (0, \pi/2)$ on both C_R and C_2 so for ρ large or small enough,

$$\left|\int_{C_R} f\right|, \left|\int_{C_2} f\right| \leq \frac{\pi}{2} \rho \frac{\sqrt{\log^2 \rho + (\pi/2)^2}}{|\rho^2-1|} \leq \frac{\pi\sqrt{2}}{2} \frac{\rho|\log \rho|}{|\rho^2-1|}.$$ Taking $\rho \to \infty$ and $\rho \to 0$ implies $\int_{C_R} f \to 0$

and $\int_{C_2} f \to 0$ respectively. Note $z = 1$ is a removable singularity of f, hence the function extends continuously over $z = 1$, so we do not need an indent around $z = 1$, hence we have $\int_{C_3} f = \int_{1+r}^R \frac{\log|t|}{t^2-1} e^{i0} dt$ which goes to K as $r \to 0, R \to \infty$. Lastly, $z = it$ on C_1 so

$$\int_{C_1} f = \int_R^{1+r} \frac{\log(it)}{(it)^2-1} i dt = \int_{1+r}^R \frac{\log|t|+i\pi/2}{t^2+1} i dt \to i\int_0^\infty \frac{\log|t|}{t^2+1} dt - \int_0^\infty \frac{\pi/2}{t^2+1} dt = -\pi^2/4 \text{ as } r \to 0$$

and $R \to \infty$. There are no singularities in the contour, so by Cauchy's theorem, we have $K - \pi^2/4 = 0$, or $K = \pi^2/4$.

Notice that integrating along the imaginary axis was productive because the function becomes real-valued on that axis due to the $z^2 - 1$ in the denominator. In general, we see that **any polynomial $z^n + c, c \in \mathbb{R}$ is real-valued on the lines $\{z \in \mathbb{C} : \arg z = \pi/n\}$ and $\{z \in \mathbb{C} : \arg z = 2\pi/n\}$**. Consequently, when integrating functions with these polynomials in the denominator, **it is useful to consider (indented) circular sector contours of angle π/n or $2\pi/n$.** In fact, this is a good rule of thumb when n is any positive real number.

(Question 5)

Based on our discussion on the previous page, evaluate the following two integrals where $n \in \mathbb{N}$, $n \geq 2$.

$$J_n = \int_0^\infty \frac{1}{x^n + 1} \, dx \qquad K_n = \int_0^\infty \frac{\log x}{x^n + 1} \, dx$$

Both integrals can actually be evaluated at once using a single application of the residue theorem.

1. What function and contour will you pick? After you answer, check with the solutions before proceeding.

2. For the contour and function chosen above, how many singularities of f are there inside the contour? Then, use the residue theorem to show that:

$$\begin{cases} K_n - K_n \cos\left(\dfrac{2\pi}{n}\right) + \dfrac{2\pi}{n} J_n \sin\left(\dfrac{2\pi}{n}\right) = \dfrac{2\pi^2}{n^2} \cos\left(\dfrac{\pi}{n}\right) \\[2mm] K_n \sin\left(\dfrac{2\pi}{n}\right) + \dfrac{2\pi}{n} J_n \cos\left(\dfrac{2\pi}{n}\right) = -\dfrac{2\pi^2}{n^2} \sin\left(\dfrac{\pi}{n}\right) \end{cases}$$

Hence find J_n and K_n.

3. Now, for $n \in \mathbb{N}$, $n \geq 2$, evaluate the integral

$$L_n = \int_0^\infty \frac{\log x}{x^n - 1} \, dx$$

Which contour and function will you pick? Keep in mind that we have already solved J_n and K_n. After you answer, check with the solutions before proceeding.

4. Using the function and contour chosen above, find L_n.

5. In this question, we have taken n to be an integer greater or equal to two, but in fact, the integrals J_n, K_n, and L_n all converge if n is a real number strictly greater than one. Assuming this fact, evaluate the integrals J_n, K_n, and L_n when $n \in \mathbb{R}$, $n > 1$ and justify your answer. You should not be calculating everything all over again.

(Solutions to Question 5)

1.
If we use an indented circular sector of angle π/n, we notice that the parameterization $z = te^{i\pi/n}$ will cause $z^n + 1$ to become $-z^n + 1$, and this will eventually lead to a big mess, and you will end up having to do more work (potentially using the residue theorem twice) to get out of it. It is faster and easier to use the indented circular sector of angle $2\pi/n$. As for which function to pick, we pick $f(z) = \log z /(z^n + 1)$. Note that, unlike with keyhole contours, we don't have to worry about our desired integrals cancelling out anywhere since we are not integrating around the same branch cut twice. We simply need to choose a branch cut for $\log z$ that does not run into our contour. Since $n \geq 2$, the contour will never be in the lower-half plane, so we can choose any branch cut that lies only in the lower-half plane, such as the ray $[0, -i\infty)$ with $\arg z \in (-\pi/2, 3\pi/2)$ for $\log z$. We could also use the principal branch $(-\pi, \pi)$ granted that for the case $n = 2$, we parameterize $z = -t + i\epsilon$ on the negative real axis instead of $z = -t$ so as to avoid the branch cut $(-\infty, 0]$.
We will use $f(z) = \log z /(z^n + 1)$ with $\arg z \in (-\pi/2, 3\pi/2)$ for $\log z$ over the following contour, where C_r and C_R are circular arcs of angle $2\pi/n$.
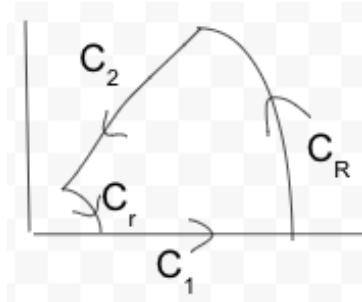
Fig. 25

2.
The roots of $z^n + 1$ are $e^{i\left(\frac{\pi}{n} + \frac{2\pi k}{n}\right)}$, so the only singularity of f in the contour is $e^{i\pi/n}$, a simple pole. Using ML-lemma, it is not hard to show $\int_{C_R} f \to 0$ and $\int_{C_r} f \to 0$ as $R \to \infty$ and $r \to 0$ respectively. We also have $\int_{C_1} f = \int_r^R \frac{\log|t|}{t^n + 1} dt \to K_n$ as $r \to 0, R \to \infty$. However, on C_2, we have $z = te^{i2\pi/n}$, so:

$$\int_{C_2} f = \int_R^r \frac{\log(te^{i2\pi/n})}{(te^{i2\pi/n})^n + 1} e^{i2\pi/n} dt = -\int_r^R \frac{\log|t| + i2\pi/n}{t^n + 1} e^{i2\pi/n} dt$$

$$\to -e^{i2\pi/n} K_n - \frac{2\pi}{n} i e^{i2\pi/n} J_n$$

as $r \to 0, R \to \infty$. Since the only pole in the contour is the simple pole $e^{i\pi/n}$, we have:

$$\text{Res}\left(f, e^{i\pi/n}\right) = \lim_{z \to e^{i\pi/n}} \frac{\log z}{nz^{n-1}} = \frac{i\pi/n}{ne^{i\pi(n-1)/n}} = \frac{i\pi/n}{-ne^{-i\pi/n}} = -\frac{i\pi}{n^2} e^{i\pi/n}$$

Then by residue theorem,

$$K_n - e^{i2\pi/n} K_n - \frac{2\pi}{n} i e^{i2\pi/n} J_n = \frac{2\pi^2}{n^2} e^{i\pi/n}$$

Writing $e^{i2\pi/n} = \cos(2\pi/n) + i\sin(2\pi/n)$ and equating real and imaginary parts yields:

$$\Rightarrow \begin{cases} K_n - K_n \cos\left(\dfrac{2\pi}{n}\right) + \dfrac{2\pi}{n} J_n \sin\left(\dfrac{2\pi}{n}\right) = \dfrac{2\pi^2}{n^2} \cos\left(\dfrac{\pi}{n}\right) \\[2ex] K_n \sin\left(\dfrac{2\pi}{n}\right) + \dfrac{2\pi}{n} J_n \cos\left(\dfrac{2\pi}{n}\right) = -\dfrac{2\pi^2}{n^2} \sin\left(\dfrac{\pi}{n}\right) \end{cases}$$

How you solve this system is up to you, but one way goes as follows: if $n = 2$ or $n = 4$, some terms become zero and the system simplifies considerably and we quickly find $J_2 = \pi/2$, $K_2 = 0$, $J_4 = \pi\sqrt{2}/4$, and $K_4 = -\pi^2\sqrt{2}/16$. If $n = 3$ or $n \geq 5$, then none of the trigonometric terms can be zero so we are justified in dividing the first equation by $\sin(2\pi/n)$, dividing the second equation by $\cos(2\pi/n)$ and then subtracting the two equations from each other to get:

$$K_n \csc\left(\frac{2\pi}{n}\right) - K_n \left(\frac{\cos 2\pi/n}{\sin 2\pi/n} + \frac{\sin 2\pi/n}{\cos 2\pi/n}\right) = \frac{2\pi^2}{n^2}\left(\frac{\cos \pi/n}{\sin 2\pi/n} + \frac{\sin \pi/n}{\cos 2\pi/n}\right)$$

After using trigonometric sum identities and simplifying,

$$\Rightarrow K_n = \frac{2\pi^2}{n^2} \frac{\cos\left(\frac{\pi}{n}\right)}{\cos\left(\frac{2\pi}{n}\right) - 1} = -\frac{\pi^2}{n^2} \csc^2\left(\frac{\pi}{n}\right) \cos\left(\frac{\pi}{n}\right)$$

Substituting back into the second equation and simplifying using double-angle identities yields $J_n = (\pi/n)\csc(\pi/n)$. These formulas hold for $n = 3$ and $n \geq 5$, but if we plug in $n = 2$ and $n = 4$, it can be verified that we indeed get the same values that were calculated earlier. Therefore, for all $n \in \mathbb{N}$, $n \geq 2$,

$$J_n = \frac{\pi}{n} \csc\left(\frac{\pi}{n}\right) \qquad\qquad K_n = -\frac{\pi^2}{n^2} \csc^2\left(\frac{\pi}{n}\right) \cos\left(\frac{\pi}{n}\right)$$

3.
You could still use a sector of angle $2\pi/n$ and the solution would be similar to the previous part, but you eventually produce the divergent integral $\int_r^R \frac{1}{x^n - 1} dx$, meaning you would have to simplify much more carefully. Furthermore, the contour would run into the pole $z = e^{i2\pi/n}$, and although we could integrate around the pole, that is more work. As it turns out, a sector of angle π/n is now advantageous since it will cause K_n and J_n to appear. Therefore we use the same contour as Fig. 25 but now C_r and C_R are sectors of angle π/n. We pick $f(z) = \log z / (z^n - 1)$ with $\arg z \in (-\pi/2, 3\pi/2)$ for $\log z$. Note $z = 1$ is a removable singularity of f, so the function extends continuously over $z = 1$, so we do not need an indent around $z = 1$. Note f has no singularities in the contour.

4.

Once again, by using ML-lemma, we have $\int_{C_R} f \to 0$ and $\int_{C_r} f \to 0$ as $R \to \infty$ and $r \to 0$ respectively. Note $z = 1$ is a removable singularity of f, so the function extends continuously over $z = 1$, so we have $\int_{C_1} f \to L_n$. Now, on C_2 we have $z = te^{i\pi/n}$ hence:

$$\int_{C_2} f = \int_R^r \frac{\log(te^{i\pi/n})}{(te^{i\pi/n})^n - 1} e^{i\pi/n} dt = \int_r^R \frac{\log|t| + i\pi/n}{t^n + 1} e^{i\pi/n} dt \to e^{\frac{i\pi}{n}} \left(K_n + \frac{i\pi}{n} J_n \right)$$

as $r \to 0, R \to \infty$. Lastly, the zeroes of $z^n - 1$ are $e^{i2\pi k}$, none of which lie inside our contour (and $z = 1$ was handled earlier) so there are no singularities inside our contour. Then by Cauchy's theorem,

$$L_n + e^{\frac{i\pi}{n}} \left(K_n + \frac{i\pi}{n} J_n \right) = 0$$

$$\Rightarrow L_n = -\left(\cos\frac{\pi}{n} + i\sin\frac{\pi}{n} \right)\left(-\frac{\pi^2}{n^2} \csc^2\left(\frac{\pi}{n}\right) \cos\left(\frac{\pi}{n}\right) + i\frac{\pi^2}{n^2} \csc\left(\frac{\pi}{n}\right) \right)$$

Equating real parts and simplifying,

$$\Rightarrow L_n = \frac{\pi^2}{n^2} \csc^2\left(\frac{\pi}{n}\right)$$

which incidentally leads to the interesting equation $L_n = (J_n)^2$.

5.

We still have $J_n = \frac{\pi}{n} \csc\left(\frac{\pi}{n}\right)$, $K_n = -\frac{\pi^2}{n^2} \csc^2\left(\frac{\pi}{n}\right) \cos\left(\frac{\pi}{n}\right)$, and $L_n = \frac{\pi^2}{n^2} \csc^2\left(\frac{\pi}{n}\right)$ when $n \in \mathbb{R}$ and $n > 1$. We now require a branch cut for z^n since n no longer has to be an integer. However, we can take the same branch cut $[0, -i\infty)$ and still use $\arg z \in (-\pi/2, 3\pi/2)$ for z^n. Then our z^n will certainly be continuous and single-valued on a domain containing our contour. Furthermore, if we parameterize this z^n on C_2 using $z = te^{i2\pi/n}$ (and note we would have to write z^n as $e^{n\log z} = |z|^n e^{in \arg z}$), we would have $\left(te^{i2\pi/n}\right)^n e^{in \arg(te^{i2\pi/n})} = t^n e^{in(2\pi/n)} = t^n$ where $\arg\left(e^{i2\pi/n}\right) = 2\pi/n$ since we chose $\arg z \in (-\pi/2, 3\pi/2)$. That is, after parameterizing, z^n still becomes t^n just like when n was a positive integer. It is not hard to see that this will occur with the parameterization $z = t$ as well. Then the integrals of $\log z /(z^n + 1)$ over C_1 and C_2 remain unchanged. The integrals of $\log z /(z^n + 1)$ over C_r and C_R can still be shown to vanish and the residue of $\log z /(z^n + 1)$ at $e^{i\pi/n}$ will still equal $-i\pi e^{i\pi/n}/n^2$ (but when calculating this residue, we have to use chain rule when differentiating z^n instead of power rule, see Appendix 3). It can be shown that the computation of L_n is, in the end, similarly unaffected, as $z = 1$ is still a removable singularity of $\log z /(z^n - 1)$ for $n \in \mathbb{R}, n > 1$ (see Appendix 1). So the formulas that we found for $n \in \mathbb{N}, n \geq 2$ are actually still valid for $n \in \mathbb{R}, n > 1$.

(Question 6)

For $a, b \in \mathbb{R}$ and $0 \leq a < 1$ and $b > 1$, evaluate the integrals:

$$J = \text{PV} \int_0^\infty \frac{1}{x^a(x^b - 1)} dx \qquad K = \int_0^\infty \frac{\log x}{x^a(x^b - 1)} dx$$

where J is the principal value of $\int_0^\infty 1/(x^a(x^b - 1))dx$. You may assume the integral of $1/(x^a(x^b - 1))$ diverges on any interval containing $x = 1$ and that $1/(x^a(x^b - 1))$ is absolutely integrable on the intervals $(0, 1 - r)$ and $(1 + r, \infty)$ for all $r > 0$. And as usual, you may assume the integral K converges. If you need to, check Appendix 1 for a refresher on principal value integrals and the "fractional residue theorem," which will also be used in this question.

1. What function and contour will you pick? After you answer, check with the solutions before proceeding. It may be helpful to show that $z = 1$ is a removable singularity of $\log z /(z^b - 1)$ for $b > 1$.

2. Using the function and contour chosen above, find J and K. We offer guided solutions below.

   a. Show $\int_{C_R} f \to 0$ and $\int_{C_r} f \to 0$ as $R \to \infty$ and $r \to 0$ respetively.

   b. Using the fractional residue theorem, show:

   $$\lim_{r \to 0} \int_{C_3} f(z)dz = \frac{2\pi^2}{b^2} e^{i2\pi(1-a)/b}$$

   c. Let $\gamma = 2\pi(1 - a)/b$. Show that:

   $$\begin{cases} (1 - \cos\gamma)K + \dfrac{2\pi}{b}J \sin\gamma + \dfrac{2\pi^2}{b^2}\cos\gamma = 0 \\[3mm] -K \sin\gamma - \dfrac{2\pi}{b}J \cos\gamma + \dfrac{2\pi^2}{b^2}\sin\gamma = 0 \end{cases}$$

   d. Hence find J and K.

(Solutions to Question 6)

1.
We use the circular sector of angle $2\pi/b$ with an indent C_3 around the pole at $z = e^{i2\pi/b}$. We pick $f(z) = \frac{\log z}{z^a(z^b-1)}$ and for $\log z$, z^a, and z^b, we use the branch cut $[0, -i\infty)$ with $\arg z \in (-\pi/2, 3\pi/2)$. Since $z = 1$ is a removable singularity of f (see Appendix 1), the function f extends continuously over $z = 1$, so we do not need an indent around $z = 1$.
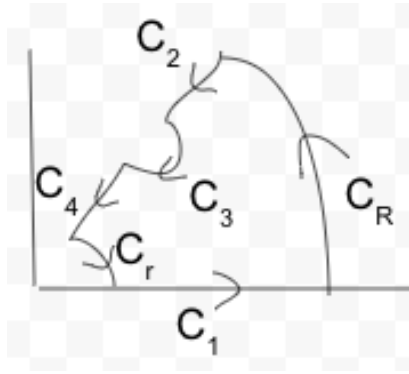

Fig. 26

2.
Keep in mind $b > 1$ and $0 \leq a < 1$. For $z = \rho e^{i\theta}$, $\theta \in (0, 2\pi/b)$ and ρ small or large enough, we have $|f(z)| = \frac{|\log|z| + i\arg z|}{|z|^a|z^b-1|} \leq \frac{\sqrt{\log^2 \rho + 4\pi^2/b^2}}{\rho^a|\rho^b-1|} \leq \frac{\sqrt{2}|\log \rho|}{\rho^a|\rho^b-1|}$ and therefore

$\left|\int_{C_r} f\right|, \left|\int_{C_R} f\right| \leq \frac{2\pi}{b} \frac{\rho^{1-a}\sqrt{2}|\log \rho|}{|\rho^b-1|} = \frac{2\sqrt{2}\pi}{b} \frac{\rho^{1-a}\log \rho}{\rho^b-1}$ (note the signs work out in such a way that we can remove the absolute value signs regardless of whether ρ is very small or very large). Since $0 < 1 - a \leq 1$, the numerator $\rho^{1-a}\log \rho$ vanishes as $\rho \to 0$ (use L'Hopital's) and the whole expression goes to 0 as $\rho \to \infty$ since $\frac{\rho^{1-a}\log \rho}{\rho^b-1} \leq \frac{\rho^{1-a}\log \rho}{\rho^b-\rho^b/2} = \frac{2\log \rho}{\rho^{b+a-1}}$ which goes to 0 as $\rho \to \infty$ by L'Hopital's (note $b + a - 1 > 0$). So $\int_{C_R} f \to 0$ and $\int_{C_r} f \to 0$. It is easy to see $\int_{C_1} f = \int_r^R \frac{\log|t|}{|t|^a(|t|^b-1)} dt$. On C_2, we parameterize $z = te^{i2\pi/b}$, hence:

$$\int_{C_2} f = \int_R^{1+r} \frac{\log(te^{i2\pi/b})}{e^{a\log(te^{i2\pi/b})}(e^{b\log(te^{i2\pi/b})} - 1)} e^{i2\pi/b} dt = -\int_{1+r}^R \frac{\log|t| + i2\pi/b}{|t|^a e^{i2\pi a/b}(|t|^b - 1)} e^{i2\pi/b} dt$$

$$= -e^{i2\pi(1-a)/b} \int_{1+r}^R \frac{\log|t|}{|t|^a(|t|^b - 1)} dt - \frac{2\pi i}{b} e^{i2\pi(1-a)/b} \int_{1+r}^R \frac{dt}{|t|^a(|t|^b - 1)}$$

Similarly, on C_4, we have:

$$\int_{C_4} f = -e^{i2\pi(1-a)/b} \int_r^{1-r} \frac{\log|t|}{|t|^a(|t|^b - 1)} dt - \frac{2\pi i}{b} e^{i2\pi(1-a)/b} \int_r^{1-r} \frac{dt}{|t|^a(|t|^b - 1)}$$

Note we cannot take limits yet since $\int_0^\infty 1/(x^a(x^b - 1))dx$ diverges.

We notice that C_3 is the arc of a circle of angle π around the simple pole $e^{i2\pi/b}$ traversed clockwise, hence by fractional residue theorem, $\int_{C_3} f \to -\pi i \operatorname{Res}(f, e^{i2\pi/b})$ as $r \to 0$. We have:

$$\operatorname{Res}(f, e^{i2\pi/b}) = \lim_{z \to e^{i2\pi/b}} \frac{\log z / z^a}{(z^b - 1)'} = \lim_{z \to e^{i2\pi/b}} \frac{\log z / z^a}{z^b (b/z)} = \frac{2\pi i}{b^2} e^{i2\pi(1-a)/b}$$

Hence $\int_{C_3} f \to \frac{2\pi^2}{b^2} e^{i2\pi(1-a)/b}$ as $r \to 0$.

There are no singularities inside the contour, hence by Cauchy's theorem,

$$\int_{C_r + C_R} f + \int_r^R \frac{\log|t|}{|t|^a(|t|^b - 1)} dt - e^{\frac{i2\pi(1-a)}{b}} \left(\int_r^{1-r} + \int_{1+r}^R \right) \frac{\log|t|}{|t|^a(|t|^b - 1)} dt$$

$$- \frac{2\pi i}{b} e^{\frac{i2\pi(1-a)}{b}} \left(\int_r^{1-r} + \int_{1+r}^R \right) \frac{dt}{|t|^a(|t|^b - 1)} + \int_{C_3} f = 0$$

We write $\int_r^R \frac{\log|t|}{|t|^a(|t|^b-1)} dt = \left(\int_r^{1-r} + \int_{1-r}^{1+r} + \int_{1+r}^R \right) \frac{\log|t|}{|t|^a(|t|^b-1)} dt$ and simplify,

$$\int_{C_r + C_R} f + \int_{1-r}^{1+r} \frac{\log|t|}{|t|^a(|t|^b - 1)} dt + \left(1 - e^{\frac{i2\pi(1-a)}{b}} \right) \left(\int_r^{1-r} + \int_{1+r}^R \right) \frac{\log|t|}{|t|^a(|t|^b - 1)} dt$$

$$- \frac{2\pi i}{b} e^{\frac{i2\pi(1-a)}{b}} \left(\int_r^{1-r} + \int_{1+r}^R \right) \frac{dt}{|t|^a(|t|^b - 1)} + \int_{C_3} f = 0$$

Now we take $r \to 0$ and $R \to \infty$. We denote $\gamma = 2\pi(1-a)/b$ for concision.

$$(1 - e^{i\gamma})K - \frac{2\pi i}{b} e^{i\gamma} J + \frac{2\pi^2}{b^2} e^{i\gamma} = 0$$

Equating real and imaginary parts,

$$\begin{cases} (1 - \cos\gamma)K + \dfrac{2\pi}{b} J \sin\gamma + \dfrac{2\pi^2}{b^2} \cos\gamma = 0 \\[2mm] -K \sin\gamma - \dfrac{2\pi}{b} J \cos\gamma + \dfrac{2\pi^2}{b^2} \sin\gamma = 0 \end{cases}$$

Solving the system (see Question 5 for a similar example) yields:

$$J = -\frac{\pi}{b} \cot\left(\frac{\pi(1-a)}{b} \right) \qquad\qquad K = \frac{\pi^2}{b^2} \csc^2\left(\frac{\pi(1-a)}{b} \right)$$

Remember, $-\frac{\pi}{b} \cot\left(\frac{\pi(1-a)}{b} \right)$ is the principal value of the integral, but the integral itself diverges. Note that our result for K is consistent with our calculation of L_n in Question 5.

(Question 7)

For $0 < b < 4$, evaluate the integral:

$$J = \int_0^\infty \frac{\log(1 + x^4)}{x^{b+1}} dx$$

1. We first construct a branch of $\log(1 + z^4)$. The laws of $\log$ motivate us to write
    $\log(1 + z^4) = \log(z - e^{i\pi/4}) + \log(z - e^{i3\pi/4}) + \log(z - e^{i5\pi/4}) + \log(z - e^{i7\pi/4})$.
    We choose the branches as follows:

Function	Branch cut	Domain of $\arg(f_k(z))$
$f_1(z) = \log(z - e^{i\pi/4})$	$\{z : z = re^{i\pi/4}, r \geq 1\}$	$(-7\pi/4, \pi/4)$
$f_2(z) = \log(z - e^{i3\pi/4})$	$\{z : z = re^{i3\pi/4}, r \geq 1\}$	$(-5\pi/4, 3\pi/4)$
$f_3(z) = \log(z - e^{i5\pi/4})$	$\{z : z = re^{i5\pi/4}, r \geq 1\}$	$(-3\pi/4, 5\pi/4)$
$f_4(z) = \log(z - e^{i7\pi/4})$	$\{z : z = re^{i7\pi/4}, r \geq 1\}$	$(-\pi/4, 7\pi/4)$

    Using the above table, draw plots of $\arg(f_k(z))$ for each $k = 1, 2, 3, 4$.

2. Define $f_1(z) + \cdots + f_4(z)$ be our branch of $\log(1 + z^4)$. Sketch a plot of
    $\arg(\log(1 + z^4))$. After you answer, check with the solutions before proceeding.

3. We integrate $f(z) = \log(1 + z^4)/z^{b+1}$ where $\log(1 + z^4)$ is the branch as defined
    above and we choose z^{b+1} with branch cut $[0, -i\infty)$ and $\arg z \in (-\pi/2, 3\pi/2)$.
    Which contour should we integrate f over? After you answer, check with the
    solutions before proceeding.

4. Hence find J. We offer guided solutions below. You can just assume $\int_{C_2} f \to 0$ as
    $r \to 0$ since proving this is rather tricky, but we will still prove it in the solutions.

    a. Show $\int_{C_R} f \to 0$ as $R \to \infty$. Then show $\int_{C_2} f \to 0$ as $r \to 0$ using ML-lemma,
        or just assume it is true and move on.

    b. Show $\int_{C_5} f \to 0$, and $\int_{C_8} f \to 0$ as $r \to 0$ using ML-lemma.

    c. Show that:
        $$\lim_{r \to 0} \int_{C_4 + C_6 + C_7 + C_9} f = -\frac{2\pi i}{b}\left(e^{-ib\pi/4} + e^{-ib3\pi/4}\right)$$

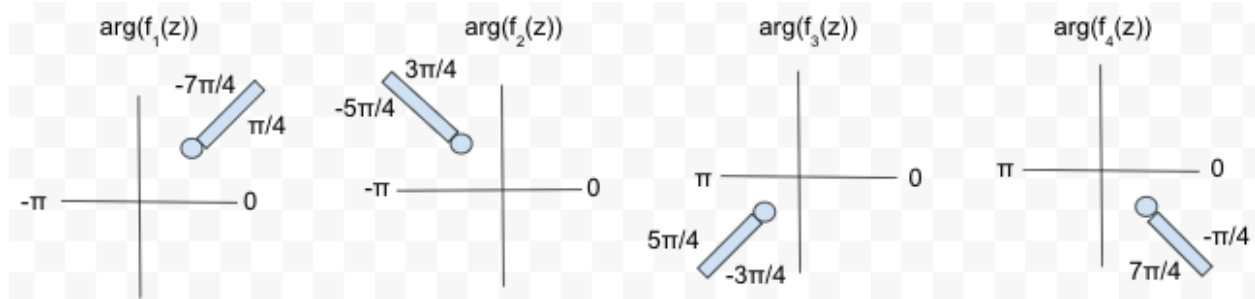   d. Hence find J.

(Solutions to Question 7)

1.

Fig. 27

2.
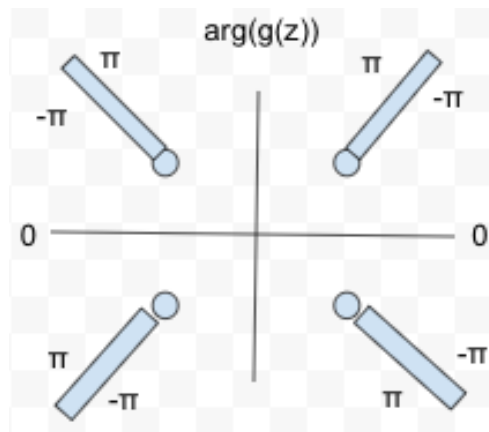We have $\arg g(z) = \arg f_1(z) + \cdots + \arg f_4(z)$. We see that $\arg g(z) \in (-\pi, \pi)$.

Fig. 28

3.
We use the following contour:
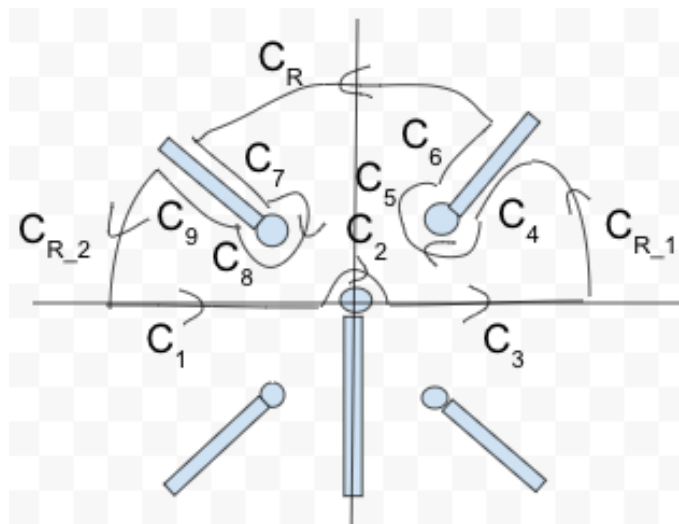
Fig. 29

Note that $e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}, 0,$ and ∞ are the branch points of f (the $\log(1 + z^4)$ term contributes $e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}, \infty$ while the z^{b+1} term contributes 0 and ∞). However, we must keep the branch cuts for $\log(1 + z^4)$ and z^{b+1} separate in our minds: for instance, $\arg(\log(1 + z^4))$ is zero on both the negative and positive real axis, but $\arg z$ (from z^{b+1}) is zero on the positive real axis and π on the negative real axis. As usual, C_2, C_5, C_9 have radius $r \to 0$ and C_R, C_{R_1}, C_{R_2} have radius $R \to \infty$.

4.

For $z = Re^{i\theta}, \theta \in (0,\pi)$ and R large enough, we have $|f(z)| = \frac{\sqrt{\log^2|1+z^4| + (\arg(1+z^4))^2}}{|z|^{b+1}} \leq$

$\frac{\sqrt{\log^2(1+R^4) + \pi^2}}{R^{b+1}} \leq \frac{\sqrt{\log^2(1+R^4) + \log^2(1+R^4)}}{R^{b+1}} = \frac{\sqrt{2}\log(1+R^4)}{R^{b+1}}$ hence $\left|\int_{C_R} f\right| \leq \frac{\pi}{2} \frac{\sqrt{2}\log(1+R^4)}{R^b}$ which

goes to 0 as $R \to \infty$ by L'Hopital's. Hence $\int_{C_R} f$ vanishes as $R \to \infty$. Similarly, the integrals over C_{R_1} and C_{R_2} vanish as $R \to \infty$. We have $z = re^{i\theta}, \theta \in (0,\pi)$ on C_2 and noting $\arg(1 + r^4 e^{i4\theta}) \in [-\theta_0, \theta_0]$ for small $\theta_0 > 0$ when r is small enough,

$$|f(z)| \leq \frac{\sqrt{\log^2(1+r^4) + (\arg(1+r^4 e^{i4\theta}))^2}}{r^{b+1}} = \frac{\sqrt{\log^2(1+r^4) + \arctan^2\left(\frac{r^4 \sin(4\theta)}{1 + r^4 \cos(4\theta)}\right)}}{r^{b+1}}$$

$$\leq \frac{\sqrt{\log^2(1+r^4) + \arctan^2\left(\frac{r^4}{1-r^4}\right)}}{r^{b+1}}$$

As $r \to 0$, this becomes a 0/0 limit, but before L'Hopital's, we try to simplify a little. We aim to show $\log(1 + x^4) \leq \arctan(x^4/(1-x^4))$ eventually as $x \to 0^+$. Both are zero when $x = 0$, so it suffices to show the derivative of $\arctan(x^4/(1-x^4))$ is greater than the derivative of $\log(1 + x^4)$ eventually as $x \to 0^+$, but this is readily apparent: we have $\left(\arctan(x^4/(1-x^4))\right)' = 4x^3/(2x^8 - 2x^4 + 1)$ which is certainly greater than $(\log(1 + x^4))' = 4x^3/(1 + x^4)$ for x small enough. Then $\log^2(1 + r^4) \leq \arctan^2\left(\frac{r^4}{1-r^4}\right)$ for r small enough, hence:

$$\leq \frac{\sqrt{\arctan^2\left(\frac{r^4}{1-r^4}\right) + \arctan^2\left(\frac{r^4}{1-r^4}\right)}}{r^{b+1}} = \frac{\sqrt{2}\arctan\left(\frac{r^4}{1-r^4}\right)}{r^{b+1}}$$

Hence $\left|\int_{C_2} f\right| \leq \pi\sqrt{2}\arctan\left(\frac{r^4}{1-r^4}\right)/r^b$ which goes to 0 as $r \to 0$ by L'Hopital's. Hence $\int_{C_2} f \to 0$ as $r \to 0$. On C_5, we have $z = e^{i\pi/4} + re^{i\theta}$. Writing $\log(1 + z^4)$ as $f_1 + \cdots + f_4$, we have:

$$|f(z)| \leq \left|\frac{\log(re^{i\theta}) + \log(\sqrt{2} + re^{i\theta}) + \log(\sqrt{2}(1+i) + re^{i\theta}) + \log(i\sqrt{2} + re^{i\theta})}{(e^{i\pi/4} + re^{i\theta})^{b+1}}\right|$$

Note $\left|e^{\frac{i\pi}{4}} + re^{i\theta}\right|^{b+1} = \left(\left(\frac{1}{\sqrt{2}} + r\cos\theta\right)^2 + \left(\frac{1}{\sqrt{2}} + r\sin\theta\right)^2\right)^{\frac{b+1}{2}} \geq \left(\left(\frac{1}{\sqrt{2}} - r\right)^2 + \left(\frac{1}{\sqrt{2}} - r\right)^2\right)^{\frac{b+1}{2}}$ for r small enough an $\theta \in [0,\pi]$,

$$\leq \frac{|\log(r)| + \log(\sqrt{2} + r) + \log(2 + r) + \log(\sqrt{2} + r)}{\sqrt{2}^{b+1}\left((1/\sqrt{2}) - r\right)^{b+1}}$$

$$\Rightarrow \left|\int_{C_5} f\right| \leq 2\pi r \frac{|\log(r)| + \log(\sqrt{2} + r) + \log(2 + r) + \log(\sqrt{2} + r)}{\sqrt{2}^{b+1}\left((1/\sqrt{2}) - r\right)^{b+1}}$$

which goes to 0 as $r \to 0$ (note $r \log r \to 0$ as $r \to 0$). Hence $\int_{C_5} f \to 0$ as $r \to 0$. A similar calculation shows $\int_{C_8} f \to 0$ as $r \to 0$. Next, we work on C_4, C_6, C_7, C_9. Recall that $\arg(\log(1+z^4)) \in (-\pi, \pi)$ (see Fig. 28) and $\arg(z^{b+1}) \in (-\pi/2, 3\pi/2)$. Then we have:

$z = te^{i\pi/4} - i\epsilon$ on C_4 hence $\int_{C_4} f = \int_R^{1+r} \frac{\log|1-t^4|+i\pi}{|t|^{b+1}e^{i(b+1)\pi/4}} e^{i\pi/4} dt$

$z = te^{i\pi/4} + i\epsilon$ on C_6 hence $\int_{C_6} f = \int_{1+r}^R \frac{\log|1-t^4|-i\pi}{|t|^{b+1}e^{i(b+1)\pi/4}} e^{i\pi/4} dt$

$z = te^{i3\pi/4} + i\epsilon$ on C_7 hence $\int_{C_7} f = \int_R^{1+r} \frac{\log|1-t^4|+i\pi}{|t|^{b+1}e^{i(b+1)3\pi/4}} e^{i3\pi/4} dt$

$z = te^{i3\pi/4} - i\epsilon$ on C_9 hence $\int_{C_9} f = \int_{1+r}^R \frac{\log|1-t^4|-i\pi}{|t|^{b+1}e^{i(b+1)3\pi/4}} e^{i3\pi/4} dt$

It can be verified that $\int_1^\infty dt/t^{b+1} = 1/b$ for $b > 0$. Then, adding it all together,

$$\int_{C_4+C_6+C_7+C_9} f = -2\pi i \left(e^{-ib\pi/4} + e^{-ib3\pi/4}\right) \int_{1+r}^R \frac{dt}{t^{b+1}}$$

$$\to -\frac{2\pi i}{b}\left(e^{-ib\pi/4} + e^{-ib3\pi/4}\right)$$

as $r \to 0, R \to \infty$. Lastly, $z = -t$ on C_1 and $z = t$ on C_2. Recall $\log(1+z^4)$ has argument 0 on the real axis but $\arg z$ (for z^{b+1}) has argument 0 and π on the positive and negative real axis respectively. Then as $r \to 0, R \to \infty$, we have $\int_{C_2} f \to \int_0^\infty \frac{\log|1+t^4|+i0}{|t|^{b+1}e^{i(b+1)0}} dt = J$ and $\int_{C_1} f \to -\int_\infty^0 \frac{\log|1+t^4|+i0}{|t|^{b+1}e^{i(b+1)\pi}} dt = -e^{-ib\pi}J$. Then by Cauchy's theorem,

$$\left(1 - e^{-ib\pi}\right)J - \frac{2\pi i}{b}\left(e^{-ib\pi/4} + e^{-ib3\pi/4}\right) = 0$$

Write $1 - e^{-ib\pi} = \left(e^{ib\pi/2} - e^{-ib\pi/2}\right)/e^{ib\pi/2} = 2ie^{-ib\pi/2}\sin(b\pi/2)$ and simplify,

$$\Rightarrow J = \frac{\pi}{b}\left(\frac{e^{ib\pi/4} + e^{-ib\pi/4}}{\sin(b\pi/2)}\right) = \frac{\pi}{b}\left(\frac{2\cos(b\pi/4)}{\sin(b\pi/2)}\right)$$

$$= \frac{\pi}{b}\left(\frac{2\cos(b\pi/4)}{2\sin(b\pi/4)\cos(b\pi/4)}\right)$$

$$= \frac{\pi}{b}\csc\left(\frac{b\pi}{4}\right)$$

(Question 8)

For $b \in \mathbb{R}\backslash\{0\}$, evaluate:

$$J = \int_0^\infty \frac{\log(1 + x^2)}{x^2(x^4 + 1)} dx$$

1. Construct an appropriate branch of $\log(1 + z^2)$. Draw a plot of $\arg(\log(1 + z^2))$. Which contour and function will you use? After you answer, check with the solutions before proceeding. It may be helpful to show that $\log(1 + z^2)/z^2$ has a removable singularity at zero.

2. Hence find J. You may assume the following result, which is obtained using partial fractions decomposition:

$$\int_1^\infty \frac{1}{t^2(t^4 + 1)} dt = 1 - \frac{\pi + \log(3 + 2\sqrt{2})}{4\sqrt{2}}$$

We offer guided solutions below.

 a. Show $\int_{C_R} f \to 0$ as $R \to \infty$ and $\int_{C_5} f \to 0$ as $r \to 0$.

 b. Hence find J.
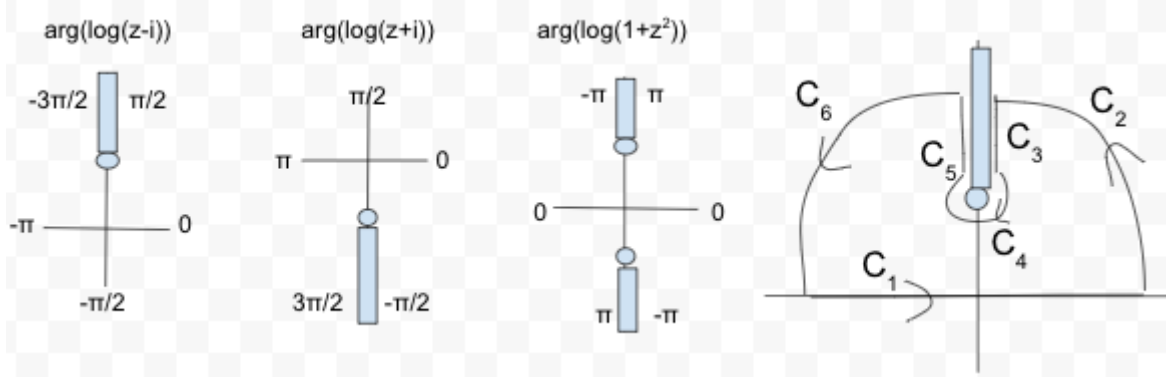
(Solutions to Question 8)

1.

Fig. 30

Choose $\log(z - i)$ with branch cut $[i, \infty)$ and $\arg(z - i) \in (-3\pi/2, \pi/2)$ and choose $\log(z + i)$ with branch cut $[-i, \infty)$ and $\arg(z + i) \in (-\pi/2, 3\pi/2)$. Then we define $\log(1 + z^2) = \log(z - i) + \log(z + i)$ to be our branch. We integrate $f(z) = \frac{\log(1+z^2)}{z^2(z^4+1)}$ over the above semicircular contour. By manually computing the Taylor series of $\log(1 + z^2)$ around 0, we see the first nonzero term is the z^2 term (you don't even have to bother finding the coefficient), hence $z = 0$ is a removable singularity of f, so we do not need an indent around zero.

2.
Using $|\log(1 + z^2)| \le |\log(z + i)| + |\log(z - i)| \le 2 \log(1 + R)$ when $z = Re^{i\theta}$, it is easy to see $\int_{C_2} f, \int_{C_6} f \to 0$ as $R \to \infty$ using ML-lemma. On C_4 we have $z = i + re^{i\theta}$, therefore

$$|f(z)| \le \frac{\log(2+r)+\log(r)}{\left|(i+re^{i\theta})^2\right|\left|\left|(i+re^{i\theta})^4\right|-1\right|} \le \frac{\log(2+r)+\log(r)}{(1-2r-r^2)(4r-6r^2-4r^3-r^4)}$$ for r small enough, hence

$\left|\int_{C_4} f\right| \le \frac{2\pi(\log(2+r)+\log(r))}{(1-2r-r^2)(4-6r-4r^2-r^3)}$ which goes to 0 as $r \to 0$. Hence $\int_{C_4} f \to 0$ as $r \to 0$. Since $\log(1 + z^2)$ has argument 0 on the real axis and the integrand is even, we see $\int_{C_1} f \to 2J$ as $r \to 0, R \to \infty$. Next we have $z = it + \epsilon$ and $z = it - \epsilon$ on C_3 and C_5 respectively, so $\int_{C_3} f = \int_R^{1+r} \frac{\log|1+(it)^2|+i\pi}{(it)^2((it)^4+1)} i\,dt$ and $\int_{C_5} f = \int_{1+r}^R \frac{\log|1+(it)^2|-i\pi}{(it)^2((it)^4+1)} i\,dt$. Then,

$$\int_{C_4+C_6} f = \int_{1+r}^R \frac{-2\pi}{t^2(t^4+1)}\,dt \to -2\pi\left(1 - \frac{\pi + \log(3 + 2\sqrt{2})}{4\sqrt{2}}\right)$$

as $r \to 0, R \to \infty$. The poles of f in the contour are $e^{i\pi/4}$ and $e^{i3\pi/4}$, both simple poles. Then since $\arg(\log(1 + z^2)) \in (-\pi, \pi)$, we have:

$$\text{Res}(f, e^{i\pi/4}) = \lim_{z \to e^{i\pi/4}} \frac{\log(1 + z^2)/z^2}{4z^3} = \frac{\log(1 + i)}{4e^{i5\pi/4}} = \frac{\log\sqrt{2} + i\pi/4}{4e^{i5\pi/4}}$$

$$\text{Res}(f, e^{i3\pi/4}) = \lim_{z \to e^{i3\pi/4}} \frac{\log(1 + z^2)/z^2}{4z^3} = \frac{\log(1 - i)}{4e^{i5\pi/4}} = \frac{\log\sqrt{2} - i\pi/4}{4e^{i15\pi/4}}$$

Then by residue theorem,

$$2J - 2\pi\left(1 - \frac{\pi + \log\left(3 + 2\sqrt{2}\right)}{4\sqrt{2}}\right) = 2\pi i\left(\frac{\log\sqrt{2} + i\pi/4}{4e^{i5\pi/4}} + \frac{\log\sqrt{2} - i\pi/4}{4e^{i15\pi/4}}\right)$$

$$J = \pi\left(1 - \frac{\pi + 2\log\left(6 + 4\sqrt{2}\right)}{8\sqrt{2}}\right)$$

(Question 9)

For $a > 0$, $b \in \mathbb{R}\setminus[0,a]$, and $n \in \mathbb{N}$, evaluate:

$$J = \int_0^a \frac{x^{\frac{n-1}{n}}(a-x)^{\frac{1}{n}}}{b-x}\,dx$$

Note that this integral could be solved using Laurent series, but we have not developed the necessary tools yet. If $n = 1$, it can be shown that $J = a + (b-a)\ln|-1 + a/b|$ using basic integration techniques, so from now on we will assume $n \geq 2$.

1. What contour and function will you choose? After you answer, check with the solutions before proceeding. Keep in mind you will have to construct the branch of your function carefully. Check Example 1 or Example 3 for a refresher.

2. Hence find J. We offer guided solutions below.

   a. Show $\int_{C_2} f \to 0$ and $\int_{C_4} f \to 0$ and $r \to 0$.

   b. Using the fractional residue theorem or otherwise, show that:

   $$\lim_{r\to 0}\int_{C_5} f = \begin{cases} 2\pi i|b^{n-1}(a-b)|^{1/n}e^{-i\pi/n} & b < 0 \\ -2\pi i|b^{n-1}(a-b)|^{1/n}e^{-i\pi/n} & b > a \end{cases}$$

   c. Find a branch of $(az-1)^{1/n}$ satisfying:

   $$\left(\frac{1}{z}\right)^{\frac{n-1}{n}}\left(a - \frac{1}{z}\right)^{\frac{1}{n}} = \frac{(az-1)^{\frac{1}{n}}}{z}$$

   d. Hence show $\int_{C_R} f \to 2\pi i e^{-i\pi/n}(-b + a/n)$ as $R \to \infty$.

   e. Hence find J.

(Solutions to Question 9)

1.

We wish to integrate $f(z) = \dfrac{z^{(n-1)/n}(a-z)^{1/n}}{b-z}$ over a dogbone contour. We therefore construct our branch of $z^{(n-1)/n}(a-z)^{1/n}$ by writing:

$$z^{\frac{n-1}{n}}(a-z)^{\frac{1}{n}} = z^{\frac{n-1}{n}}(z-a)^{\frac{1}{n}}(-1)^{\frac{1}{n}}$$

where we pick $z^{(n-1)/n}$ with branch cut $(-\infty, 0]$ and $\arg z \in (-\pi, \pi)$, we pick $(z-a)^{1/n}$ with branch cut $(-\infty, a]$ and $\arg(z-a) \in (-\pi, \pi)$, and we pick $(-1)^{1/n} = e^{-i\pi/n}$ with $\arg e^{-i\pi/n} = -\pi/n$. Then $\arg\big(z^{(n-1)/n}(a-z)^{1/n}\big) = \dfrac{n-1}{n}\arg z + \dfrac{1}{n}\arg(z-a) - \dfrac{\pi}{n}$ hence we can plot some values of $\arg\big(z^{(n-1)/n}(a-z)^{1/n}\big)$ below:
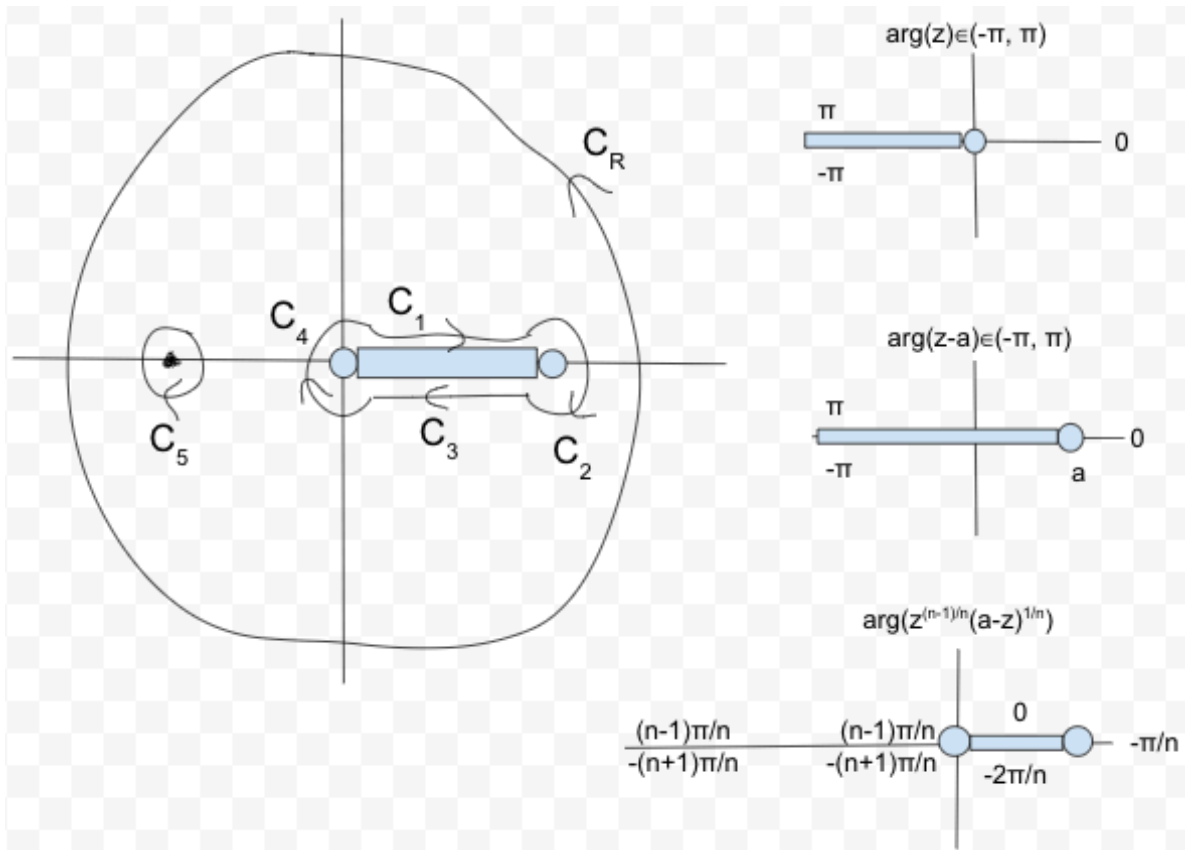
Fig. 31

Note $e^{i(n-1)\pi/n} = e^{-i(n+1)\pi/n}$, so our branch of $z^{(n-1)/n}(a-z)^{1/n}$ has branch cut $[0, a]$ and $\arg\big(z^{(n-1)/n}(a-z)^{1/n}\big) \in (-(n+1)\pi/n, 0)$. We have drawn b to be negative, but b could be anywhere in $\mathbb{R}\backslash[0, a]$, and we would just draw a circle around it small enough that it does not intersect the dogbone. The outer circle is large enough to fully contain the circle around b and the dogbone. As usual, C_2, C_4, C_5 have radius $r \to 0$ and C_R has radius $R \to \infty$.

2.

We have $z = re^{i\theta}$ on C_4 so for r small enough, $|f(z)| \leq \frac{r(a+r)}{b-r}$ hence $\left|\int_{C_4} f\right| \leq 2\pi \frac{r^2(a+r)}{b-r}$ which goes to 0 as $r \to 0$, so $\int_{C_4} f \to 0$. We have $z = a + re^{i\theta}$ on C_2 so for r small enough, $|f(z)| \leq \frac{(a+r)r}{|b-|a\pm r||}$ (which sign we take depends on how close b is to a) hence $\left|\int_{C_2} f\right| \leq 2\pi \frac{r^2(a+r)}{|b-|a\pm r||}$ which goes to 0 as $r \to 0$, so $\int_{C_2} f \to 0$. As $r \to 0$, we also find

$$\int_{C_1} f \to \int_0^a \frac{|t^{(n-1)/n}(a-t)^{1/n}|e^{i0}}{b-t} dt = J \text{ and } \int_{C_3} f \to \int_a^0 \frac{|t^{(n-1)/n}(a-t)^{1/n}|e^{-i2\pi/n}}{b-t} dt = -e^{-i2\pi/n}J.$$

Since $z = b$ is a simple pole and C_5 is a circle arc, the fractional residue theorem says:

$$\int_{C_5} f \to -2\pi i \, \text{Res}(f,b) = -2\pi i \lim_{z \to b} z^{\frac{n-1}{n}}(a-z)^{\frac{1}{n}} = -2\pi i \left| b^{\frac{n-1}{n}}(a-b)^{\frac{1}{n}} \right| e^{i \arg\left(b^{\frac{n-1}{n}}(a-b)^{\frac{1}{n}} \right)}$$

as $r \to 0$. Looking at Fig. 31, if $b > a$ then $\arg\left(b^{(n-1)/n}(a-b)^{1/n}\right) = -\pi/n$. If $b < 0$, then either $\arg\left(b^{(n-1)/n}(a-b)^{1/n}\right)$ is $(n-1)\pi/n$ or $-(n+1)\pi/n$, but both result in $e^{i \arg(b^{(n-1)/n}(a-b)^{1/n})}$ having the same value of $-e^{-i\pi/n}$. Hence,

$$\lim_{r \to 0} \int_{C_5} f = \begin{cases} 2\pi i |b^{n-1}(a-b)|^{1/n} e^{-i\pi/n} & b < 0 \\ -2\pi i |b^{n-1}(a-b)|^{1/n} e^{-i\pi/n} & b > a \end{cases}$$

It remains to tackle $\int_{C_R} f$. We have:

$$\int_{C_R} f = -2\pi i \, \text{Res}(f,\infty) = 2\pi i \, \text{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right) = 2\pi i \, \text{Res}\left(\frac{(1/z)^{\frac{n-1}{n}}(a-1/z)^{\frac{1}{n}}}{z(bz-1)}, 0\right)$$

The following steps are similar to what we did in Example 1. We aim to write $(1/z)^{(n-1)/n}(a-1/z)^{1/n} = (az-1)^{1/n}/z$, but we must choose branches for $(az-1)^{1/n}$ and $\arg z$ such that this is true. The argument of $(1/z)^{(n-1)/n}(a-1/z)^{1/n}$ still lies in the interval $(-(n+1)\pi/n, 0)$ since we have only applied inversion. By noting $1/z = \bar{z}/|z|^2$, we can plot $\arg\left((1/z)^{(n-1)/n}(a-1/z)^{1/n}\right)$ below:
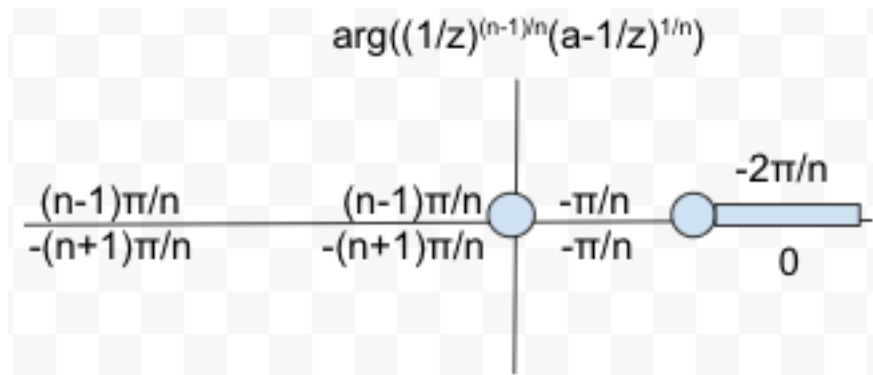

Fig. 32

We have $\arg\left((az-1)^{1/n}\right) = \arg\left((1/z)^{(n-1)/n}(a-1/z)^{1/n}\right) + \arg z$. We pick the branch of $\arg z$ satisfying $\arg z \in (0, 2\pi)$. Now we can plot some values of $\arg\left((az-1)^{1/n}\right)$,
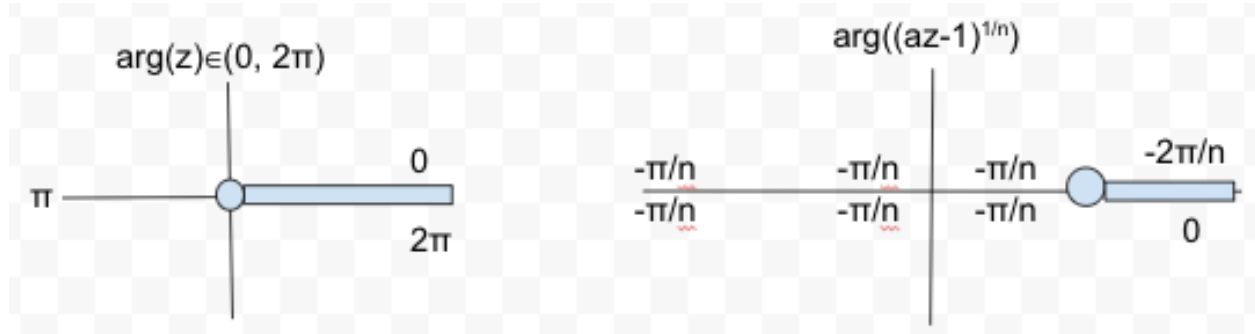

Fig. 33

So our $(az-1)^{1/n} := g(z)$ has branch cut $[1/a, \infty)$ and $\arg\left((az-1)^{1/n}\right) \in (-2\pi/n, 0)$. Note we have regained continuity at zero. Then,

$$2\pi i \operatorname{Res}\left(\frac{(1/z)^{\frac{n-1}{n}}(a-1/z)^{\frac{1}{n}}}{z(bz-1)}, 0\right) = 2\pi i \operatorname{Res}\left(\frac{(az-1)^{\frac{1}{n}}}{z^2(bz-1)}, 0\right) = 2\pi i \lim_{z \to 0} \frac{d}{dz}\left[\frac{(az-1)^{\frac{1}{n}}}{bz-1}\right]$$

$$= 2\pi i \lim_{z \to 0} \frac{d}{dz}\left[\frac{g(z)}{bz-1}\right] = 2\pi i \lim_{z \to 0} \frac{(bz-1)g(z)\frac{1}{n}\left(\frac{a}{az-1}\right) - bg(z)}{(bz-1)^2}$$

where we used chain rule to differentiate $g(z)$. Since $\arg g(z) \in (-2\pi/n, 0)$, we must have $\arg g(0) = -\pi/n$, therefore $g(0) = |g(0)|e^{i\arg g(0)} = e^{-i\pi/n}$, so our residue becomes:

$$= 2\pi i e^{-i\pi/n}\left(\frac{a}{n} - b\right)$$

There are no poles in the contour $C_R + C_1 + \cdots + C_5$, so by Cauchy's theorem, if $b < 0$,

$$J - e^{-i2\pi/n}J + 2\pi i |b^{n-1}(a-b)|^{1/n}e^{-i\pi/n} + 2\pi i e^{-i\pi/n}\left(\frac{a}{n} - b\right) = 0$$

We write $1 - e^{-i2\pi/n} = \left(e^{i\pi/n} - e^{-i\pi/n}\right)/e^{i\pi/n} = 2ie^{-i\pi/n}\sin(\pi/n)$ and simplify,

$$\Rightarrow J = \pi\left(-|b^{n-1}(a-b)|^{1/n} - \frac{a}{n} + b\right)\csc\left(\frac{\pi}{n}\right)$$

We do a similar calculation for $b > a$. Then,

$$J = \begin{cases} \pi\left(-|b^{n-1}(a-b)|^{1/n} - \dfrac{a}{n} + b\right)\csc\left(\dfrac{\pi}{n}\right) & b < 0 \\ \pi\left(|b^{n-1}(a-b)|^{1/n} - \dfrac{a}{n} + b\right)\csc\left(\dfrac{\pi}{n}\right) & b > a \end{cases}$$

Taylor and Laurent Series of Functions with Branch Cuts

Finding the Laurent series of simpler functions is sometimes not so hard. The important thing to recognize is that different branches of a function will have different Laurent series. Although it is usually easier to calculate residues using tricks and theorems, it is sometimes easier to calculate the residue at infinity by finding the Laurent series.

Example 2.1

As a simple example, let $f(z)$ and $g(z)$ both be $\log(z)$ with branch cut $(-\infty, 0]$ but f has $\arg z \in (-\pi, \pi)$ and g has $\arg z \in (\pi, 3\pi)$. We find the Laurent series of f and g centered at $z = 1$ on the disk $D = \{|z - 1| < 1\}$. Since f and g are analytic on D, the Laurent series is just a Taylor series $\sum_{k=0}^{\infty} a_k (z - 1)^k$ where $a_k = f^{(k)}(1)/k!$. Since the derivatives of $\log(z)$ are rational functions $1/z$, $-1/z^2$, etc. that do not depend on branch, the only term that will differ in the Taylor series of f and g are the constant terms. Indeed, we find that:

$$f(z) = 0 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(z - 1)^k \qquad g(z) = 2\pi i + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(z - 1)^k$$

Example 2.2

For another example, we find all Laurent series of $\sqrt{z}$ with branch cut $(-\infty, 0]$ centered at $z = 2$. Recall that there are only two such branches, which are $f_1(z) := +g(z)$ and $f_2(z) := -g(z)$ where $g(z)$ is the principal branch of $\sqrt{z}$. Each branch only has one Laurent series around $z = 2$ which must converge on $D := \{|z - 2| < 2\}$. Note the Laurent series is once again a Taylor series since $\sqrt{z}$ is analytic on D. Note that f_1 is positive on the positive real axis but f_2 is negative on the positive real axis. Then to find our Taylor series, we again compute the coefficients manually using $a_k = f_1^{(k)}(2)/k!$ to find $f_1'(z) = f_1(z)/2z$, $f_1''(z) = -f_1(z)/4z^2$, ... and $f_1(2) = \sqrt{2}$. Then,

$$f_1(z) = \sqrt{2} + \frac{\sqrt{2}}{4}(z - 2) - \frac{\sqrt{2}}{2!\,16}(z - 2)^2 + \frac{3\sqrt{2}}{3!\,64}(z - 2)^3 + \cdots$$

Then the Taylor series of f_2 is just the negative of the Taylor series of f_1,

$$f_2(z) = -\sqrt{2} - \frac{\sqrt{2}}{4}(z - 2) + \frac{\sqrt{2}}{2!\,16}(z - 2)^2 - \frac{3\sqrt{2}}{3!\,64}(z - 2)^3 - \cdots$$

Especially for functions of the form z^α, $\alpha \in \mathbb{C}$, finding Taylor series by computing derivatives is tedious. There is sometimes a better way using binomial coefficients.

Definition 2.3

For $a \in \mathbb{C}$ and $n \in \mathbb{N}_{\geq 0}$ we define the binomial coefficient "a choose n" by

$$\binom{a}{0} = 1 \qquad \binom{a}{n} = \frac{a(a - 1) \dots (a - n + 1)}{n!}, n \geq 1$$

Theorem 2.4
Binomial Theorem: Let $a \in \mathbb{C}$ and let $g(z)$ be an analytic branch of $(1 + z)^a$ defined on the open disk $D := \{|z| < 1\}$. Then for all $z \in D$, we have:

$$g(z) = g(0) \sum_{k=0}^{\infty} \binom{a}{k} z^k$$

Proof: First we determine the convergence of the series using the ratio test.

$$\lim_{k \to \infty} \left| \frac{\binom{a}{k+1} z^{k+1}}{\binom{a}{k} z^k} \right| = \lim_{k \to \infty} \frac{|a - k|}{k + 1} |z| = \lim_{k \to \infty} \frac{\sqrt{(\text{Re}(a) - k)^2 + (\text{Im}(a))^2}}{k + 1} |z| = |z|$$

Hence the series converges for $|z| < 1$, that is, on D. To prove $g(z) = g(0) \sum_{k=0}^{\infty} \binom{a}{k} z^k$ on D, let $f(z) := \sum_{k=0}^{\infty} \binom{a}{k} z^k$. Since f is a convergent power series on D, f is analytic on D and we can use term-by-term differentiation to find f',

$$f'(z) = \underbrace{\sum_{k=1}^{\infty} \binom{a}{k} k z^{k-1}}_{(1)} = \sum_{k=0}^{\infty} \binom{a}{k+1} (k + 1) z^k = \underbrace{\sum_{k=0}^{\infty} \binom{a}{k} (a - k) z^k}_{(2)}$$

where the above series also converge on D. Then,

$$(1 + z)f'(z) = f'(z) + z'f(z) = \underbrace{\sum_{k=0}^{\infty} \binom{a}{k} (a - k) z^k}_{(2)} + z \underbrace{\sum_{k=1}^{\infty} \binom{a}{k} k z^{k-1}}_{(1)}$$

$$= \sum_{k=0}^{\infty} \binom{a}{k} (a - k) z^k + \sum_{k=0}^{\infty} \binom{a}{k} k z^k = af(z)$$

Hence $(1 + z)f'(z) - af(z) = 0$ on D. Now consider $f(z)/g(z)$. Note $g(z) \neq 0$ on D. Then by writing $g(z) = e^{a \log(1+z)}$, we have:

$$\frac{d}{dz} \left[\frac{f(z)}{g(z)} \right] = \frac{g(z)f'(z) - f(z)g(z) \frac{a}{1+z}}{(g(z))^2}$$

$$= \frac{1}{g(z)(1 + z)} \left((1 + z)f'(z) - af(z) \right) = 0$$

for all $z \in D$. Then $f(z)/g(z)$ is constant on D, that is, $f(z)/g(z) = c$ on D for some fixed $c \in \mathbb{C}$. Plugging in $z = 0$ and noting $f(0) = \binom{a}{0} = 1$, we find $1/g(0) = c$. Therefore, we have $g(z) = g(0)f(z)$ on D as desired.

Example 2.5

For an example of using the binomial theorem, we return to the calculation of the residue at infinity from Example 1. Back then, we had calculated the residue of $f(z) = \sqrt{1 - z^2}/(1 + z^2)$ at infinity using $\operatorname{Res}(f, \infty) = -\operatorname{Res}\big((1/z^2)f(1/z), 0\big)$ since this residue at zero was more accessible. However, we are now able to find the Laurent series of f at infinity, allowing us to find $\operatorname{Res}(f(z), \infty)$ directly. We need the Laurent series of f that converges on $\{|z| \geq R\}$ for some $R > 0$. We will find the Laurent series of our branch of $\sqrt{1 - z^2}$ and $1/(1 + z^2)$ separately and then multiply them together. We can write any branch of $\sqrt{1 - z^2}$ in terms of some branch of $\sqrt{1 - 1/z^2}$ as follows:

$$\sqrt{1 - z^2} = \sqrt{-z^2 \left(1 - \frac{1}{z^2}\right)} = \sqrt{-1}\, z \sqrt{1 - \frac{1}{z^2}} = \pm i z \sqrt{1 - \frac{1}{z^2}}$$

Previously, we would draw a plot of $\arg \sqrt{1 - 1/z^2}$ and carefully work out which branch of $\sqrt{1 - 1/z^2}$ and which $\pm$ sign would lead to the particular branch of $\sqrt{1 - z^2}$ that we decided to use. However, we will now find all possible Laurent series for $\sqrt{1 - z^2}$ on $\{|z| \geq R\}$ and pick the one that corresponds to our particular branch. Let $w = -1/z^2$. Then for $|w| < 1$, we can write $\sqrt{1 + w}$ as a Laurent series using binomial coefficients. At $w = 0$, we have $\sqrt{1 + w} = \sqrt{1} = \pm 1$. Then by the binomial theorem,

$$\sqrt{1 - \frac{1}{z^2}} = \sqrt{1 + w} = \pm \sum_{k=0}^{\infty} \binom{1/2}{k} w^k = \pm \sum_{k=0}^{\infty} \binom{1/2}{k} \left(-\frac{1}{z^2}\right)^k$$

Therefore,

$$\sqrt{1 - z^2} = \pm i \sum_{k=0}^{\infty} \binom{1/2}{k} \frac{(-1)^k}{z^{2k-1}} = \pm i \left(z - \binom{1/2}{1}\frac{1}{z} + \binom{1/2}{2}\frac{1}{z^3} - \cdots\right)$$

So the above series converges for $|w| < 1$ which is if and only if $|z| > 1$. So we have found the possible Laurent series for $\sqrt{1 - z^2}$ that converge on $\{|z| > 1\}$. Looking at the plot of $\arg \sqrt{1 - z^2}$ in Fig. 12 (shown again below for convenience), we see that the branch of $\sqrt{1 - z^2}$ we used in Example 1 is negative imaginary on the positive real axis (note $e^{-i\pi/2} = -i$).

arg(sqrt(1-z^2)) ∈ (-3π/2, π/2)
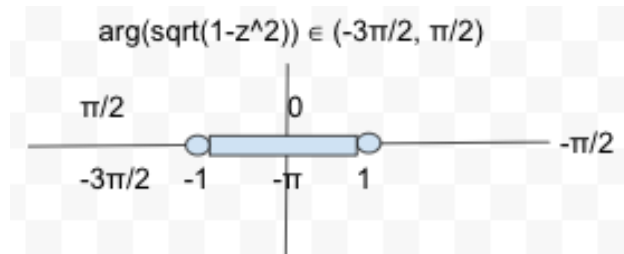
Then, by taking $z \to \infty$ along the positive real axis in our Laurent series (so all terms will vanish except the first term), it is clear that our particular branch of $\sqrt{1 - z^2}$ must have the Laurent series:

$$-i\left(z - \binom{1/2}{1}\frac{1}{z} + \binom{1/2}{2}\frac{1}{z^3} - \cdots\right)$$

which converges for $|z| > 1$.

We now must find the Laurent series of $1/(1 + z^2)$ that converges for $|z| > 1$. Luckily, this is much more straightforward. For $|z| > 1$, we have:

$$\frac{1}{1 + z^2} = \frac{1/z^2}{1 + 1/z^2} = \frac{1}{z^2}\sum_{k=0}^{\infty}\left(-\frac{1}{z^2}\right)^k = \frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{z^6} - \cdots$$

Multiplying the two Laurent series together,

$$f(z) = \frac{\sqrt{1 - z^2}}{1 + z^2} = -i\left(z - \binom{1/2}{1}\frac{1}{z} + \binom{1/2}{2}\frac{1}{z^3} - \cdots\right)\left(\frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{z^6} - \cdots\right)$$

The residue of $f(z)$ at infinity is defined as the negative of the coefficient of z^{-1} in this expansion, so $\mathrm{Res}(f, \infty) = i$, which is consistent with our previous work in Example 1.

Was it faster to use Laurent series to find the residue of f at infinity, or was it faster to calculate the residue of $\frac{1}{z^2}f\left(\frac{1}{z}\right)$ at zero like we did in Example 1? It really does depend on the integrand. If the integrand is the product of functions with relatively straightforward Laurent series, then perhaps Laurent series are the way to go. On the other hand, it is not immediately obvious how to find the Laurent series of $\left(\frac{1-z}{1+z}\right)^{1/6}$.

(Question 10)

Let $a < b$ be real numbers. Using contour integration, evaluate:

$$J = \int_a^b \sqrt{(x-a)(b-x)}\, dx$$

1. Let $c = b - a$. Show that:
$$J = \int_0^c \sqrt{x(c-x)}\, dx$$

2. Which function and contour will you choose? After you answer, check with the solutions before proceeding.

3. Show $\int_{C_1 + C_2 + C_3 + C_4} f \to 2J$ as $r \to 0$.

4. By finding the appropriate Laurent series of f, calculate $\mathrm{Res}(f, \infty)$. Hence find $\lim\limits_{R \to \infty} \int_{C_R} f$.

5. Hence find J.

(Solutions to Question 10)

1.
Use the substitution $x - a \mapsto x$.

2.
We will integrate a branch of $\sqrt{z(c-z)}$ over a dogbone contour traversed clockwise with an extra outer circle of radius $R \to \infty$ traversed counterclockwise. We construct our branch as follows: pick $\sqrt{z}$ with branch cut $(-\infty, 0]$ and $\arg z \in (-\pi, \pi)$, pick $\sqrt{z-c}$ with branch cut $(-\infty, c]$ and $\arg(z-c) \in (-\pi, \pi)$, and pick $\sqrt{-1} = -i$ with $\arg(-i) = -\pi/2$. Then we will define $f(z) = \sqrt{z}\sqrt{z-c}\sqrt{-1} = -i\sqrt{z}\sqrt{z-c}$ to be our branch of $\sqrt{z(c-z)}$, so $\arg\sqrt{z(c-z)} = \frac{1}{2}\arg z + \frac{1}{2}\arg(z-c) - \frac{\pi}{2}$. Then we plot $\arg\sqrt{z(c-z)}$ below:
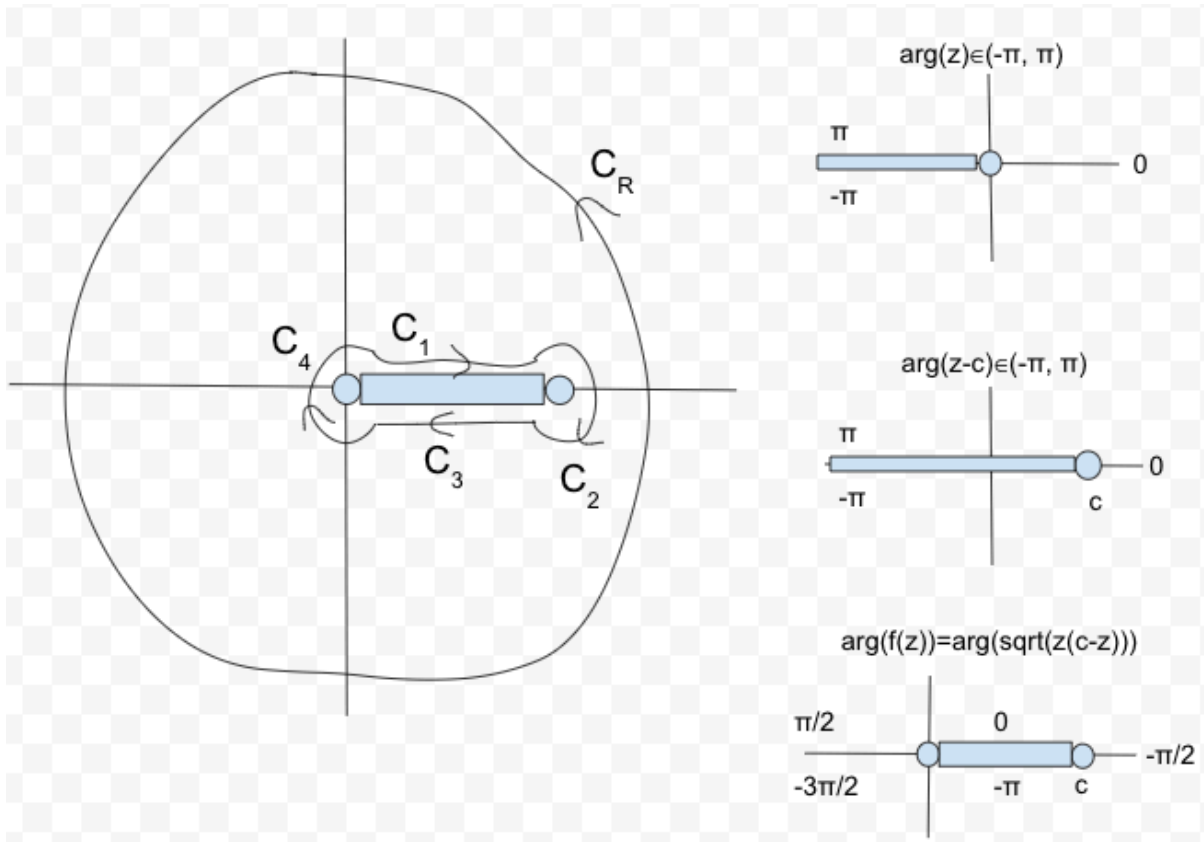

Fig. 34

So our branch of $\sqrt{z(c-z)}$ has branch cut $[0, c]$ and $\arg\sqrt{z(c-z)} \in (-3\pi/2, \pi/2)$.

3.

On C_2, we have $z = c + re^{i\theta}$, so $|f(z)| = \left|\sqrt{(c+re^{i\theta})re^{i\theta}}\right| \le \sqrt{(c+r)r}$ therefore $\left|\int_{C_2} f\right| \le 2\pi r\sqrt{(c+r)r}$ which clearly goes to 0 as $r \to 0$, so $\int_{C_2} f \to 0$ as $r \to 0$. Similarly, $\int_{C_4} f \to 0$ as $r \to 0$. On C_1, we have $z = t + i\epsilon$ so $\int_{C_1} f \to \int_{-1}^{1} \sqrt{t(c-t)}\, e^{i0} = J$ as $r \to 0$. On C_3, we have $z = t - i\epsilon$ so $\int_{C_3} f \to \int_{1}^{-1} \sqrt{t(c-t)}\, e^{-i\pi} dt = J$ as $r \to 0$.

4.

We need the Laurent series of f that converges on $\{z \ge \rho\}$ for some $\rho > 0$. For some branch of $\sqrt{1 - c/z}$, we have:

$$f(z) = \sqrt{z(c-z)} = \sqrt{-z^2\left(1 - \frac{c}{z}\right)} = \sqrt{-1}\, z\sqrt{1 - \frac{c}{z}} = \pm iz\sqrt{1 - \frac{c}{z}}$$

For $|-c/z| < 1$, that is, for $|z| > c$, we have $\sqrt{1 - \frac{c}{z}} = \pm\sum_{k=0}^{\infty}\binom{1/2}{k}\left(-\frac{c}{z}\right)^k$, therefore

$$f(z) = \pm i\sum_{k=0}^{\infty}\binom{1/2}{k}\frac{(-c)^k}{z^{k-1}} = \pm i\left(z - c\binom{1/2}{1} + c^2\binom{1/2}{2}\frac{1}{z} - c^3\binom{1/2}{3}\frac{1}{z^2} + \cdots\right)$$

Looking at Fig. 34, our f is negative imaginary on the positive real axis. By taking $z \to \infty$ along the positive real axis in our Laurent series (so all terms will vanish except the first two terms), we see that our particular branch has Laurent series

$$f(z) = -i\left(z - c\binom{1/2}{1} + c^2\binom{1/2}{2}\frac{1}{z} - c^3\binom{1/2}{3}\frac{1}{z^2} + \cdots\right)$$

Then $\text{Res}(f, \infty)$ is the negative of the coefficient of z^{-1},

$$\text{Res}(f, \infty) = i\binom{1/2}{2}c^2 = i\frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!}c^2 = -\frac{1}{8}ic^2$$

Hence $\lim_{R \to \infty}\int_{C_R} f = -2\pi i\,\text{Res}(f, \infty) = -\pi c^2/4 = -\pi(b-a)^2/4$.

5.

There are no poles in the contour $C_1 + C_2 + C_3 + C_4 + C_R$, so by Cauchy's theorem,

$$2J - \frac{\pi}{4}(b-a)^2 = 0$$

$$J = \frac{\pi}{8}(b-a)^2$$

More Practice Questions

1. For $0 < a < b$, evaluate:
$$\int_0^\infty \frac{x^{a-1}}{x^b + 1}\,dx$$

2. Let $b > 1$. By integrating a branch of $1/(z^b - 1)$ around an appropriate contour, evaluate:
$$\text{PV} \int_0^\infty \frac{1}{x^b - 1}\,dx$$

3. For $a > 0$, evaluate:
$$\int_0^\infty \frac{\log x}{\sqrt{x}(a + x)^2}\,dx$$

4. For $b > 0$, evaluate:
$$\int_0^\infty \frac{\log^2 x}{x^2 + b^2}\,dx$$

5. By integrating a branch of $\log^2 z /(z^2 - 1)$ around an appropriate contour, evaluate:
$$\int_0^\infty \frac{\log^2 x}{x^2 - 1}\,dx$$

6. For $a < b$ and $n \in \mathbb{Z}^+$, prove that:
$$\int_a^b \frac{x^n}{\sqrt{(x - a)(b - x)}}\,dx = \pi \sum_{k=0}^n (-1)^k \binom{-1/2}{k} \binom{n}{k} (b - a)^k a^{n-k}$$

7. Let $0 < a < 1$ and $a = m/n$ where $m, n \in \mathbb{Z}^+$, n is odd, and $\gcd(m, n) = 1$. Also let $b > 0$ and $c \in \mathbb{R}\backslash[-b, b]$. Prove that:[8]
$$\int_{-b}^b \frac{1}{x - c}\left(\frac{x - b}{x + b}\right)^a dx = \pi \left(\left(\frac{c - b}{c + b}\right)^a - 1\right)\csc(\pi a)$$

8. Let $b > 0$. By integrating a branch of $\log(1 + z^3) /(z^2 + b^2)$ over an appropriate contour, evaluate,
$$\int_1^\infty \frac{x^2 + b^2}{x^4 - b^2 x^2 + b^4}\,dx$$

---

[8] Essentially we want a to be a rational number in $(0, 1)$ in lowest terms with odd denominator. The odd denominator ensures $(c - b)^a$ and $(c + b)^a$ are well-defined for all $c \in \mathbb{R}\backslash[-b, b]$.

Appendix 1: Miscellaneous Review

Let z_0 be an isolated singularity of $f(z)$ and $f(z)$ has the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$$

which converges for $0 < |z - z_0| < \rho$ for some $\rho > 0$. The principal part of the series is defined to be:

$$\sum_{n=-\infty}^{-1} a_n(z - z_0)^n = \frac{a_{-1}}{z - z_0} + \frac{a_{-2}}{(z - z_0)^2} + \cdots$$

1. If there are no terms with negative powers of $z - z_0$, that is, if the principal part is zero, then we say z_0 is a removable singularity of f.

2. If $a_{-k} \neq 0$ for some $k \in \mathbb{N}$ and $a_{-n} = 0$ for all $n > k$, then we say that z_0 is a pole of f of order k. Poles of order one are called simple poles.

3. If the principal part is infinite, so there are infinitely many terms with negative powers of $z - z_0$, then z_0 is an essential singularity of f.

Example: Define $g(z) = \log z / (z^b - 1)$ for $b \in \mathbb{R}, b > 1$ where we choose any branch of $\log z$ and z^b that are zero at $z = 1$ and are defined on $\{\mathrm{Re}(z) > 0\}$. Prove that $z = 1$ is a removable singularity of g.

Answer: So our $\log z$ and $z^b - 1$ are analytic at $z = 1$, so they have Taylor expansions around $z = 1$ that converge immediately around $z = 1$. Recall that for a function f, we can calculate its Taylor coefficients using $f^{(k)}(z_0)/k!$ and therefore:

$$\log z = (z - 1) - \frac{(z - 1)^2}{2} + \frac{(z - 1)^3}{3} - \frac{(z - 1)^4}{4} + \cdots$$

$$z^b - 1 = b(z - 1) + \frac{b(b - 1)}{2}(z - 1)^2 + \frac{b(b - 1)(b - 2)}{3}(z - 1)^3 + \cdots$$

Then,

$$g(z) = \frac{\log z}{z^b - 1} = \frac{1 - \frac{1}{2}(z - 1) + \cdots}{b + \frac{b(b - 1)}{2}(z - 1) + \cdots}$$

By the uniqueness of the Laurent expansion, this must equal the Laurent series $\sum_{n=-\infty}^{\infty} a_n(z - 1)^n$ of $g(z)$ centered at $z = 1$ that converges immediately around $z = 1$. But plugging in $z = 1$ yields $1/b$, so the Laurent series is in fact defined at $z = 1$ (with $a_0 = 1/b$) and therefore has zero principal part. So $z = 1$ is a removable singularity of g.

Calculating Residues

Let z_0 be an isolated singularity of $f(z)$ and $f(z)$ has the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$

which converges for $0 < |z - z_0| < \rho$ for some $\rho > 0$. Then the residue of $f(z)$ at z_0, denoted $\mathrm{Res}(f, z_0)$, is defined to be the coefficient a_{-1} of $1/(z - z_0)$ in this expansion. Importantly, the residue at z_0 is obtained from the Laurent series that converges *immediately around* z_0.

To calculate residues, we list some useful results below.

1. If f has a removable singularity at z_0, then $\mathrm{Res}(f, z_0) = 0$.

   Proof: Since z_0 is a removable singularity, the Laurent series of f that converges immediately around z_0 has no principal part, so the residue is zero.

2. If f has a simple pole or removable singularity at z_0, then

   $$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0)f(z)$$

   Proof: Follows immediately from considering the appropriate Laurent series of f.

3. If $f(z)$ has a pole of order k at z_0, then

   $$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} \frac{\dfrac{d}{dz^{k-1}}[(z - z_0)^k f(z)]}{(k-1)!}$$

   Proof: Consider the appropriate Laurent series of f and use term-by-term differentiation. The case $k = 2$ becomes $\mathrm{Res}(f, z_0) = \lim_{z \to z_0} \frac{d}{dz}[(z - z_0)^2 f(z)]$.

4. If f and g are analytic at z_0 and g has a simple zero at z_0, then

   $$\mathrm{Res}\left(\frac{f}{g}, z_0\right) = \frac{f(z_0)}{g'(z_0)}$$

   Proof: So f/g either has a simple pole or a removable singularity at z_0. Noting $g(z_0) = 0$, we can use the second result to conclude:

   $$\mathrm{Res}\left(\frac{f}{g}, z_0\right) = \lim_{z \to z_0} (z - z_0)\frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f(z)}{\dfrac{g(z) - g(z_0)}{z - z_0}} = \frac{f(z_0)}{g'(z_0)}$$

   A useful corollary is the case $f(z) = 1$, that is, $\mathrm{Res}(1/g, z_0) = 1/g'(z_0)$ when g has a simple zero at z_0.

The Residue at Infinity

Let $f(z)$ be analytic for $|z| \geq R$ for some $R > 0$ with Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ that converges for $|z| \geq R$. Then we define $\mathrm{Res}(f, \infty) = -a_{-1}$. Note the additional negative sign.

In the case that there is a piecewise smooth closed curve C traversed counterclockwise such that $f(z)$ is meromorphic inside C but analytic outside C, then we additionally have:

$$\mathrm{Res}(f, \infty) = -\frac{1}{2\pi i} \int_C f(z) dz = -\mathrm{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right)$$

We emphasize again that C is being traversed counterclockwise. The proof of the first equality goes as follows: let C_R be a circle (traversed counterclockwise) of radius R large enough such that C is entirely contained inside C_R. Since f is analytic outside C, we have $\int_{C_R - C} f = 0$ by Cauchy's theorem, or $\int_C f = \int_{C_R} f$. Since $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ for $|z| \geq R$, we have

$$-\frac{1}{2\pi i} \int_C f(z) dz = -\frac{1}{2\pi i} \int_{C_R} f(z) dz = -\frac{1}{2\pi i} \int_{C_R} \sum_{n=-\infty}^{\infty} a_n z^n \, dz$$

Laurent series converge uniformly so we can interchange limits as needed. We parameterize $z = Re^{i\theta}$,

$$= -\frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_0^{2\pi} a_n R^n e^{in\theta} \, Rie^{i\theta} d\theta = -\frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_0^{2\pi} a_n R^n e^{in\theta} \, Rie^{i\theta} d\theta$$

Note that $\int_0^{2\pi} e^{i(n+1)\theta} \, d\theta$ is zero for all integers $n \neq -1$ but equals 2π when $n = -1$,

$$= -\frac{1}{2\pi i} 2\pi i a_{-1} = -a_{-1} = \mathrm{Res}(f, \infty)$$

To prove the second equality, we recommend an online article about the residue at infinity, see the references section.[9] But if you wish to try the proof for yourself, apply the inversion $z \mapsto 1/z$ to the integral $\int_{C_R} f(z) dz$. What will this inversion do to C_R and the isolated singularities of f inside C_R? After sorting out the integral, apply the residue theorem.

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[9] Orloff, Jeremy. *9.6: Residue at ∞*. LibreTexts, https://math.libretexts.org/Bookshelves/Analysis/Complex_Variables_with_Applications_(Orloff)/09%3A_Residue_Theorem/9.06%3A_Residue_at.

The Residue Theorem

Let C be a piecewise smooth closed curve traversed counterclockwise and suppose $f(z)$ is analytic on a region containing C except at a finite number of isolated singularities $z_1, \ldots, z_n$ that are inside C. Then,

$$\int_C f(z)dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}(f, z_j)$$

Since we have discussed the residue at infinity, we present a slightly more interesting view of the residue theorem by considering the Riemann sphere. Suppose we map the extended complex plane $\mathbb{C} \cup \{\infty\}$ to a sphere using a stereographic projection. Note that $z = \infty$ is mapped to the north pole of the sphere. If f were analytic on $\mathbb{C}$ except at the isolated singularities $z_1, \ldots, z_n$, then f is analytic on the Riemann sphere except at $z_1, \ldots, z_n$ and possibly ∞.
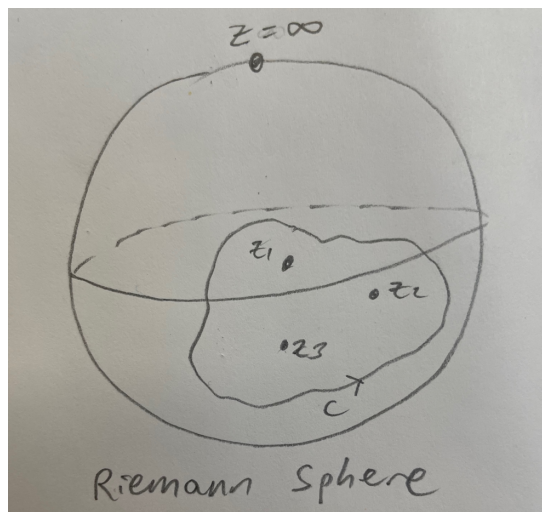

Fig. 35

If the curve C is traversed counterclockwise, then the interior of C is the region containing z_1, z_2, z_3 and the residue theorem tells us $\int_C f(z)dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}(f, z_i)$. However, if we now traversed C *clockwise*, then the interior is now the region *not* containing z_1, z_2, z_3 and the only possibly nonzero residue inside C is the one at infinity. Then the residue theorem immediately gives us the result from the previous page:

$$\operatorname{Res}(f, \infty) = \frac{1}{2\pi i} \int_{-C} f(z)dz = -\frac{1}{2\pi i} \int_C f(z)dz$$

where C is traversed counterclockwise. Note $\operatorname{Res}(f, \infty) = 0$ if f is analytic at ∞.

We will not prove the residue theorem, there is plenty of that elsewhere. However, if you need a reminder, the proof comes from integrating the appropriate Laurent series of f around a small circle around each of the isolated singularities $z_1, \ldots, z_n$.

Fractional Residue Theorem

Next is a useful result about integrating over circular arcs around simple poles. One textbook[10] calls it the fractional residue theorem, but it seems like no one else does.

(Fractional residue theorem)
Let z_0 be a simple pole of $f(z)$ and let C_r be the arc of the circle $\{|z - z_0| = r\}$ of angle α traversed counterclockwise. Then,

$$\lim_{r \to 0} \int_{C_r} f(z)\, dz = \alpha i \operatorname{Res}(f, z_0)$$

So α would be negative if the curve was traversed clockwise.

Proof: Since z_0 is a simple pole of f, we can write f as a Laurent series that converges on $\{0 < |z - z_0| < \rho\}$ for some $\rho > 0$.

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$$

Parameterizing the arc C_r with $z = z_0 + re^{i\theta}$, $\theta_0 < \theta < \theta_0 + \alpha$ where θ_0 is the starting angle,

$$\lim_{r \to 0} \int_{C_r} f(z)\, dz = \lim_{r \to 0} \int_{\theta_0}^{\theta_0 + \alpha} \left(\frac{a_{-1}}{re^{i\theta}} + a_0 + a_1 re^{i\theta} + \cdots \right) rie^{i\theta}\, d\theta$$

$$= \lim_{r \to 0} \int_{\theta_0}^{\theta_0 + \alpha} \left(a_{-1} + a_0 re^{i\theta} + a_1 r^2 e^{i2\theta} + \cdots \right) i d\theta$$

Whether you integrate term-by-term or pass the limit into the integral (note Laurent series converge uniformly, allowing these manipulations), the desired result follows.

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[10] Gamelin, Theodore W. *Complex Analysis*. Springer, 2001

Cauchy Principal Value

Consider the integral $\int_{-\infty}^{\infty}(1/x)dx$. We know this integral diverges, but if we graph $1/x$, it looks like the integral *should* equal zero since the area to the right of the line $x = 0$ is perfectly cancelled out by the (negatively signed) area to the left. The principal value of an integral allows us to assign such a value to otherwise divergent integrals.

Suppose $f(x)$ is continuous on $[a, x_0)$ and $(x_0, b]$. We define the *principal value* of the integral $\int_a^b f(x)dx$ to be

$$\text{PV} \int_a^b f(x)dx = \lim_{\epsilon \to 0} \left(\int_a^{x_0-\epsilon} f(x)dx + \int_{x_0+\epsilon}^b f(x)dx \right)$$

when the limit exists. If f is not continuous at a or b, or $a = -\infty$ or $b = \infty$, then it is required for f to be absolutely integrable on $(a, x_0 - r)$ and $(x_0 + r, b)$ for all $r > 0$ in order for the principal value to exist.

The principal value is similarly defined if f has multiple isolated discontinuities in $[a, b]$. If $f(x)$ is continuous on $[a, x_0), (x_0, x_1), \ldots, (x_{n-1}, x_n),$ and $(x_n, b]$, then

$$\text{PV} \int_a^b f(x)dx = \lim_{\epsilon_0, \ldots, \epsilon_n \to 0} \left(\int_a^{x_0-\epsilon_0} + \int_{x_0+\epsilon_0}^{x_1-\epsilon_1} + \cdots + \int_{x_n+\epsilon_n}^b \right) f(x)dx$$

when the limits exist.

Referring to the earlier example of $1/x$, it is not hard to verify that $\text{PV} \int_{-\infty}^{\infty}(1/x)dx = 0$.

The principal value of an integral equals the value of that integral when the integral converges.

Appendix 2

Passing the limit $\epsilon \to 0$ under the integral as follows:

$$\lim_{\epsilon \to 0} \int_{-1+r}^{1-r} \underbrace{\frac{\left|\sqrt{1-(t+i\epsilon)^2}\right| e^{i \arg \sqrt{1-(t+i\epsilon)^2}}}{1+(t+i\epsilon)^2}}_{h(t+i\epsilon)} \, dt = \int_{-1+r}^{1-r} \lim_{\epsilon \to 0} \frac{\left|\sqrt{1-(t+i\epsilon)^2}\right| e^{i \arg \sqrt{1-(t+i\epsilon)^2}}}{1+(t+i\epsilon)^2}$$

$$= \int_{-1+r}^{1-r} \frac{\left|\sqrt{1-t^2}\right| e^{i \arg \sqrt{1-t^2}}}{1+t^2}$$

needs some justification. The reason boils down to the fact that we are integrating over a compact set, in this case $[-1+r, 1-r]$. Let our integrand be denoted $h(t+i\epsilon)$.

Lemma 1: A continuous function over a compact set is uniformly continuous on that set.

This result is proven in most real analysis courses, and can also be shown to hold in $\mathbb{C}$. In this case, our integrand $h(t+i\epsilon)$ is indeed continuous on $[-1+r, 1-r]$ for any fixed $r > 0$ and for all $\epsilon > 0$, and is therefore uniformly continuous on $[-1+r, 1-r]$.

Lemma 2: Suppose $f(z)$ is uniformly continuous on K and $f(z+c/n)$ is defined on K for all $n \in \mathbb{N}$ and fixed $c \in \mathbb{C}$. Then $f(z+c/n) \to f$ uniformly as $n \to \infty$.

Since $f(z)$ is uniformly continuous on K, for all $\epsilon > 0$ there is $\delta > 0$ such that for all $w, w + c/n \in K$, $|w - (w + c/n)| = |c|/n < \delta$ implies $|f(w) - f(w+c/n)| < \epsilon$. In that case, given any $\epsilon > 0$, pick $N > |c|/\delta$. Then $|c|/n < |c|/N < \delta$, so we immediately have $|f(w) - f(w+c/n)| < \epsilon$ from before, hence $f(z+c/n) \to f$ uniformly as $n \to \infty$.

Then, by writing $\epsilon = 1/n$ (so $n \to \infty \Leftrightarrow \epsilon \to 0$) and by taking $c = i$ in Lemma 2, we see that $h(t+i\epsilon) \to h(t)$ uniformly on $[-1+r, 1-r]$ as $\epsilon \to 0$. Therefore, we are justified in passing the limit $\epsilon \to 0$ under the integral.

Appendix 3

Differentiating a branch of a function often requires use of the chain rule.

Let $g(z)$ be a branch of $\left(\frac{z-1}{z+1}\right)^{\frac{1}{6}}$. We cannot use power rule since the exponent is not a natural number (recall that the proof of the power rule required the exponent to be a natural number). Instead, we use chain rule as follows:

$$\frac{d}{dz}\underbrace{\left[e^{\frac{1}{6}\log\left(\frac{z-1}{z+1}\right)}\right]}_{g(z)} = \underbrace{e^{\frac{1}{6}\log\left(\frac{z-1}{z+1}\right)}}_{g(z)} \frac{d}{dz}\left[\frac{1}{6}\log\left(\frac{z-1}{z+1}\right)\right] = \frac{1}{6}g(z)\frac{\left(\frac{z-1}{z+1}\right)'}{\frac{z-1}{z+1}}$$

$$\frac{1}{6}g(z)\frac{\frac{z+1-(z-1)}{(z+1)^2}}{\frac{z-1}{z+1}} = \frac{1}{3}g(z)\frac{1}{z^2-1}$$

When we use chain rule, the branch of the differentiated function is clear: in this case, the $e^{\frac{1}{6}\log\left(\frac{z-1}{z+1}\right)}$ term in the differentiated function is the same branch of $e^{\frac{1}{6}\log\left(\frac{z-1}{z+1}\right)}$ as in the original function.

In fact, for $a \in \mathbb{C}$, we have the general result:

$$\frac{d}{dz}[z^a] = \frac{d}{dz}\left[e^{a\log z}\right] = e^{a\log z}\frac{a}{z} = z^a\frac{a}{z}$$

which looks consistent with the power rule, but we could not use the power rule to derive it. Note again that the original and differentiated function are using the same branch of z^a.

Solutions to More Practice Questions

1.
Choose the branch cut $[0, \infty)$ and $\arg z \in (0, 2\pi)$ for both z^{a-1} and z^b. We integrate $f(z) = z^{a-1}/(z^b + 1)$ over the indented circular sector of angle $2\pi/b$. Using ML-lemma, the integral over the two circular portions vanish. The only pole is $e^{i\pi/b}$, a simple pole, and the residue is $\lim\limits_{z \to e^{i\pi/b}} z^{a-1}/(z^b b/z) = -e^{i\pi a/b}/b$. Let $J = \int_0^\infty \frac{x^{a-1}}{x^b+1} dx$. Then by residue theorem,

$$\left(1 - e^{i2\pi a/b}\right)J = -2\pi i e^{i\pi a/b}/b$$

$$\Rightarrow J = \frac{\pi}{b} \csc\left(\frac{\pi a}{b}\right)$$

2.
We integrate $f(z) = 1/(z^b - 1)$ over the below contour with $\arg z \in (-\pi/2, 3\pi/2)$ for z^b, but principal branch would work too. Other functions and contours will work too.
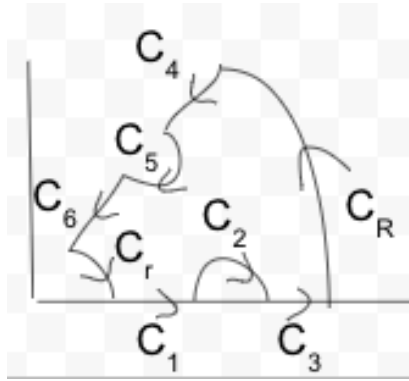

Fig. 36

The integrals over C_R and C_r vanish by ML-lemma. By fractional residue theorem, we have $\int_{C_2} f = -\pi i \, \mathrm{Res}(f, 1) = -\pi i \lim\limits_{z \to 1} \frac{1}{bz^b/z} = -\pi i/b$ and $\int_{C_5} f = -\pi i \, \mathrm{Res}\left(f, e^{i2\pi/b}\right) = -\pi i \lim\limits_{z \to e^{i2\pi/b}} \frac{1}{bz^b/z} = -(\pi i/b)e^{i2\pi/b}$. Let $J = \mathrm{PV} \int_0^\infty \frac{1}{x^b-1} dx$. Parameterize $z = te^{i2\pi/b}$ on C_4 and C_6. Then by Cauchy's theorem,

$$\int_{C_r+C_R+C_2+C_5} f + \left(\int_0^{1-r} + \int_{1+r}^R\right)\frac{dt}{t^b-1} + \left(\int_R^{1+r} + \int_{1-r}^0\right)\frac{e^{i2\pi/b}dt}{t^b-1} = 0$$

Take $r \to 0$ and $R \to \infty$,

$$-\frac{\pi i}{b} - \frac{\pi i}{b}e^{i2\pi/b} + \left(1 - e^{i2\pi/b}\right)J = 0$$

$$\Rightarrow J = -\frac{\pi}{b}\cot\left(\frac{\pi}{b}\right)$$

Note that this is consistent with our results from Question 6.

3.

Choose $\sqrt{z}$ with $\arg z \in (0, 2\pi)$ and $\log z$ with $\arg z \in (0, 2\pi)$, so both have branch cut $[0, \infty)$. We integrate the function $f(z) = \log z \, /\!\left(\sqrt{z}(a+z)^2\right)$ over the corresponding keyhole contour. Let $J = \int_0^\infty \frac{\log x}{\sqrt{x}(a+x)^2} dx$. The integral over the two circular portions vanish by ML-lemma. The only pole is at $z = -a$ which is an order two pole. So

$$\mathrm{Res}(f, -a) = \lim_{z \to -a} \frac{d}{dz}\left[\frac{\log z}{\sqrt{z}}\right] = \lim_{z \to -a} \frac{\sqrt{z}/z - \log z/(2\sqrt{z})}{\sqrt{z}^2} = \frac{-i|\sqrt{a}|/a - (\log|a| + i\pi)/(2i|\sqrt{a}|)}{-a} = \frac{-2 + \log|a| + i\pi}{2ia\sqrt{a}}.$$

Hence by residue theorem,

$$J + \int_\infty^0 \frac{\log|t| + i2\pi}{|\sqrt{t}|e^{i\pi}(a+t)^2} dt = 2\pi i \left(\frac{-2 + \log|a| + i\pi}{2ia\sqrt{a}}\right)$$

$$2J + \int_0^\infty \frac{2\pi i}{|\sqrt{t}|(a+t)^2} dt = \frac{\pi}{a\sqrt{a}}(-2 + \log|a| + i\pi)$$

Equating real and imaginary parts,

$$\int_0^\infty \frac{\log x}{\sqrt{x}(a+x)^2} dx = \frac{\pi}{2a\sqrt{a}}(-2 + \log a) \qquad \int_0^\infty \frac{dx}{\sqrt{x}(a+x)^2} = \frac{\pi}{2a\sqrt{a}}$$

4.

There are multiple options but we integrate $f(z) = \log^2 z \,/(z^2 + b^2)$ over a semicircle in the upper-half plane with indent at $z = 0$, and we choose the branch cut $[0, -i\infty)$ and $\arg z \in (-\pi/2, 3\pi/2)$ for $\log z$. Let $J = \int_0^\infty \frac{\log^2 x}{x^2 + b^2} dx$. The integral over the two circular portions vanish by ML-lemma. The only pole inside the contour is $z = ib$, and we have $\mathrm{Res}(f, ib) = \lim_{z \to ib} \frac{\log^2 z}{z + ib} = \frac{(\log|b| + i\pi/2)^2}{2ib}$. Then by residue theorem,

$$J + \int_\infty^0 \frac{(\log|-t| + i\pi)^2}{(-t)^2 + b^2}(-dt) = 2\pi i \frac{(\log|b| + i\pi/2)^2}{2ib}$$

$$J + \int_0^\infty \frac{\log^2 t + 2\pi i \log t - \pi^2}{t^2 + b^2} dt = \frac{\pi}{b}(\log|b| + i\pi/2)^2$$

$$2J + 2\pi i \int_0^\infty \frac{\log t}{t^2 + b^2} dt - \frac{\pi^3}{2b} = \frac{\pi}{b}(\log|b| + i\pi/2)^2$$

Equating real and imaginary parts,

$$\int_0^\infty \frac{\log^2 x}{x^2 + b^2} dx = \frac{\pi}{2b}\left(\frac{\pi^2}{4} + \log^2 b\right) \qquad \int_0^\infty \frac{\log x}{x^2 + b^2} dx = \frac{\pi}{2b}\log b$$

5.

Again there are multiple options but we integrate $f(z) = \log^2 z / (z^2 - 1)$ over the same contour as in Fig. 24, where we have chosen the principal branch of $\log z$ (the branch with $\arg z \in (-\pi/2, 3\pi/2)$ is also possible). Let $J = \int_0^\infty \frac{\log^2 x}{x^2 - 1} dx$. The integrals over C_2 and C_R vanish by ML-lemma. Since $z = 1$ is a removable singularity of $f(z)$, we do not need an indent around $z = 1$. Then by Cauchy's theorem,

$$0 = J + \int_\infty^0 \frac{(\log|it| + i\pi/2)^2}{(it)^2 - 1} i\, dt = J + i \int_0^\infty \frac{\log^2 t + \pi i \log t - \pi^2/4}{t^2 + 1} dt$$

Equating real parts and noting that we showed $\int_0^\infty \log t / (t^2 + 1) dt = 0$ in Question 4,

$$\int_0^\infty \frac{\log^2 x}{x^2 - 1} dx = 0$$

6.

Using the substitution $x - a \mapsto x$, the integral equals $J := \int_0^c \frac{(x+a)^n}{\sqrt{x(c-x)}} dx$ where $c = b - a$.

We use the same branch of $\sqrt{z(c-z)} := g(z)$ as constructed in Fig. 34, and we integrate $f(z) = (z + a)^n / g(z)$ over the same dogbone contour. The integrals over the circular portions vanish by ML-lemma. Then by residue theorem, $2J = 2\pi i \operatorname{Res}(f, \infty)$. Note that $1/g(z)$ is a branch of $(z(c - z))^{-1/2}$ that is positive-imaginary for z on the positive real axis. Again, there are only two branches of $(z(c - z))^{-1/2}$ with the branch cut $[0, c]$, which are negatives of each other. We have:

$$\frac{1}{g(z)} = (z(c-z))^{-1/2} = \left(-z^2\left(1 - \frac{c}{z}\right)\right)^{-1/2} = \pm i \frac{1}{z}\left(1 - \frac{c}{z}\right)^{-1/2} = \pm \frac{i}{z} \sum_{k=0}^\infty \binom{-1/2}{k}\left(-\frac{c}{z}\right)^k$$

$$= \pm i \left(\frac{1}{z} - \binom{-1/2}{1}\frac{c}{z^2} + \binom{-1/2}{2}\frac{c^2}{z^3} + \cdots\right)$$

and the series converges for $|z| > c$. Note that $\binom{-1/2}{k}$ is negative and positive when k is odd and even respectively, so all coefficients in the series of $(1 - c/z)^{-1/2}$ are in fact positive. Then our branch of $(z(c - z))^{-1/2}$ uses the positive sign in the above series, so

$$\frac{(z + a)^n}{g(z)} = i\left(\frac{1}{z} - \binom{-1/2}{1}\frac{c}{z^2} + \binom{-1/2}{2}\frac{c^2}{z^3} + \cdots\right)\sum_{k=0}^n \binom{n}{k} z^k a^{n-k}$$

By collecting terms carefully, the coefficient of z^{-1} is eventually found to be

$$i\sum_{k=0}^n (-1)^k \binom{-1/2}{k}\binom{n}{k} c^k a^{n-k}$$

Then, since $2J = 2\pi i \operatorname{Res}(f, \infty)$,

$$J = \pi \sum_{k=0}^n (-1)^k \binom{-1/2}{k}\binom{n}{k}(b - a)^k a^{n-k}$$

7.

There are many constructions, but we choose the principal branch for both $(b - z)^a$ and $(z + b)^a$ and choose $(-1)^a = e^{-i\pi a}$ with $\arg e^{-i\pi a} = -\pi a$. Then define the function $g(z) = (-1)^a \frac{(b-z)^a}{(z+b)^a} = e^{-i\pi a} \frac{(b-z)^a}{(z+b)^a}$ to be our branch of $\left(\frac{z-b}{z+b}\right)^a$ and let $f(z) = \frac{1}{z-c} g(z)$.

Since $\arg(b - z), \arg(z + b) \in (-\pi, \pi)$ and $\arg g(z) = -\pi a + a\arg(b - z) - a\arg(z + b)$, it can be shown that $f(z)$ has branch cut $[-b, b]$ and is positive just above $(-b, b)$. We will integrate f over the dogbone contour from $-b$ to b clockwise, leaving only the residues at $z = c$ and $z = \infty$ to worry about.
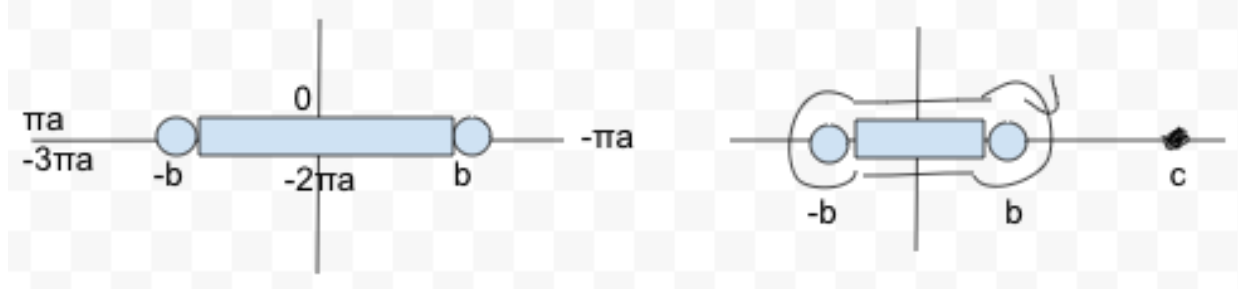

Fig. 37

Let $J = \int_{-b}^{b} \frac{1}{x-c} \left(\frac{x-b}{x+b}\right)^a dx$. The integral over the two circular portions vanish by ML-lemma. Note that:

$$\mathrm{Res}(f, \infty) = -\mathrm{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right) = -\mathrm{Res}\left(\frac{1}{z(1 - cz)}\left(\frac{1 - bz}{1 + bz}\right)^a, 0\right)$$

where $\left(\frac{1-bz}{1+bz}\right)^a = g(1/z)$. Noting that our branch of g implies $\lim_{z \to 0} g(1/z) = e^{-i\pi a}$,

$$-\mathrm{Res}\left(\frac{1}{z(1 - cz)}\left(\frac{1 - bz}{1 + bz}\right)^a, 0\right) = -\lim_{z \to 0} \frac{1}{1 - cz}\left(\frac{1 - bz}{1 + bz}\right)^a = -e^{-i\pi a}$$

Lastly, $\mathrm{Res}(f, c) = \lim_{z \to c} g(z) = \left|\left(\frac{c-b}{c+b}\right)^a\right| e^{-i\pi a}$. Then by residue theorem,

$$J + \int_{b}^{-b} \frac{1}{t-c}\left|\left(\frac{t-b}{t+b}\right)^a\right| e^{-i2\pi a} dt = 2\pi i\left(\left|\left(\frac{c-b}{c+b}\right)^a\right| e^{-i\pi a} - e^{-i\pi a}\right)$$

$$J\left(1 - e^{-i2\pi a}\right) = 2\pi i e^{-i\pi a}\left(\left|\left(\frac{c-b}{c+b}\right)^a\right| - 1\right)$$

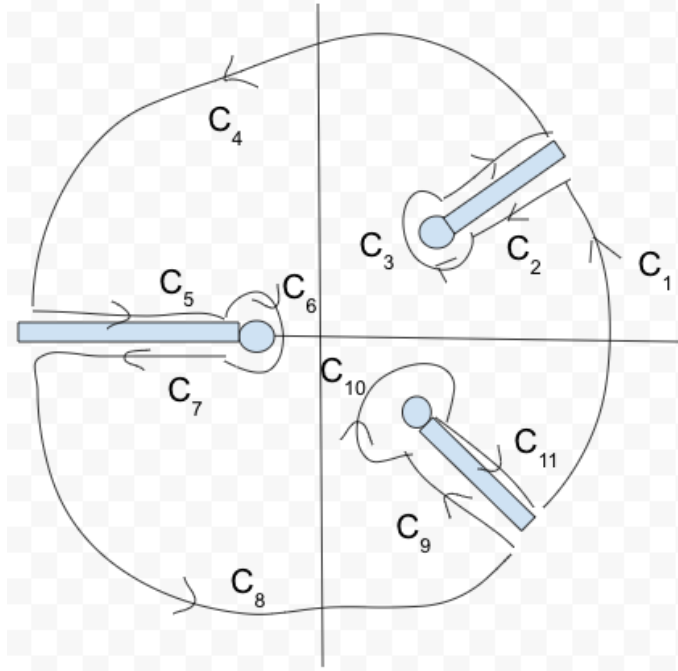$$J = \pi\left(\left(\frac{c-b}{c+b}\right)^a - 1\right)\csc(\pi a)$$

Note the absolute value sign can be removed since $(c - b)/(c + b)$ is strictly positive for all $c \in \mathbb{R}/[-b, b]$.

8.
We define f_1, f_2, f_3 as follows:

Function	Branch cut	Domain of $\arg(f_k(z))$
$f_1(z) = \log(z+1)$	$(-\infty, -1]$	$(-\pi, \pi)$
$f_2(z) = \log(z - e^{i\pi/3})$	$\{z : z = re^{i\pi/3}, r \geq 1\}$	$(-5\pi/3, \pi/3)$
$f_3(z) = \log(z - e^{i5\pi/3})$	$\{z : z = re^{i5\pi/3}, r \geq 1\}$	$(-\pi/3, 5\pi/3)$

Define $\log(1 + z^3) = f_1(z) + \cdots + f_3(z)$. Further define z^a with branch cut $[0, \infty)$ and $\arg z \in (0, 2\pi)$. We integrate $f(z) = \log(1 + z^3)/(z^2 + b^2)$ over the below contour:

The integrals over $C_1, C_3, C_5, C_7, C_9, C_{11}$ will vanish by ML-lemma, see Question 7 for a similar situation. Next, we have $\int_{C_2} f \to \int_\infty^1 \frac{\log|1+t^3|+i\pi}{t^2 e^{i2\pi/3}+b^2} e^{i\pi/3} dt$, $\int_{C_4} f \to$
$\int_1^\infty \frac{\log|1+t^3|-i\pi}{t^2 e^{i2\pi/3}+b^2} e^{i\pi/3} dt$, $\int_{C_6} f \to \int_\infty^1 \frac{\log|1+t^3|+i\pi}{t^2+b^2} (-dt)$, $\int_{C_8} f \to \int_1^\infty \frac{\log|1+t^3|-i\pi}{t^2+b^2} (-dt)$, $\int_{C_{10}} f \to$
$\int_\infty^1 \frac{\log|1+t^3|+i\pi}{t^2 e^{i10\pi/3}+b^2} e^{i5\pi/3} dt$, and $\int_{C_{12}} f \to \int_1^\infty \frac{\log|1+t^3|-i\pi}{t^2 e^{i10\pi/3}+b^2} e^{i5\pi/3} dt$. Adding all these integrals together, we find

$$\int_{C_2 + C_4 + C_6 + \cdots + C_{12}} f = -2\pi i \int_1^\infty \frac{e^{i\pi/3}}{t^2 e^{i2\pi/3} + b^2} - \frac{1}{t^2 + b^2} + \frac{e^{i5\pi/3}}{t^2 e^{i10\pi/3} + b^2} dt$$

$$-2\pi i \int_1^\infty \frac{1 + \sqrt{3}i}{t^2(-1 + \sqrt{3}i) + 2b^2} + \frac{1 - \sqrt{3}i}{t^2(-1 - \sqrt{3}i) + 2b^2} dt + \int_1^\infty \frac{2\pi i}{t^2 + b^2} dt$$

$$= -2\pi i \int_1^\infty \frac{t^2 + b^2}{t^4 - b^2 t^2 + b^4} dt + 2\pi i \left[\frac{1}{b} \arctan\left(\frac{t}{b}\right)\right]_1^\infty$$

$$= -2\pi i J + 2\pi i \left(\frac{\pi}{2b} - \frac{1}{b} \arctan \frac{1}{b} \right)$$

We now compute the residues of f inside the contour, keeping in mind that We have

$\operatorname{Res}(f, ib) = \lim\limits_{z \to ib} \frac{\log(1+z^3)}{z+ib} = \frac{\log(1-ib^3)}{2ib} = \frac{\log|1-ib^3|+i \arg(1-ib^3)}{2ib} = \frac{\log|\sqrt{1+b^6}|-i \arctan b^3}{2ib}$ and

similarly, $\operatorname{Res}(f, -ib) = \lim\limits_{z \to -ib} \frac{\log(1+z^3)}{z-ib} = \frac{\log|\sqrt{1+b^6}|+i \arctan b^3}{-2ib}$. Then by residue theorem,

$$-2\pi i J + 2\pi i \left(\frac{\pi}{2b} - \frac{1}{b} \arctan \frac{1}{b} \right) = 2\pi i \big(\operatorname{Res}(f, ib) + \operatorname{Res}(f, -ib) \big)$$

$$-J + \frac{\pi}{2b} - \frac{1}{b} \arctan \frac{1}{b} = \frac{-2i \arctan b^3}{2bi}$$

Since $\arctan x + \arctan 1/x = \pi/2$ for $x > 0$,

$$J = \frac{\arctan(b) + \arctan(b^3)}{b}$$

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