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See “differentiating_under_integral_proof” document for the proofs of the theorems used in this document.

Let $\varphi = \left(1+\sqrt{5}\right)/2$ denote the golden ratio. Define the following integral:

$$J = \int_0^1 \left(\frac{x^\varphi - 1}{\log x}\right)^2 dx$$

Prove J converges and find its value.

We use differentiation under the integral sign. Define the function:

$$J(a) = \int_0^1 \left(\frac{x^a - 1}{\log x}\right)^2 dx$$

We first show $J(a)$ converges if $a \geq 0$. By L'Hopital's, note $\lim_{x\to 1} \frac{x^a-1}{\log x} = \lim_{x\to 1} a x^a = a$ for all $a \in \mathbb{R}$ (note the case $a = 0$ must be handled separately), hence $\lim_{x\to 1}\left(\frac{x^a-1}{\log x}\right)^2 = a^2$ so there are no convergence issues at $x = 1$. To show convergence as $x \to 0$, note that if $a = 0$ then $\lim_{x\to 0}\left(\frac{x^a-1}{\log x}\right)^2 = \lim_{x\to 0}\frac{0}{\log^2 x} = 0$, and if $a > 0$ then clearly $\lim_{x\to 0}\left(\frac{x^a-1}{\log x}\right)^2 = 0$, so for $a \geq 0$ there are no convergence issues at $x = 0$. So overall, $J(a)$ converges if $a \geq 0$ (although desmos will appear to suggest $J(a)$ converges if and only if $a \geq -1/2$, but there is something quite interesting and subtle going on here, we will return to this later).

Since $J(a)$ is well-defined if $a \geq 0$, we proceed to differentiate under the integral sign, but we first would like the integral $J(a)$ to converge uniformly. We restrict the domain of $J(a)$ to $[0, M]$ for some $M > 0$ large and consequently note $\left(\frac{x^a-1}{\log x}\right)^2 \leq \left(\frac{x^M-1}{\log x}\right)^2$, which we have shown is integrable on $(0,1)$, hence $J(a)$ converges uniformly on $[0, M]$ by M-test for improper integrals. The convergence of the differentiated integral will be shown once we get there. We have

$$J'(a) = \int_0^1 \frac{\partial}{\partial a}\frac{\left(e^{a\log x}-1\right)^2}{\log^2 x}dx = \int_0^1 \frac{2\left(e^{a\log x}-1\right)e^{a\log x}\log x}{\log^2 x}dx$$

$$= \int_0^1 \frac{2e^{2a\log x} - 2e^{a\log x}}{\log x}dx$$

So $J'(a) = \int_0^1 \frac{2x^a(x^a-1)}{\log x}dx$, and by taking the limit of the integrand as $x \to 0$ and $x \to 1$ and applying the M-test just like before, it can be shown $J'(a)$ converges if $a \geq 0$ and the convergence is uniform on $[0, M]$. So everything we have done so far is justified.

$$\Rightarrow J''(a) = \int_0^1 \frac{\partial}{\partial a}\frac{2e^{2a\log x} - 2e^{a\log x}}{\log x}dx$$

$$= \int_0^1 \frac{2e^{2a\log x}(2\log x) - 2e^{a\log x}(\log x)}{\log x} dx = \int_0^1 4e^{2a\log x} - 2e^{a\log x} dx$$

$$= \int_0^1 4x^{2a} - 2x^a dx = \left[\frac{4x^{2a+1}}{2a+1} - \frac{2x^{a+1}}{a+1}\right]_0^1 = \frac{4}{2a+1} - \frac{2}{a+1}$$

So clearly $J''(a)$ converges for $a \geq 0$, since we were able to compute it. Then,

$$J'(a) = \int \frac{4}{2a+1} - \frac{2}{a+1} da = 2\ln(2a+1) - 2\ln(a+1) + c_1$$

for some $c_1 \in \mathbb{R}$. Since $a \geq 0$, we did not need the absolute value signs in the logarithm terms. Returning to the integral form of $J'(a)$, we have

$$J'(a) = \int_0^1 \frac{2x^{2a} - 2x^a}{\log x} dx \Rightarrow J'(0) = \int_0^1 \frac{2x^{2(0)} - 2x^0}{\log x} dx = 0$$

Hence $2\ln|2(0)+1| - 2\ln|0+1| + c_1 = 0$, so $c_1 = 0$.

$$J(a) = \int 2\ln(2a+1) - 2\ln(a+1)\, da$$

$$= (2a+1)(\ln(2a+1) - 1) - 2(a+1)(\ln(a+1) - 1) + c_2$$

for some $c_2 \in \mathbb{R}$. Returning to the integral form of $J(a)$, we have

$$J(a) = \int_0^1 \frac{(x^a - 1)^2}{\log^2 x} dx \Rightarrow J(0) = \int_0^1 \frac{(x^0 - 1)^2}{\log^2 x} dx = 0$$

Hence $(2(0)+1)(\ln(2(0)+1) - 1) - 2(0+1)(\ln(0+1) - 1) + c_2 = 0$, so $c_2 = -1$. Then finally,

$$J(a) = (2a+1)(\ln(2a+1) - 1) - 2(a+1)(\ln(a+1) - 1) - 1$$

$$= (2a+1)\ln(2a+1) - 2(a+1)\ln(a+1)$$

Now we simply plug in $\varphi = \left(1 + \sqrt{5}\right)/2$. After simplifying, we find

$$\int_0^1 \frac{(x^\varphi - 1)^2}{\log^2 x} dx = \sqrt{5}\ln(\varphi)$$

But now we may address the convergence of $J(a)$ for $a \geq -1/2$ suggested earlier. We have shown $J(a) = (2a+1)\ln(2a+1) - 2(a+1)\ln(a+1)$ if $a \geq 0$, but the expression $(2a+1)\ln(2a+1) - 2(a+1)\ln(a+1)$ is defined if and only if $a > -1/2$ due to the $\ln(2a+1)$ term, so it is natural to wonder if the domain of $J(a) = \int_0^1 \frac{(x^a-1)^2}{\log^2 x} dx$ can be similarly extended. This turns out to be possible, but it is harder to show $J(a)$ converges for all $a \in (-1/2, 0)$. Furthermore, it can be shown (using L'Hopital's) that $\lim\limits_{a \to -1/2^+} (2a+1)\ln(2a+1) - 2(a+1)\ln(a+1) = \ln 2$, so $\lim\limits_{a \to -1/2^+} \int_0^1 \frac{(x^a-1)^2}{\log^2 x} dx = \ln 2$. This combined with the continuity of $(2a+1)\ln(2a+1) - 2(a+1)\ln(a+1)$ on $(-1/2, 0)$ means we can essentially "fill in the hole at $a = -1/2$" and extend $J(a) =$

$\int_0^1 \frac{(x^a-1)^2}{\log^2 x} dx$ to a continuous function on the closed interval $[-1/2, \infty)$ by defining $J(-1/2) = \ln 2$, even if the integral $\int_0^1 \frac{(x^a-1)^2}{\log^2 x} dx$ does not converge at $a = -1/2$. This is the reason why desmos returns a value for the integral $\int_0^1 \frac{(x^a-1)^2}{\log^2 x} dx$ for all $a \geq -1/2$. In short, it is true that

$$\int_0^1 \frac{(x^a-1)^2}{\log^2 x} dx = (2a+1)\ln(2a+1) - 2(a+1)\ln(a+1), \qquad a > -1/2$$

Incidentally, the convergence of $J(a)$ for $a > -1/2$ could have been discovered directly from the definition of the integral. Using the substitution $x \to e^x$, we have

$$J(a) = \int_0^1 \frac{(x^a-1)^2}{\log^2 x} dx = \int_{-\infty}^0 \left(\frac{e^{ax}-1}{x}\right)^2 e^x dx = \int_{-\infty}^0 \frac{e^{(2a+1)x} - 2e^{(a+1)x} + 1}{x^2} dx$$

Notice the $2a+1$ and $a+1$ terms in the exponentials, which reflect their appearance in the expression $(2a+1)\ln(2a+1) - 2(a+1)\ln(a+1)$. From here, it is probably possible to directly prove the integral converges if and only if $a > -1/2$. Taylor series may be the way to go.

Evaluate the following integral and prove it converges:

$$J = \int_0^\infty \frac{\sin^2 x}{x^2(x^2+1)}\,dx$$

One fast option is to integrate $f(z) = \frac{1-e^{i2z}}{z^2(z^2+1)}$ (since $\cos 2x = 1 - 2\sin^2 x$) over a semicircle in the upper-half plane indented at the origin. The integral will vanish over the semicircular portion that tends to ∞ and since $z = 0$ is a simple pole of f, we are able to handle the indent at the origin. The value of J can then be extracted by taking the real part of the integral. However, in case you forgot your double angle identities, we can use differentiation under the integral sign to convert J into an easier integral.

We define the following function for all $t \in \mathbb{R}$,

$$J(t) = \int_0^\infty \frac{\sin^2(tx)}{x^2(x^2+1)}\,dx$$

To prove $J(t)$ converges on $\mathbb{R}$, note $I(t) \coloneqq \int_0^\infty \frac{\sin^2(tx)}{x^2}\,dx$ converges on $\mathbb{R}$ since $\lim_{x\to 0} \frac{\sin^2(tx)}{x^2} = t^2$ by L'Hopital's (implying $\int_0^1 \frac{\sin^2(tx)}{x^2}\,dx$ converges) and $0 \le \int_1^\infty \frac{\sin^2(tx)}{x^2}\,dx \le \int_1^\infty \frac{1}{x^2}\,dx = 1$ (implying $\int_1^\infty \frac{\sin^2(tx)}{x^2}\,dx$ converges). Then,

$$0 \le J(t) = \int_0^\infty \frac{\sin^2(tx)}{x^4+x^2}\,dx \le \int_0^\infty \frac{\sin^2(tx)}{x^2}\,dx = I(t)$$

which we have shown converges, hence $J(t)$ converges on $\mathbb{R}$. We also wish $J(t)$ to converge uniformly, so we again substitute $tx \mapsto x$ to find

$$J(t) = \int_0^\infty \frac{\sin^2(tx)}{x^4+x^2}\,dx = \int_0^\infty \frac{t^3\sin^2 x}{x^4+t^2x^2}\,dx \le \int_0^\infty \frac{t^3\sin^2 x}{t^2x^2}\,dx = \int_0^\infty t\,\frac{\sin^2 x}{x^2}\,dx = tI(1)$$

and although $tI(1)$ converges, the presence of the t term suggests the convergence of $J(t)$ may not be uniform on $\mathbb{R}$, but if we restrict the domain of $J(t)$ to some interval $[-M, M]$, then we are assured the convergence is uniform on $[-M, M]$ by the M-test for improper integrals since $\left|t\frac{\sin^2 x}{x^2}\right| \le M\frac{\sin^2 x}{x^2}$ which we have shown is integrable on $(0, \infty)$. Then,

$$J'(t) = \int_0^\infty \frac{\partial}{\partial t}\frac{\sin^2(tx)}{x^2(x^2+1)}\,dx = \int_0^\infty \frac{2\sin(tx)\cos(tx)\,x}{x^2(x^2+1)}\,dx = \int_0^\infty \frac{\sin(2tx)}{x(x^2+1)}\,dx$$

It is not hard to show this integral converges for all $t \in \mathbb{R}$ and converges uniformly on $[-M, M]$, the proof is analogous to what we did before,

$$\Rightarrow J''(t) = \int_0^\infty \frac{2x\cos(2tx)}{x(x^2+1)}\,dx = \int_0^\infty \frac{2\cos(2tx)}{x^2+1}\,dx$$

which converges since the integrand is less than $2/(x^2+1)$. However this is a standard integral[1] in an undergraduate complex analysis course and can be shown to equal πe^{-2t} (integrate $e^{2tz}/(z^2+1)$ over a semicircular contour). So $J''(t) = \pi e^{-2t}$, hence $J'(t) = \int \pi e^{-2t} dt = -\pi e^{-2t}/2 + c_1$ for some $c_1 \in \mathbb{R}$. Since $J'(0) = \int_0^\infty \frac{\sin(2(0)x)}{x(x^2+1)} dx = 0$, we have $-\pi e^{-2(0)}/2 + c_1 \Rightarrow c_1 = \pi/2$, so $J'(t) = \frac{\pi}{2}(1 - e^{-2t})$. Then $J(t) = \int \frac{\pi}{2}(1 - e^{-2t})\, dt = \frac{\pi}{2}(t + e^{-2t}/2) + c_2$ for some $c_2 \in \mathbb{R}$, but again we have $J(0) = 0$ so $c_2 = -\pi/4$. Then finally we have

$$J(t) = \frac{\pi}{4}(e^{-2t} + 2t - 1)$$

Then the desired integral is

$$J(1) = \frac{\pi}{4}(e^{-2} + 1)$$

Essentially, our differentiating under the integral removed the hassle of dealing with a pole at the origin for contour integration. This is generally a good idea for solving any integral with a troublesome pole at the origin. Of course, the integral $\int_0^\infty \frac{\cos(2tx)}{x^2+1} dx$ can be solved without complex analysis (you could use further differentiation under the integral sign, the Laplace transform, etc.)

[1] The general result is $\int_0^\infty \frac{\cos(ax)}{x^2+b^2} dx = \frac{\pi}{2b} e^{-ab}$ for $a, b > 0$. Consequently we cannot have $t = 0$ when we apply this result to our integral.

Evaluate the integral:

$$\int_0^{\frac{\pi}{2}} \frac{x}{\tan x}\,dx = \int_0^1 \frac{\arcsin x}{x}\,dx$$

Equality is seen by substituting $u = \arcsin x$. Define the function for $a \geq 0$,

$$I(a) = \int_0^{\frac{\pi}{2}} \frac{\arctan(a\tan x)}{\tan x}\,dx$$

To prove convergence, substitute $u = \tan x$,

$$\Rightarrow I(a) = \int_0^{\infty} \frac{\arctan(au)}{u(1+u^2)}\,du$$

Note $\lim_{u\to 0} \frac{\arctan(au)}{u} = \lim_{u\to 0} \frac{a}{1+a^2u^2} = a$, hence $\lim_{u\to 0} \frac{\arctan(au)}{u(1+u^2)} = a$ so there are no convergence issues as $u \to 0$, so $\int_0^1 \frac{\arctan(au)}{u(1+u^2)}\,du$ converges. But $\int_1^{\infty} \frac{\arctan(au)}{u(1+u^2)}\,du \leq \int_1^{\infty} \frac{\pi/2}{u^2}\,du = \pi/2$ so overall $I(a)$ converges if $a \geq 0$. To show uniform convergence, note that on $[0, M]$ for $M > 0$ we have $\frac{\arctan(au)}{u(1+u^2)} \leq \frac{\arctan(Mu)}{u(1+u^2)}$ (since $\arctan$ is increasing) which we have shown is integrable on $(0,\infty)$, hence the integral converges uniformly on $[0, M]$ by M-test for improper integrals. Then, differentiating under the integral,

$$I'(a) = \int_0^{\infty} \frac{du}{(1+u^2)(1+a^2u^2)}$$

This integral is readily solved with complex analysis by integrating $f(z) = \frac{1}{(1+z^2)(1+a^2z^2)}$ over the semicircular contour in the upper-half plane. Further restrict $a \neq 1$ so $f(z)$ has four distinct simple poles $\pm i, \pm i/a$. Then by residue theorem, noting the integrand is even,

$$2\int_0^{\infty} \frac{du}{(1+u^2)(1+a^2u^2)} = 2\pi i\left(\operatorname{Res}(f, i) + \operatorname{Res}\left(f, \frac{i}{a}\right)\right)$$

$$\Rightarrow \int_0^{\infty} \frac{du}{(1+u^2)(1+a^2u^2)} = \pi i\left(\frac{1}{2i(1-a^2)} + \frac{a}{2i(a^2-1)}\right) = \frac{\pi/2}{a+1}$$

Note $\frac{\pi/2}{a+1}$ is defined and equals $\pi/4$ at $a = 1$, and since $\frac{\pi/2}{a+1}$ is also continuous on $[0,\infty)$, we can extend $I'(a) = \int_0^{\infty} \frac{du}{(1+u^2)(1+a^2u^2)}$ to a continuous function on $[0,\infty)$ by defining $I'(1) = \pi/4$.

So $I'(a) = \frac{\pi/2}{a+1}$, implying $I(a) = \frac{\pi}{2}\ln(a+1) + c$ (we can ignore absolute value signs since we are only considering $a \geq 0$). Since $I(0) = \int_0^{\infty} \frac{\arctan((0)u)}{u(1+u^2)}\,du = 0$, we quickly find $c = 0$, so we indeed just have $I(a) = \frac{\pi}{2}\ln(a+1)$. Then the desired integral is

$$\int_0^{\frac{\pi}{2}} \frac{x}{\tan x}\,dx = I(1) = \frac{\pi}{2}\ln 2$$