We aim to evaluate:

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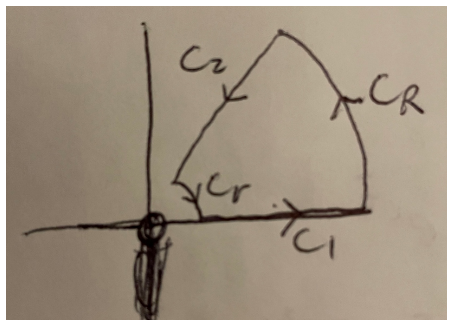
$$J = \int_0^\infty \cos(x^a)\,dx \qquad\qquad K = \int_0^\infty \sin(x^a)\,dx$$

$$L = \int_0^\infty \frac{\sin(x^a)}{x^a}\,dx$$

given that the above integrals all converge for $a > 1$.

We integrate $f(z) = e^{iz^a}$ where z^a has branch cut $[0, -i\infty)$ and $\arg z \in (-\pi/2, 3\pi/2)$ over the indented circular sector of angle $\pi/2a$.



(Attempting to use a circular sector of angle $2\pi/a$ will unfortunately lead to a $0 = 0$ situation from the residue theorem, so although the manipulations are legal, they are ultimately unhelpful)

We have $z = \rho e^{i\theta}, \theta \in (0, \pi/2a)$ on C_r (with $\rho \to 0$) and C_R (with $\rho \to \infty$). Writing z^a in polar form $z^a = |z|^a e^{ia \arg z}$, we have

$$|f(z)| = \left| e^{i|z|^a e^{ia \arg z}} \right| = e^{-|z|^a \sin(a \arg z)} = e^{-\rho^a \sin(a\theta)} \leq e^{-\rho^a\left(\frac{2a}{\pi}\theta\right)} \leq 1$$

where the second-to-last inequality follows from $\sin(a\theta)$ being above the line $\frac{2a}{\pi}\theta$ on the interval $(0, \pi/2a)$. Consequently we have $\left|\int_{C_r} f\right| \leq \frac{\pi}{2a}\rho(1)$ by ML-lemma, and this goes to 0 as $\rho \to 0$, so $\int_{C_r} f \to 0$. However this bound does not vanish as $\rho \to \infty$ so we will need a sharper estimate for $\int_{C_R} f$. Note that

$$\left|\int_{C_R} f(z)dz\right| \leq \int_0^{\frac{\pi}{2a}} |f(z)||dz| \leq \int_0^{\frac{\pi}{2a}} e^{-\rho^a\left(\frac{2a}{\pi}\theta\right)} \rho d\theta = \left[\frac{e^{-\rho^a\left(\frac{2a}{\pi}\theta\right)}}{-\rho^a\left(\frac{2a}{\pi}\right)}\rho\right]_0^{\frac{\pi}{2a}} = \frac{\pi\left(1 - e^{-\rho^a}\right)}{2a\rho^{a-1}}$$

and since $a > 1$, this indeed goes to 0 as $\rho \to \infty$. Hence $\int_{C_R} f \to 0$.

Parameterizing $z = t$ on C_1, we quickly find $\int_{C_1} f \to J + iK$. On C_2, we have $z = te^{i\pi/2a}$, so

$$\int_{C_2} f = \int_{\infty}^{0} e^{i\left(te^{i\frac{\pi}{2a}}\right)^a} e^{i\frac{\pi}{2a}} dt = -e^{i\frac{\pi}{2a}} \int_0^{\infty} e^{-t^a} dt$$

Substitute $u = t^a$, so $du = at^{a-1}dt$,

$$= -e^{i\frac{\pi}{2a}} \frac{1}{a} \int_0^{\infty} e^{-u} u^{\frac{1}{a}-1} du = -e^{i\frac{\pi}{2a}} \frac{1}{a} \Gamma\left(\frac{1}{a}\right)$$

The function f is analytic in and around the contour, so by Cauchy's theorem,

$$J + iK - e^{i\frac{\pi}{2a}} \frac{1}{a} \Gamma\left(\frac{1}{a}\right) = 0$$

Equating real and imaginary parts, we have:

$$J = \frac{1}{a}\Gamma\left(\frac{1}{a}\right)\cos\left(\frac{\pi}{2a}\right) \qquad K = \frac{1}{a}\Gamma\left(\frac{1}{a}\right)\sin\left(\frac{\pi}{2a}\right)$$

The above expressions are also the analytic continuations of the complex functions $f_1(a) = \int_0^{\infty} \cos(x^a)\, dx$ and $f_2(a) = \int_0^{\infty} \sin(x^a)\, dx$ respectively to the whole complex plane excepting the singularities of $\Gamma(1/a)$.

We now evaluate L. Using integration by parts,

$$L = \int_0^{\infty} x^{-a} \sin(x^a)\ dx = \left[\frac{x^{-a+1}}{-a+1}\sin(x^a)\right]_0^{\infty} - \int_0^{\infty} \frac{x^{-a+1}}{-a+1}\cos(x^a)\, ax^{a-1} dx$$

Since $a > 1$, we have $\lim_{x\to\infty} \frac{x^{-a+1}}{-a+1}\sin(x^a) = \lim_{x\to\infty} \frac{\sin(x^a)}{(1-a)x^{a-1}} = 0$,

$$= 0 + \frac{a}{a-1}\int_0^{\infty} \cos(x^a)\, dx$$

$$= \frac{1}{a-1}\Gamma\left(\frac{1}{a}\right)\cos\left(\frac{\pi}{2a}\right)$$

It should be noted that the integral

$$\int_0^{\infty} \frac{\cos(x^a)}{x^a} dx$$

diverges for all real numbers a. This can be seen by noting that the first term in the Taylor series of $\cos(x^a)$ centered at $x = 0$ is 1, and the integral $\int_0^{\infty} 1/x^a\, dx$ diverges for all $a \in \mathbb{R}$.