

Assuming they converge, evaluate the following integrals:

Wenzhi Tseng

$$I = \int_0^1 \frac{\ln^2 x}{x^2 - 1} dx \quad J = \int_0^{\frac{\pi}{2}} \theta \ln(\cos \theta) d\theta$$

We first solve I . Using partial fractions,

$$\int \frac{\ln^2 x}{x^2 - 1} dx = \frac{1}{2} \left(\int \frac{\ln^2 x}{x - 1} dx - \int \frac{\ln^2 x}{x + 1} dx \right)$$

Substitute $u = x - 1$ and $u = x + 1$ in the first and second integrals respectively,

$$= \frac{1}{2} \left(\int \frac{\ln^2(u + 1)}{u} du - \int \frac{\ln^2(u - 1)}{u} du \right)$$

Now use integration by parts, noting we can write $\int u du = \ln(-u)$,

$$= \left[\frac{\ln(-u) \ln^2(u + 1)}{2} - \int \frac{\ln(-u) \ln(u + 1)}{u + 1} du \right] - \left[\frac{\ln u \ln^2(u - 1)}{2} - \int \frac{\ln u \ln(u - 1)}{u - 1} du \right]$$

Undoing the substitutions in the first and third terms and substituting $u = v - 1$ and $u = 1 - v$ in the integrals,

$$= \frac{1}{2} (\ln^2 x) \ln \left(\frac{1 - x}{1 + x} \right) + \int \frac{-\ln(1 - v) \ln(v)}{v} dv - \int \frac{-\ln(1 - v) \ln(-v)}{v} dv$$

From here on we make use of polylogarithms, see appendix for a brief review. We again use integration by parts,

$$\begin{aligned} &= \frac{1}{2} (\ln^2 x) \ln \left(\frac{1 - x}{1 + x} \right) + \left[(\text{Li}_2(v)) \ln(v) - \int \frac{\text{Li}_2(v)}{v} dv \right] - \left[(\text{Li}_2(v)) \ln(-v) - \int \frac{\text{Li}_2(v)}{v} dv \right] \\ &= \frac{1}{2} (\ln^2 x) \ln \left(\frac{1 - x}{1 + x} \right) + (\text{Li}_2(v)) \ln(v) - \text{Li}_3(v) - (\text{Li}_2(v)) \ln(-v) + \text{Li}_3(v) \end{aligned}$$

Undoing the substitutions,

$$= \frac{1}{2} (\ln^2 x) \ln \left(\frac{1 - x}{1 + x} \right) + (\ln x) (\text{Li}_2(x) - \text{Li}_2(-x)) + \text{Li}_3(-x) - \text{Li}_3(x) + c, c \in \mathbb{R}$$

Let the above antiderivative be $F(x)$. So we need to compute $\lim_{x \rightarrow 1} F(x) - \lim_{x \rightarrow 0} F(x)$. Note that $\text{Li}_s(x)$ has the power series expansion $\text{Li}_s(x) = \sum_{k=1}^{\infty} x^k / k^s$ for $|x| \leq 1$, and the sum converges absolutely by ratio test for $s > 0$. Then, we are justified in the following series manipulations:

$$\text{Li}_s(x) - \text{Li}_s(-x) = \left(x + \frac{x^2}{2^s} + \frac{x^3}{3^s} + \dots \right) - \left(-x + \frac{x^2}{2^s} + \frac{x^3}{3^s} - \dots \right) = 2 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)^s}$$

Then,

$$F(x) = \frac{1}{2} (\ln^2 x) \ln \left(\frac{1 - x}{1 + x} \right) + 2(\ln x) \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)^2} - 2 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)^3} + c$$

We now compute the limits required to find $\lim_{x \rightarrow 0} F(x)$. The most troublesome limit is

$$\lim_{x \rightarrow 0} (\ln x) \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)^2} = \lim_{x \rightarrow 0} \frac{\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)^2}}{1/\ln x}$$

The sum clearly converges at $x = 1$, so it converges absolutely on $[-1, 1]$ and is thus infinitely differentiable and term-by-term differentiation is allowed and all derivatives will converge on $(0, 1)$. Then, by L'Hopital's,

$$= \lim_{x \rightarrow 0} \frac{\sum_{k=0}^{\infty} \frac{x^{2k}}{2k+1}}{-1/x \ln^2 x} = \lim_{x \rightarrow 0} \frac{\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}}{-1/\ln^2 x} = \lim_{x \rightarrow 0} \frac{\sum_{k=0}^{\infty} x^{2k}}{2/x \ln^3 x} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{x \ln^3 x}{1 - x^2} = 0$$

since $\lim_{x \rightarrow 0} x \ln^3 x = 0$ (also by repeated L'Hopital's).

It can also be shown that $\lim_{x \rightarrow 0} (\ln^2 x) \ln \left(\frac{1-x}{1+x} \right) = \lim_{x \rightarrow 1} (\ln^2 x) \ln \left(\frac{1-x}{1+x} \right) = 0$ using repeated L'Hopital's. Putting everything together, we have $\lim_{x \rightarrow 0} F(x) = 0$ and thus

$$\int_0^1 \frac{\ln^2 x}{x^2 - 1} dx = \lim_{x \rightarrow 1} F(x) - \lim_{x \rightarrow 0} F(x) = -2 \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3}$$

But notice,

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} + \sum_{k=1}^{\infty} \frac{1}{(2k)^3} = \sum_{k=1}^{\infty} \frac{1}{k^3} = \zeta(3)$$

So the second sum is just $\zeta(3)/8$. Then $\sum_{k=0}^{\infty} 1/(2k+1)^3 = 7\zeta(3)/8$, so

$$\int_0^1 \frac{\ln^2 x}{x^2 - 1} dx = -\frac{7}{4} \zeta(3)$$

Now we solve J . We will freely use the following result, which is proved (using symmetry tricks) in my document "ln(sinx) and related integrals"

$$\int_0^{\pi/2} \ln(\sin \theta) d\theta = \int_0^{\pi/2} \ln(\cos \theta) d\theta = -\frac{\pi}{2} \ln 2 \quad (1)$$

Substituting $\theta \mapsto \pi/2 - \theta$ and using law of logs,

$$J = \int_0^{\pi/2} \theta \ln(\cos \theta) d\theta = \int_0^{\pi/2} \left(\frac{\pi}{2} - \theta \right) \ln(\sin \theta) d\theta = \frac{\pi}{2} \int_0^{\pi/2} \ln(\sin \theta) d\theta - \int_0^{\pi/2} \theta \ln(\sin \theta) d\theta$$

Rearranging and using (1),

$$\Rightarrow \int_0^{\pi/2} \theta \ln(\cos \theta) d\theta + \int_0^{\pi/2} \theta \ln(\sin \theta) d\theta = -\frac{\pi^2}{4} \ln 2 \quad (2)$$

On the other hand, by laws of logs,

$$\int_0^{\frac{\pi}{2}} \theta \ln(\sin \theta) d\theta - \int_0^{\frac{\pi}{2}} \theta \ln(\cos \theta) d\theta = \int_0^{\frac{\pi}{2}} \theta \ln(\tan \theta) d\theta \quad (3)$$

Define the following function for $t \geq 0$,

$$J(t) = \int_0^{\frac{\pi}{2}} (\arctan(t \tan \theta)) \ln(\tan \theta) d\theta$$

Now differentiate under the integral. We skip over verifying the convergence/uniform convergence of the relevant integrals.

$$\Rightarrow J'(t) = \int_0^{\frac{\pi}{2}} \frac{\tan \theta}{1 + t^2 \tan^2 \theta} \ln(\tan \theta) d\theta$$

Substitute $u = \tan \theta$,

$$= \int_0^{\infty} \frac{x \ln x}{(1 + t^2 x^2)(1 + x^2)} dx$$

This integral suggests contour integration due to the polynomial in the denominator and the rather straightforward function $x \ln x$ in the numerator. We choose $\log z$ with branch cut $[0, \infty)$ and $\arg z \in (0, 2\pi)$ and integrate the function $f(z) = \frac{z \log^2 z}{(1 + t^2 z^2)(1 + z^2)}$ over the corresponding keyhole contour. For $t \neq 0$ and $t \neq 1$, note f has four simple poles $\pm i, \pm i/t$ on the imaginary axis inside the contour. By residue theorem, we find

$$\begin{aligned} & \int_0^{\infty} \frac{x(\ln x)^2}{(1 + t^2 x^2)(1 + x^2)} dx - \int_0^{\infty} \frac{x(\ln x + 2\pi i)^2}{(1 + t^2 x^2)(1 + x^2)} dx = 2\pi i \sum \text{Res} \\ & -4\pi i \int_0^{\infty} \frac{x \ln x}{(1 + t^2 x^2)(1 + x^2)} dx + 4\pi^2 \int_0^{\infty} \frac{x}{(1 + t^2 x^2)(1 + x^2)} dx = 2\pi i \frac{\ln^2 t - 2\pi i \ln t}{t^2 - 1} \end{aligned}$$

Equating real and imaginary parts yields

$$\int_0^{\infty} \frac{x \ln x}{(1 + t^2 x^2)(1 + x^2)} dx = \frac{\ln^2 t}{2(1 - t^2)} \quad \int_0^{\infty} \frac{x}{(1 + t^2 x^2)(1 + x^2)} dx = \frac{\ln t}{t^2 - 1}$$

So $J'(t) = \frac{\ln^2 t}{2(1 - t^2)}$ for $t > 0, t \neq 1$. But note $\lim_{t \rightarrow 1} \frac{\ln^2 t}{2(1 - t^2)} = 0$ by L'Hopital's so we can extend $J'(t) = \int_0^{\infty} \frac{x \ln x}{(1 + t^2 x^2)(1 + x^2)} dx$ to a continuous function on $(0, \infty)$ where we define $J'(1) = 0$.

The antiderivative of $\frac{\ln^2 t}{2(1 - t^2)}$ was computed previously,

$$\Rightarrow J(t) = -\frac{1}{2} \left(\frac{1}{2} (\ln^2 t) \ln \left| \frac{1 - t}{1 + t} \right| + (\ln t) (\text{Li}_2(t) - \text{Li}_2(-t)) - \text{Li}_3(t) + \text{Li}_3(-t) \right) + c$$

for some $c \in \mathbb{R}$. However, since $J(0) = \int_0^{\frac{\pi}{2}} (\arctan(0 \tan \theta)) \ln(\tan \theta) d\theta = 0$, taking $t \rightarrow 0$ in the above expression (also done previously) and solving for c shows that $c = 0$.

Recall our original goal was to find the value of $J(1)$ and substitute it into (3). The computation of $J(1)$ was also done previously, so we have

$$\int_0^{\frac{\pi}{2}} \theta \ln(\tan \theta) d\theta = J(1) = \frac{7}{8}\zeta(3)$$

Look familiar? That's right. Our integrals I and J are related:

$$\int_0^1 \frac{\ln^2 x}{x^2 - 1} dx = -2 \int_0^{\frac{\pi}{2}} \theta \ln(\tan \theta) d\theta = -\frac{7}{4}\zeta(3)$$

How coincidental. But now we finish this question. Returning to (3), we now have

$$\int_0^{\frac{\pi}{2}} \theta \ln(\sin \theta) d\theta - \int_0^{\frac{\pi}{2}} \theta \ln(\cos \theta) d\theta = \frac{7}{8}\zeta(3)$$

And from (2), we had

$$\int_0^{\frac{\pi}{2}} \theta \ln(\cos \theta) d\theta + \int_0^{\frac{\pi}{2}} \theta \ln(\sin \theta) d\theta = -\frac{\pi^2}{4} \ln 2$$

Solving simultaneously yields:

$$\int_0^{\frac{\pi}{2}} \theta \ln(\sin \theta) d\theta = \frac{7}{16}\zeta(3) - \frac{\pi^2}{8} \ln 2 \quad \int_0^{\frac{\pi}{2}} \theta \ln(\cos \theta) d\theta = -\frac{7}{16}\zeta(3) - \frac{\pi^2}{8} \ln 2$$

Appendix: The Polylogarithm

In its most general form, the polylogarithm $\text{Li}_s(z)$ of order s is a complex-valued function defined as a power series and extended via analytic continuation, but for the purposes of our integration problem, we will only consider $s = 2, 3, 4, \dots$ and $z \in [-1, 1]$.

For $|x| \leq 1$ and $s = 2, 3, 4, \dots$ the following power series converges absolutely by ratio test (and p-series test for $x = 1$) and is defined to be $\text{Li}_s(x)$.

$$\text{Li}_s(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^s} = x + \frac{x^2}{2^s} + \frac{x^3}{3^s} + \dots$$

Consequently we can differentiate term-by-term,

$$\begin{aligned} \frac{d}{dx} \text{Li}_{s+1}(x) &= \frac{d}{dx} \left[x + \frac{x^2}{2^{s+1}} + \frac{x^3}{3^{s+1}} + \dots \right] = 1 + \frac{x}{2^s} + \frac{x^2}{3^s} + \dots \\ &= \frac{1}{x} \left(x + \frac{x^2}{2^s} + \frac{x^3}{3^s} + \dots \right) = \frac{\text{Li}_s(x)}{x} \end{aligned}$$

So $\text{Li}_{s+1}(x)$ is an antiderivative of $\text{Li}_s(x)/x$.

We now prove the following identity for $|x| \leq 1$.

$$\text{Li}_2(x) = - \int_0^x \frac{\ln(1-v)}{v} dv$$

We first prove the integral $\int_0^x \ln(1-v)/v dv$ exists and is continuous for $x \in [-1, 1]$.

Note $\lim_{v \rightarrow 0} \ln(1-v)/v = -1$ so we can extend the integrand to a continuous function at zero. The only other point at which the integrand is undefined is $v = 1$, so it suffices to show the integral converges as $v \rightarrow 1$. Fix $r \in (0, 1)$ so the convergence at zero is not an issue and substitute $u = -\ln(1-v)$.

$$\begin{aligned} \Rightarrow - \int_r^1 \frac{\ln(1-v)}{v} dv &= \int_{-\ln(1-r)}^{\infty} \frac{u}{e^u - 1} du \\ &= \int_{-\ln(1-r)}^1 \frac{u}{e^u - 1} du + \int_1^{\infty} \frac{u}{e^u - 1} du \end{aligned}$$

Since $u/(e^u - 1)$ is continuous on $[-\ln(1-r), 1]$, the first integral converges. Since $u/(e^u - 1)$ clearly decays much faster than $1/u^2$ on $[1, \infty)$, the second integral converges. Hence $\int_0^x \ln(1-v)/v dv$ converges as $v \rightarrow 1$ and is thus continuous on $[-1, 1]$.

Now we prove the identity. Using Taylor series,

$$- \int_0^x \frac{\ln(1-v)}{v} dv = - \int_0^x \sum_{k=1}^{\infty} \frac{-v^{k-1}}{k} dv$$

Fix $x \in [0, 1)$. Since the integrand is continuous at the bounds of integration $v = 0$ and $v = x$, we can say $v \in (0, x)$ for the purposes of integration. Then the summand $-v^{k-1}/k$ is uniformly bounded by $|x|^{k-1}/k$ on the interval $[0, x]$ and $\sum_{k=1}^{\infty} |x|^{k-1}/k$ converges by ratio test, so by M-test, the series converges uniformly on $[0, x]$, allowing us to interchange the integral and sum. An analogous argument holds for $x \in (-1, 0]$.

$$= \sum_{k=1}^{\infty} \int_0^x \frac{v^{k-1}}{k} dv = \sum_{k=1}^{\infty} \left[\frac{v^k}{k^2} \right]_0^x = \sum_{k=1}^{\infty} \frac{x^k}{k^2} = \text{Li}_2(x)$$

It remains to show the identity holds at $x = \pm 1$. However this follows from the fact that $\text{Li}_2(x) = \sum_{k=1}^{\infty} x^k/k^2$ is continuous on $[-1, 1]$ (by term-by-term continuity) and $\int_0^x \ln(1-v)/v dv$ is also continuous on $[-1, 1]$ (shown previously), and since the two functions already coincide on $(-1, 1)$, continuity forces them to also coincide at the endpoints $x = \pm 1$.