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For $n = 0, 1, 2, \ldots$, find a closed form and establish a recurrence relation for I_n where:

$$I_n \coloneqq \int_0^{\frac{\pi}{2}} \sin^n x \, dx$$

We will first compute I_n when n is even. Define the following integral $\mathcal{I}_n$ for $n = 0, 1, 2, \ldots$

$$\mathcal{I}_n \coloneqq \int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx = \frac{1}{4}\int_0^{2\pi} \sin^{2n}\theta \, d\theta = \frac{1}{4}\int_0^{2\pi} \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^{2n} d\theta$$

EITHER: Using the substitution $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta = izd\theta$, the integral becomes:

$$= \frac{1}{4}\oint_{|z|=1} \left(\frac{z - z^{-1}}{2i}\right)^{2n} \frac{dz}{iz} = \frac{1}{4}\frac{1}{(2i)^{2n}i}\oint_{|z|=1} \frac{(z - z^{-1})^{2n}}{z} dz$$

Let $f(z) = (z - z^{-1})^{2n}/z = (z^2 - 1)^{2n}/z^{2n+1}$. Clearly the only pole is at $z = 0$, so by the residue theorem,

$$\mathcal{I}_n = \frac{1}{2^{2n+2}(-1)^n i}\oint_{|z|=1} \frac{(z - z^{-1})^{2n}}{z} dz = \frac{2\pi i \operatorname{Res}(f, 0)}{2^{2n+2}(-1)^n i} = \frac{\pi(-1)^n}{2^{2n+1}}\operatorname{Res}(f, 0)$$

To find $\operatorname{Res}(f, 0)$, we find the Laurent series of f around zero using binomial theorem,

$$f(z) = \frac{(z - z^{-1})^{2n}}{z} = \frac{1}{z}\sum_{k=0}^{2n}\binom{2n}{k} z^{2n-k}\left(-\frac{1}{z}\right)^k = \sum_{k=0}^{2n}(-1)^k\binom{2n}{k} z^{2n-2k-1}$$

We require the coefficient of the z^{-1} term, so we need $2n - 2k - 1 = -1 \Rightarrow k = n$. That is, the z^{-1} term occurs when $k = n$, so $\operatorname{Res}(f, 0) = \binom{2n}{n}(-1)^n$,

$$\Rightarrow \mathcal{I}_n = \frac{\pi(-1)^n}{2^{2n+1}}\binom{2n}{n}(-1)^n = \frac{\pi}{2^{2n+1}}\binom{2n}{n} = \frac{\pi(2n)!}{2^{2n+1}(n!)^2}$$

OR: Let $z = e^{i\theta}$. Then $(z + z^{-1})^{2n} = \left(e^{i\theta} + e^{-i\theta}\right)^{2n} = (2\cos\theta)^{2n}$. However, using the binomial expansion from above, we also have:

$$(z + z^{-1})^{2n} = \sum_{k=0}^{2n}\binom{2n}{k} z^{2n-k}\frac{1}{z^k} = \sum_{k=0}^{2n}\binom{2n}{k} z^{2n-2k}$$

Since $\binom{2n}{k} = \binom{2n}{2n-k}$, we see that the coefficients of the expansion are the same for $k = 0, \ldots, n-1$ and $k = 2n, \ldots, n+1$ respectively, so we group the terms accordingly.

The term when $k = n$ is $\binom{2n}{n}$ and is handled separately. If $n \geq 1$, then:

$$\sum_{k=0}^{2n}\binom{2n}{k}z^{2n-2k}=\binom{2n}{n}+\sum_{k=0}^{n-1}\binom{2n}{k}\left(z^{2k}+\frac{1}{z^{2k}}\right)$$

$$=\binom{2n}{n}+\sum_{k=0}^{n-1}\binom{2n}{k}2\cos(2k\theta)$$

Then, we have:

$$\int_0^{\frac{\pi}{2}}(2\cos\theta)^{2n}\,d\theta=\int_0^{\frac{\pi}{2}}(z+z^{-1})^{2n}\,d\theta=\int_0^{\frac{\pi}{2}}\binom{2n}{n}+\sum_{k=0}^{n-1}\binom{2n}{k}2\cos(2k\theta)\,d\theta$$

$$=\frac{\pi}{2}\binom{2n}{n}+2\sum_{k=1}^{n-1}\binom{2n}{k}\int_0^{\frac{\pi}{2}}\cos(2k\theta)\,d\theta$$

$$=\left[\frac{\sin(2n\theta)}{n}\right]_0^{\frac{\pi}{2}}+\frac{\pi}{2}\binom{2n}{n}+2\sum_{k=1}^{n-1}\binom{2n}{k}\left[\frac{\sin(2k\theta)}{2k}\right]_0^{\frac{\pi}{2}}$$

$$=0+\frac{\pi}{2}\binom{2n}{n}+0=\frac{\pi}{2}\binom{2n}{n}$$

$$\Rightarrow\int_0^{\frac{\pi}{2}}\cos^{2n}\theta\;d\theta=\frac{\pi}{2^{2n+1}}\binom{2n}{n}$$

If $n=0$, the integral evaluates to $\pi/2$, which is consistent with the above identity.

THEN: Furthermore, using the substitution $x\mapsto\pi/2-x$, we have:

$$\mathcal{J}_n=\int_0^{\frac{\pi}{2}}\sin^{2n}x\,dx=\int_0^{\frac{\pi}{2}}\cos^{2n}x\,dx$$

The integrals for when the bounds of integration are multiples of $\pi/2$ can be found using symmetry by observing that $\sin^{2n}x$ and $\cos^{2n}x$ are $\pi/2$ periodic.

Using this closed form, we can establish the recurrence relation for $\mathcal{J}_n$ for $n=1,2,\ldots$

$$\frac{\mathcal{J}_n}{\mathcal{J}_{n-1}}=\frac{\dfrac{\pi(2n)!}{2^{2n+1}(n!)^2}}{\dfrac{\pi(2n-2)!}{2^{2n-1}\big((n-1)!\big)^2}}=\frac{(2n)(2n-1)}{2^2n^2}=\frac{2n-1}{2n}$$

$$\Rightarrow\mathcal{J}_n=\frac{2n-1}{2n}\mathcal{J}_{n-1}$$

Now we establish the recurrence relation for I_n when n is odd. Define the following integral K_n for $n = 1, 2, ...$

$$K_n \coloneqq \int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta \, d\theta = \int_0^{\frac{\pi}{2}} \underbrace{\sin^{2n}\theta}_{u} \underbrace{\sin\theta}_{v'} \, d\theta$$

$$= [-\cos\theta \sin^{2n}\theta]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\cos\theta)(2n\sin^{2n-1}\theta)(\cos\theta) d\theta$$

$$= 0 + 2n\int_0^{\frac{\pi}{2}} (\sin^{2n-1}\theta)(\cos^2\theta) d\theta = 2n\int_0^{\frac{\pi}{2}} (\sin^{2n-1}\theta)(1 - \sin^2\theta) d\theta$$

$$= 2n\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta \, d\theta - 2n\int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta \, d\theta = 2nK_{n-1} - 2nK_n$$

$$\Rightarrow K_n = \frac{2n}{2n+1} K_{n-1}$$

Finding the closed form for K_n was difficult. The following attempts were made:

1. Using the closed form of $\mathcal{J}_n$ and the recurrence relations for $\mathcal{J}_n$ and K_n (failed because a relationship between $\mathcal{J}_n$ and K_n could not be found)
2. Residue theorem using the residue at $z = 0$ (failed because the residue is zero, reflecting how the function integrates to zero around the unit circle),
3. Double integration by defining $\mathcal{J}_n$ and K_n using different dummy variables and multiplying the integrals together (failed, could not get an answer out)
4. Considered trying to use residue at infinity, but that probably won't work.
5. Actual solution: Use Fourier series, or plug in enough values of n (using the recursion after the $n = 1$ case) until you see the pattern, then use induction.

$$K_0 = \int_0^{\frac{\pi}{2}} \sin\theta \, d\theta = 1$$

$$K_1 = \frac{2(1)}{2(1)+1} K_0 = \frac{2}{3}$$

$$K_2 = \frac{2(2)}{2(2)+1} K_1 = \frac{4}{5}\left(\frac{2}{3}\right)$$

$$K_3 = \frac{2(3)}{2(3)+1} K_2 = \frac{6}{7}\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)$$

Using induction and properties of double factorial, it can be proved that:

$$K_n = \frac{(2n)!!}{(2n+1)!!} = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

Using the substitution $\theta = \pi/2 - x$, we also have:

$$K_n = \int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta \, d\theta = \int_0^{\frac{\pi}{2}} \cos^{2n+1}\theta \, d\theta$$

Therefore, to summarize all our results,

$$\mathcal{J}_n = \int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx = \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx = \frac{\pi(2n)!}{2^{2n+1}(n!)^2}$$

$$K_n = \int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta \, d\theta = \int_0^{\frac{\pi}{2}} \cos^{2n+1}\theta \, d\theta = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

$$\mathcal{J}_n = \frac{2n-1}{2n}\mathcal{J}_{n-1}$$

$$K_n = \frac{2n}{2n+1}K_{n-1}$$

Which incidentally leads to the curious identity:

$$\mathcal{J}_n K_n = \frac{\pi/2}{2n+1}$$

Lastly, we solve the following integral for $n = 0, 1, 2, \dots$

$$\mathcal{L}_n = \int_0^1 x^n(1-x)^n dx$$

Using the substitution $x = \cos^2\theta$, so $dx = -2\cos\theta\sin\theta \, d\theta$,

$$\mathcal{L}_n = 2\int_0^{\frac{\pi}{2}} \cos^{2n+1}\theta \sin^{2n+1}\theta \, d\theta = 2\int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\sin 2\theta\right)^{2n+1} d\theta = \frac{1}{2^{2n}}\int_0^{\frac{\pi}{2}} \sin^{2n+1}(2\theta) \, d\theta$$

$$= \frac{1}{2^{2n-1}}\int_0^{\pi} \sin^{2n+1}\theta \, d\theta = \frac{1}{2^{2n}}\int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta \, d\theta = \frac{1}{2^{2n}}\frac{2^{2n}(n!)^2}{(2n+1)!} = \frac{(n!)^2}{(2n+1)!}$$

References:

1. https://www.jstor.org/stable/1967516?seq=1#metadata_info_tab_contents

2.

*Do **NOT** write solutions on this page.*

12. *[Maximum mark: 20]*

Consider the complex number $z = \cos\theta + \mathrm{i}\sin\theta$.

(a) Use De Moivre's theorem to show that $z^n + z^{-n} = 2\cos n\theta$, $n \in \mathbb{Z}^+$. *[2]*

(b) Expand $\left(z + z^{-1}\right)^4$. *[1]*

(c) Hence show that $\cos^4\theta = p\cos 4\theta + q\cos 2\theta + r$, where p, q and r are constants to be determined. *[4]*

(d) Show that $\cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$. *[3]*

(e) Hence find the value of $\int_0^{\frac{\pi}{2}} \cos^6\theta \, \mathrm{d}\theta$. *[3]*

The region S is bounded by the curve $y = \sin x \cos^2 x$ and the x-axis between $x = 0$ and $x = \frac{\pi}{2}$.

(f) S is rotated through 2π radians about the x-axis. Find the value of the volume generated. *[4]*

(g) (i) Write down an expression for the constant term in the expansion of $\left(z + z^{-1}\right)^{2k}$, $k \in \mathbb{Z}^+$.

(ii) Hence determine an expression for $\int_0^{\frac{\pi}{2}} \cos^{2k}\theta \, \mathrm{d}\theta$ in terms of k. *[3]*

3.

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(b) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^{2n+1}(2\theta)\, d\theta$, $n = 0, 1, \dots$.

(i) Prove that $I_n = \frac{2n}{2n+1} I_{n-1}$, $n \geq 1$. **3**

(ii) Deduce that $I_n = \frac{2^{2n}(n!)^2}{(2n+1)!}$. **3**

Let $J_n = \int_0^1 x^n (1-x)^n\, dx$, $n = 0, 1, 2, \dots$.

(iii) Using the result of part (ii), or otherwise, show that $J_n = \frac{(n!)^2}{(2n+1)!}$. **3**

(iv) Prove that $\left(2^n n!\right)^2 \leq \left(2n+1\right)!$. **2**