Assuming they converge, evaluate the following integrals: Wenzhi Tseng

$$I=\int_0^1 \frac{\ln^2 x}{x^2-1}\,dx \qquad\qquad J=\int_0^{\frac{\pi}{2}} \theta \ln(\cos\theta)\,d\theta$$

We first solve I. Using partial fractions,

$$\int \frac{\ln^2 x}{x^2-1}\,dx=\frac{1}{2}\left(\int \frac{\ln^2 x}{x-1}\,dx-\int \frac{\ln^2 x}{x+1}\,dx\right)$$

Substitute $u=x-1$ and $\mathscr{u}=x+1$ in the first and second integrals respectively,

$$=\frac{1}{2}\left(\int \frac{\ln^2(u+1)}{u}\,du-\int \frac{\ln^2(\mathscr{u}-1)}{\mathscr{u}}\,d\mathscr{u}\right)$$

Now use integration by parts, noting we can write $\int u\,du=\ln(-u)$,

$$=\left[\frac{\ln(-u)\ln^2(u+1)}{2}-\int \frac{\ln(-u)\ln(u+1)}{u+1}\,du\right]-\left[\frac{\ln \mathscr{u}\ln^2(\mathscr{u}-1)}{2}-\int \frac{\ln \mathscr{u}\ln(\mathscr{u}-1)}{\mathscr{u}-1}\,d\mathscr{u}\right]$$

Undoing the substitutions in the first and third terms and substituting $u=v-1$ and $\mathscr{u}=1-\mathscr{v}$ in the integrals,

$$=\frac{1}{2}(\ln^2 x)\ln\left(\frac{1-x}{1+x}\right)+\int \frac{-\ln(1-v)\ln(v)}{v}\,dv-\int \frac{-\ln(1-\mathscr{v})\ln(-\mathscr{v})}{\mathscr{v}}\,d\mathscr{v}$$

From here on we make use of polylogarithms, see appendix for a brief review. We again use integration by parts,

$$=\frac{1}{2}(\ln^2 x)\ln\left(\frac{1-x}{1+x}\right)+\left[\left(\mathrm{Li}_2(v)\right)\ln(v)-\int \frac{\mathrm{Li}_2(v)}{v}\,dv\right]-\left[\left(\mathrm{Li}_2(\mathscr{v})\right)\ln(-\mathscr{v})-\int \frac{\mathrm{Li}_2(\mathscr{v})}{\mathscr{v}}\,d\mathscr{v}\right]$$

$$=\frac{1}{2}(\ln^2 x)\ln\left(\frac{1-x}{1+x}\right)+\left(\mathrm{Li}_2(v)\right)\ln(v)-\mathrm{Li}_3(v)-\left(\mathrm{Li}_2(\mathscr{v})\right)\ln(-\mathscr{v})+\mathrm{Li}_3(\mathscr{v})$$

Undoing the substitutions,

$$=\frac{1}{2}(\ln^2 x)\ln\left(\frac{1-x}{1+x}\right)+(\ln x)\left(\mathrm{Li}_2(x)-\mathrm{Li}_2(-x)\right)+\mathrm{Li}_3(-x)-\mathrm{Li}_3(x)+c,c\in\mathbb{R}$$

Let the above antiderivative be $F(x)$. So we need to compute $\lim_{x\to 1}F(x)-\lim_{x\to 0}F(x)$. Note that $\mathrm{Li}_s(x)$ has the power series expansion $\mathrm{Li}_s(x)=\sum_{k=1}^{\infty}x^k/k^s$ for $|x|\le 1$, and the sum converges absolutely by ratio test for $s>0$. Then, we are justified in the following series manipulations:

$$\mathrm{Li}_s(x)-\mathrm{Li}_s(-x)=\left(x+\frac{x^2}{2^s}+\frac{x^3}{3^s}+\cdots\right)-\left(-x+\frac{x^2}{2^s}+\frac{x^3}{3^s}-\cdots\right)=2\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)^s}$$

Then,

$$F(x)=\frac{1}{2}(\ln^2 x)\ln\left(\frac{1-x}{1+x}\right)+2(\ln x)\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)^2}-2\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)^3}+c$$

We now compute the limits required to find $\lim_{x\to 0} F(x)$. The most troublesome limit is

$$\lim_{x\to 0}(\ln x)\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)^2} = \lim_{x\to 0}\frac{\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)^2}}{1/\ln x}$$

The sum clearly converges at $x = 1$, so it converges absolutely on $[-1, 1]$ and is thus infinitely differentiable and term-by-term differentiation is allowed and all derivatives will converge on $(0, 1)$. Then, by L'Hopital's,

$$= \lim_{x\to 0}\frac{\sum_{k=0}^{\infty}\frac{x^{2k}}{2k+1}}{-1/x\ln^2 x} = \lim_{x\to 0}\frac{\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}}{-1/\ln^2 x} = \lim_{x\to 0}\frac{\sum_{k=0}^{\infty}x^{2k}}{2/x\ln^3 x} = \frac{1}{2}\lim_{x\to 0}\frac{x\ln^3 x}{1-x^2} = 0$$

since $\lim_{x\to 0} x\ln^3 x = 0$ (also by repeated L'Hopital's).

It can also be shown that $\lim_{x\to 0}(\ln^2 x)\ln\left(\frac{1-x}{1+x}\right) = \lim_{x\to 1}(\ln^2 x)\ln\left(\frac{1-x}{1+x}\right) = 0$ using repeated L'Hopital's. Putting everything together, we have $\lim_{x\to 0} F(x) = 0$ and thus

$$\int_0^1 \frac{\ln^2 x}{x^2-1}\,dx = \lim_{x\to 1}F(x) - \lim_{x\to 0}F(x) = -2\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}$$

But notice,

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3} + \sum_{k=1}^{\infty}\frac{1}{(2k)^3} = \sum_{k=1}^{\infty}\frac{1}{k^3} = \zeta(3)$$

So the second sum is just $\zeta(3)/8$. Then $\sum_{k=0}^{\infty} 1/(2k+1)^3 = 7\zeta(3)/8$, so

$$\int_0^1 \frac{\ln^2 x}{x^2-1}\,dx = -\frac{7}{4}\zeta(3)$$

Now we solve J. We will freely use the following result, which is proved (using symmetry tricks) in my document "ln(sinx) and related integrals"

$$\int_0^{\frac{\pi}{2}}\ln(\sin\theta)\,d\theta = \int_0^{\frac{\pi}{2}}\ln(\cos\theta)\,d\theta = -\frac{\pi}{2}\ln 2 \qquad (1)$$

Substituting $\theta \mapsto \pi/2 - \theta$ and using law of logs,

$$J = \int_0^{\frac{\pi}{2}}\theta\ln(\cos\theta)\,d\theta = \int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2}-\theta\right)\ln(\sin\theta)\,d\theta = \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\ln(\sin\theta)\,d\theta - \int_0^{\frac{\pi}{2}}\theta\ln(\sin\theta)\,d\theta$$

Rearranging and using (1),

$$\Rightarrow \int_0^{\frac{\pi}{2}}\theta\ln(\cos\theta)\,d\theta + \int_0^{\frac{\pi}{2}}\theta\ln(\sin\theta)\,d\theta = -\frac{\pi^2}{4}\ln 2 \qquad (2)$$

On the other hand, by laws of logs,

$$\int_0^{\frac{\pi}{2}} \theta \ln(\sin\theta)\, d\theta - \int_0^{\frac{\pi}{2}} \theta \ln(\cos\theta)\, d\theta = \int_0^{\frac{\pi}{2}} \theta \ln(\tan\theta)\, d\theta \tag{3}$$

Define the following function for $t \geq 0$,

$$J(t) = \int_0^{\frac{\pi}{2}} (\arctan(t\tan\theta)) \ln(\tan\theta)\, d\theta$$

Now differentiate under the integral. We skip over verifying the convergence/uniform convergence of the relevant integrals.

$$\Rightarrow J'(t) = \int_0^{\frac{\pi}{2}} \frac{\tan\theta}{1 + t^2\tan^2\theta} \ln(\tan\theta)\, d\theta$$

Substitute $x = \tan\theta$,

$$= \int_0^{\infty} \frac{x\ln x}{(1+t^2x^2)(1+x^2)}\, dx$$

This integral suggests contour integration due to the polynomial in the denominator and the rather straightforward function $x\ln x$ in the numerator. We choose $\log z$ with branch cut $[0,\infty)$ and $\arg z \in (0, 2\pi)$ and integrate the function $f(z) = \frac{z\log^2 z}{(1+t^2z^2)(1+z^2)}$ over the corresponding keyhole contour. For $t \neq 0$ and $t \neq 1$, note f has four simple poles $\pm i, \pm i/t$ on the imaginary axis inside the contour. By residue theorem, we find

$$\int_0^{\infty} \frac{x(\ln x)^2}{(1+t^2x^2)(1+x^2)}\, dx - \int_0^{\infty} \frac{x(\ln x + 2\pi i)^2}{(1+t^2x^2)(1+x^2)}\, dx = 2\pi i \sum \mathrm{Res}$$

$$-4\pi i \int_0^{\infty} \frac{x\ln x}{(1+t^2x^2)(1+x^2)}\, dx + 4\pi^2 \int_0^{\infty} \frac{x}{(1+t^2x^2)(1+x^2)}\, dx = 2\pi i \frac{\ln^2 t - 2\pi i \ln t}{t^2 - 1}$$

Equating real and imaginary parts yields

$$\int_0^{\infty} \frac{x\ln x}{(1+t^2x^2)(1+x^2)}\, dx = \frac{\ln^2 t}{2(1-t^2)} \qquad \int_0^{\infty} \frac{x}{(1+t^2x^2)(1+x^2)}\, dx = \frac{\ln t}{t^2 - 1}$$

So $J'(t) = \frac{\ln^2 t}{2(1-t^2)}$ for $t > 0, t \neq 1$. But note $\lim_{t\to 1} \frac{\ln^2 t}{2(1-t^2)} = 0$ by L'Hopital's so we can extend $J'(t) = \int_0^{\infty} \frac{x\ln x}{(1+t^2x^2)(1+x^2)}\, dx$ to a continuous function on $(0,\infty)$ where we define $J'(1) = 0$.

The antiderivative of $\frac{\ln^2 t}{2(1-t^2)}$ was computed previously,

$$\Rightarrow J(t) = -\frac{1}{2}\left(\frac{1}{2}(\ln^2 t)\ln\left|\frac{1-t}{1+t}\right| + (\ln t)\big(\mathrm{Li}_2(t) - \mathrm{Li}_2(-t)\big) - \mathrm{Li}_3(t) + \mathrm{Li}_3(-t)\right) + c$$

for some $c \in \mathbb{R}$. However, since $J(0) = \int_0^{\frac{\pi}{2}} (\arctan(0\tan\theta)) \ln(\tan\theta)\, d\theta = 0$, taking $t \to 0$ in the above expression (also done previously) and solving for c shows that $c = 0$.

Recall our original goal was to find the value of $J(1)$ and substitute it into (3). The computation of $J(1)$ was also done previously, so we have

$$\int_0^{\frac{\pi}{2}} \theta \ln(\tan\theta)\, d\theta = J(1) = \frac{7}{8}\zeta(3)$$

Look familiar? That's right. Our integrals I and J are related:

$$\int_0^1 \frac{\ln^2 x}{x^2-1}\, dx = -2\int_0^{\frac{\pi}{2}} \theta \ln(\tan\theta)\, d\theta = -\frac{7}{4}\zeta(3)$$

How coincidental. But now we finish this question. Returning to (3), we now have

$$\int_0^{\frac{\pi}{2}} \theta \ln(\sin\theta)\, d\theta - \int_0^{\frac{\pi}{2}} \theta \ln(\cos\theta)\, d\theta = \frac{7}{8}\zeta(3)$$

And from (2), we had

$$\int_0^{\frac{\pi}{2}} \theta \ln(\cos\theta)\, d\theta + \int_0^{\frac{\pi}{2}} \theta \ln(\sin\theta)\, d\theta = -\frac{\pi^2}{4}\ln 2$$

Solving simultaneously yields:

$$\int_0^{\frac{\pi}{2}} \theta \ln(\sin\theta)\, d\theta = \frac{7}{16}\zeta(3) - \frac{\pi^2}{8}\ln 2 \qquad \int_0^{\frac{\pi}{2}} \theta \ln(\cos\theta)\, d\theta = -\frac{7}{16}\zeta(3) - \frac{\pi^2}{8}\ln 2$$

Appendix: The Polylogarithm

In its most general form, the polylogarithm $\mathrm{Li}_s(z)$ of order s is a complex-valued function defined as a power series and extended via analytic continuation, but for the purposes of our integration problem, we will only consider $s = 2, 3, 4, \ldots$ and $z \in [-1, 1]$.

For $|x| \le 1$ and $s = 2, 3, 4, \ldots$ the following power series converges absolutely by ratio test (and p-series test for $x = 1$) and is defined to be $\mathrm{Li}_s(x)$.

$$\mathrm{Li}_s(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^s} = x + \frac{x^2}{2^s} + \frac{x^3}{3^s} + \cdots$$

Consequently we can differentiate term-by-term,

$$\frac{d}{dx}\mathrm{Li}_{s+1}(x) = \frac{d}{dx}\left[x + \frac{x^2}{2^{s+1}} + \frac{x^3}{3^{s+1}} + \cdots\right] = 1 + \frac{x}{2^s} + \frac{x^2}{3^s} + \cdots$$

$$= \frac{1}{x}\left(x + \frac{x^2}{2^s} + \frac{x^3}{3^s} + \cdots\right) = \frac{\mathrm{Li}_s(x)}{x}$$

So $\mathrm{Li}_{s+1}(x)$ is an antiderivative of $\mathrm{Li}_s(x)/x$.

We now prove the following identity for $|x| \leq 1$.

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\ln(1-v)}{v} dv$$

We first prove the integral $\int_0^x \ln(1-v)/v\, dv$ exists and is continuous for $x \in [-1, 1]$. Note $\lim_{v\to 0} \ln(1-v)/v = -1$ by L'Hopital's, so we can extend the integrand to a continuous function at $v = 0$ and thus the integral converges as $v \to 0$, hence $\int_0^x \ln(1-v)/v\, dv$ converges for all $x \in [-1, 1)$. It remains to show $\int_0^1 \ln(1-v)/v\, dv$ converges since $v = 1$ is the only other point at which the integrand is undefined. Since we already know the integral converges as $v \to 0$, fix $r \in (0, 1)$ so the convergence at zero is not an issue and substitute $u = -\ln(1-v)$,

$$\Rightarrow -\int_r^1 \frac{\ln(1-v)}{v} dv = \int_{-\ln(1-r)}^{\infty} \frac{u}{e^u - 1} du$$

Since $u/(e^u - 1)$ is continuous on $[-\ln(1-r), \infty)$ and decays much faster than $1/u^2$ as $u \to \infty$, this integral converges. Hence $\int_0^1 \ln(1-v)/v\, dv$ converges. Since we have now shown $\int_0^x \ln(1-v)/v\, dv$ converges for all $|x| \leq 1$ and clearly the integrand is continuous on $(0, 1)$, we have that $\int_0^x \ln(1-v)/v\, dv$ is continuous for $x \in [-1, 1]$.

Now we prove the identity. Using Taylor series,

$$-\int_0^x \frac{\ln(1-v)}{v} dv = -\int_0^x \sum_{k=1}^{\infty} \frac{-v^{k-1}}{k} dv$$

Fix $x \in [0, 1)$. Since the integrand is continuous at the bounds of integration $v = 0$ and $v = x$, we can say $v \in (0, x)$ for the purposes of integration. Then the summand $-v^{k-1}/k$ is uniformly bounded by $|x|^{k-1}/k$ on the interval $[0, x]$ and $\sum_{k=1}^{\infty} |x|^{k-1}/k$ converges by ratio test, so by M-test, the series converges uniformly on $[0, x]$, allowing us to interchange the integral and sum. An analogous argument holds for $x \in (-1, 0]$.

$$= \sum_{k=1}^{\infty} \int_0^x \frac{v^{k-1}}{k} dv = \sum_{k=1}^{\infty} \left[\frac{v^k}{k^2}\right]_0^x = \sum_{k=1}^{\infty} \frac{x^k}{k^2} = \mathrm{Li}_2(x)$$

It remains to show the identity holds at $x = \pm 1$. However this follows from the fact that $\mathrm{Li}_2(x) = \sum_{k=1}^{\infty} x^k/k^2$ is continuous on $[-1, 1]$ (by term-by-term continuity) and $\int_0^x \ln(1-v)/v\, dv$ is also continuous on $[-1, 1]$ (shown previously), and since the two functions already coincide on $(-1, 1)$, continuity forces them to also coincide at the endpoints $x = \pm 1$.