The inequality $|\int f(x)dx| \leq \int |f(x)|dx$ follows from converting to the Riemann sum and applying triangle inequality to the terms in the Riemann sum.

Exercise 8.4.12
Assume $f(x,t)$ is continuous on the rectangle $D = [a,b] \times [c,d]$. Explain why the function $F(x) = \int_c^d f(x,t)dt$ is defined for $x \in [a,b]$.

Fix $x \in [a,b]$. Since $f(x,t)$ is continuous at $t \in [c,d]$, it is integrable on $[c,d]$, so the integral (which is being taken with respect to t) exists, that is, $F(x)$ is defined.

Theorem 8.4.5
If $f(x,t)$ is continuous on D, then $F(x) = \int_c^d f(x,t)dt$ is uniformly continuous on $[a,b]$.

Proof:
Since $f(x,t)$ is continuous on D, it is uniformly continuous on D, so for all $\epsilon > 0 \ \exists \ \delta > 0$ s.t. $\forall \ (x,t),(x_0,t_0) \in D$, if $|(x,t)-(x_0,t_0)| < \delta$ then $|f(x,t) - f(x_0,t_0)| < \epsilon/(d-c)$.
Now let $\epsilon > 0$, pick the same δ as above, let $x, y \in [a,b]$, and let $|x-y| < \delta$, which certainly implies $|(x,t)-(y,t)| < \delta$. Then,

$$|F(x) - F(y)| = \left|\int_c^d f(x,t)dt - \int_c^d f(y,t)dt\right| = \left|\int_c^d f(x,t) - f(y,t)dt\right|$$
$$\leq \int_c^d |f(x,t) - f(y,t)|dt < \int_c^d \frac{\epsilon}{d-c} dt = \epsilon$$

Theorem 8.4.6
If $f(x,t)$ and $f_x(x,t)$ are continuous on D, then $F(x) = \int_c^d f(x,t)dt$ is differentiable and $F'(x) = \int_c^d f_x(x,t)dt$.

Proof:
So $f_x(x,t)$ is continuous on D, it is uniformly continuous on D, so for all $\epsilon > 0 \ \exists \ \delta > 0$ s.t. $\forall \ (x,t),(x_0,t_0) \in D$, if $|(x,t)-(x_0,t_0)| < \delta$ then $|f_x(x,t) - f_x(x_0,t_0)| < \epsilon/(d-c)$.
Now let $\epsilon > 0$, pick the same δ as above, let $z, x \in [a,b]$, and let $|z-x| < \delta$. Then,

$$\left|\frac{F(z)-F(x)}{z-x} - \int_c^d f_x(x,t)dt\right| = \left|\frac{\int_c^d f(z,t)dt - \int_c^d f(x,t)dt}{z-x} - \int_c^d f_x(x,t)dt\right|$$
$$= \left|\int_c^d \frac{f(z,t)-f(x,t)}{z-x} - f_x(x,t)dt\right|$$

Since $f(x,t)$ is continuous on D and $f_x(x,t)$ exists on D, by MVT there exists k in between x and z such that $f(z,t) - f(x,t) = f_x(k,t)(z-x)$. But then certainly we have $|(k,t) - (x,t)| < \delta$ since $|z-x| < \delta$, therefore,

$$= \left|\int_c^d f_x(k,t) - f_x(x,t)dt\right| \leq \int_c^d |f_x(k,t) - f_x(x,t)|dt < \int_c^d \frac{\epsilon}{d-c}dt = \epsilon$$

Hence $F'(x) = \lim_{z\to x} \frac{F(z)-F(x)}{z-x} = \int_c^d f_x(x,t)dt$ as desired.

Definition 8.4.7

Given $f(x,t)$ defined on $\{(x,t) : x \in A, c \leq t\}$, assume $F(x) = \int_c^\infty f(x,t)dt$ exists for all $x \in A$. We say the integral $\int_c^\infty f(x,t)dt$ converges uniformly to $F(x)$ on A if $\forall\, \epsilon > 0$ $\exists\, M > c$ s.t. $\forall\, d \geq M$ and $\forall\, x \in A$ we have $\left|F(x) - \int_c^d f(x,t)dt\right| < \epsilon$. An immediate consequence of this definition is that the sequence of functions $F_n(x) = \int_c^{c+n} f(x,t)dt$ converges to $F(x)$ uniformly on A.

Theorem 8.4.8

If $f(x,t)$ is continuous on $D = [a,b] \times [c,\infty)$, then $F(x) = \int_c^\infty f(x,t)dt$ is uniformly continuous on $[a,b]$ provided the integral $\int_c^\infty f(x,t)dt$ converges uniformly.

Proof:

Since the integral converges uniformly, $\forall\, \epsilon > 0\ \exists\, M \geq c$ s.t. $\forall\, d \geq M$ and $\forall\, x \in [a,b]$ we have $\left|F(x) - \int_c^d f(x,t)dt\right| < \epsilon/3$.

But since $f(x,t)$ is continuous on $D' = [a,b] \times [c,d]$ for any $d \geq M$, we know $\int_c^d f(x,t)dt$ is uniformly continuous on D' by Theorem 8.4.5 so $\forall\, \epsilon > 0\ \exists\, \delta > 0$ s.t. $\forall\, (x,t), (x_0,t_0) \in D'$, if $|(x,t) - (x_0,t_0)| < \delta$ then $|f(x,t) - f(x_0,t_0)| < \epsilon/(3(d-c))$.

Now let $\epsilon > 0$, pick the same M as above, let $d \geq M$, define $D' = [a,b] \times [c,d]$, choose the above δ corresponding to the aforementioned D' and ϵ, let $x, y \in [a,b]$, and let $|x-y| < \delta$, which certainly implies $|(x,t) - (y,t)| < \delta$. Then,

$$|F(x) - F(y)| = \left|F(x) - \int_c^d f(x,t)dt + \int_c^d f(x,t)dt - \int_c^d f(y,t)dt + \int_c^d f(y,t)dt - F(y)\right|$$

$$\leq \left|F(x) - \int_c^d f(x,t)dt\right| + \left|\int_c^d f(x,t)dt - \int_c^d f(y,t)dt\right| + \left|\int_c^d f(y,t)dt - F(y)\right|$$

$$\leq \left|F(x) - \int_c^d f(x,t)dt\right| + \int_c^d |f(x,t) - f(y,t)|dt + \left|\int_c^d f(y,t)dt - F(y)\right|$$

$$< \frac{\epsilon}{3} + \int_c^d \frac{\epsilon}{3(d-c)}dt + \frac{\epsilon}{3} = \epsilon$$

Theorem 8.4.9
Assume $f(x,t)$ is continuous on $D = [a,b] \times [c,\infty)$ and $F(x) = \int_c^\infty f(x,t)dt$ exists for each $x \in [a,b]$. Further assume $f_x(x,t)$ exists and is continuous on D. Then $F'(x) = \int_c^\infty f_x(x,t)dt$ provided the integral $\int_c^\infty f_x(x,t)dt$ converges uniformly.

Proof:
Since the integral converges uniformly, $\forall\, \epsilon > 0\ \exists\, M \geq c$ s.t. $\forall\, d \geq M$ and $\forall\, x \in [a,b]$ we have $\left|\int_c^\infty f_x(x,t)dt - \int_c^d f_x(x,t)dt\right| < \epsilon/3$.
But since $f_x(x,t)$ is continuous on $D' = [a,b] \times [c,d]$ for any $d \geq M$, we know $\int_c^d f_x(x,t)dt$ is uniformly continuous on D' by Theorem 8.4.5 so $\forall\, \epsilon > 0\ \exists\, \delta > 0$ s.t. $\forall\, (x,t),(x_0,t_0) \in D'$, if $|(x,t)-(x_0,t_0)| < \delta$ then $|f(x,t) - f(x_0,t_0)| < \epsilon/(3(d-c))$.
Now let $\epsilon > 0$, pick the same M as above, let $d \geq M$, define $D' = [a,b] \times [c,d]$, choose the above δ corresponding to the aforementioned D' and ϵ, let $z, x \in [a,b]$, and let $|z-x| < \delta$. Then,

$$\left|\frac{F(z)-F(x)}{z-x} - \int_c^\infty f_x(x,t)dt\right| = \left|\frac{F(z)-F(x)}{z-x} - \int_c^d f_x(x,t)dt + \int_c^d f_x(x,t)dt - \int_c^\infty f_x(x,t)dt\right|$$

$$\leq \left|\frac{F(z)-F(x)}{z-x} - \int_c^d f_x(x,t)dt\right| + \left|\int_c^d f_x(x,t)dt - \int_c^\infty f_x(x,t)dt\right|$$

$$< \left|\frac{\int_c^\infty f(z,t)dt - \int_c^\infty f(x,t)dt}{z-x} - \int_c^d f_x(x,t)dt\right| + \frac{\epsilon}{3}$$

Since $\int_c^\infty f(x,t)dt$ exists at x and z, we can use linearity of the integral,

$$= \left|\int_c^\infty \frac{f(z,t)-f(x,t)}{z-x}dt - \int_c^d f_x(x,t)dt\right| + \frac{\epsilon}{3}$$

Since $f(x,t)$ is continuous on D and $f_x(x,t)$ exists on D, by MVT there exists k between x and z such that $f(z,t) - f(x,t) = f_x(k,t)(z-x)$. But then certainly we have $|(k,t)-(x,t)| < \delta$ since $|z-x| < \delta$, therefore,

$$= \left|\int_c^\infty f_x(k,t)dt - \int_c^d f_x(x,t)dt\right| + \frac{\epsilon}{3}$$

$$= \left|\int_c^\infty f_x(k,t) - \int_c^d f_x(k,t)dt + \int_c^d f_x(k,t)dt - \int_c^d f_x(x,t)dt\right| + \frac{\epsilon}{3}$$

$$\leq \left|\int_c^\infty f_x(k,t) - \int_c^d f_x(k,t)dt\right| + \left|\int_c^d f_x(k,t)dt - \int_c^d f_x(x,t)dt\right| + \frac{\epsilon}{3}$$

As usual, $\left|\int_c^d f_x(k,t)dt - \int_c^d f_x(x,t)dt\right| = \left|\int_c^d f_x(k,t) - f_x(x,t)dt\right| \leq \int_c^d |f_x(k,t) - f_x(x,t)|dt < \int_c^d \frac{\epsilon}{3(d-c)}dt = \epsilon/3$,

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

Cauchy Criterion for Improper Integrals

We say $\int_a^\infty f(x,t)dt$ is uniformly Cauchy on A if $\forall\, \epsilon > 0\ \exists\, M > c$ s.t. $\forall\, d_1, d_2 \geq M$ and $\forall\, x \in A$ we have $\left|\int_c^{d_1} f(x,t)dt - \int_c^{d_2} f(x,t)dt\right| < \epsilon$. Then $\int_a^\infty f(x,t)dt$ converges uniformly on A if and only if $\int_a^\infty f(x,t)dt$ is uniformly Cauchy on A.

If $\int_a^\infty f(x,t)dt$ is uniformly Cauchy on A, then by viewing $\left(\int_c^n f(x,t)dt\right)_{n=1}^\infty$ as a sequence of functions of x, we see that $\left(\int_c^{c+n} f(x,t)dt\right)_{n=1}^\infty$ is uniformly Cauchy and hence uniformly convergent on A, so $\forall\, \epsilon > 0\ \exists\, N \in \mathbb{N}$ s.t. if $n \geq N$ and $x \in A$, then $\left|\int_c^{c+n} f(x,t)dt - \int_c^\infty f(x,t)dt\right| < \epsilon/2$. But $\int_a^\infty f(x,t)dt$ being uniformly Cauchy on A also implies $\forall\, \epsilon > 0\ \exists\, M > c$ s.t. for all positive integers $n_1, n_2 \geq M$ and $\forall\, x \in A$ we have $\left|\int_c^{c+n_1} f(x,t)dt - \int_c^{c+n_2} f(x,t)dt\right| < \epsilon/2$. Now pick $M = \max\{N, c+1\}$ (so $M \geq N$ and $M > c$), let $d \geq M$, and let $x \in A$. Then,

$$\left|\int_c^\infty f(x,t)dt - \int_c^d f(x,t)dt\right| \leq \left|\int_c^\infty f(x,t)dt - \int_c^{\lceil d \rceil} f(x,t)dt\right| + \left|\int_c^{\lceil d \rceil} f(x,t)dt - \int_c^d f(x,t)dt\right|$$

where $\lceil d \rceil$ is the ceiling of d. We have $\left|\int_c^\infty f(x,t)dt - \int_c^{\lceil d \rceil} f(x,t)dt\right| < \epsilon/2$ since $\left(\int_c^{c+n} f(x,t)dt\right)_{n=1}^\infty$ is uniformly convergent and $\left|\int_c^{\lceil d \rceil} f(x,t)dt - \int_c^d f(x,t)dt\right| < \epsilon/2$ since $\int_c^\infty f(x,t)dt$ is uniformly Cauchy. Hence $\left|\int_c^\infty f(x,t)dt - \int_c^d f(x,t)dt\right| < \epsilon$ as desired. The proof that $\int_a^\infty f(x,t)dt$ is uniformly Cauchy on A if $\int_a^\infty f(x,t)dt$ converges uniformly on A is a classic $\epsilon/2$ proof with the last step being

$$\left|\int_c^{d_1} f(x,t)dt - \int_c^{d_2} f(x,t)dt\right| \leq \left|\int_c^{d_1} f(x,t)dt - \int_c^d f(x,t)dt\right| + \left|\int_c^d f(x,t)dt - \int_c^{d_2} f(x,t)dt\right|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and we leave the details to you.

Weierstrass M-Test for Improper Integrals (Ex. 8.4.16)

Let $A \subseteq \mathbb{R}$, suppose $f(x,t)$ defined on $\{(x,t) : x \in A, c \leq t\}$ satisfies $|f(x,t)| \leq g(t)$ for all $x \in A$ and $\int_a^\infty g(t)dt$ converges. Then $\int_a^\infty f(x,t)dt$ converges uniformly on A.

Since $\int_a^\infty g(t)dt$ converges, for all $\epsilon > 0\ \exists\, M > c$ s.t. $\forall\, d_1, d_2 \geq M$ we have $\left|\int_c^{d_1} g(t)dt - \int_c^{d_2} g(t)dt\right| = \left|\int_{\min\{d_1,d_2\}}^{\max\{d_1,d_2\}} g(t)dt\right| = \int_{\min\{d_1,d_2\}}^{\max\{d_1,d_2\}} g(t)dt < \epsilon$ (note that since $g(t) \geq |f(x,t)| \geq 0$, we can remove the absolute value signs). But then $\left|\int_c^{d_1} f(x,t)dt - \int_c^{d_2} f(x,t)dt\right| = \left|\int_{\min\{d_1,d_2\}}^{\max\{d_1,d_2\}} f(x,t)dt\right| \leq \int_{\min\{d_1,d_2\}}^{\max\{d_1,d_2\}} |f(x,t)|dt \leq \int_{\min\{d_1,d_2\}}^{\max\{d_1,d_2\}} g(t)dt < \epsilon$ for all $x \in A$. Hence $\int_a^\infty f(x,t)dt$ converges uniformly on A.

Sources:

Abbot, Stephen. *Understanding Analysis.* 2nd ed., Springer 2015.