Evaluate the following integrals:

$$I_1 = \int_0^1 \frac{\ln(x^2+1)}{x^2+1}\,dx \qquad I_2 = \int_0^{\frac{\pi}{4}} \ln(\cos x)\,dx \qquad I_3 = \int_0^{\frac{\pi}{4}} \ln(\sin x)\,dx$$

$$I_4 = \int_0^{\frac{\pi}{4}} x\tan x\,dx \qquad I_5 = \int_0^1 \frac{\ln\left(x+\frac{1}{x}\right)}{x^2+1}\,dx$$

We will freely use the following results:

$$\int_0^{\frac{\pi}{2}} \ln(\cos x)\,dx = \int_0^{\frac{\pi}{2}} \ln(\sin x)\,dx = -\frac{\pi}{2}\ln 2$$

$$\int_0^1 \frac{\ln x}{x^2+1}\,dx = \int_0^{\frac{\pi}{4}} \ln(\tan x)\,dx = -G$$

which are proved in the documents “ln(sinx) and related integrals” and “frac{lnx}{x^2+1} and related integrals” respectively.

Remark: Originally I had set out to solve I_5, but through manipulations, substitutions, and integration by parts, I stumbled into the other integrals, mainly I_1, I_2, and I_4. However since these integrals are pleasing in their own right, they will be solved here as well.

Before embarking on the solution, we will explain why differentiation under the integral sign cannot be used to solve I_1 even though defining functions $J_1(a) = \int_0^1 \frac{\ln(a^2x^2+1)}{x^2+1}\,dx$ (noting $J_1(0)=0$) or $J_2(a) = \int_0^1 \frac{\ln(x^2+a^2)}{x^2+1}\,dx$ (noting $J_2(0) = -2G$) seems natural. Focusing on $J_1(a)$, differentiating under the integral eventually leads to $J_1'(a) = \frac{1}{2}\left(\frac{\pi a - 4\arctan a}{a^2-1}\right)$ and the $\frac{\arctan a}{a^2-1}$ term (which appears because 1 is a bound of integration) has no good closed-form antiderivative. Even substituting $x \mapsto 1/x$ prior to differentiating under the integral will not work as 1 will still end up being a bound of integration. For the same reason, $J_2(a)$ will not work either. Differentiating under the integral would have worked if the bounds of integration were 0 and ∞.

Although I_1 was rather inaccessible, I was eventually able to solve I_2. We have:

$$\int_0^{\frac{\pi}{4}} \ln(\cos x)\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln(\cos x)\,dx = \int_0^{\frac{\pi}{2}} \ln(\cos x)\,dx = -\frac{\pi}{2}\ln 2$$

But by substituting $u = \pi/2 - x$ we see that $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln(\cos x)\,dx = \int_0^{\frac{\pi}{4}} \ln(\sin x)\,dx$, so

$$\int_0^{\frac{\pi}{4}} \ln(\cos x)\,dx + \int_0^{\frac{\pi}{4}} \ln(\sin x)\,dx = -\frac{\pi}{2}\ln 2 \qquad (1)$$

However we also have

$$\int_0^{\frac{\pi}{4}} \ln(\sin x)\,dx - \int_0^{\frac{\pi}{4}} \ln(\cos x)\,dx = \int_0^{\frac{\pi}{4}} \ln(\tan x)\,dx = -G$$

Solving simultaneously with (1) eventually yields

$$\int_0^{\frac{\pi}{4}} \ln(\cos x)\,dx = \frac{1}{2}\left(G - \frac{\pi}{2}\ln 2\right) \qquad \int_0^{\frac{\pi}{4}} \ln(\sin x)\,dx = -\frac{1}{2}\left(G + \frac{\pi}{2}\ln 2\right)$$

From here, the other integrals can be solved. Using integration by parts,

$$\int_0^{\frac{\pi}{4}} \ln(\cos x)\,dx = [x\ln(\cos x)]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} x\frac{-\sin x}{\cos x}\,dx$$

$$\frac{1}{2}\left(G - \frac{\pi}{2}\ln 2\right) = -\frac{\pi}{8}\ln 2 + \int_0^{\frac{\pi}{4}} x\tan x\,dx$$

$$\int_0^{\frac{\pi}{4}} x\tan x\,dx = \frac{1}{2}\left(G - \frac{\pi}{4}\ln 2\right)$$

Returning to I_1 and substituting $x = \tan\theta$,

$$I_1 = \int_0^1 \frac{\ln(x^2+1)}{x^2+1}\,dx = \int_0^{\frac{\pi}{4}} \ln(\sec^2\theta)\,d\theta = -2\int_0^{\frac{\pi}{4}} \ln(\cos\theta)\,d\theta = -G + \frac{\pi}{2}\ln 2$$

Lastly, to solve I_5,

$$\int_0^1 \frac{\ln\left(x + \frac{1}{x}\right)}{x^2+1}\,dx = \int_0^1 \frac{\ln\left(\frac{x^2+1}{x}\right)}{x^2+1}\,dx = \int_0^1 \frac{\ln(x^2+1)}{x^2+1}\,dx - \int_0^1 \frac{\ln x}{x^2+1}\,dx$$

$$= -G + \frac{\pi}{2}\ln 2 + G = \frac{\pi}{2}\ln 2$$

More integrals on next page

Evaluate and prove

$$I = \int_0^\infty \frac{\ln(1+x)}{1+x^2}dx$$

$$J = \int_0^{\frac{\pi}{4}} \frac{x}{(\cos x + \sin x)\cos x}dx = \int_0^1 \frac{\arctan x}{1+x}dx = \int_0^1 \frac{\ln(1+x)}{1+x^2}dx$$

Starting with I, substitute $x = \tan\theta$,

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \ln(1+\tan\theta)\,d\theta = \int_0^{\frac{\pi}{2}} \ln\left(\frac{\cos\theta + \sin\theta}{\cos\theta}\right)d\theta$$

$$= \int_0^{\frac{\pi}{2}} \ln(\cos\theta + \sin\theta)\,d\theta - \int_0^{\frac{\pi}{2}} \ln(\cos\theta)\,d\theta$$

Using trig sum identity in reverse,

$$= \int_0^{\frac{\pi}{2}} \ln\left(\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)\right)d\theta + \frac{\pi}{2}\ln 2$$

$$= \int_0^{\frac{\pi}{2}} \ln(\sqrt{2})\,d\theta + \int_0^{\frac{\pi}{2}} \ln\left(\sin\left(\theta + \frac{\pi}{4}\right)\right)d\theta + \frac{\pi}{2}\ln 2$$

Substitute $x = \theta + \pi/4$,

$$= \frac{\pi}{4}\ln 2 + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \ln(\sin x)\,dx + \frac{\pi}{2}\ln 2 \tag{1}$$

We write

$$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \ln(\sin x)\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln(\sin x)\,dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \ln(\sin x)\,dx$$

Substitute $u = \pi/2 - x$ for both integrals and note $\ln(\cos u)$ is even,

$$= \int_0^{\frac{\pi}{4}} \ln(\cos u)\,du + \int_{-\frac{\pi}{4}}^{0} \ln(\cos u)\,du = 2\int_0^{\frac{\pi}{4}} \ln(\cos u)\,du = G - \frac{\pi}{2}\ln 2$$

Hence, substituting back into (1), we have

$$\int_0^\infty \frac{\ln(1+x)}{1+x^2}dx = G + \frac{\pi}{4}\ln 2$$

It should be noted that defining $I(a) = \int_0^\infty \frac{\ln(1+ax)}{1+x^2}dx$ does not work as we eventually end up with $I'(a) = \frac{1}{a^2+1}\left(-\ln a + \frac{a\pi}{2}\right)$, which has no useful antiderivative. Now we turn our attention to J. Symmetry tricks and substitutions did not seem to work for the trigonometric integral and there is not much to do with $\arctan x/(1+x)$, so we proceed as follows: substitute $x \mapsto 1/x$ to get

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}dx = \int_1^\infty \frac{\ln\left(1+\frac{1}{x}\right)}{x^2+1}dx = \int_1^\infty \frac{\ln\left(\frac{x+1}{x}\right)}{x^2+1}dx = \int_1^\infty \frac{\ln(1+x)}{1+x^2}dx - \underbrace{\int_1^\infty \frac{\ln x}{1+x^2}dx}_{G}$$

$$\Rightarrow \int_1^\infty \frac{\ln(1+x)}{1+x^2}dx - \int_0^1 \frac{\ln(1+x)}{1+x^2}dx = G$$

On the other hand,

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}dx + \int_1^\infty \frac{\ln(1+x)}{1+x^2}dx = \int_0^\infty \frac{\ln(1+x)}{1+x^2}dx = G + \frac{\pi}{4}\ln 2$$

Solving simultaneously yields

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}dx = \frac{\pi}{8}\ln 2$$

It remains to show the integral equalities. We have

$$\int_0^{\frac{\pi}{4}} \frac{x}{(\cos x + \sin x)\cos x}dx = \int_0^{\frac{\pi}{4}} \frac{x}{\left(\frac{1}{\sec x} + \frac{\tan x}{\sec x}\right)\frac{1}{\sec x}}dx$$

$$\int_0^{\frac{\pi}{4}} \frac{x\sec^2 x}{1+\tan x}dx = \int_0^1 \frac{\arctan x}{1+x}dx$$

after the substitution $x \mapsto \arctan x$. From here, use integration by parts,

$$= [\ln(1+x)\arctan x]_0^1 - \int_0^1 \frac{\ln(1+x)}{1+x^2}dx$$

$$= \frac{\pi}{4}\ln 2 - \frac{\pi}{8}\ln 2 = \frac{\pi}{8}\ln 2 = \int_0^1 \frac{\ln(1+x)}{1+x^2}dx$$

as desired.