Assorted Algebra Exercises

Exercise 1
Let $K \lhd H$ and $H \lhd G$. Show that if K is a characteristic subgroup of H (that is, $\phi(K) = K$ for every automorphism ϕ of H), then $K \lhd G$.

Exercise 2
Let R be a ring with additive identity 0 and unity 1. Let $G = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in R \right\}$.

a. Show that G is a group under matrix multiplication and hence construct a nonabelian group H of order 27. That is, explicitly define a group H and prove it is nonabelian and has order 27.

b. Let H be the group constructed in part (a). Prove that $Z(H) \cong \mathbb{Z}/3\mathbb{Z}$. Then list all the elements of $Z(H)$.

Exercise 3
Prove that $\mathbb{Z} \times G$ is cyclic if and only if G is the trivial group.

Exercise 4
Let C_{13} be the cyclic group of order 13. How many subgroups does $C_{13} \times C_{13}$ have?

Exercise 5
Prove $F = \mathbb{F}_3[X]/\langle X^4 + 1 \rangle$ is a field. Then find the multiplicative order of $X^2 + 1$ in F.

Exercise 6
Let $\mathbb{F}_q$ be a finite field (so q is the power of a prime number) and $n \in \mathbb{N}$, $n \geq 2$.

a. By considering the linear independence of columns (or rows) of matrices in $\mathrm{GL}(n, \mathbb{F}_q)$, show that $|\mathrm{GL}(n, \mathbb{F}_q)| = (q^n - 1)(q^n - q)(q^n - q^2) \ldots (q^n - q^{n-1})$.

b. Using the fundamental theorem of homomorphisms, find the order of $\mathrm{SL}(n, \mathbb{F}_q)$.

Exercise 7
Let m, n be integers greater than 1. Prove that $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ is cyclic if and only if $\gcd(m, n) = 1$. *Hint: Recall* $\gcd(m, n)\,\mathrm{lcm}(m, n) = mn$ *for positive integers* m, n.

Exercise 8

Let D_n be the dihedral group of order $2n$ and let C_n be the cyclic group of order n. You may assume that $C_8, C_4 \times C_2, C_2 \times C_2 \times C_2, D_4$, and Q_8 are the only groups of order 8 up to isomorphism. Determine how many copies of each of these groups are inside $C_4 \times D_3$ and hence find all order 8 subgroups of $C_4 \times D_3$. Are they normal?

If students have not yet learned the Sylow theorems, they may use the hints below.

Hint 1: Consider the orders of elements and/or subgroups in $C_4 \times D_3$.

Hint 2: Note that $Q_8, D_4,$ and C_8 all contain C_4.

Exercise 9

Recall that the group $\mathbb{Z}/n\mathbb{Z}$ is generated by $k \in \mathbb{Z}/n\mathbb{Z}$ if and only if $\gcd(k,n) = 1$. Use this result to prove the following statement, which is related to Bezout's lemma.

Let m, n be positive integers such that $\gcd(m,n) = 1$. Then there exist integers $a, b \in \mathbb{Z}$ (not necessarily positive) such that $am + bn = 1$.

Exercise 10

Let F be a field. Let $G = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in F \right\}$, which we know is a group under matrix multiplication by Exercise 2.

a. Let $m \in \mathbb{Z}^+$ and $x, y, z \in F$ where F is any field such that $\mathrm{char}(F) \neq 2$. Prove that:

$$\begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}^m = \begin{bmatrix} 1 & mx & my + \frac{1}{2}(m-1)mxz \\ 0 & 1 & mz \\ 0 & 0 & 1 \end{bmatrix}$$

b. Let F be any field such that $\mathrm{char}(F) \neq 2$ and let $n \in \mathbb{Z}$. Find the necessary and sufficient conditions on n such that the map $\Phi_n: G \to G$, $\Phi_n(g) = g^n$ is

   i. a bijection.

   ii. an automorphism.

c. Let $p > 2$ be prime and take $F = \mathbb{F}_p$.

   i. Prove that all nonidentity elements of G have order p. What are all the proper subgroups of G up to isomorphism?

   ii. For $n \in \mathbb{Z}$, let $G_n = \{g^n : g \in G\}$. Prove G_n is a subgroup of G for all $n \in \mathbb{Z}$.

Exercise 1

Let $g \in G, k \in K$. Consider the inner automorphism $\phi: G \to G, \phi(t) = g^{-1}tg$. Then $\phi|_H$, which we denote φ, is an automorphism of H and thus $\varphi(K) = K$ since K is a characteristic subgroup of H. Note φ^{-1} is also an automorphism of H so $\varphi^{-1}(K) = K$ as well. Then $\varphi(gkg^{-1}) = \varphi(g)\varphi(k)\varphi(g^{-1}) = g(g^{-1}kg)g^{-1} = k$. Applying φ^{-1} to both sides, $gkg^{-1} = \varphi^{-1}(k) \in K$. Hence $K \lhd G$.

Exercise 2

Taking $a = b = c = 0$ we see the identity matrix is in G and is clearly the identity of G. To prove closure, note $\begin{bmatrix} 1 & a_1 & b_1 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a_2 & b_2 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a_2 + a_1 & b_2 + a_1c_2 + b_1 \\ 0 & 1 & c_2 + c_1 \\ 0 & 0 & 1 \end{bmatrix} \in G$. To prove inverse exists, we can take $\begin{bmatrix} 1 & a_2 + a_1 & b_2 + a_1c_2 + b_1 \\ 0 & 1 & c_2 + c_1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ in the above matrix equation and solve for a_2, b_2, c_2 to see that $\begin{bmatrix} 1 & -a_1 & a_1c_1 - b_1 \\ 0 & 1 & -c_1 \\ 0 & 0 & 1 \end{bmatrix} \in G$ is the inverse of $\begin{bmatrix} 1 & a_1 & b_1 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{bmatrix}$. Matrix multiplication is associative. Hence G is a group.

Now define $H = \left\{\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{F}_3\right\}$, which we now know is a group. Since each a, b, c could be $0, 1,$ or 2, we have $|H| = 3 \times 3 \times 3 = 27$.

Now, since $[H : Z(H)]$ divides $|H| = 27$ (by Lagrange's theorem) and $[H : Z(H)]$ is not prime (from class equation), it must be that either $[H : Z(H)] = 9 \Leftrightarrow |Z(H)| = 3$ or $[H : Z(H)] = 27 \Leftrightarrow |Z(H)| = 1$. But $|H|$ being a prime power implies $Z(H)$ is nontrivial (also from class equation), so in fact we must have $|Z(H)| = 3$. It follows that H is nonabelian and $Z(H) \cong \mathbb{Z}/3\mathbb{Z}$. Alternatively, one could show H is nonabelian by noting $\begin{bmatrix} 1 & a_1 & b_1 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a_2 & b_2 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a_2 & b_2 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a_1 & b_1 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & a_2 + a_1 & b_2 + a_1c_2 + b_1 \\ 0 & 1 & c_2 + c_1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a_1 + a_2 & b_1 + a_2c_1 + b_2 \\ 0 & 1 & c_1 + c_2 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow a_1c_2 = a_2c_1$, so if we pick numbers like $a_1 = c_2 = a_2 = 1$ and $a_2 = 2$, the corresponding matrices will not commute, hence H is nonabelian.

Then, either by observation or by playing with the equation $a_1c_2 = a_2c_1$ from before, we see that $\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in Z(H)$ for all $b \in \mathbb{F}_3$, and there are precisely three matrices of this form in H. Since we already showed $|Z(H)| = 3$, it must be that

$$Z(H) = \left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

Exercise 3
If G is trivial then clearly $\mathbb{Z} \times G$ is cyclic (since $\mathbb{Z}$ is cyclic). To prove the other implication, let $\mathbb{Z} \times G$ be cyclic and suppose towards a contradiction G is nontrivial. Then any generator of $\mathbb{Z} \times G$ must be of the form (m, g) where $\langle g \rangle = G$ and $\langle m \rangle = \mathbb{Z}$, and to see why, suppose (m, g) generates $\mathbb{Z} \times G$ but either $\langle g \rangle \neq G$ or $\langle m \rangle \neq \mathbb{Z}$. If $\langle g \rangle \neq G$, then there exists some $k \in G$ that is not a power of g, so then $(0, k)$ certainly could not be generated by (m, g), contradicting (m, g) being a generator. If $\langle m \rangle \neq \mathbb{Z}$, the same contradiction arises.
So any generator of $\mathbb{Z} \times G$ must be $(\pm 1, g)$ where $\langle g \rangle = G$ (since ± 1 are the only generators of $\mathbb{Z}$). If the generator is $(1, g)$, then consider $(2, g) \in \mathbb{Z} \times G$. So there exists $n \in \mathbb{Z}$ such that $(1, g)^n = (2, g)$, so $n(1) = 2$ and $g^n = g$. But $n(1) = 2$ implies $n = 2$, so $g^2 = g$, which implies $g = e$. But since $G = \langle g \rangle$, this implies G is trivial, a contradiction. If the generator is $(-1, g)$, then consider $(-2, g) \in \mathbb{Z} \times G$ and the same contradiction will arise.
Incidentally, a slightly stronger result says that if G and H are groups and G is infinite, then $G \times H$ is cyclic if and only if H is trivial and G is cyclic (so $G \cong \mathbb{Z}$).

Exercise 4
Since $|C_{13} \times C_{13}| = 169$, by Lagrange's theorem every nonidentity element must have order 13 and thus generates some copy of C_{13} (if an element had order 169 then the whole group would be C_{169}, which is obviously not $C_{13} \times C_{13}$). There are 168 nonidentity elements and every copy of C_{13} has 12 nonidentity elements, so in all there must be $168/12 = 14$ copies of C_{13} inside $C_{13} \times C_{13}$. Counting the trivial group and the whole group itself, there are 16 subgroups total. In general, $C_p \times C_p$ has $p + 3$ subgroups for p prime.

Exercise 5
To show F is a field, it suffices to show $f(X) = X^4 + 1$ is irreducible over $\mathbb{F}_3$. Since $f(0) = 1, f(1) = 2, f(2) = 2$, there are no roots of f in $\mathbb{F}_3$, so if there is a nontrivial factoring of f it must be of two quadratic factors. Any such factoring must be $X^4 + 1 = (X^2 + aX + 1)(X^2 + bX + 1) = X^4 + (a + b)X^3 + (ab + 2)X^2 + (a + b)X + 1$ and equating coefficients leads to $a + b = 0$ and $ab + 2 = 0$. Solving simultaneously yields $b^2 = 2$ (or $a^2 = 2$) but it is easily verified that 2 has no square roots in $\mathbb{F}_3$, so there is no quadratic factoring of f. Hence f is irreducible over $\mathbb{F}_3$, so F is the quotient of a ring by a maximal ideal, so F is a field.
We have $(X^2 + 1)^2 = X^4 + 2X^2 + 1 = 2X^2$ (since $X^4 + 2X^2 + 1 = f(X) + 2X^2$). But we also have $(2X^2)^2 = X^4 = 2$ (since $X^4 = f(X) + 2$). Then $(X^2 + 1)^8 = ((X^2 + 1)^2)^4 = (2X^2)^4 = ((2X^2)^2)^2 = 2^2 = 1$. Then the order of $X^2 + 1$ must divide 8. But we just found that $(X^2 + 1)^2 = 2X^2 \neq 1$ and $(X^2 + 1)^4 = 2 \neq 1$, so $n = 8$ is indeed the smallest integer such that $(X^2 + 1)^n = 1$. Hence $X^2 + 1$ has order 8.

Exercise 6
For a matrix A to be in $\mathrm{GL}(n, \mathbb{F}_q)$, it must be invertible, so all its columns (or rows, since row rank equals column rank) must be linearly independent. Consider the first column of A. Each entry has q options but we cannot have all entries being zero, so there are a total of $q^n - 1$ possibiltiies for the first column. The second column must be independent of the first column, which means it cannot be a scalar multiple. There are q scalars in $\mathbb{F}_q$ (including 0) so there are q scalar multiples of the first column (including the zero vector) that we cannot have as the second column, so there are $q^n - q$ possibilities for the second column. The third column cannot be a combination of the previous two columns, but again there are q multiples of each of the previous columns, so there are q^2 different linear combinations of the previous two columns, hence $q^n - q^2$ possibilities for the third column. In general, there are q^k linear combinations of q columns and thus $q^n - q^k$ possibilities for the k^{th} column, so $|\mathrm{GL}(n, \mathbb{F}_q)| = (q^n - 1)(q^n - q) \dots (q^n - q^{n-1})$. Consider now the function $\det\colon \mathrm{GL}(n, \mathbb{F}_q) \to \mathbb{F}_q^*$. It is well-defined since matrices from $\mathrm{GL}(n, \mathbb{F}_q)$ are invertible and thus have nonzero determinant, it is a homomorphism since the $\det$ function is multiplicative, and it is onto since for any $k \in \mathbb{F}_q^*$, the $n \times n$ diagonal matrix $(a_{ij}) \in \mathrm{GL}(n, \mathbb{F}_q)$ with $a_{11} = k$ and $a_{ii} = 1$ for all $i \neq 1$ has determinant k. Then $A \in \ker(\det) \Leftrightarrow \det(A) = 1 \Leftrightarrow A \in \mathrm{SL}(n, \mathbb{F}_q)$. Then by fundamental theorem of group homomorphism, we have $\mathrm{GL}(n, \mathbb{F}_q)/S\mathrm{L}(n, \mathbb{F}_q) \cong \mathbb{F}_q^*$, so

$$|\mathrm{SL}(n, \mathbb{F}_q)| = \frac{|\mathrm{GL}(n, \mathbb{F}_q)|}{|\mathbb{F}_q^*|} = \frac{(q^n - 1)(q^n - q) \dots (q^n - q^{n-1})}{q - 1}$$

Exercise 7
Suppose $\gcd(m, n) = 1$, so $\mathrm{lcm}(m, n) = mn$. We first show $(1, 1)$ generates $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. Note $(1, 1)^{mn} = \big((mn) \bmod m, (mn) \bmod n\big) = (0, 0)$ hence $o\big((1, 1)\big)$ divides mn. To show $o\big((1, 1)\big) = mn$, it suffices to prove $(1, 1)^k \neq 0$ for any divisor k of mn where $k \neq mn$. So let k be such a divisor. Case 1: If $0 < k < m$, then $(1, 1)^k = (k, k \bmod n) \neq (0, 0)$. Case 2: If $0 < k < n$, then $(1, 1)^k = (k \bmod m, k) \neq (0, 0)$. Case 3: If $k \geq m$ and $k \geq n$, suppose towards a contradiction that $(1, 1)^k = (0, 0)$. Then $(k \bmod m, k \bmod n) = (0, 0)$, so $k \bmod m = k \bmod n = 0$. Then $m | k$ and $n | k$. But since $k \geq m$ and $k \geq n$ (and $k \neq mn$ by assumption) we have $\mathrm{lcm}(m, n) \leq k < mn$. But this contradicts $\mathrm{lcm}(m, n) = mn$ from the start. Hence $o\big((1, 1)\big) = mn$, so $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ is indeed generated by $(1, 1)$ and thus cyclic (and also isomorphic to $\mathbb{Z}/(mn)\mathbb{Z}$ since they are cyclic groups of the same order).
Suppose $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ is cyclic and generated by (a, b), so $o\big((a, b)\big) = mn$. Suppose towards a contradiction that $\gcd(m, n) > 1$. Then $\mathrm{lcm}(m, n) = mn/\gcd(m, n) < mn$. Certainly $(a, b)^{\mathrm{lcm}(m,n)} = (0, 0)$ since $\mathrm{lcm}(m, n)$ is a multiple of both m and n, but then mn is no longer the smallest positive integer k such that $(a, b)^k = (0, 0)$, contradicting $o\big((a, b)\big) = mn$. So it must be that $\gcd(m, n) = 1$.

Exercise 8
We will use $\mathbb{Z}/4\mathbb{Z}$ to represent C_4 and $D_3 = \langle f, g : f^3 = g^2 = e, gf = f^2 g\rangle$. It is apparent enough that $C_4 \times \langle f^k g\rangle$ is a subgroup isomorphic to $C_4 \times C_2$ for $k = 0, 1, 2$, and there are no other copies of $C_4 \times C_2$ since D_3 has no C_4 subgroups and $\langle f^k g\rangle$, $k = 0, 1, 2$ are its only C_2 subgroups. So there are precisely three copies of $C_4 \times C_2$ in $C_4 \times D_3$.
EITHER:
Since $|C_4 \times D_3| = 24$, the order 8 subgroups $C_4 \times \langle f^k g\rangle$, $k = 0, 1, 2$ are all Sylow 2-subgroups. Since Sylow p-subgroups are conjugate (and thus isomorphic under some inner automorphism), there can be no order 8 subgroups that are not isomorphic to $C_4 \times C_2$.
OR:
To show there are no copies of $C_2 \times C_2 \times C_2$, note that all seven nonidentity elements of $C_2 \times C_2 \times C_2$ have order 2. We find that $(2, e), (2, f^k g), (0, f^k g)$, $k = 0, 1, 2$ are the only order 2 elements in $C_4 \times D_3$ and there are seven of them, so any $C_2 \times C_2 \times C_2$ must consist of these seven elements and $(0, e)$. But the collection of those seven elements is not closed since composing any two distinct reflections $f^k g$ leads to a rotation, for instance, we have $(0, g) * (0, fg) = (0, f^2)$. Hence there are no copies of $C_2 \times C_2 \times C_2$ in $C_4 \times D_3$. Of course, there are other methods.
To show there are no copies of Q_8, D_4, or C_8, note that Q_8, D_4, and C_8 all contain C_4. But any C_4 inside $C_4 \times D_3$ must be $C_4 \times H$ for some $H \leq D_3$ since D_3 has no elements of order 4. If $C_4 \times H$ were to equal Q_8, D_4, or C_8, then H would have to be order 2 for $C_4 \times H$ to have order 8, so $H = C_2$, but $C_4 \times C_2$ is not equal to Q_8, D_4, or C_8. So there are no copies of Q_8, D_4, or C_8 in $C_4 \times D_3$. Of course, C_8 can also be handled by arguing there are no elements of order 8 in $C_4 \times D_3$ (since C_4 and D_3 themselves have no elements of order 8), and Q_8 in particular can be handled with relative ease by considering its elements of order 4.
THEN:
Consequently, the only order 8 subgroups of $C_4 \times D_3$ are $C_4 \times \langle f^k g\rangle$ for $k = 0, 1, 2$, all isomorphic to $C_4 \times C_2$. None of them are normal as $(0, f^{k+1}g)(C_4 \times \langle f^k g\rangle)(0, f^{k+1}g)^{-1} = (0, f^{k+1}g)(C_4 \times \langle f^k g\rangle)(0, gf^{-k-1})$ which contains $(0, f^{k+1}gf^k ggf^{-k-1}) = (0, f^{k+2}g)$, but $(0, f^{k+2}g) \notin C_4 \times \langle f^k g\rangle$.

Exercise 9
Let $\gcd(m, n) = 1$. If $m = n$, then we must have $m = n = 1$, upon which $1m + 0n = 1$ and we are done. So suppose $m \neq n$, and without loss of generality suppose $n > m$ (so $m \in \mathbb{Z}/n\mathbb{Z}$). By Euclidean division we can write $m = qn + r$ for $q \in \mathbb{Z}$ and $0 \leq r < n$. Now consider $r + 1$, which must be in the set $\{1, 2, \dots, n\}$. If $r + 1 = n$, then $m = qn + n - 1$, so $-m + (q + 1)n = 1$ and we are done. If $r + 1 \in \{1, 2, \dots, n - 1\}$ then $r + 1 \in \mathbb{Z}/n\mathbb{Z}$, but $\gcd(m, n) = 1$ and $m \in \mathbb{Z}/n\mathbb{Z}$ together imply m generates $\mathbb{Z}/n\mathbb{Z}$, so there exists $a \in \mathbb{Z}$ such that $am = r + 1$. Then $m = qn + am - 1$, so $(a - 1)m + qn = 1$ and we are done.

Incidentally the converse statement is true and the proof is as follows: let d be a divisor of m and n. Then $d|m$ and $d|n$, so d divides $am + bn = 1$, so $d = 1$. Then $\gcd(m, n) = 1$.

Exercise 10

We prove part (a) by induction. The base case $m = 1$ is easily verified. Assuming the formula holds for $m = k$, we have:

$$\begin{bmatrix}1 & x & y\\ 0 & 1 & z\\ 0 & 0 & 1\end{bmatrix}^{k+1} = \begin{bmatrix}1 & kx & ky+(k-1)kxz/2\\ 0 & 1 & kz\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & x & y\\ 0 & 1 & z\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & x+kx & y+kxz+ky+(k-1)kxz/2\\ 0 & 1 & z+kz\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & (k+1)x & (k+1)y+k(k+1)xz/2\\ 0 & 1 & (k+1)z\\ 0 & 0 & 1\end{bmatrix}$$

completing the induction. The condition $\operatorname{char} F \neq 2$ ensures we can divide by 2.

Now for part (b). We will handle the case $\operatorname{char} F > 2$ first and the case $\operatorname{char} F = 0$ will quickly follow. So let $p = \operatorname{char} F > 2$. Note that if $n \bmod p \equiv 0$, then $\Phi_n(g) = g^n$ is the identity element for all $g \in G$ so Φ_n is not bijective. Also note that $\Phi_{-n} = \Phi_n \circ \Phi_{-1}$ for all $n > 0$, but Φ_{-1} is the inversion map which is an automorphism, hence Φ_{-n} is a bijection if and only if Φ_n is a bijection. Then it suffices to check if Φ_n is an automorphism and/or bijection for $n > 0$, $n \bmod p \neq 0$, so let n be such. We show injectivity first. Let $g_1 = \begin{bmatrix}1 & a_1 & b_1\\ 0 & 1 & c_1\\ 0 & 0 & 1\end{bmatrix}$ and $g_2 = \begin{bmatrix}1 & a_2 & b_2\\ 0 & 1 & c_2\\ 0 & 0 & 1\end{bmatrix}$ be arbitrary in G. Then $\Phi_n(g_1) = \Phi_n(g_2)$ implies $\begin{bmatrix}1 & a_1 & b_1\\ 0 & 1 & c_1\\ 0 & 0 & 1\end{bmatrix}^n = \begin{bmatrix}1 & a_2 & b_2\\ 0 & 1 & c_2\\ 0 & 0 & 1\end{bmatrix}^n$, and applying part (a) and subsequently equating entries yields $na_1 = na_2$, $nc_1 = nc_2$, and $nb_1 + (n-1)na_1c_1/2 = nb_2 + (n-1)na_2c_2/2$. Since $n \bmod p \neq 0$, we may divide by n in F, allowing us to conclude $a_1 = a_2$ and $c_1 = c_2$, which quickly implies $b_1 = b_2$. Hence $g_1 = g_2$ so Φ_n is injective. A quick calculation using part (a) shows $\begin{bmatrix}1 & a/n & \frac{b}{n} - \frac{n-1}{2n^2}ac\\ 0 & 1 & c/n\\ 0 & 0 & 1\end{bmatrix}^n = \begin{bmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{bmatrix}$ for any $\begin{bmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{bmatrix} \in G$, proving surjectivity. Hence, if $\operatorname{char} F > 2$, then Φ_n is bijective if and only if $n \bmod (\operatorname{char} F) \neq 0$.

Now for part (b)(ii). If $n \bmod p \equiv \pm 1$, then Φ_n is the identity and inversion map respectively, both of which are automorphisms. Clearly Φ_n is not an automorphism if $n \bmod p = 0$ since it would not even be bijective. Also note again that if $n > 0$, then $\Phi_{-n} = \Phi_n \circ \Phi_{-1}$ implies Φ_{-n} is an automorphism if and only if Φ_n is too. Then let $n > 0$, $n \bmod p \notin \{0, 1, p-1\}$. Let $g_1 = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix} \in G$ and $g_2 = \begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \in G$. Then $\Phi_n(g_1)\Phi_n(g_2) = \begin{bmatrix}1 & 0 & n\\ 0 & 1 & n\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & n & n\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & n & 2n\\ 0 & 1 & n\\ 0 & 0 & 1\end{bmatrix}$ which is not equal to

$$\begin{bmatrix}1 & n & 2n+(n-1)n/2\\0 & 1 & n\\0 & 0 & 1\end{bmatrix} = \Phi_n\left(\begin{bmatrix}1 & 1 & 2\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}\right) = \Phi_n(g_1g_2) \text{ given our conditions on } n.$$

Hence, if $\operatorname{char} F > 2$, then Φ_n is an automorphism if and only if $n \bmod (\operatorname{char} F) \equiv \pm 1$.
To see what happens if $\operatorname{char} F = 0$ (that is, $F \supseteq \mathbb{Q}$), note that the above proof works just the same but we no longer have to reduce anything modulo $\operatorname{char} F$. Hence if $\operatorname{char} F = 0$, then Φ_n is bijective if and only if $n \neq 0$ and Φ_n is an automorphism if and only if $n = \pm 1$.
Now for part (c)(i). The matrices in G are now over $\mathbb{F}_p$, so $|G| = p^3$. By part (a), we have

$$\begin{bmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{bmatrix}^p = \begin{bmatrix}1 & pa & pb+(p-1)pac/2\\0 & 1 & pc\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

for all $a, b, c \in \mathbb{F}_p$ (note $2 \in \mathbb{F}_p^*$ since $p > 2$). Hence the order of any element in G divides p. Then the order of any element in G must be p or 1, and it follows that every nonidentity element in G has order p. To find all the subgroups of G up to isomorphism, note that by Lagrange's theorem every nontrivial proper subgroup of G must be isomorphic to C_p or some group of order p^2. Recalling that any group of order p^2 must be abelian when p is prime, the only groups of order p^2 are $C_p \times C_p$ and C_{p^2} (classification of finite abelian groups). But we have already shown every element in G has order p, so there can be no element of order p^2, hence no copy of C_{p^2} in G. The Sylow theorems guarantee the existence of subgroups of orders p and p^2. Hence all the proper subgroups of G up to isomorphism are the trivial group, C_p, and $C_p \times C_p$.
Now for part (c)(ii). Since all nonidentity elements in G have order p, we have $g^n = g^{n+p}$ for all $g \in G$, implying $G_n = G_{n+p}$ for all $n \in \mathbb{Z}$. Then it suffices to show $G_n \leq G$ only for $n = 0, 1, \dots, p-1$. The case $n = 0$ is clear since G_0 is trivial, so let $n \in \{1, 2, \dots, p-1\}$. Then the map $\Phi_n : G \to G$, $\Phi_n(g) = g^n$ is bijective by part (b), that is, the map Φ_n simply permutes the elements of G, hence $G_n = G \leq G$, and since $n \in \{1, 2, \dots, p-1\}$ was arbitrary, we are done.