Assuming they converge, evaluate the following integrals: Wenzhi Tseng

$$I = \int_0^1 \frac{\ln^2 x}{x^2-1}\,dx \qquad\qquad J = \int_0^{\frac{\pi}{2}} \theta \ln(\cos\theta)\,d\theta$$

We first solve I. Using partial fractions,

$$\int \frac{\ln^2 x}{x^2-1}\,dx = \frac{1}{2}\left(\int \frac{\ln^2 x}{x-1}\,dx - \int \frac{\ln^2 x}{x+1}\,dx\right)$$

Substitute $u = x - 1$ and $u = x + 1$ in the first and second integrals respectively,

$$= \frac{1}{2}\left(\int \frac{\ln^2(u+1)}{u}\,du - \int \frac{\ln^2(u-1)}{u}\,du\right)$$

Now integrate by parts. Note $x \in (0,1)$ implies $u \in (-1,0)$, forcing $\int 1/u\,du = \ln(-u)$ in order for our antiderivative to be valid for $x \in (0,1)$,

$$= \left[\frac{\ln(-u)\ln^2(u+1)}{2} - \int \frac{\ln(-u)\ln(u+1)}{u+1}\,du\right] - \left[\frac{\ln u \ln^2(u-1)}{2} - \int \frac{\ln u \ln(u-1)}{u-1}\,du\right]$$

Undoing the substitutions in the first and third terms and substituting $u = v - 1$ and $u = 1 - v$ in the integrals,

$$= \frac{1}{2}(\ln^2 x)\ln\left(\frac{1-x}{1+x}\right) + \int \frac{-\ln(1-v)\ln(v)}{v}\,dv - \int \frac{-\ln(1-v)\ln(-v)}{v}\,dv$$

From here on we make use of polylogarithms, see appendix for a brief review. We again use integration by parts,

$$= \frac{1}{2}(\ln^2 x)\ln\left(\frac{1-x}{1+x}\right) + \left[\big(\mathrm{Li}_2(v)\big)\ln(v) - \int \frac{\mathrm{Li}_2(v)}{v}\,dv\right] - \left[\big(\mathrm{Li}_2(v)\big)\ln(-v) - \int \frac{\mathrm{Li}_2(v)}{v}\,dv\right]$$

$$= \frac{1}{2}(\ln^2 x)\ln\left(\frac{1-x}{1+x}\right) + \big(\mathrm{Li}_2(v)\big)\ln(v) - \mathrm{Li}_3(v) - \big(\mathrm{Li}_2(v)\big)\ln(-v) + \mathrm{Li}_3(v)$$

Undoing the substitutions,

$$= \frac{1}{2}(\ln^2 x)\ln\left(\frac{1-x}{1+x}\right) + (\ln x)\big(\mathrm{Li}_2(x) - \mathrm{Li}_2(-x)\big) + \mathrm{Li}_3(-x) - \mathrm{Li}_3(x) + c,\ c \in \mathbb{R}$$

Let the above antiderivative be $F(x)$. So we need to compute $\lim_{x\to 1} F(x) - \lim_{x\to 0} F(x)$. Note that $\mathrm{Li}_s(x)$ has the power series expansion $\mathrm{Li}_s(x) = \sum_{k=1}^{\infty} x^k/k^s$ for $|x| \le 1$, and the sum converges absolutely by ratio test for $s > 0$. Then, we are justified in the following series manipulations:

$$\mathrm{Li}_s(x) - \mathrm{Li}_s(-x) = \left(x + \frac{x^2}{2^s} + \frac{x^3}{3^s} + \cdots\right) - \left(-x + \frac{x^2}{2^s} + \frac{x^3}{3^s} - \cdots\right) = 2\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)^s}$$

Then,

$$F(x) = \frac{1}{2}(\ln^2 x)\ln\left(\frac{1-x}{1+x}\right) + 2(\ln x)\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)^2} - 2\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)^3} + c$$

We now compute the limits required to find $\lim_{x\to 0} F(x)$. The most troublesome limit is

$$\lim_{x\to 0}(\ln x)\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)^2} = \lim_{x\to 0}\frac{\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)^2}}{1/\ln x}$$

The sum clearly converges at $x = 1$, so it converges absolutely on $[-1, 1]$ and is thus infinitely differentiable and term-by-term differentiation is allowed and all derivatives will converge on $(0, 1)$. Then, by L'Hopital's,

$$= \lim_{x\to 0}\frac{\sum_{k=0}^{\infty}\frac{x^{2k}}{2k+1}}{-1/(x\ln^2 x)} = \lim_{x\to 0}\frac{\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}}{-1/\ln^2 x} = \lim_{x\to 0}\frac{\sum_{k=0}^{\infty}x^{2k}}{2/(x\ln^3 x)} = \frac{1}{2}\lim_{x\to 0}\frac{x\ln^3 x}{1-x^2} = 0$$

since $\lim_{x\to 0} x\ln^3 x = 0$ (also by repeated L'Hopital's).

It can also be shown that $\lim_{x\to 0}(\ln^2 x)\ln\left(\frac{1-x}{1+x}\right) = \lim_{x\to 1}(\ln^2 x)\ln\left(\frac{1-x}{1+x}\right) = 0$ using repeated L'Hopital's. Putting everything together, we have $\lim_{x\to 0} F(x) = 0$ and thus

$$\int_0^1 \frac{\ln^2 x}{x^2-1}\,dx = \lim_{x\to 1} F(x) - \lim_{x\to 0} F(x) = -2\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}$$

But notice,

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3} + \sum_{k=1}^{\infty}\frac{1}{(2k)^3} = \sum_{k=1}^{\infty}\frac{1}{k^3} = \zeta(3)$$

So the second sum is just $\zeta(3)/8$. Then $\sum_{k=0}^{\infty} 1/(2k+1)^3 = 7\zeta(3)/8$, so

$$\int_0^1 \frac{\ln^2 x}{x^2-1}\,dx = -\frac{7}{4}\zeta(3)$$

Now we solve J. We will freely use the following result, which is proved (using symmetry tricks) in my document "ln(sinx) and related integrals"

$$\int_0^{\frac{\pi}{2}} \ln(\sin\theta)\,d\theta = \int_0^{\frac{\pi}{2}} \ln(\cos\theta)\,d\theta = -\frac{\pi}{2}\ln 2 \tag{1}$$

Substituting $\theta \mapsto \pi/2 - \theta$ and using law of logs,

$$J = \int_0^{\frac{\pi}{2}} \theta\ln(\cos\theta)\,d\theta = \int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2}-\theta\right)\ln(\sin\theta)\,d\theta = \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\ln(\sin\theta)\,d\theta - \int_0^{\frac{\pi}{2}}\theta\ln(\sin\theta)\,d\theta$$

Rearranging and using (1),

$$\Rightarrow \int_0^{\frac{\pi}{2}} \theta\ln(\cos\theta)\,d\theta + \int_0^{\frac{\pi}{2}} \theta\ln(\sin\theta)\,d\theta = -\frac{\pi^2}{4}\ln 2 \tag{2}$$

On the other hand, by laws of logs,

$$\int_0^{\frac{\pi}{2}} \theta \ln(\sin\theta)\, d\theta - \int_0^{\frac{\pi}{2}} \theta \ln(\cos\theta)\, d\theta = \int_0^{\frac{\pi}{2}} \theta \ln(\tan\theta)\, d\theta \tag{3}$$

Define the following function for $t \geq 0$,

$$J(t) = \int_0^{\frac{\pi}{2}} (\arctan(t\tan\theta)) \ln(\tan\theta)\, d\theta$$

Now differentiate under the integral. We skip over verifying the convergence/uniform convergence of the relevant integrals.

$$\Rightarrow J'(t) = \int_0^{\frac{\pi}{2}} \frac{\tan\theta}{1 + t^2\tan^2\theta} \ln(\tan\theta)\, d\theta$$

Substitute $x = \tan\theta$,

$$= \int_0^{\infty} \frac{x\ln x}{(1+t^2x^2)(1+x^2)}\, dx$$

This integral suggests contour integration due to the polynomial in the denominator and the rather straightforward function $x\ln x$ in the numerator. We choose $\log z$ with branch cut $[0,\infty)$ and $\arg z \in (0, 2\pi)$ and integrate the function $f(z) = \frac{z\log^2 z}{(1+t^2z^2)(1+z^2)}$ over the corresponding keyhole contour. For $t \neq 0$ and $t \neq 1$, note f has four simple poles $\pm i, \pm i/t$ on the imaginary axis inside the contour. By residue theorem, we find

$$\int_0^{\infty} \frac{x(\ln x)^2}{(1+t^2x^2)(1+x^2)}\, dx - \int_0^{\infty} \frac{x(\ln x + 2\pi i)^2}{(1+t^2x^2)(1+x^2)}\, dx = 2\pi i \sum \operatorname{Res}$$

$$\Rightarrow -4\pi i \int_0^{\infty} \frac{x\ln x}{(1+t^2x^2)(1+x^2)}\, dx + 4\pi^2 \int_0^{\infty} \frac{x}{(1+t^2x^2)(1+x^2)}\, dx = 2\pi i \frac{\ln^2 t - 2\pi i \ln t}{t^2 - 1}$$

Equating real and imaginary parts yields

$$\int_0^{\infty} \frac{x\ln x}{(1+t^2x^2)(1+x^2)}\, dx = \frac{\ln^2 t}{2(1-t^2)} \qquad \int_0^{\infty} \frac{x}{(1+t^2x^2)(1+x^2)}\, dx = \frac{\ln t}{t^2 - 1}$$

So $J'(t) = \frac{\ln^2 t}{2(1-t^2)}$ for $t > 0, t \neq 1$. But note $\lim_{t\to 1} \frac{\ln^2 t}{2(1-t^2)} = 0$ by L'Hopital's so we can extend $J'(t) = \int_0^{\infty} \frac{x\ln x}{(1+t^2x^2)(1+x^2)}\, dx$ to a continuous function on $(0,\infty)$ where we define $J'(1) = 0$.

The antiderivative of $\frac{\ln^2 t}{2(1-t^2)}$ was computed previously,

$$\Rightarrow J(t) = -\frac{1}{2}\left(\frac{1}{2}(\ln^2 t)\ln\left|\frac{1-t}{1+t}\right| + (\ln t)\big(\operatorname{Li}_2(t) - \operatorname{Li}_2(-t)\big) - \operatorname{Li}_3(t) + \operatorname{Li}_3(-t)\right) + c$$

for some $c \in \mathbb{R}$. However, since $J(0) = \int_0^{\frac{\pi}{2}} (\arctan(0\tan\theta)) \ln(\tan\theta)\, d\theta = 0$, taking $t \to 0$ in the above expression (also done previously) and solving for c shows that $c = 0$.

Recall our original goal was to find the value of $J(1)$ and substitute it into (3). The computation of $J(1)$ was also done previously, so we have

$$\int_0^{\frac{\pi}{2}} \theta \ln(\tan\theta)\, d\theta = J(1) = \frac{7}{8}\zeta(3)$$

Look familiar? That's right. Our integrals I and J are related:

$$\int_0^1 \frac{\ln^2 x}{x^2-1}\, dx = -2\int_0^{\frac{\pi}{2}} \theta \ln(\tan\theta)\, d\theta = -\frac{7}{4}\zeta(3)$$

How coincidental. But now we finish this question. Returning to (3), we now have

$$\int_0^{\frac{\pi}{2}} \theta \ln(\sin\theta)\, d\theta - \int_0^{\frac{\pi}{2}} \theta \ln(\cos\theta)\, d\theta = \frac{7}{8}\zeta(3)$$

And from (2), we had

$$\int_0^{\frac{\pi}{2}} \theta \ln(\cos\theta)\, d\theta + \int_0^{\frac{\pi}{2}} \theta \ln(\sin\theta)\, d\theta = -\frac{\pi^2}{4}\ln 2$$

Solving simultaneously yields:

$$\int_0^{\frac{\pi}{2}} \theta \ln(\sin\theta)\, d\theta = \boxed{\frac{7}{16}\zeta(3) - \frac{\pi^2}{8}\ln 2} \qquad \int_0^{\frac{\pi}{2}} \theta \ln(\cos\theta)\, d\theta = \boxed{-\frac{7}{16}\zeta(3) - \frac{\pi^2}{8}\ln 2}$$

Appendix: The Polylogarithm

In its most general form, the polylogarithm $\mathrm{Li}_s(z)$ of order s is a complex-valued function defined as a power series and extended via analytic continuation, but for the purposes of our integration problem, we will only consider $s = 2, 3, 4, \ldots$ and $z \in [-1, 1]$.

For $|x| \leq 1$ and $s = 2, 3, 4, \ldots$ the following power series converges absolutely by ratio test (and p-series test for $x = 1$) and is defined to be $\mathrm{Li}_s(x)$.

$$\mathrm{Li}_s(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^s} = x + \frac{x^2}{2^s} + \frac{x^3}{3^s} + \cdots$$

Consequently we can differentiate term-by-term,

$$\frac{d}{dx}\mathrm{Li}_{s+1}(x) = \frac{d}{dx}\left[x + \frac{x^2}{2^{s+1}} + \frac{x^3}{3^{s+1}} + \cdots\right] = 1 + \frac{x}{2^s} + \frac{x^2}{3^s} + \cdots$$

$$= \frac{1}{x}\left(x + \frac{x^2}{2^s} + \frac{x^3}{3^s} + \cdots\right) = \frac{\mathrm{Li}_s(x)}{x}$$

So $\mathrm{Li}_{s+1}(x)$ is an antiderivative of $\mathrm{Li}_s(x)/x$.

We now prove the following identity for $|x| \leq 1$.

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\ln(1-v)}{v} dv$$

We first prove the integral $\int_0^x \ln(1-v)/v\, dv$ exists and is continuous for $x \in [-1, 1]$. Note $\lim_{v\to 0} \ln(1-v)/v = -1$ by L'Hopital's, so we can extend the integrand to a continuous function at $v = 0$ and thus the integral converges as $v \to 0$, hence $\int_0^x \ln(1-v)/v\, dv$ converges for all $x \in [-1, 1)$. It remains to show $\int_0^1 \ln(1-v)/v\, dv$ converges since $v = 1$ is the only other point at which the integrand is undefined. Since we already know the integral converges as $v \to 0$, it suffices to show something like $\int_{2/3}^1 \ln(1-v)/v\, dv$ converges. Substitute $u = -\ln(1-v)$, then

$$0 \leq -\int_{\frac{2}{3}}^1 \frac{\ln(1-v)}{v} dv = \int_{\ln 3}^\infty \frac{u}{e^u - 1} du \leq \int_1^\infty \frac{u}{e^u - 1} du \leq \int_1^\infty \frac{u}{e^u - \frac{1}{2}e^u} du$$

$$= 2\int_1^\infty ue^{-u} du \leq 2\int_0^\infty ue^{-u} du = 2\Gamma(1) = 2$$

Hence $\int_0^1 \ln(1-v)/v\, dv$ converges. Since we have now shown $\int_0^x \ln(1-v)/v\, dv$ converges for all $|x| \leq 1$ and clearly the integrand is continuous on $(0, 1)$, we have that $\int_0^x \ln(1-v)/v\, dv$ is continuous for $x \in [-1, 1]$.

Now we prove the identity. Using Taylor series,

$$-\int_0^x \frac{\ln(1-v)}{v} dv = -\int_0^x \sum_{k=1}^\infty \frac{-v^{k-1}}{k} dv$$

Fix $x \in (-1, 1)$. Then when considering the integral, we have $0 < |v| \leq |x|$. Then the summand $-v^{k-1}/k$ is bounded by $|x|^{k-1}/k$ on the interval $[-x, x]$ and $\sum_{k=1}^\infty |x|^{k-1}/k$ converges by ratio test, so by M-test, the series converges uniformly on $[-x, x]$, allowing us to interchange integration and summation,

$$= \sum_{k=1}^\infty \int_0^x \frac{v^{k-1}}{k} dv = \sum_{k=1}^\infty \left[\frac{v^k}{k^2}\right]_0^x = \sum_{k=1}^\infty \frac{x^k}{k^2} = \mathrm{Li}_2(x)$$

It remains to show the identity holds at $x = \pm 1$. However this follows from the fact that $\mathrm{Li}_2(x) = \sum_{k=1}^\infty x^k/k^2$ is continuous on $[-1, 1]$ (by term-by-term continuity) and $\int_0^x \ln(1-v)/v\, dv$ is also continuous on $[-1, 1]$ (shown previously), and since the two functions already coincide on $(-1, 1)$, continuity forces them to also coincide at the endpoints $x = \pm 1$.