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PVTOL maneuvers guided by a high-level nonlinear controller applied to a rotorcraft machine

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ABSTRACT

This work proposes a nonlinear controller, based on the Theory of Lyapunov, to stabilize a quadrotor when accomplishing positioning and trajectory tracking tasks restricted to a vertical plane. The maneuvers addressed here are commonly accomplished by PVTOL (Planar Vertical Take-off and Landing) vehicles due to the flight constraints: movement restricted to the **Z**-axis or to the **XZ**/**YZ** planes. The contributions of the paper are the nonlinear controller itself, the proof of stability of the equilibrium of the closed-loop system, and the proposal of an analytical solution to saturate the control signals to prevent the saturation of the physical actuators. Experimental results are also presented, which validate the proposed controller.

1. Introduction

Recently, some research effort has been devoted to the use of controllers to guide rotorcrafts in an autonomous way, creating the so-called UAVs – Unmanned Aerial Vehicles. Trajectory tracking and/or set-point stabilization continue to be great control problems for rotary-wing machines, and for that reason they are still being extensively investigated. A few years ago, the concept of planar vertical takeoff and landing (PVTOL) tasks was introduced in [11]. It has been an important benchmark for control design, aiming at stabilizing a rotorcraft in a vertical axis or plane. The difficulty lies in the fact that it is not possible to apply straightforwardly the nonlinear control techniques, such as feedback linearization or sliding mode control [26]. Thus, many control strategies have been proposed. In [21] Liouvillian systems are adopted to guide a miniature helicopter with a simplified dynamic model (PVTOL) during a trajectory tracking mission. In [9] a robust controller based on classical and adaptive pole placement techniques is proposed to control the yaw angle and the altitude of a miniature helicopter, whose dynamic model was obtained using the Euler–Lagrange equation. In [15] a pose controller is proposed, based on the linearization of the PVTOL model, and a stability analysis is performed using the theory of Lyapunov for linear systems. In [7] a path-following controller is designed and simulated applying the concept of smooth Jordan curves with vertical symmetry. In their previous work [8], such authors used C^2 closed curves in the proposition of a trajectory tracking controller for a VTOL vehicle in an iterative way. In [26] a trajectory tracking controller is proposed based on the first-order Padé approximation and Lyapunov concepts to deal with attitude measurement delays and input disturbances as a robust stabilization problem, and validated through numerical simulations. The limitation of such a proposal comes out when a high tracking accuracy is required due to the attitude approximation. Experimental results are obtained in [19] for a PVTOL vehicle controlled by a chain of cascaded integrators using bounded input. More recently, a new stabilization design for PVTOL aircrafts has been proposed [25], which has been validated through simulation. In such work an equivalent feedforward system is obtained, and a set of saturation levels is assigned to get a faster convergence during positioning tasks.

Other researchers have already proposed controllers to perform the navigation of quadrotors [3,6,12,17,1]. However, the controller proposed here is quite different from those ones: just a planar movement of the aircraft is considered here when designing the controller, whereas in those cases 3-D controllers are proposed. In particular, the controller proposed in [6] has some similarity with the one proposed here. There the controller is considered as a combination of four stages. The first one controls just the altitude of the aircraft, and the second one controls its yaw angle. The third

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stage controls the roll angle and the y position (indirectly), whereas the last one controls the pitch angle and the x position (indirectly). Combining the altitude control and the pitch angle and x-position control, one could generate a controller for PVTOL maneuvers in the **XZ** plane, like it is done here. However, two important differences should be stressed. The first one is that the controller proposed in [6] does not consider variations in the desired altitude (the desired altitude z_d is always constant), so that only positioning tasks are possible. The authors themselves tell that their objective is to stabilize the aircraft at hover, what supposes a constant altitude. By its turn, our controller considers the possibility of a time varying altitude, like it is shown in one of the experiments reported here, in which the aircraft tracks a trajectory in the **XZ** plane. In addition, the controller proposed in [6] has constant gains in the altitude control stage, for instance, which is a problem when big variations are imposed to the desired altitude because this generates large errors to be compensated by the controller. In such a case, high control signals could be generated, which can cause the saturation of the actuators of the aircraft. This would introduce an unforeseen non-linearity in the control loop, which could drive the control system out of stability. Our controller is designed to avoid this problem, for including a saturation in the control signal, using the tanh function. As our control system is proven to be stable when such non-linearity is included in the control loop, there is no risk that the control system loses stability since suitable initial conditions and reference profiles are given.

The main reason to propose our controller is to get a simpler controller to guide the aerial vehicle, although being limited in terms of degrees of freedom. Indeed, after being correctly oriented towards the goal, one can consider that the aircraft moves ahead following a vertical plane, so that only the altitude and the pitch angle should be controlled, as it is considered in this paper. Our main motivation is to associate three simple controllers to perform 3-D flights: two of them to guide the UAV in the **XZ** and **YZ** planes and the other to guide the vehicle in the **Z**-axis. However, this paper deals just with the navigation in the **XZ** plane, but the other two controllers can be derived from it.

Thus, this work aims at stabilizing a quadrotor, illustrated in Fig. 1(a), during the accomplishment of the tasks of positioning, trajectory tracking and hovering, restricted to the **XZ** plane, using a nonlinear controller based on the Theory of Lyapunov. The contribution of the paper is threefold: (a) the proposition of a single nonlinear controller suitable to accomplish all these PVTOL tasks; (b) the proof of stability of the equilibrium of the correspondent closed-loop system; and (c) an analytical solution to the problem of control signal saturation (such saturation is necessary to avoid the saturation of the physical actuators of the aircraft). To deal with such topics, Section 2 presents simplifications of the dynamic model of a quadrotor (previously proposed in [4]) to get the PVTOL model. Following, Section 3 discusses the controller proposed to stabilize the aircraft during flight. In the sequel, Section 4 presents experimental results obtained using the proposed controller to guide the rotorcraft and, finally, some concluding remarks are highlighted in Section 5.

2. The quadrotor model

According to [13,1], a rotary wing machine, like the one in Fig. 1 (a) (a quadrotor), can be represented as a cascade of four interconnected subsystems, as shown in Fig. 2. The two first blocks can be understood as a linear low-level dynamic model, whereas the two last ones correspond to the high-level dynamic model. In this work, the focus is on the high-level dynamic model of the quadrotor (the last two blocks of Fig. 2), which is obtained from the Euler–Lagrange equation, as detailed in [4,16], in a way quite similar to those in [6,17].

Aerial vehicles able to takeoff and land vertically, remain in a fixed position during a flight task, and move along a specified path in a vertical plane are called PVTOL vehicles. Such denomination was introduced in [11] as a benchmark for the design of controllers in aerospace engineering, and still represents a challenge for set-point stabilization and trajectory tracking missions, due to the fact that this system is underactuated, nonlinear, as well as non-minimum-phase when controlling some specific outputs (which is aimed at in this work).

The quadrotor used in this work is an aircraft able to perform such maneuvers, although not being a true PVTOL device. Thus, it is necessary to apply some flight restrictions to it in order to force it to navigate as a PVTOL vehicle (as shown in Fig. 1(b)). Considering a PVTOL task being accomplished in the **XZ** plane, the translational and rotational dynamic model of the aircraft is simplified to

$$u \sin\,\theta = m\ddot{x} \tag{1}$$

$$u \cos\,\theta = m\ddot{z} + mg \tag{2}$$

$$\tau_\theta = I_{yy}\ddot{\theta}, \tag{3}$$

where $u = \sum_{i=1}^{4} f_i$ and $\tau_\theta = k_1(-f_1 + f_2 + f_3 - f_4)$ are the control signals. In such a case, $\mathbf{x} = [x\ z\ \theta\ \dot{x}\ \dot{z}\ \dot{\theta}]^T$ is the vector of state variables.

Remark 1. Notice that the quadrotor is an underactuated system [4] whose four input variables are $[u_\theta\ u_\phi\ u_{\dot{\psi}}\ u_{\dot{z}}]^T$. To make it accomplish a PVTOL task in the **XZ** plane one should impose

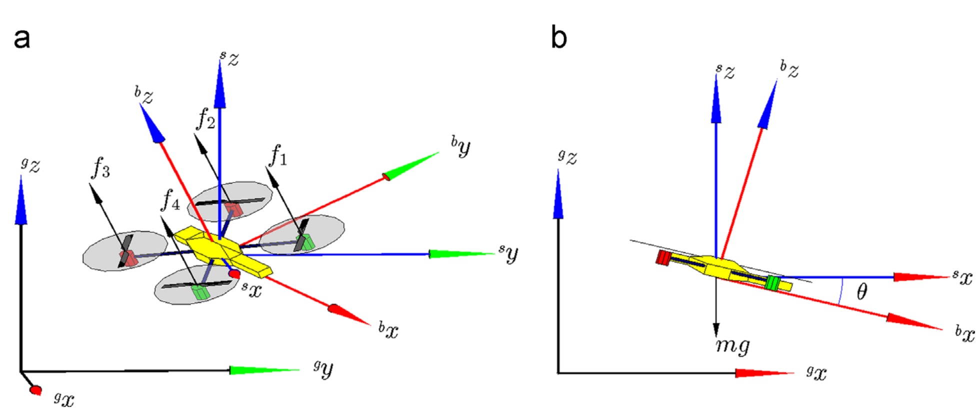

Fig. 1. Ar.Drone quadrotor: (a) 6-DOF model, with the reference system and the abstract control inputs f_i associated with it. The inertial, spatial and body frame are $\langle g\rangle$, $\langle s\rangle$ and $\langle b\rangle$, as indicated by the left superscript in the axes x, y and z. (b) The PVTOL model of the quadrotor.

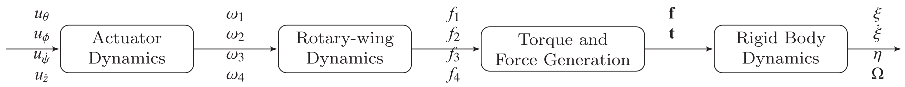

Fig. 2. Block diagram representing the dynamic model of a quadrotor.

$u_\phi \equiv 0$ and $u_{\dot\psi} \equiv 0$. This implies that only θ and z can be directly controlled. However, x can also be controlled, although indirectly, through controlling θ. This means that the nonlinear PVTOL controller designed in the sequel will only act over these three variables, leaving the other variables $\phi, \dot\psi$ and y under the control of the low level control system available onboard the aircraft, which is not dealt with in this paper.

3. The proposed Lyapunov-based controller

This section presents a position controller whose objective is to take the aircraft from any initial position to a desired one (besides having the capability of tracking a desired trajectory), i.e.

$$\mathbf{x} = [x\ z\ \theta\ \dot x\ \dot z\ \dot\theta]^T \rightarrow [x_d\ z_d\ \theta_d\ \dot x_d\ \dot z_d\ \dot\theta_d]^T = \mathbf{x}_d, \tag{4}$$

or $\tilde{\mathbf{x}} \rightarrow \mathbf{0}$, where $\tilde{\mathbf{x}} = \mathbf{x}_d - \mathbf{x}$ is the current pose error.

In order to analyze the stability of the equilibrium correspondent to the closed-loop system, the radially unbounded Lyapunov candidate function is proposed

$$\begin{aligned} V(\tilde{\mathbf{x}}, \dot{\tilde{\mathbf{x}}}) = & \underbrace{K_{z4} \ln\cosh(K_{z3}K_{z4}^{-1}\tilde z) + \frac{1}{2}\dot{\tilde z}^2}_{V1} \\ & \underbrace{+ K_{x1} \ln\cosh(K_{x2}\dot{\tilde x}) + K_{x3} \ln\cosh(K_{x2}\dot{\tilde x} + K_{x4}\tilde x) + \frac{K_{x4}}{2}\dot{\tilde x}^2}_{V_2} \\ & \underbrace{+ K_{\theta 4} \ln\cosh(K_{\theta 3}K_{\theta 4}^{-1}\tilde\theta) + \frac{1}{2}\dot{\tilde\theta}^2}_{V_3} > 0, \end{aligned} \tag{5}$$

where K_{ij} are real positive gains.

A similar Lyapunov function is adopted in [2]. The main difference is that the function in (5) by itself does not guarantee directly the asymptotic stability of the equilibrium of the closed-loop system. However, as the system discussed here is an autonomous one, the Theorem of La Salle [24] is applied to demonstrate the convergence of the states to their desired values, as discussed in the sequel, thus getting the same conclusions as [2].

The design of the controller is hereinafter split into two steps. First, the objective is to stabilize the altitude z of the aircraft, and then to control its longitudinal displacement x, through controlling its pitch angle θ.

3.1. Altitude control

Writing (2) in terms of the altitude control errors one gets

$$m(\ddot z_d - \ddot{\tilde z}) + mg = u \cos\theta \Rightarrow \ddot{\tilde z} - \ddot z_d = g - \frac{u}{m}\cos\theta, \tag{6}$$

for which the asymptotic stability of the altitude control can be guaranteed by using the trivial solution

$$u = \frac{m}{\cos\theta}(\eta_z + g), \tag{7}$$

where

$$\eta_z = \ddot z_d + K_{z1}\tanh(K_{z1}K_{z2}^{-1}\dot{\tilde z}) + K_{z3}\tanh(K_{z3}K_{z4}^{-1}\tilde z). \tag{8}$$

Then, the closed-loop system equation becomes

$$\ddot{\tilde z} + K_{z1}\tanh(K_{z1}K_{z2}^{-1}\dot{\tilde z}) + K_{z3}\tanh(K_{z3}K_{z4}^{-1}\tilde z) = 0. \tag{9}$$

Remark 2. Notice that $|\theta| \le \pi/2$, which implies a movement constraint. In this case, the vehicle cannot execute loops or aggressive maneuvers (the aim of this work is to smoothly control a quadrotor during a positioning or a trajectory tracking mission). In addition, it is worth mentioning that smooth maneuvers are commonly less demanding, in terms of energy, thus extending the total flight time.

To analyze the system stability using the theory of Lyapunov, considering the altitude control, one should obtain the first time derivative of $V_1(\tilde z, \dot{\tilde z})$ and then replace (9), thus obtaining

$$\dot V_1(\tilde z, \dot{\tilde z}) = K_{z3}\dot{\tilde z}\tanh(K_{z3}K_{z4}^{-1}\tilde z) + \dot{\tilde z}\ddot{\tilde z} = -K_{z1}\dot{\tilde z}\tanh(K_{z1}K_{z2}^{-1}\dot{\tilde z}) \le 0.$$

As $\dot V_1(\tilde z, \dot{\tilde z})$ is negative semi-definite, then $\tilde z$ and $\dot{\tilde z}$ are bounded, and it is possible to prove that $\dot{\tilde z}$ is square integrable. In other words, $\tilde z, \dot{\tilde z} \in L_\infty$ and $\dot{\tilde z} \in L_2$. Now, applying the La Salle Theorem for autonomous systems [24], when observing the dynamics of the system characterized by (9), the greatest invariant set $\mathbf{M}$ in the region

$$\mathbf{\Omega} = \left\{ \begin{bmatrix} \tilde z \\ \dot{\tilde z} \end{bmatrix} : \dot V_1(\tilde z, \dot{\tilde z}) = 0 \right\} \Rightarrow \left\{ \begin{bmatrix} \tilde z \\ \dot{\tilde z} \end{bmatrix} = \begin{bmatrix} \tilde z \\ 0 \end{bmatrix} \right\} \tag{10}$$

only takes place for $\tilde z = 0$. Therefore, the only invariant set $\mathbf{M}$ is the equilibrium $[\tilde z\ \dot{\tilde z}]^T = [0\ 0]^T$, which is, thus, asymptotically stable. In other words, $\tilde z(t), \dot{\tilde z}(t) \rightarrow 0$ when $t \rightarrow \infty$.

3.2. Pitch angle control

The next step is to stabilize the pitch angle of the rotorcraft, forcing $\theta(t) \rightarrow \theta_d(t)$, or $\tilde\theta \rightarrow 0$, with $\tilde\theta = \theta_d - \theta$. Aimed at designing a controller based on the theory of Lyapunov, one takes the first time derivative of $V_3(\tilde\theta, \dot{\tilde\theta})$ and then replaces (3), thus obtaining

$$\begin{aligned} \dot V_3(\tilde\theta, \dot{\tilde\theta}) &= K_{\theta 3}\dot{\tilde\theta}\tanh(K_{\theta 3}K_{\theta 4}^{-1}\tilde\theta) + \dot{\tilde\theta}\ddot{\tilde\theta} \\ &= \dot{\tilde\theta}\left(K_{\theta 3}\tanh(K_{\theta 3}K_{\theta 4}^{-1}\tilde\theta) + \ddot\theta_d - \frac{\tau_\theta}{I_{yy}}\right). \end{aligned} \tag{11}$$

Now, adopting the control signal

$$\tau_\theta = I_{yy}\eta_\theta, \tag{12}$$

with

$$\eta_\theta = \ddot\theta_d + K_{\theta 1}\tanh(K_{\theta 1}K_{\theta 2}^{-1}\dot{\tilde\theta}) + K_{\theta 3}\tanh(K_{\theta 3}K_{\theta 4}^{-1}\tilde\theta), \tag{13}$$

and introducing it in (11), one gets

$$\dot V_3(\tilde\theta, \dot{\tilde\theta}) = -K_{\theta 1}\dot{\tilde\theta}\tanh(K_{\theta 1}K_{\theta 2}^{-1}\dot{\tilde\theta}) \le 0, \tag{14}$$

which is negative semi-definite. Thus, $\tilde\theta, \dot{\tilde\theta} \in L_\infty$ and $\dot{\tilde\theta} \in L_2$. Analyzing the closed-loop system equation

$$\ddot{\tilde\theta} + K_{\theta 1}\tanh(K_{\theta 1}K_{\theta 2}^{-1}\dot{\tilde\theta}) + K_{\theta 3}\tanh(K_{\theta 3}K_{\theta 4}^{-1}\tilde\theta) = 0, \tag{15}$$

and applying the La Salle Theorem for autonomous systems once more, one observes that the greater invariant set $\mathbf{M}$ in

$$\mathbf{\Omega} = \left\{ \begin{bmatrix} \tilde\theta \\ \dot{\tilde\theta} \end{bmatrix} : \dot V_3(\tilde\theta, \dot{\tilde\theta}) = 0 \right\} \Rightarrow \left\{ \begin{bmatrix} \tilde\theta \\ \dot{\tilde\theta} \end{bmatrix} = \begin{bmatrix} \tilde\theta \\ 0 \end{bmatrix} \right\} \tag{16}$$

exists uniquely for $\tilde\theta = 0$. Therefore, the equilibrium $[\tilde\theta\ \dot{\tilde\theta}]^T = [0\ 0]^T$ of such a system is asymptotically stable, i.e., $\tilde\theta(t), \dot{\tilde\theta}(t) \rightarrow 0$ when $t \rightarrow \infty$.

3.3. Horizontal displacement control

As both control inputs have already been defined, in order to control the horizontal displacement one should define the profile of the pitch angle, guided by the altitude and horizontal errors. In such a case, considering (1) and (7), one has

$$\ddot{x} = (\eta_z + g)\tan(\theta_d - \tilde{\theta}) \Rightarrow \ddot{x} = (\eta_z + g)\frac{\tan\theta_d - \tan\tilde{\theta}}{1 + \tan\theta_d\ \tan\tilde{\theta}},$$

or

$$\ddot{x} - (\eta_z + g)\tan\ \theta_d = -(\eta_z + g + \ddot{x}\tan\theta_d)\tan\tilde{\theta}. \tag{17}$$

Notice that there is no external signal available to control x. Adopting

$$\theta_d = \tan^{-1}\left(\frac{\eta_x}{\eta_z + g}\right) \tag{18}$$

with

$$\eta_x = \ddot{x}_d + K_{x1}\ \tanh(K_{x2}\dot{\tilde{x}}) + K_{x3}\ \tanh(K_{x2}\dot{\tilde{x}} + K_{x4}\tilde{x}),$$

and replacing it in (17), the closed-loop system equation

$$\ddot{\tilde{x}} + K_{x1}\ \tanh(K_{x2}\dot{\tilde{x}}) + K_{x3}\ \tanh(K_{x2}\dot{\tilde{x}} + K_{x4}\tilde{x}) = \delta \tag{19}$$

is obtained, where $\delta = (\eta_z + g + \ddot{x}\ \tan\ \theta_d)\tan\tilde{\theta}$.

The signals $\eta_z(t)$ and $\tan\ \theta_d(t)$ are, by design, bounded. It was shown in Section 3.2 that the signal $\tan\ \tilde{\theta}(t)$ is bounded and tends to zero. Finally, for suitable initial conditions the signal $\ddot{x}(t) = \tan\ \theta(t)(\eta_z(t) + g)$ is bounded. Indeed, $\theta(t) - \theta_d(t) \to 0$, and since $\theta_d \in (-\pi/2, \pi/2)$, for suitable initial conditions $\tan\ \theta(t)$ is bounded as well. In conclusion, for suitable initial conditions $\theta(0)$, the signal $\delta(t)$ is bounded and tends to zero. Since $\ddot{\tilde{x}}$ is bounded, system (19) has no finite escape times (i.e., the solution exists for all $t \geq 0$).

Viewing (19) as a forced system with input δ, let $X = [x, \dot{x}]^T$, and write

$$\dot{\tilde{\mathbf{X}}} = \mathbf{A}\tilde{\mathbf{X}} + \mathbf{B}\delta + \Delta(\tilde{\mathbf{X}}),$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -K_{x_3}K_{x_4} & -K_{x_1}K_{x_2} - K_{x_2}K_{x_3} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and $\Delta(\tilde{\mathbf{X}})$ is the linearization error. Taking into account that $\mathbf{A}$ is Hurwitz, there exists a positive definite solution $\mathbf{P}$ to Lyapunov's equation $\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{I}$. Define $V = \tilde{\mathbf{X}}^T\mathbf{P}\tilde{\mathbf{X}}$. Then, there exist constants $c_1, c_2 > 0$ such that

$$\dot{V} \leq -\|\tilde{\mathbf{X}}\|_2(\|\tilde{\mathbf{X}}\|_2 - c_1\|\Delta(\tilde{\mathbf{X}})\|_2 - c_2\|\delta\|_2).$$

Since $\Delta(\tilde{\mathbf{X}})$ is a linearization error, one has

$$\frac{\|\Delta(\tilde{\mathbf{X}})\|_2}{\|\tilde{\mathbf{X}}\|_2} \to 0 \quad \text{as } \|\tilde{\mathbf{X}}\|_2 \to 0.$$

For any $\mu > 0$, there exists $r > 0$ such that $\|\Delta(\tilde{\mathbf{X}})\|_2 \leq \mu\|\tilde{\mathbf{X}}\|_2$ for all $\|\tilde{\mathbf{X}}\|_2 < r$. Pick $\mu < 1/(2c_1)$, then there exists $r > 0$ such that

$$\dot{V} \leq -\frac{1}{2}\|\tilde{\mathbf{X}}\|_2(\|\tilde{\mathbf{X}}\|_2 - 2c_2\|\delta\|_2)$$

for all $\|\tilde{\mathbf{X}}\|_2 < r$. Pick $c > 0$ and $r_1 > 0$, such that

$$\{\tilde{\mathbf{X}} : \|\tilde{\mathbf{X}}\|_2 < r_1\} \subset \Omega_c := \{\tilde{\mathbf{X}} : V(\tilde{\mathbf{X}}) < c\} \subset \{\tilde{\mathbf{X}} : \|\tilde{\mathbf{X}}\|_2 < r\}.$$

For suitable initial conditions, one has $\|\delta(t)\|_2 < r_1/(2c_2)$ for all $t \geq 0$, so that all solutions originating in Ω_c stay there for all $t \geq 0$. Moreover, as one now shows, $\tilde{\mathbf{X}}(t) \to 0$ for all such initial conditions. Let $\epsilon \in (0, r)$ be arbitrary. There exist $c^\star \in (0, c)$ and $r_2 \in (0, r_1)$ such that

$$\{\tilde{\mathbf{X}} : \|\tilde{\mathbf{X}}\|_2 < r_2\} \subset \Omega_{c^\star} \subset \{\|\tilde{\mathbf{X}}\|_2 < \epsilon\}.$$

Since $\delta(t) \to 0$, there exist $T_1 > 0$ such that, for all $t > T_1$, $\|\delta(t)\|_2 < r_2/(2c_2)$ which implies that, for all $t > T_1$,

$$\dot{V} \leq -\frac{1}{2}\|\tilde{\mathbf{X}}\|_2(\|\tilde{\mathbf{X}}\|_2 - r_2).$$

Due to the choice of r_2 and $c^\star$ above, the latter inequality implies that there exists $T_2 > T_1$ such that $\|\tilde{\mathbf{X}}(t)\|_2 < \epsilon$ for all $t > T_2$. This proves that $\tilde{\mathbf{X}}(t) \to 0$.

3.4. Selecting the gains of the controllers

The gains of the controllers are chosen in order to avoid the saturation of the control signals and to get a critically damped system response.

Before starting the gain analysis, one should have in mind that the hyperbolic tangent function (used as saturation function) can be approximated by a linear function ($\tanh\alpha \approx \alpha$) for small input values or by a threshold function ($\tanh\alpha \approx \text{sign}\ \alpha$) for great input values.

Now, considering the closed loop system of the altitude controller (9), in the linear region corresponding to small $(\tilde{z}, \dot{\tilde{z}})$ errors, the first condition $K_{z3}^2 K_{z4}^{-1} = \frac{1}{4}(K_{z1}^2 K_{z2}^{-1})^2$ should be respected to accomplish the response statement. In contrast, in the saturation region, the second condition $K_{z3} = \frac{1}{4}K_{z1}^2$ should also be regarded.

In the sequence, observing (7) and considering a maximum thrust $u_{\max}$ equal to ag, with $a \in \mathbb{R}^+$, one gets

$$|K_{z1}\ \tanh(K_{z1}K_{z2}^{-1}\tilde{z}) + K_{z3}\ \tanh(K_{z3}K_{z4}^{-1}\dot{\tilde{z}})|$$
$$\leq \left(\frac{a}{m}\cos\ \theta_{\max} - 1\right)g - \left|\ddot{\tilde{z}}_{d\ \max}\right|.$$

Notice that the right term of the inequality can be stipulated by the user, after observing the physical limitations of the mechanical system. In this work, it is selected $a = 2.5$, and a $\ddot{\tilde{z}}_{\max}$, which depends of the reference profile.

Assuming the worst case, where the error values are not small enough to disregard the hyperbolic tangent function, then the system will operate in the saturation zone. In such a case, one has the third condition, namely $K_{z1} + K_{z3} \leq ((a/m)\cos\ \theta_{\max} - 1)g - |\ddot{\tilde{z}}_{d\ \max}|$.

Consequently, defining a value for K_{z2} and manipulating the three aforementioned conditions, it is possible to attribute the control gains to the altitude control in order to obtain a critically damped response and to avoid saturation of the physical actuators.

In the sequel, a similar analysis is done for the pitch angle. Eq. (15) is used to define the two first conditions: $K_{\theta 3}^2 K_{\theta 4}^{-1} = \frac{1}{4}(K_{\theta 1}^2 K_{\theta 2}^{-1})^2$ and $K_{\theta 3} = \frac{1}{4}K_{\theta 1}^2$. The third one is obtained replacing (15) in (3), considering the saturation zone of the errors. Thus one has $K_{\theta 1} + K_{\theta 3} = \tau_{\theta\ \max}/I_{yy} + \ddot{\theta}_{d\ \max}$. Defining $K_{\theta 2}$, it is possible to have a critically damped response of θ, without saturating the controllers.

Finally, considering the natural response of (19), $\delta = 0$, it is possible to obtain $K_{x1} + K_{x3} = (1/K_{x2})\sqrt{4K_{x3}K_{x4}}$, for the linear zone, and $K_{x3} = \frac{1}{4}K_{x1}^2$, for the saturation one. Then, taking (1) and replacing (7) and (19), it is possible to obtain for the saturation zone $K_{x1} + K_{x3} \leq |\ddot{x}_{d\ \max} + (\eta_z + g)\tan\ \theta_{\max}|$, where η_z is already considered during the altitude gain attribution. Similarly, the longitudinal displacement response will approximate to a critically damped one after defining K_{x2}.

4. Experimental results

The results of two experiments run using a quadrotor, in which the controller proposed in Section 3 is programmed, are presented and discussed in the sequel. The flight missions accomplished are a trajectory tracking one and a positioning one, which are run using the same controller. The objective of such experiments is to check the stability of the closed-loop control system when the proposed controller is adopted, and to check its effectiveness to guide the aircraft during mission accomplishment as well. The experiments were run in an indoor environment in the absence of external disturbance. Moreover, the horizontal displacements were obtained through odometry (dead reckoning using the accelerometers of the IMU available onboard the aircraft), while the vertical displacement was measured using a sonar.

Table 1
UAV model parameters (moments of inertia are in [kg m^2]).

$m=0.380$ (kg)	$k_1=0.1782$ (m)	$k_2=0.0290$ (m)	$I_{xx}=9.57\times10^{-3}$	$I_{yy}=18.57\times10^{-3}$	$I_{zz}=25.55\times10^{-3}$

For the Ar.Drone Parrot quadrotor, the aircraft used as the experimental platform in this work, when using the model presented in Section 2, the parameters are those shown in Table 1. More details about the sensors available in the UAV and some suggestions of applications using this UAV can be found in [5,14]. Complementing the experimental setup, the codes are written in Matlab$^{\text{Å}}$, and run in an external computer, which is linked to the rotorcraft through a Software Development Kit (SDK) available at the Ar.Drone Parrot web site.

Remark 3. The motors of the Ar.Drone Parrot are not aligned with the ^bx- and ^by-axes. They are rotated 45° with respect to the ^bx-axis, around the ^bz-axis. As a consequence, a coupled actuation of all motors is necessary to execute a lateral or longitudinal maneuver, differently from other prototypes found in the literature.

For each experiment reported in this section, the posture state variables, defined by the position and the orientation of the aircraft during the mission are presented, as well as the abstract and the real control signals (respectively the desired forces, defined by the high-level controller, and the inputs accepted by the vehicle, defined by the low-level controller). As for the way used to convert u and τ_θ to f_i, for $i=1,\ldots,4$, and then to u_ϕ, u_θ, $u_{\dot\psi}$, $u_{\dot z}$, it is similar to the one presented in [20]. Such a step, in this work, is named the low-level control, as explained in Section 2.

The first experiment is a trajectory tracking mission in the **XZ** plane, described by $\{x_d=3\sin((\pi/60)t), z_d=1.5-\cos((\pi/60)t)\}$, resulting in an ellipsoidal shape. The desired pitch angle is given by (18). The posture variables that are not included in the PVTOL maneuvers have been set to zero. Fig. 3(f) illustrates the desired (dashed line) and traveled (solid line) paths in the **XZ** plane, with the icon representing the UAV plotted each 10 s. It is possible to observe that the rotorcraft reaches and follows the desired reference trajectory. Fig. 3(a) shows the time evolution of the Ar. Drone position in the **XZ** plane, as well as its pitch angle. Knowing that the proposed reference signals do not control the vehicle in the **Y**-axis, one can notice some errors in Fig. 3(b) as a consequence of the drifting effect associated with the inertial unit onboard the aircraft. Despite following the reference x_d and z_d in phase, the results present some tracking errors. However, such errors do not compromise the mission accomplishment as one can see in the figure. In other words, the proposed controller is able to guide the rotorcraft to follow the reference without delay.

The behavior of the pitch angle, shown in Fig. 3(a), is the most important one for analysis (it is guided by a control signal). During analysis one starts noticing the high-frequency components present in the signals $x(t)$ and $z(t)$. Thus, as a consequence of (18), the reference pitch values are not smooth, although still being suitable to guarantee the accomplishment of the programmed task. It is important to highlight that, for the quadrotor, it is troublesome to perform a positive displacement in x (which requires to increase f_2 and f_3 and decrease f_1 and f_4), while keeping the negative displacement in z (which requires to decrease all motor thrusts). Such an effect can be observed from the time instant 90 s on, in Fig. 3(a). The time evolution of the roll and yaw angles is shown in Fig. 3(b), as well as the position in the axis **Y** (notice that such variables are not managed by the high-level controller proposed in this work).

The desired values are computed considering a PVTOL maneuver that takes into account some constraints. Similar observations are valid for the yaw angle in the case of a controller designed for maneuvers in the z-axis, i.e., one can control the altitude and the yaw angle in such a case. Considering the horizontal displacement constraints, a quadrotor (or a helicopter) model becomes a second order linear system, which can be controlled using classical control techniques ([6] show it). Thus, the desired yaw value is given by such a control strategy. An experimental result is presented in the sequel to demonstrate it.

Fig. 3(c) illustrates the abstract inputs of the high-level controller. Such inputs cannot be directly applied to the Ar.Drone, thus they are considered abstract ones (or even auxiliary ones). In the figure, notice that the tendency of the curve (the forces produced by the propulsion system) increases/decreases in a sinusoidal way, as expected for the mission being accomplished. The results of the abstract/real input transformation are illustrated in Fig. 3(e). It is possible to observe that the first two references follow the evolution of the pitch and roll angles shown in Fig. 3(a) and (b), respectively. Talking about the fourth real input, it is important to mention that for $\dot z=0$ the rotorcraft is compensating its own weight, and any change will result in a higher/lower altitude. A similar analysis can be performed for the third signal $u_{\dot\psi}$ sent to the vehicle to obtain a new heading value.

In order to demonstrate the feasibility of the controller for a positioning task, Fig. 4 illustrates a flight mission whose objective is to reach the longitudinal $x_d=[0\ 2\ -2\ 0\ 3\ 0]$ m and vertical $z_d=[1\ 2\ 0.5\ 1\ 1.5\ 0.5]$ m positions, while keeping the other posture variables as zero. As for the changes in the desired position, the time interval between two subsequent ones is 20 s.

As shown in Fig. 4, the proposed controller (without any gain changing) is also capable of accomplishing positioning tasks. Notice that there is a delay to reach the target, but it is done smoothly. In this experiment, an important situation to be observed occurs between 40–50 s and 100–110 s. At these moments, the desired forces to be produced by each motor are zero, which means that the Ar.Drone starts executing an auto-rotation, which is enough to go down towards a new desired altitude (lower than the last one). This situation, however, does not compromise the accomplishment of the positioning task, as one can notice.

An important aspect of reaching the target in a smooth way is related to energy consumption. One of the key challenges that has prevented engineers from coming up with convincing aerial solutions is the "energetic cost of flying", which is orders of magnitude higher than the one correspondent to terrestrial locomotion. Moreover, compared with other types of UAVs, the quadrotor is the one with higher energy consumption, as shown in [23].

Some authors have compared control strategies developed and used for certain tasks in UAV control, focusing on performance and power consumption, to determine the amount of energy needed for computing, thus providing a rationale to introduce an energy-aware computation scheme [22]. The percentage of overall power consumption needed for computing can be as high as 20% (rotorcraft) in mini-UAVs. Therefore, suitable navigation strategies and control algorithms should be adopted whenever possible (especially during the cruising phase), switching to more accurate methods only when needed (when accurate trajectory tracking or precise vehicle stability is required). Aiming at such objectives, in [10] tracking control of a PVTOL aircraft using a technique based on linear algebra theory has been proposed in order to achieve a control law of easy implementation and less computing power, allowing high performance on-board

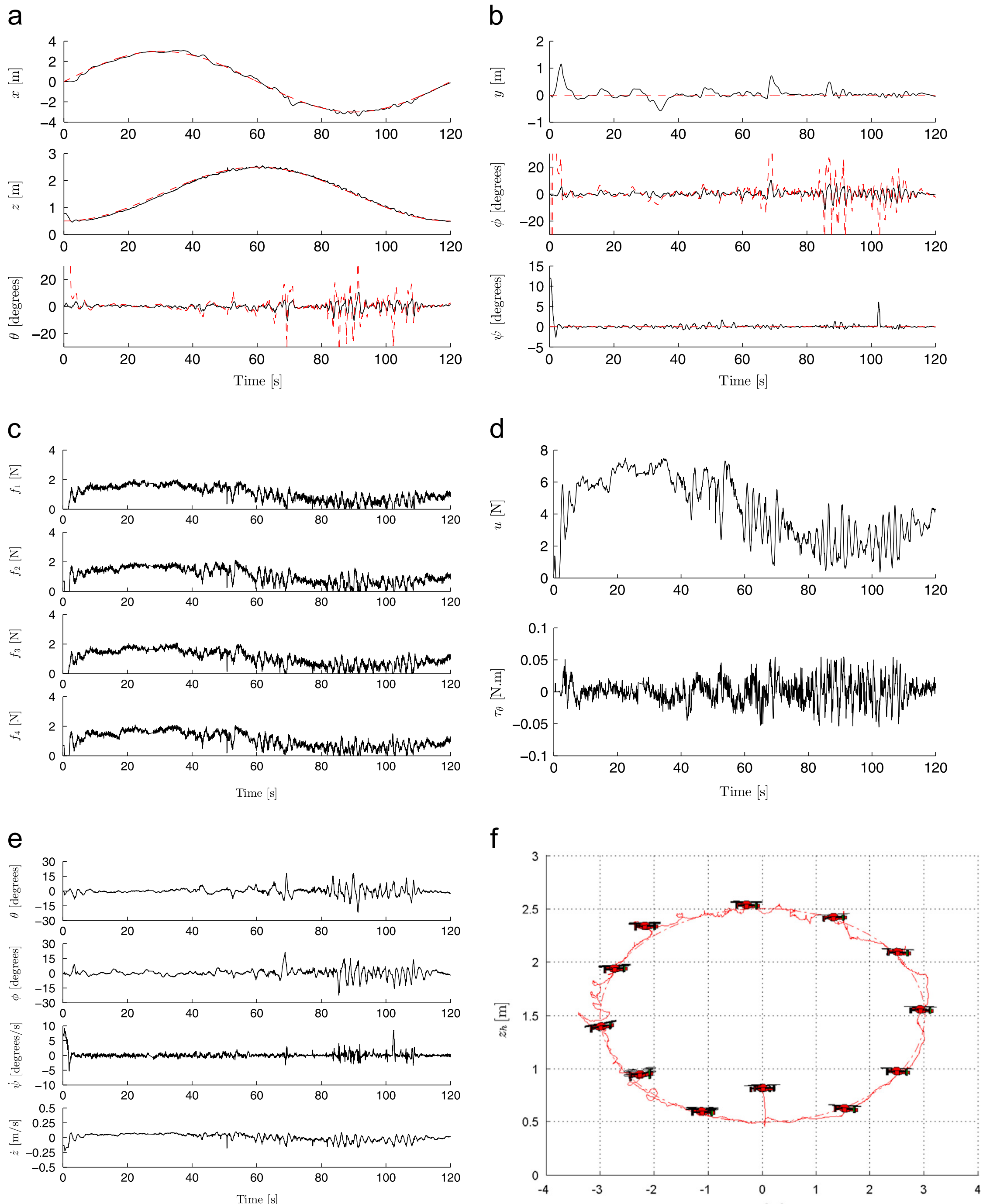

Fig. 3. Posture and control signals of the Ar.Drone Parrot during a trajectory tracking mission. (a) Planar position and pitch angle, (b) posture variables not being controlled, (c) abstract forces generated by the high-level controller, (d) designed control signals, (e) real input signals applied to the vehicle, (f) path traveled in the **XZ** plane during the PVTOL trajectory tracking task.

computing and energy-saving. In [18] a minimum-energy controller is designed and built for a class of electrically driven vehicles, whose characteristics include the smooth response of the controller.

Taking these ideas in mind, it is interesting to notice that the controller proposed in this work has the possibility of adjusting the gains K_{ij}. Therefore, a set of optimal gains can be selected to save energy, whereas a different set of gains may be adopted when aggressive maneuvers are required. This is something we are currently working on, to take into account in future works.

5. Concluding remarks

This work proposes a high-level nonlinear controller based on the Theory of Lyapunov to be applied to an unmanned aerial vehicle, in this case an autonomous quadrotor, to execute PVTOL tasks. The problem of the saturation of the control signals and an analytical solution for it are also addressed. The control law designed showed to be able to guide the aircraft while executing maneuvers of takeoff, hovering and landing in a vertical plane during the accomplishment of

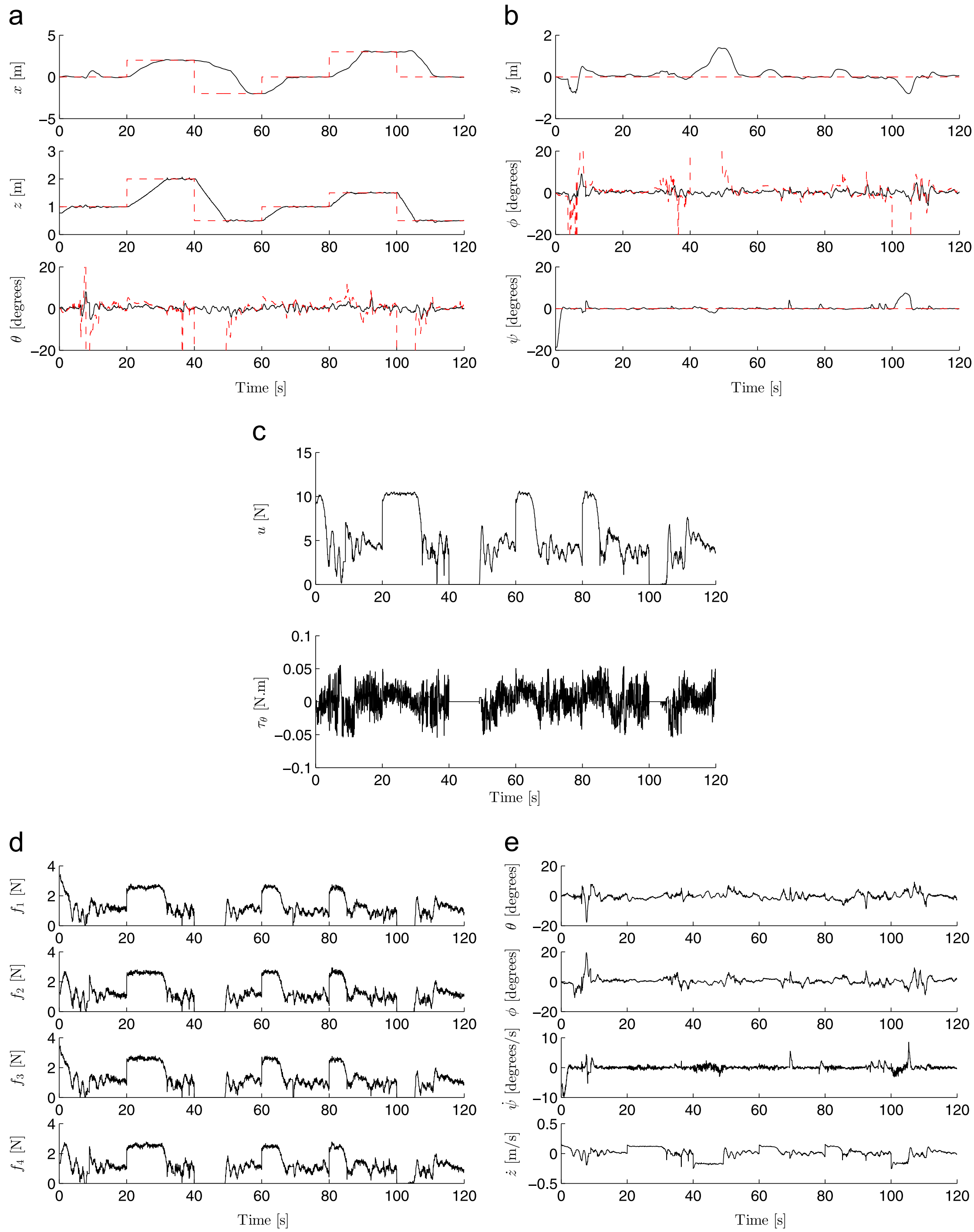

Fig. 4. Posture and control signals of the Ar.Drone Parrot during a positioning mission. (a) Planar position and pitch angle, (b) posture variables not being controlled, (c) designed control signals, (d) abstract forces generated by the high-level controller, (e) real input signals applied to the vehicle.

the tasks of positioning and trajectory tracking. Experimental results allowed assessing the performance of the proposed controller during the planned missions.

The next steps of this work are related to 3-D navigation. First, the PVTOL controller for the **XZ** plane will be switched with the controller of the **Z**-axis. In such a case, to reach a 3-D Cartesian reference, the UAV should orientate itself to a desired yaw angle, and then execute a PVTOL maneuver to perform a longitudinal displacement (considering its own reference frame). Notice that a stability analysis for switched systems should be performed in order to guarantee the asymptotic convergence of the state variables to their desired values. It could be performed using multiple Lyapunov functions, for instance. Another proposal for a 3-D navigation is to implement a control system comprising the whole set of state variables. Knowing that a quadrotor is an underactuated system (has more degrees of freedom than actuators), a suitable controller could be designed based on the Theory of Lyapunov and the partial feedback linearization technique.

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