ELT330 – Sistemas de Controle I
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Aula 13 – Linearização de Sistemas Dinâmicos no Espaço de Estados

Linearização em Representação em Espaço de Estados

Considere um sistema dinâmico onde $f_1(x_1,x_2)$ e $f_2(x_1,x_2)$ são funções não-lineares. As equações diferenciais que descrevem este sistema dinâmico podem ser escritas como:

$$\begin{cases} \dot{x}_1 = f_1(x_1,x_2) \\ \dot{x}_2 = f_2(x_1,x_2) \end{cases} \Rightarrow \begin{cases} \dfrac{dx_1}{dt} = f_1(x_1,x_2) \\ \dfrac{dx_2}{dt} = f_2(x_1,x_2) \end{cases}$$

Expandindo $f_1(x_1,x_2)$ e $f_2(x_1,x_2)$ em Série de Taylor no ponto de linearização x_0 (x_0 = P.E.) obtém-se:

$$\boxed{\begin{cases} \dfrac{dx_1}{dt} = f_1(x_1,x_2) \cong f_1(x_{10},x_{20}) + \left.\dfrac{\partial f_1}{\partial x_1}\right|_{\substack{x_1=x_{10}\\x_2=x_{20}}} (x_1 - x_{10}) + \left.\dfrac{\partial f_1}{\partial x_2}\right|_{\substack{x_1=x_{10}\\x_2=x_{20}}} (x_2 - x_{20}) \\ \dfrac{dx_2}{dt} = f_2(x_1,x_2) \cong f_2(x_{10},x_{20}) + \left.\dfrac{\partial f_2}{\partial x_1}\right|_{\substack{x_1=x_{10}\\x_2=x_{20}}} (x_1 - x_{10}) + \left.\dfrac{\partial f_2}{\partial x_2}\right|_{\substack{x_1=x_{10}\\x_2=x_{20}}} (x_2 - x_{20}) \end{cases}}$$

Estas equações (EDO´s) constituem o modelo aproximado (linearizado) do sistema não linear original.

No Ponto de Equilíbrio (P.E.) as derivadas são nulas,

$$\frac{dx_1}{dt} = f_1(x_1,x_2) = 0$$
$$\frac{dx_2}{dt} = f_2(x_1,x_2) = 0$$

Portanto:

$$\boxed{\begin{cases} \dfrac{d\overline{x_1}}{dt} \cong \overbrace{f_1(x_{10},x_{20})}^{\frac{dx_1}{dt}=0} + \overbrace{\left.\dfrac{\partial f_1}{\partial x_1}\right|_{\substack{x_1=x_{10}\\x_2=x_{20}}}}^{a_{11}} \overbrace{(x_1 - x_{10})}^{\overline{x_1}} + \overbrace{\left.\dfrac{\partial f_1}{\partial x_2}\right|_{\substack{x_1=x_{10}\\x_2=x_{20}}}}^{a_{12}} \overbrace{(x_2 - x_{20})}^{\overline{x_2}} \\ \dfrac{d\overline{x_2}}{dt} \cong \overbrace{f_2(x_{10},x_{20})}^{\frac{dx_2}{dt}=0} + \overbrace{\left.\dfrac{\partial f_2}{\partial x_1}\right|_{\substack{x_1=x_{10}\\x_2=x_{20}}}}^{a_{21}} \overbrace{(x_1 - x_{10})}^{\overline{x_1}} + \overbrace{\left.\dfrac{\partial f_2}{\partial x_2}\right|_{\substack{x_1=x_{10}\\x_2=x_{20}}}}^{a_{22}} \overbrace{(x_2 - x_{20})}^{\overline{x_2}} \end{cases}}$$

Define-se as seguintes variáveis desvio (desvio em relação ao estado estacionário):

$$\overline{\mathbf{x_1}} = (\mathbf{x_1} - \mathbf{x_{10}}) \quad \textbf{e} \quad \overline{\mathbf{x_2}} = (\mathbf{x_2} - \mathbf{x_{20}})$$

Deste modo, o modelo linearizado pode ser escrito em termos das **variáveis desvio** anteriores e na forma de **espaço de estados** como segue:

$$\boxed{\begin{cases} \overline{\dfrac{dx_1}{dt}} = a_{11}\overline{x_1} + a_{12}\overline{x_2} \\ \overline{\dfrac{dx_2}{dt}} = a_{21}\overline{x_1} + a_{22}\overline{x_2} \end{cases}}$$

Ou na **forma matricial** equivalente:

$$\begin{bmatrix} \overline{\dfrac{dx_1}{dt}} \\ \overline{\dfrac{dx_2}{dt}} \end{bmatrix} = \begin{bmatrix} \dot{\overline{x_1}} \\ \dot{\overline{x_2}} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} \overline{x_1} \\ \overline{x_2} \end{bmatrix} \Longrightarrow \dot{\overline{x}} = A.\overline{x}$$

Onde:

$$a_{11} = \left.\frac{\partial f_1}{\partial x_1}\right|_{\substack{x_1=x_{10} \\ x_2=x_{20}}} \; ; \; a_{12} = \left.\frac{\partial f_1}{\partial x_2}\right|_{\substack{x_1=x_{10} \\ x_2=x_{20}}} \; ; \; a_{21} = \left.\frac{\partial f_2}{\partial x_1}\right|_{\substack{x_1=x_{10} \\ x_2=x_{20}}} \; ; \; a_{22} = \left.\frac{\partial f_2}{\partial x_2}\right|_{\substack{x_1=x_{10} \\ x_2=x_{20}}}$$

Os modelos matemáticos para sistemas dinâmicos podem ser representados por conjuntos de Equações Diferenciais Ordinárias de 1ª ordem (**Equações de Espaço de Estados**).

Ou seja;

$$\begin{cases} \begin{bmatrix} \dot{x}_1 = \dfrac{dx_1}{dt} = f_1(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m, t); \;\; x_1(0) = x_{10} \\ \dot{x}_2 = \dfrac{dx_2}{dt} = f_2(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m, t); \;\; x_2(0) = x_{20} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dot{x}_n = \dfrac{dx_n}{dt} = f_n(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m, t); \;\; x_n(0) = x_{n0} \end{bmatrix} \\ \begin{bmatrix} y_1 = h_1(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m, t) \\ y_2 = h_2(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m, t) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ y_P = h_P(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m, t) \end{bmatrix} \end{cases}$$

Existem situações onde não se está interessado diretamente no vetor de estado, mas sim em seu efeito sobre o Vetor de Saída Y.

A linearização é útil na investigação do comportamento do sistema não linear nas vizinhanças de pontos de operação estacionária, onde diversos processos contínuos operam de fato.

Aplicando-se expansão em Série de Taylor:

$$\dot{x}_i = f_i(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m, t); \quad x_i(0) = x_{i0}$$

$$\dot{x}_i = \overbrace{f_i(x_{i0}, u_{i0})}^{\dot{x}_i = f_i|_{x_i = x_{i0}, u_i = u_{i0}} = 0} + \left.\frac{\partial f_i}{\partial x_i}\right|_{\substack{x_i = x_{i0} \\ u_i = u_{i0}}} (x_i - x_{i0}) + \cdots + \left.\frac{\partial f_i}{\partial x_n}\right|_{\substack{x_n = x_{n0} \\ u_m = u_{m0}}} (x_n - x_{n0}) +$$

$$+ \left.\frac{\partial f_i}{\partial u_i}\right|_{\substack{x_i = x_{i0} \\ u_i = u_{i0}}} (u_i - u_{i0}) + \cdots + \left.\frac{\partial f_i}{\partial u_m}\right|_{\substack{x_n = x_{n0} \\ u_m = u_{m0}}} (u_m - u_{m0})$$

Generalizando:

$$\boxed{\dot{x}_i(t) = \overbrace{f_i(\mathbf{x_0}, \mathbf{u_0})}^{=0} + \sum_{j=1}^{n} \left.\frac{\partial f_i(\mathbf{x}, \mathbf{u})}{\partial x_j}\right|_{x_0 u_0} (x_i - x_{0j}) + \sum_{j=1}^{p} \left.\frac{\partial f_i(\mathbf{x}, \mathbf{u})}{\partial u_j}\right|_{x_0 u_0} (u_i - u_{0j})}$$

Então:

$$\dot{x}_i(t) = \left.\frac{\partial f_i}{\partial x_i}\right|_{\substack{x_i = x_{i0} \\ u_i = u_{i0}}} (x_i - x_{i0}) + \cdots + \left.\frac{\partial f_i}{\partial x_n}\right|_{\substack{x_n = x_{n0} \\ u_m = u_{m0}}} (x_n - x_{n0}) +$$

$$+ \left.\frac{\partial f_i}{\partial u_i}\right|_{\substack{x_i = x_{i0} \\ u_i = u_{i0}}} (u_i - u_{i0}) + \cdots + \left.\frac{\partial f_i}{\partial u_m}\right|_{\substack{x_n = x_{n0} \\ u_m = u_{m0}}} (u_m - u_{m0})$$

Definindo as variáveis desvio;

$$\overline{\mathbf{x_i}} = (\mathbf{x_i} - \mathbf{x_{i0}}) \quad \textbf{e} \quad \overline{\mathbf{u_i}} = (\mathbf{u_i} - \mathbf{u_{i0}})$$

pode-se escrever a **Equação Linearizada** como:

$$\boxed{\overline{\dot{x}_i} = \overbrace{\left.\frac{\partial f_i}{\partial x_i}\right|_{\substack{x_i = x_{i0} \\ u_i = u_{i0}}}}^{a_{i1}} \overline{x_i} + \cdots + \overbrace{\left.\frac{\partial f_i}{\partial x_n}\right|_{\substack{x_n = x_{n0} \\ u_m = u_{m0}}}}^{a_{in}} \overline{x_n} + \overbrace{\left.\frac{\partial f_i}{\partial u_i}\right|_{\substack{x_i = x_{i0} \\ u_i = u_{i0}}}}^{b_{i1}} \overline{u_i} + \cdots + \overbrace{\left.\frac{\partial f_i}{\partial u_m}\right|_{\substack{x_n = x_{n0} \\ u_m = u_{m0}}}}^{b_{im}} \overline{u_m}}$$

Ou:

$$\overline{\dot{x}_i} = a_{i1}\overline{x_i} + \cdots + a_{in}\overline{x_n} + b_{i1}\overline{u_i} + \cdots + b_{im}\overline{u_m}$$

Onde as constantes $\mathbf{a_{in}}$ e $\mathbf{b_{im}}$ são calculadas de:

$$a_{in} = \left.\frac{\partial f_i}{\partial x_n}\right|_{\substack{x_n = x_{n0} \\ u_m = u_{m0}}} \qquad \text{e} \qquad b_{im} = \left.\frac{\partial f_i}{\partial u_m}\right|_{\substack{x_n = x_{n0} \\ u_m = u_{m0}}}$$

Após a linearização, obtém-se um conjunto de equações lineares com coeficientes constantes:

$$
\begin{cases}
\begin{bmatrix}
\dot{\overline{x_1}} = a_{11}\overline{x_1} + \cdots + a_{1n}\overline{x_n} + b_{11}\overline{u_1} + \cdots + b_{1m}\overline{u_m} \ ; \ \overline{x_1}0) = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\dot{\overline{x_n}} = a_{n1}\overline{x_1} + \cdots + a_{nn}\overline{x_n} + b_{n1}\overline{u_1} + \cdots + b_{n1m}\overline{u_m} \ ; \ \overline{x_{n1}}0) = 0
\end{bmatrix} \\
\begin{bmatrix}
\overline{y_1} = c_{11}\overline{x_1} + \cdots + c_{1n}\overline{x_n} + d_{11}\overline{u_1} + \cdots + d_{1m}\overline{u_m} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\overline{y_P} = c_{P1}\overline{x_1} + \cdots + c_{Pn}\overline{x_n} + d_{P1}\overline{u_1} + \cdots + d_{Pm}\overline{u_m}
\end{bmatrix}
\end{cases}
$$

Onde:

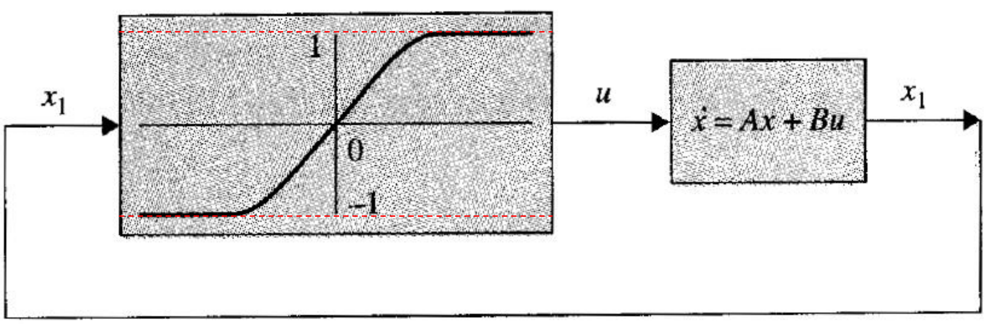
$$
\overline{y_P} = (y - y_{P0}) \quad ; \quad c_{Pm} = \left.\frac{\partial h_P}{\partial x_m}\right|_{\substack{h_P = h_{P0} \\ x_m = x_{m0}}} \quad ; \quad d_{Pm} = \left.\frac{\partial h_P}{\partial u_m}\right|_{\substack{h_P = h_{P0} \\ u_m = u_{m0}}}
$$

Em notação vetorial:

$$
\begin{cases}
\dot{\bar{x}} = A\bar{x} + B\bar{u} \quad ; \quad \overline{x_0} = 0 \\
\bar{y} = C\bar{x} + D\bar{u}
\end{cases}
$$

Os elementos das matrizes **A**, **B**, **C** e **D** são invariantes no tempo.
Na grande maioria dos casos **D = 0**, ou seja, não existe uma conexão direta dentre **u** em **y**. Ela ocorre através dos estados **x**.

Exemplo: A figura apresenta um diagrama de blocos de um sistema de controle com uma linearidade de saturação.



Linearizar o sistema em um Ponto de Operação dadas as equações de estado:

$$
\begin{cases}
\dot{x}_1(t) = f_1(x_1, x_2, u, t) = x_2(t) \\
\dot{x}_2(t) = f_2(x_1, x_2, u, t) = u(t)
\end{cases}
$$

A relação de entrada e saída da não linearidade de saturação é dada pela equação:

$$
u(t) = (1 - e^{-k|x_1(t)|}).\mathrm{sgn}[x_1(t)]
$$

Onde:

$$
sgn[x_1(t)] = \begin{cases} +1 & para \quad x_1(t) > 0 \\ -1 & para \quad x_1(t) < 0 \end{cases}
$$

Substituindo **u(t)** em **f_2(t)** tem-se:

$$
\begin{cases}
\dot{x}_1(t) = f_1(t) = x_2(t) \\
\dot{x}_2(t) = f_2(t) = (1 - e^{-k|x_1(t)|}).\,sgn[x_1(t)]
\end{cases}
$$

As equações de estado linearizadas serão:

$$\begin{cases} \overline{\dot{x}_1} \cong \overbrace{f_1(x_{10}, x_{20}, u_0)}^{(\dot{x}_1=0)} + \overbrace{\left.\frac{\partial f_1}{\partial x_1}\right|_{\substack{x_{10} \\ x_{20}}}}^{a_{11}} \overbrace{(x_1 - x_{10})}^{\bar{x_1}} + \overbrace{\left.\frac{\partial f_1}{\partial x_2}\right|_{\substack{x_{10} \\ x_{20}}}}^{a_{12}} \overbrace{(x_2 - x_{20})}^{\bar{x_2}} + \overbrace{\left.\frac{\partial f_1}{\partial u}\right|_{u_0}}^{b_{11}} \overbrace{(u - u_0)}^{\bar{u}} \\ \overline{\dot{x}_2} \cong \overbrace{f_2(x_{10}, x_{20} u_0)}^{(\dot{x}_2=0)} + \overbrace{\left.\frac{\partial f_2}{\partial x_1}\right|_{\substack{x_{10} \\ x_{20}}}}^{a_{21}} \overbrace{(x_1 - x_{10})}^{\bar{x_1}} + \overbrace{\left.\frac{\partial f_2}{\partial x_2}\right|_{\substack{x_{10} \\ x_{20}}}}^{a_{22}} \overbrace{(x_2 - x_{20})}^{\bar{x_2}} + \overbrace{\left.\frac{\partial f_2}{\partial u}\right|_{u_0}}^{b_{21}} \overbrace{(u - u_0)}^{\bar{u}} \end{cases}$$

Calculando:

$$\begin{cases} \overbrace{f_1(x_{10}, x_{20}, u_0)}^{(\dot{x}_1=0)} = 0\ ; \quad \overbrace{\left.\frac{\partial f_1}{\partial x_1}\right|_{\substack{x_{10} \\ x_{20}}}}^{a_{11}} = 0\ ; \quad (x_1 - x_{10}) = \overline{x_1}\ ; \quad \overbrace{\left.\frac{\partial f_1}{\partial x_2}\right|_{\substack{x_{10} \\ x_{20}}}}^{a_{12}} = 1\ ; \quad (x_2 - x_{20}) = \overline{x_2}\ ; \quad \overbrace{\left.\frac{\partial f_1}{\partial u}\right|_{u_0}}^{b_{11}} = 0 \\ \overbrace{f_2(x_{10}, x_{20}, u_0)}^{(\dot{x}_2=0)} = 0\ ; \quad \overbrace{\left.\frac{\partial f_2}{\partial x_1}\right|_{\substack{x_{10} \\ x_{20}}}}^{a_{21}} = ke^{-k|x_{10}|}\ ; \quad \overbrace{\left.\frac{\partial f_2}{\partial x_2}\right|_{\substack{x_{10} \\ x_{20}}}}^{a_{22}} = 0 \quad ; \quad \overbrace{\left.\frac{\partial f_2}{\partial u}\right|_{u_0}}^{b_{21}} = 1 \quad ; \quad (u - u_0) = \bar{u} \end{cases}$$

Substituindo:

$$\begin{cases} \overline{\dot{\mathrm{x}}_1} \cong 0.\overline{\mathrm{x}_1} + 1.\overline{\mathrm{x}_2} + 0.\bar{\mathrm{u}} \\ \overline{\dot{\mathrm{x}}_2} \cong (\mathrm{ke}^{-\mathrm{k}|\mathrm{x}_{10}|}).\overline{\mathrm{x}_1} + 0.\overline{\mathrm{x}_2} + 1.\bar{\mathrm{u}} \end{cases}$$

Em notação matricial:

$$\begin{bmatrix} \overline{\dot{\mathrm{x}}_1} \\ \overline{\dot{\mathrm{x}}_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \mathrm{ke}^{-\mathrm{k}|\mathrm{x}_{10}|} & 0 \end{bmatrix} . \begin{bmatrix} \overline{\mathrm{x}_1} \\ \overline{\mathrm{x}_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} . [\bar{\mathrm{u}}]$$

Exemplo) Linearizar em um Ponto de Operação o sistema não linear dado por:

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(x_1, x_2, u, t) = x_1(t) + x_2^2(t) \\ \dfrac{dx_2}{dt} = f_2(x_1, x_2, u, t) = x_1(t) + u(t) \end{cases}$$

Linearizando as equações de estado:

$$\begin{cases} \overline{\dot{x}_1} \cong \overbrace{f_1(x_{10}, x_{20}, u_0)}^{(\dot{x}_1=0)} + \overbrace{\left.\dfrac{\partial f_1}{\partial x_1}\right|_{\substack{x_{10}\\x_{20}}}}^{a_{11}} \overbrace{(x_1 - x_{10})}^{\overline{x_1}} + \overbrace{\left.\dfrac{\partial f_1}{\partial x_2}\right|_{\substack{x_{10}\\x_{20}}}}^{a_{12}} \overbrace{(x_2 - x_{20})}^{\overline{x_2}} + \overbrace{\left.\dfrac{\partial f_1}{\partial u}\right|_{u_0}}^{b_{11}} \overbrace{(u - u_0)}^{\overline{u}} \\ \overline{\dot{x}_2} \cong \overbrace{f_2(x_{10}, x_{20} u_0)}^{(\dot{x}_2=0)} + \overbrace{\left.\dfrac{\partial f_2}{\partial x_1}\right|_{\substack{x_{10}\\x_{20}}}}^{a_{21}} \overbrace{(x_1 - x_{10})}^{\overline{x_1}} + \overbrace{\left.\dfrac{\partial f_2}{\partial x_2}\right|_{\substack{x_{10}\\x_{20}}}}^{a_{22}} \overbrace{(x_2 - x_{20})}^{\overline{x_2}} + \overbrace{\left.\dfrac{\partial f_2}{\partial u}\right|_{u_0}}^{b_{21}} \overbrace{(u - u_0)}^{\overline{u}} \end{cases}$$

Calculando os coeficientes:

$$\begin{cases} \overbrace{f_1(x_{10}, x_{20}, u_0)}^{(\dot{x}_1=0)} = 0 \;; \quad \overbrace{\left.\dfrac{\partial f_1}{\partial x_1}\right|_{\substack{x_{10}\\x_{20}}}}^{a_{11}} = 1 \;; \quad (x_1 - x_{10}) = \overline{x_1} \;; \quad \overbrace{\left.\dfrac{\partial f_1}{\partial x_2}\right|_{\substack{x_{10}\\x_{20}}}}^{a_{12}} = 2x_2(t)|_{\substack{x_{10}\\x_{20}}} \;; \quad \overbrace{\left.\dfrac{\partial f_1}{\partial u}\right|_{u_0}}^{b_{11}} = 0 \\ \overbrace{f_2(x_{10}, x_{20}, u_0)}^{(\dot{x}_2=0)} = 0 \;; \quad \overbrace{\left.\dfrac{\partial f_2}{\partial x_1}\right|_{\substack{x_{10}\\x_{20}}}}^{a_{21}} = 1 \quad ; \quad \overbrace{\left.\dfrac{\partial f_2}{\partial x_2}\right|_{\substack{x_{10}\\x_{20}}}}^{a_{22}} = 0 \quad ; \quad \overbrace{\left.\dfrac{\partial f_2}{\partial u}\right|_{u_0}}^{b_{21}} = 1 \quad ; \quad (x_2 - x_{20}) = \overline{x_2} \;; \; (u - u_0) = \overline{u} \end{cases}$$

Substituindo:

$$\begin{cases} \overline{\dot{x}_1} \cong 1.\overline{x_1} + 2x_2(t)|_{\substack{x_{10}\\x_{20}}}.\overline{x_2} + 0.\overline{u} \\ \overline{\dot{x}_2} \cong 1.\overline{x_1} + 0.\overline{x_2} + 1.|\overline{u} \end{cases}$$

Em notação matricial:

$$\begin{bmatrix} \overline{\dot{x}_1} \\ \overline{\dot{x}_2} \end{bmatrix} = \begin{bmatrix} 1 & 2x_2(t)|_{\substack{x_{10}\\x_{20}}} \\ 1 & 0 \end{bmatrix} . \begin{bmatrix} \overline{x_1} \\ \overline{x_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} . [\overline{u}]$$

Exemplo) Linearizar em um Ponto de Operação o sistema não linear seguinte.

$$\begin{cases} \overset{\bullet}{x}_1(t) = f_1(x_1, x_2, u, t) = \dfrac{-1}{x_2^2(t)} \\ \overset{\bullet}{x}_2(t) = f_2(x_1, x_2, u, t) = x_1(t).u(t) \end{cases}$$

Linearizando as equações de estado:

$$\begin{cases} \overline{\overset{\bullet}{x}_1} \cong \overbrace{f_1(x_{10}, x_{20}, u_0)}^{(\overset{\bullet}{x}_1=0)} + \overbrace{\left.\dfrac{\partial f_1}{\partial x_1}\right|_{\substack{x_{10}\\x_{20}}}}^{a_{11}} \overbrace{(x_1 - x_{10})}^{\overline{x}_1} + \overbrace{\left.\dfrac{\partial f_1}{\partial x_2}\right|_{\substack{x_{10}\\x_{20}}}}^{a_{12}} \overbrace{(x_2 - x_{20})}^{\overline{x}_2} + \overbrace{\left.\dfrac{\partial f_1}{\partial u}\right|_{u_0}}^{b_{11}} \overbrace{(u - u_0)}^{\overline{u}} \\ \overline{\overset{\bullet}{x}_2} \cong \overbrace{f_2(x_{10}, x_{20}u_0)}^{(\overset{\bullet}{x}_2=0)} + \overbrace{\left.\dfrac{\partial f_2}{\partial x_1}\right|_{\substack{x_{10}\\x_{20}}}}^{a_{21}} \overbrace{(x_1 - x_{10})}^{\overline{x}_1} + \overbrace{\left.\dfrac{\partial f_2}{\partial x_2}\right|_{\substack{x_{10}\\x_{20}}}}^{a_{22}} \overbrace{(x_2 - x_{20})}^{\overline{x}_2} + \overbrace{\left.\dfrac{\partial f_2}{\partial u}\right|_{u_0}}^{b_{21}} \overbrace{(u - u_0)}^{\overline{u}} \end{cases}$$

Calculando:

$$\begin{cases} \overbrace{f_1(x_{10}, x_{20}, u_0)}^{(\overset{\bullet}{x}_1=0)} = 0\,;\ \overbrace{\left.\dfrac{\partial f_1}{\partial x_1}\right|_{\substack{x_{10}\\x_{20}}}}^{a_{11}} = 0\,;\ (x_1 - x_{10}) = \overline{x}_1\,;\ \overbrace{\left.\dfrac{\partial f_1}{\partial x_2}\right|_{\substack{x_{10}\\x_{20}}}}^{a_{12}} = \left.\dfrac{2}{x_2^3(t)}\right|_{\substack{x_{10}\\x_{20}}}\,;\ \overbrace{\left.\dfrac{\partial f_1}{\partial u}\right|_{u_0}}^{b_{11}} = 0\,;(x_2 - x_{20}) = \overline{x}_2 \\ \overbrace{f_2(x_{10}, x_{20}, u_0)}^{(\overset{\bullet}{x}_2=0)} = 0\,;\ \overbrace{\left.\dfrac{\partial f_2}{\partial x_1}\right|_{\substack{x_{10}\\x_{20}}}}^{a_{21}} = u(t)\,;\ \overbrace{\left.\dfrac{\partial f_2}{\partial x_2}\right|_{\substack{x_{10}\\x_{20}}}}^{a_{22}} = 0\,;\ \overbrace{\left.\dfrac{\partial f_2}{\partial u}\right|_{u_0}}^{b_{21}} = x_1(t)\,;(u - u_0) = \overline{u} \end{cases}$$

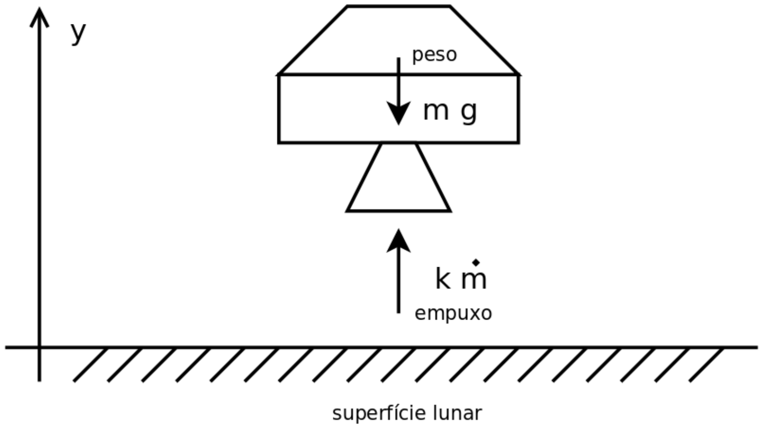
Substituindo:

$$\begin{cases} \overline{\overset{\bullet}{x}_1} \cong 0.\,\overline{x}_1 + \left.\dfrac{2}{x_2^3(t)}\right|_{\substack{x_{10}\\x_{20}}}.\,\overline{x}_2 + 0.\,\overline{u} \\ \overline{\overset{\bullet}{x}_2} \cong u(t)\big|_{u_0}.\,\overline{x}_1 + 0.\,\overline{x}_2 + x_1(t)\big|_{\substack{x_{10}\\x_{20}}}.\,\overline{u} \end{cases}$$

Em notação matricial:

$$\begin{bmatrix} \overline{\overset{\bullet}{x}_1} \\ \overline{\overset{\bullet}{x}_2} \end{bmatrix} = \begin{bmatrix} 0 & \left.\dfrac{2}{x_2^3(t)}\right|_{\substack{x_{10}\\x_{20}}} \\ u(t)\big|_{u_0} & 0 \end{bmatrix} . \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ x_1(t)\big|_{\substack{x_{10}\\x_{20}}} \end{bmatrix} . \begin{bmatrix} \overline{u} \end{bmatrix}$$

Exemplo) O pouso suave de uma nave na lua pode ser modelado como mostra o esquema a seguir.



O empuxo gerado pelo propulsor é proporcional a $\dot{\mathbf{m}}$, onde **m** é a massa do módulo lunar. A dinâmica do sistema pode ser representada por $\mathbf{m}\ddot{\mathbf{y}} = -\mathbf{k}\dot{\mathbf{m}} - \mathbf{mg}$, onde **g** é a constante gravitacional da superfície lunar. Definindo os estados $\mathbf{x_1 = y}$, $\mathbf{x_2} = \dot{\mathbf{y}}$, $\mathbf{x_3} = \mathbf{m}$ e a entrada $\mathbf{u} = \dot{\mathbf{m}}$, linearize a **equação no espaço de estados** em um **Ponto de Operação** para o sistema.

As esquações de espaço de estados serão dadas por:

$$\begin{cases} \dot{x}_1(t) = f_1(x_1, x_2, x_3, u, t) \\ \dot{x}_2(t) = f_2(x_1, x_2, x_3, u, t) \\ \dot{x}_3(t) = f_3(x_1, x_2, x_3, u, t) \end{cases}$$

Fazendo;

$$\mathrm{x}_1 = \mathrm{y} \Rightarrow \dot{x}_1 = \dot{y}\,; \quad como\; x_2 = \dot{y} \Rightarrow \dot{x}_1 = x_2\,;$$

Sabe-se que:

$$x_2 = \dot{y} \Rightarrow \dot{x}_2 = \ddot{y} \Rightarrow \dot{x}_2 = -\frac{k}{m}.\dot{m} - \frac{m}{m}.g \Rightarrow \boxed{\dot{x}_2 = -\frac{k}{m}.\dot{m} - g}$$

Substituindo:

$$\mathrm{u} = \dot{\mathrm{m}} \Rightarrow \boxed{\dot{\mathrm{x}}_2 = -\frac{\mathrm{k}}{\mathrm{m}}\mathrm{u} - \mathrm{g}}$$

Como $\mathbf{x_3} = \mathbf{m}$:

$$\dot{\mathrm{x}}_3 = \dot{\mathrm{m}} \Rightarrow \boxed{\dot{\mathrm{x}}_3 = \mathrm{u}}$$

As equações no espaço de estados serão:

$$\begin{cases} \dot{x}_1 = f_1(x_1,x_2,x_3,u,t) = 0x_1 + x_2 + 0x_3 + 0u \\ \dot{x}_2 = f_2(x_1,x_2,x_3,u,t) = 0x_1 + 0x_2 + 0x_3 - \dfrac{k}{m}.u - g \Rightarrow \boxed{\textit{Não Linear!}} \\ \dot{x}_3 = f_3(x_1,x_2,x_3,u,t) = 0x_1 + 0x_2 + 0x_3 + u \end{cases}$$

Matricialmente:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \overbrace{\left(-\dfrac{k}{m} - g\right)}^{\text{Como???}} \\ 1 \end{bmatrix} . [u]$$

Linearizando:

$$\begin{cases} \overline{\dot{x}_1} = \overbrace{f_1(x_{10},x_{20},x_{30},u_0)}^{(\dot{x}_1=0)} + \overbrace{\left.\dfrac{\partial f_1}{\partial x_1}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{11}} \overbrace{(x_1 - x_{10})}^{\bar{x}_1} + \overbrace{\left.\dfrac{\partial f_1}{\partial x_2}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{12}} \overbrace{(x_2 - x_{20})}^{\bar{x}_2} + \overbrace{\left.\dfrac{\partial f_1}{\partial x_3}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{13}} \overbrace{(x_3 - x_{30})}^{\bar{x}_3} + \overbrace{\left.\dfrac{\partial f_1}{\partial u}\right|_{u_0}}^{b_{11}} \overbrace{(u - u_0)}^{\bar{u}} \\ \overline{\dot{x}_2} \cong \overbrace{f_2(x_{10},x_{20},x_{30},u_0)}^{(\dot{x}_2=0)} + \overbrace{\left.\dfrac{\partial f_2}{\partial x_1}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{21}} \overbrace{(x_1 - x_{10})}^{\bar{x}_1} + \overbrace{\left.\dfrac{\partial f_2}{\partial x_2}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{22}} \overbrace{(x_2 - x_{20})}^{\bar{x}_2} + \overbrace{\left.\dfrac{\partial f_2}{\partial x_3}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{23}} \overbrace{(x_3 - x_{30})}^{\bar{x}_3} + \overbrace{\left.\dfrac{\partial f_2}{\partial u}\right|_{u_0}}^{b_{21}} \overbrace{(u - u_0)}^{\bar{u}} \\ \overline{\dot{x}_3} \cong \overbrace{f_3(x_{10},x_{20},x_{30},u_0)}^{(\dot{x}_3=0)} + \overbrace{\left.\dfrac{\partial f_3}{\partial x_1}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{31}} \overbrace{(x_1 - x_{10})}^{\bar{x}_1} + \overbrace{\left.\dfrac{\partial f_3}{\partial x_2}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{32}} \overbrace{(x_2 - x_{20})}^{\bar{x}_2} + \overbrace{\left.\dfrac{\partial f_3}{\partial x_3}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{33}} \overbrace{(x_3 - x_{30})}^{\bar{x}_3} + \overbrace{\left.\dfrac{\partial f_3}{\partial u}\right|_{u_0}}^{b_{31}} \overbrace{(u - u_0)}^{\bar{u}} \end{cases}$$

Calculando:

$$\begin{cases} \overbrace{f_1(x_{10},x_{20},x_{30},u_0)}^{(\dot{x}_1=0)} = 0\,;\ \overbrace{\left.\dfrac{\partial f_1}{\partial x_1}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{11}} = 0\,;\ \overbrace{\left.\dfrac{\partial f_1}{\partial x_2}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{12}} = 1\,;\ \overbrace{\left.\dfrac{\partial f_1}{\partial x_3}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{13}} = 0\,;\ \overbrace{\left.\dfrac{\partial f_1}{\partial u}\right|_{u_0}}^{b_{11}} = 0 \\ \overbrace{f_2(x_{10},x_{20},x_{30},u_0)}^{(\dot{x}_2=0)} = 0\,;\ \overbrace{\left.\dfrac{\partial f_2}{\partial x_1}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{21}} = 0\,;\ \overbrace{\left.\dfrac{\partial f_2}{\partial x_2}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{22}} = 0\,;\ \overbrace{\left.\dfrac{\partial f_2}{\partial x_3}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{23}} = 0\,;\ \overbrace{\left.\dfrac{\partial f_2}{\partial u}\right|_{u_0}}^{b_{21}} = -\left.\dfrac{k}{x_3}\right|_{x_{30}} \\ \overbrace{f_3(x_{10},x_{20},x_{30},u_0)}^{(\dot{x}_3=0)} = 0\,;\ \overbrace{\left.\dfrac{\partial f_3}{\partial x_1}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{31}} = 0\,;\ \overbrace{\left.\dfrac{\partial f_3}{\partial x_2}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{32}} = 0\,;\ \overbrace{\left.\dfrac{\partial f_3}{\partial x_3}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{33}} = 0\,;\ \overbrace{\left.\dfrac{\partial f_3}{\partial u}\right|_{u_0}}^{b_{31}} = 1 \end{cases}$$

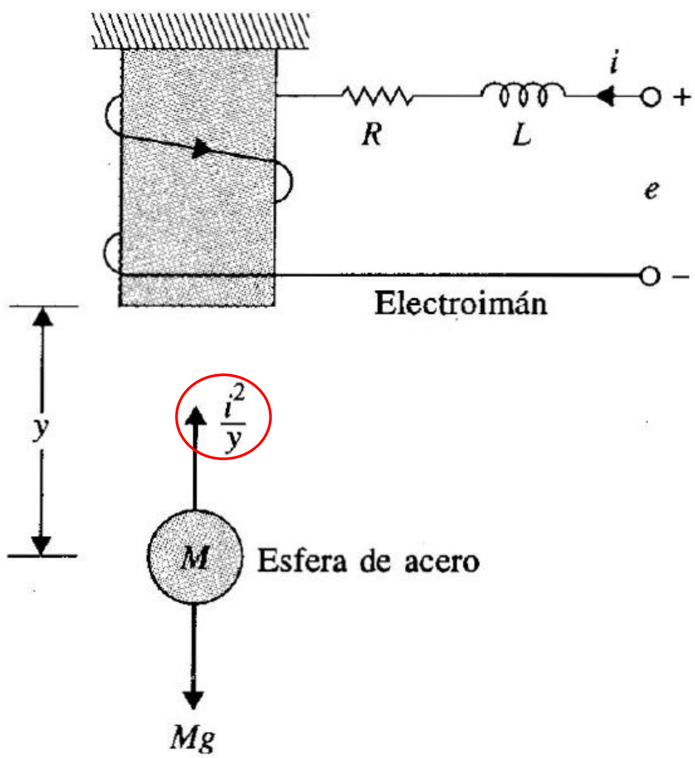
Substituindo:

$$\begin{cases} \dot{\bar{x}}_1 = 0 + 0.\bar{x}_1 + 1.\bar{x}_2 + 0.\bar{x}_3 + 0.\bar{u} \\ \dot{\bar{x}}_2 = 0 + 0.\bar{x}_1 + 0.\bar{x}_2 + 0.\bar{x}_3 - \dfrac{k}{m}.\bar{u} \\ \dot{\bar{x}}_3 = 0 + 0.\bar{x}_1 + 0.\bar{x}_2 + 0.\bar{x}_3 + 1.\bar{u} \end{cases}$$

Em notação matricial:

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -\dfrac{k}{m} \\ 1 \end{bmatrix} . \begin{bmatrix} \bar{u} \end{bmatrix}$$

Exemplo) Considere um sistema de suspensão magnética aplicado a uma esfera de aço. O objetivo deste sistema é controlar a corrente do eletroímã para que a esfera de aço fique suspensa a uma distância fixa **y** do eletroímã. A figura a seguir ilustra este sistema.



Determinar as equações de espaço de estados para este sistema.

As equações dinâmicas do sistema são dadas por:

$$\begin{cases} M\dfrac{d^2y(t)}{dt^2} = Mg - \dfrac{i(t)^2}{y(t)} \\ e = L\dfrac{di(t)}{dt} + Ri(t) \end{cases}$$

Variáveis de estado: $\boxed{x_1 = y(t)}$; $\boxed{x_2 = \dot{y}(t) \Rightarrow \dot{x}_1 = x_2}$; $\boxed{x_3 = i(t)}$ e $\boxed{u = e(t)}$

Daí:

$$\dot{x}_2 = \ddot{y}(t) = \frac{1}{M}(Mg - \frac{i(t)^2}{y(t)}) \Rightarrow \boxed{\dot{x}_2 = g - \frac{x_3^2}{Mx_1}} \quad e \quad \dot{x}_3 = \frac{di(t)}{dt} = \frac{1}{L}(e - Ri(t)) \Rightarrow \boxed{\dot{x}_3 = -\frac{Rx_3}{L} + \frac{1}{L}u}$$

As equações de estado serão:

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, x_3, u, t) = x_2 \\ \dot{x}_2 = f_2(x_1, x_2, x_3, u, t) = g - \dfrac{x_3^2}{Mx_1} \quad (\textbf{\textit{Não linear}} \text{ !!!}) \\ \dot{x}_3 = f_1(x_1, x_2, x_3, u, t) = -\dfrac{R}{L}x_3 + \dfrac{1}{L}u \end{cases}$$

A equação $\dot{x}_2 = g - \frac{x_3^2}{Mx_1}$ **não é linear!**

Linearizando:

$$\begin{cases} \overline{\dot{x}_1} = \overbrace{f_1(x_{10}, x_{20}, x_{30}, u_0)}^{(\dot{x}_1=0)} + \overbrace{\left.\dfrac{\partial f_1}{\partial x_1}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{11}} \overbrace{(x_1 - x_{10})}^{\bar{x}_1} + \overbrace{\left.\dfrac{\partial f_1}{\partial x_2}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{12}} \overbrace{(x_2 - x_{20})}^{\bar{x}_2} + \overbrace{\left.\dfrac{\partial f_1}{\partial x_3}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{13}} \overbrace{(x_3 - x_{30})}^{\bar{x}_3} + \overbrace{\left.\dfrac{\partial f_1}{\partial u}\right|_{u_0}}^{b_{11}} \overbrace{(u - u_0)}^{\bar{u}} \\ \overline{\dot{x}_2} \cong \overbrace{f_2(x_{10}, x_{20}, x_{30}, u_0)}^{(\dot{x}_2=0)} + \overbrace{\left.\dfrac{\partial f_2}{\partial x_1}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{21}} \overbrace{(x_1 - x_{10})}^{\bar{x}_1} + \overbrace{\left.\dfrac{\partial f_2}{\partial x_2}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{22}} \overbrace{(x_2 - x_{20})}^{\bar{x}_2} + \overbrace{\left.\dfrac{\partial f_2}{\partial x_3}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{23}} \overbrace{(x_3 - x_{30})}^{\bar{x}_3} + \overbrace{\left.\dfrac{\partial f_2}{\partial u}\right|_{u_0}}^{b_{21}} \overbrace{(u - u_0)}^{\bar{u}} \\ \overline{\dot{x}_3} \cong \overbrace{f_3(x_{10}, x_{20}, x_{30}, u_0)}^{(\dot{x}_3=0)} + \overbrace{\left.\dfrac{\partial f_3}{\partial x_1}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{31}} \overbrace{(x_1 - x_{10})}^{\bar{x}_1} + \overbrace{\left.\dfrac{\partial f_3}{\partial x_2}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{32}} \overbrace{(x_2 - x_{20})}^{\bar{x}_2} + \overbrace{\left.\dfrac{\partial f_3}{\partial x_3}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{33}} \overbrace{(x_3 - x_{30})}^{\bar{x}_3} + \overbrace{\left.\dfrac{\partial f_3}{\partial u}\right|_{u_0}}^{b_{31}} \overbrace{(u - u_0)}^{\bar{u}} \end{cases}$$

Calculando:

$$
\begin{cases}
\overbrace{f_1(x_{10},x_{20},x_{30},u_0)}^{(\dot{x}_1=0)} = 0 \;;\; \overbrace{\left.\dfrac{\partial f_1}{\partial x_1}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{11}} = 0 \;;\; \overbrace{\left.\dfrac{\partial f_1}{\partial x_2}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{12}} = 1 \;;\; \overbrace{\left.\dfrac{\partial f_1}{\partial x_3}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{13}} = 0 \;;\; \overbrace{\left.\dfrac{\partial f_1}{\partial u}\right|_{u_0}}^{b_{11}} = 0 \\
\overbrace{f_2(x_{10},x_{20},x_{30},u_0)}^{(\dot{x}_2=0)} = 0 \;;\; \overbrace{\left.\dfrac{\partial f_2}{\partial x_1}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{21}} = \dfrac{x_{30}^2}{Mx_{10}^2} =;\; \overbrace{\left.\dfrac{\partial f_2}{\partial x_2}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{22}} = 0 \;;\; \overbrace{\left.\dfrac{\partial f_2}{\partial x_3}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{23}} = -\dfrac{2x_{30}}{Mx_{10}} \;;\; \overbrace{\left.\dfrac{\partial f_2}{\partial u}\right|_{u_0}}^{b_{21}} = 0 \\
\overbrace{f_3(x_{10},x_{20},x_{30},u_0)}^{(\dot{x}_3=0)} = 0 \;;\; \overbrace{\left.\dfrac{\partial f_3}{\partial x_1}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{31}} = 0 \;;\; \overbrace{\left.\dfrac{\partial f_3}{\partial x_2}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{32}} = 0 \;;\; \overbrace{\left.\dfrac{\partial f_3}{\partial x_3}\right|_{\substack{x_{10}\\x_{20}\\x_{30}}}}^{a_{33}} = -\dfrac{R}{L} \;;\; \overbrace{\left.\dfrac{\partial f_3}{\partial u}\right|_{u_0}}^{b_{31}} = \dfrac{1}{L}
\end{cases}
$$

Substituindo:

$$
\begin{cases}
\overline{\dot{x}_1} = 0 + 0.\,\overline{x_1} + 1.\,\overline{x_2} + 0.\,\overline{x_3} + 0.\,\overline{u} \\
\overline{\dot{x}_2} = 0 + \dfrac{x_{30}^2}{Mx_{10}^2}.\,\overline{x_1} + 0.\,\overline{x_2} - \dfrac{2x_{30}}{Mx_{10}}.\,\overline{x_3} - 0.\,\overline{u} \\
\overline{\dot{x}_3} = 0 + 0.\,\overline{x_1} + 0.\,\overline{x_2} - \dfrac{R}{L}.\,\overline{x_3} + \dfrac{1}{L}.\,\overline{u}
\end{cases}
$$

Em notação matricial:

$$
\begin{bmatrix} \overline{\dot{x}_1} \\ \overline{\dot{x}_2} \\ \overline{\dot{x}_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \left(\dfrac{x_{30}^2}{Mx_{10}^2}\right) & 0 & \left(-\dfrac{2x_{30}}{Mx_{10}}\right) \\ 0 & 0 & \left(-\dfrac{R}{L}\right) \end{bmatrix} . \begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \left(\dfrac{1}{L}\right) \end{bmatrix} . \begin{bmatrix} \overline{u} \end{bmatrix}
$$

Precisamos dos valores de x_{10} e de x_{30} para a linearização.
Este sistema tem o ponto de equilíbrio (P.E.) quando:

$$\boxed{x_{10} = y_0(t) = \text{constante}}$$

Então:

$$\frac{dy_0(t)}{dt} = 0 \quad e \quad \frac{d^2y_0(t)}{dt^2} = 0$$

O valor de $x_{30} = i_0(t)$ é calculado pela equação a seguir:

$$M\frac{d^2y(t)}{dt^2} = Mg - \frac{i\,(t)^2}{y(t)} \Rightarrow M\frac{d^2y_0(t)}{dt^2} = Mg - \frac{i_0(t)^2}{y_0} \Rightarrow M.(0) = Mg - \frac{x_{30}^2}{x_{10}} \Rightarrow$$
$$\Rightarrow \boxed{x_{30} = \sqrt{Mgx_{10}}}$$

Substituindo na equação matricial:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ \dfrac{x_{30}^2}{Mx_{10}^2} & 0 & -\dfrac{2x_{30}}{Mx_{10}}\\ 0 & 0 & -\dfrac{R}{L}\end{bmatrix}.\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \dfrac{1}{L}\end{bmatrix}.[u] \Rightarrow$$

$$\Rightarrow \boxed{\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ \dfrac{g}{x_{10}} & 0 & -2\sqrt{\dfrac{g}{Mx_{10}}}\\ 0 & 0 & -\dfrac{R}{L}\end{bmatrix}.\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \dfrac{1}{L}\end{bmatrix}.[u]}$$

Considerando os valores a seguir:

g = 9,8 m/s² (aceleração da gravidade)
M = 1 Kg (massa da esfera)
R = 1 Ω (resistência)
L = 0,01 H (auto-indutância)

E para um **Ponto de Operação $x_{10} = y_0 = 0,5$ m**, tem-se:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ \dfrac{9,81}{0,5} & 0 & -2\sqrt{\dfrac{9,81}{1.(0,5)}}\\ 0 & 0 & -\dfrac{1}{0,01}\end{bmatrix}.\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \dfrac{1}{0,01}\end{bmatrix}.[u] \Rightarrow$$

$$\Rightarrow \boxed{\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 19,62 & 0 & -8,86\\ 0 & 0 & -100\end{bmatrix}.\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 100\end{bmatrix}.[u]}$$