2.1 Consider the memoryless system with characteristics shown in Fig 2.19, in which ***u*** denotes the input and ***y*** the output. Which of them is a linear system? Is it possible to introduce a new output so that the system in Fig 2.19(b) is linear?

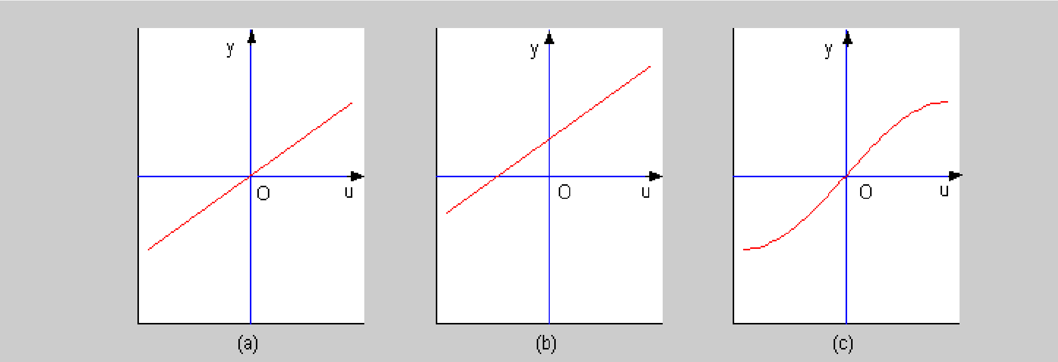


Figure 2.19

Translation: 考虑具有图 2.19 中表示的特性的无记忆系统。其中 ***u*** 表示输入，***y*** 表示输出。下面哪一个是线性系统？可以找到一个新的输出，使得图 2.19(b)中的系统是线性的吗？

Answer: The input-output relation in Fig 2.1(a) can be described as:

$$y = a * u$$

Here ***a*** is a constant. It is a memoryless system. Easy to testify that it is a linear system.
The input-output relation in Fig 2.1(b) can be described as:

$$y = a * u + b$$

Here ***a*** and ***b*** are all constants. Testify whether it has the property of additivity. Let:

$$y_1 = a * u_1 + b$$

$$y_2 = a * u_2 + b$$

then:

$$(y_1 + y_2) = a * (u_1 + u_2) + 2 * b$$

So it does not has the property of additivity, therefore, is not a linear system.
But we can introduce a new output so that it is linear. Let:

$$z = y - b$$

$$z = a * u$$

z is the new output introduced. Easy to testify that it is a linear system.
The input-output relation in Fig 2.1(c) can be described as:

$$y = a(u) * u$$

a(u) is a function of input ***u***. Choose two different input, get the outputs:

$$y_1 = a_1 * u_1$$

$$y_2 = a_2 * u_2$$

Assure:

$$a_1 \neq a_2$$

then:

$$(y_1 + y_2) = a_1 * u_1 + a_2 * u_2$$

So it does not has the property of additivity, therefore, is not a linear system.

2.2 The impulse response of an ideal lowpass filter is given by

$$g(t) = 2\omega \frac{\sin 2\omega(t - t_0)}{2\omega(t - t_0)}$$

for all ***t***, where ***w*** and ***to*** are constants. Is the ideal lowpass filter causal? Is is possible to built the filter in the real world?

Translation: 理想低通滤波器的冲激响应如式所示。对于所有的 ***t***，***w*** 和 ***to***，都是常数。理想低通滤波器是因果的吗？现实世界中有可能构造这种滤波器吗？

Answer: Consider two different time: ***ts*** and ***tr***, ***ts < tr***, the value of *g(ts-tr)* denotes the output at time ***ts***, excited by the impulse input at time ***tr***. It indicates that the system output at time ***ts*** is dependent on future input at time ***tr***. In other words, the system is not causal. We know that all physical system should be causal, so it is impossible to built the filter in the real world.

2.3 Consider a system whose input ***u*** and output ***y*** are related by

$$y(t) = (P_a u)(t) := \begin{cases} u(t) & for \quad t \leq a \\ 0 & for \quad t > a \end{cases}$$

where ***a*** is a fixed constant. The system is called a truncation operator, which chops off the input after time ***a***. Is the system linear? Is it time-invariant? Is it causal?

Translation: 考虑具有如式所示输入输出关系的系统，***a*** 是一个确定的常数。这个系统称作截断器。它截断时间 ***a*** 之后的输入。这个系统是线性的吗？它是定常的吗？是因果的吗？

Answer: Consider the input-output relation at any time ***t, t<=a***:

$$y = u$$

Easy to testify that it is linear.

Consider the input-output relation at any time ***t, t>a***:

$$y = 0$$

Easy to testify that it is linear. So for any time, the system is linear.

Consider whether it is time-invariable. Define the initial time of input ***to***, system input is ***u(t)***, ***t>=to***. Let ***to<a***, so It decides the system output ***y(t)***, ***t>=to***:

$$y(t)=\begin{cases}u(t) & for \quad t_0 \le t \le a \\ 0 & for \quad other \quad t\end{cases}$$

Shift the initial time to ***to+T***. Let ***to+T>a*** , then input is ***u(t-T)***, ***t>=to+T***. System output:

$$y'(t)=0$$

Suppose that ***u(t)*** is not equal to 0, ***y'(t)*** is not equal to ***y(t-T)***. According to the definition, this system is not time-invariant.

For any time ***t***, system output ***y(t)*** is decided by current input ***u(t)*** exclusively. So it is a causal system.

2.4 The input and output of an initially relaxed system can be denoted by ***y=Hu***, where ***H*** is some mathematical operator. Show that if the system is causal, then

$$P_a y = P_a Hu = P_a HP_a u$$

where ***Pa*** is the truncation operator defined in Problem 2.3. Is it true ***PaHu=HPau***?

Translation: 一个初始松弛系统的输入输出可以描述为：***y=Hu***，这里 ***H*** 是某种数学运算，说明假如系统是因果性的，有如式所示的关系。这里 ***Pa*** 是题 2.3 中定义的截断函数。***PaHu=HPau*** 是正确的吗？

Answer: Notice ***y=Hu***, so:

$$P_a y = P_a Hu$$

Define the initial time **0**, since the system is causal, output y begins in time **0**.

If ***a*<=0**,then ***u=Hu***. Add operation ***PaH*** in both left and right of the equation:

$$P_a Hu = P_a HP_a u$$

If ***a*>0**, we can divide ***u*** to 2 parts:

$$p(t)=\begin{cases}u(t) & for \quad 0 \le t \le a \\ 0 & for \quad other \quad t\end{cases}$$

$$q(t)=\begin{cases}u(t) & for \quad t > a \\ 0 & for \quad other \quad t\end{cases}$$

u(t)=p(t)+q(t). Pay attention that the system is casual, so the output excited by ***q(t)*** can't affect that of ***p(t)***. It is to say, system output from **0** to ***a*** is decided only by ***p(t)***. Since ***PaHu*** chops off ***Hu*** after time ***a***, easy to conclude ***PaHu=PaHp(t)***. Notice that ***p(t)=Pau***, also we have:

$$P_a Hu = P_a HP_a u$$

It means under any condition, the following equation is correct:

$$P_a y = P_a Hu = P_a HP_a u$$

PaHu=HPau is false. Consider a delay operator ***H***, ***Hu(t)=u(t-2)***, and ***a*=1**, ***u(t)*** is a step input begins at time **0,** then ***PaHu*** covers from **1** to **2**, but ***HPau*** covers from **1** to **3**.

2.5 Consider a system with input ***u*** and output ***y***. Three experiments are performed on the system using the inputs ***u1(t)***, ***u2(t)*** and ***u3(t)*** for ***t>=0***. In each case, the initial state ***x(0)*** at time ***t=0*** is the same. The corresponding outputs are denoted by ***y1,y2*** and ***y3***. Which of the following statements are correct if ***x(0)<>0***?

1. If ***u3=u1+u2***, then ***y3=y1+y2***.
2. If ***u3=0.5(u1+u2)***, then ***y3=0.5(y1+y2)***.
3. If ***u3=u1-u2***, then ***y3=y1-y2***.

Translation: 考虑具有输入 ***u*** 输出 ***y*** 的系统。在此系统上进行三次实验，输入分别为 ***u1(t)***, ***u2(t)*** 和 ***u3(t)***，***t>=0***。每种情况下，零时刻的初态 ***x(0)***都是相同的。相应的输出表示为 ***y1,y2*** 和 ***y3***。在 ***x(0)***不等于零的情况下，下面哪种说法是正确的？

Answer: A linear system has the superposition property:

$$\left.\begin{array}{r}\alpha_1 x_1(t_0)+\alpha_2 x_2(t_0)\\ \alpha_1 u_1(t)+\alpha_2 u_2(t), t\ge t_0\end{array}\right\}\rightarrow \alpha_1 y_1(t)+\alpha_2 y_2(t), t\ge t_0$$

In case 1:

$$\alpha_1 = 1 \qquad \alpha_2 = 1$$

$$\alpha_1 x_1(t_0)+\alpha_2 x_2(t_0) = 2x(0) \neq x(0)$$

So ***y3<>y1+y2***.

In case 2:

$$\alpha_1 = 0.5 \qquad \alpha_2 = 0.5$$

$$\alpha_1 x_1(t_0)+\alpha_2 x_2(t_0) = x(0)$$

So ***y3=0.5(y1+y2)***.

In case 3:

$$\alpha_1 = 1 \qquad \alpha_2 = -1$$

$$\alpha_1 x_1(t_0)+\alpha_2 x_2(t_0) = 0 \neq x(0)$$

So ***y3<>y1-y2***.

2.6 Consider a system whose input and output are related by

$$y(t)=\begin{cases} u^2(t)/u(t-1) & if \quad u(t-1)\neq 0 \\ 0 \quad if \quad u(t-1)=0 \end{cases}$$

for all ***t***.

Show that the system satisfies the homogeneity property but not the additivity property.

Translation: 考虑输入输出关系如式的系统,证明系统满足齐次性,但是不满足可加性.

Answer: Suppose the system is initially relaxed, system input:

$$p(t)=\alpha u(t)$$

a is any real constant. Then system output ***q(t)***:

$$q(t)=\begin{cases} p^2(t)/p(t-1) & if \quad p(t-1)\neq 0 \\ 0 \quad if & p(t-1)=0 \end{cases}$$

$$=\begin{cases} \alpha u^2(t)/u(t-1) & if \quad u(t-1)\neq 0 \\ 0 \quad if & u(t-1)=0 \end{cases}$$

So it satisfies the homogeneity property.

If the system satisfies the additivity property, consider system input ***m(t)*** and ***n(t)***, ***m(0)***=1, ***m(1)***=2; ***n(0)***=-1, ***n(1)***=3. Then system outputs at time **1** are:

$$r(1)=m^2(1)/m(0)=4$$

$$s(1)=n^2(1)/n(0)=-9$$

$$y(1)=[m(1)+n(1)]^2/[m(0)+n(0)]=0$$

$$\neq r(1)+s(1)$$

So the system does not satisfy the additivity property.

2.7 Show that if the additivity property holds, then the homogeneity property holds for all rational numbers ***a*** . Thus if a system has "continuity" property, then additivity implies homogeneity.

Translation: 说明系统如果具有可加性，那么对所有有理数 ***a*** 具有齐次性。因而对具有某种连续性质的系统，可加性导致齐次性。

Answer: Any rational number ***a*** can be denoted by:

$$a=m/n$$

Here ***m*** and ***n*** are both integer. Firstly, prove that if system input-output can be described as following:

$$x\rightarrow y$$

then:

$$mx\rightarrow my$$

Easy to conclude it from additivity.

Secondly, prove that if a system input-output can be described as following:

$$x\rightarrow y$$

then:

$$x/n\rightarrow y/n$$

Suppose:

$$x/n\rightarrow u$$

Using additivity:

$$n*(x/n)=x\rightarrow nu$$

So:

$$y=nu$$

$$u=y/n$$

It is to say that:

$$x/n \to y/n$$

Then:

$$x*m/n \to y*m/n$$

$$ax \to ay$$

It is the property of homogeneity.

2.8 Let ***g(t,T)=g(t+a,T+a)*** for all ***t,T*** and ***a***. Show that ***g(t,T)*** depends only on ***t-T***.

Translation: 设对于所有的 ***t,T*** 和 ***a***，***g(t,T)=g(t+a,T+a)***。说明 ***g(t,T)***仅依赖于 ***t-T***。

Answer: Define:

$$x = t + T \qquad\qquad y = t - T$$

So:

$$t = \frac{x+y}{2} \qquad\qquad T = \frac{x-y}{2}$$

Then:

$$\begin{aligned} g(t,T) &= g\left(\frac{x+y}{2}, \frac{x-y}{2}\right) \\ &= g\left(\frac{x+y}{2} + a, \frac{x-y}{2} + a\right) \\ &= g\left(\frac{x+y}{2} + \frac{-x+y}{2}, \frac{x-y}{2} + \frac{-x+y}{2}\right) \\ &= g(y,0) \end{aligned}$$

So:

$$\frac{\partial g(t,T)}{\partial x} = \frac{\partial g(y,0)}{\partial x} = 0$$

It proves that ***g(t,T)*** depends only on ***t-T***.

2.9 Consider a system with impulse response as shown in Fig2.20(a). What is the zero-state response excited by the input ***u(t)*** shown in Fig2.20(b)?

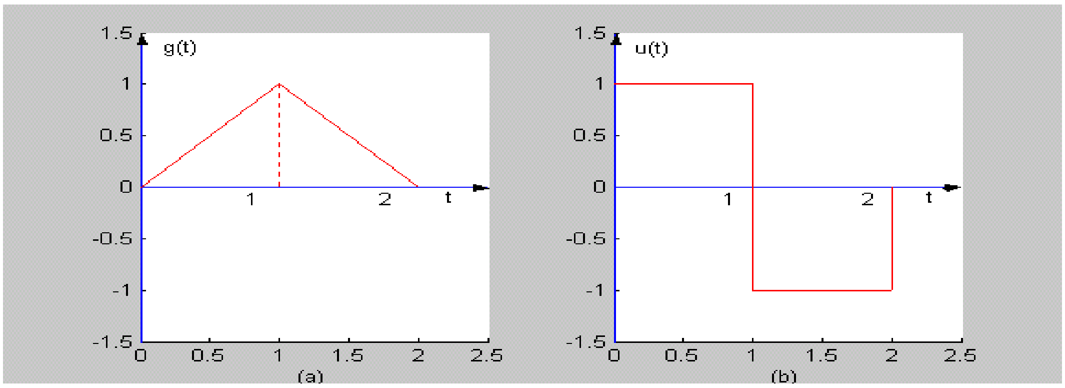


Fig2.20

Translation: 考虑冲激响应如图 2.20(a)所示的系统，由如图 2.20(b)所示输入 ***u(t)***激励的零状态响应是什么？

Answer: Write out the function of ***g(t)*** and ***u(t)***:

$$g(t)=\begin{cases} t & 0\le t\le 1 \\ 2-t & 1\le t\le 2 \end{cases}$$

$$u(t)=\begin{cases} 1 & 0\le t\le 1 \\ -1 & 1\le t\le 2 \end{cases}$$

then y(t) equals to the convolution integral:

$$y(t)=\int_0^t g(r)u(t-r)dr$$

If **0=<*t*=<1**, **0=<*r*=<1**, **0<=*t*-*r*<=1**:

$$y(t)=\int_0^t rdr=\frac{t^2}{2}$$

If **1<=*t*<=2**:

$$y(t)=\int_0^{t-1} g(r)u(t-r)dr+\int_{t-1}^{1} g(r)u(t-r)dr+\int_1^t g(r)u(t-r)dr$$

$$=y_1(t)+y_2(t)+y_3(t)$$

Calculate integral separately:

$$y_1(t)=\int_0^{t-1} g(r)u(t-r)dr \qquad 0\le r\le 1 \quad 1\le t-r\le 2$$

$$=\int_0^{t-1} -rdr=\frac{-(t-1)^2}{2}$$

$$y_2(t)=\int_{t-1}^{1} g(r)u(t-r)dr \qquad 0\le r\le 1 \quad 0\le t-r\le 1$$

$$=\int_{t-1}^{1} rdr=\frac{1}{2}-\frac{(t-1)^2}{2}$$

$$y_3(t)=\int_1^t g(r)u(t-r)dr \qquad 1\le r\le 2 \quad 0\le t-r\le 1$$

$$=\int_1^t (2-r)dr=2(t-1)-\frac{t^2-1}{2}$$

$$y(t)=y_1(t)+y_2(t)+y_3(t)=-\frac{3}{2}t^2+4t-2$$

2.10 Consider a system described by

$$\ddot{y}+2\dot{y}-3y=\dot{u}-u$$

What are the transfer function and the impulse response of the system?

Translation: 考虑如式所描述的系统，它的传递函数和冲激响应是什么？

Answer: Applying the Laplace transform to system input-output equation, supposing that the System is initial relaxed:

$$s^2Y(s)+2sY(s)-3Y(s)=sY(s)-Y(s)$$

System transfer function:

$$G(s)=\frac{U(s)}{Y(s)}=\frac{s-1}{s^2+2s-3}=\frac{1}{s+3}$$

Impulse response:

$$g(t)=L^{-1}[G(s)]=L^{-1}[\frac{1}{s+3}]=e^{-3t}$$

2.11 Let ***y(t)*** be the unit-step response of a linear time-invariant system. Show that the impulse response of the system equals ***dy(t)/dt.***

Translation: ***y(t)***是线性定常系统的单位阶跃响应。说明系统的冲激响应等于 ***dy(t)/dt***.

Answer: Let ***m(t)*** be the impulse response, and system transfer function is ***G(s)***:

$$Y(s)=G(s)*\frac{1}{s}$$

$$M(s)=G(s)$$

$$M(s)=Y(s)*s$$

So:

$$m(t)=dy(t)/dt$$

2.12 Consider a two-input and two-output system described by

$$D_{11}(p)y_1(t)+D_{12}(p)y_2(t)=N_{11}(p)u_1(t)+N_{12}(p)u_2(t)$$

$$D_{21}(p)y_1(t)+D_{22}(p)y_2(t)=N_{21}(p)u_1(t)+N_{22}(p)u_2(t)$$

where ***Nij*** and ***Dij*** are polynomials of ***p:=d/dt***. What is the transfer matrix of the system?

Translation: 考虑如式描述的两输入两输出系统， ***Nij*** 和 ***Dij*** 是 ***p:=d/dt*** 的多项式。系统的传递矩阵是什么？

Answer: For any polynomial of ***p***, ***N(p)***, its Laplace transform is ***N(s)***.

Applying Laplace transform to the input-output equation:

$$D_{11}(s)Y_1(s)+D_{12}(s)Y_2(s)=N_{11}(s)U_1(s)+N_{12}(s)U_2(s)$$

$$D_{21}(s)Y_1(s)+D_{22}(s)Y_2(s)=N_{21}(s)U_1(s)+N_{22}(s)U_2(s)$$

Write to the form of matrix:

$$\begin{bmatrix} D_{11}(s) & D_{12}(s) \\ D_{21}(s) & D_{22}(s) \end{bmatrix}\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix}=\begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix}\begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix}=\begin{bmatrix} D_{11}(s) & D_{12}(s) \\ D_{21}(s) & D_{22}(s) \end{bmatrix}^{-1}\begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix}\begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

So the transfer function matrix is:

$$G(s)=\begin{bmatrix} D_{11}(s) & D_{12}(s) \\ D_{21}(s) & D_{22}(s) \end{bmatrix}^{-1}\begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix}$$

By the premising that the matrix inverse:

$$\begin{bmatrix} D_{11}(s) & D_{12}(s) \\ D_{21}(s) & D_{22}(s) \end{bmatrix}^{-1}$$

exists.

2.11 Consider the feedback systems shows in **Fig2.5**. Show that the unit-step responses of the positive-feedback system are as shown in **Fig2.21(a)** for ***a*=1** and in **Fig2.21(b)** for ***a*=0.5**. Show also that the unit-step responses of the negative-feedback system are as shown in **Fig 2.21(c)** and **2.21(d)**, respectively, for ***a*=1** and ***a*=0.5**.

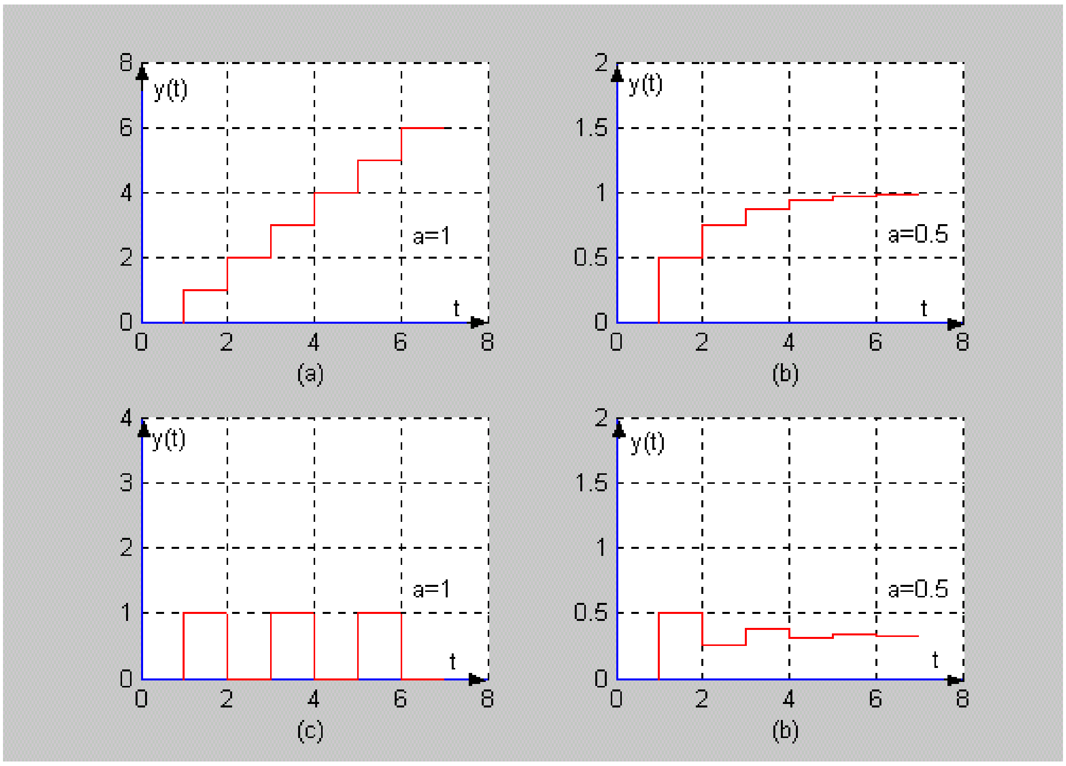


Fig 2.21

Translation: 考虑图 2.5 中所示反馈系统。说明正反馈系统的单位阶跃响应，当 ***a*=1** 时，如图 **2.21(a)**所示。当 ***a*=0.5** 时，如图 **2.21(b)**所示。说明负反馈系统的单位阶跃响应如图 **2.21(c)**和 **2.21(b)**所示，相应地，对 ***a*=1** 和 ***a*=0.5**。

Answer: Firstly, consider the positive-feedback system. It's impulse response is:

$$g(t) = \sum_{i=1}^{\infty} a^i \delta(t-i)$$

Using convolution integral:

$$y(t) = \sum_{i=1}^{\infty} a^i r(t-i)$$

When input is unit-step signal:

$$y(n) = \sum_{i=1}^{n} a^i$$

$$y(t) = y(n) \qquad n \le t \le n+1$$

Easy to draw the response curve, for ***a*=1** and ***a*=0.5**, respectively, as **Fig 2.21(a)** and **Fig 2.21(b)** shown.

Secondly, consider the negative-feedback system. It's impulse response is:

$$g(t) = -\sum_{i=1}^{\infty} (-a)^i \delta(t-i)$$

Using convolution integral:

$$y(t) = -\sum_{i=1}^{\infty} (-a)^i r(t-i)$$

When input is unit-step signal:

$$y(n) = -\sum_{i=1}^{n} (-a)^i$$

$$y(t) = y(n) \qquad n \le t \le n+1$$

Easy to draw the response curve, for ***a*=1** and ***a*=0.5**, respectively, as **Fig 2.21(c)** and **Fig 2.21(d)** shown.

2.14 Draw an op-amp circuit diagram for

$$\dot{x} = \begin{bmatrix} -2 & 4 \\ 0 & 5 \end{bmatrix} x + \begin{bmatrix} 2 \\ -4 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & 10 \end{bmatrix} x - 2u$$

2.15 Find state equations to describe the pendulum system in **Fig 2.22**. The systems are useful to model one- or two-link robotic manipulators. If θ, θ_1 and θ_2 are very small, can you consider the two systems as linear?

Translation: 试找出图 2.22 所示单摆系统的状态方程。这个系统对研究一个或两个连接的机器人操作臂很有用。假如角度都很小时，能否考虑系统为线性？

Answer: For **Fig2.22(a)**, the application of Newton's law to the linear movements yields:

$$f\cos\theta - mg = m\frac{d^2}{dt^2}(l\cos\theta) = ml(-\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta)$$

$$u - f\sin\theta = m\frac{d^2}{dt^2}(l\sin\theta) = ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)$$

Assuming θ and $\dot{\theta}$ to be small, we can use the approximation $\sin\theta = \theta$, $\cos\theta = 1$.

By retaining only the linear terms in θ and $\dot{\theta}$, we obtain $f = mg$ and:

$$\ddot{\theta} = -\frac{g}{l}\theta + \frac{1}{ml}u$$

Select state variables as $x_1 = \theta$, $x_2 = \dot{\theta}$ and output $y = \theta$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -g/l & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1/ml \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

For **Fig2.22(b)**, the application of Newton's law to the linear movements yields:

$$f_1\cos\theta_1 - f_2\cos\theta_2 - m_1 g = m_1\frac{d^2}{dt^2}(l_1\cos\theta_1)$$

$$= m_1 l_1(-\ddot{\theta}_1\sin\theta_1 - \dot{\theta}_1^2\cos\theta_1)$$

$$f_2\sin\theta_2 - f_1\sin\theta_1 = m_1\frac{d^2}{dt^2}(l_1\sin\theta_1)$$

$$= m_1 l_1(\ddot{\theta}_1\cos\theta_1 - \dot{\theta}_1^2\sin\theta_1)$$

$$f_2\cos\theta_2 - m_2 g = m_2\frac{d^2}{dt^2}(l_1\cos\theta_1 + l_2\cos\theta_2)$$

$$= m_2 l_1(-\ddot{\theta}_1\sin\theta_1 - \dot{\theta}_1^2\cos\theta_1) + m_2 l_2(-\ddot{\theta}_2\sin\theta_2 - \dot{\theta}_2^2\cos\theta_2)$$

$$u - f_2\sin\theta_2 = m_2\frac{d^2}{dt^2}(l_1\sin\theta_1 + l_2\sin\theta_2)$$

$$= m_2 l_1(\ddot{\theta}_1\cos\theta_1 - \dot{\theta}_1^2\sin\theta_1) + m_2 l_2(\ddot{\theta}_2\cos\theta_2 - \dot{\theta}_2^2\sin\theta_2)$$

Assuming θ_1, θ_2 and $\dot{\theta}_1$, $\dot{\theta}_2$ to be small, we can use the approximation $\sin\theta_1=\theta_1$, $\sin\theta_2=\theta_2$, $\cos\theta_1=1$, $\cos\theta_2=1$. By retaining only the linear terms in θ_1, θ_2 and $\dot{\theta}_1$, $\dot{\theta}_2$, we obtain $f_2 = m_2 g$, $f_1 = (m_1+m_2)g$ and:

$$\ddot{\theta}_1 = -\frac{(m_1+m_2)g}{m_1 l_1}\theta_1 + \frac{m_2 g}{m_1 l_1}\theta_2$$

$$\ddot{\theta}_2 = \frac{(m_1+m_2)g}{m_1 l_2}\theta_1 - \frac{(m_1+m_2)g}{m_1 l_2}\theta_2 + \frac{1}{m_2 l_2}u$$

Select state variables as $x_1=\theta_1$, $x_2=\dot{\theta}_1$, $x_3=\theta_2$, $x_4=\dot{\theta}_2$ and output $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(m_1+m_2)g/m_1 l_1 & 0 & m_2 g/m_1 l_1 & 0 \\ 0 & 0 & 0 & 1 \\ (m_1+m_2)g/m_1 l_2 & 0 & -(m_1+m_2)g/m_2 l_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 l_2 \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

2.17 The soft landing phase of a lunar module descending on the moon can be modeled as shown in **Fig2.24**. The thrust generated is assumed to be proportional to the derivation of **m**, where **m** is the mass of the module. Then the system can be described by

$$m\ddot{y} = -k\dot{m} - mg$$

Where ***g*** is the gravity constant on the lunar surface. Define state variables of the system as:

$$x_1 = y,\ x_2 = \dot{y},\ x_3 = m,\ y = \dot{m}$$

Find a state-space equation to describe the system.

Translation: 登月舱降落在月球时，软着陆阶段的模型如图 **2.24** 所示。产生的冲激力与 **m** 的微分成正比。系统可以描述如式所示形式。**g** 是月球表面的重力加速度常数。定义状态变量如式所示，试图找出系统的状态空间方程描述。

Answer: The system is not linear, so we can linearize it.

Suppose:

$$y = -gt^2/2 + \bar{y} \qquad \dot{y} = -gt + \dot{\bar{y}} \qquad \ddot{y} = -g + \ddot{\bar{y}}$$

$$m = m_0 + \bar{m} \qquad \dot{m} = \dot{\bar{m}}$$

So:

$$(m_0 + \bar{m})\,(-g + \ddot{\bar{y}}) = -k\dot{\bar{m}} - (m_0 + \bar{m})g$$

$$-m_0 g - \bar{m}g + m_0\ddot{\bar{y}} = -k\dot{\bar{m}} - m_0 g - \bar{m}g$$

$$m_0\ddot{\bar{y}} = -k\dot{\bar{m}}$$

Define state variables as:

$$\bar{x}_1 = \bar{y},\ \bar{x}_2 = \dot{\bar{y}},\ \bar{x}_3 = \bar{m},\ \bar{y} = \dot{\bar{m}}$$

Then:

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -k/m_0 \\ 1 \end{bmatrix} \bar{u}$$

$$\bar{y} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix}$$

2.19 Find a state equation to describe the network shown in **Fig2.26**.Find also its transfer function.

Translation: 试写出描述图 **2.26** 所示网络的状态方程，以及它的传递函数。

Answer: Select state variables as:

x_1: Voltage of left capacitor

x_2: Voltage of right capacitor

x_3: Current of inductor

Applying Kirchhoff's current law:

$$x_3 = (x_2 - L\dot{x}_3)/R$$

$$u = C\dot{x}_1 + x_1 R$$

$$u = C\dot{x}_2 + x_3$$

$$y = x_2$$

From the upper equations, we get:

$$\dot{x}_1 = -x_1/CR + u/C = -x_1 + u$$

$$\dot{x}_2 = -x_3/C + u/C = -x_3 + u$$

$$\dot{x}_3 = x_2/L - Rx_3/L = x_2 - x_3$$

$$y = x_2$$

They can be combined in matrix form as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Use MATLAB to compute transfer function. We type:

```
A=[-1,0,0;0,0,-1;0,1,-1];
B=[1;1;0];
C=[0,1,0];
D=[0];
[N1,D1]=ss2tf(A,B,C,D,1)
```

Which yields:

```
N1 =
0        1.0000        2.0000        1.0000
D1 =
1.0000        2.0000        2.0000        1.0000
```

So the transfer function is:

$$\hat{G}(s) = \frac{s^2 + 2s + 1}{s^3 + 2s^2 + 2s + 1} = \frac{s+1}{s^2 + s + 1}$$

2.20 Find a state equation to describe the network shown in Fig2.2. Compute also its transfer matrix.

Translation: 试写出描述图 **2.2** 所示网络的状态方程，计算它的传递函数矩阵。

Answer: Select state variables as **Fig 2.2** shown. Applying Kirchhoff's current law:

$$u_1 = R_1 C_1 \dot{x}_1 + x_1 + x_2 + L_1 \dot{x}_3 + R_2 x_3$$

$$C_1 \dot{x}_1 + u_2 = C_2 \dot{x}_2$$

$$x_3 = C_2 \dot{x}_2$$

$$u_1 = R_1 (x_3 - u_2) + x_1 + x_2 + L_1 \dot{x}_3 + R_2 x_3$$

$$y = L_1 \dot{x}_3 + R_2 x_3$$

From the upper equations, we get:

$$\dot{x}_1 = x_3 / C_1 - u_2 / C_1$$

$$\dot{x}_2 = x_3 / C_2$$

$$\dot{x}_3 = -x_1 / L_1 - x_2 / L_1 - (R_1 + R_1) x_3 / L_1 + u_1 / L_1 + R_1 u_2 / L_1$$

$$y = -x_1 - x_2 - R_1 x_3 + u_1 + R_1 u_2$$

They can be combined in matrix form as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1/C_1 \\ 0 & 0 & 1/C_2 \\ -L_1 & -L_1 & -(R_1 + R_2)/L_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & -1/C_1 \\ 0 & 0 \\ 1/L_1 & R_1 / L_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = \begin{bmatrix} -1 & -1 & -R_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & R_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Applying Laplace Transform to upper equations:

$$\hat{y}(s) = \frac{L_1 s + R_2}{L_1 s + R_1 + R_1 + 1/C_1 s + 1/C_2 s} \hat{u}_1(s)$$

$$+ \frac{(L_1 s + R_2)(R_1 + 1/C_2 s)}{L_1 s + R_1 + R_1 + 1/C_1 s + 1/C_2 s} \hat{u}_2(s)$$

$$\hat{G}(s) = \begin{bmatrix} \dfrac{L_1 s + R_2}{L_1 s + R_1 + R_1 + 1/C_1 s + 1/C_2 s} & \dfrac{(L_1 s + R_2)(R_1 + 1/C_2 s)}{L_1 s + R_1 + R_1 + 1/C_1 s + 1/C_2 s} \end{bmatrix}$$

2.18 Find the transfer functions from u to y_1 and from y_1 to y of the hydraulic tank system shown in **Fig2.25**. Does the transfer function from u to y equal the product of the two transfer functions? Is this also true for the system shown in **Fig2.14**?

Translation: 试写出图 **2.25** 所示水箱系统从u到y_1的传递函数和从y_1到y的传递函数。从u到y的传递函数等于两个传递函数的乘积吗？这对图 **2.14** 所示系统也是正确的吗？

Answer: Write out the equation about u, y_1 and y:

$$y_1 = x_1 / R_1$$

$$y_2 = x_2 / R_2$$

$$A_1 dx_1 = (u - y_1) / dt$$

$$A_2 dx_2 = (y_1 - y) / dt$$

Applying Laplace transform:

$$\hat{y}_1 / \hat{u} = 1/(1 + A_1 R_1 s)$$

$$\hat{y} / \hat{y}_1 = 1/(1 + A_2 R_2 s)$$

$$\hat{y} / \hat{u} = 1/(1 + A_1 R_1 s)(1 + A_2 R_2 s)$$

So:

$$\hat{y} / \hat{u} = (\hat{y}_1 / \hat{u})\ (\hat{y} / \hat{y}_1)$$

But it is not true for **Fig2.14**, because of the loading problem in the two tanks.

3.1consider Fig3.1 ,what is the representation of the vector $\underline{x}$ with respect to the basis $(q_1 \quad i_2)$? What is the representation of $\underline{q_1}$ with respect ro $(i_2 \quad q_2)$?图 3.1 中,向量$\underline{x}$关于$(q_1 \quad i_2)$的表示是什么? $\underline{q_1}$关于$(i_2 \quad q_2)$的表示有是什么?

If we draw from $\underline{x}$ two lines in parallel with $\underline{i_2}$ and $\underline{q_1}$, they tutersect at $\frac{1}{3}q_1$ and $\frac{8}{3}\underline{i_2}$ as shown , thus the representation of $\underline{x}$ with respect to the basis $(q_1 \quad i_2)$ is $\begin{bmatrix} \frac{1}{3} & \frac{8}{3} \end{bmatrix}'$, this can be verified from $\underline{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = [\underline{q_2} \quad \underline{i_2}]\begin{bmatrix} \frac{1}{3} \\ \frac{8}{3} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1/3 \\ 8/3 \end{bmatrix}$

To find the representation of $\underline{q_1}$ with respect to $(i_2 \quad q_2)$,we draw from $\underline{q_1}$ two lines in parallel with $\underline{q_2}$ and $\underline{i_2}$, they intersect at -2$\underline{i_2}$ and $\frac{3}{2}\underline{q_2}$, thus the representation of $\underline{q_1}$ with respect to $(i_2 \quad q_2)$, is $\begin{bmatrix} -2 & \frac{3}{2} \end{bmatrix}'$, this can be verified from

$$\underline{q_1} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = [\underline{i_2} \quad \underline{q_2}]\begin{bmatrix} -2 \\ 3/2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 3/2 \end{bmatrix}$$

3.2 what are the 1-morm ,2-norm , and infinite-norm of the vectors $\underline{x_1} = [2 \quad -3 \quad 1]'$, $\underline{x_2} = [1 \quad 1 \quad 1]'$, 问向量 $\underline{x_1} = [2 \quad -3 \quad 1]'$, $\underline{x_2} = [1 \quad 1 \quad 1]'$的 1-范数,2-范数和$\infty-$范数是什么?

$$\|\underline{x_1}\|_1 = \sum_{i=1}^{3}|x_{1i}| = 2+3+1 = 6 \qquad \|\underline{x_2}\|_1 = 1+1+1 = 3$$

$$\|\underline{x_1}\|_2 = \sqrt{\underline{x_1'}\underline{x_1}} = (2^2+(-3)^2+1^2)^{1/2} = \sqrt{14} \qquad \|\underline{x_2}\|_2 = (1^2+1^2+1^2)^{1/2} = \sqrt{3}$$

$$\|\underline{x_1}\|_\infty = \max x_i|x_{1i}| = 3 \qquad \|\underline{x_2}\|_\infty = \max x_i|x_{2i}| = 1$$

3.3 find two orthonormal vectors that span the same space as the two vectors in problem 3.2 ,求与题 3.2 中的两个向量张成同一空间的两个标准正交向量.

Schmidt orthonormalization praedure ,

$\underline{u_1} = \underline{x_1} = [2 \quad -3 \quad 1]'$ $\qquad \underline{q_1} = \underline{u_1} \big/ \|\underline{u_1}\| = \frac{1}{\sqrt{14}}[2 \quad -3 \quad 1]'$

$\underline{u_2} = \underline{x_2} - (\underline{q_1'}\underline{x_2})\underline{q_1} = \underline{x_2}$ $\qquad \underline{q_2} = \underline{u_2} \big/ \|\underline{u_2}\| = \frac{1}{\sqrt{3}}[1 \quad 1 \quad 1]'$

The two orthomormal vectors are $\underline{q_1} = \frac{1}{\sqrt{14}}\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \qquad \underline{q_2} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

In fact , the vectors $\underline{x_1}$ and $\underline{x_2}$ are orthogonal because $\underline{x_1'}\underline{x_2} = \underline{x_2'}\underline{x_1} = 0$ so we can only normalize the two vectors $\underline{q_1} = \underline{x_1} \big/ \|\underline{x_1}\| = \frac{1}{\sqrt{14}}\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \qquad \underline{q_2} = \underline{x_2} \big/ \|\underline{x_2}\| = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

3.4 comsider an $n \times m$ matrix A with $n \ge m$, if all colums of A are orthonormal , then $A'A = I_m$, what can you say abort AA' ? 一个$n \times m$阶矩阵A($n \ge m$), 如果A的所有列都是标准正交的,则$A'A = I_m$问AA'是怎么样?

Let $A = [\underline{a_1}\ \underline{a_2} \cdots \underline{a_m}] = [a_{ij}]_{n \times m}$, if all colums of A are orthomormal , that is

$$\underline{a_i'}\underline{a_i} = \sum_{i=1}^{n} a_{li}a_{li} = \begin{cases} 0 & if \quad i \ne j \\ 1 & if \quad i = j \end{cases}$$

$$A'A = \begin{bmatrix} \underline{a_1'} \\ \underline{a_2'} \\ \vdots \\ \underline{a_m'} \end{bmatrix} [\underline{a_i'}\ \underline{a_2'} \cdots \underline{a_m'}] = \begin{bmatrix} \underline{a_1'}\underline{a_1} & \underline{a_1'}\underline{a_2} \cdots \underline{a_1'}\underline{a_m} \\ & \cdots\cdots \\ \underline{a_m'}\underline{a_1} & \underline{a_m'}\underline{a_2} \cdots \underline{a_m'}\underline{a_m} \end{bmatrix} = \begin{bmatrix} 1 & 0 \cdots & 0 \\ 0 & 1 \ddots & 0 \\ 0 & 0 \cdots & 1 \end{bmatrix} = I_m$$

then $A'A = [\underline{a_i'}\ \underline{a_2'} \cdots \underline{a_m'}] \begin{bmatrix} \underline{a_1'} \\ \underline{a_2'} \\ \vdots \\ \underline{a_m'} \end{bmatrix} = \sum_{i=1}^{m} \underline{a_i}\underline{a_i'} = (\sum_{i=1}^{m} \underline{a_{il}}\underline{a_{jl}})_{n \times n}$

$$in \quad general \qquad \sum_{i=1}^{m} \underline{a_{il}}\underline{a_{jl}} = \begin{cases} \ne 0 & if \quad i \ne j \\ \ne 1 & if \quad i = j \end{cases}$$

if A is a symmetric square matrix , that is to say , n=m $a_{il}\underline{a_{jl}}$ for every $i, l = 1, 2 \cdots n$

then $A'A = I_m$

3.5 find the ranks and mullities of the following matrices 求下列矩阵的秩和化零度

$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $A_2 = \begin{bmatrix} 4 & 1 & -1 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ $A_3 = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{Bmatrix}$

$rank(A_1) = 2 \quad rank(A_2) = 3 \quad rank(A_3) = 3$

$Nullity(A_1) = 3 - 2 = 1 \quad Nullity(A_2) = 3 - 3 = 0 \quad Nullity(A_3) = 4 - 3 = 1$

3.6 Find bases of the range spaces of the matrices in problem 3.5 求题 3.5 中矩阵值域空间的基

the last two columns of A_1 are linearly independent , so the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ can be used as a basis of the range space of A_1 , all columns of A_2 are linearly independent ,so the set $\left\{ \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right\}$ can be used as a basis of the range spare of A_2

let $A_3 = \lfloor \underline{a_1} \quad \underline{a_2} \quad \underline{a_3} \quad \underline{a_4} \rfloor$,where $\underline{a_i}$ denotes of A_3 $\underline{a_1} and$ $\underline{a_2} and$ $\underline{a_3} and$ $\underline{a_4}$ are linearly independent , the third colums can be expressed as $\underline{a_3} = -\underline{a_1} + 2\underline{a_2}$, so the set $\{ a_1 \quad \underline{a_2} \quad \underline{a_3} \}$can be used as a basis of the range space of A_3

3.7 consider the linear algebraic equation $\begin{bmatrix} 2 & -1 \\ -3 & 3 \\ -1 & 2 \end{bmatrix} \underline{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \underline{-x}$ it has three equations and two unknowns does a solution $\underline{x}$ exist in the equation ? is the solution unique ? does a solution exist if $\underline{y} = [1 \quad 1 \quad 1]'$?

线性代数方程$\begin{bmatrix} 2 & -1 \\ -3 & 3 \\ -1 & 2 \end{bmatrix} \underline{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \underline{-x}$有三个方程两个未知数，问方程是否有解?若有解是否唯一?若$\underline{y} = [1 \quad 1 \quad 1]'$方程是否有解?

Let $A=\begin{bmatrix} 2 & -1 \\ -3 & 3 \\ -1 & 2 \end{bmatrix}=\begin{bmatrix}\underline{a}_1 & \underline{a}_2\end{bmatrix}$, clearly $\underline{a}_1$ and $\underline{a}_2$ are linearly independent , so rank(A)=2 , $\underline{y}$ is the sum of $\underline{a}_1$ and $\underline{a}_2$,that is rank([A $\underline{y}$])=2, rank(A)= rank([A $\underline{y}$]) so a solution $\underline{x}$ exists in A $\underline{x}=\underline{y}$

Nullity(A)=2- rank(A)=0

The solution is unique $\underline{x}=\begin{bmatrix}1 & 1\end{bmatrix}'$ if $\underline{y}=\begin{bmatrix}1 & 1 & 1\end{bmatrix}'$, then rank([A $\underline{y}$])=3 $\neq$ rank(A), that is to say there doesn't exist a solution in A $\underline{x}=\underline{y}$

3.8 find the general solution of $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}\underline{x}=\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ how many parameters do you have ?

求方程$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}\underline{x}=\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$的同解,同解中用了几个参数?

Let $A=\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\underline{y}=\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ we can readily obtain rank(A)= rank([A $\underline{y}$])=3 so this $\underline{y}$ lies in the range space of A and $\underline{x_p}=\begin{bmatrix}-1 & 0 & o & 1\end{bmatrix}'$ is a solution Nullity(A)=4-3=1 that means the dimension of the null space of A is 1 , the number of parameters in the general solution will be 1 , A basis of the null space of A is $\underline{n}=\begin{bmatrix}1 & -2 & 1 & o\end{bmatrix}'$ thus the general solution of A $\underline{x}=\underline{y}$ can be expressed an $\underline{x}=\underline{x_p}+\alpha\underline{n}=\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}+\alpha\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ for any real α

α is the only parameter

3.9 find the solution in example 3.3 that has the smallest Euclidean norm 求例 3 中具有最小欧氏范数的解,

the general solution in example 3.3 is $\underline{x}=\begin{bmatrix}0\\-4\\0\\0\end{bmatrix}+\alpha_1\begin{bmatrix}1\\1\\-1\\0\end{bmatrix}+\alpha_2\begin{bmatrix}0\\2\\0\\-1\end{bmatrix}=\begin{bmatrix}\alpha_1\\\alpha_1+2\alpha_2-4\\-\alpha_1\\-\alpha_2\end{bmatrix}$ for any real α_1 and α_2

the Euclidean norm of $\underline{x}$ is

$$\|\underline{x}\|=\sqrt{\alpha_1^2+(\alpha_1+2\alpha_2-4)^2+(-\alpha_1)^2+(-\alpha_2)^2}$$
$$=\sqrt{3\alpha_1^2+5\alpha_2^2+4\alpha_1\alpha_2-8\alpha_1-16\alpha_2+16}$$

$$\begin{matrix}\dfrac{\partial\|\underline{x}\|}{2\alpha_1}=0\Rightarrow 3\alpha_1+2\alpha_2-4=0\\ \dfrac{\partial\|\underline{x}\|}{2\alpha_2}=0\Rightarrow 5\alpha_2+2\alpha_1-8=0\end{matrix}\Rightarrow\begin{matrix}\alpha_1=\dfrac{4}{11}\\ \alpha_2=\dfrac{16}{11}\end{matrix}\Rightarrow\underline{x}=\begin{bmatrix}\frac{4}{11}\\-\frac{8}{11}\\-\frac{4}{11}\\-\frac{16}{11}\end{bmatrix}$$ has the smallest Euclidean norm ,

3.10 find the solution in problem 3.8 that has the smallest Euclidean norm 求题 3.8 中欧氏范数最小的解,

$\underline{x}=\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}+\alpha\begin{bmatrix}1\\-2\\1\\0\end{bmatrix}$ for any real α

$$\|\underline{x}\|=\sqrt{(\alpha-1)^2+(-2\alpha)^2+\alpha^2+1}=\sqrt{6\alpha^2-2\alpha+2}$$

the Euclidean norm of $\underline{x}$ is $\dfrac{\partial\|\underline{x}\|}{\partial\alpha}=0\Rightarrow 12\alpha-2=0\Rightarrow\alpha=\dfrac{1}{6}\Rightarrow\underline{x}=\begin{bmatrix}-\frac{5}{6}\\-\frac{3}{6}\\\frac{1}{6}\\1\end{bmatrix}$ has the smallest Euclidean norm

3.11 consider the equation $\underline{x}\{n\}=A^n\underline{x}[0]+A^{n-1}\underline{b}u[0]+A^{n-2}\underline{b}u[1]+\cdots+A\underline{b}u[n-2]+\underline{b}u[n-1]$

where A is an $n\times n$ matrix and $\underline{b}$ is an $n\times 1$ column vector ,under what conditions on A and $\underline{b}$ will there exist $u[o], u[1], \cdots u[n-1]$ to meet the equation for any $\underline{x}[n], and\, \underline{x}[0]$?

令 A 是 $n\times n$ 的矩阵, $\underline{b}$ 是 $n\times 1$ 的列向量,问在 A 和 $\underline{b}$ 满足什么条件时,存在 $u[o], u[1], \cdots u[n-1]$,对所有的 $\underline{x}[n], and\, \underline{x}[0]$,它们都满足方程 $\underline{x}\{n\} = A^n \underline{x}[0] + A^{n-1}\underline{b}u[0] + A^{n-2}\underline{b}u[1] + \cdots + A\underline{b}u[n-2] + \underline{b}u[n-1]$,

write the equation in this form $\underline{x}\{n\} = A^n \underline{x}[0] = [\underline{b}, A\underline{b} \cdots A^{n-1}\underline{b}]\begin{bmatrix} u[n-1] \\ u[n-2] \\ \vdots \\ u[0] \end{bmatrix}$

where $[\underline{b}, A\underline{b} \cdots A^{n-1}\underline{b}]$ is an $n\times n$ matrix and $\underline{x}\{n\} - A^n \underline{x}[0]$ is an $n\times 1$ column vector , from the equation we can see , $u[o], u[1], \cdots u[n-1]$ exist to meet the equation for any $\underline{x}[n], and\, \underline{x}[0]$,if and only if $\rho[\underline{b}, A\underline{b} \cdots A^{n-1}\underline{b}] = n$ under this condition , there will exist $u[o], u[1], \cdots u[n-1]$ to meet the equation for any $\underline{x}[n], and\, \underline{x}[0]$.

3.12 given $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\underline{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \end{bmatrix}$ $\overline{\underline{b}} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$ what are the representations of A with respect to the basis $\{\underline{b} \quad A\underline{b} \quad A^2\underline{b} \quad A^3\underline{b}\}$ and the basis $\{\overline{\underline{b}} \quad A\overline{\underline{b}} \quad A^2\overline{\underline{b}} \quad A^3\overline{\underline{b}}\}$, respectively? 给定 $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\underline{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \end{bmatrix}$ $\overline{\underline{b}} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$ 请问 A 关于 $\{\underline{b} \quad A\underline{b} \quad A^2\underline{b} \quad A^3\underline{b}\}$ 和基 $\{\overline{\underline{b}} \quad A\overline{\underline{b}} \quad A^2\overline{\underline{b}} \quad A^3\overline{\underline{b}}\}$ 的表示分别是什么? $A\underline{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, A^2\underline{b} = A\cdot A\underline{b} = \begin{bmatrix} 1 \\ 4 \\ 4 \\ 1 \end{bmatrix}, A^3\underline{b} = A\cdot A^2\underline{b} = \begin{bmatrix} 24 \\ 32 \\ 16 \\ 1 \end{bmatrix}$

$$A^4\underline{b} = -8\underline{b} + 20A\underline{b} - 18A^2\underline{b} + 7A^3\underline{b}$$

we have $\therefore A[\underline{b} \quad A\underline{b} \quad A^2\underline{b} \quad A^3\underline{b}] = [\underline{b} \quad A\underline{b} \quad A^2\underline{b} \quad A^3\underline{b}] = \begin{bmatrix} 0 & 0 & 0 & -8 \\ 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & -18 \\ 0 & 0 & 1 & 7 \end{bmatrix}$ thus the representation of A with respect to the basis $\{\underline{b} \quad A\underline{b} \quad A^2\underline{b} \quad A^3\underline{b}\}$ is $\overline{A} = \begin{bmatrix} 0 & 0 & 0 & -8 \\ 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & -18 \\ 0 & 0 & 1 & 7 \end{bmatrix}$

$$A\underline{\overline{b}} = \begin{bmatrix} 4 \\ 7 \\ 6 \\ 1 \end{bmatrix}, A^2\underline{\overline{b}} = \begin{bmatrix} 15 \\ 20 \\ 12 \\ 1 \end{bmatrix}, A^3\underline{\overline{b}} = \begin{bmatrix} 50 \\ 52 \\ 24 \\ 1 \end{bmatrix}, A^4\underline{\overline{b}} = \begin{bmatrix} 152 \\ 128 \\ 48 \\ 1 \end{bmatrix}$$

$$A^4\underline{\overline{b}} = -8\underline{\overline{b}} + 20A\underline{\overline{b}} - 18A^2\underline{\overline{b}} + 7A^3\underline{\overline{b}}$$

$\therefore A[\underline{\overline{b}} \quad A\underline{\overline{b}} \quad A^2\underline{\overline{b}} \quad A^3\underline{\overline{b}}] = [\underline{\overline{b}} \quad A\underline{\overline{b}} \quad A^2\underline{\overline{b}} \quad A^3\underline{\overline{b}}] = \begin{bmatrix} 0 & 0 & 0 & -8 \\ 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & -18 \\ 0 & 0 & 1 & 7 \end{bmatrix}$ thus the representation of A with respect to the basis $\{\underline{\overline{b}} \quad A\underline{\overline{b}} \quad A^2\underline{\overline{b}} \quad A^3\underline{\overline{b}}\}$ is $\overline{A} = \begin{bmatrix} 0 & 0 & 0 & -8 \\ 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & -18 \\ 0 & 0 & 1 & 7 \end{bmatrix}$

3.13 find Jordan-form representations of the following matrices

写出下列矩阵的 jordan 型表示:

$$A_1 = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix}. A_3 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}. A_4 = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix}.$$

the characteristic polynomial of A_1 is $\Delta_1 = \det(\lambda I - A_1) = (\lambda-1)(\lambda-2)(\lambda-3)$ thus the eigenvelues of A_1 are 1 ,2 , 3 , they are all distinct . so the Jordan-form representation of A_1 will be diagonal .

the eigenvectors associated with $\lambda = 1, \lambda = 2, \lambda = 3$,respectively can be any nonzero solution of

$$A_1\underline{q_1} = \underline{q_1} \Rightarrow \underline{q_1} = [1 \quad 0 \quad 0]'$$

$A_1\underline{q_2} = 2\underline{q_2} \Rightarrow \underline{q_2} = [4 \quad 1 \quad 0]'$ thus the jordan-form representation of A_1 with respect to

$$A_1\underline{q_3} = 3\underline{q_3} \Rightarrow \underline{q_3} = [5 \quad 0 \quad 1]'$$

$\{\underline{q_1} \quad \underline{q_2} \quad \underline{q_3}\}$ is $\hat{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

the characteristic polynomial of A_2 is

$\Delta_2(\lambda) = \det(\lambda I - A_2) = (\lambda^3 + 3\lambda^2 + 4\lambda + 2 = (\lambda+1)(\lambda+1+i)(\lambda+1-i)$ A_2 has eigenvalues $-1, \quad -1+j \quad and \quad -1-j$ the eigenvectors associated with $-1, \quad -1+j \quad and \quad -1-j$ are , respectively

$[1 \quad -1 \quad 1]', [1 \quad j-1 \quad -2j]' \, and \, [1 \quad -1-j \quad 2j]'$ the we have

$$Q = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1+j & -1-j \\ 0 & -2j & 2j \end{bmatrix} \quad and \quad \hat{A}_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1+j & 0 \\ 0 & 0 & -1-j \end{bmatrix} = Q^{-1}A_2Q$$

the characteristic polynomial of A_3 is $\Delta_3(\lambda) = \det(\lambda I - A_3) = (\lambda-1)^2(\lambda-2)$ theus the eigenvalues of A_3 are 1 ,1 and 2 , the eigenvalue 1 has multiplicity 2 , and nullity $(A_3 - I) = 3 - rank(A_3 - I) = 3 - 1 = 2$ the A_3 has two tinearly independent eigenvectors associated with 1 ,

$$\begin{aligned} &(A_3 - I)\underline{q} = \underline{0} \Rightarrow \underline{q_1} = [1 \quad 0 \quad 0]' \quad \underline{q_2} = [0 \quad 1 \quad 0]' \\ &(A_3 - I)\underline{q_3} = \underline{0} \Rightarrow \underline{q_3} = [1 \quad 0 \quad -1]' \end{aligned}$$

thus we have

$$Q = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad and \quad \hat{A}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = Q^{-1}A_3Q$$

the characteristic polynomial of A_4 is $\Delta_4(\lambda) = \det(\lambda I - A_4) = \lambda^3$ clearly A_4 has lnly one distinct eigenvalue 0 with multiplicity 3 , Nullity(A_4 -0I)=3-2=1 , thus A_4 has only one independent eigenvector associated with 0 , we can compute the generalized , eigenvectors

$$\begin{aligned} A_1\underline{v_1} &= \underline{0} \Rightarrow \underline{v_1} = [1 \quad 0 \quad 0]' \\ A_1\underline{v_2} &= \underline{v_1} \Rightarrow \underline{v_2} = [0 \quad 4 \quad -5]' \\ A_1\underline{v_3} &= \underline{v_2} \Rightarrow \underline{v_3} = [0 \quad -3 \quad 4]' \end{aligned}$$

of A_4 from equations below, then the representation of A_4 with respect to the basis $\{\underline{v_1} \quad \underline{v_2} \quad \underline{v_3}\}$ is

$$\hat{A}_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = Q^{-1}A_4Q \quad where \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & -3 \\ 0 & -5 & 4 \end{bmatrix}$$

3.14 consider the companion-form matrix $A=\begin{bmatrix}-\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0\end{bmatrix}$ show that its characterisic polynomial is given by $\Delta(\lambda)=\lambda^4+\alpha_1\lambda^3+\alpha_2\lambda^3+\alpha_3\lambda+\alpha_4$

show also that if λ_i is an eigenvalue of A or a solution of $\Delta(\lambda)=0$ then

$\begin{bmatrix}\lambda_i^3 & \lambda_i^3 & \lambda_i & 1'\end{bmatrix}$is an eigenvector of A associated with λ_i

证明友矩阵 A 的特征多项式$\Delta(\lambda)=\lambda^4+\alpha_1\lambda^3+\alpha_2\lambda^3+\alpha_3\lambda+\alpha_4$并且,如果$\lambda_i$ 是λ_i的一个特征值，$\Delta(\lambda)=0$的一个解，那么向量$\begin{bmatrix}\lambda_i^3 & \lambda_i^3 & \lambda_i & 1'\end{bmatrix}$是 A 关于$\lambda_i$的一个特征向量

proof:

$$\Delta(\lambda)=\det(\lambda I-A_2)=\det\begin{bmatrix}\lambda+\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ -1 & \lambda & 0 & 0 \\ 0 & -1 & \lambda & 0 \\ 0 & 0 & -1 & \lambda\end{bmatrix}$$

$$=(\lambda+\alpha_1)\det\begin{bmatrix}\lambda & 0 & 0 \\ -1 & \lambda & 0 \\ 0 & -1 & \lambda\end{bmatrix}+\det\begin{bmatrix}\alpha_2 & \alpha_3 & \alpha_4 \\ -1 & \lambda & 0 \\ 0 & -1 & \lambda\end{bmatrix}$$

$$=\lambda^4+\alpha_1\lambda^3+\alpha_2\det\begin{bmatrix}\lambda & 0 \\ -1 & \lambda\end{bmatrix}+\det\begin{bmatrix}\alpha_3 & \alpha_4 \\ -1 & \lambda\end{bmatrix}$$

$$=\lambda^4+\alpha_1\lambda^3+\alpha_2\lambda_2+\alpha_3\lambda+\alpha_4$$

if λ_i is an eigenvalue of A , that is to say , $\Delta(\lambda_i)=0$ then we have

$$\begin{bmatrix}-\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\lambda_i^3 \\ \lambda_i^2 \\ \lambda_i \\ 1\end{bmatrix}=\begin{bmatrix}-\alpha_1\lambda_i^3-\alpha_2\lambda_i^2-\alpha_3\lambda_i-\alpha_4 \\ \lambda_i^2 \\ \lambda_i \\ 1\end{bmatrix}=\begin{bmatrix}\lambda_i^4 \\ \lambda_i^3 \\ \lambda_i^2 \\ \lambda_i\end{bmatrix}=\lambda_I\begin{bmatrix}\lambda_i^3 \\ \lambda_i^2 \\ \lambda_i \\ 1\end{bmatrix}$$

that is to say $\begin{bmatrix}\lambda_i^3 & \lambda_i^3 & \lambda_i & 1'\end{bmatrix}$ is an eigenvetor oa A associated with λ_i

3.15 show that the vandermonde determinant $\begin{bmatrix} \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ equals $\Pi_{1\le i<j\le 4}(\lambda_j-\lambda_i)$, thus we conclude that the matrix is nonsingular or equivalently , the eigenvectors are linearly independent if all eigenvalues are distinct ,证明 vandermonde 行列式 $\begin{bmatrix} \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ 为 $\Pi_{1\le i<j\le 4}(\lambda_j-\lambda_i)$，因此如果所有的特征值都互不相同则该矩阵非奇异，或者等价地说，所有特征向量线性无关,

proof:

$$\det\begin{bmatrix} \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \det\begin{bmatrix} 0 & \lambda_2^2(\lambda_2-\lambda_1) & \lambda_3^2(\lambda_3-\lambda_1) & \lambda_4^2(\lambda_4-\lambda_1) \\ 0 & \lambda_2(\lambda_2-\lambda_1) & \lambda_3(\lambda_3-\lambda_1) & \lambda_4(\lambda_4-\lambda_1) \\ 0 & \lambda_2-\lambda_1 & \lambda_3-\lambda_1 & \lambda_4-\lambda_1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \det\begin{bmatrix} \lambda_2^2(\lambda_2-\lambda_1) & \lambda_3^2(\lambda_3-\lambda_1) & \lambda_4^2(\lambda_4-\lambda_1) \\ \lambda_2(\lambda_2-\lambda_1) & \lambda_3(\lambda_3-\lambda_1) & \lambda_4(\lambda_4-\lambda_1) \\ \lambda_2-\lambda_1 & \lambda_3-\lambda_1 & \lambda_4-\lambda_1 \end{bmatrix}$$

$$= -\det\begin{bmatrix} 0 & \lambda_3(\lambda_3-\lambda_1)(\lambda_3-\lambda_2) & \lambda_4(\lambda_4-\lambda_1)(\lambda_4-\lambda_2) \\ 0 & (\lambda_3-\lambda_1)(\lambda_3-\lambda_2) & (\lambda_4-\lambda_1)(\lambda_4-\lambda_2) \\ \lambda_2-\lambda_1 & \lambda_3-\lambda_1 & \lambda_4-\lambda_1 \end{bmatrix}$$

$$= -\det\begin{bmatrix} \lambda_3(\lambda_3-\lambda_1)(\lambda_3-\lambda_2) & \lambda_4(\lambda_4-\lambda_1)(\lambda_4-\lambda_2) \\ (\lambda_3-\lambda_1)(\lambda_3-\lambda_2) & (\lambda_4-\lambda_1)(\lambda_4-\lambda_2) \end{bmatrix}\cdot(\lambda_2-\lambda_1)$$

$$= -(\lambda_3-\lambda_1)[\lambda_3(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)-\lambda_4(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)]$$

$$= (\lambda_2-\lambda_1)(\lambda_4-\lambda_2)(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)$$

$$= \Pi_{1\le i<j\le 4}$$

let a, b, c and d be the eigenvalues of a matrix A , and they are distinct Assuming the matrix is singular , that is abcd=0 , let a=0 , then we have

$$\det\begin{bmatrix} 0 & b^3 & c^3 & d^3 \\ 0 & b^2 & c^2 & d^2 \\ 0 & b & c & d \\ 1 & 1 & 1 & 1 \end{bmatrix} = \det\begin{bmatrix} b^3 & c^3 & d^3 \\ b^2 & c^2 & d^2 \\ b & c & d \end{bmatrix} = -bcd\cdot\det\begin{bmatrix} b^2 & c^2 & d^2 \\ b & c & d \\ 1 & 1 & 1 \end{bmatrix} = -bcd(d-c)(d-b)(c-b)$$

and from vandermonde determinant

$$\det\begin{bmatrix} a^3 & b^3 & c^3 & d^3 \\ a^2 & b^2 & c^2 & d^2 \\ a & b & c & d \\ 1 & 1 & 1 & 1 \end{bmatrix} = \det\begin{bmatrix} 0 & b^3 & c^3 & d^3 \\ 0 & b^2 & c^2 & d^2 \\ 0 & b & c & d \\ 1 & 1 & 1 & 1 \end{bmatrix} = (d-c)(d-b)(c-a)(b-a) = bcd(d-c)(d-b)$$

so we can see $\det\begin{bmatrix} 0 & b^3 & c^3 & d^3 \\ 0 & b^2 & c^2 & d^2 \\ 0 & b & c & d \\ 1 & 1 & 1 & 1 \end{bmatrix} = -\det\begin{bmatrix} 0 & b^3 & c^3 & d^3 \\ 0 & b^2 & c^2 & d^2 \\ 0 & b & c & d \\ 1 & 1 & 1 & 1 \end{bmatrix}$,

that is to say $bcd(d-c)(d-b) = 0 \Rightarrow a, b, c, and\ d$ are not distinet

this implies the assumption is not true , that is , the matrix is nonsingular let $\underline{q}_i$ be the eigenvectors of A , $A\underline{q}_1 = a\underline{q}_1 \quad A\underline{q}_2 = a\underline{q}_2 \quad A\underline{q}_3 = a\underline{q}_3 \quad A\underline{q}_4 = a\underline{q}_4$

$$\begin{cases} \alpha_1\underline{q}_1 + \alpha_2\underline{q}_2 + \alpha_3\underline{q}_3 + \alpha_4\underline{q}_4 = 0 \\ a\alpha_1\underline{q}_1 + b\alpha_2\underline{q}_2 + c\alpha_3\underline{q}_3 + d\alpha_4\underline{q}_4 = 0 \\ a^2\alpha_1\underline{q}_1 + b^2\alpha_2\underline{q}_2 + c^2\alpha_3\underline{q}_3 + d^2\alpha_4\underline{q}_4 = 0 \\ a^3\alpha_1\underline{q}_1 + b^3\alpha_2\underline{q}_2 + c^3\alpha_3\underline{q}_3 + d^3\alpha_4\underline{q}_4 = 0 \end{cases} \Rightarrow \begin{pmatrix} \alpha_1\underline{q}_1 & \alpha_2\underline{q}_2 & \alpha_3\underline{q}_3 & \alpha_4\underline{q}_4 \end{pmatrix} \begin{pmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & d^2 & d^3 \\ 1 & d & d^2 & d^3 \end{pmatrix} = \underline{0}_{4\times 4}$$

$$so \quad \begin{matrix} \alpha_i\underline{q}_i = \underline{0}_{n\times 1} \\ and\underline{q}_i \neq \underline{0}_{n\times 1} \end{matrix} \quad \alpha_i = 0 \Rightarrow \underline{q}_i \quad linenrly \quad independent$$

3.16 show that the companion-form matrix in problem 3.14 is nonsingular if and only if $\alpha_4 \neq 0$, under this assumption , show that its inverse equals

$$A^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1/\alpha_4 & -\alpha_1/\alpha_4 & -\alpha_2/\alpha_4 & -\alpha_3/\alpha_4 \end{bmatrix}$$ 证明题 3.14 中的友矩阵非奇异当且仅当 $\alpha_4 \neq 0$，且矩阵的逆为 $A^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1/\alpha_4 & -\alpha_1/\alpha_4 & -\alpha_2/\alpha_4 & -\alpha_3/\alpha_4 \end{bmatrix}$

proof: as we know , the chacteristic polynomial is

$\Delta(\lambda) = \det(\lambda I - A) = \lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \alpha_3\lambda + \alpha_4$ so let $\lambda = 0$, we have

$\det(A)=(-1)^4\det(A)=\det(A)=\alpha_4$

A is nonsingular if and only if

$\alpha_4 \neq 0$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1/\alpha_4 & -\alpha_1/\alpha_4 & -\alpha_2/\alpha_4 & -\alpha_3/\alpha_4 \end{bmatrix} \cdot \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$$

$$\begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1/\alpha_4 & -\alpha_1/\alpha_4 & -\alpha_2/\alpha_4 & -\alpha_3/\alpha_4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$$

$$\therefore A^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1/\alpha_4 & -\alpha_1/\alpha_4 & -\alpha_2/\alpha_4 & -\alpha_3/\alpha_4 \end{bmatrix}$$

3.17 consider $A=\begin{bmatrix} \lambda & \lambda T & \lambda t T^2/2 \\ 0 & \lambda & \lambda t T \\ 0 & 0 & \lambda \end{bmatrix}$ with $\lambda \neq 0$ and T>0 show that $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}'$ is an generalized eigenvector of grade 3 and the three columns of $Q=\begin{bmatrix} \lambda^2 T^2 & \lambda T^2/2 & 0 \\ 0 & \lambda t T & 0 \\ 0 & 0 & 0 \end{bmatrix}$ constitute a chain of generalized eigenvectors of length 3 , venty $Q^{-1}AQ=\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda t & 1 \\ 0 & 0 & \lambda \end{bmatrix}$

矩阵 A 中 $\lambda \neq 0$,T>0，证明 $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}'$ 是 3 级广义特征向量，并且矩阵 Q 的 3 列组成长度是 3 的广义特征向量链,验证 $Q^{-1}AQ=\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda t & 1 \\ 0 & 0 & \lambda \end{bmatrix}$

Proof : clearly A has only one distinct eigenvalue λ with multiplicity 3

$$(A-\lambda I)^2\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0&0&\lambda^2T^2\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}\lambda^2T^2\\0\\0\end{bmatrix}\neq \underline{0}$$

$$(A-\lambda I)^3\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0&0&\lambda^2T^2\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&\lambda T&\lambda T^2/2\\0&0&\lambda T\\0&0&0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}=\underline{0}$$

these two equation imply that $\begin{bmatrix}0&0&1\end{bmatrix}'$ is a generalized eigenvctor of grade 3 ,

$$(A-\lambda I)\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0&\lambda T&\lambda T^2/2\\0&0&\lambda T\\0&0&0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}\lambda^2T^2\\\lambda T\\0\end{bmatrix}$$

and

$$(A-\lambda I)\begin{bmatrix}\lambda^2T^2\\\lambda T\\0\end{bmatrix}=\begin{bmatrix}0&\lambda T&\lambda T^2/2\\0&0&\lambda T\\0&0&0\end{bmatrix}\begin{bmatrix}\lambda^2T^2\\\lambda T\\0\end{bmatrix}=\begin{bmatrix}\lambda^2T^2\\0\\0\end{bmatrix}$$

that is the three columns of Q consititute a chain of generalized eigenvectors of length 3

$$AQ=\begin{bmatrix}\lambda&\lambda T&\lambda T^2/2\\0&\lambda&\lambda T\\0&0&\lambda\end{bmatrix}\begin{bmatrix}\lambda^2T^2&\lambda T^2/2&0\\0&\lambda T&0\\0&0&1\end{bmatrix}=\begin{bmatrix}\lambda^3T^2&3\lambda^2T^2/2&\lambda T^2/2\\0&\lambda T&\lambda T\\0&0&\lambda\end{bmatrix}$$

$$Q\begin{bmatrix}\lambda&1&0\\0&\lambda&1\\0&0&\lambda\end{bmatrix}=\begin{bmatrix}\lambda^2T^2&\lambda T^2/2&0\\0&\lambda T&0\\0&0&1\end{bmatrix}\begin{bmatrix}\lambda&1&0\\0&\lambda&1\\0&0&\lambda\end{bmatrix}=\begin{bmatrix}\lambda^3T^2&3\lambda^2T^2/2&\lambda T^2/2\\0&\lambda^2T&\lambda T\\0&0&\lambda\end{bmatrix}$$

$$Q^{-1}AQ=\begin{bmatrix}\lambda&1&0\\0&\lambda&1\\0&0&\lambda\end{bmatrix}$$

3.18 Find the characteristic polynomials and the minimal polynomials of the following matrices
求下列矩阵的特征多项式和最小多项式,

$$\begin{bmatrix}\lambda_1&1&0&0\\0&\lambda_1&1&0\\0&0&\lambda_1&0\\0&0&0&\lambda_1\end{bmatrix}(a)\quad\begin{bmatrix}\lambda_1&1&0&0\\0&\lambda_1&1&0\\0&0&\lambda_1&0\\0&0&0&\lambda_1\end{bmatrix}(b)\quad\begin{bmatrix}\lambda_1&1&0&0\\0&\lambda_1&0&0\\0&0&\lambda_1&0\\0&0&0&\lambda_1\end{bmatrix}(c)\quad\begin{bmatrix}\lambda_1&0&0&0\\0&\lambda_1&0&0\\0&0&\lambda_1&0\\0&0&0&\lambda_1\end{bmatrix}(d)$$

$(a)\Delta_1(\lambda)=(\lambda-\lambda_1)^3(\lambda-\lambda_2) \qquad \Psi_1(\lambda)=(\lambda-\lambda_1)^3(\lambda-\lambda_2)$
$(b)\Delta_2(\lambda)=(\lambda-\lambda_1)^4 \qquad \Psi_2(\lambda)=(\lambda-\lambda_1)^3$
$(c)\Delta_3(\lambda)=(\lambda-\lambda_1)^4 \qquad \Psi_3(\lambda)=(\lambda-\lambda_1)^2$
$(d)\Delta_4(\lambda)=(\lambda-\lambda_1)^4 \qquad \Psi_4(\lambda)=(\lambda-\lambda_1)$

3.19 show that if λ is an eigenvalue of A with eigenvector $\underline{x}$ then $f(\lambda)$ is an eigenvalue of $f(A)$ with the same eigenvector $\underline{x}$

证明如果λ是 A 的关于λ的特征向量,那么$f(\lambda)$是$f(A)$的特征值, $\underline{x}$是$f(A)$关于$f(\lambda)$的特征向量,

proof let A be an $n\times n$ matrix , use theorem 3.5 for any function $f(x)$ we can define $h(x)=\beta_0+\beta_1 x+\cdots+\beta_{n-1}x^{n-1}$ which equals $f(x)$ on the spectrum of A

if λ is an eigenvalue of A , then we have $f(\lambda)=h(\lambda)$ and

$$f(A)=h(A)\quad \begin{aligned} &f(\lambda)\because A\underline{x}\lambda\underline{x}\Rightarrow A^2\underline{x}=\lambda A\underline{x}=\lambda^2\underline{x},\quad A^3\underline{x}=\lambda^3\underline{x},\cdots A^k\underline{x}=\lambda^k\underline{x}\\ &\therefore f(A)\underline{x}=h(A)\underline{x}=(\beta_0 I+\beta_1 A+\cdots+\beta_{n-1}A^{n-1})\underline{x}\\ &\qquad=\beta_0 x+\beta_1\lambda x+\cdots+\beta_{n-1}\lambda^{n-1}x\\ &\qquad=h(\lambda)\underline{x}=f(\lambda)\underline{x}\end{aligned}$$

which implies that $f(\lambda)$ is an eigenvalue of $f(A)$ with the same eigenvector $\underline{x}$

3.20 show that an $n\times n$ matrix has the property $A^k=\underline{0}$ for $k\geq m$ if and only if A has eigenvalues 0 with multiplicity n and index m of less , such a matrix is called a nilpotent matrix

证明$n\times n$的矩阵在$A^k=\underline{0}$当且仅当 A 的 n 重 0 特征值指数不大于 m ,这样的矩阵被称为归零矩阵,

proof : if A has eigenvalues 0 with multiplicity n and index M or less then the Jordan-form representation of A is $\hat{A}=\begin{bmatrix} J_1 & & \\ & J_2 \ddots & \\ & & J_3\end{bmatrix}$ where $J_i=\begin{bmatrix} 0 & 1\ddots & \\ & 0\ddots & 1\\ & & 0\end{bmatrix}_{n_i\times n_i}$ $\displaystyle\sum_{i=1}^{l}n_i=n$ $\max_i n_i\leq m$

from the nilpotent property , we have $J_i^k=\underline{0}\quad for\quad k\geq n_i$ so if

$k \ge m \ge \max_i n_i$, $J_i^k = \underline{0}$ $\quad all \quad i = 1,2,\cdots,l$ and $\hat{A} = \begin{bmatrix} J^k{}_1 & & \\ & J^k{}_2 \ddots & \\ & & J^k{}_3 \end{bmatrix} = \underline{0}$

then $A^k = \underline{0}$ $\quad (f(A) = 0 \quad \underline{if \quad and\ only\ if\ f(\hat{A})} = \underline{0})$

If $A^k = \underline{0} \quad for \quad k \ge m,$ then $\hat{A}^k = \underline{0} \quad for \quad k \ge m,$ where

$$\hat{A} = \begin{bmatrix} J_1 & & \\ & J_2 \ddots & \\ & & J_3 \end{bmatrix}, J_i = \begin{bmatrix} \lambda_i & 1 \ddots & \\ & \lambda_i \ddots & 1 \\ & & \lambda_i \end{bmatrix}_{n_i \times n_i} \quad \sum_{i=1}^{l} n_i = n$$

So we have

$$J^k{}_i = \begin{bmatrix} \lambda_i{}^k & k\lambda_i{}^{k-1} \cdots & \dfrac{k!}{(k-n_i+1)(n_i-1)!}\lambda_i{}^{k-n_i+1} \\ & \lambda_i{}^k \ddots & \vdots \\ & & \lambda_i{}^k \end{bmatrix} = \underline{0} \ i = 1,2,\cdots l$$

$\therefore \lambda_i = 0, \quad and \quad n_i \le m \quad for \quad i = 1,2,\cdots l$

which implies that A has only one distinct eigenvalue o with multiplicity n and index m or less ,

3.21 given $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, find $A^{10}, A^{103} \quad and \quad e^{At}$ 求A的函数 $A^{10}, A^{103} \quad and \quad e^{At}$,

the characteristic polynomial of A is $\Delta_2(\lambda) = \det(\lambda I - A) = \lambda(\lambda-1)^2$

let $h(\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2$ on the spectrum of A , we have

$$f(0) = h(0) \quad 0^{10} = \beta_0$$
$$f(1) = h(1) \quad 1^{10} = \beta_0 + \beta_1 + \beta_2$$
$$f'(1) = h'(1) \quad 10 \cdot 1^{10} = \beta_1 + 2\beta_2$$

the we have $\beta_0 = 0, \quad \beta_2 = -8 \quad \beta_2 = 9$

$$A^{10} = -8A + 9A^2$$
$$= -8\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

the compute

$$A^{103} \quad 0^{103} = \beta_0 \qquad \beta_0 = 0$$
$$1^{103} = \beta_0 + \beta_1 + \beta_2 \Rightarrow \beta_1 = -101$$
$$103 \cdot 1^{103} = \beta_1 + 2\beta_2 \qquad \beta_2 = 102$$

$$A^{103} = -101A + 102A^2$$
$$= -101\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + 102\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 102 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^0 = \beta_0 \qquad \beta_0 = 1$$
to compute e^{At}: $\quad e^t = \beta_0 + \beta_1 + \beta_2 \Rightarrow \beta_1 = 2e^t - te^t - 2$
$$te^t = \beta_1 + 2\beta_2 \qquad \beta_2 = te^t - e^t + 1$$

$$Ae^{At} = \beta_0 I + \beta_1 A + \beta_2 A^2$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (2e^t - e^t - 2)\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + (2e^t - e^t + 1)\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} e^t & e^t - 1 & te^t - e^t + 1 \\ 0 & 1 & e^t - 1 \\ 0 & 0 & e^t \end{bmatrix}$$

3.22 use two different methods to compute e^{At} for A_1 and A_4 in problem 3.13

用两种方法计算题 3.13 中A_1和A_4 的函数e^{A_1t}, e^{A_2t}

method 1 : the Jordan-form representation of A_1 with respect to the basis

$\{[1 \quad 0 \quad 0]' \quad [4 \quad 1 \quad 0]' \quad [5 \quad 0 \quad 1]'\}$ is $\hat{A}_1 = diag\{1, \quad 2, \quad 3\}$

$$Ca \quad \therefore e^{A_1t} = Qe^{\hat{A}_1t}Q^{-1}, \quad where \quad Q = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Cb \qquad = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}\begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

Cc

$$Cd \qquad = \begin{bmatrix} e^t & 4e^{2t} & 5e^{3t} \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}\begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & 4(e^{3t} - e^t) & 5(e^{3t} - e^t) \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}$$

$$\hat{A}_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = Qe^{A_4 t}Q \quad where \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & -3 \\ 0 & -5 & 4 \end{bmatrix}$$

$$\therefore e^{A_4 t} = Qe^{A_4 t}Q^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & -5 & 4 \end{bmatrix} \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4t + 5t^2/2 & 3t + 2t^2 \\ 0 & 20t + 1 & 16t \\ 0 & -25t & -20t + 1 \end{bmatrix}$$

method 2: the characteristic polynomial of A_1 is $\Delta(\lambda) = (\lambda - 1)(\lambda - 2)(\lambda - 3).$ let

$h(\lambda) = \beta_0 + \beta_1 \lambda + \beta_2 \lambda^2$ on the spectrum of A_1 , we have

$$\begin{aligned} f(1) = h(1) \quad & e^t = \beta_0 + \beta_1 + \beta & & \beta_0 = 3e^t - 3e^{3t} + e^{3t} \\ f(2) = h(2) \quad & e^{2t} = \beta_0 + 2\beta_1 + 3\beta_2 & \Rightarrow & \beta_2 = -\frac{5}{2}e^t + 4e^{2t} + -\frac{3}{2}e^{3t} \\ f'(3) = h(3) \quad & e^{3t} = \beta_0 + 3\beta_1 + 9\beta_{2\,2} & & \beta_3 = \frac{1}{2}(e^t - 2e^{2t} + e^{3t}) \end{aligned}$$

$$e^{A_1 t} = \beta_0 I + \beta_1 A_2 + \beta_2 A_1^2$$

$$= \begin{bmatrix} 3e^t - 3e^{2t} + e^{3t} & 0 & 0 \\ 0 & 3e^t - 3e^{2t} + e^{3t} & 0 \\ 0 & 0 & 3e^t - 3e^{2t} + e^{3t} \end{bmatrix} + (-\frac{5}{2}e^t + 4e^{2t} - \frac{3}{2}e^{3t}) \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$+ \frac{1}{2}(e^t - 2e^{2t} + e^{3t}) \begin{bmatrix} 1 & 12 & 40 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 4(e^{2t} - e^t) & 5(e^{3t} - e^t) \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}$$

the characteristic polynomial of A_4 is $\Delta(\lambda) = \lambda^3$, let $h(\lambda) = \beta_0 + \beta_1 \lambda + \beta_2 \lambda^2$ on the spectrum of A_4 ,we have

$$f(0) = h(0) : 1 = \beta_0$$
$$f'(0) = h'(0) : t = \beta_1$$
$$f''(0) = h''(0) : t^2 = 2\beta_2$$

$$e^{A_4 t} = \beta_0 I + \beta_1 A_4 + \beta_1 A_4^2$$

thus $$= I + t\begin{bmatrix} 0 & 4 & 3t+2t^2 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix} + t^2/2\begin{bmatrix} 0 & 5 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4t+5t^2/2 & 3t+2t^2 \\ 0 & 20t+1 & 16t \\ 0 & -25t & -20t+1 \end{bmatrix}$$

3.23 Show that functions of the same matrix ; that is $f(A)g(A)=g(A)f(A)$ consequently we have $Ae^{At}=e^{At}A$ 证明同一矩阵的函数具有可交换性，即 $f(A)g(A)=g(A)f(A)$ 因此有 $Ae^{At}=e^{At}A$ 成立

proof: let $\begin{matrix} g(A)=\beta_0 I+\beta_1 A+\cdots+\beta_{n-1}A^{n-1} \\ f(A)=\alpha_0 I+\alpha_1 A+\cdots+\alpha_{n-1}A^{n-1} \end{matrix}$ (n is the order of A)

$$f(A)g(A)=\alpha_0\beta_0 I+(\alpha_0\beta_1+\alpha_1\beta_0)A+\cdots+\sum_{i=0}^{n-1}\alpha_i\beta_{n-1-i}A^{n-1}+\sum_{i=1}^{n}\alpha_i\beta_{n-i}A^n+\cdots+\alpha_{n-1}\beta_{n-1}A^{2n-2}$$

$$=\sum_{k=0}^{n-1}(\sum_{i=0}^{k}\alpha_i\beta_{k-i})A^k+\sum_{k=n}^{2n-2}(\sum_{i=k-n+1}^{k}\alpha_i\beta_{k-i})A^k$$

$$f(A)g(A)=\sum_{k=0}^{n-1}(\sum_{i=0}^{k}\alpha_i\beta_{k-i})A^k+\sum_{k=n}^{2n-2}(\sum_{i=k-n+1}^{k}\alpha_i\beta_{k-i})A^k=f(A)g(A)$$

let $f(A)=A,\quad g(A)=e^{At}$ then we have $Ae^{At}=e^{At}A$

3.24 let $C=\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}$, find a B such that $e^B=C$ show that of $\lambda_i=0$ for some I ,then B does not exist

let $C=\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}$, find a B such that $e^B=C$ Is it true that ,for any nonsingular c ,there exists a matrix B such that $e^B=C$?

令 $C=\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}$ 证明若 $\lambda_1\cdot\lambda_1\cdot\lambda_1=0$,则不存在 B 使 $e^B=C$

若 $C=\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}$,是否对任意非奇异 C 都存在 B 使，$e^B=C$?

Let $f(\lambda)=\ln e^{\lambda}=\lambda$ so $f(B)=\ln e^B=B$

$$B=\ln C=\begin{bmatrix}\ln\lambda_1 & 0 & 0\\ 0 & \ln\lambda_2 & 0\\ 0 & 0 & \ln\lambda_3\end{bmatrix}$$ where $\lambda_1>0,\quad i=1,2,3$, if $\lambda=0$ for some i ,

$\ln\lambda_1$ does not exist

for $C=\begin{bmatrix}\lambda & 1 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix}$ we have $B=\ln C=\begin{bmatrix}\ln\lambda & \ln'\lambda & 0\\ 0 & \ln\lambda & 0\\ 0 & 0 & \ln\lambda\end{bmatrix}=\begin{bmatrix}\ln\lambda & 1/\lambda & 0\\ 0 & \ln\lambda & 0\\ 0 & 0 & \ln\lambda\end{bmatrix}$,

where $\lambda>0,\quad if\quad \lambda\le 0$ then $\ln\lambda$ does mot exist , so B does mot exist , we can conclude that , it is mot true that , for any nonsingular C THERE EXISTS a B such that $e^k=c$

3.25 let $(sI-A)=\frac{1}{\Delta(s)}Adj(sI-A)$ and let m(s) be the monic greatest common divisor of all entries of Adj(Si-A) , Verify for the matrix A_2 in problem 3.13 that the minimal polynomial of A equals $\Delta(s)/m(s)$

令，$(sI-A)=\frac{1}{\Delta(s)}Adj(sI-A)$，并且令 m(s)是 Adj(Si-A)的所有元素的第一最大公因子，利用题 3.13 中 A_2验证 A 的最小多项式为$\Delta(s)/m(s)$

verification :

$$A_3=\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}\qquad \hat{A}_3=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}$$

$$\Delta(s)=(s-1)^2(s-2)\qquad \varphi(s)=(s-1)\backslash(s-2)$$

$$sI-A=\begin{bmatrix}S-1 & 0 & 1\\ 0 & S-1 & 0\\ 0 & 0 & s-2\end{bmatrix},\quad Adj(sI-A)=\begin{bmatrix}(s-1)(s-2) & 0 & -(s-1)\\ 0 & (s-1)(s-2) & 0\\ 0 & 0 & (s-1)^2\end{bmatrix}$$

$$so\qquad m(s)=s-1$$

we can easily obtain that $\varphi(s)=\Delta(s)/m(s)$

3.26 Define $(sI-A)^{-1}=\frac{1}{\Delta(s)}\left[R_0s^{n-1}+R_1s^{n-2}+\cdots+R_{n-2}s+R_{n-1}\right]$ where $\Delta(s)=\det(sI-A):=s^2+\alpha_1s^{n-1}+\alpha_2s^{n-2}+\cdots+\alpha_n\quad and\quad R_i$ are constant matrices theis definition is valid because the degree in s of the adjoint of (sI-A) is at most n-1 , verify

$$\alpha_1 = -\frac{tr(AR_0)}{1} \qquad R_0 = I$$

$$\alpha_2 = -\frac{tr(AR_1)}{2} \qquad R_1 = AR_0 + \alpha_1 I = A + \alpha_1 I$$

$$\alpha_3 = -\frac{tr(AR_1)}{2} \qquad R_2 = AR_1 + \alpha_2 I = A^2 + \alpha_1 A + \alpha_2 I$$

$$\vdots$$

$$\alpha_{n-1} = -\frac{tr(AR_1)}{n-1} \qquad R_{n-1} = AR_{n-2} + \alpha_{n-1} I = A^{n-1} + \alpha_1 A^{n-2} + \cdots + \alpha_{n-2} A + \alpha_{n-1} I$$

$$\alpha_n = -\frac{tr(AR_1)}{n} \qquad \underline{0} = AR_{n-1} + \alpha_n I$$

where tr stands for the trase of a matrix and is defined as the sum of all its diagonal entries this process of computing $\alpha_i \quad and \quad R_i$ is called the leverrier algorithm

定义 $(SI - A)^{-1} := \frac{1}{\Delta(s)}[R_0 S^{n-1} + R_1 S^{n-2} + \cdots + R_{n-2} S + R_{n-1}]$ 其中 $\Delta(s)$ 是 A 的特征多项式 $\Delta(s) = \det(sI - A) := s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_n \quad and \quad R_i$ 是常数矩阵,这样定义是有效的, 因为 SI-A 的伴随矩阵中 S 的阶次不超过 N-1 验证

$$\alpha_1 = -\frac{tr(AR_0)}{1} \qquad R_0 = I$$

$$\alpha_2 = -\frac{tr(AR_1)}{2} \qquad R_1 = AR_0 + \alpha_1 I = A + \alpha_1 I$$

$$\alpha_3 = -\frac{tr(AR_1)}{2} \qquad R_2 = AR_1 + \alpha_2 I = A^2 + \alpha_1 A + \alpha_2 I$$

$$\vdots$$

$$\alpha_{n-1} = -\frac{tr(AR_1)}{n-1} \qquad R_{n-1} = AR_{n-2} + \alpha_{n-1} I = A^{n-1} + \alpha_1 A^{n-2} + \cdots + \alpha_{n-2} A + \alpha_{n-1} I$$

$$\alpha_n = -\frac{tr(AR_1)}{n} \qquad \underline{0} = AR_{n-1} + \alpha_n I$$

其中矩阵的迹 tr 定义为其对角元素之和, 这种计算 α_i 和 R_i 的程式被称为 leverrier 算法.

verification:

$$\begin{aligned}
&(SI - A)[R_0 S^{n-1} + R_1 S^{n-2} + \cdots + R_{n-2} S + R_{n-1}] \\
&= R_0 S^n + (R_1 - AR_0) S^{n-1} + (R_2 - AR_1) S^{n-2} + \cdots (R_{n-1} - AR_{n-2}) S + AR_{n-1} \\
&= I(S^n + \alpha_1 S^{n-1} + \alpha_2 S^{n-2} + \cdots + \alpha_{n-1} S + \alpha_n) \\
&= I\Delta(S) \\
&\textit{which} \quad \textit{implies}
\end{aligned}$$

$$(SI-A)^{-1} := \frac{1}{\Delta(s)}[R_0 S^{n-1} + R_1 S^{n-2} + \cdots + R_{n-2} S + R_{n-1}]$$'

where $\Delta(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_n$

3.27 use problem 3.26 to prove the cayley-hamilton theorem

利用题 3.26 证明 cayley-hamilton 定理

proof:

$$\begin{aligned}
&R_0 = I \\
&R_1 - AR_0 = \alpha_1 I \\
&R_2 - AR_1 = \alpha_2 I \\
&\vdots \\
&R_{n-1} - AR_{n-2} = \alpha_{n-1} I \\
&-AR_{n-1} = \alpha_n I
\end{aligned}$$

multiplying ith equation by A^{n-i+1} yields ($i = 1,2\cdots n$)

$$\begin{aligned}
&A^n R_0 = A^n \\
&A^{n-1} R_1 - A^n R_0 = \alpha_1 A^{n-1} \\
&A^{n-2} R_2 - A^{n-1} R_1 = \alpha_2 A^{n-2} \\
&\vdots \\
&AR_{n-1} - A^2 R_{n-2} = \alpha_{n-1} A \\
&-AR_{n-1} = \alpha_n I
\end{aligned}$$

then we can see $\begin{aligned} &A^n + \alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \cdots + \alpha_{n-1} A + \alpha_n I \\ &= A^n R_0 + + A^{n-1} R_1 - A^n R_0 + \cdots + AR_{n-1} - A^2 R_{n-2} - AR_{n-1} = \underline{0} \end{aligned}$ that is

$\Delta(A) = \underline{0}$

3.28 use problem 3.26 to show

$$(sI-A)^{-1}$$
$$= \frac{1}{\Delta(s)}\left[A^{n-1} + (s+\alpha_1)A^{n-2} + (s^2 + \alpha_1 s + \alpha_2)A^{n-3} + \cdots + (s^{n-1} + \alpha_1 s^{n-2} + \cdots + \alpha_{n-1})I\right]$$

利用题 3.26 证明上式,

Proof:

$$(SI-A)^{-1} := \frac{1}{\Delta(s)}[R_0 S^{n-1} + R_1 S^{n-2} + \cdots + R_{n-2} S + R_{n-1}]$$
$$= \frac{1}{\Delta(s)}[S^{n-1} + (A+\alpha_1 I)S^{n-2} + (A^2 + \alpha_1 A + \alpha_2 I)S^{n-3} + \cdots$$
$$(A^{n-2} + \alpha_1 A^{n-3} + \cdots + \alpha_{n-3} A + \alpha_{n-4})S + A^{n-1} + \alpha_1 A^{n-2} + \cdots + \alpha_{n-2} A + \alpha_{n-1} I]$$
$$= \frac{1}{\Delta(s)}\left[A^{n-1} + (s+\alpha_1)A^{n-2} + (s^2 + \alpha_1 s + \alpha_2)A^{n-3} + \cdots + (s^{n-1} + \alpha_1 s^{n-2} + \cdots + \alpha_{n-1})I\right]$$

another : let $\alpha_0 = 1$

$$(SI-A)^{-1} := \frac{1}{\Delta(s)}[R_0 S^{n-1} + R_1 S^{n-2} + \cdots + R_{n-2} S + R_{n-1}]$$
$$= \frac{1}{\Delta(s)}\sum_{i=i}^{n-1} R_i S^{n-1-i} = \frac{1}{\Delta(s)}\sum_{i=0}^{n-1} S^{n-1-i} \sum_{i=0}^{n-1} \alpha_{i-l} A^{L}$$
$$= \frac{1}{\Delta(s)}\sum_{i=0}^{N-1} \alpha_i S^{n-1-I} A^0 + \sum_{i=0}^{n-1} \alpha_i S^{n-1-I} A + \cdots +$$
$$S^{n-1} + (A+\alpha_1 I)S^{n-2} + (A^2 + \alpha_1 A + \alpha_2 I)S^{n-3} + \cdots$$
$$(A^{n-2} + \alpha_1 A^{n-3} + \cdots + \alpha_{n-3} A + \alpha_{n-4})S + A^{n-1} + \alpha_1 A^{n-2} + \cdots + \alpha_{n-2} A + \alpha_{n-1} I]$$
$$= \frac{1}{\Delta(s)}\left[A^{n-1} + (s+\alpha_1)A^{n-2} + (s^2 + \alpha_1 s + \alpha_2)A^{n-3} + \cdots + (s^{n-1} + \alpha_1 s^{n-2} + \cdots + \alpha_{n-1})I\right]$$

3.29 let all eigenvalues of A be distinct and let $\underline{q_i}$ be a right eigenvector of A associated with λ_i that is $A\underline{q_i} = \lambda_I \underline{q_i}$ define $Q = \lfloor \underline{q_1} \quad \underline{q_2} \quad \cdots \quad \underline{q_n} \rfloor$ and define

$$P; = Q^{-1} =: \begin{bmatrix} \underline{p_1} \\ \underline{p_2} \\ \vdots \\ \underline{p_n} \end{bmatrix},$$

where $\underline{p_i}$ is the ith row of P , show that $\underline{p_i}$ is a left eigenvector of A associated with λ_i, that is $\underline{p_i} A = \lambda_i \underline{p_i}$ 如果 A 的所有特征值互不相同, $\underline{q_i}$ 是关于 λ_i 的一个右特征向量,即

$A\underline{q_i} = \lambda_I \underline{q_i}$,定义 $Q = \lfloor \underline{q_1} \quad \underline{q_2} \quad \cdots \quad \underline{q_n} \rfloor$ 并且 $P; = Q^{-1} =: \begin{bmatrix} \underline{p_1} \\ \underline{p_2} \\ \vdots \\ \underline{p_n} \end{bmatrix}$

其中 $\underline{p_i}$ 是 P 的第 I 行, 证明 $\underline{p_i}$ 是 A 的关于 λ_i 的一个左特征向量,即 $\underline{p_i} A = \lambda_i \underline{p_i}$

Proof: all eigenvalues of A are distinct , and $\underline{q_i}$ is a right eigenvector of A associated with λ_i,

and $Q = \begin{bmatrix} \underline{q_1} & \underline{q_2} & \cdots & \underline{q_n} \end{bmatrix}$ so we know that $\hat{A} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 \ddots & \\ & & \lambda_3 \end{bmatrix} = Q^{-1}AQ = PAP^{-1}$

$\therefore PA = \hat{A}P$ That is

$:\begin{bmatrix} \underline{p_1} \\ \underline{p_2} \\ \vdots \\ \underline{p_n} \end{bmatrix} A = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} \underline{p_1} \\ \underline{p_2} \\ \vdots \\ \underline{p_n} \end{bmatrix} \Rightarrow \begin{bmatrix} \underline{p_1}A \\ \underline{p_2}A \\ \vdots \\ \underline{p_n}A \end{bmatrix} = \begin{bmatrix} \lambda_1 \underline{p_1} \\ \lambda_2 \underline{p_2} \\ \vdots \\ \lambda_n \underline{p_n} \end{bmatrix}$ so $\underline{p_i}A = \lambda_i \underline{p_i}$, that is , $\underline{p_i}$ is a left eigenvector of A associated with λ_i

3.30 show that if all eigenvalues of A are distinct , then $(SI - A)^{-1}$ can be expressed as $(SI - A)^{-1} = \sum \frac{1}{s - \lambda_i} \underline{q_i}\,\underline{p_i}$ where $\underline{q_i}$ and $\underline{p_i}$ are right and left eigenvectors of A associated with λ_i 证明若 A 的所有特征值互不相同，则 $(SI - A)^{-1}$ 可以表示为 $(SI - A)^{-1} = \sum \frac{1}{s - \lambda_i} \underline{q_i}\,\underline{p_i}$ 其中 $\underline{q_i}$ 和 $\underline{p_i}$ 是 A 的关于 λ_i 的右特征值和左特征值,

Proof: if all eigenvalues of A are distinct , let $\underline{q_i}$ be a right eigenvector of A associated with λ_i,

then $Q = \begin{bmatrix} \underline{q_1} & \underline{q_2} & \cdots & \underline{q_n} \end{bmatrix}$ is nonsingular , and $:Q^{-1} = \begin{bmatrix} \underline{p_1} \\ \underline{p_2} \\ \vdots \\ \underline{p_n} \end{bmatrix}$ where is a left eigenvector of A associated with λ_i,

$$\sum \frac{1}{s - \lambda_i} \underline{q_i}\,\underline{p_i}(SI - A)$$

$$= \sum \frac{1}{s - \lambda_i}(s\underline{q_i}\,\underline{p_i} - \underline{q_i}\,\underline{p_i}A) = \sum \frac{1}{s - \lambda_i}(s\underline{q_i}\,\underline{p_i} - \underline{q_i}\lambda_i \underline{p_i})$$

$$= \sum \frac{1}{s - \lambda_i}(s - \lambda_i)(\underline{q_i}\,\underline{p_i}) = \sum \underline{q_i} p_i$$

That is $(SI - A)^{-1} = \sum \frac{1}{s - \lambda_i} \underline{q_i}\,\underline{p_i}$

3.31 find the M to meet the lyapunov equation in (3.59) with $A=\begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}$ $B=3$ $C=\begin{bmatrix}3\\3\end{bmatrix}$ what are the eigenvalues of the lyapunov equation ? is the lyapunov equation singular ? is the solution unique ?

已知A,B,C,求M 使之满足(3.59) 的lyapunov 方程的特征值, 该方程是否奇异>解是否唯一?

$\because AM+MB=C$

$\therefore \begin{bmatrix}3 & 1\\ -2 & 1\end{bmatrix}M=C\Rightarrow M=\begin{bmatrix}0\\3\end{bmatrix}$

$$\Delta_A(\lambda)=\det(\lambda I-A)=(\lambda+1-j)(\lambda+1+j) \quad \Delta_B(\lambda)=\det(\lambda I-B)=\lambda+3$$

The eigenvalues of the Lyapunov equation are

$\eta_1=-1+j+3=2+j)$ $\eta_2=-1-j+3=2-j$

The lyapunov equation is nonsingular M satisfying the equation

3.32 repeat problem 3.31 for $A=\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}$ $B=1$ $C_1=\begin{bmatrix}3\\3\end{bmatrix}$ $C_2=\begin{bmatrix}3\\-3\end{bmatrix}$ with two different C ,用本题给出的 A, B, C, 重复题 3.31 的问题,

$\because AM+MB=C$

$\therefore \begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}M=C_1\Rightarrow No \quad solution$

$\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}M=C_2\Rightarrow M=\begin{bmatrix}\alpha\\ 3-\alpha\end{bmatrix} for \quad any \quad \alpha$

$\Delta_A(\lambda)=(\lambda+1)^2 \quad \Delta_B(\lambda)=\lambda-1$

the eigenvalues of the lyapunov equation are $\eta_1=\eta_2=-1+1=0$ the lyapunov equation is singular because it has zero eigenvalue if C lies in the range space of the lyapunov equation , then solution exist and are not unique

,

3.33 check to see if the following matrices are positive definite or senidefinite 确定下列矩阵是否正定或者正半定, $\begin{bmatrix}2 & 3 & 2\\ 3 & 1 & 0\\ 2 & 0 & 2\end{bmatrix}$ $\begin{bmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ -1 & 0 & 2\end{bmatrix}$ $\begin{bmatrix}a_1a_1 & a_1a_2 & a_1a_3\\ a_2a_1 & a_2a_2 & a_2a_3\\ a_3a_1 & a_3a_2 & a_3a_3\end{bmatrix}$

1. $\because \det\begin{bmatrix}2 & 3\\3 & 1\end{bmatrix} = 2-9=-7<0 \quad \therefore \begin{bmatrix}2 & 3 & 2\\3 & 1 & 0\\2 & 0 & 2\end{bmatrix}$ is not positive definite , nor is pesitive semidefinite

2. $\because \det\begin{bmatrix}\lambda & 0 & -1\\0 & \lambda & 0\\1 & 0 & \lambda-2\end{bmatrix} = \lambda(\lambda-1+\sqrt{2})(\lambda-1-\sqrt{2})$ it has a negative eigenvalue $1-\sqrt{2}$,

so the second matrix is not positive definite , neither is positive demidefinte ,,

3 the third matrix 's prindipal minors $a_1a_1 \geq 0, a_2a_2 \geq 0, a_3a_3 \geq 0,$`

$$\det\begin{bmatrix}a_1a_1 & a_1a_2\\a_2a_1 & a_2a_2\end{bmatrix}=0 \quad \det\begin{bmatrix}a_1a_1 & a_1a_3\\a_3a_1 & a_3a_3\end{bmatrix}=0 \quad \det\begin{bmatrix}a_2a_2 & a_2a_3\\a_3a_2 & a_3a_3\end{bmatrix}=0 \quad \det\begin{bmatrix}a_1a_1 & a_1a_2 & a_1a_3\\a_2a_1 & a_2a_2 & a_2a_3\\a_3a_1 & a_3a_2 & a_3a_3\end{bmatrix}=0$$

that is all the principal minors of the thire matrix are zero or positive , so the matrix is positive semidefinite ,

3.34 compute the singular values of the following matrices 计算下列矩阵的奇值,

$$\begin{bmatrix}-1 & 0 & 1\\2 & -1 & 0\end{bmatrix} \quad \begin{bmatrix}-1 & 2\\2 & 4\end{bmatrix}$$

$$\begin{bmatrix}-1 & 0 & 1\\2 & -1 & 0\end{bmatrix}\begin{bmatrix}-1 & 2\\0 & -1\\1 & 0\end{bmatrix} = \begin{bmatrix}2 & -2\\-2 & 5\end{bmatrix}$$

$$\Delta(\lambda) = (\lambda-2)(\lambda-5)-4 = \lambda^2-7\lambda+6 = (\lambda-1)(\lambda-6)$$

the eigenvalues of $\begin{bmatrix}-1 & 0 & 1\\2 & -1 & 0\end{bmatrix}\begin{bmatrix}-1 & 2\\0 & -1\\1 & 0\end{bmatrix}$ are 6 and 1 , thus the singular values of

$\begin{bmatrix}-1 & 0 & 1\\2 & -1 & 0\end{bmatrix}$ are $\sqrt{6}$ and 1 ,

$$\begin{bmatrix}-1 & 2\\2 & 4\end{bmatrix}\begin{bmatrix}-1 & 2\\2 & 4\end{bmatrix} = \begin{bmatrix}5 & 6\\6 & 20\end{bmatrix}$$

$\Delta(\lambda) = (\lambda-5)(\lambda-20)-36 = (\lambda-\frac{25}{2}+\frac{3}{2}\sqrt{41})(\lambda-\frac{25}{2}-\frac{3}{2}\sqrt{41})$ the eigenvalues of

$\begin{bmatrix}-1 & 2\\2 & 4\end{bmatrix}\begin{bmatrix}-1 & 2\\2 & 4\end{bmatrix}$ are $\frac{25}{2}+\frac{3}{2}\sqrt{41} \quad and \quad \frac{25}{2}-\frac{3}{2}\sqrt{41}$, thus the singular values of

$\begin{bmatrix} -1 & 2 \\ 2 & 4 \end{bmatrix}$ are $(\frac{25}{2}+\frac{3}{2}\sqrt{41})^{1/2}=4.70 \quad and \quad (\frac{25}{2}-\frac{3}{2}\sqrt{41})^{1/2}=1.70$

3.35 if A is symmetric , what is the ralatimship between its eigenvalues and singular values ? 对称矩阵 A 的特征值与奇异值之间有什么关系?

If A is symmetric , then $A'A = AA' = A^2$ LET $\underline{q_i}$ be an eigenvector of A associated with eigenvaue λ_i that is , $A\underline{q_i} = \lambda_i \underline{q_i}, (i = 1,3,\cdots n)$ thus we have

$A^2 \underline{q_i} = A\lambda_i \underline{q_i} = \lambda_i A\underline{q_i} = \lambda_i^2 \underline{q_i} (i = 1,3,\cdots n)$

Which implies λ_i^2 is the eigenvalue of A^2 $for \quad i = 1,2,\cdots n$, (n is the order of A)

So the singular values of A are $|\lambda_i| \; for \quad i = 1,2,\cdots n$ where λ_i is the eigenvalue of A

3.36 show $\det\left(I_n + \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}\begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}\right) = 1 + \sum_{m=1}^{n} a_m b_m$

证明上式成立

let $A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ $B = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}$ A is $n \times 1$ and B is $1 \times n$

use (3.64) we can readily obtain

$$\det\left(I_n + \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}\begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}\right) = \det\left(I_1 + \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}\right)$$

$$= 1 + \sum_{m \le 1}^{n} a_m b_m$$

3.37 show (3.65) 证明(3.65)
proof: let

$$N=\begin{bmatrix}\sqrt{s}I_m & A\\ \underline{0} & \sqrt{s}I_n\end{bmatrix}\quad Q=\begin{bmatrix}\sqrt{s}I_m & \underline{0}\\ B & \sqrt{s}I_n\end{bmatrix}\quad P=\begin{bmatrix}\sqrt{s}I_m & -A\\ -B & \sqrt{s}I_n\end{bmatrix}$$

then we have

$$NP=\begin{bmatrix}sI_m-AB & \underline{0}\\ \sqrt{s}B & sI_n\end{bmatrix}\quad QP=\begin{bmatrix}sI_m & -\sqrt{s}A\\ \underline{0} & sI_n-BA\end{bmatrix}$$

because

$$\det(NP)=\det N\det P=s\det P$$
$$\det(QP)=\det Q\det P=s\det P$$

we have det(NP)=det(QP)

And
$$\det(NP)=\det(sI_m-AB)\det(sI_n)=S^n\det(sI_m-AB)$$
$$\det(QP)=\det(sI_m)\det(sI_n-BA)=S^m\det(sI_n-BA)$$

3.38 Consider $A\underline{x}=\underline{y}$, where A is $m\times n$ and has rank m is $(A'A)^{-1}A'\underline{y}$ a solution ? if not , under what condition will it be a solution? Is $A'(AA')^{-1}\underline{y}$ a solution ?

$m\times n$ 阶矩阵 A 秩为 m，$(A'A)^{-1}A'\underline{y}$ 是不是方程 $A\underline{x}=\underline{y}$ 的解？如果不是，那么在什么条件下，他才会成为该方程的解？$A'(AA')^{-1}\underline{y}$ 是不是方程的解？

A is $m\times n$ and has rank m so we know that $m\le n$, and $A'A$ is a square matrix of $m\times m$

rankA=m , rank($A'A)\le rankA=m\le n$,

So if $m\ne n$,then rank($A'A$)<n , $(A'A)^{-1}$ does mot exist m and $(A'A)^{-1}A'\underline{y}$ isn't a solution if $A\underline{x}=\underline{y}$

If m=n , and rankA=m , so A is nonsingular , then we have rank($A'A$)=rank(A)=m , and A $(A'A)^{-1}A'\underline{y}$=A $A^{-1}(A')^{-1}A'\underline{y}=\underline{y}$ that is $(A'A)^{-1}A'\underline{y}$ is a solution ,

∵ RankA=M

∴ Rank($A'A$)=m $A'A$ is monsingular and $(A'A)^{-1}$ exists , so we have

$AA'(AA')^{-1}\underline{y}=\underline{y}$, that is , $A'(AA')^{-1}\underline{y}$ is a solution of $A\underline{x}=\underline{y}$,

PROBLEMS OF CHAPTER 4

4.1 An oscillation can be generated by 一个振荡器可由下式描述：

$$\dot{X}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}X$$

试证其解为：

Show that its solution is $X(t)=\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}X(0)$

Proof: $X(t)=e^{At}X(0)=e^{\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}t}X(0)$,the eigenvalues of A are j,-j;

Let $h(\lambda)=\beta_0+\beta_1\lambda$ .If $h(\lambda)=e^{\lambda t}$,then on the spectrum of A,then

$$\begin{aligned}h(j)&=\beta_0+\beta_1 j=e^{jt}=\cos t+j\sin t\\ h(-j)&=\beta_0-\beta_1 j=e^{-jt}=\cos t-j\sin t\end{aligned}\quad \text{then}\quad \begin{aligned}\beta_0&=\cos t\\ \beta_1&=\sin t\end{aligned}$$

so $h(A)=\beta_0 I+\beta_1 A=\cos t\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\sin t\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}=\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}$

$$X(t)=e^{At}X(0)=e^{\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}t}X(0)=\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}X(0)$$

4.2 Use two different methods to find the unit-step response of 用两种方法求下面系统的单位阶跃响应：

$$\dot{X}=\begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}X+\begin{bmatrix}1\\ 1\end{bmatrix}U$$

$$Y=\begin{bmatrix}2 & 3\end{bmatrix}X$$

Answer: assuming the initial state is zero state.

method1:we use (3.20) to compute

$$(sI-A)^{-1}=\begin{bmatrix}s & -1\\ 2 & s+2\end{bmatrix}^{-1}=\frac{1}{s^2+2s+2}\begin{bmatrix}s+2 & 1\\ -2 & s\end{bmatrix}$$

then $e^{At}=L^{-1}((sI-A)^{-1})=\begin{bmatrix}\cos t+\sin t & \sin t\\ -2\sin t & \cos t-\sin t\end{bmatrix}e^{-t}$ and

$$Y(s)=C(sI-A)^{-1}BU(s)=\frac{5s}{(s^2+2s+2)s}=\frac{5}{(s^2+2s+2)}$$

then $y(t)=5e^{-t}\sin t$ for t>=0

method2:

$$
\begin{aligned}
y(t) &= C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\
&= C\int_0^t e^{A(t-\tau)}d\tau B = CA^{-1}(e^{At}-e^0)B \\
&= C\begin{bmatrix}-1 & -0.5\\ 1 & 0\end{bmatrix}\begin{bmatrix}e^{-t}\cos t+2e^{-t}\sin t-1\\ e^{-t}\cos t-3e^{-t}\sin t-1\end{bmatrix} \\
&= 5\sin te^{-t}
\end{aligned}
$$

for t>=0

4.3 Discretize the state equation in Problem 4.2 for T=1 and T=π.离散化习题4.3中的状态方程，T分别取1和π

Answer:

$$
X[k+1] = e^{AT}X[k] + \left(\int_0^T e^{A\alpha}d\alpha\right)BU[k]
$$
$$
Y[k] = CX[k] + DU[k]
$$

For T=1,use matlab:

[ab,bd]=c2d(a,b,1)

ab =0.5083 0.3096

-0.6191 -0.1108

bd =1.0471

-0.1821

$$
X[k+1] = \begin{bmatrix}0.5083 & 0.3096\\ -0.6191 & -0.1108\end{bmatrix}X[k] + \begin{bmatrix}1.0471\\ -0.1821\end{bmatrix}U[k]
$$
$$
Y[k] = \begin{bmatrix}2 & 3\end{bmatrix}X[k]
$$

for T=π ,use matlab:

[ab,bd]=c2d(a,b,3.1415926)

ab =-0.0432 0.0000

-0.0000 -0.0432

bd =1.5648

-1.0432

$$
X[k+1] = \begin{bmatrix}-0.0432 & 0\\ 0 & -0.0432\end{bmatrix}X[k] + \begin{bmatrix}1.5648\\ -1.0432\end{bmatrix}U[k]
$$
$$
Y[k] = \begin{bmatrix}2 & 3\end{bmatrix}X[k]
$$

4.4 Find the companion-form and modal-form equivalent equations of 求系统的等价友形和模式规范形。

$$
\dot{X} = \begin{bmatrix}-2 & 0 & 0\\ 1 & 0 & 1\\ 0 & -2 & -2\end{bmatrix}X + \begin{bmatrix}1\\ 0\\ 1\end{bmatrix}U
$$

$$
Y = \begin{bmatrix}1 & -1 & 0\end{bmatrix}X
$$

Answer: use [ab ,bb,cb,db,p]=canon(a,b,c,d,'companion)

We get the companion form

```
ab =   0      0     -4
       1      0     -6
       0      1     -4
bb =   1
       0
       0
cb = 1     -4      8
db =0
p =1.0000     1.0000           0
    0.5000     0.5000     -0.5000
     0.2500           0     -0.2500
```

$$\dot{X}=\begin{bmatrix}0 & 0 & -4\\ 1 & 0 & -6\\ 0 & 1 & -4\end{bmatrix}X+\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}U$$

$$Y=\begin{bmatrix}1 & -4 & 8\end{bmatrix}X$$

use use [ab ,bb,cb,db,p]=canon(a,b,c,d) we get the modal form

```
ab = -1      1      0
     -1     -1      0
      0      0     -2
bb = -3.4641
           0
      1.4142
cb = 0     -0.5774     0.7071
db = 0
p = -1.7321     -1.7321     -1.7321
          0      1.7321           0
      1.4142          0           0
```

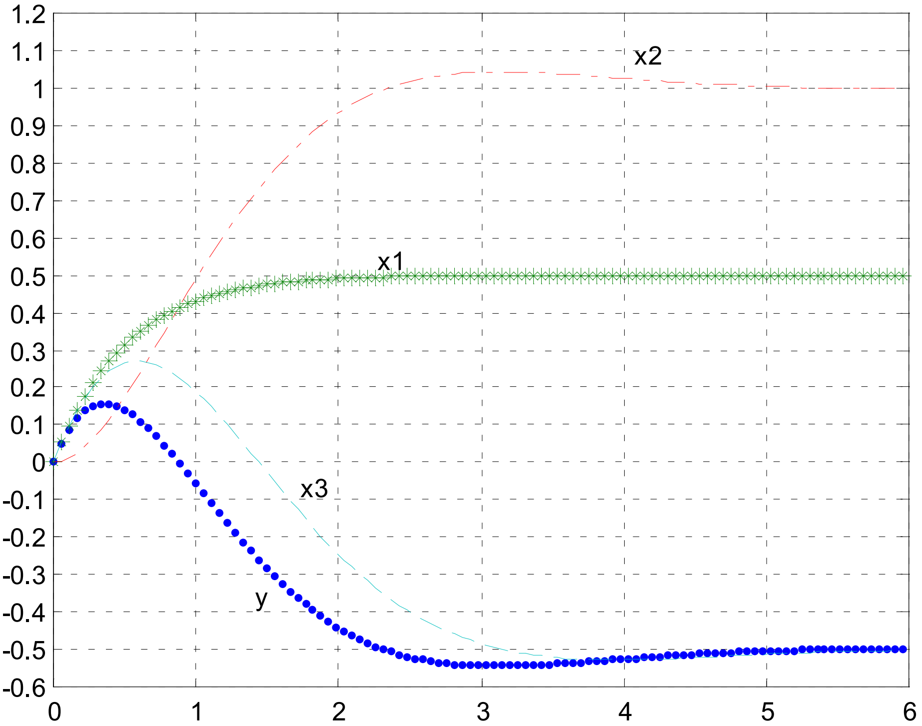
$$\dot{X}=\begin{bmatrix}-1 & 1 & 0\\ -1 & -1 & 0\\ 0 & 0 & -2\end{bmatrix}X+\begin{bmatrix}-3.4641\\ 0\\ 1.4142\end{bmatrix}U$$

$$Y=\begin{bmatrix}0 & -0.5774 & 0.7071\end{bmatrix}X$$

4.5 Find an equivalent state equation of the equation in Problem 4.4 so that all state variables have their larest magnitudes roughly equal to the largest magnitude of the output.If all signals are required to lie inside ± 10 volts and if the input is a step function with magnitude a,what is the permissible largest a?找出习题 4.4 中方程的等价状态方程使所有状态变量的最大量几乎等于输出的最大值。如果所有的信号需要在±10 伏以内，输入为大小为 a 的的阶跃函数。求所允许的最大的 a 值。

Answer: first we use matlab to find its unit-step response .we type

```
a=[-2 0 0;1 0 1;0 -2 -2]; b=[1 0 1]';c=[1 -1 0]; d=0;
[y,x,t]=step(a,b,c,d)
plot(t,y,'.',t,x(:,1),'*',t,x(:,2),'-.',t,x(:,3),'--')
```



so from the fig above.we know max(|y|)=0.55, max(|x1|=0.5;max(|x2|)=1.05,and max(|x3|)=0.52

for unit step input.Define $\bar{x}_1 = x_1, \bar{x}_2 = 0.5x_2, \bar{x}_3 = x_3$, then

$$\dot{\bar{X}} = \begin{bmatrix} -2 & 0 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & -4 & -2 \end{bmatrix} \bar{X} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 1 & -2 & 0 \end{bmatrix} \bar{X}$$

the largest permissible a is 10/0.55=18.2

4.6 Consider $\dot{X} = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix} X + \begin{bmatrix} b_1 \\ \bar{b}_2 \end{bmatrix} U \qquad Y = \begin{bmatrix} c_1 & \bar{c}_1 \end{bmatrix} X$

where the overbar denotes complex conjugate.Verify that the equation can be transformed into

$$\dot{\bar{X}} = \bar{A}\bar{X} + \bar{B}u \qquad y = \bar{C}_1 \bar{X}$$

with $\bar{A} = \begin{bmatrix} 0 & 1 \\ -\lambda\bar{\lambda} & \lambda + \bar{\lambda} \end{bmatrix}$ $\bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\bar{C}_1 = \begin{bmatrix} -2\operatorname{Re}(\bar{\lambda} b_1 c_1) & 2\operatorname{Re}(b_1 c_1) \end{bmatrix}$,by using the transformation $X = Q\bar{X}$ with $Q = \begin{bmatrix} -\bar{\lambda} b_1 & b_1 \\ -\lambda \bar{b}_1 & \bar{b}_1 \end{bmatrix}$。试验证以上变换成立。

Answer: let $X = Q\overline{X}$ with $Q = \begin{bmatrix} -\overline{\lambda} b_1 & b_1 \\ -\lambda \overline{b}_1 & \overline{b}_1 \end{bmatrix}$,we get

$$Q\dot{\overline{X}} = \begin{bmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{bmatrix} Q\overline{X} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} U \qquad Y = \begin{bmatrix} c_1 & \overline{c}_1 \end{bmatrix} Q\overline{X} \qquad Q^{-1} = \frac{1}{(\lambda - \overline{\lambda}) b_1 \overline{b}_1} \begin{bmatrix} \overline{b}_1 & -b_1 \\ \lambda \overline{b}_1 & -\overline{\lambda} b_1 \end{bmatrix}$$

so

$$\dot{\overline{X}} = Q^{-1} \begin{bmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{bmatrix} Q\overline{X} + Q^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} U$$

$$= \frac{1}{(\lambda - \overline{\lambda}) b_1 \overline{b}_1} \begin{bmatrix} \overline{b}_1 & -b_1 \\ \lambda \overline{b}_1 & -\overline{\lambda} b_1 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{bmatrix} \begin{bmatrix} -\overline{\lambda} b_1 & b_1 \\ -\lambda \overline{b}_1 & \overline{b}_1 \end{bmatrix} \overline{X} + \frac{1}{(\lambda - \overline{\lambda}) b_1 \overline{b}_1} \begin{bmatrix} \overline{b}_1 & -b_1 \\ \lambda \overline{b}_1 & -\overline{\lambda} b_1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} U$$

$$= \frac{1}{(\lambda - \overline{\lambda}) b_1 \overline{b}_1} \begin{bmatrix} \overline{b}_1 \lambda & -b_1 \overline{\lambda} \\ \lambda^2 \overline{b}_1 & -\overline{\lambda}^2 b_1 \end{bmatrix} \begin{bmatrix} -\overline{\lambda} b_1 & b_1 \\ -\lambda \overline{b}_1 & \overline{b}_1 \end{bmatrix} \overline{X} + \frac{1}{(\lambda - \overline{\lambda}) b_1 \overline{b}_1} \begin{bmatrix} 0 \\ (\lambda - \overline{\lambda}) b_1 \overline{b}_1 \end{bmatrix} U$$

$$= \begin{bmatrix} 0 & 1 \\ -\lambda \overline{\lambda} & \lambda + \overline{\lambda} \end{bmatrix} \overline{X} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U = \overline{A}\overline{X} + \overline{B} U$$

$$Y = \begin{bmatrix} c_1 & \overline{c}_1 \end{bmatrix} Q\overline{X} = \begin{bmatrix} c_1 & \overline{c}_1 \end{bmatrix} \begin{bmatrix} -\overline{\lambda} b_1 & b_1 \\ -\lambda \overline{b}_1 & \overline{b}_1 \end{bmatrix} \overline{X} = \begin{bmatrix} -\overline{\lambda} b_1 c_1 - \lambda \overline{b}_1 \overline{c}_1 & c_1 b_1 + \overline{c}_1 \overline{b}_1 \end{bmatrix} \overline{X} = \overline{C}_1 \overline{X}$$

4.7 Verify that the Jordan-form equation

$$\dot{X} = \begin{bmatrix} \lambda & 1 & & & & \\ & \lambda & 1 & & & \\ & & \lambda & 0 & & \\ & & & \overline{\lambda} & 1 & \\ & & & & \overline{\lambda} & 1 \\ & & & & & \overline{\lambda} \end{bmatrix} X + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \overline{b}_1 \\ \overline{b}_2 \\ \overline{b}_3 \end{bmatrix} U$$

$$y = \begin{bmatrix} c_1 & c_2 & c_3 & \overline{c}_1 & \overline{c}_2 & \overline{c}_3 \end{bmatrix} X$$

can be transformed into

$$\dot{\overline{X}} = \begin{bmatrix} \overline{A} & I_2 & 0 \\ 0 & \overline{A} & I_2 \\ 0 & 0 & \overline{A} \end{bmatrix} \overline{X} + \begin{bmatrix} \overline{B} \\ \overline{B} \\ \overline{B} \end{bmatrix} u \qquad y = \begin{bmatrix} \overline{C}_1 & \overline{C}_2 & \overline{C}_3 \end{bmatrix} \overline{X}$$

where $\overline{A}, \overline{B}, and\ \overline{C}_i$ are defined in Problem 4.6 and I_2 is the unit matrix of order 2.

试验证变换成立，其中 $\overline{A}, \overline{B}, and\ \overline{C}_i$ 是习题 4.6 中定义的形式。I_2 为二阶单位阵。

PROOF: Change the order of the state variables from [x1 x2 x3 x4 x5 x6]’ to [x1 x4 x2 x5 x3 x6]’
And then we get

$$\dot{\overline{X}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_4 \\ \dot{x}_2 \\ \dot{x}_5 \\ \dot{x}_3 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} \lambda & & 1 & & & \\ & \overline{\lambda} & & 1 & & \\ & & \lambda & & 1 & \\ & & & \overline{\lambda} & & 1 \\ & & & & \lambda & \\ & & & & & \overline{\lambda} \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \\ x_2 \\ x_5 \\ x_3 \\ x_6 \end{bmatrix} + \begin{bmatrix} b_1 \\ \overline{b}_1 \\ b_2 \\ \overline{b}_2 \\ b_3 \\ \overline{b}_3 \end{bmatrix} = \begin{bmatrix} \overline{A} & I_2 & 0 \\ 0 & \overline{A} & I_2 \\ 0 & 0 & \overline{A} \end{bmatrix} \overline{X} + \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \\ \overline{B}_3 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & \overline{c}_1 & c_2 & \overline{c}_2 & c_3 & \overline{c}_3 \end{bmatrix} \overline{X} = \begin{bmatrix} \overline{C}_1 & \overline{C}_2 & \overline{C}_3 \end{bmatrix} \overline{X}$$

4.8 Are the two sets of state equations

$$\dot{X} = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} U \qquad y = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} X$$

and

$$\dot{X} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} U \qquad y = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} X$$

equivalent?Are they zero-state equivalent?上面两组状态方程等价吗？它们的零状态等价吗？

Answer:

$$\hat{G}_1(s) = C(sI - A)^{-1} B = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} s-2 & -1 & -2 \\ 0 & s-2 & -2 \\ 0 & 0 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}}{(s-2)^2 (s-1)} \begin{bmatrix} (s-2)(s-1) & (s-1) & 2(s-1) \\ 0 & (s-2)(s-1) & 2(s-2) \\ 0 & 0 & (s-2)^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{(s-2)^2}$$

$$\hat{G}_2(s) = C(sI - A)^{-1} B = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} s-2 & -1 & -1 \\ 0 & s-2 & -1 \\ 0 & 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}}{(s-2)^2 (s+1)} \begin{bmatrix} (s-2)(s+1) & (s+1) & (s-1) \\ 0 & (s-2)(s+1) & (s-2) \\ 0 & 0 & (s-2)^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{(s-2)^2}$$

obviously, $\hat{G}_1(s) = \hat{G}_2(s)$,so they are zero-state equivalent but not equivalent.

4.9 Verify that the transfer matrix in (4.33)has the following realization:

$$\dot{X}=\begin{bmatrix} -\alpha_1 I_q & I_q & 0 & \cdots & 0 \\ -\alpha_2 I_q & 0 & I_q & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -\alpha_{r-1} I_q & 0 & 0 & \cdots & I_q \\ -\alpha_r I_q & 0 & 0 & \cdots & 0 \end{bmatrix} X+\begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_{r-1} \\ N_r \end{bmatrix} U$$

$$Y=\begin{bmatrix} I_q & 0 & 0 & \cdots & 0 \end{bmatrix} X$$

This is called the observable canonical form realization and has dimension rq.It is dual to (4.34).
验证式(4.33)具有如上的实现。这叫做能观标准型实现，维数为 rq。它与式(4.34)对偶。
Answer:

$$\begin{aligned}
&define \\
&C(sI-A)^{-1}=[Z_1 \quad Z_2 \quad \cdots \quad Z_r] \\
&then \\
&C=[Z_1 \quad Z_2 \quad \cdots \quad Z_r](sI-A) \\
&we\ get \\
&sZ_i=AZ_i, i=2\cdots r \Rightarrow Z_i=\frac{Z_{i-1}}{s}, i=2\cdots r \\
&sZ_1=I_q-\sum_{i=1}^{r} Z_i\alpha_i = I_q-\sum_{i=1}^{r}\frac{Z_1\alpha_i}{s^{i-1}} \quad then \\
&Z_1=\frac{s^{r-1}}{d(s)} I_q, Z_2=\frac{s^{r-2}}{d(s)} I_q, \cdots, Z_r=\frac{1}{d(s)} I_q \\
&then \\
&C(sI-A)^{-1}B=\frac{1}{d(s)}(s^{r-1}N_1+\cdots+N_r)
\end{aligned}$$

this satisfies (4.33).

4.10 Consider the 1×2 proper rational matrix 考虑如下有理正则矩阵

$$\begin{aligned}
\hat{G}(s)=&\begin{bmatrix} d_1 & d_2 \end{bmatrix}+\frac{1}{s^4+\alpha_1 s^3+\alpha_2 s^2+\alpha_3 s+\alpha_4} \\
&\times\begin{bmatrix} \beta_{11}s^3+\beta_{21}s^2+\beta_{31}s+\beta_{41} & \beta_{12}s^3+\beta_{22}s^2+\beta_{32}s+\beta_{42} \end{bmatrix}
\end{aligned}$$

Show that its observable canonical form realization can be reduced from Problem 4.9 as
试证习题 4.9 种系统的能观标准形实现能降低维数如下表示：

$$\dot{X}=\begin{bmatrix} -\alpha_1 & 1 & 0 & 0 \\ -\alpha_2 & 0 & 1 & 0 \\ -\alpha_3 & 0 & 0 & 1 \\ -\alpha_4 & 0 & 0 & 0 \end{bmatrix} X+\begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \\ \beta_{41} & \beta_{42} \end{bmatrix}$$

$$y=\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} X+\begin{bmatrix} d_1 & d_2 \end{bmatrix} U$$

Answer: In this case,r=4,q=1,so

$$I_q=1, N_1=[\beta_{11} \quad \beta_{12}], N_2=[\beta_{21} \quad \beta_{22}], N_3=[\beta_{31} \quad \beta_{32}], N_4=[\beta_{41} \quad \beta_{42}]$$

its observable canonical form realization can be reduced from 4.9 to 4.10

4.11 Find a realization for the proper rational matrix 求下面正则有理传递函数矩阵的一种实现。

$$\hat{G}(s)=\begin{bmatrix}\dfrac{2}{s+1} & \dfrac{2s-3}{(s+1)(s+2)}\\ \dfrac{s-2}{s+1} & \dfrac{s}{s+2}\end{bmatrix}=\begin{bmatrix}\dfrac{2}{s+1} & \dfrac{2s-3}{(s+1)(s+2)}\\ \dfrac{-3}{s+1} & \dfrac{-2}{s+2}\end{bmatrix}+\begin{bmatrix}0&0\\1&1\end{bmatrix}$$

$$=\frac{1}{s^2+3s+2}\left[s\begin{bmatrix}2&2\\-3&-2\end{bmatrix}+\begin{bmatrix}4&-3\\-6&-2\end{bmatrix}\right]+\begin{bmatrix}0&0\\1&1\end{bmatrix}$$

so the realization is

$$\dot{X}=\begin{bmatrix}-3&0&-2&0\\0&-3&0&-2\\1&0&0&0\\0&1&0&0\end{bmatrix}X+\begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix}U$$

$$Y=\begin{bmatrix}2&2&4&-3\\-3&-2&-6&-2\end{bmatrix}X+\begin{bmatrix}0&0\\1&1\end{bmatrix}U$$

4.12 Find a realization for each column of $\hat{G}(s)$ in Problem 4.11,and then connect them.as shown in fig4.4(a).to obtain a realization of $\hat{G}(s)$.What is the dimension of this realization ?Compare this dimension with the one in Problem 4.11.求习题 4.11 中 $\hat{G}(s)$ 每一列的实现，在如图 4.4(a)所示把它们连结得到 $\hat{G}(s)$ 的一种实现。比较其与习题 4.11 的维数。

Answer:

$$\hat{G}_1(s)=\frac{1}{s+1}\begin{bmatrix}2\\s-2\end{bmatrix}=\frac{1}{s+1}\begin{bmatrix}2\\-3\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}$$

$$\dot{X}_1=-X_1+U_1$$

$$Y_{C1}=\begin{bmatrix}2\\-3\end{bmatrix}X_1+\begin{bmatrix}0\\1\end{bmatrix}U_1$$

$$\hat{G}_2(s)=\frac{1}{(s+1)(s+2)}\begin{bmatrix}2s-3\\-2s-2\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}=\frac{1}{(s+1)(s+2)}\left[s\begin{bmatrix}2\\-2\end{bmatrix}+\begin{bmatrix}-3\\-2\end{bmatrix}\right]+\begin{bmatrix}0\\1\end{bmatrix}$$

$$\dot{X}_2=\begin{bmatrix}-3&-2\\1&0\end{bmatrix}X_2+\begin{bmatrix}1\\0\end{bmatrix}U_2$$

$$Y_{C2}=\begin{bmatrix}2&-3\\-2&-2\end{bmatrix}X_2+\begin{bmatrix}0\\1\end{bmatrix}U_2$$

These two realizations can be combined as

$$\dot{X} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & -2 \\ 0 & 1 & 0 \end{bmatrix} X + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 2 & 2 & -3 \\ -3 & -2 & -2 \end{bmatrix} X + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} U$$

the dimension of the realization of 4.12 is 3,and of 4.11 is 4.

4.13 Find a realization for each row of $\hat{G}(s)$ in Problem 4.11 and then connect them,as shown in Fig.4.4(b),to obtain a realization of $\hat{G}(s)$.What is the dimension of this realization of this realization?Compare this dimension with the ones in Problems 4.11 and 4.12. 求习题 4.11 中 $\hat{G}(s)$ 每一行的实现，在如图 4.4(b)所示把它们连结得到 $\hat{G}(s)$ 的一种实现。比较其与习题 4.11、4.12 的维数。

Answer:

$$\hat{G}_1(s) = \frac{1}{(s+1)(s+2)}\begin{bmatrix} 2s+4 & 2s-3 \end{bmatrix} = \frac{1}{s^2+3s+2}\left[s\begin{bmatrix} 2 & 2 \end{bmatrix} + \begin{bmatrix} 4 & -3 \end{bmatrix} \right]$$

$$\dot{X}_1 = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} X_1 + \begin{bmatrix} 2 & 2 \\ 4 & -3 \end{bmatrix} U_1$$

$$Y_{C1} = \begin{bmatrix} 1 & 0 \end{bmatrix} X_1 + \begin{bmatrix} 0 & 0 \end{bmatrix} U_1$$

$$\hat{G}_2(s) = \frac{1}{(s+1)(s+2)}\begin{bmatrix} -3(s+2) & -2(s+1) \end{bmatrix} + \begin{bmatrix} 1 & 1 \end{bmatrix} = \frac{1}{(s+1)(s+2)}\left[s\begin{bmatrix} -3 & -2 \end{bmatrix} + \begin{bmatrix} -6 & -2 \end{bmatrix} \right] + \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$\dot{X}_2 = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} X_2 + \begin{bmatrix} -3 & -2 \\ -6 & -2 \end{bmatrix} U_2$$

$$Y_{C2} = \begin{bmatrix} 1 & 0 \end{bmatrix} X_2 + \begin{bmatrix} 1 & 1 \end{bmatrix} U_2$$

These two realizations can be combined as

$$\dot{X} = \begin{bmatrix} -3 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & -2 & 0 \end{bmatrix} X + \begin{bmatrix} 2 & 2 \\ 4 & -3 \\ -3 & -2 \\ -6 & -2 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} X + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} U$$

the dimension of this realization is 4,equal to that of Problem 4.11.so the smallest dimension is of Problem 4.12.

4.14 Find a realization for $\hat{G}(s) = \begin{bmatrix} \frac{-(12s+6)}{3s+34} & \frac{22s+23}{3s+34} \end{bmatrix}$

求 $\hat{G}(s)=\begin{bmatrix}\dfrac{-(12s+6)}{3s+34} & \dfrac{22s+23}{3s+34}\end{bmatrix}$ 的一种实现

Answer:

$$\hat{G}(s)=\begin{bmatrix}\dfrac{-(12s+6)}{3s+34} & \dfrac{22s+23}{3s+34}\end{bmatrix}=\begin{bmatrix}-4 & 22/3\end{bmatrix}+\frac{1}{s+34/3}\begin{bmatrix}130/3 & -679/9\end{bmatrix}$$

$$\dot{X}=-\frac{34}{3}X+\begin{bmatrix}130/3 & -679/9\end{bmatrix}U$$

$$Y=X+\begin{bmatrix}-4 & 22/3\end{bmatrix}U$$

4.16 Find fundamental matrices and state transition matrices for 求下列状态方程的基本矩阵和状态转移矩阵：

$$\dot{X}=\begin{bmatrix}0 & 1\\ 0 & t\end{bmatrix}X \qquad and \qquad \dot{X}=\begin{bmatrix}-1 & e^{2t}\\ 0 & -1\end{bmatrix}X$$

Answer:

for the first case:
$$\dot{x}_1=x_2 \Rightarrow x_1(t)=\int_0^t x_2(t)dt+x_1(0)$$
$$\dot{x}_2=tx_2 \Rightarrow d\ln x_2(t)=tdt \Rightarrow x_2(t)=x_2(0)e^{0.5t^2}$$

we have $X(0)=\begin{bmatrix}1\\0\end{bmatrix} \Rightarrow X(t)=\begin{bmatrix}1\\0\end{bmatrix}$ $\quad and \quad$ $X(0)=\begin{bmatrix}0\\1\end{bmatrix} \Rightarrow X(t)=\begin{bmatrix}\int_0^t e^{0.5\tau^2 d\tau}\\ e^{0.5t^2}\end{bmatrix}$

The two initial states are linearly independent.thus

$$X(t)=\begin{bmatrix}1 & \int_0^t e^{0.5\tau^2 d\tau}\\ 0 & e^{0.5t^2}\end{bmatrix} \qquad and$$

$$\Phi(t,t_0)=\begin{bmatrix}1 & \int_0^t e^{0.5\tau^2 d\tau}\\ 0 & e^{0.5t^2}\end{bmatrix}\begin{bmatrix}1 & \int_0^{t_0} e^{0.5\tau^2 d\tau}\\ 0 & e^{0.5t_0^2}\end{bmatrix}^{-1}=\begin{bmatrix}1 & e^{-0.5t_0^2 d\tau}\int_{t_0}^t e^{0.5\tau^2 d\tau}\\ 0 & e^{0.5(t^2-t_0^2)}\end{bmatrix}$$

for the second case:

$$\dot{x}_1=-x_1+e^{2t}x_2(t)=-x_1+e^{t}x_2(0)\Rightarrow$$
$$x_1(t)=e^{-\int_0^t dt}[\int_0^t x_2(0)e^{t}e^{\int dt}dt+c]=0.5x_2(0)(e^{t}-e^{-t})+x_1(0)e^{-t}$$
$$\dot{x}_2=-x_2 \Rightarrow x_2(t)=x_2(0)e^{-t}$$

we have

$$X(0)=\begin{bmatrix}1\\0\end{bmatrix} \Rightarrow X(t)=\begin{bmatrix}e^{-t}\\0\end{bmatrix} \qquad and \qquad X(0)=\begin{bmatrix}0\\1\end{bmatrix} \Rightarrow X(t)=\begin{bmatrix}0.5(e^{t}-e^{-t})\\ e^{-t}\end{bmatrix}$$

the two initial states are linearly independent;thus

$$X(t)=\begin{bmatrix} e^{-t} & 0.5(e^{t}-e^{-t}) \\ 0 & e^{-t} \end{bmatrix} \qquad and$$

$$\Phi(t,t_0)=\begin{bmatrix} e^{-t} & 0.5(e^{t}-e^{-t}) \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} e^{-t_0} & 0.5(e^{t_0}-e^{-t_0}) \\ 0 & e^{-t_0} \end{bmatrix}^{-1}$$

$$=\begin{bmatrix} e^{t_0-t} & 0.5e^{-t}(e^{t_0}-e^{-3t_0})+0.5(e^{t}-e^{-t})e^{t_0} \\ 0 & e^{t_0-t} \end{bmatrix}$$

4.17 Show $\partial\Phi(t_0,t)/\partial t=-\Phi(t_0,t)A(t).$

试证 $\partial\Phi(t_0,t)/\partial t=-\Phi(t_0,t)A(t).$

Proof: first we know

$$\frac{\partial\Phi(t,t_0)}{\partial t}=A(t)\Phi(t,t_0)$$

then

$$\frac{\partial\Phi(t,t)}{\partial t}=\frac{\partial(\Phi(t,t_0)\Phi(t_0,t))}{\partial t}=\frac{\partial\Phi(t,t_0)}{\partial t}\Phi(t_0,t)+\Phi(t,t_0)\frac{\partial\Phi(t_0,t)}{\partial t}$$

$$=A(t)\Phi(t,t_0)\Phi(t_0,t)+\Phi^{-1}(t_0,t)\frac{\partial\Phi(t_0,t)}{\partial t}=A(t)+\Phi^{-1}(t_0,t)\frac{\partial\Phi(t_0,t)}{\partial t}$$

$$=0$$

$$\Rightarrow \partial\Phi(t_0,t)/\partial t=-\Phi(t_0,t)A(t)$$

4.18 Given $A(t)=\begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}$,show $\det\Phi(t,t_0)=\exp\left[\int_{t_0}^{t}(a_{11}(\tau)+a_{22}(\tau))d\tau\right]$

已知 A(t),试证 $\det\Phi(t,t_0)=\exp\left[\int_{t_0}^{t}(a_{11}(\tau)+a_{22}(\tau))d\tau\right]$

Proof:

$$\begin{bmatrix} \dot\phi_{11} & \dot\phi_{12} \\ \dot\phi_{21} & \dot\phi_{22} \end{bmatrix}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

so

$$\frac{\partial}{\partial t}(\det\Phi)=\frac{\partial}{\partial t}(\phi_{11}\phi_{22}-\phi_{21}\phi_{12})=\dot\phi_{11}\phi_{22}-\dot\phi_{21}\phi_{12}+\phi_{11}\dot\phi_{22}-\phi_{21}\dot\phi_{12}$$

$$=(a_{11}\phi_{11}+a_{12}\phi_{21})\phi_{22}-(a_{11}\phi_{12}+a_{12}\phi_{22})\phi_{21}+\phi_{11}(a_{21}\phi_{12}+a_{22}\phi_{22})-\phi_{12}(a_{21}\phi_{11}+a_{22}\phi_{21})$$

$$=(a_{11}+a_{22})(\phi_{11}\phi_{22}-\phi_{21}\phi_{12})=(a_{11}+a_{22})\det\Phi$$

so

$$\det\Phi(t,t_0)=ce^{\int_{t_0}^{t}(a_{11}(\tau)+a_{22}(\tau))d\tau} \qquad with \quad \Phi(t_0,t_0)=I$$

we get

$$c=1$$

then $\det\Phi(t,t_0)=e^{\int_{t_0}^{t}(a_{11}(\tau)+a_{22}(\tau))d\tau}$

4.20 Find the state transition matrix of 求状态转移矩阵

$$\dot{x} = \begin{bmatrix} -\sin t & 0 \\ 0 & -\cos t \end{bmatrix} x$$

Answer:

$$\begin{aligned} \dot{x}_1 &= -\sin t \cdot x_1 \Rightarrow x_1(t) = x_1(0)e^{\cos t - 1} \\ \dot{x}_2 &= -\cos t x_2 \Rightarrow x_2(t) = x_2(0)e^{-\sin t} \end{aligned}$$

then we have

$$X(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow X(t) = \begin{bmatrix} e^{-1+\cos t} \\ 0 \end{bmatrix} \quad \text{and} \quad X(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow X(t) = \begin{bmatrix} 0 \\ e^{-\sin t} \end{bmatrix}$$

the two initial states are linearly independent;thus

$$X(t) = \begin{bmatrix} e^{-1+\cos t} & 0 \\ 0 & e^{-\sin t} \end{bmatrix} \quad \text{and}$$

$$\Phi(t,t_0) = \begin{bmatrix} e^{-1+\cos t} & 0 \\ 0 & e^{-\sin t} \end{bmatrix} \begin{bmatrix} e^{-1+\cos t_0} & 0 \\ 0 & e^{-\sin t_0} \end{bmatrix}^{-1} = \begin{bmatrix} e^{\cos t - \cos t_0} & 0 \\ 0 & e^{-\sin t + \sin t_0} \end{bmatrix}$$

4.21 Verify that $X(t) = e^{At}Ce^{Bt}$ is the solution of 试验证 $X(t) = e^{At}Ce^{Bt}$ 为下面方程的解：

$$\dot{X} = AX + XB \qquad x(0) = C$$

Verify:first we know $\frac{\partial}{\partial t}e^{At} = Ae^{At} = e^{At}A$,then

$$\begin{aligned} &\dot{X} = \frac{\partial}{\partial t}(e^{At}Ce^{Bt}) = (Ae^{At})Ce^{Bt} + e^{At}C(e^{Bt}B) = AX + XB \\ &X(0) = e^0 Ce^0 = C \\ &Q.E.D \end{aligned}$$

4.23 Find an equivalent time-invariant state equation of the equation in Problem 4.20.
求习题 4.20 中方程的等价时不变状态方程。

Answer: let $P(t) = X^{-1}(t) = \begin{bmatrix} e^{1-\cos t} & 0 \\ 0 & e^{\sin t} \end{bmatrix}$ and $\bar{x}(t) = P(t)x(t)$

Then $\bar{A}(t) = 0 \quad \dot{\bar{x}}(t) = 0$

4.24 Transform a time-invariant (A,B,C) into $(0,\bar{B}(t),\bar{C}(t))$ is by a time-varying equation transformation.求把(A,B,C)转换成$(0,\bar{B}(t),\bar{C}(t))$的时变状态转移方程。

Answer: since (A,B,C)is time-invariant,then

$$X(t) = e^{At}$$

$$P(t) = X^{-1}(t)$$
$$\overline{B}(t) = X^{-1}(t)B = e^{-At}B$$
$$\overline{C}(t) = CX(t) = Ce^{At}$$
$$\Phi(t,t_0) = X(t)X^{-1}(t_0) = e^{A(t-t_0)}$$

4.25 Find a time-varying realization and a time-invariant realization of the inpulse response $g(t) = t^2 e^{\lambda t}$.求冲击响应的时变实现及时不变实现。

Answer:

$$g(t,\tau) = g(t-\tau) = (t-\tau)^2 e^{\lambda(t-\tau)} = (t^2 - 2t\tau + \tau^2)e^{\lambda t - \lambda\tau}$$
$$= \begin{bmatrix} t^2 e^{\lambda t} & -2te^{\lambda t} & e^{\lambda t} \end{bmatrix} \begin{bmatrix} e^{-\lambda\tau} \\ \tau e^{-\lambda\tau} \\ \tau^2 e^{-\lambda\tau} \end{bmatrix}$$

the time-varying realization is

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} e^{-\lambda t} \\ te^{-\lambda t} \\ t^2 e^{-\lambda t} \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} t^2 e^{\lambda t} & -2te^{\lambda t} & e^{\lambda t} \end{bmatrix} x(t)$$

$$\hat{g}(s) = L[t^2 e^{\lambda t}] = \frac{2}{s^3 - 3\lambda s^2 + 3\lambda s - \lambda^3}$$
$$u\sin g \quad (4.41), we\ get\ the\ time-inviant\ realization:$$
$$\dot{x} = \begin{bmatrix} 3\lambda & -3\lambda^2 & \lambda^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$
$$y = \begin{bmatrix} 0 & 0 & 2 \end{bmatrix} x$$

4.26 Find a realization of $g(t,\tau) = \sin t(e^{-(t-\tau)})\cos\tau$. Is it possible to find a time-invariant state equation realization?求 $g(t,\tau) = \sin t(e^{-(t-\tau)})\cos\tau$ 的一种实现，问能否找到一种时不变状态方程实现。

Answer: clearly we can get the time-varying realization of

$g(t,\tau) = \sin t(e^{-(t-\tau)})\cos\tau = (\sin te^{-t})(e^{\tau}\cos\tau) = M(t)N(\tau)$ using Theorem 4.5

we get the realization.

$$\dot{x}(t) = N(t)u(t) = e^t \cos tu(t)$$
$$y(t) = \sin te^{-t} x(t)$$

but we can't get g(t) from it,because $g(t,\tau) \neq g(t-\tau)$,so it's impossible to find a time-invariant state eqution realization.

5.1

Is the network shown in Fing5.2 BIBO stable ? If not , find a bounded input that will excite an unbound output

图 5.2 的网络 BIBO 是否稳定?如果不是.请举出 一般激励无界输出的有界输入.

From Fig5.2 .we can obtain that

$x = u - y$

$y = x$

Assuming zero initial and applying Laplace transform then we have

$$\hat{y}(s) = \frac{\hat{\hat{y}}(s)}{\hat{u}(s)} = \frac{s}{s^2+1}$$

$$g(t) = L^{-1}[g(s)] = L^{-1}[\frac{s}{s^2+1}] = \cos t$$

because

$$\int_0^\infty |g(t)|\,dt = \int_0^\infty |\cos t|\,dt = \int_0^{\frac{x}{2}} \cos t dt + \sum_{k=0}^{\infty} \int_{\frac{2k+1}{2}\pi}^{\frac{2k+3}{2}\pi} (-1)^{k+1} \cos t dt$$

$$=1+\sum_{k=0}^{\infty}(-1)^{k+1}[\sin(2\pi+\tfrac{3}{2}\pi)-\sin(k\pi+\tfrac{1}{2}\pi)$$

$$=1+\sum_{k=0}^{\infty}(-1)^{k+1}(-1)^{k}[\sin\tfrac{3}{2}\pi-\sin\tfrac{1}{2}\pi]$$

$$=1+\sum_{k=0}^{\infty}(-1)^{2k+1}(-1)^{2k+1}\cdot(-2)$$

$$=1+2\sum_{k=0}^{\infty}1=\infty$$

Which is not bounded .thus the network shown in Fin5.2 is not BIBO stable ,If u(t)=sint ,then we have

y(t)= $\int_0^t \cos(t-\tau)\sin\tau d\tau = \int_0^t \cos\tau \sin(t-\tau)\ d\tau = \frac{1}{2}\int_0^t \sin t d\tau = \frac{1}{2} t \sin t$

we can see u(t)is bounded ,and the output excited by this input is not bounded

5.2

consider a system with an irrational function $\hat{y}(s)$ .show that a necessary condition for the system to be BIBO stable is that |g(s)| is finite for all Res≥ 0

一个系统的传递函数是 S 的非有理式。证明该系统 BIBO 稳定的一个必要条件是对所有 Res≥0. |g(s)|有界。

Proof let s=$\lambda + j\zeta$, then

$$\hat{g}(s) = \hat{g}(\lambda + j\zeta) = \int_0^\infty g(t)e^{-\lambda t}e^{-j\zeta t}dt = \int_0^\infty g(t)e^{-\lambda t}\cos t dt - j\int_0^\infty g(t)e^{-\lambda t}\sin\zeta t dt$$

If the system is BIBO stable ,we have

$\int_0^\infty |g(t)|\,dt \le M < +\infty$. for some constant M

which implies shat $\int_0^\infty g(t)dt$ is convergent .

For all Res=$\lambda \ge 0$, We have

|g(t) $e^{-\lambda t}\cos\zeta t \mid \le \mid g(t) \mid$

|g(t0 $e^{-\lambda t}\sin\zeta t \mid \le \mid g(t) \mid$

so $\int_0^\infty g(t)e^{-\lambda t}\cos tdt$ and $\int_0^\infty g(t)e^{-\lambda t}\sin\zeta tdt$ are also convergent , that is .

$$\int_0^\infty g(t)e^{-\lambda t}\cos\zeta t = N < +\infty$$

$$\int_0^\infty g(t)e^{-\lambda t}\sin\zeta tdt = L < +\infty$$

where N and L are finite constant. Then

$|\hat{g}(s)| = (N^2 + (-L)^2)^{\frac{1}{2}} = (N^2 + L^2)^{\frac{1}{2}}$ is finite ,for all **Res** ≥ 0.

Another Proof, If the system is BIBO stable ,we have $\int_0^\infty |g(t)|dt \le M < +\infty$. for some constant M .And

.

For all Res ≥ 0. We obtain $|\hat{g}(s)| = \int_0^\infty |g(t)e^{-(\mathrm{Re}\,s)t}dt \le \int_0^\infty |g(t0|dt \le M < +\infty$.

That is . $|\hat{g}(s)|$ is finite for all Res≥ 0

5.3

Is a system with impulse g(t)=$\frac{1}{t+1}$ BIBO stable ? how about g(t)=te^{-t} for **t**≥ 0?

脉冲响应为个 g(t)=$\frac{1}{t+1}$ 的系统是否 BIBO 稳定? $g(t) = te^{-t}, t \ge 0$的系统又如何?

Becaus $\int_0^\infty |\frac{1}{t+1}|dt = \int_0^\infty \frac{1}{t+1}dt = \ln(t+1) = \infty$

$$\int_0^\infty |te^{-t}|dt = \int_0^\infty te^{-t}dt = (-te^{-t} - e^{-t})|_0^\infty = 1 < +\infty$$

the system with impulse response g(t)=$\frac{1}{t+1}$ is not BIBO stable ,while the system with impulse response g(t)=te^{-t} for $t \ge 0$ is BIBO stable.

5．4 Is a system with transfer function $\hat{g}(s) = \frac{e^{-2s}}{(s+1)}$ BIBO stable

一个系统的传递函数为 $\hat{g}(s) = \frac{e^{-2s}}{(s+1)}$，问该系统是否 BIBO 稳定。

Laplace transform has the property .

If $g(t) = f(t-a)$. then $\hat{g}(s) = e^{-\alpha s}\hat{f}(s)$

We know $L^{-1}[\frac{1}{s+1}] = e^{-t}, t \ge 0$. Thus $L^{-1}[e^{-\alpha s}\frac{1}{s+1}] = e^{-(t-2)}, t \ge 2$

So the impulse response of the system is $g(t) = e^{-(t-2)}$. for $t \ge 2$

$$\int_0^\infty |g(t)|dt == \int_2^\infty e^{-(t-2)}dt = 1 < +\infty$$

the system is BIBO stable

5.5 Show that negative-feedback shown in Fig . 2.5(b) is BIBO stable if and only if and the gain a has a magnitude less than 1.For a=1 ,find a bounded input $r(t)$ that will excite an unbounded output.

证明图 2.5(b) 中的负反馈系统 BIBO 稳定当且增到 a 的模小于 1。对于 a=1 情况，找出一有界输入 r(t), 她所产生的输出是无界的。

Poof ,If r(t)= $\delta(t)$ .then the output is the impulse response of the system and equal

$$g(t) = a\delta(t-1) - a^3\delta(r-3) - a^4\delta(t-4) + \ldots = \sum_{t=1}^{\infty}(-1)^{i+1}a^i\delta(t-i)$$

the impulse is definded as the limit of the pulse in Fig.3.2 and can be considered to be positive .thus we have

$$|g(t)| = \sum_{t=1}^{\infty}|a|^i\ \delta(t-i)$$

and $\int_0^\infty |g(t)|\,dt = \sum_{t=1}^{\infty}|a|^i \int_0^\infty \delta(t-i)dt = \sum_{t=1}^{\infty}|a|^i = \begin{cases} \infty & if \quad |a| \geq 0 \\ \frac{|a|}{(1-|a|)} < +\infty & if \quad |a| < 1 \end{cases}$

which implies that the negative-feedback system shown in Fig.2.5(b) is BIBO stable if and only if the gain a has magitude less than 1.

For $a=1$ .if we choose $r(t)=\sin(t\pi)$. clearly it is bounded the output excited by $r(t)=\sin(t\pi)$ is

$$y(t) = \int_0^t g(t-\tau)r(\tau)d\tau = \sum_{i+1}^{\infty}(-1)^{i+1}\sin(t\pi - i\pi) = \sum_{i=1}^{\infty}(-1)^{i+1}\sin(t\pi - i\pi) = \sum_{i=1}^{\infty}(-1)^{i+1}\sin(t\pi - i\pi) = \sum_{i=1}^{\infty}(-1)^{i+1}(-1)^i\sin(t\pi)$$

$$= -\sin(t\pi)\sum_{i=1}^{\infty}1$$

And y(t) is not bounded

5.6 consider a system with transfer function $\hat{g}(s) = \frac{(s-2)}{(s+1)}$。 what are the steady-state responses by u (t)=3.for $t \geq 0$ .and by u (t)=sin2t. for $t \geq 0$?

一个系统的传递函数是 $\hat{g}(s) = \frac{(s-2)}{(s+1)}$，分别由 $u(t)=3.\ t \geq 0$ 和 $u(t)=\sin 2t.\, t \geq 0$ 所产生的稳态响应是什么？

The impulse response of the system is $g(t) = L^{-1}[\frac{s-2}{s+1}] = L^{-1}[`-\frac{3}{s+1}] = \delta(t) - 3e^{-t}.\quad t \geq 0$。

And we have

$$\int_0^\infty |g(t)|\,dt = \int_0^\infty |\delta(t) - 3e^{-t}|\,dt \leq \int_0^\infty (|\delta(t)| + |3e^{-t}|)dt = \int_0^\infty \delta(t)dt + \int_0^\infty 3e^{-t}dt = 1+3 = 4 < +\infty$$

so the system is BIBO stable

using Theorem 5.2 ,we can readily obtain the steady-state responses

if u (t)=3for $t \geq 0$.,then as $t \to \infty, y(t)$ approaches $\hat{g}(0)\cdot 3 = -2\cdot 3 = -6$

if u (t)=sin2t for $t \geq 0$, then ,as $t \to \infty.\ y(t)$ approaches

$$|\hat{g}(j\tau)|\sin(\tau t + \angle\hat{g}(j\tau)) = \frac{2\sqrt{10}}{5}\sin(\tau t + \arctan 3) = 1.26\sin(\tau t + 1.25)$$

5.7 consider $\dot{x}=\begin{bmatrix}-1 & 10\\ 0 & 1\end{bmatrix}x+\begin{bmatrix}-2\\ 0\end{bmatrix}u$

y=[-2 3] $x-2u$ is it BIBO stable ?

问上述状态方程描述的系统是否 BIBO 稳定？

The transfer function of the system is

$$\hat{g}(s)=\begin{bmatrix}-2 & 3\end{bmatrix}\begin{bmatrix}s+1 & -10\\ 0 & \leq -1\end{bmatrix}^{-1}\begin{bmatrix}-2\\ 0\end{bmatrix}-2=\begin{bmatrix}-2 & 3\end{bmatrix}\begin{bmatrix}\frac{1}{s+1} & \frac{10}{(s+1)(s-1)}\\ 0 & \frac{1}{s-1}\end{bmatrix}\begin{bmatrix}-2\\ 0\end{bmatrix}-2$$

$$=\frac{4}{s+1}-2=\frac{-2s+2}{s+1}$$

The pole of is -1.which lies inside the left-half s-plane, so the system is BIBO stable

5.8 consider a discrete-time system with impulse response

$g[k]=k(0.8)^k$ for $k\geq 0,$ is the system BIBO stable ?

一个离散时间系统的脉冲响应序列是 $g[k]=k(0.8)^k$ ， $k\geq 0,$ 。问该系统是否 BIBO 稳定？

Because

$$\sum_{k=0}^{\infty}|g[k]|=\sum_{k=0}^{\infty}k(0.8)^k=\frac{1}{0.2}(1-0.8)\cdot\sum_{k=0}^{\infty}k(0.8)^k =$$

$$\frac{1}{0.2}[\sum_{k=0}^{\infty}k\cdot(0.8)^k-\sum_{k=0}^{\infty}k\cdot(0.8)^{k+1}]=\frac{1}{0.2}\sum_{k=0}^{\infty}0.8^k=\frac{1}{0.2}\cdot\frac{0.8}{1-0.8}=20<+\infty$$

G[k]is absolutely sumable in $[0,\infty]$. The discrete-time system is BIBO stable .

5.9 Is the state equation in problem 5.7 marginally stable? Asymptotically stable？题 5.7 中的状态方程描
述的系统是否稳定？是否渐进稳定？

The characteristic polynomial

$$\Delta(\lambda)=\det(\lambda I-A)=\det\begin{bmatrix}\lambda+1 & -10\\ 0 & \lambda-1\end{bmatrix}=(\lambda+1)(\lambda-1)$$

the eigenralue Ihas positive real part ,thus the equation is not marginally stable neither is Asymptotically stable.

5.10 Is the homogeneous stable equation $\dot{x}=\begin{bmatrix}-1 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}x$ .marginally stable ? Asymptotically stable?

齐次状态方程 $\dot{x}=\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\underline{x}$，是否限界稳定？

The characteristic polynomial is $\Delta(\lambda)=\det(\lambda I-A)=\det\begin{bmatrix} \lambda+1 & 0 & -1 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}=\lambda^2(\lambda+1)$

And the minimal polynomial is $\psi(\lambda)=\lambda(\lambda+1)$.

The matrix has eigenvalues 0 , 0 . and –1 . the eigenvalue 0 is a simple root of the minimal .thus the equation is marginally stable

The system is not asymptotically stable because the matrix has zero eigenvalues

5 。11 Is the homogeneous stable equation $\dot{x}=\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\underline{x}$ marginally stable ? asymptotically stable ?

齐次状态方程 $\dot{x}=\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\underline{x}$ 是否限界稳定？是否渐进稳定？

The characteristic polynomial $\Delta(\lambda)=\det\begin{bmatrix} \lambda+1 & 0 & -1 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda \end{bmatrix}=\lambda^2(\lambda+1)$.

And the minimal polynomial is $\psi(\lambda)=\lambda^2(\lambda+1)$ ..the matrix has eigenvalues 0. 0. and –1 . the eigenvalue 0 is a repeated root of the minimal polymial $\Psi(\lambda)$. so the equation is not marginally stable ,neither is asymptotically stable .

5.12 Is the discrete-time homogeneous state equation

$$\underline{x}[k+1]=\begin{bmatrix} 0.9 & 0 & -1 \\ 0 & 1 & -0 \\ 0 & 0 & 1 \end{bmatrix}\underline{x[k]}$$

marginally stable ?Asymptotically stable ?

离散时间齐次状态方程 $\underline{x}[k+1]=\begin{bmatrix} 0.9 & 0 & -1 \\ 0 & 1 & -0 \\ 0 & 0 & 1 \end{bmatrix}\underline{x[k]}$ 是否限界稳定？是否渐进稳定？

The characteristic of the system matrix is $\Delta(\lambda)=(\lambda-1)^2(\lambda-0.9)$, .and the minimal polynomial is $\psi(\lambda)=(\lambda-1)(\lambda-0.9)$, the equation is not asymptotically stable because the matrix has eigenvalues 1. whose magnitudes equal to 1 .

the equation is marginally stable because the matrix has all its eigenvalues with magnitudes less than or to 1 and the eigenvalue 1 is simple root of the minimal polynomial . $\psi(\lambda)$.

5.13 Is the discrete-time homogeneous state equation $\underline{x}[k+1]=\begin{bmatrix}0.9 & 0 & -1\\ 0 & 1 & -0\\ 0 & 0 & 1\end{bmatrix}\underline{x[k]}$ marginally stable ? asymptotically stable ?

离散时间齐次状态方程 $\underline{x}[k+1]=\begin{bmatrix}0.9 & 0 & -1\\ 0 & 1 & -0\\ 0 & 0 & 1\end{bmatrix}\underline{x[k]}$ 是否稳定？是否渐进稳定？

Its characteristic polynomial is $\Delta(\lambda)=(\lambda-1)^2(\lambda-0.9)$, and its minimal polynomial is $\psi(\lambda)=(\lambda-1)^2(\lambda-0.9)$, the matrix has eigenvalues 1 .1 .and 0.9 . the eigenvalue 1 with magnitude equal to 1 is a repeated root of the minimal polynimial $\psi(\lambda)$. thus the equation is not marginally stable and is not asymptotically stable .

5.14 Use theorem 5.5 to show that all eigenvalues of $A=\begin{bmatrix}0 & 1\\ -0.5 & -1\end{bmatrix}$ have negative real parts .

用定理 5.5 证明 A 的特征值都具有负实部.

Poof :For any given definite symmetric matrix N where $N=\begin{bmatrix}a & b\\ b & c\end{bmatrix}$ with a>0 .and $ac-b^2>0$

The lyapunov equation $A'M+MA=-N$ can be written as

$$\begin{bmatrix}0 & -0.5\\ 1 & -1\end{bmatrix}\begin{bmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix}+\begin{bmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix}\begin{bmatrix}0 & 1\\ -0.5 & -1\end{bmatrix}=\begin{bmatrix}-a & -b\\ -b & -c\end{bmatrix}$$

that we have $\begin{bmatrix}-0.5(m_{12}+m_{21}) & m_{11}-m_{12}-0.5m_{22}\\ m_{11}-m_{21}-0.5m_{22} & m_{12}+m_{21}-2m_{22}\end{bmatrix}=\begin{bmatrix}-a & -b\\ -b & -c\end{bmatrix}$

thus $\because$ $\begin{aligned}m_{11}&=1.5a+0.25c-b\\ m_{12}&=m_{21}=a\\ m_{22}&=a+0.5c\end{aligned}$ $M=\begin{bmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix}$ is the unique solution of the lyapunor equation and M is symmetric and $\begin{matrix}a>0\\ ac-b^2>0\end{matrix}\Rightarrow\begin{matrix}a>0\\ c>0\end{matrix}$

$$\therefore (1.5a+0.25c)-ac=\frac{1}{16}[(6a+c)^2-16ac]=\frac{1}{16}(36a^2+c^2-4ac)=\frac{1}{16}[(2a-c)^2+32a^2]\geq 0$$

$$\left.\begin{matrix}(1.5a+0.25c)^2\geq ac>b^2\\ a>0,\quad c>0\end{matrix}\right\}\Rightarrow m_{11}=1.5a+0.25c-b>0$$

$$\det\begin{bmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix}=m_{11}m_{22}-m_{12}m_{21}=(1.5a+0.25c-b)(a+-.5c)-a^2$$

$$=\frac{1}{8}(4a^2+8ac+c^2-8ab-4bc)$$

$$=\frac{1}{8}[(2a-2b)^2+(c-2b)^2+8(ac-b^2)]>0$$

we see is positive definite

because for any given positive definite symmetric matrix N, The lyapunov equation $A'M+MA=-N$ has a unique symmetric solution M and M is positive definite , all eigenvalues of A have negative real parts ,

5.15 Use theorem 5.D5show that all eigencalues of the A in Problem 5.14 magnitudes less than 1.

Poof : For any given positive definite symmetric matrix N , we try to find the solution of discrete Lyapunov equation M

$M-A'MA=N$ Let $N=\begin{bmatrix} a & b \\ b & c \end{bmatrix}$. where a>0 and $ac-b^2>0$

and M is assumed as $M=\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$,then

$$M-A'MA=\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}-\begin{bmatrix} 0 & -0.5 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix}$$

$$=\begin{bmatrix} m_{11}-0.25m_{22} & m_{12}+0.5(m_{21}-m_{22}) \\ m_{21}+0.5(m_{12}-m_{22}) & -m_{11}+m_{12}+m_{21} \end{bmatrix}$$

$$=\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

this $m_{11}=\frac{8}{5}a+\frac{3}{5}c-\frac{4}{5}b,\quad m_{12}=m_{21}=\frac{4}{5}a+\frac{4}{5}c-\frac{2}{5}b,\quad m_{22}=\frac{12}{5}a+\frac{12}{5}c-\frac{16}{5}b$

$M=\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ is the unique symmetric solution of the discrete Lyapunov equation

$M-A'MA=N$, And we have $\left.\begin{matrix} (8a+3c)^2-16ac=64a^2+9c^2+32ac \\ \left.\begin{matrix} a>0 \\ ac-b^2 \end{matrix}\right) \Rightarrow \begin{matrix} a>0 \\ ca>0 \end{matrix} \end{matrix}\right\}$

$$\Rightarrow (8a+3c)^2>16ac>16b\Rightarrow 8a+3c-4b>0\Rightarrow m_{11}=\frac{8}{5}a+\frac{3}{5}c-\frac{4}{5}b>0$$

$$\det\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}=m_{11}m_{22}-m_{12}m_{21}=\frac{1}{25}[(8a+3(-4b)(12a+12c-16b)-(4a+4c-2b)^2]$$

$$=\frac{4}{5}(4a^2+c^2+3b^2+5ac-8ab-4bc)$$

$$=\frac{4}{5}[(2a-2b)^2+(c-2b)^2+5(ac-b^2)]>0$$

we see M is positive definite .use theorem 5.D5,we can conclude that all eigenvalues of A have magnitudes less than 1.

5.16 For any distinct negative real λ_i and any nonzero real a_i ,show that the matrix

$$M=\begin{bmatrix} -\frac{a_1{}^2}{2\lambda_1} & -\frac{a_1a_2}{\lambda_1+\lambda_2} & -\frac{a_1a_3}{\lambda_1+\lambda_3} \\ -\frac{a_2a_1}{\lambda_2+\lambda_1} & -\frac{a_2^2}{2\lambda_2} & -\frac{a_2a_3}{\lambda_2+\lambda_3} \\ -\frac{a_3a_1}{\lambda_3+\lambda_1} & -\frac{a_3a_2}{\lambda_3+\lambda_2} & -\frac{a_3^2}{2\lambda_3} \end{bmatrix}$$ is positive definite .

正明上定义的矩阵 M 是正宗的 （ λ_i 为相异的负实数， a_i 为非零实数）。

Poof : $\lambda_i, i=1,2,3.$ are distinct negatire real numbers and a_i are nonzero real numbers , so we have

$(1),-\frac{a_1^2}{2\lambda 1}>0$

(2) . $\det\begin{bmatrix} -\frac{a_1{}^2}{2\lambda_1} & -\frac{a_1a_2}{\lambda_1+\lambda_2} \\ -\frac{a_2a_1}{\lambda_2+\lambda_1} & -\frac{a_2^2}{2\lambda_2} \end{bmatrix}=\frac{a_1^2a_2^2}{4\lambda_1\lambda_2}-\frac{a_1^2a_2^2}{(\lambda_1+\lambda_2)^2}=a_1^2a_2^2(\frac{1}{4\lambda_1\lambda_2}-\frac{1}{(\lambda_1+\lambda_2)^2})$

$\because (\lambda_1+\lambda_2)^2-4\lambda_1\lambda_2=\lambda_1^2+\lambda_2^2+2\lambda_1\lambda_2-4\lambda_1\lambda_2=(\lambda_1-\lambda_2)^2$ 〉

$\therefore (\lambda_1+\lambda_2)^2>4\lambda_1\lambda_2 \quad and \quad \frac{1}{4\lambda_1\lambda_2}>\frac{1}{(\lambda_1+\lambda_2)^2}$

$$\det\begin{bmatrix} -\frac{a_1{}^2}{2\lambda_1} & -\frac{a_1a_2}{\lambda_1+\lambda_2} \\ -\frac{a_2a_1}{\lambda_2+\lambda_1} & -\frac{a_2^2}{2\lambda_2} \end{bmatrix} = a_1^2a_2^2(\frac{1}{4\lambda_1\lambda_2}-\frac{1}{(\lambda_1+\lambda_2)^2})>0$$

(3)

$$\det M = -a_1^2 a_2^2 a_3^2 \begin{bmatrix} -\frac{1}{2\lambda_1} & -\frac{1}{\lambda_1+\lambda_2} & -\frac{1}{\lambda_1+\lambda_3} \\ -\frac{1}{\lambda_2+\lambda_1} & -\frac{1}{2\lambda_2} & -\frac{1}{\lambda_2+\lambda_3} \\ -\frac{1}{\lambda_3+\lambda_1} & -\frac{1}{\lambda_3+\lambda_2} & -\frac{1}{2\lambda_3} \end{bmatrix}$$

$$= -a_1^2 a_2^2 a_3^2 \det \begin{bmatrix} \frac{1}{2\lambda_1} & \frac{1}{\lambda_1+\lambda_2} & \frac{1}{\lambda_1+\lambda_3} \\ 0 & \frac{1}{2\lambda_2} - \frac{2\lambda_3}{(\lambda_2+\lambda_1)^2} & \frac{1}{\lambda_2+\lambda_3} - \frac{2\lambda_1}{(\lambda_1+\lambda_2))(\lambda_1+\lambda_{3)}} \\ 0 & \frac{1}{\lambda_2+\lambda_3} - \frac{2\lambda_1}{(\lambda_1+\lambda_2))(\lambda_1+\lambda_{3)}} & \frac{1}{2\lambda_3} - \frac{2\lambda_1}{(\lambda_1+\lambda_2)^2} \end{bmatrix}$$

$$= -\frac{1}{2\lambda_1} a_1^2 a_2^2 a_3^2 \det \left[\left(\frac{1}{2\lambda_2} - \frac{2\lambda_1}{(\lambda_2+\lambda_1)^2}\right) \quad \left(\frac{1}{2\lambda_3} - \frac{2\lambda_1}{(\lambda_3+\lambda_1)^2}\right) - \left(\frac{1}{\lambda_2+\lambda_3} - \frac{2\lambda_1}{(\lambda_1+\lambda_2))(\lambda_1+\lambda_3)}\right)^2 \right]$$

$$= -\frac{1}{2\lambda_1} a_1^2 a_2^2 a_3^2 \det \left[\frac{1}{4\lambda_2\lambda_3} - \frac{\lambda_1}{\lambda_3(\lambda_1+\lambda_2)} - \frac{\lambda_1}{\lambda_2(\lambda_1+\lambda_3)} - \frac{1}{(\lambda_2+\lambda_3)^2} + \frac{4\lambda_1}{(\lambda_1+\lambda_2))(\lambda_1+\lambda_3)(\lambda_{1=2}+\lambda_3)} \right]$$

$$\because \quad -\frac{1}{2\lambda_1} a_1^2 a_2^2 a_3^2 > 0, \quad \frac{1}{4\lambda_2\lambda_3} - \frac{\lambda_1}{\lambda_3(\lambda_1+\lambda_2)} > 0. \quad -\frac{\lambda_1}{\lambda_3(\lambda_1+\lambda_2)} > 0, \quad -\frac{\lambda_1}{\lambda_2(\lambda_1+\lambda_3)} > 0,$$

$$\frac{4\lambda_1}{(\lambda_1+\lambda_2))(\lambda_1+\lambda_3)(\lambda_{1=2}+\lambda_3)} > 0.$$

$\therefore \det M < 0$

From (1), (2) and (3) , We can conclude that M is positive definite

5.17 A real matrix M (not necessarily symmetric) is defined to be positive definite if $\underline{X}' M \underline{X} > 0$ for any non zero $\underline{X}$ .Is it true that the matrix M is positive definite if all eigenvalues of Mare real and positive or you check its positive definiteness ?

一个实矩阵 M 被定义为正定的，如果对任意非零 $\underline{X}$ 都有 $\underline{X}' M \underline{X} > 0$。如果 M 的所有特征值都是正实数，是不是就有 M 正定/ 或者如果 M 的所有主子式都是正，M 是不是正定呢？如果不是，要如何确定它的正定性呢？

If all eigenvalues of M are real and positive or if all its leading pricipal minors are positive ,the matrix M may not be positive definite .the exampal following can show it .

Let $M = \begin{bmatrix} 1 & -8 \\ 0 & 1 \end{bmatrix}$, Its eigenvalues 1,1 are real and positive and all its leading pricipal minors are positive , but for non zero $\underline{X = \begin{bmatrix} 1 & 2 \end{bmatrix}'}$ we can see $X'MX = -11 < 0$. So M is not positive definite .

Because $X'MX$ is a real number . we have $(\underline{X}' M \underline{X})' = \underline{X}' M' \underline{X} = \underline{X}' M \underline{X}$, and

$$\underline{X}'M\underline{X} = \underline{\frac{1}{2}\underline{X}'}M\underline{X} + \frac{1}{2}\underline{X}'M'\underline{X} = \underline{X}'[\frac{1}{2}(M+M')]\underline{X}$$

This $\underline{X}'M\underline{X} > 0$, *if and only if* $\underline{X}'[\frac{1}{2}(M+M')]\underline{X}$>0, that is , M is positive and the matrix $\frac{1}{2}(M+M')$ is symmetric , so we can use theorem 3.7 to cheek its positive definiteness .

5.18 show that all eigenvalues of A have real parts less than $-\mu<0$ if and only if ,for any given positive definite symmetric matrix N , the equation $A'M+MA+2\mu M=-N$ has a unique symmetric solution M and M is positive definite .

证明 A 的所有特征值的实部小于$-\mu<0$ 当且对任意给定的正定对称阵 N。方程 $A'M+MA+2\mu M=-N$ 有唯一解 M，且 M 是正定的。

Proof : the equation can be written as $(A+\mu I)'+M(A+\mu I)=-N$

Let $B=A+\mu I$, then the equation becomes $B'M+MB=-N$. So all eigenvalues of B have megative parts if and only if ,for any given positive definite symmetric matrix N, the equation $B'M+MB=-N$ has a unique symmetric solution m and is positive definitive .

And we know $B=A+\mu I$, so $\det(\lambda I-B)=\det(\lambda I-A-\mu I)=\det((\lambda-\mu)I-A)$, that is , all eigenvalues of A are the eigenvalues of B subtracted $-\mu$. So all eigenvalues of A are real parts less than $-\mu<0$ .if and only all eigenvalues of B have negative real parts , eguivalontly if and only if , for any given positive symmetric matrix N the equation $A'M+MA+2\mu M=-N$ has a unique symmetric solution M and M is positive definite .

5.19 show that all eigenvalues of A have magnitudes less than ρ if and only if , for any given positive definite symmetric matrix N ,the equation $\rho^2M-A'MA=\rho^2N$. Has a unique symmetric solution M and M is positive definite .

证明 A 的所有特征值的模都小于ρ 当且仅当对任意给定的对称正定 N ， 方程 $\rho^2M-A'MA=\rho^2N$ 有唯一解 M ，且 M 是对称正定阵。

Proof : the equation can be written as $M-(c)'M(\rho^{-1}A)=N$ let $B=\rho^{-1}A$, then $\det(\lambda I-B)=\det(\lambda I-\rho^{-1}A)=\rho^{-1}\det(\lambda I\rho-A)$. that is ,all eigenvalues of A are the eigenvalues B multiplied by ρ ,so all eigenvalue of B have msgritudes less than 1 . equivalently if and only if , for any given positive definite symmetric matrix N the equation $\rho^2M-A'MA=\rho^2N$ has a unique symmetric solution M and M is positive definite .

5．20 Is a system with impulse response $g(t,\tau)=e^{-2|t|-|\tau|}$, for $t\geq\tau$, BIBO stable ? how about $g(t,\tau)=\sin(e^{-(t-\tau)})\cos t$?

脉冲响应为$g(t,\tau)=e^{-2|t|-|\tau|}$，$t\geq\tau$， 的系统是否 BIBO 稳定？ $g(t,\tau)=\sin(e^{-(t-\tau)})\cos t$? 的系统是否 BIBO 稳定？

$$\int_{t_0}^{t}|e^{-2|t|-|t|}|\,d\tau=\int_{t_0}^{t}e^{-2|t|-|t|}d\tau=\int_{t_0}^{t}e^{-|t|}d\tau \quad 、$$

$$=\begin{cases} e^{-2t}(e^{-t_0}-e^{-t}) & if \quad t_0\ge 0 \\ e^{-2t}(2-e^{t_0}-e^{-t}) \quad if \quad t_0<0 \quad and \quad t>0 & \le \quad 2 \quad <+ \quad \infty \\ e^{2t}(2-e^{t_0}-e^{-t}) \quad if \quad t_0<0 \quad and \quad t<0 & \end{cases}$$

$$|\sin(e^{-(t-\tau)})\cos t|=|\sin t\cdot e^{-(t-\tau)}\cdot|\cos t|\le|\sin t|\cdot(1-e^{t_0-t})$$

$$\int_{t_0}^{t}|\sin t|\cdot(e^{-(t-\tau)})d\tau=|\sin t|\cdot e^{-t}\int_{t_0}^{t}e^{\tau}d\tau=|\sin t|\cdot(1-e^{-(t_0-t)})$$

$$\because \quad t_0\le t, \qquad \therefore \quad e^{t_0-t}\in(0,1]$$

$$\int_{t_0}^{t}|\sin t|\cdot(e^{-(t-\tau)})d\tau\le 0<+\infty \qquad so$$

$$\int_{t_0}^{t}|\sin t\cdot(e^{-(t-\tau)})\cos\tau|\le|\sin t|\cdot e^{-t}\int_{t_0}^{t}|\sin t|\cdot e^{-(t_0-t)}\le 1<+\infty$$

Both systems are BIBO stable .

5.21 Consider the time-varying equation $\dot{x}=2t\tau+u \qquad y=e^{-t^2x}$ is the equation stable ? marginally stable ? asymptotically stable ?

时变方程 $\dot{x}=2t\tau+u \qquad y=e^{-t^2x}$ 是否 BIBO 稳定 限界稳定？ 渐进稳定？

$\dot{x}(t0=2tx(t) \quad\Rightarrow x(t)=x(0)e^{t_2} \quad\Rightarrow \underline{\overline{x}}(t)=e^{t_2}$ is a fundamental matrix

$\Rightarrow$ its stable transition matrix is $\phi(t,t_0)=e^{t^2-t_0}$

$\Rightarrow$ Its impulse response is $g(t,\tau)=e^{-t^2}\cdot e^{t^2-\tau^2}=e^{-\tau^2}$

$$\int_{t_0}^{t}g(t,\tau)d\tau=\int_{t_0}^{t}e^{-\tau^2}d\tau\le\int_{-\infty}^{+\infty}e^{-\tau^2}d\tau\cdot=2e^{-\tau^2}d\tau=\int_{0}^{+\infty}e^{-\tau^2}d\tau=2\cdot\frac{\pi}{2}=\pi<+\infty$$

so the equation is BIBO stable .

$|\phi(t,t_0)|=|e^{t^2-t_0}|=e^{t^2-t_0}, \qquad t\ge t_0$ is not bounded ,that is . there does not exist a finite constant M such that $\phi(t,t_0)\le M<+\infty$. so the equation is not marginally stable and is not asymptotically stable .

5.22 show that the equation in problem 5.21 can be transformed by using $\overline{x}=p(t)x, \qquad$ with $p(t)=e^{-t^2}$, into $\dot{\overline{x}}=0.\overline{x}+e^{-t^2}u \qquad y=\overline{x}$ is the equation BIBO stable ?marginally stable ? asymptotically ? is the transformation a lyapunov transformation ?

证 明 题 5 。 21 方 程 是 可 通 过 $\overline{x}=p(t)x, \qquad$ with $p(t)=e^{-t^2}$ 变 换 成 $\dot{\overline{x}}=0.\overline{x}+e^{-t^2}u \qquad y=\overline{x}$ 该方程是否 BIBO 稳定？界限稳定？渐进稳定？该变换是不是 lyapunov 变换 ？

proof : we have found a fundamental matrix of the equation in problem 5.21 .is $\underline{\overline{x}}(t)=e^{t^2}$. so from theorem 4.3 we know , let p（t）$=\underline{\overline{x}}^{-1}(t)=e^{-t^2}$. and $\overline{x}=p(t)x,$, then we have

$\overline{A}(t)=0$

$\overline{B}(t)=\underline{\overline{x}}^{-1}(t)\overline{B}(t)=e^{-t^2}. \qquad \overline{C}(t)=C(t)\underline{\overline{B}}(t)=e^{-t^2}e^{t^2}=1 \qquad . \quad \overline{D}(t)=D(t)=0$

and the equation can be transformed into $\qquad \dot{\overline{x}}=0.\overline{x}+e^{-t^2}u \qquad y=\overline{x}$

the impulse response is invariant under equivalence transformation , so is the BIBO stability . that is marginally stable . $\dot{\bar{x}} \to = 0.\bar{x} \Rightarrow \bar{x}(t) = \bar{x}(0) \Rightarrow \underline{\bar{x}}(t) = 1$ is a fundamental matrix $\Rightarrow$ the state transition matrix is $\phi(t, t_0) = 1$ $|\phi(t, t_0)| = 1 \le 1 < +\infty$ so the equation is marginally stable . $t \to \infty, |\phi(t, t_0)| \to 0$ (不趋于零)so the equation is not asymptotically stable . from theorem 5.7 we know the transformation is mot a lyapunov transformation

5.23 is the homogeneous equation $\underline{\dot{x}} = \begin{bmatrix} -1 & 0 \\ -e^{-3t} & 0 \end{bmatrix} \underline{\dot{x}}$ *for* $t_0 \ge 0$ marginally stable ? asymptotically stable?

齐次方程 $\underline{\dot{x}} = \begin{bmatrix} -1 & 0 \\ -e^{-3t} & 0 \end{bmatrix} \underline{\dot{x}}$, $t_0 \ge 0$ 是否限界稳定？渐进稳定？

$$\underline{\dot{x}} = \begin{bmatrix} -1 & 0 \\ -e^{-3t} & 0 \end{bmatrix} \underline{x} \Rightarrow \begin{matrix} \dot{x}_1 = -x_1 \\ \dot{x}_2 = -e^{-3t} x_1 \end{matrix} \Rightarrow \begin{matrix} x_1(t) = x_1(0) e^{-t} \\ x_2(t) = \frac{1}{4} x_1(0) e^{-4t} + x_2(0) - \frac{1}{4} x_1(0) \end{matrix}$$

$$\Rightarrow \underline{\bar{x}} = \begin{bmatrix} e^{-t} & 0 \\ \frac{1}{4} e^{-4t} & 1 \end{bmatrix}$$

is a fundamental matrix of the equation . $\Rightarrow \Phi(t) = \underline{\bar{X}}(t) \underline{\bar{X}}^{-1}(t_0) = \begin{bmatrix} e^{-t+t_0} & 0 \\ \frac{1}{4} e^{-4t+t_0} & 1 \end{bmatrix}$ is the state transition of the equation. $\|\phi(t, t_0)\|_\infty = \max\left\{ e^{-t+t_0}, 1 + \frac{1}{4} e^{-4t+t_0} - \frac{1}{4} e^{-3t_0} \right\}$ for all $t_0 \ge 0$ and $t \ge t_0$ $0 \le e^{-t+t_0} \le 1;$ $0 \le \frac{1}{4} e^{-4t+t_0} \le \frac{1}{4}$ $0 \le e^{-t+t_0} \le 1;$ $-\frac{1}{4} \le \frac{1}{4} e^{-3t_0} \le 0,$ *so* $\frac{3}{4} \le 1 + \frac{1}{4} e^{-4t+t_0} - \frac{1}{4} e^{-3t_0} \le \frac{5}{4}$. $\|\phi(t, t_0)\|_\infty \le \frac{5}{4} < +\infty$ the equation is marginally stable .if $t - t_0$ is fixed to some constant , then $\|\phi(t, t_0)\| \to 0$ (不趋于零)。 *as* $t \to \infty,$ so the equation is not asymptotically stable .

for all t and t_0 with $t \ge t_0$. we have $0 \le e^{-t+t_0} \le 1;$ $0 \le \frac{1}{4} e^{-4t+t_0} \le \frac{1}{4}$. every entry of $\phi(t, t_0)$ is bounded , so the equation is marginally stable .

because the (2,2)th entry of $\phi(t, t_0)$ does not approach zero as $t \to \infty,$ the equation is not asymptotically .

6.1 is the stable equation $\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mu$ controllable ? observable ?题

$$y = [1 \quad 2 \quad 1]\underline{x}$$

述状态方程是否可控？ 是否可观？

$$\rho([B \quad AB \quad A^2B]) = \rho(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}) = 3$$

thus the state equation is controllable ,

$$\rho\ (\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}) = \rho(\begin{bmatrix} 1 & 2 & 0 \\ -1 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix}) = 1$$

but it is not observable .

6.2 is the state equation $\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 2 & -1 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \underline{u}$ controllable ? observable?

$$y = [1 \quad 0 \quad 1]\underline{x}$$

述状态方程是否可控？ 是否可观？

$\rho(B) = 2. \quad n - 2 = 3 - 2 = 1,$ using corollary , we have

$\rho([B \quad AB]) = \rho(\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}) = 3$,Thus the state equation is controllable

$\rho(\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}) = \rho(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -1 \\ 0 & -2 & 4 \end{bmatrix}) = 3$. Thus the state equation is observable

6.3 Is it true that the rank of [B AB $\cdots A^{n-1}B]$ equals the rank of $[AB \quad A^2B \cdots A^nB]$? If not ,under what condition well it be true ?

ρ([B A$[AB \quad A^2B \cdots A^nB]$B $\cdots A^{n-1}B]$]=ρ()是否成立？若否，问在什么条件下该成立？

It is not true that the rank of [B AB $\cdots A^{n-1}B]$ equals the rank of $[AB \quad A^2B \cdots A^nB]$.

If A is nonsingular {(ρ ($[AB \quad A^2B \cdots A^nB]$)= ρ (A [B AB $\cdots A^{n-1}B]$]= ρ ([B

AB$\cdots A^{n-1}B]$(see equation (3.62)) then it will be true

6.4 show that the state equation $\underline{\dot{x}} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\underline{x} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix}\underline{u}$ is controllable if and only if the pair

$(A_{22}\ \ A_{21})$ IS controllable 证明题述状态方程可控当且仅当$(A_{22}\ \ A_{21})$可控?

Proof :

6.5 find a state equation to describe the network shown in fig.6.1 and then check its controllability and observability

求图 6.1 中网络的一个状态方程,是否可控?可观?

The state variables x_1 and x_2 are choosen as shown ,then we have $\begin{array}{l} \dot{x}_1 + x_1 = u \\ \dot{x}_2 + x_2 = 0 \\ y = -x_2 + 2u \end{array}$ thus

a state equation describing the network can be es pressed as $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}U$ checking the controllability and observability : $y - [0 \quad -1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 2u$

$$\rho([B \quad AB] = \rho(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}) = 1$$

$$\rho(\begin{bmatrix} C \\ CA \end{bmatrix}) = \rho(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}) = 1$$

The equation is neither controllable nor obserbable

6.6 find the controllability index and obserbability index of the state equation in problems 6,1 and 6.2

求题 6.1 和 6.2 中状态方程的可控性指数和可观性指数.

The state equation in problem 6.1 is $\begin{array}{l} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}U \\ y - [0 \quad -1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 2u \end{array}$ the equation is controllable ,so

we have $n/p \le \mu \le \min(\bar{n}, n-p+1) \le n-p+1.\ where \quad n=3,\ \ p = rank(b) = 1$

$\therefore n/p \le \mu \le n-p+1.\ \ ie \quad 3 \le \mu \le 3 \quad\ \mu = 3$

$$\because \rho([\underline{c}]) = \rho\left(\begin{bmatrix} \underline{C} \\ \underline{C}A \end{bmatrix}\right)\rho\left(\begin{bmatrix} \underline{C} \\ \underline{C}A \\ \underline{C}A^2 \end{bmatrix}\right) = 1, \quad \therefore\ V = 1$$

The controllability index and cbservability index of the state eqyation in problem 6.1 are 3 and 1 , respectively .the state equation in problem 6.2 is $\underline{\dot{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \underline{u}$

$y - [1 \quad 0 \quad 1] \underline{x}$

The state equation in is both controllable and observable ,so we have $\therefore n/p \le \mu \le n-p+1. \ and \quad n/p \le \nu \le n-p+1$

$where \quad n=3, \ \ p = rank(B) = 2\,. \quad q = rank\underline{c} = 1$

$\therefore 3/2 \le \mu \le 2 \ \ .3 \le \mu \le 3 \quad \mu = 2, \quad \nu = 3$

The controllability index and cbservabolity index of the state equation in problem 6.2 are 2 and 3 ,respectively rank(c)=n .the state equation is controllable so $n/p \le \mu \le n-p+1. \, where \ p = rank(B) = n \Rightarrow \mu = 1$

6.7 what is the controllability index of the state equation $\underline{\dot{x}} = A\underline{X} + IU$ WHERE I is the unit matrix ?

状态方程$\underline{\underline{\dot{x} = A\underline{X}}} + IU$的可控性指数是什么？

Solution . $C = [B \quad AB \quad A^2B \cdots A^{n-1}B] = [I \quad A \quad A^2 \cdots A^{n-1}]$

C is an $n \times n^2$ matrix. So rank(C)$\le n$. And it is clearly that the first n columns of C are linearly independent. This rank(C)$\le n$ .and $u_i = 1 \ for \ i = 0, 1, \cdots n$

The controllability index of the state equation $\underline{\dot{x}} = A\underline{X} + IU$ is $u = \max(u_1, u_2, \cdots, u_n) = 1$

6.8 reduce the state equation $\underline{\dot{x}} = \begin{bmatrix} -1 & 4 \\ 3 & -1 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \qquad y = [1 \quad 1] \underline{x}$. To a controllable one . is the reduced equation observable ?

将题述状态方程可控的，降维后方程是否可观？

Solution :because $\rho(C) = \rho([B \quad AB]) = \rho(\begin{bmatrix} 1 & 3 \\ 3 & 3 \end{bmatrix}) = 1 < 2$

The state equation is not controllable . choosing $Q = P^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and let $\underline{\bar{x}} = p\underline{x}$ then we have $\overline{A} = PAP^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & -5 \end{bmatrix}$

$\overline{B} = PB = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $\overline{C} = CP^{-1} = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix}$

thus the state equation can be reduced to a controllable subequation

$\dot{\overline{x}}_c = 3\overline{x}_c + u \qquad \overline{y} = 2\overline{x}_c$ and this reduced equation is observable

6.9 reduce the state equation in problem 6.5 to a controllable and observable equation . 将题 6.5 中的状态方程降维为可控且可观方程.

Solution : the state equation in problem 6.5 is

$$\dot{\underline{x}} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\underline{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}U$$
$$y - [0 \quad -1]x + 2u$$

it is neither controllable nor observable

from the form of the state equation ,we can reaelily obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}U$$
$$y - [0 \quad -1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 2u$$

thus the reduced controllable state equation can be independent of x_c . so the equation can be further reduced to y=2u .

6.10 reduce the state equation
$$\dot{\underline{x}} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}\underline{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}u$$
$$y = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \end{bmatrix}\underline{x}$$
to a controllable and observable equation 将状态方程降维为可控可观方程。

Solution the state equation is in Jordan-form , and the state variables associated with λ_1 are independent of the state variables associated with λ_2 thus we can decompose the state equation into two independent state equations .and reduce the two state equation controllable and observable equation , respectively and then combine the two controllable and observable equations associated with λ_1 and

λ_2 . Into one equation ,that is we wanted .

Decompose the equation . $\underline{\dot{x}_1} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix} \underline{x_1} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \qquad y = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \underline{x_1}$

$$\underline{\dot{x}_2} = \begin{bmatrix} \lambda_2 & 1 \\ 0 & \lambda_2 \end{bmatrix} \underline{x_2} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 0 & 1 \end{bmatrix} \underline{x_2}$$

Where $\underline{x} = \begin{bmatrix} \underline{x_1} \\ \underline{x_2} \end{bmatrix}, \qquad y = y_1 + y_2$

Deal with the first sub equation .c

Choosing $Q = P^{-1} := \begin{bmatrix} 0 & 1 & 0 \\ 1 & \lambda_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *and let* $\underline{\overline{x_1}} = p\underline{x_1}$ then we have

$$\overline{A_1} = PA_1P^{-1} = \begin{bmatrix} -C & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & \lambda_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -\lambda^2{}_1 & 1 \\ 1 & 2\lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

$$\overline{B_1} = PB_1 = \begin{bmatrix} -\lambda_{11} & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \overline{C_1} = C_1P^{-1} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & C & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & 1 \end{bmatrix}$$

Thus the sub equation can be reduced to a controllable one

$$\dot{\underline{\overline{x}}}_{1c} = \begin{bmatrix} 0 & -\lambda_1^2 \\ 1 & \lambda_1 \end{bmatrix} \underline{\overline{x}}_{1c} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$\overline{y}_1 = \begin{bmatrix} 1 & 2\lambda_1 \end{bmatrix} \underline{\overline{x}}_{1c}$$

$$\because \quad \rho(O) = \rho\left(\begin{bmatrix} 1 & \lambda_1 \\ \lambda_1 & \lambda_1^2 \end{bmatrix}\right) = 1 < 2$$

the reduced controllable equation is not observable choosing $P = \begin{bmatrix} 1 & \lambda_1 \\ 0 & 1 \end{bmatrix}$ .and let ,

$$\overline{\overline{A}}_1 = \begin{bmatrix} 1 & \lambda_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -\lambda_1^2 \\ 1 & 2\lambda_1 \end{bmatrix} \begin{bmatrix} 1 & -\lambda_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{bmatrix}$$

$\underline{\overline{\overline{x}}}_{1c} = p\underline{\overline{x}}_{1c}$ then we have $\overline{\overline{B}}_1 = \begin{bmatrix} 1 & \lambda_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\overline{\overline{C_1}} = \begin{bmatrix} 1 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 & -\lambda_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

thus we obtain that reduced controllable and observable equation of the sub equation associated with

$$\lambda_1 : \quad \dot{x}_{1co} = \lambda_1 x_{1c0} + u$$
$$\bar{y}_1 = x_{1c0}$$

deal with the second sub equation :

$\rho(C_2) = \rho(\begin{bmatrix} 0 & 1 \\ 1 & \lambda_2 \end{bmatrix}) = 2$ it is controllable

$\rho(O_2) = \rho(\begin{bmatrix} 0 & 1 \\ 0 & \lambda_2 \end{bmatrix}) = 1 < 2$ It is not observable choosing $p = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and let

$\underline{\bar{x}_2} = p\underline{x_2}$ then we have

$$\overline{A_2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_2 & 1 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_2 & 0 \\ 1 & \lambda_2 \end{bmatrix}$$
$$\overline{B_2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\overline{C_2} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

thus the sub equation can be reduced to a controllable and observable

$$\dot{x}_{ico} = \lambda_2 \underline{x_{ico}} + u$$
$$\overline{y_2} = \underline{x_{ico}}$$

combining the two controllable and observable state equations . we obtain the reduced controllable and observable state equation of the original equation and it is

$$\dot{\underline{x}}_{co} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \underline{x}_{co} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$\bar{y} = \begin{bmatrix} 1 & 1 \end{bmatrix} \underline{x}_{co}$$

6.11 consider the n-dimensional state equation $\begin{matrix} \dot{\underline{x}} = A\underline{x} + B\underline{u} \\ \underline{y} = c\underline{x} + o\underline{u} \end{matrix}$ the rank of its controllability matrix is assumed to be $n_1 < n$. Let Q_1 be an $n \times n_1$ matrix whose columns are any n_1 linearly independent columns of the controllability matrix let p_1 be an $n_1 \times n$ matrix such that $p_1 Q_1 = In_1$,where In_1 is the unit matrix of order n_1 .show that the following n_1-dimensional state equation

$$\begin{aligned}\dot{\underline{\bar{x}}}_1 &= P_1 A Q_1 \underline{\bar{x}}_1 + P_1 B \underline{u} \\ \underline{\bar{y}} &= C Q_1 \underline{\bar{x}}_1 + D\underline{u}\end{aligned}$$

is controllable and has the same transfer matrix as the original state equation

n 维状态方程 $\begin{aligned}\dot{\underline{x}} &= A\underline{x} + B\underline{u} \\ \underline{y} &= c\underline{x} + o\underline{u}\end{aligned}$ 。假设共可控性矩阵的秩为 $n_1 < n$，令 Q_1 为可控性矩阵中任意 n_1 个线性无关组成的 $n \times n_1$ 矩阵，而 p_1 是 P 发矩阵且 $p_1 Q_1 = In_1$，其中 In_1 是 n_1 阶单位阵。证明下列 n_1 维状态方程 $\begin{aligned}\dot{\underline{\bar{x}}}_1 &= P_1 A Q_1 \underline{\bar{x}}_1 + P_1 B \underline{u} \\ \underline{\bar{y}} &= C Q_1 \underline{\bar{x}}_1 + D\underline{u}\end{aligned}$ 可控且与原状态方程具有相同的传函。

Let $P = Q^{-1} := \begin{bmatrix} P_1 \\ \hline P_2 \end{bmatrix}$, where P_1 is an $n_1 \times n$ matrix and P_2 is an $(n-n_1) \times n$ one ,and we have

$$QP = Q_1 P_1 + Q_2 P_2 = I_n \qquad PQ = \begin{bmatrix} P_1 Q_1 & P_1 Q_2 \\ P_2 Q_1 & P_2 Q_2 \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 \\ 0 & I_{n-n_1} \end{bmatrix} = I_n \quad \text{that is} \quad \begin{aligned} P_1 Q_1 &= I_{n_1} \\ P_2 Q_2 &= I_{n-n_1} \end{aligned}$$

Then the equivalence trans formation $\underline{\bar{x}} = p\underline{x}$ will transform the original state equation into

$$\begin{aligned}\begin{bmatrix} \dot{\underline{\bar{x}}}_c \\ \dot{\underline{\bar{x}}}_{\bar{c}} \end{bmatrix} &= \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} A \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} \underline{\bar{x}}_c \\ \underline{\bar{x}}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} B\underline{u} & y &= C\begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} \underline{\bar{x}}_c \\ \underline{\bar{x}}_{\bar{c}} \end{bmatrix} + D\underline{u} \\ &= \begin{bmatrix} P_1 A Q_1 & P_1 A Q_2 \\ P_2 A Q_1 & P_2 A Q_2 \end{bmatrix} \begin{bmatrix} \underline{\bar{x}}_c \\ \underline{\bar{x}}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} p_1 B \\ p_2 B \end{bmatrix} \underline{u} & &= \begin{bmatrix} CQ_1 & CQ_2 \end{bmatrix} \begin{bmatrix} \underline{\bar{x}}_c \\ \underline{\bar{x}}_{\bar{c}} \end{bmatrix} + D\underline{u} \\ &= \begin{bmatrix} \overline{A_C} & \overline{A_{12}} \\ 0 & \overline{A_{\bar{C}}} \end{bmatrix} \begin{bmatrix} \underline{\bar{x}}_c \\ \underline{\bar{x}}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \overline{B}_c \\ 0 \end{bmatrix} \underline{u} & &= \begin{bmatrix} \overline{C}_c & \overline{C}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \underline{\bar{x}}_c \\ \underline{\bar{x}}_{\bar{c}} \end{bmatrix} + D\underline{u}\end{aligned}$$

where $\overline{A}_c = P_1 A Q_1.\ \ \overline{B}_c = P_1 B.\ \ \overline{C}_C = CP_1,\ \ and\ \ \overline{A}_c\ is\ n_1 \times n_1$ the reduced n_1-dimensional state equation $\begin{aligned}\dot{\underline{\bar{x}}}_1 &= P_1 A Q_1 \underline{\bar{x}}_1 + P_1 B \underline{u} \\ \underline{\bar{y}} &= C Q_1 \underline{\bar{x}}_1 + D\underline{u}\end{aligned}$ is controllable and has the sane transfer matrix as the original state equation

6.12 In problem 6.11 the reduction procedure reduces to solving for P_1 in $P_1 Q_1 = I.$ how do you solve P_1?

题 6.11 中的降维过程为求解 $P_1 Q_1 = I.$ 中的 P_1，该如何求 P_1?

Solution : from the proof of problem 6.11 we can solve P_1 in this way

$\rho(C) = \rho[AB \quad A^2B \cdots A^nB] = n_1 < n$

we fprm the $n < n$ matrix $Q = ([q_1 \cdots q_{n_1} \cdots q_{n]}$, where the first n_1 columns are any n_1 linearly independent columns of C , and the remaining columns can aribitrarily be chosen as Q is nonsingular .

Let $P = Q^{-1} := \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$, where P_1 is an $n_1 \times n$ matrix and $p_1 Q_1 = In_1$

6.13 Develop a similar statement as in problem 6.11 for an unobservable state equation .对一不可观状态方程，如题 6.11 般低哦降维为可观方程.

Solution : consider the n-dimensional state equation $\begin{aligned} \dot{\underline{x}} &= A\underline{x} + B\underline{u} \\ \underline{y} &= C\underline{x} + D\underline{u} \end{aligned}$ the rank of its observability matrix is assumed to be $n_1 < n$ .let P_1 be an $n_1 \times n$ matrix whose rows are any n_1 linearly independent rows of the observability matrix . let Q_1 be an $n \times n_1$ matrix such that $p_1 Q_1 = In_1$ .where In_1 is the unit matrix of order n_1 , then the following n_1-dimensional state equation

$$\begin{aligned} \dot{\underline{\bar{x}}}_1 &= P_1 A Q_1 \underline{\bar{x}}_1 + P_1 B \underline{u} \\ \underline{\bar{y}} &= C Q_1 \underline{\bar{x}}_1 + D\underline{u} \end{aligned}$$

is observable and has the same transfer matrix as the original state equation

6.14 is the Jordan-form state equation controllable and observable ?

下 Jordan-form 状态方程是不是可控，可观？

$$\dot{\underline{x}} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \underline{x} + \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \underline{u} \quad \underline{y} = \begin{bmatrix} 2 & 2 & 1 & 3 & -1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \underline{x}$$

$$\text{solution} : \rho(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}) = 3 \qquad and \qquad \rho(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}) = 2$$

so the state equation is controllable ,however ,it is not observable because

$$\rho(\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}) = 2 \neq 3$$

6.15 is it possible to find a set of b_{ij} and a set of c_{ij} such that the state equation is controllable ? observable ?是否可能找到一组b_{ij}和一组c_{ij}使如下状态方程可控？可观？

$$\underline{\dot{x}} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \underline{x} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \\ b_{51} & b_{52} \end{bmatrix} \underline{u} \qquad \underline{y} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} \end{bmatrix} \underline{u}$$

solution : it is impossible to find a set of b_{ij} such that the state equation is observable ,because

$$\rho(\begin{bmatrix} b_{21} & b_{22} \\ b_{41} & b_{42} \\ b_{51} & b_{52} \end{bmatrix}) \leq 2 < 3$$

it is possible to find a set of c_{ij} such that the state equation is observable .because

$$\rho(\begin{bmatrix} C_{11} & C_{13} & C_{15} \\ C_{21} & C_{23} & C_{25} \\ C_{31} & C_{33} & C_{35} \end{bmatrix}) \leq 3$$

The equality may hold for some set of c_{ij} such as $\begin{bmatrix} C_{11} & C_{13} & C_{15} \\ C_{21} & C_{23} & C_{25} \\ C_{31} & C_{33} & C_{35} \end{bmatrix} = I$ the state equation is observable if and only if $\rho(\begin{bmatrix} C_{11} & C_{13} & C_{15} \\ C_{21} & C_{23} & C_{25} \\ C_{31} & C_{33} & C_{35} \end{bmatrix}) = 36$

6.16 consider the state equation

$$\underline{\dot{x}} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & \beta_1 & 0 & 0 \\ 0 & -\beta_1 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & \beta_2 \\ 0 & 0 & 0 & \beta_2 & \alpha_1 \end{bmatrix} \underline{x} + \begin{bmatrix} b_1 \\ b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \end{bmatrix} u$$

$$y = [C_1 \quad C_{11} \quad C_{12} \quad C_{21} \quad C_{22}] \underline{x}$$

it is the modal form discussed in (4.28) . it has one real eigenvalue and two pairs of complex conjugate eigenvalues . it is assumed that they are distinct , show that the state equation is controllable if and only if $b_1 \neq 0; \quad b_{i1} \neq 0 \quad or \quad b_{i2} \neq 0 \quad for \quad i = 1, 2$ it is observable if and only if $c_1 \neq 0; \quad c_{i1} \neq 0 \quad or \quad c_{i2} \neq 0 \quad for \quad i = 1, 2$

题中的状态方程是（4.82）中讨论的规范型。它有一个实特征值和两对复特征值，并假定它们都是互不相同的。证明该状态方程可控当且仅当

$b_1 \neq 0; \quad b_{i1} \neq 0 \quad or \quad b_{i2} \neq 0 \quad for \quad i = 1, 2$ 可观当且仅当

$c_1 \neq 0; \quad c_{i1} \neq 0 \quad or \quad c_{i2} \neq 0 \quad for \quad i = 1, 2$

proof:；controllability and observability are invariant under any equivalence transformation . so we introduce a nonsingular matrix is transformed into Jordan form.

$$p = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & -0.5j & 0 & 0 \\ 0 & 0.5 & 0.5j & 0 & 0 \\ 0 & 0 & 0 & 0.5 & -0.5j \\ 0 & 0 & 0 & 0.5 & 0.5j \end{bmatrix}, \quad p^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & i & -i & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & i & -j \end{bmatrix}$$

$$\overline{A} = PAP^{-1} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 + J\beta_1 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 - J\beta_1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 + J\beta_2 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 - J\beta_2 \end{bmatrix} \qquad \overline{B} = PB = \begin{bmatrix} b_1 \\ 0.5(b_{11} - jb_{12}) \\ 0.5(b_{11} + jb_{12}) \\ 0.5(b_{211} - jb_{22}) \\ 0.5(b_{21} + jb_{22}) \end{bmatrix}$$

$$\overline{C} = CP^{-1} = \begin{bmatrix} C_1 & C_{11} + jC_{12} & C_{11} - jC_{12} & C_{21} + jC_{22} & C_{22} - jC_{22} \end{bmatrix}$$

Using theorem 6.8 the state equation is controllable if and only if

$b_1 \neq 0; \quad 0.5(b_{11} \pm jb_{12}) \neq 0; \quad 0.5(b_{21} \pm jb_{22}) \neq 0;$ equivalently if and only if

$b_1 \neq 0; \quad b_{i1} \neq 0; \quad or \quad b_{i2} \neq 0; \qquad for \quad i = 1, \quad 2$

The state equation is observable if and only if

$c_{11} \neq 0; \quad c_{11} \pm jc_{12} \neq 0; \quad c_{21} \pm jc_{22} \neq 0;$ equivalently , if and only if

$c_1 \neq 0; \quad c_{i1} \neq 0 \quad or \quad c_{i2} \neq 0 \quad for \quad i = 1, 2$

6.17 find two- and three-dimensional state equations to describe the network shown in fig 6.12 . discuss their controllability and observability

找到图 6.12 所示网络的两维和三维状态方程描述。讨论它们的可控和可观性，

solution: the network can be described by the following equation :

$$\frac{u+x_1}{2}=\dot{x}_2-2\dot{x}_1; \qquad \dot{x}_2=3\dot{x}_3; \qquad x_3=-x_1-x_2; \qquad y=x_3$$

they can be arranged as

$$\dot{x}_1=-\frac{2}{11}x_1-\frac{2}{11}u$$
$$\dot{x}_2=\frac{3}{22}x_1+\frac{3}{22}u$$
$$\dot{x}_3=\frac{1}{22}x_1+\frac{1}{22}u$$
$$y=x_3=-x_1-x_2$$

so we can find a two-dimensional state equation to describe the network as

following : $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-\frac{2}{11} & 0\\ \frac{3}{22} & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}-\frac{2}{11}\\ \frac{3}{22}\end{bmatrix}u \qquad y=x_3=-x_1-x_2$

and it is not controllable because $\rho(c)=\rho(\begin{bmatrix}-\frac{2}{11} & \left(\frac{2}{11}\right)^2\\ \frac{3}{22} & -\frac{2}{11}-\frac{3}{22}\end{bmatrix})=1<2$

however it is observable because $\rho(o)=\rho(\begin{bmatrix}-1 & -1\\ \frac{1}{22} & 0\end{bmatrix})=2$

we can also find a three-dimensional state equation to describe

it : $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-\frac{2}{11} & 0 & 0\\ \frac{3}{22} & 0 & 0\\ \frac{1}{22} & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}-\frac{2}{11}\\ \frac{3}{22}\\ \frac{1}{22}\end{bmatrix}u \qquad y=\begin{bmatrix}0 & 0 & 1\end{bmatrix}x$ and it is neither controllable nor

observable because $\rho(\begin{bmatrix}B & AB & A^2B\end{bmatrix})=1<3 \qquad \rho(\begin{bmatrix}C\\ CA\\ CA^2\end{bmatrix})=2<3$

6．18 check controllability and observability of the state equation obtained in problem 2.19 . can uou give a physical interpretation directly from the network

检验题 2.19 中所得状态方程的可控性和可观性。能否直接从该网络给出一个物理解释？

Solution : the state equation obtained in problem 2.19 is

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{bmatrix}=\begin{bmatrix}1&0&0\\0&0&-1\\0&1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}1\\1\\0\end{bmatrix}u \qquad y=\begin{bmatrix}0&1&0\end{bmatrix}\underline{x}$$

It is controllable but not observable because

$$\rho(\begin{bmatrix}B & AB & A^2B\end{bmatrix})=\rho\left(\begin{bmatrix}1&1&1\\1&0&-1\\0&1&1\end{bmatrix}\right)=3 \qquad \rho(\begin{bmatrix}C\\CA\\CA^2\end{bmatrix})=\rho\left(\begin{bmatrix}0&1&0\\0&0&-1\\0&-1&-1\end{bmatrix}\right)\le 2$$

A physical interpretation I can give is that from the network , we can that if $u=0 \quad x_1(0)=a\neq 0$

and $\quad x_2(0)=0$ then $y\equiv 0$ that is any $\underline{x}(0)=\begin{bmatrix}a & 0 & *\end{bmatrix}^T$ and $u(t)\equiv 0$ yield the same output

$Y(t)\equiv 0$ thus there is no way to determine the initial state uniquely and the state equation describing

the network is not observable and the input is has effect on all of the three variables . so the state equation describing the network is controllable

6.19 continuous-time state equation in problem 4.2 and its discretized equations in problem 4.3 with sampling period T=1 and π. Discuss controllability and observability of the discretized equations .
考虑题 4.2 中的连续时间状态以及题 4.3 中它的离散化状态方程（T=1 and π）．讨论离散状态方程的可控性和可观性

solution : the continuous-time state equation in problem 4.2 is $\dot{\underline{x}}=\begin{bmatrix}0&1\\-2&-2\end{bmatrix}\underline{x}+\begin{bmatrix}1\\1\end{bmatrix}u \quad y=\begin{bmatrix}2&3\end{bmatrix}\underline{x}$

the discretized
equation :

$$\underline{x}[k+1]=\begin{bmatrix}(\cos T+\sin T)e^{-T} & \sin Te^{-T}\\ -2\sin Te^{-T} & (\cos T-\sin T)e^{-T}\end{bmatrix}\underline{x}[k]+\begin{bmatrix}\frac{3}{2}-(\frac{3}{2}conT+\frac{1}{2}\sin T)e^{-T}\\ 1+(2\sin T+conT)e^{-T}\end{bmatrix}u[k]$$

$$y[k]=\begin{bmatrix}2&3\end{bmatrix}\underline{x}[k]$$

T=1 $\quad A_d=\begin{bmatrix}0.5083 & 0.3096\\-0.6191 & -0.1108\end{bmatrix} \qquad B_d=\begin{bmatrix}1.0471\\-0.1821\end{bmatrix}$

T= π: $\quad A_d=\begin{bmatrix}0.0432 & 0\\0 & 0.0432\end{bmatrix} \qquad B_d=\begin{bmatrix}1.5648\\-1.0432\end{bmatrix}$

The xontinuous-time state equation is controllable and observable , and the sustem it describes is single-input so the drscretized equation is controllable and observable if and only if

$|I_m[\lambda_i-\lambda j]|\neq 2\pi m/2 \quad for \quad m=1,2\cdots$ whenever $R_e[\lambda_i-\lambda j]=0$ the eigenvalues of a are

–1+j and –1-j . so $|I_m[\lambda_i-\lambda j]|=2$ for T=1 . $2\pi m\neq 2 \quad for \quad any \quad m=1,2\cdots$.so the discretized equation is neither controllable nor observable . for T= π 2m=1 .so the descretized equation is controllable and observable .

6.20 check controllability and observabrlity of $\underline{\dot{x}} = \begin{bmatrix} 0 & 1 \\ 0 & t \end{bmatrix}\underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u \quad y = \begin{bmatrix} 0 & 1 \end{bmatrix}\underline{x}$ 检验可控性和观性。

Solution : $M_1 = -A(t)M_0\begin{bmatrix} 1 \\ -t \end{bmatrix}$ the determinant of the matrix $\begin{bmatrix} M_0 & M_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -t \end{bmatrix}$ is 1. which is nonzero for all t . thus the state equation is controllable at every t。

Its state transition matrix is $\Phi(t,t_0) = \begin{bmatrix} 1 & -e^{0.5t^2}\int_{t_0}^{t} e^{0.5\tau^2} d\tau \\ 0 & e^{0.5(t^2-t_0^2)} \end{bmatrix}$ and

$$C(\tau)\Phi(t,t_0) = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -e^{0.5\tau^2}\int_{t_0}^{t} e^{0.5\tau^2} d\vartheta \\ 0 & e^{0.5(\tau^2-t_0^2)} \end{bmatrix} = \begin{bmatrix} 0 & e^{0.5(\tau^2-t_0^2)} \end{bmatrix}$$

$$W_0(t_0,t_1) = \int_{t_0}^{t_1} \begin{bmatrix} 0 \\ e^{0.5(\tau^2-t_0^2)} \end{bmatrix} \cdot \begin{bmatrix} 0 & e^{0.5(\tau^2-t_0^2)} \end{bmatrix} d\tau$$

$$= \int_{t_0}^{t_1} \begin{bmatrix} 0 & 0 \\ 0 & e^{\tau^2-t_0^2} \end{bmatrix} d\tau$$

We see . $W_0(t_0,t_1)$ is singular for all t_0 and t , thus the state equation is not observable at any t_0 .

6.21 check controllability and observability of $\underline{\dot{x}} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\underline{x} + \begin{bmatrix} 1 \\ e^{-t} \end{bmatrix}u \quad y = \begin{bmatrix} 0 & e^{-t} \end{bmatrix}\underline{x}$ solution :

$x_1(t) = x_1(0) \qquad x_2(t) = x_2(0)e^{-t}$ its state transition matrix is

$$\Phi(t,t_0) = \underline{X}(t)\underline{X}^{-1}(t_0) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{t_0} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-t+t_0} \end{bmatrix} \text{ and}$$

$$\Phi(t,\tau)B(\tau) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-t+\tau} \end{bmatrix}\begin{bmatrix} 1 \\ e^{-\tau} \end{bmatrix} = \begin{bmatrix} 1 \\ e^{-t} \end{bmatrix}$$

$$C(\tau)\Phi(\tau,t_0) = \begin{bmatrix} 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{-\tau+t_0} \end{bmatrix} = \begin{bmatrix} 0 & e^{-2\tau+t_0} \end{bmatrix}$$

WE compute $W_c(t_0,t) = \int_{t_0}^{t} \begin{bmatrix} 1 \\ e^{-t} \end{bmatrix}\begin{bmatrix} 1 & e^{-t} \end{bmatrix} d\tau = \int_{t_0}^{t} \begin{bmatrix} 1 & e^{-t} \\ e^{-t} & e^{-2t} \end{bmatrix} d\tau = \begin{bmatrix} t-t_0 & e^{-t}(t-t_0) \\ e^{-t}(t-t_0) & e^{-2t}(t-t_0) \end{bmatrix}$

Its determinant is identically zero for all to and t_0 , so the state equation is not controllable at any t_0

$$W_c(t_0,t)=\int_{t_0}^{t}\begin{bmatrix}1\\ e^{-2\tau+t_0}\end{bmatrix}\begin{bmatrix}1 & e^{-2\tau+t_0}\end{bmatrix}d\tau=\int_{t_0}^{t}\begin{bmatrix}0 & 0\\ 0 & e^{-4\tau+2t_0}\end{bmatrix}d\tau$$ its determinant is identically zero for all

t_0 and t_1 , so the state equation is not observable at any t_0.

6.22 show shat ($(A(t),B(t))$ is controllable at t_0 if and only if $(-A'(t),B'(t))$ is observable at t_0

证明$(A(t),B(t))$在t_0可控当且仅当$(-A'(t),B'(t))$在t_0可观。

Proof: $\because \Phi(t,t_0)\Phi(t_0,t)=I$ where $\Phi(t,t_0)$ is the transition matrix of $\underline{\dot{x}}(t)=A(t)\underline{x}(t))$

$$\therefore \frac{\partial}{\partial t}\Phi(t,t_0)\Phi(t_0,t)=\frac{\partial\Phi(t,t_0)}{\partial t}\Phi(t_0,t)+\Phi(t,t_0)\frac{\partial\Phi(t_0,t)}{\partial t}$$
$$=A(t)\Phi(t,t_0)\Phi(t_0,t)+\Phi(t,t_0)\frac{\partial}{\partial t}\Phi(t_0,t)=A(t)+\Phi(t,t_0)\frac{\partial}{\partial t}\Phi(t_0,t)=0$$

There we have $\frac{\partial\Phi(t_0,t)}{\partial t}=-\Phi(t_0,t)+A(t)$ and. $\frac{\partial\Phi(t_0,t)}{\partial t}=-A'(t)+\Phi'(t_0,t)$

So $\Phi'(t_0,t)$ is the transition matrix of $\underline{\dot{x}}(t)=A'(t)\underline{x}(t))$ $(A(t),B(t))$ is controllable at t_0 if and only if there exists a finite $t_1>t_0$ such that $W_c(t_0,t)=\int_{t_0}^{t_1}\Phi(t_1,\tau)B(\tau)B'(\tau)\Phi'(t_1,\tau)d\tau$ is nonsingular

$(-A'(t),B'(t))$ is observable at t_0 if and only if there exists a finite $t_1>t_0$ such that

$W_c(t_0,t)=\int_{t_0}^{t_1}\Phi(t_0,\tau)B(\tau)B'(\tau)\Phi'(t_0,\tau)d\tau$ is nonsingular.

For time-invariant systems , show that （A,B）is controllable if and only if (-A,B) is controllable , Is true foe time-varying systems ?
证明对时不变系统，（A,B）可控当且仅当（-A,B）可控。 对时变系统是否有相同结论？
Proof: for time-invariant system (A,B) is controllable if and only if

$\rho(C_1)=\rho\begin{bmatrix}B & AB\cdots A^{n-1}B\end{bmatrix}=n$ (assuming a is $n\times n$

(-A,B) is controllable if and only if $\rho(C_2)=\rho\begin{bmatrix}B & AB\cdots A^{n-1}B\end{bmatrix}=n$

we know that any column of a matrix multiplied by nonzero constant does not change the rank of the matrix . so $\rho(C_1)=\rho(C_2)$ the conditions are identically the same , thus we conclude that (A,B)is controllable if and only if (-A,B)is controllable .for time-systems, this is not true .

7.1 Given $g(s)=\frac{s-1}{(s^2-1)(s+2)}$. Find a three-dimensional controllable realization check its observalility

找 $g(s)=\frac{s-1}{(s^2-1)(s+2)}$ 的一三维可控性突现，判断其可观性。

Solution : $g(s)=\frac{s-1}{(s^2-1)(s+2)}\frac{s-1}{(s^2+2s^2-s-2)}$ using (7.9) we can find a three-dimensional controllable realization as following

$\underline{\dot{x}}=\begin{bmatrix}-2 & 1 & 2\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\underline{x}+\begin{bmatrix}1\\0\\0\end{bmatrix}u \qquad y=\begin{bmatrix}0 & 1 & -1\end{bmatrix}\underline{x}$ this realization is not observable because (s-1)and $(s^2-1)(s+2)$ are not coprime

7.2 find a three-dimensional observable realization for the transfer function in problem 7.1 check its controllability

找题 7.1 中 $g(s)=\frac{s-1}{(s^2-1)(s+2)}$ 的一三维可控性突现，判断其可观性。

Solution : using (7.14) , we can find a 3-dimensional observable realization for the transfer function : $\underline{\dot{x}}=\begin{bmatrix}-2 & 1 & 0\\ 1 & 0 & 1\\ 2 & 0 & 0\end{bmatrix}\underline{x}+\begin{bmatrix}0\\1\\-1\end{bmatrix}u \qquad y=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\underline{x}$ and this tealization is not controllable because (s-1) and $(s^2-1)(s+2)$ are not coprime

7.3 Find an uncontrollable and unobservable realization for the transfer function in problem 7.1 find also a minimal realization
求题 7.1 中传函的一个不可控且不可观突现以及一个最小突现。

Solution : $g(s)=\frac{s-1}{(s^2-1)(s+2)}=\frac{1}{(s+1)(s+2)}=\frac{1}{s^2+3s+2}$ a minimal realization is ,

$\underline{\dot{x}}=\begin{bmatrix}-3 & -2\\ 1 & 0\end{bmatrix}\underline{x}+\begin{bmatrix}1\\0\end{bmatrix}u \qquad y=\begin{bmatrix}0 & 1\end{bmatrix}\underline{x}$ an uncontrollable and unobservable realization is $\underline{\dot{x}}=\begin{bmatrix}-3 & -2 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\underline{x}+\begin{bmatrix}1\\0\\0\end{bmatrix}u \qquad y=\begin{bmatrix}0 & 1 & 0\end{bmatrix}\underline{x}$

7．4 use the Sylvester resultant to find the degree of the transfer function in problem 7.1
用 Sylvester resultant 求题 7.1 中传函的阶
solution :

$$s = \begin{bmatrix} -2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -2 & -1 & 0 & 0 \\ \alpha & 0 & -1 & 1 & -2 & -1 \\ 1 & 0 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

rank(s)=5. because all three D-columns of s are linearly independent . we conclude that s has only two linearly independent N-columns . thus deg $\hat{y}(s) = 2$

7.5 use the Sylvester resultant to reduce $\dfrac{2s-1}{4s^2-1}$ to a coprime fraction 用 Sylvester resultant 将 $\dfrac{2s-1}{4s^2-1}$ 化简为既约分式。

Solution : $s = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 2 & -1 & -1 \\ 4 & 0 & 0 & 2 \\ 0 & 0 & 4 & 0 \end{bmatrix}$ $rank(s) = 3, \quad u = 1 \quad s_1 = s$

null $(s_1) = \begin{bmatrix} -\dfrac{1}{2} & \dfrac{1}{2} & 0 & 1 \end{bmatrix}^T$ thus we have $\overline{N}(s) = \dfrac{1}{2} + 0 \cdot s$ $\quad \overline{D}(s) = \dfrac{1}{2} + s$ and

$$\frac{2s-1}{4s^2-1} = \frac{\frac{1}{2}}{s + \frac{1}{2}} = \frac{1}{2s+1}$$

7.6 form the Sylvester resultant of $\hat{g}(s) = \dfrac{s+2}{(s^2+2s)}$ by arranging the coefficients of $\overline{N}(s)$ and $\overline{D}(s)$ in descending powers of s and then search linearly independent columns in order from left to right . is it true that all D-columns are linearly independent of their LHS columns ?is it true that the degree of $\hat{g}(s)$ equals the number of linearly independent N-columns?

$g(s) = \dfrac{s+2}{(s^2+2s)}$ 将 $\overline{N}(s)$ 和 $\overline{D}(s)$ 按 s 的降幂排列形成 Sylvester resultant ，然后从左至右找出线性无关列。所有的 D-列是否都与其 LHS 线性无关？$\hat{g}(s)$ 的阶数是否等于线性无关 N-列的数回？

Solution ; $\hat{g}(s)=\dfrac{s+2}{(s^2+2s)}:=\dfrac{D(s)}{N(s)}$ $\deg\hat{g}(s)=1$

$$\overline{D}(s)=D_2S^2+D_1S+D_0$$
$$N(s)=N_2S^2+N_1S+N_0$$

from the Sylvester resultant :

$s=\begin{bmatrix}1&0&0&0\\2&1&1&0\\0&2&2&1\\0&0&0&2\end{bmatrix}$ $rank(s)=3$ $u=2$ (the number of linearly independent N-columns

7.7 consider $\hat{g}(s)=\dfrac{\beta_1 s+\beta_2}{(s^2+\alpha_1 s+\partial_2)}=\dfrac{D(s)}{N(s)}$ and its realization

$\underline{\dot{x}}=\begin{bmatrix}-\alpha_1&-\alpha_2\\1&0\end{bmatrix}\underline{x}+\begin{bmatrix}1\\0\end{bmatrix}u$ $y=[\beta_1\quad\beta_2]\underline{x}$

show that the state equation is observable if and only if the Sylvester resultant of $D(s)$ and $N(s)$ is nonsingular . 考虑 $\hat{g}(s)=\dfrac{\beta_1 s+\beta_2}{(s^2+\alpha_1 s+\partial_2)}=\dfrac{D(s)}{N(s)}$ 及其实现 $\underline{\dot{x}}=\begin{bmatrix}-\alpha_1&-\alpha_2\\1&0\end{bmatrix}\underline{x}+\begin{bmatrix}1\\0\end{bmatrix}u$ $y=[\beta_1\quad\beta_2]\underline{x}$ 证明该状态方程可观及其当且仅当 $D(s)$ and $N(s)$ 的 Sylvester resultant 非奇异

proof: $\underline{\dot{x}}=\begin{bmatrix}-\alpha_1&-\alpha_2\\1&0\end{bmatrix}\underline{x}+\begin{bmatrix}1\\0\end{bmatrix}u$ $y=[\beta_1\quad\beta_2]\underline{x}$ is a controllable canonical form realization of $\hat{g}(s)$ from theorem 7.1 , we know the state equation is observable if and only id $D(s)$ and $N(s)$ are coprime

from the formation of Sylvester resultant we can conclude that $D(s)$ and $N(s)$ are coprime if and only if the Sylvester resultant is nonsingular thus the state equation is observable if and only if the Sylvester resultant of $D(s)$ and $N(s)$ is nonsingular

7.8 repeat problem 7.7 for a transfer function of degree 3 and its controllable-form realization , 对一

个 3 阶传函及其可控型实现重复题 7.7

solution : a transfer function of degree 3 can be expressed as $\hat{g}(s)=\dfrac{\beta_1 s^3+\beta_1 s^2+\beta_2 s+\beta_3}{(s^3+\alpha_1 s^2+\partial_2 s+\alpha_3)}=\dfrac{N(s)}{D(s)}$

where $D(s)$ *and* $N(s)$ are coprime

writing $\hat{g}(s)$ as $\hat{g}(s)=\dfrac{(\beta_0-\alpha_1\beta_0)s^2+(\beta_2-\alpha_2\beta_0)s+\beta_2+\alpha_3\beta_0}{s^3+\alpha_1 s^2+\alpha_2 s+\alpha_3)}$, we can obtain a controllable canonical form realization of $\hat{g}(s)$

$$\underline{\dot{x}}=\begin{bmatrix}-\alpha_1 & -\alpha_2 & -\alpha_3\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\underline{x}+\begin{bmatrix}1\\0\\0\end{bmatrix}u$$

$$y=[\beta_1-\alpha_1\beta_0 \quad \beta_2-\alpha_2\beta_0 \quad \beta_3-\alpha_3\beta_0]\underline{x}+\beta_0 u$$

the Sylvester resultant of $D(s)$ *and* $N(s)$ is nonsingular because $D(s)$ *and* $N(s)$ are coprime , thus the state equation is observable .

7.9 verify theorem 7.7 fir $\hat{g}(s)=\dfrac{1}{(s+1)^2}$.对 $\hat{g}(s)=\dfrac{1}{(s+1)^2}$ 验定定理 7.7

verification : $\hat{g}(s)=\dfrac{1}{(s+1)^2}$ is a strictly proper rational function with degree 2, and it can be expanded into an infinite power series as

$$\hat{g}(s)=0\cdot s^{-1}+s^{-2}-2s^{-3}+3s^{-4}-4s^{-5}+5s^{-6}+\cdots s^{-1}$$

$$\rho T(1,1)=\rho([0])=0 \rho T(1,2)=\rho T(2,1)=1$$
$$\rho T(2,2)=\rho T(2+K,2+L)=2 \quad for \quad veyer \quad k,l=1,2\cdots$$

7.10 use the Markov parameters of $\hat{g}(s)=\dfrac{1}{(s+1)^2}$ to find on irreducible companion-form realization

用马尔科夫参数求 $\hat{g}(s)=\dfrac{1}{(s+1)^2}$ 的一个不可简约的有型实现

solution

$$\hat{g}(s)=0\cdot s^{-1}+s^{-2}+(-2)s^{-3}+3s^{-4}+(-4)s^{-5}+5s^{-6}+\cdots$$
$$\rho T(2,2)=\rho\begin{bmatrix}0 & 1\\ 1 & -2\end{bmatrix}=2$$

deg $\hat{g}(s)$ =2

$$\therefore \quad A=\widetilde{T}(2,2)T^{-1}(2,2)=\begin{bmatrix}1 & -2\\ -2 & 3\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & -2\end{bmatrix}^{-1}=\begin{bmatrix}1 & -2\\ -2 & 3\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}$$

$$\underline{b}=\begin{bmatrix}0\\ 1\end{bmatrix} \qquad \underline{c}=\begin{bmatrix}1 & 0\end{bmatrix}$$

the triplet $(A,\underline{b},\underline{c})$ is an irreducible com-panion-form realization

7.11 use the Markov parameters of $\hat{g}(s)=\dfrac{1}{(s+1)^2}$ to find an irreducible balanced-form realization

用 $\hat{g}(s)$ 的马尔科夫参数求它的一个不可简约的均衡型实现.

Solution : $T(2,2)=\begin{bmatrix}0 & 1\\ 1 & -2\end{bmatrix}$

Using Matlab I type

$$\begin{aligned}
&t=[0 \quad 1;1 \quad -2]tt=[1 \quad -2;-2 \quad 3]\\
&[k, \quad s, \quad l]=svd(t), \qquad sl=sqrt(s);\\
&o=k*sl;c=sl*l';\\
&A=inv(o)*\vartheta*inv(c)\\
&b=c(;,1),c=O(1,1)
\end{aligned}$$

This yields the following balanced realization

$$\underline{\dot{x}}=\begin{bmatrix}-1.7071 & 0.7071\\ -0.7071 & -0.2929\end{bmatrix}\underline{x}+\begin{bmatrix}0.5946\\ -0.5946\end{bmatrix}$$

$$y=\begin{bmatrix}-0.5946 & -0.5946\end{bmatrix}\underline{x}$$

7.12

Show that the two state equation $\underline{\dot{x}}=\begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix}\underline{x}+\begin{bmatrix}1\\ 0\end{bmatrix}u \quad y=\begin{bmatrix}2 & 2\end{bmatrix}\underline{x}$ and $\underline{\dot{x}}=\begin{bmatrix}2 & 0\\ -1 & -1\end{bmatrix}\underline{x}+\begin{bmatrix}1\\ 2\end{bmatrix}u \quad y=\begin{bmatrix}2 & 0\end{bmatrix}\underline{x}$ are realizations of $(2s+2)/(s^2-s-2)$, are they minimal realizations ? are they algebraically equivalent?

证明以上两状态方程都是 $(2s+2)/(s^2-s-2)$ 的突现,他们是否最小突现?是否代数等价?

Proof : $\dfrac{(2s+2)}{(s^2-s-2)}=\dfrac{2(s+1)}{(s+1)(s-2)}=\dfrac{2}{s-2}$

$$[2 \quad 2]\begin{bmatrix} s-2 & -1 \\ 0 & s-1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = [2 \quad 2]\begin{bmatrix} \frac{1}{s-2} & \frac{1}{(s-1)(s-2)} \\ 0 & \frac{1}{s-1} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{2}{s-2}$$

$$[2 \quad 0]\begin{bmatrix} s-2 & 0 \\ 1 & s+1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = [2 \quad 0]\begin{bmatrix} \frac{1}{s-2} & 0 \\ -\frac{1}{(s+1)(s-2)} & \frac{1}{s+1} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{2}{s-2}$$

thus the two state equations are realizations of $(2s+2)\big/(s^2-s-2)$ the degree of the transfer function is 1 , and the two state equations are both two-dimensional .so they are nor minimal realizations . they are not algebraically equivalent because there does not exist a nonsingular matrix p such that

$$P\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}P^{-1} = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}$$

$$P\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$[2 \quad 2]P^{-1} = [2 \quad 0]$$

7.13 find the characteristic polynomials and degrees of the following proper rational matrices

$$\hat{G}_1 = \begin{bmatrix} \frac{1}{s} & \frac{s+3}{s+1} \\ \frac{1}{s+3} & \frac{s}{s+1} \end{bmatrix} \qquad \hat{G}_1 = \begin{bmatrix} \frac{1}{(s+1)^2} & \frac{1}{(s+1)(s+2)} \\ \frac{1}{s+2} & \frac{1}{(s+1)(s+2)} \end{bmatrix} \qquad \hat{G}_3 = \begin{bmatrix} \frac{1}{(s+1)^2} & \frac{s+3}{s+2} & \frac{1}{s+5} \\ \frac{1}{s+2} & \frac{s+1}{s+4} & \frac{1}{s} \end{bmatrix}$$

note that each entry of $\hat{G}_s(s)$ has different poles from other entries

求正则有理矩阵$\hat{G}_1(s)$，$\hat{G}_2(s)$和$\hat{G}_3(s)$的特征多项式和阶数.

Solution : the matrix $\hat{G}_1(s)$ has $\frac{1}{s}$, $\frac{s+3}{s+1}$, $\frac{1}{s+3}$, $\frac{s}{s+1}$ and $\det\hat{G}_1(s)$=0 as its minors

thus the characteristic polynomial of $\hat{G}_1(s)$ is s(s+1)(s+3) and $\hat{G}_1(s)$=3 the matrix $\hat{G}_2(s)$ has

$\frac{1}{(s+1)^2}$, $\frac{1}{(s+1)(s+2)}$, $\frac{1}{s+2}$, $\frac{1}{(s+1)(s+2)}$. $\det\hat{G}_2(s) = \frac{-s^2-s+1}{(s+1)^3(s+2)^2}$ as

its minors , thus the characteristic polynomial of $\hat{G}_2(s) = (s+1)^3(s+2)^2$ *and* $\delta\hat{G}_2(s) = 5$

Every entry of $\hat{G}_3(s)$ has poles that differ from those of all other entries , so the characteristic

polynomial of $\hat{G}_3(s)$ is $s(s+1)^2(s+2)(s+3)^2(s+4)(s+5)$ and $\delta\hat{G}_3(s) = 8$

7.14 use the left fraction $\hat{G}(s)=\begin{bmatrix} s & 1 \\ -s & s \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ to form a generalized resultant as in (7.83), and then search its linearly independent columns in order from left to right ,what is the number of linearly independent N-columns ?what is the degree of $\hat{G}(s)$? Find a right coprime fraction lf $\hat{G}(s)$,is the given left fraction coprime?用 $\hat{G}(s)$ 的左分式构成(7.83)中所示的广义终结阵,然后从左到右找出其线性无关列.线性无关的 N-列的数目是多少> $\hat{G}(s)$ 的阶是多少?求出 $\hat{G}(s)$ 的一个既约右分式.题给的左分式是否既约?

Solution : $\hat{G}(s)=\begin{bmatrix} s & 1 \\ -s & s \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ -1 \end{bmatrix}=: D^{-1}(s)\overline{N}(s)$

Thus we have

$$\overline{D}(s)=\begin{bmatrix} s & 1 \\ -s & s \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}s$$

$$\overline{N}(s)=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

And the generalized resultant

$$S=\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix} \text{ rank s=5 ,}$$

the number of linearly independent N-colums is l. that is u=1,

$$null(s)=\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
$$-N_0 \quad D_0 \quad -N_1 \; D_1$$

So we have

$$D(S)=0+1\cdot S=S$$

$$N(S)=\begin{bmatrix} +1 \\ 0 \end{bmatrix}+\begin{bmatrix} 0 \\ 0 \end{bmatrix}s=\begin{bmatrix} +1 \\ 0 \end{bmatrix}$$

$$\hat{G}(s)=\begin{bmatrix} 1 \\ 0 \end{bmatrix}s^{-1}$$

$$\deg=\hat{G}(s)=u=1$$

the given left fraction is not coprime .

7.15 are all D-columns in the generalized resultant in problem 7.14 linearly independent .pf their LHS columns ?Now in forming the generalized resultant ,the coefficient matrices of $D(S)$ and $N(S)$ are arranged in descending powers of s , instead of ascending powers of s as in problem 7.14 .is it true that all D-columns are linearly independent of their LHS columns? Doe's he degree of $\hat{G}(s)$ equal the number of linearly independent N-columns ?does theorem 7.M4hold ?题 7.14 中的广义终结阵中是否所有的 D-列都与其 LHS 列线性无关?现将 $D(S)$ 和 $N(S)$ 的系数矩阵按 S 得降幂.排列构成另一种形式的广义终结阵.在这样的终结阵中是否所有的 D-列也与其 LHS 列线性无关? $\hat{G}(s)$ 的阶数是否等于线性无关 N-列的数目?定理 7.M4 是否成立?

Solution : because $D_1 \neq 0$, ALL THE D-columns in the generalized resultant in problem7.14 are linearly independent of their LHS columns

Now forming the generalized resultant by arranging the coefficient matrices of $\overline{D}(S)$ and $\overline{N}(S)$ in descending powers of s :

$$S = \begin{bmatrix} \overline{D}_1 & \overline{N}_1 & 0 & 0 \\ \overline{D}_0 & \overline{N}_0 & \overline{D}_1 & \overline{N}_1 \\ 0 & 0 & \overline{D}_0 & \overline{N}_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} rank \qquad S = 5$$

we see the $\overline{D}_0$ -column in the second colums block is linearly dependent of its LHS columns .so it is not true that all D-columns are linearly independent of their LHS columns .

the number of linearly independent $\overline{N}$ -columns is 2 and the degree of $\hat{G}(s)$ is I as known in problem 7.14 , so the degree of $\hat{G}(s)$ does not equal the number of linearly independent $\overline{N}$ -columns , and the theorem 7.M4 does not hold .

7.16, use the right coprime fraction of $\hat{G}(s)$ obtained in problem 7.14 to form a generalized tesultant as in (7.89). search its linearly independent rows in order from top to bottom , and then find a left coprime fraction of $\hat{G}(s)$ 用题 7.14 中得到的 $\hat{G}(s)$ 的既约右分式,如式(7.89)构造一个广义终结阵, 从上至下找出其线性无关行, 求出 $\hat{G}(s)$ 的一个既约左分式.

Solution : $\hat{G}(s) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} s^{-1} \qquad D(s) = s \qquad N(s) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

The generalized resultant

$$T=\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\\0&0&1\\0&1&0\\0&0&0\end{bmatrix} \qquad rank \quad T=3 \quad \gamma_1=1, \quad \gamma_2=0$$

$$mull(\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix})=[0 \quad 0 \quad 1]'$$

$$mull(\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\\0&1&0\end{bmatrix})=[-1 \quad 0 \quad 0 \quad 1]'$$

$$\therefore \quad [-\overline{N}_0 \quad \overline{D}_0 \quad -\overline{N}_0 \quad \overline{D}_0]=\begin{bmatrix}-1&0&0&0&1&0\\0&0&1&0&0&0\end{bmatrix}$$

$$\overline{D}(s)=\begin{bmatrix}0&0\\0&1\end{bmatrix}+\begin{bmatrix}1&0\\0&0\end{bmatrix}s=\begin{bmatrix}s&0\\0&1\end{bmatrix}$$

$$\overline{N}(s)=\begin{bmatrix}+1\\0\end{bmatrix}$$

thus a left coprime fraction of $\begin{bmatrix}s^2+0.5s & 0.5s\\ -0.5 & s-0.5\end{bmatrix}$ is $\hat{G}(s)=\begin{bmatrix}s&0\\0&1\end{bmatrix}^{-1}\begin{bmatrix}1\\0\end{bmatrix}$

7.17 find a right coprime fraction of $\hat{G}(s)=\begin{bmatrix}\frac{s^2+1}{s^3} & \frac{2s+1}{s^2}\\ \frac{s+2}{s^2} & \frac{2}{s}\end{bmatrix}$ snd then a minimal realization

求 $\hat{G}(s)$ 的一个既约右分式及一个最小实现]

solution : $\hat{G}(s)=\begin{bmatrix}\frac{s^2+1}{s^3} & \frac{2s+1}{s^2}\\ \frac{s+2}{s^2} & \frac{2}{s}\end{bmatrix}=:\overline{D}^{-1}(S)\overline{N}(S)$

where

$$\overline{D}(s)=\begin{bmatrix}s^3&0\\0&s^2\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}+\begin{bmatrix}0&0\\0&0\end{bmatrix}s+\begin{bmatrix}0&0\\0&1\end{bmatrix}s^2+\begin{bmatrix}1&0\\0&0\end{bmatrix}s^3$$

$$\overline{N}(s)=\begin{bmatrix}s^2+1 & s(2s+1)\\ s+2 & 2s\end{bmatrix}=\begin{bmatrix}1&0\\2&0\end{bmatrix}+\begin{bmatrix}0&1\\1&2\end{bmatrix}s+\begin{bmatrix}1&2\\0&0\end{bmatrix}s^2$$

the generalized resultant is

$$S=\begin{bmatrix}0&0&1&0&0&0&0&0&0&0&0&0\\0&0&2&0&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&1&&0&0&0&0\\0&0&1&2&0&0&2&0&0&0&0&0\\0&0&1&2&0&0&0&1&0&0&1&0\\0&1&0&0&0&0&1&2&0&0&2&0\\1&0&0&0&0&0&1&2&0&0&0&1\\0&0&0&0&0&1&0&0&0&0&1&2\\0&0&0&0&1&0&0&0&0&0&1&2\\0&0&0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0\end{bmatrix}$$

rank $s=9, \qquad \mu_1=2,\mu_2=1$

the monic null vectors of the submatrices that consist of the primary dependent $\overline{N}_2$-columns and $\overline{N}_1$-columns are , respectively

$$Z2=\begin{bmatrix}-2.5 & -2.5 & 0 & -0.5 & 0 & 0 & 0.5 & 1\end{bmatrix}'$$
$$Z1=\begin{bmatrix}-0.5 & -2.5 & 0 & -0.5 & -1 & -1 & 0.5 & 0 & 0 & 1\end{bmatrix}$$

$$\therefore \quad \begin{bmatrix}-N_0\\D_0\\-N_1\\D_1\\N_2\\D_2\end{bmatrix}=\begin{bmatrix}-0.5 & -2.5 & 0 & -0.5 & -1 & -1 & 0.5 & 0 & 0 & 0 & 1 & 0\\-2.5 & -2.5 & 0 & -0.5 & 0 & 0 & 0.5 & 1 & 0 & 0 & 0 & 0\end{bmatrix}'$$

$$D(s)=\begin{bmatrix}0 & 0\\-0.5 & -0.5\end{bmatrix}+\begin{bmatrix}0.5 & 0.5\\0 & 1\end{bmatrix}s+\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}s^2=\begin{bmatrix}s^2+0.5s & 0.5s\\-0.5 & s-0.5\end{bmatrix}$$

$$N(s)=\begin{bmatrix}0.5 & 2.5\\2.5 & 2.5\end{bmatrix}+\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}s=\begin{bmatrix}s+0.5 & 2.5\\s+2.5 & 2.5\end{bmatrix}$$

thus a right coprime fraction of $\hat{G}(s)$ is $\hat{G}(s)=\begin{bmatrix}s+0.5 & 2.5\\s+2.5 & 2.5\end{bmatrix}\begin{bmatrix}s^2+0.5s & 0.5s\\-0.5 & s-0.5\end{bmatrix}$

we define $H(s)=\begin{bmatrix}s^2 & 0\\0 & s\end{bmatrix} \qquad L(s)=\begin{bmatrix}s & 0\\1 & 0\\0 & 1\end{bmatrix}$

then we have

$$D(s)=\begin{bmatrix}1 & 0.5\\ 0 & 1\end{bmatrix}H(s)+\begin{bmatrix}0.5 & 0 & 0\\ 0 & -0.5 & -0.5\end{bmatrix}L(s)$$

$$N(s)=\begin{bmatrix}1 & 0.5 & 2.5\\ 1 & 2.5 & 2.5\end{bmatrix}L(s)$$

$$D_{hc}^{-1}=\begin{bmatrix}1 & 0.5\\ 0 & 1\end{bmatrix}^{-1}=\begin{bmatrix}1 & -0.5\\ 0 & 1\end{bmatrix}$$

$$D_{hc}^{-1}D_{ic}=\begin{bmatrix}1 & -0.5\\ 0 & 1\end{bmatrix}\begin{bmatrix}0.5 & 0 & 0\\ 0 & -0.5 & -0.5\end{bmatrix}=\begin{bmatrix}0.5 & 0.25 & 0.25\\ 0 & -0.5 & -0.5\end{bmatrix}$$

thus a minimal realization of $\hat{G}(s)$ is

$$\underline{\dot{x}}=\begin{bmatrix}-0.5 & -0.25 & -0.25\\ 1 & 0 & 0\\ 0 & 0.5 & 0.5\end{bmatrix}\underline{x}+\begin{bmatrix}1 & -0.5\\ 0 & 0\\ 0 & 1\end{bmatrix}\underline{u}$$

$$\underline{y}=\begin{bmatrix}1 & 0.5 & 2.5\\ 1 & 2.5 & 2.5\end{bmatrix}\underline{x}$$

8.1 given $\underline{\dot{x}} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 1 \end{bmatrix} \underline{x}$

find the state feedback gain $\underline{k}$ so that the feedback system has –1 and –2 as its eigenvalues .

compute $\underline{k}$ directly with using any equivalence transformation 对题给状态方程，不用等价交换，

直接求状态反馈增益$\underline{k}$使状态反馈系统的特征值为-1,-2.

Solution : introducing state feedback $u = r - \begin{bmatrix} k_1 & k_2 \end{bmatrix} \underline{x}$, we can obtain

$$\underline{\dot{x}} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \underline{x} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} r$$

the new A-matrix has characteristic polynomial

$$\begin{aligned} \Delta_f(s) &= (s - 2 + k_1)(s - 1 + 2k_2) - (-1 - 2k_1)(1 - k_2) \\ &= s^2 + (k_1 + 2k_2 - 3)s + (k_1 - 5k_2 + 3) \end{aligned}$$

the desired characteristic polynomial is $(s+1)(s+2) = s^2 + 3s + 2$,equating

$k_1 + 2k_2 - 3 = 3$ and $k_1 - 5k_2 + 3 = 2$, yields

$k_1 = 4 \quad and \quad k_2 = 1$ so the desired state feedback gain $\underline{k}$ is $\underline{k} = \begin{bmatrix} 4 & 1 \end{bmatrix}$

8.2 repeat problem 8.1 by using (8.13), 利用式(8.13)重解题 8.1.

solution :

$$\begin{aligned} &\Delta(s) = (s-2)(s-1) + 1 = s^2 - 3s + 1 \\ &\Delta_f(s) = (s+1)(s+2) = s^2 + 3s + 2 \\ &\underline{\bar{k}} = \begin{bmatrix} 3 - (-3) & 2 - 3 \end{bmatrix} = \begin{bmatrix} 6 & -1 \end{bmatrix} \\ &\overline{C}^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \qquad \overline{C} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \\ &D = \begin{bmatrix} \underline{b} & A\underline{b} \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix} \quad C^{-1} = \begin{bmatrix} -1/7 & 4/7 \\ 2/7 & -1/7 \end{bmatrix} \quad C \; is \; non\sin gular, \quad so \; (A, b) \; is \quad can\,trollable \end{aligned}$$

using (8.13)

$$\underline{k} = \underline{\bar{k}}\overline{C}C^{-1} = \begin{bmatrix} 6 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/7 & 4/7 \\ 2/7 & -1/7 \end{bmatrix} = \begin{bmatrix} 4 & 1 \end{bmatrix}$$

8.3 Repeat problem 8.1 by solving a lyapunov equation

用求解 lyapunov 方程的方法重解题 8.1.

solution : (A,b) is controllable

selecting $F=\begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix} and\ \underline{\bar{k}}=\begin{bmatrix}1 & 1\end{bmatrix}$ then

$$AT-TF=\underline{b}\underline{\bar{k}}\Rightarrow T=\begin{bmatrix}0 & \frac{1}{13}\\ 1 & \frac{9}{13}\end{bmatrix}\quad T^{-1}=\begin{bmatrix}-9 & 1\\ 13 & 0\end{bmatrix}$$

$$\underline{k}=\underline{\bar{k}}T^{-1}=\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}-9 & 1\\ 13 & 0\end{bmatrix}=\begin{bmatrix}4 & 1\end{bmatrix}$$

8.4 find the state foodback gain for the state equation $\dot{\underline{x}}=\begin{bmatrix}1 & 1 & -2\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}\underline{x}+\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}u$

so that the resulting system has eigenvalues –2 and –1+j1 . use the method you think is the simplest by hand to carry out the design 求状态反馈增益. 使题给状态方程的状态反馈系统具有-2 和-1+j1 特征值/

solution : (A,b) is controllable

$$\Delta(s)=(s-1)^3=s^3-3s^2+3s-1$$
$$\Delta_f(s)=(s+2)(s+1+j1)(s+1-j1)=s^3+4s^2+6s+4$$
$$\underline{\bar{k}}=\begin{bmatrix}4-(-3) & 6-3 & 4-(-1)\end{bmatrix}=\begin{bmatrix}7 & 3 & 5\end{bmatrix}$$
$$\bar{C}^{-1}=\begin{bmatrix}1 & -3 & 3\\ 0 & 1 & -3\\ 0 & 0 & 1\end{bmatrix}\qquad \bar{C}=\begin{bmatrix}1 & 3 & 6\\ 0 & 1 & 3\\ 0 & 0 & 1\end{bmatrix}$$
$$C=\begin{bmatrix}1 & -1 & -2\\ 0 & 1 & 2\\ 1 & 1 & 1\end{bmatrix}\quad C^{-1}=\begin{bmatrix}1 & 1 & 0\\ -2 & -3 & 2\\ 1 & 2 & -1\end{bmatrix}$$

Using(8.13) $\underline{k}=\underline{\bar{k}}\bar{C}C^{-1}\begin{bmatrix}7 & 3 & 5\end{bmatrix}\begin{bmatrix}1 & 3 & 6\\ 0 & 1 & 3\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\ -2 & -3 & 2\\ 1 & 2 & -1\end{bmatrix}=\begin{bmatrix}15 & 47 & -8\end{bmatrix}$

8.4 Consider a system with transfer function $\hat{g}(s)=\dfrac{(s-1)(s+2)}{(s+1)(s-2)(s+3)}$ is it possible to

change the transfer function to $\hat{g}_f(s)=\dfrac{(s-1)}{(s+2)(s+3)}$ by state feedback? Is the resulting system BIBO stable ?asymptotically stable?

是否能够通过状态反馈将传函 $\hat{g}(s)$ 变为 $\hat{g}_f(s)$?所得系统是否 BIBO 稳定?是否逐渐稳定?

Solution : $\hat{g}_f(s)=\dfrac{(s-1)}{(s+2)(s+3)}=\dfrac{(s-1)(s+2)}{s+2)^2(s+3)}$

We can easily see that it is possible to change $\hat{g}(s)$ to $\hat{g}_f(s)$ by state feedback and the resulting system is asymptotically stable and BIBO stable .

8.6 consider a system with transfer function $\hat{g}(s)=\dfrac{(s-1)(s+2)}{(s+1)(s-2)(s+3)}$ is it possible to change the transfer function to $\hat{g}_f(s)=\dfrac{1}{s+3}$ by state feedback ? is the resulting system BIBO stable? Asymptotically stable ?

能否通过状态反馈将传函 $\hat{g}(s)$ 变为 $\hat{g}_f(s)$?所得系统是否 BIBO 稳定?是否逐渐稳定?

Solution ; $\hat{g}_f(s)=\dfrac{1}{s+3}=\dfrac{(s-1)(s+2)}{(s-1)(s+2)(s+3)}$

It is possible to change $\hat{g}(s)$ to $\hat{g}_f(s)$ by state feedback and the resulting system is asymptotically stable and BIBO stable .. however it is not asymptotically stable .

8.7 consider the continuous-time state equation

$$\dot{\underline{x}}=\begin{bmatrix}1&1&-2\\0&1&1\\0&0&1\end{bmatrix}\underline{x}+\begin{bmatrix}1\\0\\1\end{bmatrix}u \quad y=\begin{bmatrix}2&0&0\end{bmatrix}\underline{x}$$

let $u=pr-\underline{k}\underline{x}$, find the feedforward gain p and state feedback gain $\underline{k}$ so that the resulting system has eigenvalues –2 and $-1\pm j1$ and will track asymptotically any step reference input

对题给连续时间状态方程, 令 $u=pr-\underline{k}\underline{x}$ 求前馈增益 p 和状态反馈增益 $\underline{k}$ 使所得系统特征值为-2 和 $-1\pm j1$, 并且能够渐进跟踪任意阶跃参考输入.

Solution :

$$\hat{g}(s)=\underline{c}(sI-A)\underline{b}=\begin{bmatrix}2 & 0 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{s-1} & \frac{1}{(s-1)^2} & \frac{1}{(s-1)^3}-\frac{1}{(s-1)^2} \\ 0 & \frac{1}{s-1} & \frac{1}{(s-1)^2} \\ 0 & 0 & \frac{1}{s-1}\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}=\frac{2s^2-8s+8}{s^3-3s^2+3s-1}$$

$$\Delta_f(s)=s^3+4s^2+6s-+4$$

$$\therefore \quad P=\frac{\alpha_3}{\beta_3}=\frac{4}{8}=0.5$$

$$\underline{k}=\begin{bmatrix}15 & 47 & -8\end{bmatrix}$$

(obtained at problem 8.4)

8.8 Consider the discrete-time state equation

$$\underline{\dot{x}}[k+1]=\begin{bmatrix}1 & 1 & -2\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}\underline{x}+\begin{bmatrix}1\\0\\1\end{bmatrix}u[k]$$

$$y[k]=\begin{bmatrix}2 & 0 & 0\end{bmatrix}\underline{x}[k]$$

find the state feedback gain so that the resulting system has all eigenvalues at x=0 show that for any initial state the zero-input response of the feedback system becomes identically zero for $k \geq 3$

对题给离散时间状态方程,求状态反馈增益使所得系统所有特征值都在 Z=0, 证明对任意初始状态,反馈系统的零输入响应均为零($k \geq 3$)

solution ; (A,b) is controllable

$$\Delta(z)=z^3-3s+3z-1$$

$$\Delta_f(z)=z^3$$

$$\underline{\bar{k}}=\begin{bmatrix}3 & -3 & 1\end{bmatrix}$$

$$\underline{k}=\bar{C}C^{-1}\underline{\bar{k}}=\begin{bmatrix}3 & -3 & 1\end{bmatrix}\begin{bmatrix}1 & 3 & 6\\0 & 1 & 3\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\-2 & -3 & 2\\1 & 2 & -1\end{bmatrix}=\begin{bmatrix}1 & 5 & 2\end{bmatrix}$$

The state feedback equation becomes

$$\underline{\dot{x}}[k+1]=\begin{bmatrix}1 & 1 & -2\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}-\begin{bmatrix}1\\0\\1\end{bmatrix}\begin{bmatrix}1 & 5 & 2\end{bmatrix}\underline{x}[k]+\begin{bmatrix}1\\0\\1\end{bmatrix}r[k]=\begin{bmatrix}0 & -4 & -4\\0 & 1 & 1\\-1 & -5 & -1\end{bmatrix}\underline{x}[k]+\begin{bmatrix}1\\0\\1\end{bmatrix}r[k]$$

$$y[k]=\begin{bmatrix}2 & 0 & 0\end{bmatrix}\underline{x}[k]$$

denoting the mew-A matrix as $\bar{A}$, we can present the zero-mput response of the feedback system as following $y_{zi}[k]=\underline{C}\bar{A}^k\underline{x}[0]$

compute $\overline{A}^{k}$

$$\overline{A}=Q\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}Q^{-1}$$

$$\overline{A}^{k}=Q\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}^{K}Q^{-1}$$

using the nilpotent property

$$\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}^{K}=0\quad for\quad k\geq 3$$

so we can readily obtain $y_{zi}[k]=\underline{C}0\underline{x}[0]=0$ for $k\geq 3$

consider the discret-time state equation in problem 8.8 let for $u[k]=pr[k]-\underline{k}\underline{x}[k]$ where p is a feedforward gain for the $\underline{k}$ in problem 8.8 find a gain p so that the output will track any step reference input , show also that y[k]=r[k] for $k\geq 3$,thus exact tracking is achieved in a finite number of sampling periods instead of asymptotically . this is possible if all poles of the resulting system are placed at z=0 ,this is called the dead-beat design

对题8.8中的离散时间状态方程,令$u[k]=pr[k]-\underline{k}\underline{x}[k]$,其中p 是前馈增益,对题8.8中所得$\underline{k}$,求一增益p 使输出能够跟踪任意阶跃参考输入,并证明$k\geq 3$时 y[k]=r[k].实际的跟踪是在无穷多采样周期中得到的,而不是渐进的. 当反馈系统的所有标点都被配置在Z=0时,这样的跟踪是可以做到的,这就是所谓的最小[拍设计.

Solution ;

$$\hat{g}(z)=\frac{2z^{2}-8s+8}{z^{3}-3s^{2}+3z-1}\quad \Delta(z)=z^{3}\quad u[k]=pr[k]-\underline{k}\underline{x}[k]$$

$$\hat{g}_{f}(z)=p\frac{z^{2}-8z+8}{z^{3}}$$

(A,b) is controllable all poles of $\hat{g}_{f}(z)$ can be assigned to lie inside the section in fig 8.3(b) under this condition , if the reference input is a step function with magnitude a , then the output y[k] will approach the constant $\hat{g}_{f}(1)\cdot a\quad as\quad k\to+\infty$

thus in order for y[k] to track any step reference input we need $\hat{g}_{f}(1)=1$

$\hat{g}_f(1) = 2p = 1 \Rightarrow p = 0.5$

the resulting system can be described as

$$\underline{x}[k] = \begin{bmatrix} 0 & -4 & -4 \\ 0 & 1 & 1 \\ -1 & -5 & -1 \end{bmatrix} \underline{x}[k] + \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \end{bmatrix} r[k]$$

$$= \overline{A}\underline{x}[k] + \overline{\underline{b}}r[k]$$

$$y[k] = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \underline{x}[k]$$

the response excited by r[k] is

$$y[k] = \underline{C}\overline{A}^K \underline{x}[0] + \sum_{M=0}^{K-1} \underline{C}\overline{A}^{K-1-m} \overline{\underline{b}}r[m]$$

$$as \quad we \quad know, \quad \overline{A}^K = 0 \quad for \quad k \geq 3, \quad so$$

$$y[k] = \underline{c}\overline{\underline{b}}r[k-1] + \underline{c}\overline{A}\overline{\underline{b}}r[k-2] + \underline{c}\overline{A}^2 \overline{\underline{b}}r[k-3]$$

$$= r[k-1] - 4r[k-2] + 4r[k-3] \quad for \quad k \geq 3$$

for any step reference input r[k]=a the response is

y[k]=(1-4+4) a =a =r[k]   for $k \geq 3$

8.10 consider the uncontrollable state equation $\dot{\underline{x}} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} u$ is it possible to find a gain $\underline{k}$ so that the equation with state feedback $u = r - \underline{k}\underline{x}$ has eigenvalues –2, -2 , -1 ,-1 ? is it possible to have eigenvalues –2, -2 ,–2 ,-1 ? how about –2 ,-2, –2 ,-2 ? is the equation stabilizable?对题给不可控状态方程,是否可以求出一增益$\underline{k}$使具有反馈$u = r - \underline{k}\underline{x}$的方程具有特征值-2. –2, -1, -1, ?特征值-2, -2, -2,-1?特征值-2,-2,-2,-2?该方程是否镇定?

Solution: the uncontrollable state equation can be transformed into

$$\begin{bmatrix} \dot{\overline{x}}_c \\ \dot{\overline{x}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -4 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \overline{x}_c \\ \overline{x}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$= \begin{bmatrix} \overline{A}_c & 0 \\ 0 & \overline{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \overline{x}_c \\ \overline{x}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \overline{b}_c \\ 0 \end{bmatrix} u$$

Where the transformation matrix is $P^{-1} = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & 4 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{bmatrix}$

$(\overline{A_C}, \underline{\overline{b}}_c)$ is controllable and $(\overline{A_{\overline{C}}})$ is stable , so the equation is stabilizable . the eigenvalue of $(\overline{A_{\overline{C}}})$,say –1 , is not affected by the state feedback ,while the eigenvalues of the controllable sub-state equation can be arbitrarily assigned in pairs .

so by using state feedback ,it is possible for the resulting system to have eigenvalues –2 ,-2 , -1, -1, also eigenvalues –2, -2, -2, -1.

It is impossible to have –2, -2, -2, -2, as it eigenvalues because state feedback can not affect the eigenvalue –1.

8.11 design a full-dimensional and a reduced-dimensional state estimator for the state equation in problem 8.1 select the eigenbalues of the estimators from $\{-3 \quad -2 \pm j2\}$

对题 8.1 中的状态方程，设计一全维状态估计器和一将维估计器.估计器的特征值从 $\{-3 \quad -2 \pm j2\}$中选取.

Solution : the eigenvalues of the full-dimensional state estimator must be selected as $-2 \pm j2$,

the A-, $\underline{b}-$ and $\underline{c-}$matxices in problem 8.1 are

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \underline{c} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

the full-dimensional state estimator can be written as $\dot{\underline{\hat{x}}} = (A - \underline{l}\underline{c}) = \underline{\hat{x}} + \underline{b}u + \underline{l}y$

let $\underline{l} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$ then the characteristic polynomial of $\dot{\underline{\hat{x}}} = (A - \underline{l}\underline{c})$ is

$$\begin{aligned} \Delta'(s) &= (s - 2 + l_1)(s - 1 + l_2) + (1 + l_2)(1 - l_1) \\ &= s^2 + (l_1 + l_2 - 3)s + 3 - 2l_1 - l_2 \\ &= (s+2)^2 + 4 = s^2 + 4s + 8 \\ \Rightarrow \quad & l_1 = -12 \qquad l_2 = 19 \end{aligned}$$

thus a full-dimensional state estimor with eigenvalues $-2 \pm j2$ has been designed as following

$$\dot{\underline{\hat{x}}} = \begin{bmatrix} 14 & 13 \\ -20 & -18 \end{bmatrix} \underline{\hat{x}} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u + \begin{bmatrix} -12 \\ 19 \end{bmatrix} y$$

designing a reduced-dimensional state estimator with eigenvalue –3 :

(1) select |x| stable matrix F=-3

(2) select |x| vector L=1, (F,L) IS controllable

(3) solve T=: TA-FT=$l\underline{c}$

$$T = \begin{bmatrix} t_1 & t_2 \end{bmatrix}$$

$$\text{let T}: \begin{bmatrix} t_1 & t_2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} + 3\begin{bmatrix} t_1 & t_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$\Rightarrow T = \begin{bmatrix} \frac{3}{21} & \frac{4}{21} \end{bmatrix}$$

(4) THE l-dimensional state estimutor with eigenvalue –3 is

$$\dot{z} = -3x + \frac{13}{21}u + y$$

$$\hat{\underline{x}} = \begin{bmatrix} 1 & 1 \\ \frac{5}{21} & \frac{4}{21} \end{bmatrix}^{-1} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} -4 & 21 \\ 5 & -21 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$$

8.12 consider the state equation in problem 8.1 .compute the transfer function from r to y of the state feedback system compute the transfer function from r to y if the feedback gain is applied to the estimated state of the full-dimensional estimator designed in problem 8.11 compute the transfer function from r to y if the feedback gain is applied to the estimated state of the reduced-dimensional state estimator also designed in problem 8.11 are the three overall transfer functions the same ?

对题 8.1 中的状态方程,计算状态反馈系统从 r 到 y 的传送函数.若将反馈增益应用到题 8.11 中所设计的全维状态估计器所估计的估计状态,计算从 r 到 y 的传送函数,若将反馈增益应用到题 8.11 中所设计的降维状态估计器的估计状态，计算从 r 到 y 的传送函数，这三个传送函数是否相同?

Solution

$$\dot{\underline{x}} = A\underline{x} + \underline{b}u \quad y = \underline{c}x \quad u = r - \underline{k}\underline{x}$$

$$(1) \quad \therefore \hat{y}_{x\to y} = \underline{c}(sI - A + \underline{b}\underline{k})^{-1}\underline{b}$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{s+2} & 0 \\ \frac{9}{(s+1)(s+2)} & \frac{1}{s+1} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{3s-4}{(s+1)(s+2)}$$

$$\dot{\underline{x}} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\underline{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\underline{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} r - \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 4 & 1 \end{bmatrix}\hat{\underline{x}}$$

$$= \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\underline{x} - \begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix}\hat{\underline{x}} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} r$$

(2) $$\dot{\hat{\underline{x}}} = \begin{bmatrix} 14 & 13 \\ -20 & -18 \end{bmatrix}\hat{\underline{x}} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u + \begin{bmatrix} -12 \\ 19 \end{bmatrix} y$$

$$= \begin{bmatrix} 14 & 13 \\ -20 & -18 \end{bmatrix}\hat{\underline{x}} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} r - \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 4 & 1 \end{bmatrix}\hat{\underline{x}} + \begin{bmatrix} -12 \\ 19 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix}\underline{x}$$

$$= \begin{bmatrix} 10 & 12 \\ -28 & -20 \end{bmatrix}\hat{\underline{x}} + \begin{bmatrix} -12 & -12 \\ 19 & 19 \end{bmatrix}\underline{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix}\underline{x}$$

$$\therefore \begin{bmatrix} \dot{\underline{x}} \\ \dot{\hat{\underline{x}}} \end{bmatrix} = \begin{bmatrix} 2 & 1 & -4 & -1 \\ -1 & 1 & -8 & -2 \\ -12 & -12 & 10 & 12 \\ 19 & 19 & -28 & -20 \end{bmatrix}\begin{bmatrix} \underline{x} \\ \hat{\underline{x}} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 1 \\ 2 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} \underline{x} \\ \hat{\underline{x}} \end{bmatrix}$$

$$\therefore \hat{g}_{r\to y} = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} s-2 & -1 & -4 & 1 \\ 1 & s-1 & -8 & 2 \\ -12 & 12 & 10 & -12 \\ 19 & -19 & 28 & s+20 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{3s^3+8s^2+8s-32}{s^4+7s^3+22s^2+32s+16} = \frac{(3s-4)(s^2+4s+8)}{(s+1)(s+2)(s^2+4s+8)} = \frac{3s-4}{(s+1)(s+2)}$$

$$\dot{\underline{x}}=\begin{bmatrix}2&1\\-1&1\end{bmatrix}\underline{x}+\begin{bmatrix}1\\2\end{bmatrix}u=\begin{bmatrix}2&1\\-1&1\end{bmatrix}\underline{x}-\begin{bmatrix}4&1\\8&2\end{bmatrix}\hat{\underline{x}}+\begin{bmatrix}1\\2\end{bmatrix}r-\begin{bmatrix}1\\2\end{bmatrix}r$$

$$=\begin{bmatrix}2&1\\-1&1\end{bmatrix}\underline{x}-\begin{bmatrix}4&1\\8&2\end{bmatrix}\hat{\underline{x}}\begin{bmatrix}-4&21\\5&-21\end{bmatrix}\begin{bmatrix}y\\z\end{bmatrix}+\begin{bmatrix}1\\2\end{bmatrix}r$$

$$=\begin{bmatrix}2&1\\-1&1\end{bmatrix}\underline{x}-\begin{bmatrix}-11\\-22\end{bmatrix}\begin{bmatrix}1&1\end{bmatrix}\underline{x}-\begin{bmatrix}63\\126\end{bmatrix}z+\begin{bmatrix}1\\2\end{bmatrix}r$$

(3) $$=\begin{bmatrix}13&12\\21&23\end{bmatrix}\underline{x}+\begin{bmatrix}63\\126\end{bmatrix}z+\begin{bmatrix}1\\2\end{bmatrix}r$$

$$\dot{z}=-3z+\frac{13}{21}(r-\begin{bmatrix}4&1\end{bmatrix}\begin{bmatrix}-4&21\\5&-21\end{bmatrix}\begin{bmatrix}y\\z\end{bmatrix})+\begin{bmatrix}1&1\end{bmatrix}\underline{x}$$

$$=\begin{bmatrix}\frac{164}{21}&\frac{164}{21}\end{bmatrix}\underline{x}-42z+\frac{13}{21}r$$

$$y=\begin{bmatrix}1&1\end{bmatrix}\underline{x}$$

$$\therefore\begin{bmatrix}\dot{\underline{x}}\\\dot{z}\end{bmatrix}=\begin{bmatrix}13&12&-63\\21&23&-126\\\frac{164}{21}&\frac{164}{21}&-42\end{bmatrix}\begin{bmatrix}\underline{x}\\z\end{bmatrix}+\begin{bmatrix}1\\2\\\frac{13}{21}\end{bmatrix}r$$

$$y=\begin{bmatrix}1&1&0\end{bmatrix}\begin{bmatrix}\underline{x}\\z\end{bmatrix}$$

$$\therefore\hat{g}_{r\to y}(s)=\begin{bmatrix}1&1&0\end{bmatrix}\begin{bmatrix}s-13&1-2&63\\21&s-23&126\\-\frac{164}{21}&-\frac{164}{21}&s+42\end{bmatrix}^{-1}+\begin{bmatrix}1\\2\\\frac{13}{21}\end{bmatrix}$$

$$=\frac{3s^2+5s-12}{s^3+6s^2+11s+6}=\frac{(3s-4)(s+3)}{(s+1)(s+2)(s+3)}=\frac{3s-4}{(s+1)(s+2)}$$

these three overall transfer functions are the same , which verifies the result discussed in section 8.5.

8.13 let $A=\begin{bmatrix}0&1&0&0\\0&0&1&0\\-3&1&2&3\\2&1&0&0\end{bmatrix}B=\begin{bmatrix}0&0\\0&0\\1&2\\0&2\end{bmatrix}$ find two different constant matrices k such that

(A-BK) has eigenvalues $-4\pm3j$ and $-5\pm4j$ 求两个不同的常数阵使(A-BK)具有特征值

$-4\pm3j$ 和 $-5\pm4j$

solution : (1) select a 4×4 matrix $F=\begin{bmatrix}-4 & -3 & 0 & 0\\ 3 & -4 & 0 & 0\\ 0 & 0 & -5 & -4\\ 0 & 0 & 4 & -5\end{bmatrix}$

the eigenvalues of F are $-4\pm 3j$ and $-5\pm 4j$

(2)if $\overline{k_1}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$, $(F,\overline{k_1})$ is observable

$$AT_1-T_1F=B\overline{K_1}\Rightarrow T_1=\begin{bmatrix}-0.0013 & -0.0059 & 0.0005 & -0.0042\\ -0.0126 & 0.0273 & -0.0193 & 0.0191\\ 0.1322 & -0.0714 & 0.1731 & -0.0185\\ 0.0006 & -0.0043 & 0.2451 & -0.1982\end{bmatrix}$$

$$K_1=\overline{K}_1T_1^{-1}=\begin{bmatrix}-606.2000 & -168.0000 & -14.2000 & -2.0000\\ 371.0670 & 119.1758 & 14.8747 & 2.2253\end{bmatrix}$$

(3)IF $\overline{K}_2=\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}$ $(F,\overline{k_2})$ is observable

$$AT_2-T_2F=B\overline{K_2}\Rightarrow T_2=\begin{bmatrix}-0.0046 & -0.0071 & 0.0024 & -0.0081\\ -0.0399 & 0.0147 & -0.0443 & 0.0306\\ 0.2036 & 0.0607 & 0.3440 & 0.0245\\ 0.0048 & -0.0037 & 0.2473 & 0.2007\end{bmatrix}$$

$$K_2=\overline{K}_2T_2^{-1}=\begin{bmatrix}-252.9824 & -55.2145 & -0.0893 & -3.2509\\ 185.5527 & 59.8815 & 7.4593 & 2.5853\end{bmatrix}$$