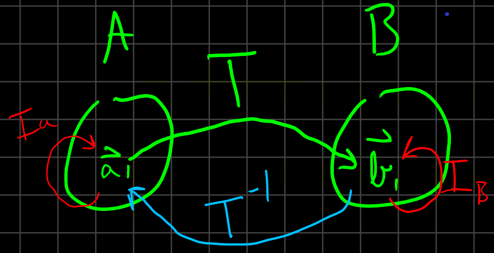
Section 8 Inverse of a transformation



$T : A \to B$

T is invertible
T has an inverse

T^{-1} Rules:

1. T^{-1} is unique
2. $\vec{a}_1$ maps to $\vec{b}_1$
3. $\vec{b}_1$ maps to only 1 $\vec{a}_1$

$$\underbrace{T^{-1}(\underbrace{T(\vec{a}_1)}_{\vec{b}_1})}_{\vec{a}_1} = \underbrace{I_A(\vec{a}_1)}_{\vec{a}_1}$$

$$\underbrace{T(\underbrace{T^{-1}(\vec{b}_1)}_{\vec{a}_1})}_{\vec{b}_1} = \underbrace{I_B(\vec{b}_1)}_{\vec{b}_1}$$

$I : A \to A \qquad I_A(\vec{a}_1) = \vec{a} =$

$T(\vec{a}) = [\quad]\vec{a}$

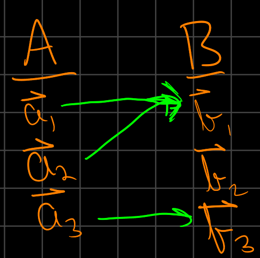
$T^{-1}(\vec{b}) = \underbrace{[\quad]}_{\text{unique}}\vec{b}$

Identity I transformation

$I : B \to B \; ; \; I_B(\vec{b}_1) = \vec{b}_1$

T is surjective ("onto") every $\vec{b}$ is being mapped to

T is injective/one-to-one:
only one $\vec{a}$ is mapping to a given $\vec{b}$



Not surjective ($\vec{b}_2$ is not mapped)
Not injective (2 $\vec{a}$'s are mapped to $\vec{b}_1$)

$\vec{a}_1 \to \vec{b}_1$
$\vec{a}_2 \to \vec{b}_2$
$a_3 \to \vec{b}_2$

✓ Surjective (every $\vec{b}$ is being mapped to)
Not injective

Quiz 45: Inverse of a transformation

Question 1:

Question: If the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is mapping vectors in A to vectors in B, then...

Answer choices:

A T is surjective

B T is injective

C T is both surjective and injective

D T is neither surjective nor injective

○ A

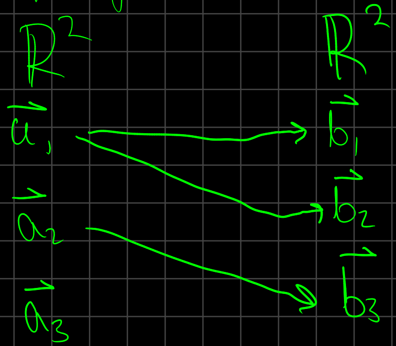
○ B

○ C

} Not surjective
($\vec{b}_3$) is unmapped

Not Injective
(2 a's map to $\vec{b}_1$)

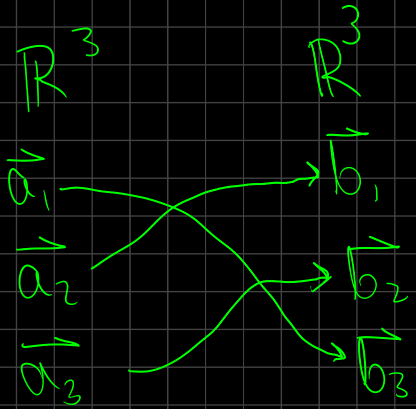
Q2 mapping of vectors A to B:



√ surjective
(all b's are mapped)

Not injective
(1 to 2) relation exists

Q3



(Both surjective & injective)
"onto" "1 to 1"

8.116) Invertibility of the matrix-vector product

$T(\vec{x}) = A\vec{x}$ A is the Transformation matrix $T: \mathbb{R}^n \to \mathbb{R}^n$

T can only be invertible when A is a square matrix

Square Eg

$$\begin{bmatrix}3\times3\end{bmatrix}\begin{bmatrix}3\times1\end{bmatrix} = \begin{bmatrix}3\times1\end{bmatrix}$$

$\mathbb{R}^3$ $\mathbb{R}^3$

eg: Rectangular Matrix

$$A = \begin{bmatrix}3 & 1 & 0\\ 0 & 2 & -2\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 1/3 & 0\\ 0 & 1 & -1\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & 1/3\\ 0 & 1 & -1\end{bmatrix} \quad \text{rref}(A)$$

$$\begin{array}{cccc} \text{pivot} & \text{pivot} & \text{free} & \\ x_1 & x_2 & x_3 & \end{array}$$

$$\text{nullspace} = \left[\begin{array}{ccc|c}1 & 0 & 1/3 & 0\\ 0 & 1 & -1 & 0\end{array}\right] \qquad \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_3\begin{bmatrix}-1/3\\1\\1\end{bmatrix}$$

$x_1 = -1/3\,x_3$ $x_2 = x_3$

$$N(A) = \text{span}\left(\begin{bmatrix}-1/3\\1\\1\end{bmatrix}\right)$$

Taller than wide matrix is not surjective

$$B = \begin{bmatrix}3 & 0\\ 1 & 2\\ 0 & -2\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 2\\ 3 & 0\\ 0 & -2\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 2\\ 0 & -6\\ 0 & -2\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}1 & 2\\ 0 & 1\\ 0 & -2\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 1\end{bmatrix}$$

$$C(B) = \text{span}\left(\begin{bmatrix}3\\1\\0\end{bmatrix}, \begin{bmatrix}0\\2\\-2\end{bmatrix}\right)$$

, can't span $\mathbb{R}^3$ with only 2 vectors, can't map to all vectors in $\mathbb{R}^3$

(This matrix T is not invertible)

$$\begin{bmatrix}-1/3\\1\\1\end{bmatrix} \quad \begin{bmatrix}-2/3\\2\\2\end{bmatrix} \quad \begin{bmatrix}0\\0\\0\end{bmatrix}$$

} All of these map to the zero vector

$\to \vec{0}$ thus this transformation is not injective. More cols than rows leaves free cols.

T: if $m > n$ then our transformation is not surjective

$m \times n$

if $n > m$ then our transformation is not injective

Quiz 46: Invertibility from the matrix-vector product

Q1

Say whether or not the transformation T is surjective or injective:

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \vec{x}$$

rref:

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Q3

$$T = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$m > n$, thus this matrix is <u>not</u> surjective

Section 8: Inverse transformations are linear

$[A \mid I] \rightarrow [I \mid A^{-1}]$

(1) Linear transformation T

(2) T is invertible

then $\Rightarrow$ T^{-1} is a linear transformation

$$T^{-1}(\vec{a}+\vec{b}) = T^{-1}(\vec{a}) + T^{-1}(\vec{b})$$: closed under addition

$$T^{-1}(c\vec{a}) = cT^{-1}(\vec{a})$$: closed under scalar multiplication

$$T(\vec{x}) = A\vec{x}$$

$$T^{-1}(\vec{x}) = A^{-1}\vec{x}$$

Let's say

$$T(\vec{x}) = A\vec{x}$$

$$T^{-1}(T(\vec{x})) = A^{-1}A\vec{x} = I\vec{x}$$

Quiz 47: Inverse transformations are linear

Q1 Find B^{-1}

$$B^{-1} = \begin{bmatrix} -3/10 & -1/10 \\ -1/5 & -2/5 \end{bmatrix}$$

$-\frac{5}{20} + -1/20$ $\qquad -\frac{2}{4} - -\frac{12}{4} = \frac{10}{4}$ $\qquad \frac{1}{4}\left(-\frac{2}{5}\right)$

$$B = \begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix}$$

given $[B \mid I]$

we can obtain rref to arrive at $[I \mid B^{-1}]$

$$\left[\begin{array}{cc|cc} -4 & 1 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{array}\right]$$

$-\frac{1}{4}R_1 \to R_1$

$$\left[\begin{array}{cc|cc} 1 & -1/4 & -1/4 & 0 \\ 2 & -3 & 0 & 1 \end{array}\right] \Rightarrow$$

$2R_1 - R_2 \to R_2$

$$\left[\begin{array}{cc|cc} 1 & -1/4 & -1/4 & 0 \\ 0 & 5/2 & -2/4 & -1 \end{array}\right] \Rightarrow$$

$2/5 R_2 \to R_2$

$$\left[\begin{array}{cc|cc} 1 & -1/4 & -1/4 & 0 \\ 0 & 1 & -1/5 & -2/5 \end{array}\right] \Rightarrow$$

$R_1 + 1/4 R_2 \to R_1$

$$\left[\begin{array}{cc|cc} 1 & 0 & -3/10 & -1/10 \\ 0 & 1 & -1/5 & -2/5 \end{array}\right]$$

Q2 Find M^{-1}

$3R_1 + R_2 \to R_2$
$4R_1 - R_3 \to R_3$

$-10/6 \qquad -\frac{5}{8} \cdot -\frac{18}{14} \qquad \frac{9}{3} - \frac{10}{3} = -\frac{1}{3}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 4 & -2 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 6 & 10 & 3 & 1 & 0 \\ 0 & 10 & 12 & 4 & 0 & -1 \end{array}\right]$$

$12 - \left(10 \cdot \frac{5}{3}\right)$

$\frac{36}{3} - \frac{50}{3} = -\frac{14}{3}$

$1/6 R_2 \to R_2$ $\qquad R_1 - 2R_2 \to R_1$ $\qquad R_3 - 10R_2 \to R_3$ $\qquad -3/14 R_3 \to R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5/3 & 1/2 & 1/6 & 0 \\ 0 & 10 & 12 & 4 & 0 & -1 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -1/3 & 0 & -1/3 & 0 \\ 0 & 1 & 5/3 & 1/2 & 1/6 & 0 \\ 0 & 0 & -14/3 & -1 & -5/3 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/14 & -3/14 & 1/14 \\ 0 & 1 & 0 & 1/7 & -3/7 & -5/14 \\ 0 & 0 & 1 & 3/14 & 5/14 & 3/14 \end{array}\right]$$

with above i can deduce correct answer given options. stop here

8.120) Matrix Inverses, and invertible and singular matrices

Matrix inverses is essentially matrix division

$\frac{3}{3} = 1$ $\quad 3 \cdot \frac{1}{3} = 1$

$x \cdot \frac{1}{x^{1}} = 1$

$x \cdot x^{-1} = 1$

$A \cdot A^{-1} = I$

determinate

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$|A| = ad - bc$

singular check:

$ad - bc = 0$

$ad = bc$

$\frac{a}{b} = \frac{c}{d}$

Formula for determinate of a 2x2 matrix

$|A| \neq 0$

$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

If $|A| != 0$ then the matrix is singular (Not invertible)

Example, find the determinate:

$A = \begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix}$; then $A^{-1} = \frac{1}{(-2 \cdot 1) - (0 \cdot 4)} \begin{bmatrix} 1 & -0 \\ -4 & -2 \end{bmatrix}$

$A^{-1} = \frac{-1}{2} \begin{bmatrix} 1 & 0 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} -1/2 & 0 \\ 2 & 1 \end{bmatrix}$

Quiz 48: Matrix inverses; & invertible & singular matrices

Q1) Are these matrices inverses of each other?

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \qquad A^{-1} = \frac{1}{2\cdot1 - 5\cdot3}\begin{bmatrix} 1 & -5 \\ -3 & 2 \end{bmatrix} = -\frac{1}{13}\begin{bmatrix} 1 & -5 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -1/13 & 5/13 \\ 3/13 & -2/13 \end{bmatrix}$$

$$B = \begin{bmatrix} -1/13 & 5/13 \\ 3/13 & -2/13 \end{bmatrix}$$

$B = A^{-1}$; thus A & B are inverse of each other

8.122) Solving Systems with inverse matrices

Example:

$7x + 5y = -4$

$-6x + 3y = -33$

$$\underbrace{\begin{bmatrix} 7 & 5 \\ -6 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} -4 \\ -33 \end{bmatrix}}_{\vec{b}}$$

$A\vec{x} = \vec{b}$

$A^{-1}A\vec{x} = A^{-1}\vec{b}$

$I\vec{x} = A^{-1}b$

$\vec{x} = A^{-1}\vec{b}$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{(7\cdot3)-(5\cdot-6)} \cdot \begin{bmatrix} 3 & -5 \\ 6 & 7 \end{bmatrix}$$

$$= \frac{1}{51}\begin{bmatrix} 3 & -5 \\ 6 & 7 \end{bmatrix} = A^{-1} = \begin{bmatrix} 3/51 & -5/51 \\ 6/51 & 7/51 \end{bmatrix}$$

$\frac{153}{51}$

231
24
255

Check:

$A^{-1}\vec{b} = \vec{x}$

$$\begin{bmatrix} 3/51 & -5/51 \\ 6/51 & 7/51 \end{bmatrix}\begin{bmatrix} -4 \\ -33 \end{bmatrix} = \begin{bmatrix} -12/51 + \frac{165}{51} \\ \frac{-24}{51} - \frac{231}{51} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{153}{51} \\ \frac{-255}{51} \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Quiz 49: Solving Systems with Inverse Matrices

Q1 Use an inverse matrix to find a solution to the system:

$$3x + 12y = 51$$
$$-2x + 6y = -6$$

let: $A = \begin{bmatrix} 3 & 12 \\ -2 & 6 \end{bmatrix}$; find A^{-1} where $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$A^{-1} = \frac{1}{(3\cdot 6)-(12\cdot -2)}\begin{bmatrix} 6 & -12 \\ 2 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{42}\begin{bmatrix} 6 & -12 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1/7 & -2/7 \\ 1/21 & 1/14 \end{bmatrix}$$

$$\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = \begin{bmatrix} 1/7 & -2/7 \\ 1/21 & 1/14 \end{bmatrix}\begin{bmatrix} 51 \\ -6 \end{bmatrix} = \begin{bmatrix} 51/7 + \frac{12}{7} \\ 51/21 + -6/14 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{63}{7} \\ 17/7 - 3/7 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \end{bmatrix} = \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Q2 Use an inverse matrix to find the solution to the system:

$$y - 5x = -15$$
$$3x + 8y = 95$$

$$A = \begin{bmatrix} -5 & 1 \\ 3 & 8 \end{bmatrix}$$

$$A^{-1}\vec{b} = \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} -8/43 & 1/43 \\ 3/43 & 5/43 \end{bmatrix}\begin{bmatrix} -15 \\ 95 \end{bmatrix} = \begin{bmatrix} \frac{-8}{43}\cdot -15 + \frac{1}{43}\cdot 95 \\ \frac{-45}{43} + \frac{5}{43}\cdot 95 \end{bmatrix} = \begin{bmatrix} \frac{120}{43} + \frac{95}{43} \\ \frac{-45}{43} + \frac{475}{43} \end{bmatrix}$$

$95 \cdot 5 = 475$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{(-5\cdot 8)-(1\cdot 3)}\begin{bmatrix} 8 & -1 \\ -3 & -5 \end{bmatrix} = \frac{1}{-43}\begin{bmatrix} 8 & -1 \\ -3 & -5 \end{bmatrix} = \begin{bmatrix} -8/43 & 1/43 \\ 3/43 & 5/43 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} \frac{215}{43} \\ \frac{430}{43} \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

Q3 Find solution to the system:

$$4x + 8y = -20$$
$$-12x - 3y = -66$$

$$A = \begin{bmatrix} 4 & 8 \\ -12 & -3 \end{bmatrix}$$

$66 \cdot 4 = 264$ $66 \cdot 8 = 528$ $240 + 264 = 504$ $84 \div 3 = 28$ $528 + 60 = 588$ $588 \div 84 = 7$

$$A^{-1} = \frac{1}{(4\cdot -3)-(8\cdot -12)}\begin{bmatrix} -3 & -8 \\ 12 & 4 \end{bmatrix} = \frac{1}{84}\begin{bmatrix} -3 & -8 \\ 12 & 4 \end{bmatrix}$$

$$\begin{bmatrix} \frac{-3}{84} & \frac{-8}{84} \\ \frac{12}{84} & \frac{4}{84} \end{bmatrix}\begin{bmatrix} -20 \\ -66 \end{bmatrix} = \begin{bmatrix} \frac{-3}{84}\cdot -20 + \frac{-8}{84}\cdot -66 \\ 12/84 \cdot -20 + 4/84 \cdot -66 \end{bmatrix} = \begin{bmatrix} \frac{60}{84} + \frac{528}{84} \\ -\frac{240}{84} - \frac{264}{84} \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \end{bmatrix} = \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

✓