7.9.5) Functions & Transformations

eg: $f(x) = x^2$: inputs & outputs

function : real #s

x	x^2
0	0
1	1
2	4
3	9
⋮	⋮

$f: x \mapsto x^2$

f: x "mapping" to x^2

eg: vector valued function

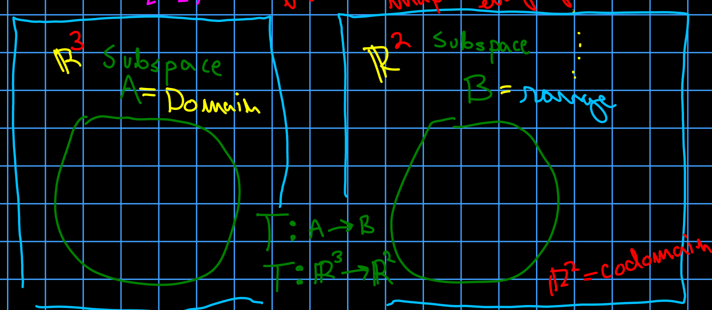
$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 3x_1 \end{bmatrix}$$

transformation : matrices

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$
$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \end{bmatrix}$
$\begin{bmatrix} -5 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 7 \\ -15 \end{bmatrix}$
⋮	⋮

* Sometimes B = codomain & range but not if it doesn't map to everything

$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$

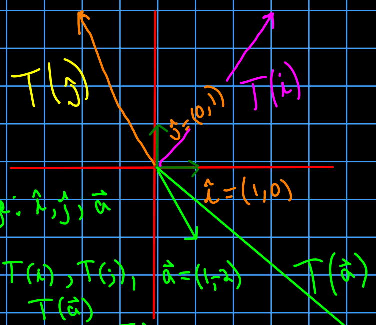


All of $\mathbb{R}^2$

All of $\mathbb{R}^3$

7.97 Transformation Matrixes and the image of the subset

- A 2x2 transformation matrix tells you where the standard basis vectors, $\hat{i}$ and $\hat{j}$ are moving



Preimage: $\hat{i}, \hat{j}, \vec{a}$

Image: $T(\hat{i}), T(\hat{j}), T(\vec{a})$ aka $T\hat{i}, T\hat{j}, T\vec{a}$ $\vec{a}=(1,-2)$

$$T\left(\begin{bmatrix}1\\-2\end{bmatrix}\right): \underset{2\times 2}{\begin{bmatrix}3 & -2\\2 & 4\end{bmatrix}}\underset{2\times 1}{\begin{bmatrix}1\\-2\end{bmatrix}} = \begin{bmatrix}(3\cdot 1)+(-2)(-2)\\(2\cdot 1)+(4\cdot -2)\end{bmatrix} = \underline{\begin{bmatrix}7\\-2\end{bmatrix}}$$

$$T = \begin{array}{c} \begin{array}{cc} \hat{i} & \hat{j} \end{array} \\ \begin{bmatrix}1 & -2\\0 & 4\end{bmatrix} \end{array}$$

$$T = \begin{bmatrix}3 & -2\\2 & 4\end{bmatrix}$$

$\mathbb{R}^3$

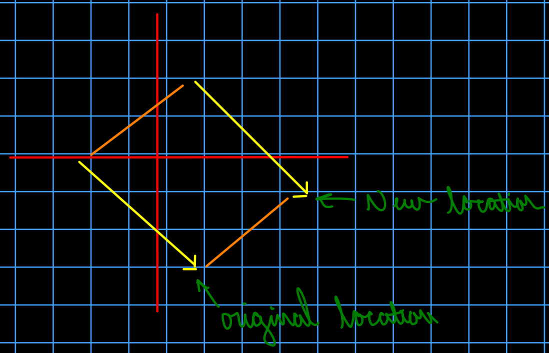
$$T = \begin{array}{c} \begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \end{array} \\ \begin{bmatrix}1 & 2 & 3\\1 & 2 & 3\\1 & 2 & 3\end{bmatrix} \end{array}$$

$\hat{i}: (1,0,0) \to (1,1,1)$

$\hat{j}: (0,1,0) \to (2,2,2)$

$\hat{k}: (0,0,1) \to (3,3,3)$

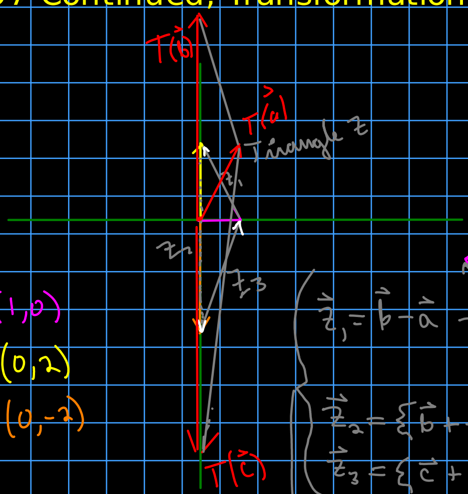
Section 7: Transformations



Example:

$$\left\{\begin{matrix}\text{reflect}\\ \text{stretch}\\ \text{Rotate}\end{matrix}\right\} \longrightarrow \left[\begin{matrix}\text{Transformation}\\ \text{Matrix}\end{matrix}\right]$$

7.97 Continued, Transformation matrices and the image of the subset



$\vec{a} = (1,0)$

$\vec{b} = (0,2)$

$\vec{c} = (0,-2)$

$$T = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

$$T\vec{a} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad T\vec{c} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \end{bmatrix}$$

$$T\vec{b} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

$\vec{z}_1 = \vec{b} - \vec{a} \rightarrow \{\vec{a} + t(\vec{b} - \vec{a}) \mid 0 \le t \le 1\}$

"start at $\vec{a}$, all points along vector z_1

$\vec{z}_2 = \{\vec{b} + t(\vec{c} - \vec{b}) \mid 0 \le t \le 1\}$

$\vec{z}_3 = \{\vec{c} + t(\vec{a} - \vec{c}) \mid 0 \le t \le 1\}$

→ Pre-image of the subset

$T(\vec{a} + t(\vec{b} - \vec{a}))$ ← Image of the subset

$T\vec{a} + T(\vec{b} - \vec{a})$

$T\vec{a} + tT(\vec{b} - \vec{a})$

$T\vec{a} + t(T\vec{b} - T\vec{a})$

Quiz 38: Transformation matrices and the image of the subset

Q1) If $\vec{a} = (-4, 2)$ becomes $\vec{b}$ after undergoing a transformation by matrix Q, find $\vec{b}$.

$$Q = \begin{bmatrix} 11 & 1 \\ 0 & -6 \end{bmatrix}$$

$$T\vec{a} \rightarrow Q\vec{a} \rightarrow \begin{bmatrix} 11 & 1 \\ 0 & -6 \end{bmatrix}\begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} -42 \\ -12 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} -42 \\ -12 \end{bmatrix}$$

Q2) What are the vertices of the transformation of the polygon given by

$\vec{p}_1 =$ (-2, 1)
$\vec{p}_2 =$ (1, 3)
$\vec{p}_3 =$ (2, -2)
$\vec{p}_4 =$ (-3, -1)

After transformation by matrix P?

$$P = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}$$

$$P\vec{p}_1 = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$P\vec{p}_2 = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$$

$$P\vec{p}_3 = \ldots \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$$

$$P\vec{p}_4 = \ldots \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

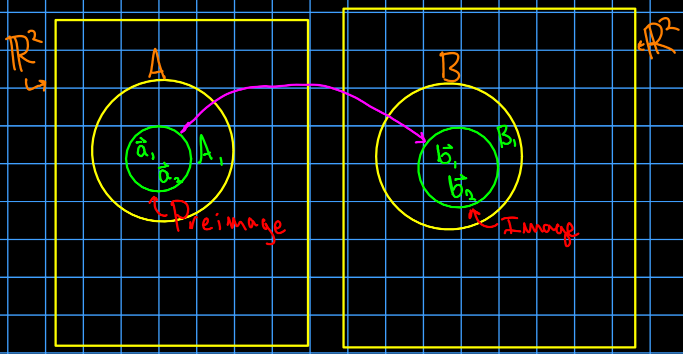
Q3) Transform triangle w/vertices:
(−3, 0)
(1, 2)
(1, −2)
by S

$$\begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \end{bmatrix}$$

$$\ldots \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$\ldots \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

7.99) Preimage, image, and the kernel



A & B are sets of vectors

Which particular vectors in subset A_1 map to B_1

eg:

$$\begin{bmatrix}4&0\\-2&3\end{bmatrix}\vec{a}_1 = b_1 = \begin{bmatrix}0\\0\end{bmatrix}$$

$$'' \quad '' \cdot \vec{a}_2 = b_2 = \begin{bmatrix}4\\1\end{bmatrix}$$

$$T = \begin{bmatrix}4&0\\-2&3\end{bmatrix}$$

$T = A \to B$

$T = \mathbb{R}^2 \to \mathbb{R}^2$

Work backwards:

$$\left[\begin{array}{cc|c}4&0&0\\-2&3&0\end{array}\right] \to \left[\begin{array}{cc|c}1&0&0\\0&3&0\end{array}\right] \to \left[\begin{array}{cc|c}1&0&0\\0&1&0\end{array}\right]$$

$$\left[\begin{array}{cc|c}4&0&4\\-2&3&1\end{array}\right] \to \left[\begin{array}{cc|c}1&0&1\\0&3&9\end{array}\right] \to \left[\begin{array}{cc|c}1&0&1\\0&1&3\end{array}\right]$$

KER(T) : All of the vectors that map to $\vec{0}$ in the image

$b_{1x} = 0$

$b_{1y} = 0$

$\vec{a}_1 = \begin{bmatrix}0\\0\end{bmatrix}$

$b_{2x} = 1$

$b_{2y} = 3$

$\vec{a}_2 = \begin{bmatrix}1\\1\end{bmatrix}$

Quiz 39) Preimage, image, and the kernel.

Q1) Find the preimage of A_1 of the subset B_1 under the transformation

$T : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad B_1 = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$

$T(\vec{x}) = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Augment T with vectors of B_1 to find a_1 and a_2 of A_1

$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 3 & -2 \end{array}\right]$ $\quad \left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 3 & 5 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & -2/3 \end{array}\right]$ $\quad \left[\begin{array}{cc|c} 1 & 0 & 14/3 \\ 0 & 1 & 5/3 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & 0 & 1/3 \\ 0 & 1 & -2/3 \end{array}\right]$

$a_1 = \begin{bmatrix} 1/3 \\ -2/3 \end{bmatrix}$ $\quad a_2 = \begin{bmatrix} 14/3 \\ 5/3 \end{bmatrix}$

$A_1 = \left\{ \begin{bmatrix} 1/3 \\ -2/3 \end{bmatrix}, \begin{bmatrix} 14/3 \\ 5/3 \end{bmatrix} \right\}$

Q2) Find the preimage A of the subset B_1 under the transformation

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$B_1 = \left\{ \begin{bmatrix} -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$

$T(\vec{x}) = \begin{bmatrix} -2 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Augment T with b_1, b_2 to find x_1, x_2:

$\left[\begin{array}{cc|c} -2 & 0 & -3 \\ 1 & 4 & 0 \end{array}\right]$ obtain rref(T)

$-\frac{1}{2}R_1 \to R_1$

$R_1 - R_2 \to R_2$

$\left[\begin{array}{cc|c} 1 & 0 & 3/2 \\ 0 & -4 & 3/2 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & 0 & 3/2 \\ 0 & 1 & -3/8 \end{array}\right]$

$\left\{ a_1 = \begin{bmatrix} 3/2 \\ -3/8 \end{bmatrix} \right.$

$\left[\begin{array}{cc|c} -2 & 0 & 2 \\ 1 & 4 & 2 \end{array}\right]$ obtain rref(A)

$\left[\begin{array}{cc|c} -2 & 0 & 2 \\ 1 & 4 & 2 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 1 & 4 & 2 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & -4 & -3 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 3/4 \end{array}\right]$

$A_1 = \left\{ \begin{bmatrix} 3/2 \\ -3/8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3/4 \end{bmatrix} \right\}$

Q3) Find the preimage of A_1 of the subset B_1 under the transformation

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$$B_1 = \left\{ \begin{bmatrix} 5 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\}$$

$$T(\vec{x}) = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A_1 = \left\{ \begin{bmatrix} 8/3 \\ -7/6 \end{bmatrix}, \begin{bmatrix} -5/3 \\ 2/3 \end{bmatrix} \right\}$$

Augment T with $\vec{b}_1$ & $\vec{b}_2$ to find $\vec{a}_1$ & $\vec{a}_2$

$$\left[\begin{array}{cc|c} 1 & -2 & 5 \\ 1 & 4 & -2 \end{array}\right]$$

$\frac{44}{6} =$

$$\left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & -6 & 7 \end{array}\right]$$

$\frac{30}{6} - 2\left(\frac{7}{6}\right)$

$\frac{16}{6} = \frac{8}{3}$

$$\left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 1 & -7/6 \end{array}\right]$$

$R_1 + 2R_2 \to R_1$

$$\left[\begin{array}{cc|c} 1 & 0 & 8/3 \\ 0 & 1 & -7/6 \end{array}\right]$$

$$a_1 = \begin{bmatrix} 8/3 \\ -7/6 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -2 & -3 \\ 1 & 4 & 1 \end{array}\right]$$

$R_1 - R_2 \to R_2$

$$\left[\begin{array}{cc|c} 1 & -2 & -3 \\ 0 & -6 & -4 \end{array}\right]$$

$-\frac{9}{3} + \frac{4}{3}$

$-\frac{1}{6}R_2 \to R_2$

$$\left[\begin{array}{cc|c} 1 & -2 & -3 \\ 0 & 1 & 2/3 \end{array}\right]$$

$3 + \frac{2}{3}$

$R_1 + 2R_2 \to R_1$

$$\left[\begin{array}{cc|c} 1 & 0 & -5/3 \\ 0 & 1 & 2/3 \end{array}\right]$$

$\frac{4}{3} + \frac{9}{3}$

$13/3$

$$a_2 = \begin{bmatrix} -5/3 \\ 2/3 \end{bmatrix}$$

Linear transformation as matrix-vector products

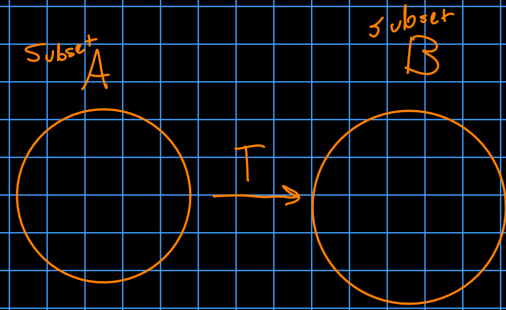
We have been dealing with linear transformations, and will continue to work with them.

① $T: \mathbb{R}^n \to \mathbb{R}^m$

Linear transformation properties:

② $T(\vec{a}+\vec{b}) = T\vec{a} + T\vec{b}$

③ $T(c\vec{a}) = cT(\vec{a})$



These 3 properties make a linear transformation

$$\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}\begin{bmatrix}a_1\\ a_2\end{bmatrix} = \begin{bmatrix}b_1\\ b_2\end{bmatrix}$$

Eg $\mathbb{R}^2 \to \mathbb{R}^3$:

$$T\left(\begin{bmatrix}a_1\\ a_2\end{bmatrix}\right) = \begin{bmatrix}-a_1 + 2a_2\\ a_2 - 3a_1\\ a_1 - a_2\end{bmatrix} \to \begin{bmatrix}-1 & 2\\ -3 & 1\\ 1 & -1\end{bmatrix}$$

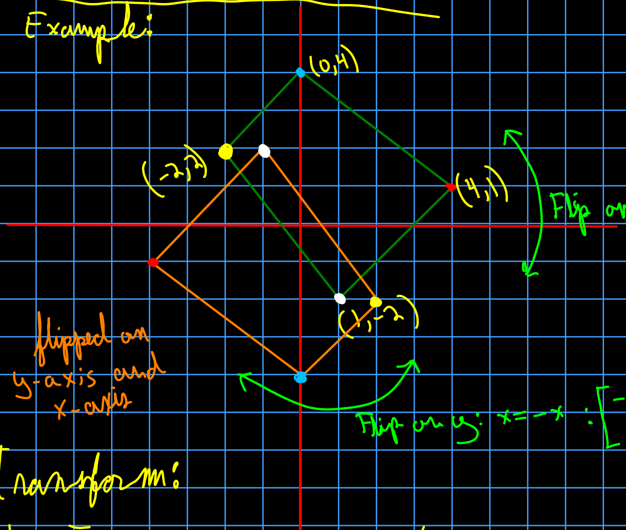
$$T\begin{bmatrix}1\\ 2\end{bmatrix} \to \begin{bmatrix}3\\ -1\\ -1\end{bmatrix}$$

$$T = \begin{bmatrix}-1 & 2\\ -3 & 1\\ 1 & -1\end{bmatrix}$$

Transform from $\mathbb{R}^2$: $\hat{i}: (1,0)$ $\hat{j}: (0,1)$

$$T\left(\begin{bmatrix}1\\ 0\end{bmatrix}\right) = \begin{bmatrix}-1\\ -3\\ 1\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\ 1\end{bmatrix}\right) = \begin{bmatrix}2\\ 1\\ -1\end{bmatrix}$$

Example:



Flip on y-axis: $y = -y$: $\begin{bmatrix}x\\ -y\end{bmatrix}$

flipped on y-axis and x-axis

Flip on y: $x = -x$: $\begin{bmatrix}-x\\ y\end{bmatrix}$

$$T\left(\begin{bmatrix}1\\ 0\end{bmatrix}\right) = \begin{bmatrix}-1\\ 0\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\ 1\end{bmatrix}\right) = \begin{bmatrix}0\\ -1\end{bmatrix}$$

$$\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}4\\ 1\end{bmatrix} = \begin{bmatrix}-4\\ -1\end{bmatrix} \qquad \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0\\ 4\end{bmatrix} = \begin{bmatrix}0\\ -4\end{bmatrix}$$

flipped rectangle

Transform:

1st: Flip across y-axis

2nd: Flip across x-axis

roll into 1 Transformation axis: $T = \begin{bmatrix}-x\\ -y\end{bmatrix}$

$$\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}-2\\ 2\end{bmatrix} = \begin{bmatrix}2\\ -2\end{bmatrix} \qquad \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1\\ -2\end{bmatrix} = \begin{bmatrix}-1\\ 2\end{bmatrix}$$

Quiz 40) Linear transformations as matrix-vector products

Q1) Use a matrix-vector product to reflect the square with vertices (-3, 2), (4,2), (4, -5) and (-3, -5) over the x-axis.

$T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ } flip y values to flip square on the x-axis

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -3 \\ -5 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

→ Answer

Q2) Use a matrix-vector product to double the width of a rectangle with vertices:
(3, -6), (3, 1), (-1, 1), (-1, -6)

"Doubling the width" means 2x, thus the transformation matrix is as follows:

$$T = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -6 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

$$\text{"} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

$$\text{"} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\text{"} \begin{bmatrix} -1 \\ -6 \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$$

Answer

Q3) Reflect the coords of the parallelogram over the y-axis, and compress it vertically by a factor of 3.
Given the vectors (1,1), (0, -4), (-4, -4), and (-3, 1)

invert x, $\frac{y}{3}$: gives us transformation matrix T:

$$T = \begin{bmatrix} -1 & 0 \\ 0 & 1/3 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1/3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1/3 \end{bmatrix}$$

$$\text{"} \begin{bmatrix} 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ -4/3 \end{bmatrix}$$

$$\text{"} \begin{bmatrix} -4 \\ -4 \end{bmatrix} = \begin{bmatrix} 4 \\ -4/3 \end{bmatrix}$$

$$\text{"} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1/3 \end{bmatrix}$$

Answer

7.103) Linear transformation as rotations

$\mathbb{R}^2$

$$Rot_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$\mathbb{R}^3$

$$Rot_{\theta \text{ around } x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$Rot_{\theta \text{ around } y} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$Rot_{\theta \text{ around } z} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Rotating_\theta(\vec{a} + \vec{b}) = rot_\theta(\vec{a}) + rot_\theta(\vec{b})$$

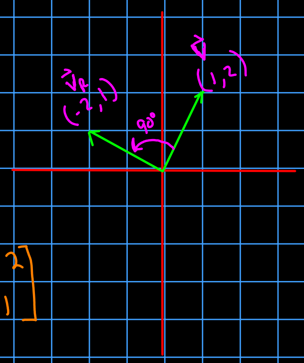
$$Rot_\theta(c\vec{a}) = C\,rot_\theta(\vec{a})$$

$\theta = 90°$

$$rot_{90°} = \begin{bmatrix} \cos_{90°} & -\sin_{90°} \\ \sin_{90°} & \cos_{90°} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$Rot\ 90°$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$



Quiz 41) Linear transformations as rotations

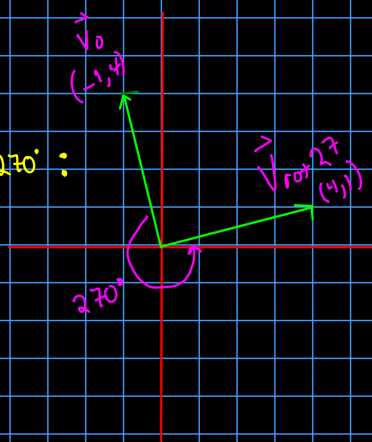
Q1) Find the rotation of x = (-1, 4) by an angle of θ = 270°

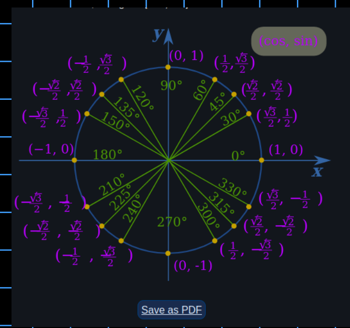
given formula:

$$rot_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

find for $\theta = 270°$:

$$rot_{270°} = \begin{bmatrix} \cos 270° & -\sin 270° \\ \sin 270° & \cos 270° \end{bmatrix}$$





Where, given an unit circle @ 270°:

$\cos(270°) = 0$

$\sin(270°) = -1$

$$rot_{270°} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{matrix} 4 \\ 1 \end{matrix}$$

Q2) find the rotation of $\vec{x} = (2, 0, -3)$ by an angle of $\theta = 60^\circ$ about the x-axis.

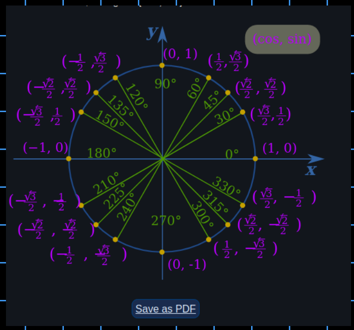
given the Transformation matrix T for rotation about the x-axis in $\mathbb{R}^3$:

given in unit circle: $\cos 60^\circ = 1/2$, $\sin 60 = \sqrt{3}/2$

$$\text{Rot}_{\theta \text{ around } x}: \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 60^\circ & -\sin\theta \\ 0 & \sin 60^\circ & \cos\theta \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{bmatrix}$$

$$T(\vec{x}) = \underset{3 \times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{bmatrix}} \underset{3 \times 1}{\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}} = \begin{bmatrix} 2 \\ \frac{3\sqrt{3}}{2} \\ -\frac{3}{2} \end{bmatrix}$$

Q3) Find the rotation of $\vec{x} = (-2, 3, -1)$ by an angle of $\theta = 225^\circ$ about the z-axis.



- Recall the transformation matrix for rotation about the z-axis:

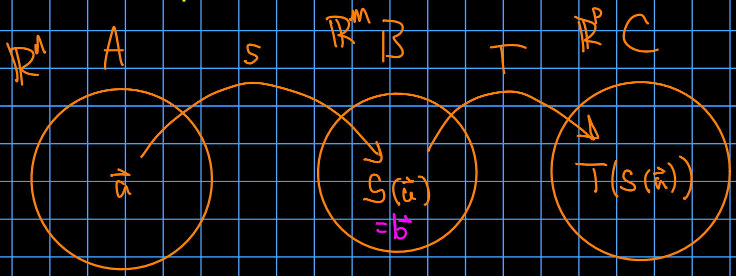
$$T_{rot\,\theta\,around\,z} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\cos 225^\circ = -\frac{\sqrt{2}}{2}$$

$$\sin 225^\circ = -\frac{\sqrt{2}}{2}$$

$$\begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} (-2)(-\sqrt{2}/2) + (3)(\sqrt{2}/2) \\ (-2)(-\sqrt{2}/2) - (3)(\sqrt{2}/2) \\ 0(-2) + 0(3) + (1)(-1) \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{5\sqrt{2}}{2} \\ -\sqrt{2}/2 \\ -1 \end{bmatrix}}_{\text{Answer}}$$

7.109) Compositions of linear transformations



$n = m = p$
or
$n \neq m \neq p$

$S: A \to B$
$T: B \to C$

"some" matrix A or B:

$m \times n$: $S(\vec{a}) = A\vec{a}$

$p \times m$: $T(\vec{b}) = B\vec{b}$

All together is the composition = $\circ$ symbol

$T \circ S(\vec{a})$
or
$T(S(\vec{a}))$

composition:
1) can write as matrix-vector products

$$T \circ S(\vec{a}) = T(S(\vec{a})) = T(A\vec{a}) = BA\vec{a} = \underline{C\vec{a}}$$

Eg

$$S(\vec{a}) = \begin{bmatrix} 2 & 0 \\ -1 & 4 \end{bmatrix}\vec{a} \qquad \vec{a}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$T(\vec{b}) = \begin{bmatrix} -1 & -2 \\ 3 & 0 \end{bmatrix}\vec{b}$$

$$\underline{T(S(\vec{a}))} = \begin{bmatrix} -1 & -2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -1 & 4 \end{bmatrix}(\vec{a})$$

$$= \begin{bmatrix} 0 & -8 \\ 6 & 0 \end{bmatrix}\vec{a}$$

$$= \begin{bmatrix} 0 & -8 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\underline{T(S(\vec{a})) = \begin{bmatrix} -16 \\ 18 \end{bmatrix}}$$

Quiz 44: Compositions of linear transformations

Q1) If S : X -> Y and T : Y -> Z , then what is $T(S(\vec{x}))$?

Where:

$$S(\vec{x}) = \begin{bmatrix} -x_1 + x_2 \\ 3x_1 \end{bmatrix} = \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{bmatrix} -1 & 1 \\ 3 & 0 \end{bmatrix} \end{array} \qquad \vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} 2x_1 - x_2 \\ -2x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & -2 \end{bmatrix}$$

$$T(S(\vec{x})) = \begin{bmatrix} 2 & -1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} -5 & 2 \\ -6 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} -5 & 2 \\ -6 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ -6 \end{bmatrix} = T(S(\vec{x})) = \vec{z}$$

Q2) If S : X -> Y and T : Y -> Z, the what is T(S($\vec{x}$))?

Where:

$$S(\vec{x}) = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & -4 \\ -1 & 1 & 0 \end{bmatrix} \qquad T(\vec{x}) = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 4 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix}$$

$$T(S(\vec{x})) = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & -4 \\ -1 & 1 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} -6 & 3 & 0 \\ -1 & 1 & 8 \\ -4 & 4 & 0 \end{bmatrix} \vec{x}$$

$$= \begin{bmatrix} -6 & 3 & 0 \\ -1 & 1 & 8 \\ -4 & 4 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} -24 \\ -6 \\ -24 \end{bmatrix} = T(S(\vec{x})) = \begin{bmatrix} -24 \\ -6 \\ -24 \end{bmatrix}$$

Q3 " " ": $\vec{x} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$

$$T(S(\vec{x})) = \begin{bmatrix} 0 & 3 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & -5 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 & 0 & 6 \\ 0 & 0 & 10 \\ 4 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -12 \\ -20 \\ 20 \end{bmatrix} = T(S(\vec{x}))$$