Q3

find $-\frac{1}{2}\cdot A + 3B$

$$A = \begin{bmatrix} 4 & -2 & 0 \\ -8 & 10 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 0 & -3 \\ 5 & -5 & 0 \end{bmatrix}$$

$$-\frac{1}{2}\cdot A = -\frac{1}{2}\begin{bmatrix} 4 & -2 & 0 \\ -8 & 10 & -5 \end{bmatrix} \qquad 3\cdot B = 3\begin{bmatrix} -1 & 0 & -3 \\ 5 & -5 & 0 \end{bmatrix}$$

$$-\frac{1}{2}\cdot A = \begin{bmatrix} -2 & 1 & 0 \\ 4 & -5 & \frac{5}{2} \end{bmatrix} \qquad = \begin{bmatrix} -3 & 0 & -9 \\ 15 & -15 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 0 \\ 4 & -5 & 5/2 \end{bmatrix} + \begin{bmatrix} -3 & 0 & -9 \\ 15 & -15 & 0 \end{bmatrix} = \begin{bmatrix} -5 & 1 & -9 \\ 19 & -20 & 5/2 \end{bmatrix}$$

Quiz 3:11 zero matrices

Draw a $O_{4\times 2}$ matrix:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Dimensions of $M + O = M$?

$$M = \begin{bmatrix} -5 & 0 & 3 & 1 \\ 8 & -15 & -4 & 7 \end{bmatrix} =$$

$M_{2\times 4}$

Section 3:32 Matrix Multiplication

→ $A\cdot B \neq B\cdot A$ (Not communicative)

→ Rule: The number of cols in 1^{st} matrix must equal # of rows in 2^{nd} matrix to multiply.

$2\cdot1 + 3\cdot -1 = -1$

~~$-1\cdot0 + 0\cdot1 = -1$~~

~~$-1\cdot0 + 0\cdot-1$~~

$-1\cdot0 + 0\cdot0 = -1$

$-1\cdot1 + 0\cdot-1 = -1$

$2\times4 \quad 4\times2 = 2\times2$ resulting matrix

$3\times2 \cdot 2\cdot4 = 3\times4$ resulting matrix

Properties of Matrices:

✗ commutative $AB \neq BA$

✓ Associative $(AB)C = A(BC)$

✓ Distributive $C(A-B) = CA - CB$

$(A-B)C = AC - BC$

$CA - CB \neq AC - BC$

✗ communicative ✗

$$\begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 0 \\ -3 & 12 \end{bmatrix}$$

$2\cdot1 + 0\cdot-1$

Quiz 12 matrix multiplication

Q1 Multiply A & B:

$$\begin{bmatrix} 5 & 2 \\ 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} 9 & 1 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} 5\cdot9+2\cdot6 & 5\cdot1+2\cdot-1 \\ 0\cdot9+-2\cdot6 & 0\cdot1+-2\cdot-1 \end{bmatrix} = \begin{bmatrix} 57 & 3 \\ -12 & 2 \end{bmatrix}$$

Q2 find $A\cdot B$ $\quad 2\times3 \quad 3\times2$

$$\begin{bmatrix} 7 & 2 & -4 \\ -5 & 10 & 3 \end{bmatrix} \cdot \begin{bmatrix} 7 & 1 \\ 7 & 2 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} 71 & -13 \\ 29 & 33 \end{bmatrix} ✓$$

$7\cdot7 + 2\cdot7 + -4\cdot-2 = 49 + 14 + 8$

$7 + 4 + -24 = -13$

$-35 + 70 + -6 =$

$-5 + 20 + 18$

Q3 find $AB + AC$:

$$\begin{bmatrix} 3 & -1 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 17 & 3 \\ -3 & 14 \end{bmatrix}$$

$$AB = \begin{bmatrix} 17 & 3 \\ -3 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 26 & 8 \end{bmatrix}$$

$$AC = \begin{bmatrix} 0 & -2 \\ 26 & 8 \end{bmatrix}$$

✓

$$AB + AC = \begin{bmatrix} 17 & 3 \\ -3 & 14 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 26 & 8 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 23 & 22 \end{bmatrix}$$

Quiz 13: Identity Matrixes

Q1 $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

Q2 $A_{2\times4}$: Dimensions of I given $I \cdot A$?

$I_{2\times2} \; A_{2\times4}$ ✓

I is a 2×2 matrix ✓

Q3 find I and product of $I \cdot A$ if

$$\underset{2\times3}{A} = \begin{bmatrix} -3 & 7 & 1 \\ 2 & 8 & 4 \end{bmatrix}$$

$$I_{2\times2} \cdot A_{2\times3}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \; ; \; I \cdot A = \begin{bmatrix} -3 & 7 & 1 \\ 2 & 8 & 4 \end{bmatrix}$$

3.37: The elimination Matrix:

$A^{-1} A = I$ (A must be a square matrix)

$A A^{-1} = I$ (inverse matrix IS the elimination matrix)

$EA = I$

Example, give rref:

$\begin{bmatrix} 2 & 4 \\ 0 & -3 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\frac{1}{2}R_1 \to R_1 \quad -\frac{1}{3}R_2 \to R_2 \quad \cancel{R_1 - \frac{3}{3}R_2} \to R_1$

$R_1 - 2 \cdot R_2 \to R_1$ $\quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} & \\ & \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1/2 & 2/3 \\ 0 & -1/3 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & \cancel{0}\,2-2 \\ 0 & \cancel{-4/3}\,0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$1/2 R_1 \longrightarrow R_1 \longrightarrow \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$-1/3 R_2 \longrightarrow R_2 \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1/3 \end{bmatrix}$$

$$R_1 - 2R_2 \longrightarrow R_1 \longrightarrow \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 0 \\ 0 & -1/3 \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \right) = A^{-1} \quad (E)$$

$$\cancel{\begin{bmatrix} 1 & 2/3 \\ 0 & -1/3 \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 2/3 \\ 0 & -1/3 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1/3 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & -1/3 \end{bmatrix} = \begin{bmatrix} 1/2 & 2/3 \\ 0 & -1/3 \end{bmatrix}$$

(Quiz 14)

Q1 Which elimination matrix accomplishes this row operation?

$$-2R_3 + R_1 \longrightarrow R_1$$

$$E = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q2 Which elimination matrix accomplishes the row operations

$$\tfrac{1}{2} \cdot R_2 \longrightarrow R_2 \quad \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$-R_2 + R_1 \longrightarrow R_1 \quad \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

(Queue the multiplactions in reverse)

$$\cancel{E = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1/2 \end{bmatrix}}$$

$$E = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix}$$

Q3 find the elimination matrix for A.

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad -R_1 \longrightarrow R_1 : \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad -3R_3 + R_2 \longrightarrow R_2$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad 2R_2 + R_1 \longrightarrow R_1 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$-R_1 \longrightarrow R_1 : \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$-3R_3 + R_2 \longrightarrow R_2 : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$2R_2 + R_1 \longrightarrow R_1 : \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix}1&2&0\\0&1&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}1&0&0\\0&1&-3\\0&0&1\end{bmatrix}\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)=E$$

$$\begin{bmatrix}1&2&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}-1&0&0\\0&1&-3\\0&0&1\end{bmatrix}=E$$

$$\begin{bmatrix}-1&2&-6\\0&1&-3\\0&0&1\end{bmatrix}=E$$

$$\begin{bmatrix}-1&2&-6\\0&1&-3\\0&0&1\end{bmatrix}=E$$

4:50 Linear Independence in 2 dimensions

$\mathbb{R}^2$: 2, L.I., in $\mathbb{R}^2$

$\mathbb{R}^3$: 3, L.I., in $\mathbb{R}^3$

$\mathbb{R}^n$: n, L.I., in $\mathbb{R}^n$

$$2\vec{v}=2\begin{bmatrix}2\\2\end{bmatrix}=\begin{bmatrix}4\\4\end{bmatrix}=\vec{w}$$

$\therefore \vec{v}, \vec{w}$ not L.I.

$$\underbrace{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}0\\6\end{bmatrix}}_{\text{L.I.}}$$

Equation to determine if L.I.

$$c_1\begin{bmatrix}a\\b\end{bmatrix}+c_2\begin{bmatrix}y\\z\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

break into 2 equations:

$$c_1 a + c_2 y = \emptyset$$

$$c_1 b + c_2 z = \emptyset$$

4:39 Matrices as vectors

→ A vector is defined by its length & direction.
(magnitude)

$\vec{a} = \begin{bmatrix}3\\4\end{bmatrix}$ column vector

$\vec{b} = [3, 4, 5]$ row vector

→ Column matrix: any # of rows, but only 1 col

→ Row matrix: any # of cols, but only 1 row

$$K = \begin{bmatrix} 1 & 0 & 0 & 3 \\ -2 & 1 & 1 & 4 \end{bmatrix}$$

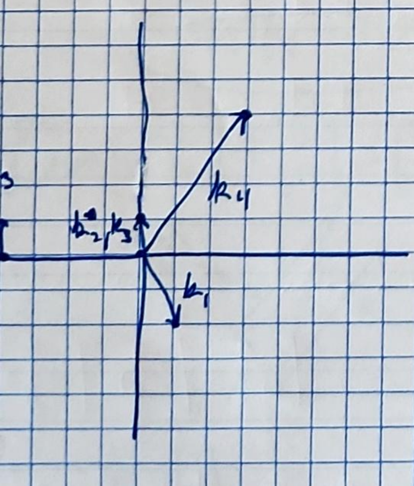
→ set of column vectors:

$k_1 = \begin{bmatrix}1\\-2\end{bmatrix}$ $k_2 = \begin{bmatrix}0\\1\end{bmatrix}$ $k_3 = \begin{bmatrix}0\\1\end{bmatrix}$ $k_4 = \begin{bmatrix}3\\4\end{bmatrix}$ $\Big\}$ 2 components, $\in$ exist in $\mathbb{R}^2$

→ set of row vecs:

$k_1 = [1, 0 \quad 0 \quad 3]$

$k_2 = [-2 \quad 1 \quad 1 \quad 4]$ $\Big\}$ 4 components $\in$ exist in $\mathbb{R}^4$



Quiz 15: Vectors

$$K = \begin{bmatrix} 1 & -1 & 1 & 4 \\ -2 & 1 & 0 & -1 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

Q1 How many row & col vectors are in Matrix K?

- 3 row vectors
- 4 col vectors

Q2 Name the column vectors of A:

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 4 & -2 & 8 & 4 \\ 5 & 6 & -2 & -3 \end{bmatrix}$$

$$\vec{A}_1 = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix},\ \vec{A}_2 = \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix},\ \vec{A}_3 = \begin{bmatrix} 3 \\ 8 \\ -2 \end{bmatrix},\ \vec{A}_4 = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$$

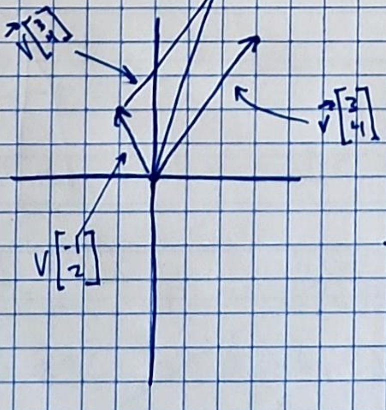
Q3 Name the row vectors of A:

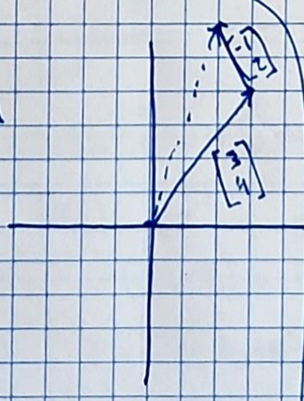
$$\vec{a}_1 = \begin{bmatrix} 2 & 1 & 3 & 1 \end{bmatrix};\ \vec{a}_2 = \begin{bmatrix} 4 & -2 & 8 & 4 \end{bmatrix};\ \vec{a}_3 = \begin{bmatrix} 5 & 6 & -2 & 3 \end{bmatrix}$$

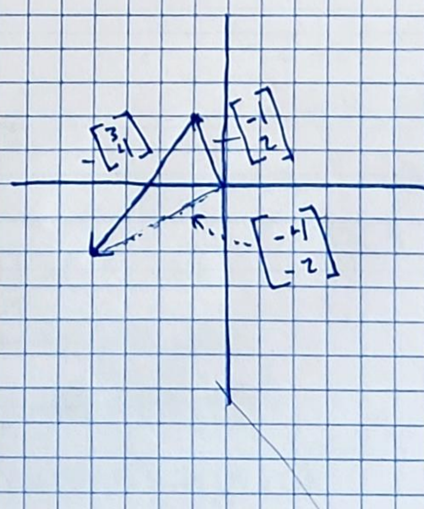
4:44 Vector Operations

Vector Addition: $\begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

Subtraction: $\begin{bmatrix} -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$







→ Vector Multiplication

Dot product

Dot product is the sum of product of the components

$\vec{a} = (-1, 2)$

$\vec{b} = (3, 4)$

$\vec{a} \cdot \vec{b} = (-1)(3) + (2)(4)$

$-3 + 8$

$= 5$

To express dot product as matrix: must multiply row matrix by column matrix:

$A \cdot B = \underset{1 \times 2}{\begin{bmatrix} -1 & 2 \end{bmatrix}} \underset{2 \times 1}{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} = \begin{bmatrix} 5 \end{bmatrix}$

$BA = \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \end{bmatrix}$

$\boxed{\text{Quiz 16: Vector Operations}}$

-1, 9

-11, 13

5, 9

Q1 find:

$\vec{a} + \vec{b} = (2, 2)$

$\vec{a} = (-2, 3)$

$\vec{b} = (4, -1)$

Q2 find $\vec{a} - \vec{b}$

$\vec{a} - \vec{b} = (-2, 3) - (4, -1) = (-6, 4)$

$2\vec{c} + 3\vec{a} = (-8, 11)$

$\vec{d} + 4\vec{b} = (19, 2)$

$(-8, 11) - (19, 2)$

$-27, 9$

Q3 Find the sum of $2\vec{c} + 3\vec{a} - \vec{d} + 4\vec{b} = (11, 4)$

$= 2\vec{c} = 2(-1, 1) = (-2, 2)$

$3\vec{a} = 3(-2, 3) = (-6, 9)$

$\vec{d} = (3, -2)$

$4\vec{b} = 4(4, -1) = (16, -4)$

$2\vec{c} + 3\vec{a}$

$(-2, 2) + (-6, 9) = (-8, 10)$

$\cancel{(-8, 10) + (3, -2) = (-5, 8)}$

$\cancel{(-5, 8) + (16, -4) = (11, 5)}$

$\vec{d} + 4\vec{b} = (3, -2) + (16, 4) = (19, 2)$

$2\vec{c} + 3\vec{a} = (-8, 11)$

$\vec{d} = (3, -2)$

$2\vec{c} + 3\vec{a} - \vec{d} = (-8, 11) - (3, -2)$

$= (-11, 13)$

$4\vec{b} = (16, -4)$

$2\vec{c} + 3\vec{a} - \vec{d} + 4\vec{b} = (-11, 13) + (16, -4) = \boxed{(5, 9)}$ ✓

4:46 Unit Vectors

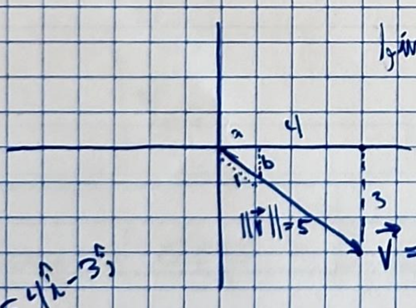
Every Vector:
- direction
- length (magnitude)

→ Unit vector: Any vector w/ a length of 1

$\vec{u}$ $\hat{u}$ ← unit vector has hat symbol $\hat{i}$ $\hat{j}$

---

finding the unit vector



$\vec{v} = 4\hat{i} - 3\hat{j}$

length symbol $= \| \;\; \|$

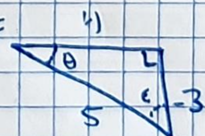
$\|\vec{v}\| = \sqrt{4^2 + (-3)^2}$

$= \sqrt{16 + 9}$

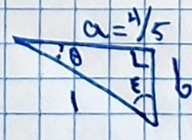
$= 5$

find a & b

big triangle =



little triangle



→ Thus:

$\frac{4}{5} = \frac{a}{1}$

$a = 4/5$

$-\frac{3}{5} = \frac{b}{1}$ $b = -3/5$

$\vec{u} = \left(\frac{4}{5}, -\frac{3}{5}\right)$: length of 1

Another way to solve:

formula: $\vec{U} = \frac{1}{\|\vec{V}\|}\vec{V}$

$$\vec{U} = \frac{1}{5}\begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

$$\vec{U} = \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}$$

Let:

$\vec{V} = (4, 3)$

Many ways to express:

$\vec{V} = \begin{bmatrix} 4 \\ -3 \end{bmatrix} = [4 \quad -3]$

$\vec{V} = 4\hat{i} - 3\hat{j}$

$$\vec{V} = (1, 2, 3) = (1\cdot\hat{i} + 2\cdot\hat{j} + 3\cdot\hat{k})$$

$$\|\vec{V}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$

$$\hat{U} = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\hat{U} = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}$$

Basis Vectors:

$\mathbb{R}^2$ $\hat{i}, \hat{j}$	$\mathbb{R}^3$ $\hat{i}, \hat{j}, \hat{k}$
2d plane	3d plane

$\rightarrow$ Standard basis vectors

In $\mathbb{R}^2$: $i = (1, 0)$

$j = (0, 1)$

In $\mathbb{R}^3$: $i = (1\ 0\ 0)$

$j = (0\ 1\ 0)$

$k = (0\ 0\ 1)$

Quiz 17: Unit Vectors & Basis Vectors

Q1 Find the unit vector in the direction of $\vec{a} = (5, -9)$

$$\|\vec{a}\| = \sqrt{25+81}$$

$$\|\vec{a}\| = \sqrt{106}$$

$$\hat{u} = \frac{1}{\|\vec{a}\|} \cdot \vec{a} = \frac{1}{\sqrt{106}} \begin{bmatrix} 5 \\ -9 \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} \frac{5}{\sqrt{106}} \\ \frac{-9}{\sqrt{106}} \end{bmatrix}$$

Q2 Find the unit vector in the direction of $\vec{v} = (2, -1, 4)$

$$\|\vec{v}\| = \sqrt{2^2 + (-1)^2 + (4)^2}$$

$$= \sqrt{4+1+16}$$

$$\|\vec{v}\| = \sqrt{21}$$

$$\vec{u} = \frac{1}{\sqrt{21}} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} \frac{2}{\sqrt{21}} \\ \frac{-1}{\sqrt{21}} \\ \frac{4}{\sqrt{21}} \end{bmatrix}$$

Q3 Represent $\vec{x} = (-2, 8, -4)$ with the standard basis vectors.

$$\vec{x} = -2\hat{i} + 8\hat{j} - 4\hat{k}$$

(4.48) Linear Combinations & Span (Done on-line)

4.50 continued):

Try:

$$c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}0\\6\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$c_1 1 + c_2 \cdot 0 = 0$

$c_1 = 0$

$c_1 1 + c_2 6 = 0$

$c_2 = 0/6$

$c_2 = 0 \therefore$ L.I.

Try:

$$c_1\begin{bmatrix}2\\2\end{bmatrix} + c_2\begin{bmatrix}4\\4\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$c_1 2 + c_2 4 = 0$ $\quad c_1 + 2c_2 = 0$

$\cancel{c_1 2 + c_2 4 = 0}$ $\quad c_1 = -2c_2$

$\cancel{c_2 = -\frac{1}{2}c_1}$

∞ combinations can make this True. $\therefore$ Not L.I.

$c_1 = -2$, $c_2 = 1$

$-2 = -2 \cdot 1 \rightarrow$ one example

Try:

unnecessary vector to illustrate L.I. in $\mathbb{R}^2$

$$c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}0\\6\end{bmatrix} + c_3\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$c_1 1 + 2c_3 = 0 \longrightarrow c_1 = -2c_3$

$c_1 + 6 \cdot c_2 + 3c_3 = 0 \rightarrow c_1 = -6c_2 - 3c_3$

$-2c_3 = -6c_2 - 3c_3$

$c_3 = -6c_2$

$c_2 = 1$; $c_3 = -6$

$c_2 = 2$; $c_3 = -12$

Quiz 19 Linear Independence in 2d.

Q1. (A) A set of 2-d vectors is Linearly Independent

Q2 Which vector set is L.I.?

(A) $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Try: $c_1 + 3 \cdot c_2 = 0$

$c_1 = -3c_2$

Let $c_1 = 3$: $3 = -3c_2$

$c_2 = -1$

∞ examples, thus not L.I.

(B) $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$

Try:

$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$c_1 + 3c_2 = 0$

$c_1 = -3c_2$

$2c_1 - 6c_2 = 0$

$-6c_2 - 6c_2 = 0$

$-12c_2 = 0$

$c_2 = 0$

$c_1 = -3 \cdot 0$

$c_1 = 0$

both are $\emptyset$

Thus L.I. ✓

Q2 Which vector set is L.I.?

Try D:

$$\left[\begin{array}{ccc|c} 7 & 3 & -6 & 0 \\ -1 & 2 & 4 & 0 \\ 4 & -1 & 2 & 0 \end{array}\right]$$

$-R_2 \leftrightarrow R_1$

$$\left[\begin{array}{ccc|c} 1 & -2 & -4 & 0 \\ 7 & 3 & -6 & 0 \\ 4 & -1 & 2 & 0 \end{array}\right]$$

$R_2 - 7R_1 \rightarrow R_2$ & $R_3 - 4 \cdot R_1 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & -2 & -4 & 0 \\ 0 & 17 & 22 & 0 \\ 0 & 7 & 18 & 0 \end{array}\right]$$

$\frac{1}{17} \cdot R_2 \rightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & -2 & -4 & 0 \\ 0 & 1 & 22/17 & 0 \\ 0 & 7 & 18 & 0 \end{array}\right]$$

$2R_2 + R_1 \rightarrow R_1$ & $R_3 - 7R_2 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & -24/17 & 0 \\ 0 & 1 & 22/17 & 0 \\ 0 & 0 & 152/17 & 0 \end{array}\right]$$

This is correct

Final steps:

$\frac{17}{152} R_3 \rightarrow R_3$

$R_2 - \frac{22}{17} R_3 \rightarrow R_2$

$R_1 + \frac{24}{17} R_3 \rightarrow R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & -24/17 & 0 \\ 0 & 1 & 22/17 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

D is L.I. $\quad c_1 = 0$; $c_2 = 0$; $c_3 = 0$

$18 \times 17 = 126 + 180 = 306$

$154 \div 17$: 153

$34 \times 2 = 68$

$\frac{44}{17} - \frac{68}{17} = -24/17$

$22 \times 7 = 154$

$18 - \frac{154}{17}$

$306 - 154 = 152$

(Q3) Which vector set is L.I.?

(A) $-3C_1 + 9C_2 - C_3 = \phi \longrightarrow C_3 = -3C_1 + 9C_2$

$5C_1 - 15C_2 + 6C_3 = 0$

$5C_1 - 15C_2 + 6(-3C_1 + 9C_2) = 0$

$5C_1 - 15C_2 - 18C_1 + 54C_2 = 0$

$-13C_1 = -39C_2$

$C_1 = 3C_2$

Many ∞ examples to make this true, thus Not L.I.

- Short Cut: 3 vectors for $\mathbb{R}^2$, unneeded 3^{rd} vector ϕ L.I.

(D) $\vec{a} = \begin{bmatrix}-3\\5\end{bmatrix}$, $\vec{b} = \begin{bmatrix}9\\15\end{bmatrix}$

Try:

$-3C_1 + 9C_2 = 0 \longrightarrow C_1 = 3C_2$

$5C_1 + 15C_2 = 0$

$\downarrow$ sub. $C_1 = 3C_2$

$15C_2 + 15C_2 = 0$

$C_2 = 0/30$

$C_2 = \phi$

$\downarrow$ sub $C_2 = \phi$

$-3C_1 + 9(0) = 0$

$-3C_1 = 0$

$C_1 = 0$

→ both are equal to zero and this is the only solution, thus this set of vectors is L.I.

4.52: Linear Independence in 3 dimensions

In order to span any space:

$\longrightarrow \mathbb{R}^n$: n vectors ; Linear Independence, in $\mathbb{R}^n$

Example:

Is the set of vectors L.I?:

$$\begin{bmatrix}1\\1\\1\end{bmatrix} \begin{bmatrix}0\\-3\\2\end{bmatrix} \begin{bmatrix}1\\0\\3\end{bmatrix}$$

*Figure 1:

$$C_1\begin{bmatrix}1\\1\\1\end{bmatrix} + C_2\begin{bmatrix}0\\-3\\2\end{bmatrix} + C_3\begin{bmatrix}1\\0\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

→ Solve using a matrix:

$$\left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\1 & -3 & 0 & 0\\1 & 2 & 3 & 0\end{array}\right]$$

$R_2 - R_1 \longrightarrow R_2$ & $R_3 - R_1 \longrightarrow R_3$

$$\left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\0 & -3 & -1 & 0\\0 & 2 & 2 & 0\end{array}\right]$$

$-\frac{1}{3}R_2 \longrightarrow R_2$

$$\left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\0 & 1 & 1/3 & 0\\0 & 2 & 2 & 0\end{array}\right]$$

$R_3 - 2R_2 \longrightarrow R_3$

$$\left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\0 & 1 & 1/3 & 0\\0 & 0 & 4/3 & 0\end{array}\right]$$

$\frac{3}{4}R_3 \longrightarrow R_3$

$$\left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\0 & 1 & 1/3 & 0\\0 & 0 & 1 & 0\end{array}\right]$$

$R_1 - R_3 \longrightarrow R_1$

& $R_2 - \frac{1}{3}R_3 \longrightarrow R_2$

~~1 0 0 0~~

~~0 0 0~~

$$\begin{array}{ccc} C_1 & C_2 & C_3 \end{array}$$
$$\left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{array}\right]$$

$\left.\begin{array}{l}C_1 = 0\\C_2 = 0\\C_3 = 0\end{array}\right\}$ → only combination that makes figure 1 True. Thus this vector set is L.I. These 3 vectors span $\mathbb{R}^3$

Quiz 20: Linear Independence in 3 dimensions

Q1 Which vector set is L.I.?

(A)
$$\begin{array}{ccc} c_1 & c_2 & c_3 \end{array}$$
$$\left[\begin{array}{ccc|c} 4 & -2 & -10 & 0 \\ 2 & 1 & -5 & 0 \\ 2 & 1 & 5 & 0 \end{array}\right]$$

$-5 - -\frac{10}{4}$

$-\frac{20}{4} + \frac{10}{4} = \cancel{30/4}\ -\frac{10}{4}$

$\frac{20}{4} + \frac{10}{4} = \frac{30}{4}$

$\frac{1}{4}R_1 \longrightarrow R_1$

$$\left[\begin{array}{ccc|c} 1 & -1/2 & -5/4 & 0 \\ 2 & 1 & -5 & 0 \\ 2 & 1 & 5 & 0 \end{array}\right] \longrightarrow$$

~~Notice: Row 2 & 3 are same vector. Thus this set of vectors~~

$R_2 - 2R_1 \longrightarrow R_2$

$R_3 - 2R_1 \longrightarrow R_3$

~~$1 \quad 1/2 \quad -5/4 \quad | \quad 0$~~

~~$0 \quad 2 \quad -10/4 \quad 30/4 \quad -10/4 \quad | \quad 0$~~

$$\left[\begin{array}{ccc|c} 1 & 1/2 & -5/4 & 0 \\ 0 & 2 & -10/4 & 0 \\ 0 & 2 & 30/4 & 0 \end{array}\right]$$

$\frac{1}{2}R_2 \longrightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & 1/2 & -5/4 & 0 \\ 0 & 1 & -5/4 & 0 \\ 0 & 2 & 30/4 & 0 \end{array}\right]$$

$-\frac{10}{4} + \frac{5}{4}$

$R_1 - 1/2 R_2 \longrightarrow R_1$ & ~~$R_3 + 6R_2 \longrightarrow R_3$~~

$R_3 - 2R_2 \longrightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & -5/8 & 0 \\ 0 & 1 & -5/4 & 0 \\ 0 & 0 & 10 & 0 \end{array}\right]$$

$\frac{30}{4} + \frac{10}{4}$

$\frac{1}{10}R_3 \longrightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & -5/8 & 0 \\ 0 & 1 & -5/4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$R_2 + \frac{5}{4}R_3 \longrightarrow R_2$

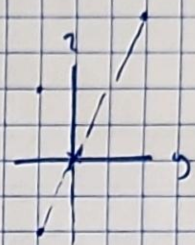
$R_1 + 5/8 R_3 \longrightarrow R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\longrightarrow$ This equation (A) is Linearly Independent as $c_1 = 0$, $c_2 = 0$, & $c_3 = 0$

(Q8) Which set is L.I.

Try D:
$$\left[\begin{array}{ccc|c} 7 & 3 & -6 & 0 \\ -1 & 2 & 4 & 0 \\ 4 & -1 & 2 & 0 \end{array}\right]$$



$$\left[\begin{array}{ccc|c} 1 & 3/7 & -6/7 & 0 \\ -1 & 2 & 4 & 0 \\ 4 & -1 & 2 & 0 \end{array}\right]$$

$R_1 - R_2 \rightarrow R_2$ $\quad R_3 - 4R_1 \rightarrow R_3$

$-\frac{6}{7} - \frac{28}{7}$

$$\left[\begin{array}{ccc|c} 1 & 3/7 & -6/7 & 0 \\ 0 & -11/7 & -5 & 0 \\ 0 & -19/7 & 38/7 & 0 \end{array}\right]$$

$-1 - \frac{12}{7}$

$\frac{14}{7} + \frac{24}{7} = \frac{38}{7}$

$-7/11 R_2 \rightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & 3/7 & -6/7 & 0 \\ 0 & 1 & 35/11 & 0 \\ 0 & -19/17 & 38/7 & 0 \end{array}\right]$$

$\begin{array}{r} 35 \\ \times 7 \\ \hline 105 \end{array}$

$-\frac{6}{7} - \frac{15}{11}$

$-\frac{66}{77} - \frac{105}{77}$

11

$R_1 - 3/7 R_2 \rightarrow R_1$

$1 \quad 0 \quad -171/77$

Hard to solve. Getting some massive fractions. Visually, D is the answer considering graphing the y & z values of the vectors

Q7 Repeat:

Try C: $\left[\begin{array}{ccc|c} 7 & 3 & 6 & 0 \\ -1 & 2 & 4 & 0 \\ 4 & -1 & -2 & 0 \end{array}\right] \xrightarrow{-R_2 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 1 & -2 & -4 & 0 \\ 7 & 3 & 6 & 0 \\ 4 & -1 & -2 & 0 \end{array}\right]$

$R_2 - 7R_1$

$\left[\begin{array}{ccc|c} 1 & -2 & -4 & 0 \\ 0 & 17 & 34 & 0 \\ 4 & -1 & -2 & 0 \end{array}\right] \xrightarrow{R_3 - 4R_1 \to R_3} \left[\begin{array}{ccc|c} 1 & -2 & -4 & 0 \\ 0 & 17 & 34 & 0 \\ 0 & 7 & 14 & 0 \end{array}\right] \xrightarrow{\frac{1}{7}R_3 \leftrightarrow R_2} \cancel{\left[\begin{array}{ccc|c} 1 & -2 & -4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 17 & 34 & 0 \end{array}\right]}$

$\frac{1}{17}R_2 \to R_2$

$\left[\begin{array}{ccc|c} 1 & -2 & -4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 7 & 14 & 0 \end{array}\right] \xrightarrow{R_1 + 2R_2 \to R_1 \ \& \ R_3 - 7R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$c_1 = 0$; $c_2 + 2 \cdot c_3 = 0$

$\cancel{c_2 + 2c_3}$ $\boxed{c_2 = -2c_3}$

∞ possibilities make this true, thus these vectors aren't L.I.

Quiz 20

Question 2

Try (D) again:

$$\left[\begin{array}{ccc|c} 7 & 3 & -6 & 0 \\ -1 & 2 & 4 & 0 \\ 4 & -1 & 2 & 0 \end{array}\right] \xrightarrow{-R_2 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 1 & -2 & -4 & 0 \\ 7 & 3 & -6 & 0 \\ 4 & -1 & 2 & 0 \end{array}\right]$$

$7 \cdot R_1 - R_2 \rightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & -2 & -4 & 0 \\ 0 & -17 & -22 & 0 \\ 4 & -1 & 2 & 0 \end{array}\right] \xrightarrow{4 \cdot R_1 - R_3 \rightarrow R_3} \left[\begin{array}{ccc|c} 1 & -2 & -4 & 0 \\ 0 & -17 & -22 & 0 \\ 0 & -7 & -18 & 0 \end{array}\right]$$

$-\frac{1}{17} R_2 \rightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & -2 & -4 & 0 \\ 0 & 1 & 22/17 & 0 \\ 0 & -7 & -18 & 0 \end{array}\right] \longrightarrow$$

Looking at the answer key my prior attempt was on the right track, refer to that.

matrix D is L.I.

Quiz 20

Q3 Try and see if L.I. :

$$B=\left[\begin{array}{ccc|c} -8 & -3 & -6 & 0 \\ 4 & -6 & 1 & 0 \\ 2 & -9 & 3 & 0 \end{array}\right] \xrightarrow{-\frac{1}{8}R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 3/8 & 3/4 & 0 \\ 4 & -6 & 1 & 0 \\ 2 & -9 & 3 & 0 \end{array}\right]$$

$4 \cdot R_1 - R_2 \to R_2$ $\qquad$ $2R_1 - R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & 3/8 & 3/4 & 0 \\ 0 & 15/2 & 2 & 0 \\ 2 & -9 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3/8 & 3/4 & 0 \\ 0 & 15/2 & 2 & 0 \\ 0 & 39/4 & -3/2 & 0 \end{array}\right]$$

Scratch:

$\frac{12}{8}+6$

$\frac{3}{2}+\frac{12}{2}=\frac{15}{2}$

$3-1=2$

$\frac{3}{4}+\frac{36}{4}$

$\frac{3}{2}-\frac{6}{2}=-3/2$

$\frac{2}{15}R_2 \to R_2$

$$\left[\begin{array}{ccc|c} 1 & 3/8 & 3/4 & 0 \\ 0 & 1 & 4/15 & 0 \\ 0 & 39/4 & -3/2 & 0 \end{array}\right]$$

→ This is the answer. not continuing because the common denominators are too large to bother.

...

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

→ rref $c_1=0$; $c_2=0$; $c_3=0$
~~thus~~ this is a L.I. vector set.

[4.5²] Linear Subspaces

$\mathbb{R}^2$: all 2D vectors $\vec{v} = (x, y)$

$\mathbb{R}^3$: all 3D vectors $\vec{v} = (x, y, z)$

$\vdots$

$\mathbb{R}^N$: all ND vectors $\vec{v} = (V_1, V_2, V_3, \dots, V_n)$

Subspace rules:

① Must include zero vector : $\vec{0} \in S$

② Closed under scalar multiplication : $c\vec{x} \in S$

③ Closed under addition $\vec{x} + \vec{y} \in S$

In re: to ②, example: $\overset{\text{set}}{V} = \{V_1, V_2, V_3, V_4 \dots\}$

$c \cdot V_3$ must be in set of vectors V

$\mathbb{R}^2$, line through origin, $\vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ } 3 subspaces

Closed under
scalar multiplication

example: $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x + y = 0 \right\}$ → It follows $x = -y$, $-x = y$

Subspace of $\mathbb{R}^2$

$= \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \dots \right\}$

$V = \begin{bmatrix} x \\ -x \end{bmatrix} \quad \begin{bmatrix} -y \\ y \end{bmatrix}$

$c \begin{bmatrix} x \\ -x \end{bmatrix} = \begin{bmatrix} cx \\ -cx \end{bmatrix}$ $\quad cx + (-cx) = 0$

$cx - cx = 0 \rightarrow 0 = 0$ ✓

$c \begin{bmatrix} -y \\ y \end{bmatrix} = \begin{bmatrix} -cy \\ cy \end{bmatrix}$ $\quad -cy + cy = 0$

$0 = 0$ ✓

Closed under
✓ Addition

$\begin{bmatrix} x \\ -x \end{bmatrix} + \begin{bmatrix} -y \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ -x + y \end{bmatrix}$

$= \begin{bmatrix} x - y \\ y - x \end{bmatrix}$

$x + y = 0$

$x - y + y - x = 0$

$0 + 0 = 0$

$0 = 0$

Quiz 21: Linear Subspaces

Q1) Which part of the definition of a subspace is redundant?

A. "The set includes the zero vector"

Q2) Which of these are possible subspace of $\mathbb{R}^2$?

A: $\mathbb{R}^2$

B: $\vec{0} = (0,0)$

C: A line through $(0,0)$

$\rightarrow$ All of these

Q3) Is V a subspace of $\mathbb{R}^2$?

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; x = 1 \right\}$$

Let's choose a few vectors:

$\vec{V}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ & $\vec{V}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$\vec{V}_1 + \vec{V}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \notin$ of V

$3\vec{V}_1 = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \notin$ of V

(4:56 Spans as subspaces)

① Closed under scalar multiplication

② Closed under addition

$$\frac{\mathbb{R}^2}{\mathbb{R}^2}$$

$\vec{0} = (0,0)$

Any line through $(0,0)$