Section 6: Matrix-Vector Products

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80 : multiplying a matrix by a vector

$C(A^T)$ Row space		Col space	$C(A)$
	Null space $N(A)$	Left Nullspace $N(A^T)$	

Every matrix has these four spaces.

AB

$A = \begin{bmatrix} 1 & 0 \\ -2 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix}$

$3\times 2 \qquad 2\times 1$

$\vec{b} = (5, -1)$

$A\cdot\vec{b} = \begin{bmatrix} 1\cdot 5 + 0\cdot -1 \\ -2\cdot 5 + 4\cdot -1 \\ 0\cdot 5 + 1\cdot -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -14 \\ -1 \end{bmatrix}$

$\vec{w}B = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & 0 \\ 0 & -2 \end{bmatrix}$

$1\times 3 \qquad 3\times 2$

$= \begin{bmatrix} 1\cdot 3 + 0\cdot 1 + -2\cdot 0 & 1\cdot -4 + 0\cdot 0 + -2\cdot -2 \end{bmatrix}$

$\vec{w}B = \begin{bmatrix} 3 & 0 \end{bmatrix}$

Quiz 31

Q1: Find the matrix-vector product, $A\vec{v}$

$$\begin{bmatrix} -1 & 5 & 4 \\ 3 & 2 & 7 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$$

$3 \times 3 \quad 3 \times 1$

$\vec{v} = (-2, 0, 4)$

$$= \begin{bmatrix} -1 \cdot -2 + 5 \cdot 0 + 4 \cdot 4 \\ 3 \cdot -2 + 0 + 7 \cdot 4 \\ -1 \cdot -2 + 0 + 1 \cdot 4 \end{bmatrix} = \begin{bmatrix} 18 \\ 22 \\ 6 \end{bmatrix} = A\vec{v}$$

Q2: Find the matrix-vector product, $M\vec{v}$

$$M = \begin{bmatrix} -5 & -3 & 1 & 6 \\ 0 & 4 & -2 & 1 \end{bmatrix}$$

$\vec{v} = (1, -3, 5, -4)$

$$M\vec{v} = \begin{bmatrix} -5 & -3 & 1 & 6 \\ 0 & 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 5 \\ -4 \end{bmatrix} = \begin{bmatrix} (-5)(1) + (-3)(-3) + (1)(5) + (6)(-4) \\ (0)(1) + (4)(-3) + (-2)(5) + (1)(-4) \end{bmatrix}$$

$2 \times 4 \quad 4 \times 1$

$$= \begin{bmatrix} -5 + 9 + 5 - 24 \\ 0 - 12 - 10 - 4 \end{bmatrix} = \begin{bmatrix} -15 \\ -26 \end{bmatrix} = M\vec{v}$$

Q3: find the matrix-vector product, $\vec{v} \cdot M$

$$\vec{v} \cdot M = \underset{1 \times 2}{\begin{bmatrix} -2 & 1 \end{bmatrix}} \underset{2 \times 3}{\begin{bmatrix} -4 & -5 & 6 \\ 8 & 3 & -4 \end{bmatrix}} = \begin{bmatrix} (-2)(-4)+(1)(8) & (-2)(-5)+(1)(3) & (-2)(6)+(1)(-4) \end{bmatrix}$$

$$\begin{bmatrix} 8+8 & 10+3 & -12-4 \end{bmatrix}$$

$$\vec{v} M = \begin{bmatrix} 16 & 13 & -16 \end{bmatrix}$$

6.82: The null space and $A\vec{x}=\vec{0}$

This equation defines the null space. If you have a matrix A. Any vector $\vec{x}$ that s.t. $A x = 0$, then $\vec{x}$ is in the nullspace of A

null space is a subspace

3 Criterion to say nullspace is a subspace

Subspace:

1. Zero vector

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{1,1}\cdot 0 + a_{1,2}\cdot 0 \\ a_{2,1}\cdot 0 + a_{2,2}\cdot 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2. Closed under addition

$\vec{x}_1$ $[A\vec{x}_1 = \vec{0}]$

$A\vec{x}_2 = \vec{0}$

$$A\vec{x}_1 + A\vec{x}_2 = \vec{0} + \vec{0}$$
$$A\vec{x}_1 + A\vec{x}_2 = \vec{0}$$
$$A(\vec{x}_1 + \vec{x}_2) = \vec{0}$$

Satisfies $A\vec{x} = \vec{0}$

3. Closed under scalar multiplication

x_1 $\quad c\vec{x}_1$

$$A(c\vec{x}_1) = \vec{0}$$
$$cA\vec{x}_1 = \vec{0}$$

sub $\vec{0}$

$$c\vec{0} = \vec{0}$$
$$\vec{0} = \vec{0}$$

Example: $x\begin{bmatrix} 4 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$A = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix}$ → Same setup. Test for Linear Dependence of Columns

$$\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4x_1 - 2x_2 = 0$$
$$2x_1 - x_2 = 0$$

$$\left[\begin{array}{cc|c} 4 & -2 & 0 \\ 2 & -1 & 0 \end{array}\right] \xrightarrow[R_2 \leftrightarrow R_1]{} \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 4 & -2 & 0 \end{array}\right] \xrightarrow{\tfrac{1}{2}R_1 \to R_1} \begin{bmatrix} 1 & -\tfrac{1}{2} & 0 \\ 4 & -2 & 0 \end{bmatrix}$$

$4R_1 - R_2 \to R_2$

$$\begin{bmatrix} 1 & -\tfrac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - \tfrac{1}{2}x_2 = 0$$
$$x_1 = \tfrac{1}{2}x_2$$

$$\underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \ldots}_{= N(A)}$$

Quiz 32: The null space an $A\vec{x}=\vec{O}$

$$A = \begin{bmatrix} 1 & -4 & 3 \\ 2 & 4 & 2 \\ 1 & -5 & 0 \end{bmatrix}$$

Q1) Is $\vec{x} = (-5, 1, 3)$ in the null space of A?

Suppose this is True, then:

$$\underbrace{\overset{A}{\begin{bmatrix} 1 & -4 & 3 \\ 2 & 4 & 2 \\ -1 & -5 & 0 \end{bmatrix}}}_{3\times 3} \underbrace{\overset{\vec{x}}{\begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}}}_{3\times 1} = \overset{\vec{O}}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$ is True

Verify:

$$\begin{bmatrix} (1)(-5) + (-4)(1) + (3)(3) \\ (2)(-5) + (4)(1) + (2)(3) \\ (-1)(-5) + (-5)(1) + (0)(3) \end{bmatrix} = \begin{bmatrix} -5 - 4 + 9 \\ -10 + 4 + 6 \\ 5 - 5 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\checkmark$ thus, $\vec{x}$ is in the $N(A)$ by the definition of $A\vec{x} = \vec{O}$

Q2: Which vectors are in the null space of A?

$A\vec{x}=\vec{0}$

$$\begin{bmatrix} -3 & 1 & 9 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2\times 3 \qquad 3\times 1$

$-3x_1 + x_2 + 9x_3 = 0$

$x_1 + x_2 + x_3 = 0$

$R_2 \leftrightarrow R_1$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -3 & 1 & 9 & 0 \end{array}\right] \xrightarrow{3R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 4 & 12 & 0 \end{array}\right] \xrightarrow{1/4 R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right] \xrightarrow{R_1 - R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right]$

$x_1 = 2x_3$

$x_2 = -3x_3$

Answer is D

Try D $\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

$2\cdot 1 + 0(-3) + (1)(-2) = 0$ ✓

$2\cdot 0 + (-3)(1) + (1)(3) = 0$ ✓

Q3) Which of the vectors is in the null space of A?

$$A = \begin{bmatrix} 5 & 3 & 1 & 5 \\ -10 & -2 & 1 & -3 \\ -5 & 1 & 2 & 4 \\ 7 & 1 & -1 & -2 \end{bmatrix}$$

Try $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} (5)(1) + (3)(0) + (1)(1) + 5(1) \\ \cdots \\ \\ \end{bmatrix} = \begin{bmatrix} 11 \\ \vdots \\ \end{bmatrix}$$

Stop here as $11 \neq 0$

Try $\vec{x} = \begin{bmatrix} -1 \\ 3 \\ -4 \\ 0 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} (5)(-1) + (3)(3) + (1)(-4) + (5)(0) \\ (-10)(-1) + (-2)(3) + (1)(-4) + (-3)(0) \\ (-5)(-1) + (1)(3) + (2)(-4) + (4)(0) \\ (7)(-1) + (1)(3) + (-1)(-4) + (-2)(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus $\vec{x} = (-1, 3, -4, 0)$ is in the $N(A)$

6.84 Null space of a matrix:

$$A = \begin{bmatrix} 2 & 0 & -1 & 2 \\ -4 & 0 & 2 & -4 \\ -6 & 0 & -3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{A} \quad \vec{x} = \vec{0}$$

$3 \times 4 \quad 4 \times 1 = 3 \times 1$

Rewrite A:

$2 \cdot R_1 + R_2 \to R_2$

$$\begin{bmatrix} 2 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ -6 & 0 & -3 & -6 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 2 & 0 & -1 & 2 \\ -6 & 0 & -3 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$3R_1 + R_2 \to R_2$

$$\to \begin{bmatrix} 2 & 0 & -1 & 2 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\frac{1}{2}R_1 \to R_1$

$$\to \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 1 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$-\frac{1}{6}R_2 \to R_2$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\frac{1}{2}R_2 + R_1 \to R_1$

$$\xrightarrow{\text{rref}(A)} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot cols

free cols

$x_4 = -x_1 + 0x_2$

$x_1 + x_4 = 0 \to x_1 = -x_4$

$x_3 = 0$

solve for pivot variables

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$N(A) = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

Quiz 33: Null space of a matrix

Q1) Find the null space of matrix A

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 0 & -1 & 4 \\ 6 & -6 & -12 \end{bmatrix}$$

$$A\vec{x} = \vec{0}$$

$$\underset{3\times 3}{\begin{bmatrix} 1 & -1 & -2 \\ 0 & -1 & 4 \\ 6 & -6 & -12 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{6R_1 - R_3 \to R_3} \begin{bmatrix} 1 & -1 & -2 \\ 0 & -1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-R_2 \to R_2} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 + R_2 \to R_1} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \; \mathrm{rref}(A)$$

Free cols

Pivot cols

$$x_1 - 6x_3 = 0$$

$$x_1 = 6x_3$$

$$x_2 = 4x_3$$

$$x_3 \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$$

$$N(A) = \mathrm{Span}\left(\begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}\right)$$

Q2: Find the null space of M:

$$M = \begin{bmatrix} 1 & 2 & -3 & 5 \\ 2 & 4 & -6 & 10 \\ -3 & -6 & 9 & -15 \\ 4 & 1 & -12 & 6 \end{bmatrix} \xrightarrow[\substack{3R_1+R_3 \to R_3 \\ 2R_1-R_2 \to R_2}]{4R_1-R_4 \to R_4} \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 14 \end{bmatrix} \xrightarrow[\frac{1}{7}R_2 \to R_2]{R_2 \leftrightarrow R_4} \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 2R_2 \to R_1} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

rref (A)

pivot cols

free cols

$x_1 - 3x_3 + x_4 = 0$

Solve for pivot vars:

$x_1 = 3x_3 - x_4$

$x_2 = -2x_4$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$$N(M) = \text{span}\left(\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right)$$

Q3: Find the null space of B:

$3R_1 - R_3 \to R_3$

$2R_1 - R_2 \to R_2$

$$B = \begin{bmatrix} 1 & 1 & -2 & 5 \\ 2 & 2 & -4 & 10 \\ 3 & 3 & -6 & 15 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{rref}(B)$$

free cols

pivot cols

solve for pivot variable:

$$x_1 = -x_2 + 2x_3 - 5x_4$$

Rewrite as linear combination of free variables:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Longrightarrow \boxed{N(B) = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)}$$

6.86 The column spac of $A\vec{x}=\vec{b}$

$N(A)$ = nullspace of A

$C(A)$ = colspace of A = linear linear combinations of A == span of columns of A

$N(A)=\vec{0}$ columns of A L.I. → Basis

$N(A)=\vec{0},\vec{v}_1$ " " " L.D. → Does not form a basis

$A=\begin{bmatrix}2 & 4\\3 & 6\end{bmatrix}$ $\quad C(A)=\text{span}\left(\begin{bmatrix}2\\3\end{bmatrix},\begin{bmatrix}4\\6\end{bmatrix}\right)$

e.g. $\begin{bmatrix}2 & 4\\3 & 6\end{bmatrix}\to\begin{bmatrix}1 & 2\\3 & 6\end{bmatrix}\to\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}$ rref A

$\begin{bmatrix}2 & 4\\3 & 6\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}6\\9\end{bmatrix}$

$$\begin{array}{cc}\text{pivot} & \text{free}\\ x_1 & x_2\end{array}$$

$\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

Basis for $C(A)=\begin{bmatrix}2\\3\end{bmatrix}$

$C(A)=\text{span}\left(\begin{bmatrix}2\\3\end{bmatrix}\right)$

$x_1\begin{bmatrix}2\\3\end{bmatrix}+x_2\begin{bmatrix}4\\6\end{bmatrix}=\begin{bmatrix}6\\9\end{bmatrix}$ if $\vec{b}\in C(A)$, then $\vec{x}$ exists as a solution

$x_1+2x_2=0$ $\quad \begin{bmatrix}x_1\\x_2\end{bmatrix}=x_2\begin{bmatrix}-2\\1\end{bmatrix}$

$x_1=-2x_2$

Find basis

Set $x_2=0$ since it's a free var

$x_1\begin{bmatrix}2\\3\end{bmatrix}+x_2\begin{bmatrix}4\\6\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ $\quad x_1\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$x_1\begin{bmatrix}2\\3\end{bmatrix}+0\begin{bmatrix}4\\6\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ $\quad x_1=0$ basis

since we found another vector in the Null Space $\begin{bmatrix}-2\\1\end{bmatrix}$, that means the columns of matrix A are linearly dependent. Thus the columns of A do not form a basis for A, but A is spanned by them

$N(A)=N(\text{rref}(A))=\text{span}\left(\begin{bmatrix}-2\\1\end{bmatrix}\right)$

Quiz 34: The column space of Ax=b

Q1: Find the null space, then find the column space of A

$A\vec{x} = \vec{0}$

$$A \Rightarrow \begin{bmatrix} 1 & -5 & 2 & 4 \\ 3 & 0 & 1 & 2 \\ 1 & -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{0}$$

$4 \times 4 \qquad 4 \times 1$

$-\frac{2}{3} + \frac{1}{3} \cdot \frac{8}{3}$ $\qquad$ $\frac{2}{3} - \frac{1}{3} \cdot \frac{8}{3}$

$-1/3$

$-\frac{5}{3} + \frac{6}{3}$ $\qquad$ $-\frac{10}{3} + \frac{12}{3}$ $\qquad$ $-\frac{3}{4}\left(-\frac{8}{3}\right)$

$3R_1 - R_2 \to R_2$
$R_1 - R_3 \to R_3$

$-1/15 R_2 \to R_2$

$5R_2 + R_1 \to R_1$
$4R_2 + R_3 \to R_3$

$-\frac{3}{4}R_3 \to R_3$

$R_1 - 1/3 R_3 \to R_1$
$R_2 + 1/3 R_3 \to R_2$

Free col

$$\begin{bmatrix} 1 & -5 & 2 & 4 \\ 0 & -15 & 5 & 10 \\ 0 & -4 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -5 & 2 & 4 \\ 0 & 1 & -1/3 & -2/3 \\ 0 & -4 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1/3 & 2/3 \\ 0 & 1 & -1/3 & -2/3 \\ 0 & 0 & -4/3 & -8/3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1/3 & 2/3 \\ 0 & 1 & -1/3 & -2/3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

rref(A)

$x_1 = 0$
$x_2 = 0$
$x_3 = -2x_4$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$N(A) = \operatorname{Span}\left(\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}\right)$$

- The column space is all the linear combinations of the column vectors.

$$C(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}\right)$$

Q2: Find the column space of M in terms of its basis:

$$M = \begin{bmatrix} -1 & 2 & 6 & 5 \\ 0 & 3 & -7 & 9 \\ 3 & -6 & -18 & -15 \end{bmatrix}$$

obtain rref:

$3R_1 + R_3 \longrightarrow R_3$ $\qquad$ $-R_1 \longrightarrow R_1$, $\quad \frac{1}{3}R_2 \longrightarrow R_2$

$$\begin{bmatrix} -1 & 2 & 6 & 5 \\ 0 & 3 & -7 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & -6 & -5 \\ 0 & 1 & -7/3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

First & second columns are the pivot columns

thus, these are the basis of M:

$$C(M) = \mathrm{Span}\left(\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix} \right)$$

Row Echelon [NOT rref!]

$$A=\begin{bmatrix}1 & -2 & 4 & -5\\ 0 & 3 & 5 & 7\\ -3 & 6 & 3 & 9\\ 2 & -4 & -2 & -6\end{bmatrix} \xrightarrow[2R_1-R_4\to R_4]{3R_1+R_3\to R_3} \begin{bmatrix}1 & -2 & 4 & -5\\ 0 & 3 & 5 & 7\\ 0 & 0 & 15 & -6\\ 0 & 0 & 10 & -4\end{bmatrix} \xrightarrow{\frac{1}{3}R_2\to R_2} \begin{bmatrix}1 & -2 & 4 & -5\\ 0 & 1 & 5/3 & 7/3\\ 0 & 0 & 15 & -6\\ 0 & 0 & 10 & -4\end{bmatrix}$$

$$\begin{bmatrix}1 & -2 & 4 & -5\\ 0 & 1 & 5/3 & 7/3\\ 0 & 0 & 1 & -2/5\\ 0 & 0 & 10 & -4\end{bmatrix} \xrightarrow{10R_3-R_4\to R_4} \begin{bmatrix}1 & -2 & 4 & -5\\ 0 & 1 & 5/3 & 7/3\\ 0 & 0 & 1 & -2/5\\ 0 & 0 & 0 & 0\end{bmatrix}$$

In terms of the basis, cols 1, 2, 3 are the pivots

$$C(A)=\text{span}\left(\begin{bmatrix}1\\0\\-3\\2\end{bmatrix}, \begin{bmatrix}-2\\3\\6\\-4\end{bmatrix}, \begin{bmatrix}4\\5\\3\\-2\end{bmatrix}\right)$$

6.88 Solving $A\vec{x}=\vec{b}$

$A\vec{x}=\vec{0} \rightarrow$ complimentary Solution
$A\vec{x}=\vec{b} \rightarrow$ particular solution
} general solution

$A\vec{x}_n + A\vec{x}_p = \vec{0} + \vec{b}$

$A(\vec{x}_n + \vec{x}_p) = \vec{b}$

general solution $= \vec{x} = \vec{x}_n + \vec{x}_p$ (what we're looking for)

Example:

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 2 & 3 & 5 & b_2 \\ 4 & 6 & 10 & b_3 \end{array}\right] \longrightarrow \text{get rref:}$$

$2R_1 - R_2 \rightarrow R_2$

$4R_1 - R_3 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 0 & 1 & -5 & 2b_1-b_2 \\ 0 & 2 & -10 & 4b_1-b_3 \end{array}\right] \rightarrow$$

$R_1 - 2R_2 \rightarrow R_1$

$2R_2 - R_3 \rightarrow R_3$

$b_1 - 4b_1 + 2b_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 10 & -3b_1+2b_2 \\ 0 & 1 & -5 & 2b_1-b_2 \\ 0 & 0 & 0 & b_3-2b_2 \end{array}\right]$$

$2(2b_1 - b_2)$

$4b_1 - 2b_2 - 4b_1 + b_3$

$0 = b_3 - 2b_2$

$b_3 = 2b_2$

eg: $b_2 = 1$
then $b_3 = 2$
$b_1 = 0$
} Sub

Find Complimentary solution:

augment with zero vector

$$\left[\begin{array}{ccc|c} 1 & 0 & 10 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + 10x_3 = 0$

$x_2 - 5x_3 = 0$

$x_1 = -10x_3$

$x_2 = 5x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -10 \\ 5 \\ 1 \end{bmatrix}$$

$$\vec{x}_n = c_1 \begin{bmatrix} -10 \\ 5 \\ 1 \end{bmatrix}$$

$b_1 = 0$
$b_2 = 1$
Sub: $b_3 = 2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 10 & 2 \\ 0 & 1 & -5 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + 10x_3 = 2$

$x_2 - 5x_3 = -1$

} set free var to zero x_3

$x_1 = 2$

$x_2 = -1$

$x_3 = 0$

$$\vec{x}_p = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -10 \\ 5 \\ 1 \end{bmatrix}$$

Quiz 35: Solving $A\vec{x} = \vec{b}$

Q1) Find the general solution to $A\vec{x} = \vec{b}$

obtain rref: $\frac{1}{3}R_1 \to R_1$, $2R_1 + R_3 \to R_3$, $R_1 + R_2 \to R_2$

$2R_2 + R_1 \to R_1$, $4R_2 + R_3 \to R_3$

$$A = \left[\begin{array}{ccc|c} 3 & -6 & 6 & b_1 \\ -3 & 7 & -9 & b_2 \\ -6 & 8 & 0 & b_3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 2 & b_1/3 \\ 0 & 1 & -3 & b_1 + b_2 \\ 0 & -4 & 12 & b_3 + 2(b_1) \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -4 & \frac{1}{3}b + 2(b_1 + b_2) \\ 0 & 1 & -3 & b_1 + b_2 \\ 0 & 0 & 0 & 4(b_1 + b_2) + 2b_1 + b_3 \end{array}\right]$$

$\frac{1}{3}b + 2(b_1+b_2) \to \frac{7}{3}b_1 + 2b_2$

solve in terms of pivots:

find complimentary s/tn.

pivot cols, free cols

$1x_1 - 4x_3 = 0$

$x_1 = 4x_3$

$x_2 - 3x_3 = 0$

$x_2 = 3x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} \Longrightarrow X_h = c_1 \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} *$$

complimentary s/tn.

Given: $\vec{b} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

$b_1 = \frac{7}{3} \cdot 1 + 2(-1) = \frac{1}{3}$

$b_2 = 1 + (-1) = 0$

$b_3 = 6(1) + 4(-1) + -2 = 0$

$$\vec{X}_p = \begin{bmatrix} \frac{1}{3} \\ 0 \\ 0 \end{bmatrix} *$$

General Solution: * $$\vec{X} = \begin{bmatrix} 1/3 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

Q2) Find the general solution to $A\vec{x} = \vec{b}$

augment w/ $\vec{b}$

$$A=\begin{bmatrix} 1 & -1 & 3 & -1 & 3 \\ -1 & 0 & 0 & 2 & 1 \\ 0 & 1 & -3 & -1 & -4 \end{bmatrix} = \left[\begin{array}{ccccc|c} 1 & -1 & 3 & -1 & 3 & b_1 \\ -1 & 0 & 0 & 2 & 1 & b_2 \\ 0 & 1 & -3 & -1 & -4 & b_3 \end{array}\right]$$

$R_2 + R_1 \rightarrow R_2$

$$\Rightarrow \left[\begin{array}{ccccc|c} 1 & -1 & 3 & -1 & 3 & b_1 \\ 0 & -1 & 3 & 1 & 4 & b_2+b_1 \\ 0 & 1 & -3 & -1 & -4 & b_3 \end{array}\right]$$

$-R_2 \rightarrow R_2$

$$\Rightarrow \left[\begin{array}{ccccc|c} 1 & -1 & 3 & -1 & 3 & b_1 \\ 0 & 1 & -3 & -1 & -4 & -b_2-b_1 \\ 0 & 1 & -3 & -1 & -4 & b_3 \end{array}\right]$$

$R_1 + R_2 \rightarrow R_1$
$R_3 - R_2 \rightarrow R_3$

$$\Rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -2 & -1 & b_1-b_2-b_1 \Rightarrow -b_2 \\ 0 & 1 & -3 & -1 & -4 & -b_1-b_2 \\ 0 & 0 & 0 & 0 & 0 & b_3-(-b_2-b_1) \Rightarrow b_3+b_2+b_1 \end{array}\right]$$

rref(A)

$$\Rightarrow \begin{array}{ccccc} \wedge & \wedge & \wedge & \wedge & \wedge \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -2 & -1 & -b_2 \\ 0 & 1 & -3 & -1 & -4 & -b_1-b_2 \\ 0 & 0 & 0 & 0 & 0 & b_3+b_2+b_1 \end{array}\right]$$

$\wedge$ = pivot columns
$\wedge$ = free columns

Particular solution where $\vec{b} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ (given)

$b_1 = 1$ $\quad x_1 = -b_2 = 2$
$b_2 = -2$ $\quad x_2 = -b_1 - b_2 = 1$
$b_3 = 1$ $\quad x_3 = b_3 + b_2 + b_1 = 0$

plug into general sltn

solve system of equations in terms of pivots

$x_1 - 2x_4 - x_5 = 0$ $\qquad x_2 - 3x_3 - x_4 - 4x_5 = 0$

$x_1 = 2x_4 + x_5$ $\qquad x_2 = 3x_3 + x_4 + 4x_5$

$$x_3 = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad x_4 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad x_5 = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Complimentary Solution:

$$X_n = c_1\begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Find Particular, set free vars to zero:

$x_1 - 2(0) - 0 = 2$

$x_2 - 3(0) - 1(0) - 4(0) = 1$

free vars were all set to zero:

$x_3 = 0 \quad x_4 = 0 \quad x_5 = 0$

Thus $\vec{X}_p = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

General sltn:

$$\vec{X} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_1\begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Quiz 3, Dimensionality, nullity, and rank

Q1) find the nullity of M.

1st obtain rref(M):

$3R_1 + R_2 \to R_2$

$5R_1 + R_3 \to R_3$

$-R_3 \leftrightarrow R_2$

$$M = \begin{bmatrix} 1 & -2 & 3 & -1 & 2 \\ -3 & 6 & -9 & 3 & -6 \\ -5 & 9 & -7 & 4 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 8 & 1 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & \overbrace{3 & -1 & 2} \\ 0 & 1 & -8 & -1 & -10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3 free cols thus:

$$N(M) = 3$$

Q2) find the rank of X.

$$X = \begin{bmatrix} 1 & 3 & -6 \\ 1 & -5 & 9 \\ -2 & 10 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & -6 \\ 0 & 8 & -15 \\ 0 & 16 & -16 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -6 \\ 0 & 1 & -15/8 \\ 0 & 0 & -46 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & -6 \\ 0 & 1 & -15/8 \\ 0 & 0 & 1 \end{bmatrix}$$

Q3 Find the nullity and rank of A:

$$A = \begin{bmatrix} 1 & 3 & -2 & -1 & 0 \\ 0 & 1 & 0 & 5 & -3 \\ 0 & -1 & 1 & -6 & -4 \\ 0 & 1 & 0 & 5 & -3 \end{bmatrix}$$

$2R_1 - R_2 \rightarrow R_2$

$R_1 - R_3 \rightarrow R_3$

$R_1 - R_4 \rightarrow R_4$

$R_2 + R_3 \rightarrow R_3$ $\quad R_2 - R_4 \rightarrow R_4$

$$\begin{bmatrix} 1 & 3 & -2 & -1 & 0 \\ 0 & 1 & 0 & 5 & -3 \\ 0 & 0 & 1 & -1 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3 pivot cols : $r(A) = 3$

2 free cols : $N(A) = 2$

Q3) Find the general solution to $A\vec{x} = \vec{b}$

$$A = \left[\begin{array}{cccc|c} 2 & 3 & 4 & -4 & b_1 \\ 2 & 3 & 8 & -10 & b_2 \\ 6 & 9 & 16 & -18 & b_3 \end{array}\right] \quad \text{with } \vec{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\frac{1}{2} b_1 - 2\left(\frac{b_2 - b_1}{4}\right)$ $\qquad -\frac{4}{2} - 2\left(-\frac{3}{2}\right) \Rightarrow -2+3$

$3R_1 - R_3 \to R_3$; $R_1 - R_2 \to R_2$

$$\left[\begin{array}{cccc|c} 2 & 3 & 4 & -4 & b_1 \\ 0 & 0 & -4 & 6 & b_1 - b_2 \\ 0 & 0 & -4 & 6 & 3b_1 - b_3 \end{array}\right] \Rightarrow$$

$\frac{1}{2} R_1 \to R_1$

$$\left[\begin{array}{cccc|c} 1 & 3/2 & 2 & -2 & \frac{1}{2} b_1 \\ 0 & 0 & -4 & 6 & b_1 - b_2 \\ 0 & 0 & -4 & 6 & 3b_1 - b_3 \end{array}\right] \Rightarrow$$

$R_2 - R_3 \to R_3$

$-\frac{1}{4} R_2 \to R_2 \Longrightarrow R_1 - 2R_2 \to R_1$

$$\left[\begin{array}{cccc|c} 1 & 3/2 & 0 & 1 & b_1 - \frac{1}{2} b_2 \\ 0 & 0 & 1 & -3/2 & -\left(\frac{b_1 - b_2}{4}\right) \\ 0 & 0 & 0 & 0 & b_1 - b_2 - (3b_1 - b_3) \end{array}\right]$$

↑ rref(A)

$$x_n = x_2 \begin{bmatrix} -3/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad x_4 \begin{bmatrix} -1 \\ 0 \\ 3/2 \\ 1 \end{bmatrix}$$

given $\vec{b} = (1, -1, 1)$:

$b_1 - \frac{1}{2} b_2 = 3/2$; $\frac{b_2 - b_1}{4} = -1/2$; $-2b_1 - b_2 + b_2 = 0$

set free vars to ϕ:

$x_1 + 3/2(0) + 1(0) = 3/2$ $\qquad x_1 = 3/2$

$x_3 - 3/2(0) = -1/2$ $\qquad x_3 = -1/2$

$$X_p = \begin{bmatrix} 3/2 \\ 0 \\ -1/2 \\ 0 \end{bmatrix}$$

^ free column

^ pivot columns

Solve for pivot variables:

$x_1 = -3/2 x_2 - x_4$

$x_3 = \frac{3}{2} x_4$

Put free column in these terms: ←

$$x_2 \begin{bmatrix} -3/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad x_4 \begin{bmatrix} -1 \\ 0 \\ 3/2 \\ 1 \end{bmatrix}$$

These → are complementary soltn.

6.90 Diminsionality, nullity, and rank

$\frac{1}{2}R_1 \to R_1$
$R_2 - R_1 \to R_2$

$$A = \begin{bmatrix} 2 & 0 & 4 & -2 \\ 1 & 3 & -1 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 3 & -3 & 1 \\ 0 & -2 & 1 & 1 \end{bmatrix}$$

$\frac{1}{3}R_2 \to R_2$
$R_3 + 2R_2 \to R_3$
$-R_3 \to R_3$

$$\Rightarrow \begin{array}{cccc} & & & \\ \end{array}\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{5}{3} \end{bmatrix}$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$

$R_1 - 2R_3 \to R_1$
$R_2 + R_3 \to R_2$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 & \frac{7}{3} \\ 0 & 1 & 0 & -\frac{4}{3} \\ 0 & 0 & 1 & -\frac{5}{3} \end{bmatrix}}_{rref(A)}$$

$-\frac{3}{3} - 2\left(-\frac{5}{3}\right)$

$-\frac{3}{3} + \frac{10}{3}$

$\frac{1}{3} - \frac{5}{3}$

Dimensions: 3 x 4

Dimensionality is different: the # of basis vectors to span a space

Nullity: Dimension of the null space of A

nullity is the number of free columns in rref of a matrix

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -7/3 \\ 4/3 \\ 5/3 \\ 1 \end{bmatrix}$$

nullity(A) = 1 because x_4 is only free vector

Dimension: Column space
"Rank"

Dim(C(A)), rank A

$r = \text{rank}(A) = 3$

$\text{Dim}(C(A)) = r$

$\text{Dim}(N(A)) = n - r$