② Elimination Method:

$$6x + 3y = 27$$
$$-5x + 2y = 0$$

First equation by 2
Multiply 2nd by 3

$2(6x + 3y) = 27 \cdot 2 \rightarrow 12x + 6y = 54$

$3(-5x) + 3(2y) = 0 \rightarrow -15x + 6y = \emptyset$

$\rightarrow$
$$12x + 6y = 54$$
$$-(-15x + 6y) = 0$$
$$27x = 54$$
$$\downarrow$$
$$x = 2 \ \checkmark$$

sub 2 for x: $6(2) + 3y = 27$

$$3y = 15$$
$$y = 5 \ \checkmark$$

$$\boxed{(x, y) = (2, 5)}$$

sub 2 for x: $-5(2) + 2y = 0$

$$2y = 10$$
$$y = 5 \ \checkmark$$

③ Graphing Method

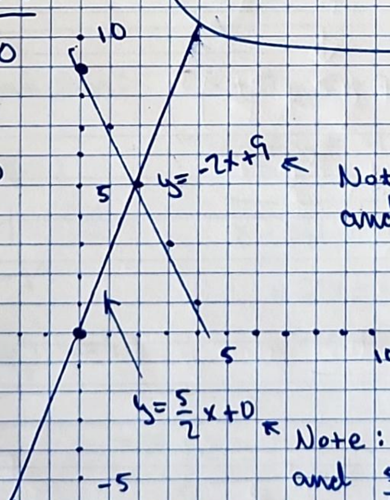
- put in slope intercept form (solve for y):

- $6x + 3y = 27$

$$3y = 27 - 6x$$
$$y = 9 - 3x$$
$$y = -2x + 9$$

- $-5x + 2y = 0$

$$2y = 5x$$
$$y = \frac{5}{2}x + 0$$



Note: 9 is the y-intercept, and $\left(\frac{-2}{1}\right)$ is the "rise over run"

Note: $\emptyset$ is the y-intercept, and $\frac{5}{2}$ is the "rise over run"

$(x, y) = (2, 5)$

- Knowing these 3 methods: (Substitution, Elimination, Graphing) in a linear system is same as solving in a matrix system.

Quiz: Linear systems:

Q1 Use substitution to find the unique solution:

$$\boxed{\begin{aligned} y &= x+7 \\ x+2y &= -16 \end{aligned}}$$

sub $(x+7)$: $x+2(x+7) = -16$

$3x + 14 = -16$

$3x = -30$

$x = -10$

sub -10 for x into original equations:

$y = -10 + 7$ | $-10 + 2y = -16$

$y = -3$ | $2y = -6$

$y = -3$

$(x, y) = (-10, -3)$

Q2 Use elimination: $x - 3y = -7 \xrightarrow{\text{mult. by 2:}} 2x - 6y = -14$

$2x - 3y = 4$

$2x - 6y = -14$

$-(2x - 3y) = 4$

$-3y = -18$

$y = 6$

$x - 3(6) = -7$ | $2x - 3(6) = 4$

$x - 18 = -7$ | $2x - 18 = 4$

$x = 11$ | $2x = 22$

$x = 11$

$\boxed{(x, y) = (11, 6)}$

[redacted]

Section 2: Operations On 1 Matrix

Three Methods:

① **Substitution Method:**

- Two Unknowns:

Solve for:
$$\boxed{\begin{aligned} 6x + 3y &= 27 \\ -5x + 2y &= 0 \end{aligned}}$$

$$6x = 27 - 3y$$
$$\downarrow$$
$$x = \frac{27 - 3y}{6}$$
$$\downarrow$$

solved for $x = \frac{9}{2} - \frac{y}{2}$

<u>sub</u> x: $-5x + 2y = 0$

$$-5\left(\frac{9}{2} - \frac{y}{2}\right) + 2y = 0$$

$$-\frac{45}{2} + \frac{5y}{2} + 2y = 0$$
$$\downarrow$$
$$2\left(-\frac{45}{2} + \frac{5y}{2} + 2y\right) = 0(2)$$

$$-45 + 5y + 4y = 0$$
$$9y = 45$$
$$y = 5$$

sub y into both original equations:

$$6x + 3(5) = 27$$
$$6x = 12$$
$$x = 2$$

$$-5x + 2(5) = 0$$
$$-5x = -10$$
$$x = 2$$

Thus: $\left\{\begin{aligned} y &= 5 \\ x &= 2 \end{aligned}\right.$

$$(x, y) = (2, 5)$$

② Elimination Method:

$x + 3y = 27$

$x + 2y = 0$

irst equation by 2

ltiply 2nd by 3

$2(6x + 3y) = 27 \cdot 2 \rightarrow 12x + 6y = 54$

$3(-5x) + 3(2y) = 0 \rightarrow -15x + 6y = \emptyset$

$\rightarrow$

$12x + 6y = 54$

$-(-15x + 6y) = 0$

$27x = 54$

$\downarrow$

$x = 2$ ✓

sub 2 for x: $6(2) + 3y = 27$

$3y = 15$

$y = 5$ ✓

$\boxed{(x, y) = (2, 5)}$

sub 2 for x: $-5(2) + 2y = 0$

$2y = 10$

$y = 5$ ✓

① Graphing Method

- put in slope intercept form (solve for y):

$6x + 3y = 27$

$3y = 27 - 6x$

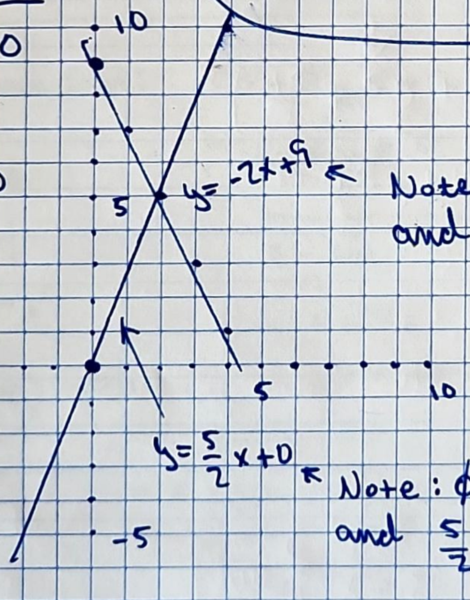
$y = 9 - 2x$

$y = \frac{-2}{1}x + 9$

$-5x + 2y = 0$

$2y = 5x$

$y = \frac{5}{2}x + 0$



Note: 9 is the y-intercept, and $\left(\frac{-2}{1}\right)$ is the "rise over run"

Note: $\emptyset$ is the y-intercept, and $\frac{5}{2}$ is the "rise over run"

$(x, y) = (2, 5)$

- Knowing these 3 methods: Substitution, Elimination, Graphing in a linear system is same as solving in a matrix system.

Quiz: Linear systems:

Q1 Use substitution to find the unique solution:

$$\begin{cases} y = x + 7 \\ x + 2y = -16 \end{cases}$$

sub $(x+7)$: $x + 2(x+7) = -16$

$3x + 14 = -16$

$3x = -30$

$x = -10$

sub -10 for x into original equations:

$y = -10 + 7$ | $-10 + 2y = -16$

$y = -3$ | $2y = -6$

$y = -3$

$(x, y) = (-10, -3)$

Q2 Use elimination: $x - 3y = -7 \xrightarrow{\text{mult. by 2:}} 2x - 6y = -14$

$2x - 3y = 4$

$2x - 6y = -14$

$-(2x - 3y) = 4$

$-3y = -18$

$y = 6$

$x - 3(6) = -7$ | $2x - 3(6) = 4$

$x - 18 = -7$ | $2x - 18 = 4$

$x = 11$ | $2x = 22$

$x = 11$

$\boxed{(x, y) = (11, 6)}$

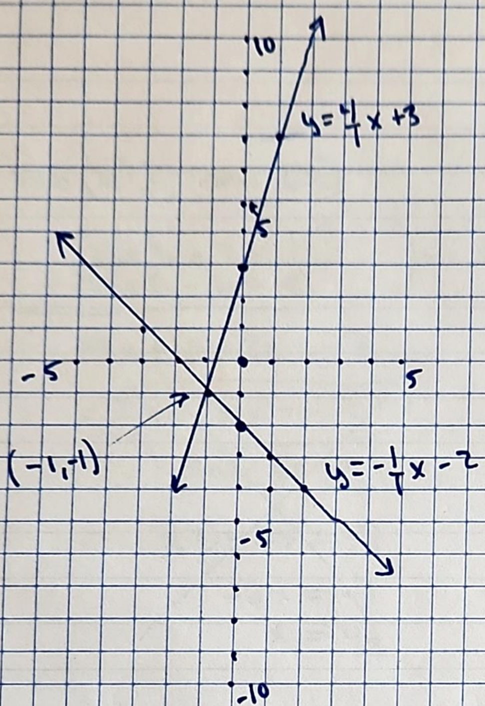
Q3 Graph: (and find unique Solution for:)

$4x - y = -3$ $\qquad$ $x + y = -2$

$y - 3 = 4x$

$y = \frac{4}{1}x + 3$ $\qquad$ $y = -\frac{1}{1}x - 2$

slope intercept form



$(x, y) = (-1, -1)$

Section 2: Quiz 2

Q1 Solve system of equations:

[1] $x + y - z = 4$
[2] $x - y - z = 2$
[3] $x + 2y + z = 1$

~~add.~~ Subtract [2] from [1]:

$$\begin{array}{r} x + y - z = 4 \\ -(x - y - z = 2) \\ \hline 0 + 2y \quad = 2 \end{array}$$

$y = 1$

sub. (2[1]) from [3]

$$\begin{array}{r} x + 2y + z = 1 \\ -(2x + 2y - 2z = 8) \\ \hline -3x + 0 + 3z = -7 \end{array}$$

$3z - 3x = -7$

sub $z = -2$

$3(-2) - 3x = -7$

$-3x = -1$

$x = 1/3$

sub [3] from [1]

$$\begin{array}{r} x + y - z = 4 \\ -(x + 2y + z = 1) \\ \hline 0 - y - 2z = 3 \end{array}$$

sub $y = 1$

$-1 - 2z = 3$

$-2z = 4$

$z = -2$

sub $y = 1$; $z = -2$

[1] $x + 1 + 2 = 4$ ✓
$x = 1$

[2] $x - 1 + 2 = 2$ ✓

$x = 1$

[3] $x + 2 - 2 = 1$ ✓

$x = 1$

$(x, y, z) = (1, 1, -2)$

Q2 Solve the linear system:

$[1]\quad x - y + z = -6$

$[2]\quad 3x - 4y - z = -4$

$[3]\quad -2x + 3y + 4z = 14$

Try: $[2]\cdot 2$

$[3]\cdot 3$

$2\cdot[2]\quad 6x - 8y - 2z = -8$

$3\cdot[3]\quad [-6x + 9y + 12z = 42]$ + add

$0 + y + 10z = 42$

$10z + y = 42$

$y = 42 - 10z$

~~sub~~ add [2] to [1]:

$$x - y + z = -6$$
$$+(3x - 4y - z = -4)$$

$[4]\quad 4x - 5y = -10$

$4x = 10 + 5y$

$x = \frac{10}{4} + \frac{5}{4}y$

try $[2]\cdot 4$

$4\cdot[2]\quad 12x - 16y - 4z = -16$

~~sub~~ add $4\cdot[2]$ from [3]:

$$-2x + 3y + 4z = 14$$
$$+(12x - 16y - 4z = -16)$$

$[6]\quad 10x - 13y = -2$

$10\left(\frac{10}{4} + \frac{5}{4}y\right) - 13y = -2$

$\frac{100}{4} + \frac{50}{4}y - 13y = -2$

$[4]\quad 4x - 5y = -10$

$[6]\quad 10x - 13y = -2$

Solve [4] for y:

$-5y = -10 - 4x$

$[7]\quad y = \frac{4}{5}x + 2$

Sub [7] into [6]

$10x - 13\left(\frac{4}{5}x + 2\right) = -2$

$10x - \frac{52}{5}x - 26 = -2$

$5\left(10x - \frac{52}{5}x - 26\right) = -2(5)$

~~$50x - 260$~~

$50x - 52x - 130 = -10$

$-2x = 120$

$x = -60$

Sub $x = -60$ into [7]

$y = \frac{4}{5}(-60) + 2$

$= 4(-12) + 2$

$= -48 + 2 = y = -46$

Sub $x = -60$ into [1] $y = -46$

$-60 + 46 + z = -6$

$z = 8$

$(x, y, z) = (-60, -46, 8)$

(Q3) Solve the linear system:

[1] $x + 2z = 3$

[2] $3x - 2y + z = -11$

[3] $2x + y + 3z = 9$

add 2·[3] to [2] to eliminate y:

$$\begin{array}{r} 3x - 2y + z = -11 \\ +\,(4x + 2y + 6z = 18) \\ \hline 7x + 7z = 7 \\ x + z = 1 \end{array}$$

[4] $\rightarrow x = 1 - z$

sub [4] into [1] to solve for z

$$(1 - z) + 2z = 3$$

$$1 + z = 3$$

[5] $z = 2$

sub [5] into [1] to solve for x

$$x + 2(2) = 3$$

[6] $x = -1$

sub [5] & [6] into [1], [2], [3] to ~~validate~~ solve for y

[1] $-1 + 2(2) = 3$ ✓

[2] $-3 - 2y + 2 = -11$

$$-2y = -10$$

$$y = 5$$ ✓

$(x, y, z) = (-1, 5, 2)$

sub x, y, z into [3]

$$-2 + 5 + 6 = 9$$ ✓

2.12 Matrix Dimensions & Entries

- Naming convention capital letters
- Consider an array of values, a simple arrangement of data

$$A = \begin{bmatrix} 2 & 6 \\ -1 & 0 \end{bmatrix}$$ think of same as $2x = 6$, $-1x = 0$

- Dimensions: Rows by Columns

2 x 2 matrix

$$A = \begin{bmatrix} 2 & 6 \\ -1 & 0 \end{bmatrix}$$

2 x 4 matrix

$$B = \begin{bmatrix} 9 & -2 & 1 & 8 \\ 0 & 0 & -1 & 4 \end{bmatrix}$$

3 x 1 matrix

$$C = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}$$

$$D = \begin{bmatrix} -3 & 4 & 0 & 6 \end{bmatrix}$$

1 x 4 matrix

$a_{1,2} = 6$ $b_{2,2} = 0$

$a_{2,1} = -1$ $b_{1,4} = 8$

Section 2: Quiz 3

Q1 $K = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 2 & 5 & 6 & -2 \end{bmatrix}$ Dimensions of K? 2×4

Q2 $B = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}$ Find $b_{2,1}$: $b_{2,1} = 0$

3 Give dimensions & find $m_{3,2}$:

$M = \begin{bmatrix} 1 & 3 & 7 \\ 0 & -1 & 2 \\ 9 & 4 & 6 \end{bmatrix}$

$M = 3 \times 3$ matrix

$m_{3,2} = 4$

Section 2.14: Representing Systems w/ matrices

$\left.\begin{array}{lr} 6x + 3y = 27 & [1] \\ -5x + 2y = 0 & [2] \end{array}\right) \to$ Linear System

$\begin{array}{cc} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 6 & 3 \\ -5 & 2 \end{bmatrix} \end{array}$ $\to$ This is ~~a~~ the Coefficient Matrix

$\begin{array}{cc} & \begin{array}{ccc} x & y & c \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \left[\begin{array}{cc|c} 6 & 3 & 27 \\ 5 & 2 & 0 \end{array}\right] \end{array}$ $\to$ This is the augmented Matrix

— Given a Linear System:

[1] $3x - 2y + z = 2$
[2] $-x - 5y + 4z = 1$
[3] $x + 4y - 6z = -9$

— Corresponding Augmented Matrix

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 2 \\ -1 & -5 & 4 & 1 \\ 1 & 4 & -6 & -9 \end{array}\right]$$

A mixed-up Linear System:

$$2y = 2 - 3x$$
$$-x - 5y + 4z = 1$$
$$9 + 6z = y$$

$$\left.\begin{aligned} 3x + 2y + \varnothing z &= 2 \\ -x - 5y + 4z &= 1 \\ \varnothing x - y + 6z &= -9 \end{aligned}\right) \text{fixed}$$

$$\begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc|c} 3 & 2 & 0 & 2 \\ -1 & -5 & 4 & 1 \\ \varnothing & -1 & 6 & -9 \end{array}\right] \end{array}$$

Section 2: Quiz 4

Q1 Represent the system w/ an augmented Matrix called B:

$$4x + 2y = 8$$
$$-2x + 7y = 11$$
$$\Downarrow$$
$$\left[\begin{array}{cc|c} 4 & 2 & 8 \\ -2 & 7 & 11 \end{array}\right]$$

Q2 " " " " " " G:

$$a - 3b + 9c + 6d = 4$$
$$8a + 6c = 9d + 15$$
$$\downarrow$$
$$8a + \varnothing b + 6c - 9d = 15$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & -3 & 9 & 6 & 4 \\ 8 & \varnothing & 6 & -9 & 15 \end{array}\right]$$

(Q3) Represent linear system w/ an augmented matrix called ~~A~~: N:

$$6a + 4b - c = 9$$
$$5b = -6a + 7c - 6$$
$$3c = 14 - 2a$$

$\xrightarrow{\text{fix up:}}$

$$6a + 4b - c = 9$$
$$6a + 5b - 7c = -6$$
$$2a + 0b + 3c = 14$$

$$\left[\begin{array}{ccc|c} 6 & 4 & -1 & 9 \\ 6 & 5 & -7 & -6 \\ 2 & 0 & 3 & 14 \end{array}\right]$$

Section 2.16 Simple row operations

[1] $3x - 2y + z = 2$

[2] $-x - 5y + 4z = 1$

[3] $x + 4y - 6z = -9$

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 2 \\ -1 & -5 & 4 & 1 \\ 1 & 4 & -6 & -9 \end{array}\right]$$

① Switching [1] & [2]

$R_1 \leftrightarrow R_2$

$$\left[\begin{array}{ccc|c} -1 & -5 & 4 & 1 \\ 3 & -2 & 1 & 2 \\ 1 & 4 & -6 & -9 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

or

$R_2 \leftrightarrow R_1$

② Scaling Row Operation

multiply 3 thru [2]

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 2 \\ -3 & -15 & 12 & 3 \\ 1 & 4 & -6 & -9 \end{array}\right]$$

Multiply by 3:

$3R_2 \rightarrow R_2$

undo:

$\frac{1}{3}R_2 \rightarrow R_2$

③ Adding row operations:

④ We can also subtract

$R_2 + R_3 \longrightarrow R_2$ same as: $R_2 - (-R_3) \rightarrow R_2$

$$\left[\begin{array}{ccc|c} 3 & -2 & -1 & 2 \\ 0 & -1 & -2 & -8 \\ 1 & 4 & -6 & -9 \end{array}\right]$$

① Switching Rows ② Scaling Rows ③ Adding Rows

Section 2: Quiz 5

Q1 Perform $R_1 \leftrightarrow R_2$ on:

$$\begin{bmatrix} 1 & -2 & 5 \\ 6 & 7 & 0 \\ 7 & 4 & 9 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 6 & 7 & 0 \\ 1 & -2 & 5 \\ 7 & 4 & 9 \end{bmatrix} \checkmark$$

Q2 Perform $2R_2 \leftrightarrow 4R_3$ on:

$$\begin{bmatrix} 6 & 1 & 5 & -8 \\ -2 & 3 & 7 & 9 \\ 5 & -2 & 0 & 1 \end{bmatrix} \xrightarrow[4R_3]{2R_2} \begin{bmatrix} 6 & 1 & 5 & -8 \\ -4 & 6 & 14 & 18 \\ 20 & -8 & 0 & 4 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 6 & 1 & 5 & -8 \\ 20 & -8 & 0 & 4 \\ -4 & 6 & 14 & 18 \end{bmatrix} \checkmark$$

Q3 Perform $3R_1 + R_3 \to R_1$ $\begin{bmatrix} 7 & 8 & -2 & 0 \\ 5 & 1 & 6 & 13 \\ 4 & -7 & 3 & 9 \end{bmatrix}$

$3R_1 \to R_1$

$$\begin{bmatrix} 21 & 24 & -6 & 0 \\ 5 & 1 & 6 & 13 \\ 4 & -7 & 3 & 9 \end{bmatrix}$$

$R_1 + R_3 \to R_1$

$$\begin{bmatrix} 25 & 17 & -3 & 9 \\ 5 & 1 & 6 & 13 \\ 4 & -7 & 3 & 9 \end{bmatrix} \checkmark$$

2.18 Pivot Entries & Row-Echelon forms

→ Row Echelon:

1. Pivots are 1
2. Zero rows at the bottom
3. Staircase Pivots

→ Reduced Row-Echelon form:

4. Pivots alone in its own column

$$\left[\begin{array}{ccc|c} -1 & 0 & 2 & 1 \\ 3 & 1 & 1 & -2 \\ 1 & -4 & 0 & 2 \end{array}\right]$$

→ Pivots are first non-zero entry in each row, thus the pivots are $(-1, 3, 1)$

*"Pivots are 1"

Row 1 multiply by -1: $-1\cdot R_1 \to R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 3 & 1 & 1 & -2 \\ 1 & -4 & 0 & 2 \end{array}\right]$$

$R_2 - 3\cdot R_1 \to R_2$

~~$3\cdot R_1 - R_2 \to R_2$~~

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 7 & 1 \\ 1 & -4 & 0 & 2 \end{array}\right]$$

$R_3 - R_1 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 7 & 1 \\ 0 & -4 & 2 & 3 \end{array}\right]$$

$R_3 + 4\cdot R_2 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & 30 & 7 \end{array}\right]$$

$\frac{1}{30}\cdot R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & 1 & 7/30 \end{array}\right]$$

Pivots $(1, 1, 1)$

Row echelon form "Ref"

Reduced REF:

$R_1 + 2\cdot R_3 \to R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -8/15 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & 1 & 7/30 \end{array}\right]$$

$R_2 - 7\cdot R_3 \to R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -8/15 \\ 0 & 1 & 0 & -19/30 \\ 0 & 0 & 1 & 7/30 \end{array}\right]$$

$\frac{30}{30} - 49/30 = \frac{-19}{30}$

rref

$x = -8/15$

$y = -19/30$

$z = 7/30$

Pivots $(1, 1, 30)$

Section 2: Quiz 6

Q3 What is the rref of:

$$\begin{bmatrix} 3 & 9 & 15 & 96 \\ -4 & -12 & -18 & -118 \end{bmatrix}$$

$-36+36$

$-54+60$ $\quad$ $-54+45$

$-354+384$

$3 \cdot R_2 + 4 \cdot R_1 \to R_2$

$$\begin{bmatrix} 3 & 9 & 15 & 96 \\ 0 & 0 & \cancel{9}\ 6 & 30 \end{bmatrix}$$

$-\frac{1}{3} \cdot R_2 \to R_2$

$$\begin{bmatrix} 3 & 9 & 15 & 96 \\ 0 & 0 & -3 & -10 \end{bmatrix}$$

$\frac{1}{3} \cdot R_1 \to R_1$

$$\begin{bmatrix} 1 & 3 & 5 & 32 \\ 0 & 0 & -3 & -10 \end{bmatrix}$$

$-\frac{1}{9} \cdot R_2 \to R_2$

$$\begin{bmatrix} 3 & 9 & 15 & 96 \\ 0 & 0 & 1 & -10/3 \end{bmatrix}$$

$\frac{1}{3} R_1 \to R_1$

$$\begin{bmatrix} 1 & 3 & 5 & 32 \\ 0 & 0 & 1 & -10/3 \end{bmatrix}$$

$R_1 - 5 \cdot R_2 \to R_1$

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\frac{96}{3} - -\frac{50}{3}$ $\quad$ $3 \overline{)146}$ $\quad 12 \quad 26$

$\frac{146}{3} \to 146$

$\frac{1}{3} \cdot R_1 \to R_1$

$\frac{1}{6} R_2 \to R_2$

$$\begin{bmatrix} 1 & 3 & 5 & 32 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$R_1 - 5 \cdot R_2 \to R_1$

$$\begin{bmatrix} 1 & 3 & 0 & 7 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$ ✓

Section 2.19 Gauss-Jordan Elimination

Algorithm: "Gaussian Elimination"

① optional: Pull out any scalars from each row in the matrix

② If the first entry in the first row is zero, swap it with another row

③ Multiply first row by scalar to make leading entry equal to 1.

④ Add scaled multiples of the ~~1st~~ cur row to every other row in the matrix until every entry in 1st column is $\emptyset$ (besides the first row)

⑤ Go back to step 2 until matrix is in row echelon form. (Next col)

Ex

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 2 \\ -1 & -5 & 4 & 1 \\ 1 & 4 & -6 & -9 \end{array}\right]$$

$R_3 \leftrightarrow R_1$

$$\left[\begin{array}{ccc|c} 1 & 4 & -6 & -9 \\ -1 & -5 & 4 & 1 \\ 3 & -2 & 1 & 2 \end{array}\right]$$

$R_3 - 3 \cdot R_1 \to R_3$ & $R_2 + R_1 \to R_2$

$$\left[\begin{array}{ccc|c} 1 & 4 & -6 & -9 \\ 0 & -1 & -2 & -8 \\ 0 & -14 & 19 & 29 \end{array}\right]$$

$-1 \cdot R_2 \to R_2$

$$\left[\begin{array}{ccc|c} 1 & 4 & -6 & -9 \\ 0 & 1 & 2 & 8 \\ 0 & -14 & 19 & 29 \end{array}\right]$$

$R_1 - 4 \cdot R_2 \to R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & -14 & -41 \\ 0 & 1 & 2 & 8 \\ 0 & -14 & 19 & 29 \end{array}\right]$$

$14 \cdot R_2 + R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & -14 & -41 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 47 & 141 \end{array}\right]$$

step 5

$\frac{1}{47}R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & -14 & -41 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$14 \cdot R_3 + R_1 \to R_1$ & $R_2 - 2 \cdot R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

rref ✓

$x = 1$

$y = 2$

$z = 3$

Quiz 7: Gaussian Elimination

Q1 Solve w/ Gaussian Elimination

$x + 3y = 13$

$2x + 4y = 16$

↓ Matrix form

$$\left[\begin{array}{cc|c} 1 & 3 & 13 \\ 2 & 4 & 16 \end{array}\right]$$

$\frac{1}{2} \cdot R_2 \to R_2$ ↓ factor out Scalars

$$\left[\begin{array}{cc|c} 1 & 3 & 13 \\ 1 & 2 & 8 \end{array}\right]$$

$R_2 - R_1 \to R_2$

$$\left[\begin{array}{cc|c} 1 & 3 & 13 \\ 0 & -1 & -5 \end{array}\right]$$

$-1 \cdot R_2 \to R_2$

$$\left[\begin{array}{cc|c} 1 & 3 & 13 \\ 0 & 1 & 5 \end{array}\right]$$

$R_1 - 3 \cdot R_2 \to R_1$

$$\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 5 \end{array}\right]$$

$(x, y) = (-2, 5)$

Q2 Solve using Gaussian-Elimination w/ a rref:

$x + 4z = 11$

$x - y + 4z = 6$ $\quad R_1 \leftrightarrow R_3$

$2x + 9z = 25$

$$\left[\begin{array}{ccc|c} 1 & -1 & +4 & 6 \\ 2 & 0 & 9 & 25 \\ 1 & 0 & 4 & 11 \end{array}\right]$$

Try $R_2 - 2 \cdot R_1 \to R_2$

$2 \cdot R_1 - R_2 \to R_2$

$$\left[\begin{array}{ccc|c} 1 & -1 & 4 & 6 \\ 0 & +2 & +1 & +13 \\ 1 & 0 & 4 & 11 \end{array}\right]$$

$R_3 - R_1 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & -1 & 4 & 6 \\ 0 & -2 & -1 & -13 \\ 0 & 1 & 0 & 11 \end{array}\right]$$

$R_2 \longleftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & -1 & 4 & 6 \\ 0 & 1 & 0 & 11 \\ 0 & -2 & -1 & 13 \end{array}\right]$$

$R_1 + R_2 \longrightarrow R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 17 \\ 0 & 1 & 0 & 11 \\ 0 & -2 & -1 & 13 \end{array}\right]$$

$R_3 + 2 \cdot R_2 \longrightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 17 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & -1 & 35\ \cancel{13} \end{array}\right]$$

$\cancel{-1 \cdot R_3} \longrightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 6 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & -13 \end{array}\right]$$

$\cancel{4 \cdot R_3 - R_1} \longrightarrow R_1 \quad R_1 - 4 \cdot R_3 \longrightarrow R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 58 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & -13 \end{array}\right]$$

$$\begin{array}{cccc} \cancel{1} & \cancel{0} & \cancel{4} & \cancel{17} \\ 0 & \cancel{1} & \cancel{0} & \cancel{11} \\ 0 & \cancel{0} & 1 & \cancel{-35} \end{array}$$

$1 \quad 0 \quad 0$

Start over:

$$x + 0y + 4z = 11$$
$$x - y + 4z = 6$$
$$2x + 0y + 9z = 25$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 11 \\ 1 & -1 & 4 & 6 \\ 2 & 0 & 9 & 25 \end{array}\right]$$

$R_2 - R_1 \longrightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 11 \\ 0 & -1 & 0 & -5 \\ 2 & 0 & 9 & 25 \end{array}\right]$$

$R_3 - 2 \cdot R_1 \longrightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 11 \\ 0 & -1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$-1 \cdot R_2 \longrightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 11 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$\cancel{4 \cdot R_3 - R_1 \longrightarrow R_3}$

$R_1 - 4 \cdot R_3 \longrightarrow R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$(x, y, z) = (-1, 5, 3)$ ✓

Q3

$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 15 \\ 1 & 1 & 3 & 10 \\ 2 & 1 & 2 & 9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 15 \\ 0 & -1 & -2 & -5 \\ 2 & 1 & 2 & 9 \end{array}\right]$$

$R_3 - 2\cdot R_1 \rightarrow R_3$

$\cancel{1 \quad 2 \quad 5 \quad 15}$

$\cancel{0 \quad -1 \quad -2 \quad -5}$

$0 \quad -1 \quad -8$

$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 15 \\ 0 & -1 & -2 & -5 \\ 0 & -3 & -8 & -21 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 15 \\ 0 & 1 & 2 & 5 \\ 0 & -3 & -8 & -21 \end{array}\right]$$

$R_1 - 2\cdot R_2 \rightarrow R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 2 & 5 \\ 0 & -3 & -8 & -21 \end{array}\right]$$

~~$3R_2 + R_3 \rightarrow R_2$~~

$3\cdot R_2 + R_3 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & -2 & -6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$(x, y, z) = (2, -1, 3)$ ✓

Section 2.22 : Number of Solutions to the Linear System

One solution	No Solutions	Many Solutions
$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & A \\ 0 & 1 & 0 & b \\ 0 & 0 & 0 & c \end{array}\right]$	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 0 & \emptyset \end{array}\right]$

Ex:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -3 \\ 4 & 4 & -2 & 1 \\ -2 & 2 & -4 & 6 \end{array}\right]$$

$R_2 - 4 \cdot R_1 \longrightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -3 \\ 0 & 8 & -10 & 13 \\ -2 & 2 & -4 & 6 \end{array}\right]$$

$2 \cdot R_1 + R_3 \longrightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -3 \\ 0 & 8 & -10 & 13 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \infty \text{ Many solutions}$$

$\frac{1}{8} \cdot R_2 \longrightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -3 \\ 0 & 1 & -5/4 & 13/8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 + R_2 \longrightarrow R_1$

$\cancel{-\frac{24}{8}} + \frac{13}{8} = -\frac{11}{8}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3/4 & -11/8 \\ 0 & 1 & -5/4 & 13/8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Pivot col   Pivot   Free col

x pivot var

y pivot var

z is free var

$x = -\frac{3}{4}z - 11/8$

$y = \frac{5}{4}z + 13/8$

x & y depend on value for z

Section 2 Quiz 8

Q1 How many solutions for the following linear system?

$$\left[\begin{array}{ccc|c} -1 & -2 & 3 & -30 \\ -2 & -3 & -5 & 22 \\ 1 & 5 & 5 & -11 \end{array}\right]$$

$R_3 \leftrightarrow R_1$ & $R_2 + 2 \cdot R_1 \rightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & 5 & 5 & -11 \\ -2 & -3 & -5 & 22 \\ -1 & -2 & 3 & -30 \end{array}\right]$$

$2 \cdot R_1 + R_2 \rightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & 5 & 5 & -11 \\ 0 & 7 & 5 & 0 \\ -1 & -2 & 3 & -30 \end{array}\right]$$

$R_3 + R_1 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 5 & 5 & -11 \\ 0 & 7 & 5 & 0 \\ 0 & 3 & 8 & -41 \end{array}\right]$$

$\frac{1}{7} \cdot R_2 \rightarrow R_2$ & $\cancel{7 \cdot R_1 - 5 R_2 \rightarrow R_1}$

$$\left[\begin{array}{ccc|c} 1 & 5 & 5 & -11 \\ 0 & 1 & 5/7 & 0 \\ 0 & 3 & 8 & -41 \end{array}\right]$$

$\cancel{7 \cdot R_1 - 5 \cdot R_2 \rightarrow R_1}$

$R_1 - 5 \cdot R_2 \rightarrow R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{10}{7} & -11 \\ 0 & 1 & 5/7 & 0 \\ 0 & 3 & 8 & -41 \end{array}\right]$$

$\frac{35}{7} - \frac{25}{7} = \frac{10}{7}$

$R_3 - 3 \cdot R_2 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{10}{7} & -11 \\ 0 & 1 & 5/7 & 0 \\ 0 & 0 & 41/7 & -41 \end{array}\right] \quad \frac{56}{7} - \frac{15}{7}$$

$\frac{7}{41} \cdot R_3 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 10/7 & -11 \\ 0 & 1 & 5/7 & 0 \\ 0 & 0 & 1 & -7 \end{array}\right]$$

$\cancel{R_1 - 2 \cdot R_2 \rightarrow R_1}$

$\cdot \frac{7}{7}$

Stopping here. In theory can remove $\frac{10}{7}$ & $5/7$ to get rref of matrix by $\pm$ a fraction of R_3.

→ Looks to be only 1 solution

Q1 (again)

$$\left[\begin{array}{ccc|c} -1 & -2 & 3 & -30 \\ -2 & -3 & -5 & 22 \\ 1 & 5 & 5 & -11 \end{array}\right]$$

$-R_1 \rightarrow R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 30 \\ -2 & -3 & -5 & 22 \\ 1 & 5 & 5 & -11 \end{array}\right]$$

$2 \cdot R_1 + R_2 \rightarrow R_2$ &

$R_3 - R_1 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 30 \\ 0 & 1 & -11 & 82 \\ 0 & 3 & 8 & -41 \end{array}\right]$$

$30 - 164$

$R_1 - 2R_2 \rightarrow R_1$ & $3 \cdot R_2 - R_3 \rightarrow R_3$

$$\begin{array}{r} 246 \\ 41 \\ \hline 287 \end{array}$$

$-33 - 8$

$$\left[\begin{array}{ccc|c} 1 & 0 & 19 & -134 \\ 0 & 1 & -11 & 82 \\ 0 & 0 & -41 & 287 \end{array}\right]$$

$$41 \overline{)\,287\,}\ \ 7$$
287

$-\frac{1}{41} \cdot R_3 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 19 & -134 \\ 0 & 1 & -11 & 82 \\ 0 & 0 & 1 & -7 \end{array}\right]$$

$82 - 77$

$R_2 + 11 \cdot R_3 \rightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 19 & -134 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -7 \end{array}\right]$$

R_1

$R_1 - 19 \cdot R_3 \rightarrow R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -7 \end{array}\right] \quad -134 + 133$$

$(x, y, z) = (-1, 5, -7)$

$\therefore$ there is only 1 solution ✓

Q2 one solution, zero solution, or many solutions?

$$\left[\begin{array}{ccc|c} 3 & 9 & -3 & 24 \\ 1 & -3 & 11 & -2 \\ -2 & 5 & -20 & -5 \end{array}\right]$$

$\frac{1}{3}R_1 \rightarrow R_1$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 8 \\ 1 & -3 & 11 & -2 \\ -2 & 5 & -20 & -5 \end{array}\right]$$

$R_1 - R_2 \rightarrow R_2$ & $2 \cdot R_1 + R_3 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 8 \\ 0 & 6 & -12 & 10 \\ 0 & 11 & -22 & 11 \end{array}\right]$$

$\frac{1}{6}R_2 \rightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 8 \\ 0 & 1 & -2 & 5/3 \\ 0 & 11 & -22 & 11 \end{array}\right]$$

$8 - \frac{15}{3} \quad \frac{24}{3} - \frac{15}{3}$

$-22 - (11 \cdot -2)$

$R_1 - 3 \cdot R_2 \rightarrow R_1$ & $R_3 - 11 \cdot R_2 \rightarrow R_3$

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 3 \\ 0 & 1 & -2 & 5/3 \\ 0 & 0 & 0 & -22/3 \end{array}\right]$$

$11 - 11 \cdot 5/3$

$\frac{33}{3} - \frac{55}{3} = -\frac{22}{3}$

$0 \neq -22/3$

$\therefore$ No Solutions ✓

Q3 $1, \phi,$ or many solutions?

$$\left[\begin{array}{cccc|c} 1 & 2 & -3 & 7 & 4 \\ 3 & -1 & -2 & 0 & 12 \\ -2 & 5 & 0 & 2 & 5 \\ 2 & 3 & -5 & 11 & 8 \end{array}\right]$$

$3 \cdot R_1 - R_2 \to R_2$ & $2 \cdot R_1 + R_3 \to R_3$

& $2 \cdot R_1 - R_4 \to R_4$

$$\left[\begin{array}{cccc|c} 1 & 2 & -3 & 7 & 4 \\ 0 & 7 & -7 & 21 & 0 \\ 0 & 9 & -6 & 16 & 13 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right]$$

$\frac{1}{7} \cdot R_2 \to R_2$

$$\left[\begin{array}{cccc|c} 1 & 2 & -3 & 7 & 4 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 9 & -6 & 16 & 13 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right]$$

$R_1 - 2 \cdot R_2 \to R_1$ & $9 \cdot R_2 - R_3 \to R_3$

& $R_2 - R_4 \to R_4$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & -3 & 11 & -13 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since the entire row is all zeros, this has ∞ many solutions

$\phi z = \phi$

z

③ Operations on ℤ matrixes

→ Adding & Subtracting
  ↳ Dimensions must match

Eg:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ G & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

$$X + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

$$X = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$X = \begin{bmatrix} 6-1 & 8-2 \\ 10-3 & 12-4 \end{bmatrix}$$

$$X = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

(A, B, C are matrixes:

Addition:

$$A + B = B + A$$ (commutative)

$$(A+B)+C = A+(B+C)$$ ← Associative

Subtraction:

$$A - B \neq B - A$$

$$(A-B)-C \neq A-(B-C)$$

Section 3: Quiz 9

Q1 Add:

$$\begin{bmatrix} 4 & -3 & 6 \\ 8 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 1 \\ 11 & 4 & -9 \end{bmatrix} = \begin{bmatrix} 7 & -3 & 7 \\ 19 & 6 & -8 \end{bmatrix} \checkmark$$

Q2 Subtract:

$$\begin{bmatrix} 8 & 1 & 3 \\ 6 & -4 & 5 \\ 0 & 1 & 9 \end{bmatrix} - \begin{bmatrix} 6 & 12 & 5 \\ 5 & 1 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 8-6 & 1-12 & 3-5 \\ 6-5 & -4-1 & 5-0 \\ 0-(-2) & 1-7 & 9-2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -11 & -2 \\ 1 & -5 & 5 \\ 2 & -6 & 7 \end{bmatrix}$$

Q3 Solve for matrix X

$$\begin{bmatrix} 8 & 2 \\ 7 & 9 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = X + \begin{bmatrix} 5 & 7 \\ -5 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 6 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -1 \\ 4 & 8 \end{bmatrix} = X + \begin{bmatrix} 5 & 7 \\ -5 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 6 & -4 \end{bmatrix}$$

$$X = \begin{bmatrix} 6 & -1 \\ 4 & 8 \end{bmatrix} - \begin{bmatrix} 7 & 7 \\ 1 & 5 \end{bmatrix}$$

$$X = \begin{bmatrix} -1 & -8 \\ 3 & 3 \end{bmatrix} \checkmark$$

Section 3: 29 : Scalar Multiplication

→ Multiply entire matrix by a scalar:

$$2\begin{bmatrix}1 & 2 & 3\\0 & 1 & 1\end{bmatrix} = \begin{bmatrix}2 & 4 & 6\\0 & 2 & 2\end{bmatrix}$$

↓ division (multiply by reciprocal)

$$\frac{1}{2}\begin{bmatrix}2 & 4 & 6\\0 & 2 & 2\end{bmatrix} = \begin{bmatrix}1 & 2 & 3\\0 & 1 & 1\end{bmatrix}$$

$$1\begin{bmatrix}1 & 2 & 3\\0 & 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 2 & 3\\0 & 1 & 1\end{bmatrix}$$

$$0\begin{bmatrix}1 & 2 & 3\\0 & 1 & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$ zero matrix

---

Q1 Use scalar multiplication to simplify:

$$4\begin{bmatrix}5 & 2 & 1\\-2 & 4 & 7\end{bmatrix} \longrightarrow \begin{bmatrix}20 & 8 & 4\\-8 & 16 & 28\end{bmatrix}$$

Q2 Solve:

$$3\begin{bmatrix}7 & 1\\8 & 3\end{bmatrix} + X = -4\begin{bmatrix}0 & -5\\-2 & 3\end{bmatrix} \longrightarrow \begin{bmatrix}21 & 3\\24 & 9\end{bmatrix} + X = \begin{bmatrix}0 & 20\\8 & -12\end{bmatrix}$$

$$\cancel{X = \begin{bmatrix}0 & 20\\8 & -12\end{bmatrix} - \begin{bmatrix}21 & 3\\24 & 9\end{bmatrix}}$$

$$\cancel{X = \begin{bmatrix}21 & 23\\32 & 21\end{bmatrix}} \qquad X = \begin{bmatrix}-21 & 17\\-16 & -3\end{bmatrix}$$

$$X = \begin{bmatrix}0 & 20\\8 & -12\end{bmatrix} - \begin{bmatrix}21 & 3\\24 & 9\end{bmatrix}$$

$$X = \begin{bmatrix}-21 & 17\\-16 & -21\end{bmatrix}$$