THE SPECTRAL TOPOLOGY IN RINGS

by

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Declaration

By submitting this dissertation, I declare that the entirety of the work contained herein is my own original work and that I am the owner of the copyright thereof (unless explicitly otherwise stated) and that I have not previously submitted it or part thereof for obtaining any other qualification(s).

Date: January 2022

FJ Wessels

Dedication

To my Teachers:

Mnr. Haasbroek:	Wie aan ons die stappe van langdeëling vir die eerste keer verduidelik het deur om van komieklike ”karate-moves” gebruik te maak [Laerskool Villieria - 2003]
Mnr. Naudé:	Wie aan ons die wetenskaplike metode vir die eerste keer verduidelik het en my belangstelling vir ondersoek en navorsing geprikkel het [Laerskool Tini Vorster - 2004 tot 2005]
Mev. Horn:	My English teacher and guardian from Hoërskool Hugenote who encouraged me to rise above my circumstances [Hoërskool Hugenote - 2006 to 2010]
Mev. Vermaak:	Thank you for your encouragement in expressing my most creative thoughts [Hoërskool Hugenote - 2009 to 2010]
Mev. Loock:	Wie bo haar eie omstandighede gestyg het om ander te leer hoe om dieselfde te bereik [Hoërskool Hugenote - 2009 to 2010]
Mev. Roodt:	Wie imaginêre getalle en Calculus aan ons bekend gestel het [Hoërskool Hugenote - 2008]
Mev. Nortjé:	Wie Euklidiese meetkunde op ’n begrypbare grondslag vir die eerste keer aan ons oorgedra het [Hoërskool Hugenote - 2009]

To my Lecturers:

Prof. Pretorius:	Wie hoër-orde konsepte van Calculus op 'n eenvoudige, begrypbare wyse aan 'n gehoor kon oordra [Universiteit van Pretoria - 2012]
Prof. Sauer:	Wie die streng wiskundige metodes van parsiële differensiaalvergelykings die eerste keer aan ons verduidelik het [Universiteit van Pretoria - 2014]
Prof. Cinti:	Who played a great influence in my learning of the methods and techniques of partial differential equations via correspondence [University of Johannesburg - 2018]
Dr. Braadvedt:	Who has been a great coordinator, facilitator, mentor and go-to person. I wish I took your course on measure theory. [University of Johannesburg - 2018 to 2019]
Prof. Brits:	Wie die vak Funksionele Analise voor my oë op die skryfbord ontsyfer het - 'n ongelooflike ervaring en voorreg om u klas te kon bywoon [University of Johannesburg - 2018]
Prof. Raubenheimer:	To have witnessed his mathematical collaborative brilliance first-hand has been a great privilege for me throughout my learning experiences [University of Johannesburg]
Dr. Swartz:	I am the tool, he the mechanic Together to invent the engine I the builder, he the engineer Together to construct a machine I the magic, he the wizard To produce the spell of truth

To those who understand little of my interest in mathematics but much of my journey to success:

Vanessa:	(Mother) The source of nurture in my life. I am proud to be your son
Louis:	(Father) Who, despite having difficulties of your own, taught me the importance of balance in life
Ingrid:	(Sister) Hoewel ons hemelsbreëd verskil, kon ons net sowel ’n tweeling gewees het met al die liefde wat ons deel
Marincy:	(Ouma) My liefde vir u strek verder as die landlyn Kaap toe
Tiny & Shmally:	My kitty meauw meauws

Ek het julle oneindig-maal lief

Darryn Jacobs:	As ek aan jou dink, dink ek aan ’n broer
Sean Lubane:	To my dearest friend who, despite having been temporarily homeless during senior year, showed persistence and determination in finishing what he started
Jerome & Amber Bastick:	Ware vriendskap, die jare saam julle is vir my baie kosbaar
Bryce Mapole:	Whose leadership qualities and inspirational character influenced my contribution to Africa
Celine de Villiers:	The first person to believe in my capabilities
Dawie, Leticia & Louis:	Dankie vir julle bystand gedurende baie moeilike tye
Gagie:	Your positivity shines bright
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To My Future Wife:	Please answer your DM’s

I dedicate this dissertation to you

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Introduction

A careful study of Hilbert [22] and Cauchy's [6] original work reveals that the word *characteristique* in French translates to "eigenschaften" in German ("eienskap" in Afrikaans) which may be interpreted as *intrinsic* or *property* in the sense that it associates a(n) (set of) eigenvalue(s) λ [thought of as the seed] belonging to an operator T [thought of as the stem]. It is due to this operator-eigenvalue tug'o war that most of the modern theory has become so well developed to model physical problems leading to insightful results. The word "integralgleichung" is used frequently throughout the text which translates to English as *integral equation*. In his article Hilbert discusses the symmetric kernel on page 52, characteristic determinant equations on page 53, linear combinations and algebraic solutions to the integral equations on page 57, convergent power series solution of Fredholm determinant expression on page 58, orthogonality and eigenfunction solutions of integral equations on page 67, essentially the foundation for classical operator theory.

Long before the power of vector space algebra was postulated into existence and became common practice, the eigenvalue method was one of the tools used to study systems of linear equations as roots of determinant equations, but made some of its first appearance in the studies of differential and integral equations originating from variational calculus during the celestial era of mathematics, giving birth to operator theory. Operator theory can be chronologically tracked through Leonard Euler (1707 - 1783), Jean le Rond d'Alembert (1717 - 1783), Joseph Louis Lagrange (1736 - 1813), Daniel (1700 - 1782) & the three Bernoulli brothers (1710 - 1790, 1744 - 1807, 1759 - 1789), Pierre Simon Laplace (1749 - 1827), Joseph Fourier (1768 - 1830), Johann Peter Gustav Lejeune Dirichlet (1805 - 1859), who all played a big part in pursuit of an even bigger picture - *Modelling, analysing, solving and generalising functional equations.*

Sometimes eigenvalues are referred to as the roots of characteristic equations, those equations deeply associated with its homogeneous auxiliary equations and in modern mathematics related to the spectrum, the set of values in a scalar field associated with algebraic invertibility of an operator expression of the form $T - \lambda I$. In fact, eigenvalues arise naturally in the generalization of operator equations as the following examples illustrate:

Example 0.0.0.1 Consider the homogeneous linear system of differential equations:

$$\tfrac{d}{dt}\overline{x}(t) = A\overline{x}(t)$$

where $\overline{x}(t)$ is a column of functionals dependent on t and A is the coefficient matrix for the system. It is an elementary exercise to verify upon substitution that $\overline{x}(t) = v(t)e^{\lambda t}$ reduces the equation to $\lambda\overline{v} = A\overline{v}$ not surprisingly, an ordinary eigenvalue problem which may be written more recognizably as $(A - \lambda I)\overline{v} = \overline{0}$ with the remaining task to compute the eigenvalues λ and eigenvectors $\overline{v}$. □

Example 0.0.0.2 Consider the classical homogeneous Fredholm integral equation:

$$g(x) = \mu \int_a^b K(x,t)g(t)\ dt$$

Let $\mu = \frac{1}{\lambda}$ and define the integral transform by $(Tg)(x) := \displaystyle\int_a^b K(x,t)g(t)\ dt$ so that the above may be written more compactly as $g(x) = \frac{1}{\lambda}Tg(x)$. Multiplying λ throughout gives $\lambda g(x) = Tg(x)$ which can be rearranged as $Tg(x) - \lambda g(x) = 0$ and finally factoring $g(x)$, the equation is recognizable as an ordinary eigenvalue problem $(T - \lambda I)g(x) = 0$. □

Example 0.0.0.3 Consider the operator equation $(Tu)(x,t) = (Su)(x,t)$ for an unknown two variable function $u(x,t)$ dependent on position x along the closed interval $[0,l]$ and time t, described on the domain $\mathscr{D}(u) = \{(x,t) : x \in [0,l], t \geq 0\}$.

If we assign $T = \frac{\partial}{\partial t}$ and $S = \alpha\frac{\partial}{\partial x}$ then we recover the familiar transport equation.

If we assign $T = \frac{\partial}{\partial t}$ and $S = \alpha^2\frac{\partial^2}{\partial x^2}$ then we recover the familiar heat equation.

If we assign $T = \frac{\partial^2}{\partial t^2}$ and $S = \alpha^2\nabla^2$ then we recover the familiar wave equation.

We wish to find a product solution of the form $u(x,t) = f(x)g(t)$ so that the equation becomes separable $\frac{Tg}{g} = \frac{Sf}{f}$ in which case this reduces to the recognizable ordinary eigenvalue problems:

$$Tg = \lambda g \quad \text{and} \quad Sf = \lambda f$$

but as it turns out, separability of operator equations is related to the spectral theorem. The applicability of the technique is intrinsic to the coordinate system in which the equation is modelled and depends on the symmetry properties of the equation. □

In each example above, the equations were reducible to an ordinary linear eigenvalue problem. The foundations of the solvability theory for such problems had already mostly been explored by researchers such as the three prodigies, Johann Carl Friedrich Gauss (1777 - 1855) (in his *Disquisitiones Arithmeticae*), Évariste Galois (1811 - 1832), and Niels Hendrik Abel (1802 - 1829). This is the insight which motivated mathematicians to study operator equations collectively as a subject. However, a great deal of the behaviour of operator equations is due to the underlying structure space on which the equation is defined. For example continuity, convergence, invertibility, integrability and differentiability are all in one way or another, dependent, not only on the form of the equation, but also largely on the underlying algebraic and topological aspects of the space.

The demand for clear insight about the underlying structure space for operator equations further stimulated research in the fields of topology and algebra. The story diverts to set theory and the axiomatization of mathematics. Georg Ferdinand Ludwig Philipp Cantor's (1845 - 1918) set theory had been digested by the mathematics community and axiomatized by Ernst Zermelo (1871 - 1953) in his 1908 paper *Untersuchengen über die Grundlagen der Mengenlehre* [”mengenlehre” in German translates to *set theory* in English] while Giuseppe Peano fully axiomatized linear spaces in his 1888 book, titled *Calcolo*. Cantor had already loosely defined the notions of open, closed and derived sets, influencing René-Louis Baire (1874 - 1932), Émile Borel (1871 - 1956) and Henri Lebesgue (1875 - 1941) to write his phenomenal piece, titled *Sur l'approximation des fonctions*, thereby extending the ideas of George Friedrich Bernhard Riemann (1826 - 1866) with strong reliance on set theory, consequently giving rise to the subjects of measure and integration theory as it is taught in current times.

Set theory shed new light on the concept of a function which inspired Maurice René Fréchet (1878 - 1973) to fabricate the axioms of a metric space, in doing so, extending Leibniz and Newton's classical calculus of functions to topology, roughly phrased from the mouth of contemporary mathematical dialect. Felix Hausdorff (1868 - 1942) extended much of topology and set theory by introducing the concepts of neighbourhoods, neighbourhood systems and partially ordered sets, associated with Zorn's lemma, in his 1914 piece *Grundzüge der Mengenlehre*. By now, the *duality principle* seeped its way through to function spaces found in the works of Hans Hahn (1879 - 1934), Frigyes Riesz (1880 - 1956), Stefan Banach (1892 - 1945), Juluisz Schauder (1899 - 1943), Giulio Ascoli (1843 - 1896) and Cesare Arzelá (1847 - 1912) who all played roles on elaborating the concepts of Bernard Bolzano (1781 - 1848), Karl Theodor Wilhelm Weierstrass (1815 - 1897) and other contemporaries to sequences of functions (boundedness, continuity, convergence, compactness), representations of linear functionals, functional domain extension, normed spaces, completeness of spaces, etc...

The power of the *duality principle* was once again emphasised when Kazimierz Kuratowski (1896 - 1980) defined his closure algebra, a dual approach for constructing a topology on any collection, in his 1922 paper [25], a discovery that would become extremely important to the marriage of algebra and topology. In the words of Herman Weyl (1885 - 1955) - "...the angel of topology and the devil of abstract algebra fight for the soul... " A substantial amount of modern mathematical content for algebraic topology is due to the team of writers, under the name Nicolas Bourbaki (1934 - 1935) in their series of textbooks.

For the past couple of decades, attempts have been made (Oscar Zariski (1899 - 1986)) to find an analytic expression which generates a topology on a ring. The first question that would come to mind for the majority of people is "why do we want such a structure?"

To answer this question we heavily rely on the motto *modelling, analysing, solving and generalising functional equations* to argue our point of view that it is convenient to know the topological and algebraic features of an operator all at once, since this will allow us to decide whether it may have certain desirable properties such as being (left/right (nearly)) invertible, open, continuous, quasinilpotent, etc...

This dissertation celebrates the work of Robin Harte and Dragana Cvetković-ilić who recently succeeded in the task of defining a topology on a ring via a Kuratowski closure operation, leading to new insights for operators - the main theme of this dissertation. Below is a brief overview of the content of various chapters.

In Chapter 1 we define basic concepts that we need to understand the content that follows.

In Chapter 2 we prove that the construction of the authors is a Kuratowski closure operation. We also discuss some properties of the closure operation.

In Chapter 3 we compare the spectral topology with the norm topology on a Banach algebra. We also look at additional properties, including the structure of neighbourhoods of the spectral topology.

In Chapter 4 we look at how the spectral closure enables us to define a concept of quasinilpotent that applies to a general ring.

In Chapter 5 we look at how the spectral closure intervenes in concepts of generalized invertibility.

In Chapter 6 we look at how the spectral closure intervenes with Fredholm Theory relative to a Banach algebra homomorphism.

In Chapter 7 we look at how the spectral closure intervenes in the concepts of Bass Stable Rank of a ring.

In Chapter 8 we give a brief summary of what was achieved, as well as highlight some questions raised by the study.

✠

Chapter 1

Rings, topologies, Banach algebras

1.1 Introduction

This chapter provides an overview of all the prerequisites needed for our discussions in coming chapters. The main mathematical structures we will encounter are rings, topologies, topologocal rings and Banach algebras. In this chapter these structures are defined and discussed in the detail necessary to make the theory that follows understandable. Groups and vector spaces are considered to be understood and are not defined.

1.2 Rings

Definition 1.2.0.1 ([15], p. 95) A *ring* is a nonempty set R on which there are defined two binary operations, $+$ and $\cdot$, called addition and multiplication respectively, which satisfy the following axioms:

R1. with respect to $+$, R is an abelian group.

R2. $\cdot$ is associative: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in R$.

R3. The following distributive laws are satisfied (for all $a, b, c \in R$)

$$a \cdot (b + c) = a \cdot b + a \cdot c \qquad \text{and} \qquad (b + c) \cdot a = b \cdot a + c \cdot a.$$

If R is a ring with operations of $+$ and $\cdot$ we will also use the triple $\langle R, +, \cdot \rangle$ to represent the ring structure.

If R is a ring and $a \cdot b = b \cdot a$ for all $a, b \in R$, then R is called *commutative*.

If R is a ring then by R1 it has a unique *additive identity*, which we denote by 0, which satisfies the condition that $0 \cdot a = a \cdot 0 = 0$ for all $a \in R$.

Let R be a ring and suppose there exists $1 \in R$ with the property that for all $a \in R$

$$1 \cdot a = a = a \cdot 1.$$

We will call such an element an *identity element*, *multiplicative identity* or simply an *identity* of the ring. A ring with an identity is referred to as a *ring with unity* or a *unital ring*. It is not necessary for a ring to have an identity, but if it does then the identity is unique. This is easily seen to be the case as follows. Suppose 1 and $1'$ are identities of the ring. Then it is clear (from the properties of 1 and $1'$) that we must have

$$1 = 1 \cdot 1' = 1'.$$

If a ring R contains an identity with respect to multiplication, we will write 1_R if we need to emphasize the ring R. In this dissertation every ring will have an identity. From this point the expression 'Let R be a ring' will mean 'Let R be a ring with identity'.

If $\langle R, +, \cdot \rangle$ is a ring then we will usually suppress the multiplication sign, so that for $a, b \in R$, ab will have the same meaning as $a \cdot b$. The symbol for multiplication will be used only if we feel readability will be enhanced by its use.

Let R be a ring. An element $a \in R$, is called *left* (respectively *right*) *invertible* if there exists $b \in R$ such that $ba = 1$ (respectively $ab = 1$). The element b is called a *left* (respectively *right*) *inverse* of a.

If an element $a \in R$ is both left and right invertible, then the left and right inverses for a coincide, as we shall now illustrate. If a is left invertible and right invertible then there exist $b, c \in R$ such that $ba = 1$ and $ac = 1$. So

$$b = b \cdot 1 = b \cdot (a \cdot c) = (b \cdot a) \cdot c = 1 \cdot c = c. \tag{1.1}$$

In this last case a is called *invertible* and its left (and right) inverse is called an *inverse*. This argument also shows that the inverse of an element, if it exists, must be unique. This is so because an inverse is both a left inverse and a right inverse. The inverse of a is denoted by a^{-1}. The sets of left invertible, right invertible and invertible elements in R are denoted by R_l^{-1}, R_r^{-1} and R^{-1} respectively.

Example 1.2.0.2 ([15], p. 97) Let $R = \{0\}$, and let us define addition and multiplication by:

$$0 + 0 = 0 \quad \text{and} \quad 0 \cdot 0 = 0.$$

Then the axioms R1 - R3 are satisfied in a trivial way. R is a ring called the *trivial ring*. Note that R is commutative with identity 0. Note also that in R, we have $1_R = 0_R$. □

In what follows we will always assume that an arbirary ring is not the trivial ring.

An element a of a ring R is said to be *nilpotent* if there exists some $n \in \mathbb{N}$ such that $a^n = 0$.

Remark 1.2.0.3 Suppose $G, H \subseteq R$ and $K \subseteq R^{-1}$ for a ring R. We will use the following notations:

(a) $K^{-1} = \{k^{-1} : k \in K\}$.

(b) $G + H = \{g + h : g \in G, h \in H\}$.

(c) $G \cdot H = \{g \cdot h : g \in G, h \in H\}$.

(d) $a + G = \{a + g : g \in G\}$.

□

The following lemma lists a number of properties of invertible elements which we will need to refer to.

Lemma 1.2.0.4 Let R be a ring. Then

(a) If $a, b \in R^{-1}$ then $ab \in R^{-1}$.

(b) If $ab \in R^{-1}$ then $a \in R_r^{-1}$ and $b \in R_l^{-1}$.

(c) If $ab \in R^{-1}$ and $b \in R^{-1}$ then $a \in R^{-1}$.

(d) If $a \in R^{-1}$ and $b \notin R^{-1}$ then $ab \notin R^{-1}$.

□

Remark 1.2.0.5 We can say more about the structure of the set of invertibles in a ring. If $\langle R, +, \cdot \rangle$ is a ring, then $\langle R^{-1}, \cdot \rangle$ is a group. We will refer to this group as the *group of units* or *the invertible group* of the ring R. □

Next, we define types of rings encountered throughout the dissertation:

Definition 1.2.0.6 ([13], p. 252) Let R be a ring and suppose that $a, b \in R$. Suppose also that $a \neq 0$ and $b \neq 0$, but that $a \cdot b = 0$. Then we call a and b *divisors of zero*. In particular, a is a *left* divisor of zero and b is a *right* divisor of zero.

Definition 1.2.0.7 ([13], p. 254) An *integral domain* is a commutative ring with unity containing no divisors of zero.

Example 1.2.0.8 ([14], p. 249) The ring of integers is an integral domain. □

Proposition 1.2.0.9 ([15], p. 101) Let R be an integral domain, $a, b, c \in R, a \neq 0$. Suppose that $a \cdot b = a \cdot c$. Then $b = c$. □

Definition 1.2.0.10 A *division ring* is a ring with the property that every nonzero element is invertible.

Definition 1.2.0.11 A *Boolean ring* is a ring R with the property that $r^2 = r$ for every $r \in R$.

Remark 1.2.0.12 If R is a Boolean ring, then $R^{-1} = \{1\}$. To see this suppose $r \in R^{-1}$. Then

$$r = r \cdot 1 = r \cdot (r \cdot r^{-1}) = r^2 \cdot r^{-1} = r \cdot r^{-1} = 1.$$

□

It is well known that $\mathbb{Z}$, together with the usual addition and multiplication of integers is a ring. In what follows we will assume some of the basic properties of this ring. This includes the fact that it is a ring with unity and some of the order theoretic properties of the ring. Developing these from scratch would take us too far from the main focus of this dissertation. The details of these properties of the ring of integers is developed in Chapter 2 of [15].

Example 1.2.0.13 $\mathbb{Z}$ with usual addition $+$ and multiplication $\cdot$ is a ring. We show that $\mathbb{Z}^{-1} = \{-1, 1\}$. Consider arbitrary $a, b \in \mathbb{Z}$ such that $ab = 1$. We show that either $a = b = 1$ or $a = b = -1$.

Clearly $a \neq 0$ and $b \neq 0$. We prove the statement above using a proof by cases argument, based on the possible values for a. These values are:

$$a > 1, \qquad a < -1, \qquad a = 1, \qquad a = -1.$$

Suppose that $a > 1$. If $b > 1$ then $ab > 1$, contradicting $ab = 1$. If $b < -1$ then $ab < -1$, contradicting $ab = 1$. If $b = 1$, then $ab > 1$, again contradicting $ab = 1$. Finally, if $b = -1$, then $ab < -1$, again leading to a contradiction.

The argument for $a < -1$ follows a similar pattern, with signs and inequalities reversed.

Suppose that $a = 1$. Then we must have $b = 1$, for suppose $b \neq 1$. Then $ab = 1 \cdot b = b \neq 1$, contradicting that $ab = 1$.

Finally, suppose that $a = -1$. Then necessarily $b = -1$ since otherwise $ab = (-1) \cdot b = -b \neq 1$. $\square$

Example 1.2.0.14 ([15], p. 100) Let R and S be any two rings, and let $R \times S$ denote the Cartesian product of R and S as sets. Let $(r, s), (r', s') \in R \times S$. Then we define addition and multiplication in $R \times S$ by

$$(r, s) + (r', s') = (r + r', s + s'),$$
$$(r, s)(r', s') = (rr', ss')$$

With $+$ and $\cdot$ defined as above we have that $\langle R \times S, +, \cdot \rangle$ is a ring. This structure is sometimes referred to as the *direct sum* of R and S and denoted by $R \oplus S$. Checking that $R \oplus S$ is a ring is a routine exercise. So, instead of checking every condition (R1 - R3) we look at only the details relating to the zero element and invertibility in the product ring, $R \oplus S$, since those details will be pertinent for the discussions to come.

Let 0_R and 0_S be the zero elements in R and S respectively, and let 1_R and 1_S be the multiplicative identities in R and S respectively.

First we show that $(0_R, 0_S)$ is the zero element in the product ring. To see this, let $(r, s) \in R \oplus S$. Then

$$(r, s) \cdot (0_R, 0_S) = (r \cdot 0_R, s \cdot 0_S) = (0_R, 0_S)$$

and

$$(0_R, 0_S) \cdot (r, s) = (0_R \cdot r, 0_S \cdot s) = (0_R, 0_S).$$

Also,

$$(0_R, 0_S) + (r, s) = (0_R + r, 0_S + s) = (r, s)$$

and

$$(r, s) + (0_R, 0_S) = (r + 0_R, s + 0_S) = (r, s).$$

Hence $(0_R, 0_S)$ is the zero element in $R \oplus S$. Next,

$$(r, s) \cdot (1_R, 1_S) = (r \cdot 1_R, s \cdot 1_S) = (r, s)$$

and

$$(1_R, 1_S) \cdot (r, s) = (1_R \cdot r, 1_S \cdot s) = (r, s).$$

Hence $(1_R, 1_S)$ is the multiplicative identity in $R \oplus S$.

Next, let R^{-1} and S^{-1} be the sets of invertibles in R and S respectively. Then

$$(R \oplus S)^{-1} = R^{-1} \times S^{-1}.$$

To see this, let $(r, s) \in (R \oplus S)^{-1}$. Then there must be $(r', s') \in R \oplus S$ such that

$$(r', s') \cdot (r, s) = (1_R, 1_S)$$

and

$$(r, s) \cdot (r', s') = (1_R, 1_S).$$

These equations imply that $r' = r^{-1}$ and $s' = s^{-1}$. Hence $(r, s) \in R^{-1} \times S^{-1}$, which means $(R \oplus S)^{-1} \subseteq R^{-1} \times S^{-1}$.

Conversely, let $(r, s) \in R^{-1} \times S^{-1}$. Then $r \in R^{-1}$ and $s \in S^{-1}$. Hence the inverses r^{-1} and s^{-1} exist. The equations

$$(r, s)(r^{-1}, s^{-1}) = (1_R, 1_S)$$

and

$$(r^{-1}, s^{-1})(r, s) = (1_R, 1_S)$$

prove that $(r, s) \in (R \oplus S)^{-1}$. Hence $R^{-1} \times S^{-1} \subseteq (R \oplus S)^{-1}$. We have proved that

$$(R \oplus S)^{-1} = R^{-1} \times S^{-1}.$$

$\square$

Lemma 1.2.0.15 ([7], p. 86) Let R be a ring and $a, b \in R$. Then

$$1 - ab \in R^{-1} \iff 1 - ba \in R^{-1} \tag{1.2}$$

$\square$

Remark 1.2.0.16 Let R be a ring and $a, b \in R$. We note that condition (1.2) in Lemma 1.2.0.15 is equivalent to the condition

$$1 - ab \notin R^{-1} \iff 1 - ba \notin R^{-1} \tag{1.3}$$

$\square$

1.2.1 Ideals

Definition 1.2.1.1 ([15], p. 114) Let R be a ring. A subset S of R is called a *subring* of R if S is a ring with respect to the operations of addition and multiplication inherited from R.

Theorem 1.2.1.2 ([15], p. 115) Let S be a nonempty subset of a ring R. Then S is a subring of R if and only if for $a, b \in S$, we have $a - b \in S$ and $ab \in S$. $\square$

Definition 1.2.1.3 ([15], p. 117) Let R be a ring and let I be a subring of R. We say that I is a *left (right) ideal* of R if for all $r \in R$ and $a \in I$ we have that $ra \in I$ ($ar \in I$). An ideal I which is both a left and right ideal of R, is simply called an *ideal* of R.

For any ring R, R itself is an ideal of R, called the *improper ideal*. All other ideals of R are *proper ideals*. For any ring R, the singleton set $\{0\}$ is an ideal, called the *trivial ideal*. All other ideals of R are *non-trivial ideals*.

Lemma 1.2.1.4 Let R be a commutative ring with identity and I an ideal of R. If $I \cap R^{-1} \neq \emptyset$ then $I = R$.

Proof: Let R and I be as described and suppose that $a \in I \cap R^{-1}$. We have that $I \subseteq R$. We show that $R \subseteq I$. Since $a \in R^{-1}$, a has an inverse, a^{-1} in R. Since I is an ideal we have $1 = a^{-1}a \in I$. Let $r \in R$ be arbitrary. Then $r = r \cdot 1 \in I$, hence $R \subseteq I$. Hence $I = R$. $\square$

Definition 1.2.1.5 ([13], p. 322) A *maximal left (right) ideal* of a ring R is a left (right) ideal I of R different from R such that there is no proper left (right) ideal of R properly containing I. An ideal which is both a maximal left ideal and a maximal right ideal is simply called a maximal ideal.

Proposition 1.2.1.6 ([15], p. 118) Let R be a ring and I an ideal of R. Denote by R/I the set of *cosets* of the form $a + I$ where $a \in R$. Define addition and multiplication of cosets in R/I by:

$$(a + I) + (b + I) = (a + b) + I \qquad \text{and} \qquad (a + I) \cdot (b + I) = a \cdot b + I.$$

With these definitions R/I is a ring called the *quotient ring* of R modulo I. $\square$

Remark 1.2.1.7 The ring R/I partitions the ring R into a set of pairwise disjoint equivalence classes. The set $a + I$ is the equivalence class containing the element a. We will also use the equivalent notation $[a]$ for the equivalence class of a. Hence $[a] = a + I$. If $[a], [b]$ are elements in R/I, then $[a] = [b] \iff a - b \in I$. $\square$

Theorem 1.2.1.8 ([32], p. 347) (Krull's Theorem) Let R be a ring with unity and let I be a proper (left/right) ideal of R (respectively). Then there is a maximal (left/right) ideal of R containing I (respectively). $\square$

1.2.2 The Jacobson radical

We define the notions relating to an important ideal, called the *Jacobson radical*.

Definition 1.2.2.1 ([26], p. 50) Let R be a ring and $\mathcal{M}_l$ be the collection of all maximal left ideals of R. The *Jacobson radical* of R is the intersection of all maximal left ideals of R:

$$\operatorname{Rad} R = \bigcap_{I \in \mathcal{M}_l} I$$

Strictly speaking, $\operatorname{Rad} R$ is the left Jacobson radical. Similarly, the right Jacobson radical of R is defined by intersecting the maximal right ideals $I \in \mathcal{M}_r$ of R. It turns out, fortuitously, that the left and right Jacobson radicals coincide, so the distinction is, after all, unnecessary.

The following proposition is used to characterize the Jacobson radical of a ring R in terms of its group of invertible elements, R^{-1}.

Proposition 1.2.2.2 ([26], Lemma 4.1 - p. 50) Let R be a ring. Then

$$\operatorname{Rad} R = \{a \in R : 1 - Ra \subseteq R^{-1}\}$$

□

Definition 1.2.2.3 ([26], p. 52) Let R be a ring. If $\operatorname{Rad} R = \{0\}$, then R is said to be a *semisimple* ring.

Example 1.2.2.4 $\langle \mathbb{Z}, +, \cdot \rangle$ is semisimple. To see why this is the case, recall that $x \in \operatorname{Rad} \mathbb{Z}$ if and only if $1 - xr \in \mathbb{Z}^{-1}$ for all $r \in \mathbb{Z}$. From Example 1.2.0.13 we have $\mathbb{Z}^{-1} = \{-1, 1\}$ so that $x \in \operatorname{Rad} \mathbb{Z}$ if and only if for all $r \in R$ we have that $1 - xr = -1$ or $1 - xr = 1$, which simplifies to $xr = 0$ or $xr = 2$. Upon substitution of $r = \pm 1$ we get $x = 2$ and $x = -2$, an impossibility, or $x = 0$ necessarily. □

Definition 1.2.2.5 ([26], p. 279) A ring with a unique maximal ideal is called *local*.

It is not true in general that a ring has a unique maximal ideal. For example the ring of integers $\mathbb{Z}$ has infinitely many maximal ideals. If $I = p\mathbb{Z}$ where p is prime, then I is a maximal ideal in $\mathbb{Z}$ ([14], page 269, Exercise 9).

Proposition 1.2.2.6 Let R be a local ring. Then $R = R^{-1} \cup \operatorname{Rad} R$.

Proof: The fact that $R^{-1} \cup \operatorname{Rad} R \subseteq R$ is obvious. It remains to show that $R \subseteq R^{-1} \cup \operatorname{Rad} R$. Consider arbitrary $a \in R$. If $a \in R^{-1}$, we are done. So suppose that $a \notin R^{-1}$. Then either $a \notin R_l^{-1}$ or $a \notin R_r^{-1}$. Without loss of generality, suppose that $a \notin R_r^{-1}$. Then aR is a proper right ideal of R. By Theorem 1.2.1.8, aR is contained in a maximal right ideal. But $\operatorname{Rad} R$ is the unique maximal right ideal, hence $a = a \cdot 1 \in aR \subseteq \operatorname{Rad} R$ and the result follows. The case in which $a \notin R_l^{-1}$ is similar. □

Proposition 1.2.2.7 Let R be a commutative ring and let $I = R \setminus R^{-1}$. If I is an ideal then I is the unique maximal ideal of R.

Proof: Suppose that R is a commutative ring with identity and that $I = R \setminus R^{-1}$ is an ideal. Notice that since $1 \notin I$, we know that I is a proper ideal. To see that I is maximal, suppose that there exists an ideal J such that $I \subseteq J \subseteq R$. Suppose also that $I \neq J$. Hence there exists $a \in J$ such that $a \notin I$. But then $a \in R^{-1}$. Hence $J \cap R^{-1} \neq \emptyset$. By Lemma 1.2.1.4 we have that $J = R$, and so I must be maximal.

To see that I is unique, suppose that J is another maximal ideal of R. Then J must be a proper ideal, so that $J \cap R^{-1} = \emptyset$. That means that $J \subset R \setminus R^{-1} = I$. Since J is a maximal ideal it cannot be properly contained in any other proper ideal, hence $J = I$, and I is unique. $\square$

Example 1.2.2.8 ([33], p. 3) Let R be a commutative ring. Denote by $R[[x]]$ the set of all infinite expressions of the form

$$a_0 + a_1x + a_2x^2 + \cdots + a_ix^i + \cdots = \sum_{i=0}^{\infty} a_ix^i \tag{1.4}$$

with coefficients a_i coming from the ring R. In (1.4), x is an indeterminate, i.e. x is not in R and is not a solution of any algebraic equation with coefficients in R. Also, even though the symbol $'+'$ is used, the expression does not really represent addition. The expression in (1.4) is actually a convenient way to express the infinite sequence

$$(a_n)_{n\in\mathbb{N}} = (a_0, a_1, a_2, \ldots, a_i, \ldots). \tag{1.5}$$

The alternative notation (1.4) is sometimes preferable. We will use both notations.

If (a_n) and (b_n) are two elements of $R[[x]]$ then $(a_n) = (b_n)$ if and only if $a_i = b_i$ for all $i \in \mathbb{N}$.

Let $\sum_{i=0}^{\infty} a_ix^i, \sum_{i=0}^{\infty} b_ix^i \in R[[x]]$. Then we define on $R[[x]]$ operations of addition and multiplication as:

$$\sum_{i=0}^{\infty} a_ix^i + \sum_{i=0}^{\infty} b_ix^i = \sum_{i=0}^{\infty}(a_i + b_i)x^i$$

and

$$\Big(\sum_{i=0}^{\infty} a_i x^i\Big) \cdot \Big(\sum_{j=0}^{\infty} b_j x^j\Big) = \sum_{k=0}^{\infty} c_k x^k$$

where, for each integer $k \geqslant 0$,

$$c_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0 \tag{1.6}$$

With these operations $R[[x]]$ becomes a ring, called the *ring of formal power series over* R. In this ring, the additive and multiplicative identities are given by

$$(0, 0, 0, \dots) \text{ and } (1, 0, 0, \dots)$$

respectively. □

Lemma 1.2.2.9 ([26], p. 8) Let $R[[x]]$ be the ring of all formal power series over R in the indeterminate x as described in Example 1.2.2.8. Then an arbitrary formal power series $\sum_{i=0}^{\infty} a_i x^i$ is invertible in $R[[x]]$ if and only if its leading coefficient a_0 is invertible in R.

Proof:

Suppose $\sum_{i=0}^{\infty} a_i x^i$ is invertible in $R[[x]]$. Then there exists $\sum_{j=0}^{\infty} b_j x^j \in R[[x]]$ such that

$$\Big(\sum_{i=0}^{\infty} a_i x^i\Big) \cdot \Big(\sum_{j=0}^{\infty} b_j x^j\Big) = \sum_{k=0}^{\infty} c_k x^k \tag{1.7}$$

and

$$\Big(\sum_{j=0}^{\infty} b_j x^j\Big) \cdot \Big(\sum_{i=0}^{\infty} a_i x^i\Big) = \sum_{k=0}^{\infty} c_k x^k \tag{1.8}$$

where $c_0 = 1$ and $c_k = 0$ for $k > 0$. From (1.6) we must have that $1 = c_0 = a_0 \cdot b_0$ and $1 = c_0 = b_0 \cdot a_0$. This means that $a_0 \in R^{-1}$, as required.

Conversely, suppose that $\sum_{i=0}^{\infty} a_i x^i \in R[[x]]$ and that $a_0 \in R^{-1}$. Let $\sum_{j=0}^{\infty} b_j x^j \in R[[x]]$, such that $b_0 = a_0^{-1}$, and $b_j = 0$ for $j > 0$. Then it is easy to see that the inverse of $\sum_{i=0}^{\infty} a_i x^i$ is $\sum_{j=0}^{\infty} b_j x^j$.

□

Example 1.2.2.10 The ring $R = \mathbb{C}[[z]]$ is a local ring.

Proof: We can write $\mathbb{C}[[z]] = (\mathbb{C}[[z]])^{-1} \bigcup \mathbb{C}[[z]] \setminus (\mathbb{C}[[z]])^{-1}$. The ring $\mathbb{C}$ is a division ring, and so by Lemma 1.2.2.9 we can write

$$\mathbb{C}[[z]] \setminus (\mathbb{C}[[z]])^{-1} = \left\{ \sum_{i=0}^{\infty} a_i z^i \in \mathbb{C}[[z]] : a_0 = 0 \right\}$$

We will show that $I = \left\{ \sum_{i=0}^{\infty} a_i z^i \in \mathbb{C}[[z]] : a_0 = 0 \right\}$ is an ideal in $\mathbb{C}$. So, let $\sum_{i=0}^{\infty} a_i z^i, \sum_{j=0}^{\infty} b_j z^j \in I$. Then $a_0 = b_0 = 0$, hence $a_0 - b_0 = 0$, and $a_0 b_0 = 0$. This means that $\sum_{i=0}^{\infty} a_i z^i - \sum_{j=0}^{\infty} b_j z^j \in I$ and
$\left(\sum_{i=0}^{\infty} a_i z^i\right) \cdot \left(\sum_{j=0}^{\infty} b_j z^j\right) \in I$. Hence, by Theorem 1.2.1.2, I is a subring of $\mathbb{C}[[z]]$.

To see that I is an ideal, let $\sum_{i=0}^{\infty} a_i z^i \in \mathbb{C}[[z]]$ and $\sum_{j=0}^{\infty} b_j z^j \in I$. Then $b_0 = 0$ and so $a_0 b_0 = 0$ and $b_0 a_0 = 0$. This means that $\left(\sum_{i=0}^{\infty} a_i z^i\right) \cdot \left(\sum_{j=0}^{\infty} b_j z^j\right) \in I$ and $\left(\sum_{j=0}^{\infty} b_j z^j\right) \cdot \left(\sum_{i=0}^{\infty} a_i z^i\right) \in I$. Hence I is an ideal. By Proposition 1.2.2.7, I is the unique maximal ideal in $\mathbb{C}$, and so $\mathbb{C}$ is a local ring. □

Proposition 1.2.2.11 If R is a local ring, then the quotient ring $R/\operatorname{Rad} R$ is a division ring.

Proof: Suppose R is a local ring. Then by Proposition 1.2.2.6 we can write R as $R = R^{-1} \cup \operatorname{Rad} R$. For $r \in R$ we denote its equivalence class as $[r] = r + \operatorname{Rad} R$. Suppose $[r] \neq [0]$. Then $r = r - 0 \notin \operatorname{Rad} R$. Hence $r \in R^{-1}$, so there exists $s \in R$ such that $rs = sr = 1$. But then $[r][s] = [rs] = [1]$ and $[s][r] = [sr] = 1$. This shows that every nonzero element of the ring $R/\operatorname{Rad} R$ is invertible. Hence $R/\operatorname{Rad} R$ is a division ring. □

1.2.3 Homomorphisms

Definition 1.2.3.1 ([15], p. 121) let R and S be rings. A *ring homomorphism from R to S* is a function $f : R \to S$ such that for $a, b \in R$, we have

H1. $f(a + b) = f(a) + f(b)$,

H2. $f(a \cdot b) = f(a) \cdot f(b)$.

Definition 1.2.3.2 ([15], p. 125) Let R and S be rings and $f : R \to S$ be a ring homomorphism. Then we define the *kernel* of f as

$$\ker f = f^{-1}(\{0\}) = \{r \in R : f(r) = 0\}.$$

Proposition 1.2.3.3 ([15], p. 124) Let R and S be rings, and let f be a homomorphism from R onto S. If R has an identity 1, then S has an identity $f(1)$. □

Proposition 1.2.3.4 ([15], p. 125) Let R and S be rings and let $f : R \to S$ be a ring homomorphism. Then the kernel of f is a two sided ideal of R. □

Remark 1.2.3.5 Let R be a ring and I a two sided ideal of R. The map

$$\begin{aligned} \pi_I : R &\to R/I \\ a &\mapsto a + I \end{aligned}$$

is a homomorphism ([14], p. 282) called the *canonical homomorphism* from R to R/I.

The canonical homomorphism from R to $R/\operatorname{Rad} R$ will be denoted by π. Hence for $a \in R$, $\pi(a) = a + \operatorname{Rad} R = [a]$. □

1.3 Topologies

1.3.1 General notions

Definition 1.3.1.1 ([36], p. 23) A *topology* on a set X is a collection τ of subsets of X, called the *open sets*, satisfying:

T1. Any union of elements of τ belongs to τ,

T2. Any finite intersection of elements of τ belongs to τ,

T3. $\emptyset$ and X belong to τ.

$\langle X, \tau \rangle$ is called a topological space, abbreviated X, when the topology τ is understood.

Given two topologies τ_1 and τ_2 on the same underlying set X, we say τ_1 is *weaker*, *smaller* or *coarser* than τ_2, alternatively τ_2 is *stronger*, *larger* or *finer* than τ_1 if and only if $\tau_1 \subseteq \tau_2$.

Definition 1.3.1.2 ([36], p. 27) If X is a topological space and $A \subseteq X$, then the *interior* of A in X is the set $\mathrm{int}(A) = \bigcup\{G \subseteq X : G \text{ open in } X \text{ and } G \subseteq A\}$.

An element belonging to $\mathrm{int}(A)$ is called an *interior point* of A.

We will encounter the following examples of topological spaces.

Example 1.3.1.3 Let X be any set and let $\tau = \{\emptyset, X\}$. Then τ is a topology on X, called the *trivial (indiscrete)* topology. Clearly it is coarser than any other topology on X. $\square$

Example 1.3.1.4 Let X be any set and let τ be the collection of all subsets of X. Then τ is a topology on X, called the *discrete* topology on X. It is the strongest topology on X. $\square$

Definition 1.3.1.5 ([36], p. 24) Let $\langle X, \tau \rangle$ be a topological space. Then $A \subseteq X$ is said to be *closed* if its *complement* is open, i.e. if $X \setminus A \in \tau$.

The following theorem is a consequence of De Morgan's laws in conjunction with the obvious duality between the notions of open set and closed set.

Theorem 1.3.1.6 ([36], p. 24) If $\mathscr{F}$ is the collection of closed sets in a topological space X, then

F1. Any intersection of members of $\mathscr{F}$ belongs to $\mathscr{F}$,

F2. Any finite union of members of $\mathscr{F}$ belongs to $\mathscr{F}$,

F3. X and $\emptyset$ both belong to $\mathscr{F}$.

Conversely, given a set X, and a family $\mathscr{F}$ of subsets of X that satisfies conditions F1, F2 and F3, the collection of complements of members of $\mathscr{F}$ is a topology on X in which the family of closed sets is just $\mathscr{F}$. □

Definition 1.3.1.7 ([36], p. 25) If $\langle X, \tau \rangle$ is a topological space and $A \subseteq X$, then the *closure* of A in X is the set

$$\mathrm{cl}_\tau(A) = \bigcap \{K \subseteq X : K \text{ closed in } \langle X, \tau \rangle \text{ and } A \subseteq K\}.$$

If the topology is understood, we will omit the subscript τ and write $\mathrm{cl}(A)$ to mean $\mathrm{cl}_\tau(A)$.

By property F1 from Theorem 1.3.1.6, for $A \subseteq X$, the set $\mathrm{cl}_\tau(A)$ is closed. It is the smallest closed set containing A in the sense that it is contained in every closed set containing A.

In Definition 1.3.1.1, we defined a topology by specifying the open sets. Often topologies are defined by using a *closure operation*, described next.

Definition 1.3.1.8 ([36], p. 25) Let X be a set and let

$$k : \wp(X) \to \wp(X)$$

be an operation assigning to each $A \subseteq X$ the subset $k(A) \subseteq X$ satisfying the following properties for all $A, B \subseteq X$ and $\emptyset$:

K1. $A \subseteq k(A)$

K2. $k(k(A)) = k(A)$

K3. $k(A \cup B) = k(A) \cup k(B)$

K4. $k(\emptyset) = \emptyset$.

Then k is called a *Kuratowski closure operation* on X.

Theorem 1.3.1.9 ([36], p. 25) Suppose X is a set and $k : \wp(X) \to \wp(X)$ is a Kuratowski closure operation on the set X. For $A \subseteq X$ we define:

K5. A is closed in X if and only if $k(A) = A$.

The result is a topology on X, whose closure operation is just the closure operation we started with.

Proof: Suppose that X is a set and that $k : \wp(X) \to \wp(X)$ is a Kuratowski closure operation on X. We define

$$\mathscr{F}_0 = \{A \subseteq X : k(A) = A\}$$

We show that $\mathscr{F}_0$ satisfies the conditions of Theorem 1.3.1.6 and hence defines a topology on X.

First we show that $A \subseteq B \implies k(A) \subseteq k(B)$. If $A \subseteq B$ then $B = A \cup (B \setminus A)$. From K3 we have that $k(B) = k(A) \cup k(B \setminus A)$, from which it follows that $k(A) \subseteq k(B)$. Hence we have shown that

$$A \subseteq B \implies k(A) \subseteq k(B) \tag{1.9}$$

Next, suppose that $F_\lambda \in \mathscr{F}_0$ for each $\lambda \in \Lambda$. Then for each $\lambda \in \Lambda$ we have:

$$\bigcap_{\lambda \in \Lambda} F_\lambda \subseteq F_\lambda \tag{1.10}$$

We apply (1.9) above to (1.10) to give us

$$k\left(\bigcap_{\lambda \in \Lambda} F_\lambda\right) \subseteq k(F_\lambda) \tag{1.11}$$

From (1.11) and the fact that $k(F_\lambda) = F_\lambda$ for all $\lambda \in \Lambda$ we have

$$k\left(\bigcap_{\lambda \in \Lambda} F_\lambda\right) \subseteq \bigcap_{\lambda \in \Lambda} k(F_\lambda) = \bigcap_{\lambda \in \Lambda} F_\lambda \tag{1.12}$$

From K1 we have that

$$\bigcap_{\lambda \in \Lambda} F_\lambda \subseteq k\left(\bigcap_{\lambda \in \Lambda} F_\lambda\right) \tag{1.13}$$

From (1.12) and (1.13) we have that

$$\bigcap_{\lambda \in \Lambda} F_\lambda = k\left(\bigcap_{\lambda \in \Lambda} F_\lambda\right)$$

which shows that F1 is satisfied by $\mathscr{F}_0$.

Next, suppose that $F_1, \cdots, F_n \in \mathscr{F}_0$. Then by K3 and induction we have that:

$$\begin{aligned} k(F_1 \cup \cdots \cup F_n) &= k(F_1) \cup \cdots \cup k(F_n) \\ &= F_1 \cup \cdots \cup F_n. \end{aligned}$$

Hence $\mathscr{F}_0$ satisfies F2.

Finally, by K4 we have that $\emptyset \in \mathscr{F}_0$ and by K1 we have that $X \in \mathscr{F}_0$, so that $\mathscr{F}_0$ satisfies F3. Since the collection $\mathscr{F}_0$ satisfies the conditions of Theorem 1.3.1.6, $\mathscr{F}_0$ defines a topology on X, in which $\mathscr{F}_0$ is the collection of closed sets.

It remains to show that the closure of a set in the generated topology is simply the closure operation we initially started with. What we need to show is that if $A \subseteq X$, then $k(A)$ is the smallest closed set containing A.

First we note that by K2, we have that $k(k(A)) = k(A)$, hence we know that $k(A)$ is a closed set in the generated topology. From K1, we have that $k(A)$ is a closed set that contains A. Next, let K be any element from $\mathscr{F}_0$ containing A. Then from (1.9)

$$A \subseteq K \implies k(A) \subseteq k(K) = K.$$

Hence $k(A)$ is the smallest element of $\mathscr{F}_0$ containing A. □

Example 1.3.1.10 ([36], p. 26) Let X be an infinite set and for $A \subseteq X$ we define $k_0 : \wp(X) \to \wp(X)$ as:

$$k_0(A) = \begin{cases} A & \text{if A is finite} \\ X & \text{if A is infinite} \end{cases}$$

The properties K1 to K4 can be verified for the operation k_0, and it is a Kuratowski closure operation on X. The generated topology is called the *co-finite topology*. It has as closed sets, all sets A such that $k_0(A) = A$, hence the closed sets are $X, \emptyset$ and $A \subseteq X$ such that A is finite. The open sets are $X, \emptyset$ and $A \subseteq X$ such that $X \setminus A$ is finite. □

Definition 1.3.1.11 ([36], p. 38) If $\langle X, \tau \rangle$ is a topological space, a *base* for τ is a collection $\mathscr{B} \subseteq \tau$ such that

$$\tau = \left\{ \bigcup_{B \in \mathscr{C}} B : \mathscr{C} \subseteq \mathscr{B} \right\}$$

Definition 1.3.1.12 ([36], p. 39) If $\langle X, \tau \rangle$ is a topological space, a *subbase* for τ is a collection $\mathscr{D} \subseteq \tau$ such that the collection of all finite intersections of elements from $\mathscr{D}$ forms a base for τ.

Definition 1.3.1.13 ([36], p. 31) If X is a topological space and $x \in X$, a *neighbourhood* of x is a set U which contains an open set V containing x. The collection $\mathcal{N}_x$ of all neighbourhoods of x is the *neighbourhood system* at x.

Definition 1.3.1.14 ([36], p. 35) An *accumulation point (cluster point)* of a set A in a topological space X is a point $x \in X$ such that each neighbourhood of x contains some point of A, other than x. The set $\operatorname{der} A$ is the set of accumulation points of A.

Theorem 1.3.1.15 ([36], p. 35) Let $\langle X, \tau \rangle$ be a topological space, and let $A \subseteq X$. Then

$$\mathrm{cl}_\tau(A) = A \cup \operatorname{der}_\tau A.$$

$\square$

When the topology is understood, the symbol $\overline{A}$ is often used to represent the closure of A.

Definition 1.3.1.16 ([36], p. 70) A sequence (x_n) in a topological space X is said to *converge* to $x \in X$ written $x_n \xrightarrow[n]{\infty} x$ if and only if for each neighbourhood U of x, there exists $N \in \mathbb{N}$ such that $n \geq N \implies x_n \in U$.

Lemma 1.3.1.17 Let X be a set with topologies τ and σ such that $\mathrm{cl}_\tau(A) \subseteq \mathrm{cl}_\sigma(A)$ for every $A \subseteq X$. Then $\sigma \subseteq \tau$.

Proof: Let $A \in \sigma$. Then $X \setminus A$ is closed in X with respect to σ giving $X \setminus A = \mathrm{cl}_\sigma(X \setminus A)$. By assumption $\mathrm{cl}_\tau(X \setminus A) \subseteq \mathrm{cl}_\sigma(X \setminus A) = X \setminus A$, hence $\mathrm{cl}_\tau(X \setminus A) = X \setminus A$. Hence $A \in \tau$, and so $\sigma \subseteq \tau$. $\square$

1.3.2 Product and quotient topologies

Definition 1.3.2.1 ([36], p. 52) Let $\{\langle X_i, \tau_i \rangle \ : i \in I\}$ be any collection of topological spaces, with index set I. Denote by $X = \prod_i X_i = \{t = \langle x_i \ : i \in I \rangle : x_i \in X_i\}$ the *Cartesian product* of $X_i's$, the set of all functions t defined on the index set I such that the value of the function at a particular index i is an element of X_i.

The map $p_j : \prod_i X_i \to X_j$ defined by $p_j(t) = x_j$ returns the j^{th} coordinate for the tuple $t = \langle x_i \ : i \in I \rangle$ and is called the j^{th} *projection map*.

Definition 1.3.2.2 ([36], p. 53) The *product topology* on $X = \prod_i X_i$ is obtained by taking as a base for the open sets, sets of the form $\prod_i U_i$, where

P1. U_i is open in X_i, for each $i \in I$.

P2. For all but finitely many coordinates, $U_i = X_i$.

P1 can be replaced by

P1′. $U_i \in \mathscr{B}_i$ where for each i, $\mathscr{B}_i$ is a (fixed) base for the topology of X_i.

Also, notice that the set $\prod_i U_i$, where $U_i = X_i$ except for $i = i_1, \ldots, i_n$ can be written

$$\prod_i U_i = p_{i_1}^{-1}(U_{i_1}) \cap \cdots \cap p_{i_n}^{-1}(U_{i_n})$$

Thus the product topology is precisely that topology which has for a subbase the collection $\{p_i^{-1}(U_i) : i \in I, U_i \text{ is open in } X_i\}$. Again, the sets U_i can be restricted to come from some fixed base (in fact, in this case, subbase) in X_i.

Wherever necessary, we will denote the product topology by $\tau^\times$ and the product topological space by $\langle X, \tau^\times \rangle$, where $X = \prod_i X_i$. Hereafter, $X = \prod_i X_i$ is always assumed to be endowed with the product topology if each X_i is a topological space.

Proposition 1.3.2.3 ([36], p. 54) The product topology $\tau^\times$ on X is the weakest topology on X with respect to which all the projection functions $p_i : X \to X_i$ defined by $p_{j_0}(\langle x_i : i \in I \rangle) = x_{j_0}$ for each $j_0 \in I$ are continuous. $\square$

Definition 1.3.2.4 ([36], p. 59) Let $\langle X, \tau \rangle$ be a topological space, Y a set and $g : X \to Y$ be an onto map. Then the collection τ_g of subsets of Y defined by

$$\tau_g = \{G \subseteq Y : g^{-1}(G) \text{ is open in } X\}$$

is a topology on Y, called the *quotient topology* induced on Y by g. When Y is given some such topology, it is called a *quotient space* of X, and the inducing map is called the *quotient map*.

Theorem 1.3.2.5 ([36], p. 59) If X and Y are topological spaces and $f : X \to Y$ is continuous and either open or closed, then the topology τ on Y is the quotient topology τ_f. $\square$

1.3.3 Separation

Definition 1.3.3.1 ([36], p. 85) A topological space X is a T_0-*space* (or the topology on X is T_0) if and only if whenever x and y are distinct points in X, there is an open set containing one and not the other.

Definition 1.3.3.2 ([36], p. 86) A topological space X is a T_1-*space* if and only if whenever x and y are distinct points in X, there is a neighbourhood of each not containing the other.

Definition 1.3.3.3 ([36], p. 86) A topological space X is a T_2-*space* or a *Hausdorff* space if and only if whenever x and y are distinct points in X, there are disjoint open sets U and V in X with $x \in U$ and $y \in V$.

Remark 1.3.3.4 It follows from Definitions 1.3.3.1, 1.3.3.2 and 1.3.3.3 that every T_2 space is a T_1 space and every T_1 space is a T_0 space, but the converse statements are not true in general.

For example, the set $X = \{a, b\}$ equipped with the topology $\tau = \{\emptyset, \{a\}, X\}$ is a T_0 space which is not T_1.

Similarly an infinite set equipped with the co-finite topology is a T_1 space, since singleton sets are closed. Such a topology is not T_2, since no two nonempty open sets are disjoint.

Proposition 1.3.3.5 ([36], p. 86) If X is a topological space, then the following are equivalent:

(a) X is T_1,

(b) each one-point set in X is closed,

(c) each subset of X is the intersection of the open sets containing it. $\square$

1.3.4 Continuity

Definition 1.3.4.1 ([36], p. 44) Let X and Y be topological spaces. A function $f : X \to Y$ is said to be continuous at x_0 if and only if for each neighbourhood V of $f(x_0)$ in Y there is a neighbourhood U of x_0 in X such that $f(U) \subseteq V$. We say that f is *continuous on* X if and only if f is continuous at each $x_0 \in X$.

Proposition 1.3.4.2 ([36], p. 44) Let X and Y be topological spaces and

$$f : X \to Y$$

Then the following are equivalent:

(a) f is continuous,

(b) for each open set H in Y, $f^{-1}[H]$ is open in X,

(c) for each closed set K in Y, $f^{-1}[K]$ is closed in X,

(d) for each $E \subset X$, $f[\mathrm{cl}_X\, E] \subset \mathrm{cl}_Y\, f[E]$. □

Definition 1.3.4.3 ([36], p. 46) If X and Y are topological spaces, a function f from X to Y is a *homeomorphism* if and only if f is one to one, onto, continuous and has a continuous inverse, f^{-1}. In this case, we say that X and Y are homeomorphic.

Evidently, a continuous map $f : X \to Y$ is a homeomorphism if and only if there is a continuous map $g : Y \to X$ such that the compositions $g \circ f$ and $f \circ g$ are the identity maps on X and Y respectively.

1.3.5 Compactness

Definition 1.3.5.1 ([36], p. 104) Let X be a topological space. A *cover* (or *covering*) of X is a collection $\mathcal{A}$ of subsets of X whose union is X. A subcover of a cover $\mathcal{A}$, is a subcollection $\mathcal{A}'$ of $\mathcal{A}$ which is a cover. An *open cover* is a cover consisting of open sets.

Definition 1.3.5.2 ([36], p. 116) Let X be a topological space. X is *compact* if and only if every open cover of X has a finite subcover.

1.4 Topological groups and topological rings

Definition 1.4.0.1 ([35], p. 13) A topology τ on a group G, denoted multiplicatively, is a *group topology* and G, furnished with τ, is a *topological group* if the following conditions hold:

TG1. $(x, y) \mapsto xy$ is continuous from $G \times G$ to G,

TG2. $x \mapsto x^{-1}$ is continuous from G to G,

where G is given topology τ and $G \times G$ carries the Cartesian product topology $\tau^{\times}$ determined by τ.

Definition 1.4.0.2 ([35], p. 1) A topology τ on a ring R is a *ring topology* and R, furnished with τ, is a *topological ring* if the following conditions hold:

TR1. $(x, y) \mapsto x + y$ is continuous from $R \times R$ to R,

TR2. $x \mapsto -x$ is continuous from R to R,

TR3. $(x, y) \mapsto xy$ is continuous from $R \times R$ to R,

where R is given topology τ and $R \times R$ carries the Cartesian product topology $\tau^{\times}$ determined by τ.

Remark 1.4.0.3 Notice that every topological ring is also a topological group with respect to addition and we call $- : A \to A$ defined by $x \mapsto -x$ the inversion map. $\square$

Theorem 1.4.0.4 ([35], p. 14) Let G be a topological group and let $a \in G$. The functions $x \to -x$, $x \to a + x$ and $x \to x + a$ are homeomorphisms from G to G. Consequently, for any $X \subseteq G$, we have $\overline{-X} = -\overline{X}$, $\overline{a + X} = a + \overline{X}$, $\overline{X + a} = \overline{X} + a$. Also, for any open (closed) subset $P \subseteq G$ we have that , $-P, a + P$ are open (closed). $\square$

Lemma 1.4.0.5 ([35], p. 15) Let G be a topological group. If V is a neighbourhood of zero, so is $-V$. $\square$

1.5 Banach algebras

Definition 1.5.0.1 ([24], p. 59) Let V be a vector space over a field $\mathbb{K}$. A *norm* on V is a map $\|\cdot\| : V \to R^+$ that satisfies the following properties, for all $x, y \in V$ and $\alpha \in \mathbb{K}$:

N1. $\|x\| \geqslant 0$,

N2. $\|x\| = 0 \iff x = 0$,

N3. $\|\alpha x\| = |\alpha|\|x\|$,

N4. $\|x + y\| \leqslant \|x\| + \|y\|$.

A vector space V with a norm $\|\cdot\|$ defined on it is called a *normed* vector space, denoted by $\langle V, \|\cdot\|\rangle$. One now constructs (or induces) a *metric* using the norm, by defining the distance between $a, b \in V$ as

$$d(a, b) = \|a - b\|.$$

A *Banach space* is a normed vector space which is complete in the metric induced by the norm.

Definition 1.5.0.2 An *algebra* is a vector space A over a field $\mathbb{K}$, with a multiplication operation such that for all $x, y, z \in A$ and $\lambda \in \mathbb{K}$:

BA1. $x(yz) = (xy)z$,

BA2. $(x + y)z = xz + yz$,

BA3. $x(y + z) = xy + xz$,

BA4. $\lambda(xy) = (\lambda x)y = x(\lambda y)$.

If, in addition, A is a Banach space for a norm $||\cdot||$ and satisfies the norm inequality $||xy|| \leq ||x|| \cdot ||y||$, for all $x, y \in A$, we say that A is a *Banach algebra*.

If the field $\mathbb{K}$ is either the set of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$, the Banach algebra is called a *real* or *complex* Banach algebra, respectively.

An *identity* of A is an element $1 \in A$ such that for all $x \in A$ we have that $1x = x1 = x$. We use 1_A to represent the identity, when emphasizing the Banach algebra A. If a Banach algebra has an identity, it is unique and the Banach algebra is called *unital*.

The proof that the identity is unique is essentially the same as the proof that the identity in a ring is unique. Left and right invertible (and invertible) elements in a Banach algebra are defined as they are in a ring.

We make a normed space $\langle X, ||\cdot||\rangle$ into a topological space, via the concept of an open ball as follows. For $x_0 \in X$ and $\epsilon \in \mathbb{R}^+$, we define the *open ball of radius* ϵ, *centered at* x_0 as

$$B(x_0, \epsilon) := \{x \in X : ||x - x_0|| < \epsilon\}.$$

Next, we define an arbitrary set in the space $\langle X, ||\cdot||\rangle$ to be open if it contains a ball about each of its points.

Theorem 1.5.0.3 ([3], p. 35) Suppose that A is a Banach algebra with identity 1. If $x \in A$ and $||x|| < 1$ then $1 - x \in A^{-1}$ and

$$(1-x)^{-1} = \sum_{k=0}^{\infty} x^k \quad \text{where } x^0 = 1.$$

□

Theorem 1.5.0.4 ([3], p. 36) Suppose that A is a Banach algebra and that a is invertible. If $||x - a|| < \dfrac{1}{||a^{-1}||}$, then x is invertible. Moreover the mapping $x \mapsto x^{-1}$ is a homeomorphism from A^{-1} onto A^{-1}. □

Remark 1.5.0.5 Theorem 1.5.0.4 proves that the set of invertible elements in a Banach algebra is an open set in the topology induced by the norm. □

Proposition 1.5.0.6 ([8], p. 3549) In a Banach algebra A, we have:

$$\mathrm{cl}_{||\cdot||}(A_l^{-1}) \cap A_r^{-1} = A^{-1} = A_l^{-1} \cap \mathrm{cl}_{||\cdot||}(A_r^{-1})$$

□

1.5.1 Spectral theory

Definition 1.5.1.1 ([20], p. 341) Let A be a unital complex Banach algebra. The *spectrum* of $a \in A$ is defined as the set

$$\sigma(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin A^{-1}\}.$$

Definition 1.5.1.2 ([20], p. 352) Let A be a unital complex Banach algebra. The *spectral radius* of $a \in A$ is defined as

$$\rho(a) = \max\{|\lambda| : \lambda \in \sigma(a)\}.$$

It follows easily that $\rho(a) = 0$ if and only if $\sigma(a) = \{0\}$. In fact, we have the characterization:

Theorem 1.5.1.3 ([3], Theorem 3.2.8 p. 38) Let A be a unital complex Banach algebra and let $a \in A$. Then

(a) $\lambda \to (\lambda 1 - a))^{-1}$ is analytic on $\mathbb{C} \setminus \sigma(a)$ and goes to zero at infinity,

(b) $\sigma(a)$ is compact and non empty,

(c) $\rho(a) = \lim\limits_{n\to\infty} \|a^n\|^{\frac{1}{n}}$.

□

Definition 1.5.1.4 ([20], p. 251) If A is a normed algebra and $a \in A$, then a is *quasinilpotent* if $\|a^n\|^{\frac{1}{n}} \xrightarrow[n]{\infty} 0$. The set of quasinilpotent elements in A is $\mathrm{QN}_{\|\cdot\|}(A)$.

Proposition 1.5.1.5 ([20], p. 354) An element a in a Banach algebra is quasinilpotent if and only if $\sigma(a) = \{0\}$. □

Definition 1.5.1.6 ([16], p. 72) An element a of a Banach algebra A, is called a *left (right) topological zero divisor* if and only if there exists a sequence (x_n) in A such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $ax_n \to 0 \quad (x_n a \to 0)$. An element which is both a left and right topological zero divisor, is said to be a *(two-sided) topological zero divisor*.

Proposition 1.5.1.7 ([28], p. 6) Let A be a unital Banach algebra with invertible group A^{-1}. If $a \in A$ is any element such that $a \in \partial A^{-1}$ then a is a topological divisor of zero. $\square$

Example 1.5.1.8 Any quasinilpotent element in a Banach algebra is a topological zero divisor.

Proof: Suppose $a \in \mathrm{QN}_{\|\cdot\|}(A)$. Then from Proposition 1.5.1.5 we have that $\sigma(a) = \{0\}$. It follows that $a \notin A^{-1}$. Let (λ_n) be any sequence in $\mathbb{C}$ such that $\lambda_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \lambda_n = 0$. Then clearly $(a - \lambda_n)$ is a sequence in A^{-1} such that $\lim\limits_{n\to\infty} (a - \lambda_n) = a$. Hence it follows that $a \in \partial A^{-1}$. By Proposition 1.5.1.7 a is a topological divisor of zero. $\square$

Example 1.5.1.9 ([3], p. 19) We construct a counterexample to show that, in the infinite dimensional case, quasinilpotence does not imply nilpotence. Define the unit square $S = [0,1] \times [0,1] \subseteq \mathbb{R}^2$ with Lebesgue measure η. Then the Riemann-Liouville operator on Hilbert space $\mathcal{L}^2(S, \eta)$ is defined by:

$$(V^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt$$

where α is a complex number with positive real part

If we let $\alpha = 1$, we recover the Volterra operator:

$$(Vf)(x) = \textstyle\int_0^x f(t)dt$$

with the kernel on S defined by $K(x,t) = \begin{cases} 1 & \text{if} \quad x \geq t \\ 0 & \text{if} \quad x < t \end{cases}$

By the famous Arzelá-Ascoli theorem, we have that since V is equicontinuous and bounded, consequently compact, it must therefore be quasinilpotent. However, by direct calculation, we can verify that if $f = 1$ identically, then V is not nilpotent:

$$V(x) = \int_0^x 1\ dt = t\,\Big|_0^x = x - 0 = x \neq 0 \qquad \Longrightarrow \qquad V^\alpha f \neq 0 \text{ for all } \alpha \in \mathbb{N}$$

1.5.2 Basic Operator Theory

In this section we briefly discuss some basic notions in Operator Theory. Our aim is to have the notions of Fredholm and Weyl operators available to us for later discussions.

Let X and Y be Complex Banach spaces. By an *operator* $T : X \to Y$ we mean a bounded linear mapping. The set of all bounded linear mappings is denoted by $\mathcal{BL}(X, Y)$. We denote the range of T by $\operatorname{ran} T$.

If $X = Y$ then $\mathcal{BL}(X, Y)$ is denoted by $\mathcal{BL}(X)$.

Let $T \in \mathcal{BL}(X)$. Then we say T has *finite rank* if $\dim(T(X)) < \infty$. T is called a *compact* operator on X if $\overline{T(U)}$ is compact, where U is the closed unit ball of X. The finite rank operators in $\mathcal{BL}(X)$ form an ideal, denoted by $\mathcal{F}(X)$, and the compact operators form a closed ideal, denoted by $\mathcal{K}(X)$.

Example 1.5.2.1 ([4], p. 3) Let X be a Banach space. Denote by $l_\infty(X)$ the linear space of all bounded sequences (x_n) of elements $x_n \in X$ with the supremum norm:

$$\|(x_n)\| = \sup\{\|x_n\| : n \in \mathbb{N}\}.$$

$\mathrm{m}(X)$ is the linear subspace of $l_\infty(X)$ consisting of those sequences every subsequence of which contains a convergent subsequence. It is elementary to check that $l_\infty(X)$ is a Banach space and that $\mathrm{m}(X)$ is a closed subspace of $l_\infty(X)$.

Let $\hat{X}$ denote the quotient space $l_\infty(X)/\,\mathrm{m}(X)$ and if $T \in \mathcal{BL}(X)$ let $\hat{T}$ denote the operator on $\hat{X}$ defined by

$$\hat{T}((x_n) + \mathrm{m}(X)) = (Tx_n) + \mathrm{m}(X).$$

□

Definition 1.5.2.2 ([28], p. 149) Let X, Y be Banach spaces, and let $T \in \mathcal{BL}(X, Y)$. We say that

(a) T is *upper semi-Fredholm* if $\operatorname{ran} T$ is closed and $\dim \ker T < \infty$;

(b) T is *lower semi-Fredholm* if $\operatorname{codim} \operatorname{ran} T = \dim Y/\operatorname{ran} T < \infty$;

(c) T is *Fredholm* if $\dim \ker T < \infty$ and $\operatorname{codim} \operatorname{ran} T < \infty$.

Definition 1.5.2.3 ([28], p. 150) Let X, Y be Banach spaces and let $T \in \mathcal{BL}(X, Y)$. Suppose also that T has closed range. Then we define:

$$\alpha(T) = \dim \ker T \qquad \text{and} \qquad \beta(T) = \operatorname{codim} \operatorname{ran} T.$$

We also define the *index* of T as:

$$\iota(T) = \alpha(T) - \beta(T).$$

We denote the class of Fredholm operators acting between Banach spaces X to Y by $\Phi(X, Y)$ and if $X = Y$ we simply write $\Phi(X)$. Clearly if $T \in \Phi(X, Y)$, then $\iota(T) \in \mathbb{Z}$. A Fredholm operator for which $\iota(T) = 0$ is called *Weyl*. We denote the class of Weyl operators acting between Banach spaces X to Y by $\Omega(X, Y)$ and if $X = Y$ we simply write $\Omega(X)$. It's obvious that $\Omega(X, Y) \subseteq \Phi(X, Y)$.

Definition 1.5.2.4 ([34], p. 251) Let X and Y be Banach spaces. Given $T \in \mathcal{BL}(X, Y)$, we say that an operator $S \in \mathcal{BL}(Y, X)$ is a pseudoinverse or generalized inverse of T if and only if $T = TST$. The operator T is called relatively regular.

The following result is called Atkinson's Theorem

Theorem 1.5.2.5 ([4], p. 3) Let X be a Banach space over $\mathbb{C}$. For $T \in \mathcal{BL}(X)$ the following statements are equivalent

(a) $T \in \Phi(X)$,

(b) $T + \mathcal{F}(X) \in \Big(\mathcal{BL}(X)/\mathcal{F}(X)\Big)^{-1}$,

(c) $T + \mathcal{K}(X) \in \Big(\mathcal{BL}(X)/\mathcal{K}(X)\Big)^{-1}$,

(d) $\hat{T} \in \mathcal{BL}(\hat{X})^{-1}$.

✠

Chapter 2

The spectral topology in rings

2.1 Introduction

In this chapter we present an example of a Kuratowski closure operation, called the *spectral closure*, which gives rise to a topology on a ring, called the *spectral topology*. We discuss some of the properties of this topology. We look at some examples of the topologies on different types of rings. We also look at the product spaces and quotient spaces of rings with the spectral topology.

2.1.1 Algebraic closure

In [8] the authors of that article (Harte and Cvetkovic-Ilić) made a first attempt at defining an operation that defines a topology on a ring.

Definition 2.1.1.1 ([8], p. 3547) Let R be a ring and $K \subseteq R$. We define the *algebraic closure* of K by:

$$\mathrm{cl}_{\mathrm{alg}}(K) = \{a \in R : \text{ for all } b \in R \text{ there exists } a' \in K : \ 1 - b(a - a') \in R^{-1}\}$$

Proposition 2.1.1.2 Let R be a ring and $a \in R$. Then $\mathrm{cl}_{\mathrm{alg}}(\{a\}) = a + \mathrm{Rad}\, R$.

Proof: We show that $\mathrm{cl}_{\mathrm{alg}}(\{a\}) \subseteq a + \mathrm{Rad}\, R$ and $a + \mathrm{Rad}\, R \subseteq \mathrm{cl}_{\mathrm{alg}}(\{a\})$.

To see the first inclusion, let $c \in \mathrm{cl}_{\mathrm{alg}}(\{a\})$. Then for every $b \in R$ there exists $a' \in \{a\}$ such that $1 - b(c - a') \in R^{-1}$. Since $\{a\}$ is a singleton, this is equivalent to saying that: For every $b \in R$ we have that $1 - b(c - a) \in R^{-1}$. Hence $1 - R(c - a) \subseteq R^{-1}$. This means that $c - a \in \mathrm{Rad}\, R$. So there exists $d \in \mathrm{Rad}\, R$ s.t. $c - a = d$. Hence $c = a + d$, or $c \in a + \mathrm{Rad}\, R$ as required.

To prove the second inclusion, let $b \in a + \operatorname{Rad} R$. Then there exists $c \in \operatorname{Rad} R$ such that $b = a+c$. This means that $c = b-a \in \operatorname{Rad} R$. Hence $1-R(b-a) \subseteq R^{-1}$. Hence, for every $d \in R$, we have that $1 - d(b-a) \in R^{-1}$. This means that $b \in \mathrm{cl}_{\mathrm{alg}}(\{a\})$. Hence the second inclusion holds and the proposition is proved. □

Remark 2.1.1.3 The algebraic closure has almost all the properties of a topological closure, failing only at K3 (see [8], page 3548). The definition in the next section is essentially a generalization of the the definition of algebraic closure. As we shall see it satisfies all the conditions of a closure operator.

2.2 The spectral topology on a ring

2.2.1 The spectral closure

Definition 2.2.1.1 ([9], p. 268) Let R be a ring, $K \subseteq R$. The *spectral closure* of K is:

$$\mathrm{CL}(K) = \{a \in R : \forall \text{ finite } J \subseteq R \ \exists\ a' \in K : 1 - J(a-a') \subseteq R^{-1}\} \tag{1}$$

Our first concern is to show the equivalence of (1) with (2):

$$\mathrm{CL}(K) = \{a \in R : \forall \text{ finite } J, L \subseteq R \ \exists\ a' \in K : 1 - J(a-a')L \subseteq R^{-1}\} \tag{2}$$

For the purpose of the argument, denote the set in (1) by $\mathrm{CL}_1(K)$ and the set in (2) by $\mathrm{CL}_2(K)$. We show that $\mathrm{CL}_1(K) \subseteq \mathrm{CL}_2(K)$ and $\mathrm{CL}_2(K) \subseteq \mathrm{CL}_1(K)$.

For the first inclusion, let $a \in \mathrm{CL}_1(K)$ and let $H, L \subseteq R$ such that H, L are finite. Then the product set $H \cdot L$ is finite. Since $a \in \mathrm{CL}_1(K)$ there exists $a' \in K$ such that $1 - H \cdot L(a-a') \subseteq R^{-1}$. This means that $1 - h \cdot l(a-a') \in R^{-1}$ for all $h \in H, l \in L$. By Lemma 1.2.0.15, this means that $1 - l(a-a')h \in R^{-1}$ for all $h \in H, l \in L$. Hence $1 - L(a-a')H \subseteq R^{-1}$, which means that $a \in \mathrm{CL}_2(K)$. Hence $\mathrm{CL}_1(K) \subseteq \mathrm{CL}_2(K)$.

For the reverse inclusion, let $a \in \mathrm{CL}_2(K)$. Then for all finite $J, L \subseteq R$ there exists $a' \in K$ such that $1 - J(a-a')L \subseteq R^{-1}$. In particular, consider $L = \{1\}$. Then $1-J(a-a') = 1-J(a-a')\{1\} \subseteq R^{-1}$. Hence $a \in \mathrm{CL}_1(K)$, so that $\mathrm{CL}_2(K) \subseteq \mathrm{CL}_1(K)$.

Combining the two inclusions gives $\mathrm{CL}_1(K) = \mathrm{CL}_2(K)$.

If we want to emphasize that the spectral closure of a subset K is taken in a specific ring R, we indicate this explicitly by use of a subscript, as in $\mathrm{CL}_R(K)$, but omit the subscript whenever no confusion is possible.

2.2.2 Basic properties of the spectral closure

Lemma 2.2.2.1 Let R be a ring, and let $A \subseteq B \subseteq R$. Then $\mathrm{CL}(A) \subseteq \mathrm{CL}(B)$.

Proof: Suppose $A \subseteq B$ and let $a \in \mathrm{CL}(A)$. Also, suppose that $J \subseteq R, J$ finite. Since $a \in \mathrm{CL}(A)$ there exists $a' \in A$ such that $1 - J(a - a') \subseteq R^{-1}$. But $A \subseteq B$, so $a' \in B$. Hence $a \in \mathrm{CL}(B)$, as required. $\square$

Proposition 2.2.2.2 Let R be a ring. Then $\mathrm{CL}(R^{-1}) = R^{-1}\,\mathrm{CL}(R^{-1})$.

Proof: To see that $\mathrm{CL}(R^{-1}) \subseteq R^{-1}\,\mathrm{CL}(R^{-1})$, let $a \in \mathrm{CL}(R^{-1})$. Then $a = 1 \cdot a \in R^{-1}\,\mathrm{CL}(R^{-1})$, and hence $\mathrm{CL}(R^{-1}) \subseteq R^{-1}\,\mathrm{CL}(R^{-1})$.

To see the reverse inclusion, let $a \in R^{-1}\,\mathrm{CL}(R^{-1})$. Then $a = a_1a_2$, where $a_1 \in R^{-1}$ and $a_2 \in \mathrm{CL}(R^{-1})$. Let $J \subseteq R, J$ finite and arbitrary. Then Ja_1 is also finite, and since $a_2 \in \mathrm{CL}(R^{-1})$ there exists $a_2' \in R^{-1}$ such that $1 - Ja_1(a_2 - a_2') \in R^{-1}$, hence $1 - J(a_1a_2 - a_1a_2') \in R^{-1}$. This gives $1 - J(a - a_1a_2') \in R^{-1}$. By part (a) of Lemma 1.2.0.4, we have that $a_1a_2' \in R^{-1}$. Hence $a \in \mathrm{CL}(R^{-1})$, or $R^{-1}\,\mathrm{CL}(R^{-1}) \subseteq \mathrm{CL}(R^{-1})$. Hence the result follows. $\square$

Proposition 2.2.2.3 ([9], p. 268) Let R be a ring. Then

$$\mathrm{CL}(\{a\}) = a + \operatorname{Rad} R \text{ for all } a \in R.$$

Proof: We show that

$$a + \operatorname{Rad} R \subseteq \mathrm{CL}(\{a\}) \text{ for all } a \in R. \tag{1}$$

and

$$\mathrm{CL}(\{a\}) \subseteq a + \operatorname{Rad} R \text{ for all } a \in R. \tag{2}$$

To see (1), let $b \in a + \operatorname{Rad} R$. Then $b = a + c$ for some $c \in \operatorname{Rad} R$. Let $J \subseteq R, J$ finite. Then

$$\begin{aligned} 1 - J(b-a) &\subseteq 1 - R(b-a) && \text{(since } J \subseteq R\text{)} \\ &\subseteq R^{-1} && \text{(by Proposition 1.2.2.2).} \end{aligned}$$

This last containment shows that $b \in \mathrm{CL}(\{a\})$ as required.

To prove (2), let $b \in \mathrm{CL}(\{a\})$. Let $x \in R$. Then $\{x\} \subseteq R, \{x\}$ finite. Since $b \in \mathrm{CL}(\{a\})$ we must have that

$$1 - \{x\}(b - a) \subseteq R^{-1}.$$

Since this last containment holds for all $x \in R$, we must have that $1 - R(b-a) \subseteq R^{-1}$. By Proposition 1.2.2.2 we have that $b - a \in \operatorname{Rad} R$. Hence

$$b - a = r, \text{ for some } r \in \operatorname{Rad} R, \text{ or } b \in a + \operatorname{Rad} R,$$

as required. □

Proposition 2.2.2.4 ([9], p. 268) Let R be a ring and $K, H \subseteq R$. Then the spectral closure is compatible with the ring operations:

(a) $\mathrm{CL}(K) + \mathrm{CL}(H) \subseteq \mathrm{CL}(K + H)$

(b) $\mathrm{CL}(K) \cdot \mathrm{CL}(H) \subseteq \mathrm{CL}(K \cdot H)$

Proof:

(a) Suppose that $x \in \mathrm{CL}(K)$ and $y \in \mathrm{CL}(H)$. We show that $x + y \in \mathrm{CL}(K + H)$.

Let $J \subseteq R, J$ finite. Since $x \in \mathrm{CL}(K)$ there exists $x' \in K$ such that $1 - J(x - x') \subseteq R^{-1}$. Let $G = 1 - J(x - x')$. Then $G \subseteq R^{-1}$ and G is finite. Then $G^{-1}J$ is also finite, and since $y \in \mathrm{CL}(H)$ there exists $y' \in H$ such that $1 - G^{-1}J(y - y') \subseteq R^{-1}$. Hence we have, for $j \in J$:

$$\begin{aligned} 1 - j[(x + y) - (x' + y')] &= 1 - j[x + y - x' - y'] \\ &= 1 - j[x - x' + y - y'] \\ &= 1 - j[(x - x') + (y - y')] \\ &= 1 - j(x - x') - j(y - y'). \end{aligned}$$

Hence

$$\begin{aligned} 1 - j[(x + y) - (x' + y')] &\in G - J(y - y') \\ &\subseteq G[1 - G^{-1}J(y - y')] \\ &\subseteq R^{-1} \qquad \text{(using part (a) of Lemma 1.2.0.4).} \end{aligned}$$

Hence $1 - J[(x + y) - (x' + y')] \subseteq R^{-1}$. It is clear that $x' + y' \in K + H$ so that (a) is proved.

(b) Suppose $x \in \mathrm{CL}(K)$ and $y \in \mathrm{CL}(H)$. We show that $xy \in \mathrm{CL}(KH)$. Let $J \subseteq R$, J finite. Since $x \in \mathrm{CL}(K)$ there exists $x' \in K$ such that $1 - J(x - x')y \subseteq R^{-1}$. Let $D = 1 - J(x - x')y$. Then $D \subseteq R^{-1}$, and since D is finite, we have D^{-1} is also finite. Also $\{x'\} \subseteq R$ and is finite. Hence $D^{-1}Jx' \subseteq R$ and is finite. Since $y \in \mathrm{CL}(H)$ we have that there exists $y' \in H$ such that $1 - D^{-1}Jx'(y - y') \subseteq R^{-1}$. Hence for $j \in J$:

$$\begin{aligned} 1 - j(xy - x'y') &= 1 - j(xy) + j(x'y') \\ &= 1 - j(xy) + j(x'y) - j(x'y) + j(x'y') \\ &= 1 - jxy + jx'y - jx'y + jx'y' \\ &= 1 - j(xy - x'y) - j(x'y - x'y') \\ &= 1 - j(x - x')y - jx'(y - y'). \end{aligned}$$

Hence

$$\begin{aligned} 1 - j(xy - x'y') &\in D - Jx'(y - y') \\ &\subseteq D[1 - D^{-1}Jx'(y - y')] \\ &\subseteq R^{-1} \qquad \text{(using part (a) of Lemma 1.2.0.4).} \end{aligned}$$

Hence $1 - J(xy - x'y') \subseteq R^{-1}$. It is clear that $x'y' \in KH$, and so (b) is proved □

Lemma 2.2.2.5 Let R be a ring. Then for $V \subseteq R$, we have:

(a) $a \in \mathrm{CL}(V) \implies -a \in \mathrm{CL}(-V)$,

(b) $\mathrm{CL}(-V) = -\,\mathrm{CL}(V)$.

Proof: To prove (a), suppose that $a \in \mathrm{CL}(V)$ and let J be an arbitrary finite subset of R. Let $J' = -J$. Then J' is also finite. Since $a \in \mathrm{CL}(V)$, there exists $a' \in V$ such that $1 - J'(a - a') \subseteq R^{-1}$. Then $1 + J'(-a + a') \subseteq R^{-1}$. Hence

$$1 + J'(-a - (-a')) \subseteq R^{-1} \tag{2.1}$$

Let $b = -a'$. Then $b \in -V$. From (2.1) we have that

$$1 - J(-a - b) \subseteq R^{-1}$$

Since J was arbitrary and finite, we have that $-a \in \mathrm{CL}(-V)$, as desired.

To prove (b) we show that $\mathrm{CL}(-V) \subseteq -\mathrm{CL}(V)$ and $-\mathrm{CL}(V) \subseteq \mathrm{CL}(-V)$. For the first inclusion, suppose that $a \in \mathrm{CL}(-V)$. From part (a), we have that $-a \in \mathrm{CL}(-(-V)) = \mathrm{CL}(V)$. Hence $-a \in \mathrm{CL}(V)$, so $a \in -\mathrm{CL}(V)$. Hence the first inclusion is proved. For the reverse inclusion, suppose that $a \in -\mathrm{CL}(V)$. Then $-a \in \mathrm{CL}(V)$ so by part (a) we have that $a \in \mathrm{CL}(-V)$, as desired. Hence the second inclusion is proved and $\mathrm{CL}(-V) = -\mathrm{CL}(V)$. $\square$

2.2.3 The spectral closure on products of rings

In the next result we extend the spectral closure to the direct sum of two rings.

Proposition 2.2.3.1 Let R and S be rings. and let $E_1 \subseteq R$ and $E_2 \subseteq S$. Then

$$\mathrm{CL}_{R\oplus S}(E_1 \oplus E_2) = \mathrm{CL}_R(E_1) \oplus \mathrm{CL}_S(E_2)$$

Proof: We prove the statement by showing that

$$(e_1, e_2) \in \mathrm{CL}_{R\oplus S}(E_1 \oplus E_2) \iff (e_1, e_2) \in \mathrm{CL}_R(E_1) \oplus \mathrm{CL}_S(E_2).$$

So, $(e_1, e_2) \in \mathrm{CL}_{R\oplus S}(E_1 \oplus E_2) \iff$

for finite $J \oplus K \subseteq R \oplus S$ there exists $(e_1', e_2') \in E_1 \oplus E_2$ such that

$$(1,1) - (J \oplus K)\Big[(e_1, e_2) - (e_1', e_2')\Big] \subseteq (R \oplus S)^{-1} = R^{-1} \oplus S^{-1} \iff$$

$J \subseteq R, J$ finite and $e_1' \in E_1$ and $1 - J(e_1 - e_1') \subseteq R^{-1}$

and

$K \subseteq S, K$ finite and $e_2' \in E_2$ and $1 - K(e_2 - e_2') \subseteq S^{-1} \iff$

$$(e_1, e_2) \in \mathrm{CL}(E_1) \oplus \mathrm{CL}(E_2)$$

$\square$

2.3 The spectral topology

Theorem 2.3.0.1 ([9], p. 268) On a ring R, the spectral closure generates a topology, called *the spectral topology*, defined as

$$\tau = \{K \subseteq R : \mathrm{CL}(R \setminus K) = R \setminus K\}.$$

Proof: We show that $\mathrm{CL}(\cdot) : \wp(R) \to \wp(R)$ satisfies the Kuratowski closure axioms, K1 through K4 of Definition 1.3.1.8.

To show that K1 holds, suppose that $K \subseteq R$. Let $J \subseteq R$, J finite and let $x \in K$. Then $1 - J(x-x) = 1 - \{0\} = \{1\} \subseteq R^{-1}$. Hence $x \in \mathrm{CL}(K)$. So $K \subseteq \mathrm{CL}(K)$, and K1 holds.

To show that K2 holds, notice that from K1 and Lemma 2.2.2.1 we have $\mathrm{CL}(K) \subseteq \mathrm{CL}(\mathrm{CL}(K))$. So it remains to show the reverse inclusion, that $\mathrm{CL}(\mathrm{CL}(K)) \subseteq \mathrm{CL}(K)$. To do so, let $a \in \mathrm{CL}(\mathrm{CL}(K))$. We show that $a \in \mathrm{CL}(K)$. Let $J \subseteq R, J$ finite. Then there exists $a' \in \mathrm{CL}(K)$ such that $1 - J(a-a') \subseteq R^{-1}$. Let $G = 1 - J(a - a')$. Then $G \subseteq R^{-1}$. Since J is finite, G is finite, and since $G \subseteq R^{-1}$, the set G^{-1} is well defined. Also, since G is finite, so is G^{-1}, and so is $G^{-1}J$.

Since $a' \in \mathrm{CL}(K)$ and $G^{-1}J$ is finite, there exists $a'' \in K$ such that

$$1 - G^{-1}J(a' - a'') \subseteq R^{-1}.$$

Next we have for $j \in J$:

$$\begin{aligned} 1 - j(a - a'') &= 1 - j(a - a' + a' - a'') \\ &= 1 - j(a - a') - j(a' - a''). \end{aligned}$$

Hence

$$\begin{aligned} 1 - j(a - a'') &\in G - J(a' - a'') \\ &\subseteq G(1 - G^{-1}J(a' - a'')). \end{aligned}$$

Since $G \subseteq R^{-1}$ and $1 - G^{-1} \cdot J(a' - a'') \subseteq R^{-1}$, by part (a) of Lemma 1.2.0.4, we have that $G(1 - G^{-1} \cdot J(a' - a'')) \subseteq R^{-1}$, which means that $1 - J(a - a'') \subseteq R^{-1}$. Hence $a \in \mathrm{CL}(K)$, giving the reverse inclusion, that $\mathrm{CL}(\mathrm{CL}(K)) \subseteq \mathrm{CL}(K)$, and so K2 holds.

To prove K3 holds, we show $\mathrm{CL}(K) \cup \mathrm{CL}(H) \subseteq \mathrm{CL}(K \cup H)$ and $\mathrm{CL}(K \cup H) \subseteq \mathrm{CL}(K) \cup \mathrm{CL}(H)$.

Since $K \subseteq K \cup H$ and $H \subseteq K \cup H$, we have from Lemma 2.2.2.1 that $\mathrm{CL}(K) \subseteq \mathrm{CL}(K \cup H)$ and $\mathrm{CL}(H) \subseteq \mathrm{CL}(K \cup H)$. If two sets are both subsets of the same set, then so is the union of the two sets, giving $\mathrm{CL}(K) \cup \mathrm{CL}(H) \subseteq \mathrm{CL}(K \cup H)$.

For the reverse inclusion, suppose that $x \in \mathrm{CL}(K \cup H) \setminus \mathrm{CL}(H)$. Then we have that $x \in \mathrm{CL}(K \cup H)$ and $x \notin \mathrm{CL}(H)$. Since $x \notin \mathrm{CL}(H)$, there exists $L \subseteq R, L$ finite, such that for all $h \in H$ we have

$$1 - L(x - h) \nsubseteq R^{-1} \tag{2.2}$$

Let $J \subseteq R, J$ finite and arbitrary. Since $x \in \mathrm{CL}(K \cup H)$, and $J \cup L \subseteq R$ is finite there exists $x' \in K \cup H$ with $1 - L(x - x') \subseteq 1 - (J \cup L)(x - x') \subseteq R^{-1}$. Since $x' \in K \cup H$, we have that $x' \in K$ or $x' \in H$. But $x' \in H$ would contradict (2.2), hence $x' \in K$. So we have, for the arbitrary finite $J \subseteq R$ that there exists $x' \in K$ with the property that

$$1 - J(x - x') \subseteq 1 - (J \cup L)(x - x') \subseteq R^{-1}.$$

So $x \in \mathrm{CL}(K)$, and so $\mathrm{CL}(K \cup H) \subseteq \mathrm{CL}(K) \cup \mathrm{CL}(H)$. Hence the spectral closure operation satisfies property K3.

To show that K4 holds, notice that it follows vacuously that $\mathrm{CL}(\emptyset) = \emptyset$ from definition $\mathrm{CL}(\emptyset) = \{a \in A : \ \forall \text{ finite } J \subseteq R \ \exists \ a' \in \emptyset : \ 1 - J(a - a') \subseteq R^{-1}\}$.

By Theorem 1.3.1.9, $\mathrm{CL}(\cdot)$ generates a topology on R defined by

$$\tau = \{K \subseteq R : \mathrm{CL}(R \setminus K) = R \setminus K\}$$

which we will call the *spectral topology* on the ring R.

$\square$

Theorem 2.3.0.2 Let R be a ring and τ be the spectral topology on R. Let $R \oplus R$ be endowed with the product topology induced by the spectral topology on R. Then

(a) The map $\cdot : R \oplus R \to R$ is continuous.

(b) The map $+ : R \oplus R \to R$ is continuous.

(c) The map $- : R \to R$, that maps $r \in R$ to its additive inverse, is continuous.

(d) $\langle R, \tau \rangle$ is $T_1 \iff R$ is semisimple.

Proof:

(a) To see that multiplication is continuous, we use part (d) of Theorem 1.3.4.2. So we let $E_1 \subseteq R$ and $E_2 \subseteq R$. We will show that

$$\cdot \left[\mathrm{CL}_{R\oplus R}\Big(E_1 \oplus E_2\Big)\right] \subseteq \mathrm{CL}_R\left(\cdot \Big[E_1 \oplus E_2\Big]\right) \tag{2.3}$$

First, from Proposition 2.2.3.1 we have $\mathrm{CL}_{R\oplus R}(E_1 \oplus E_2) = \mathrm{CL}_R(E_1) \oplus \mathrm{CL}_R(E_2)$.

Next, we note that

$$\begin{aligned} \cdot\Big[E_1 \oplus E_2\Big] &= \{\cdot((e_1, e_2)) : (e_1, e_2) \in E_1 \oplus E_2\} \\ &= \{e_1 \cdot e_2 : e_1 \in E_1, e_2 \in E_2\} \\ &= E_1 \cdot E_2 \end{aligned}$$

So, to prove (2.3) is equivalent to proving:

$$\cdot\Big[\mathrm{CL}_R(E_1) \oplus \mathrm{CL}_R(E_2)\Big] \subseteq \mathrm{CL}_R\Big(E_1 \cdot E_2\Big)$$

or

$$\mathrm{CL}_R(E_1) \cdot \mathrm{CL}_R(E_2) \subseteq \mathrm{CL}_R\Big(E_1 \cdot E_2\Big).$$

This is what we proved in part (b) of Proposition 2.2.2.4. Hence multiplication is continuous.

(b) To see that addition is continuous, we use part (d) of Theorem 1.3.4.2. So we let $E_1 \subseteq R$ and $E_2 \subseteq R$. We will show that

$$+ \left[\mathrm{CL}_{R\oplus R}\Big(E_1 \oplus E_2\Big)\right] \subseteq \mathrm{CL}_R\left(+ \Big[E_1 \oplus E_2\Big]\right) \tag{2.4}$$

First, from Proposition 2.2.3.1 we have $\mathrm{CL}_{R\oplus R}(E_1 \oplus E_2) = \mathrm{CL}_R(E_1) \oplus \mathrm{CL}_R(E_2)$.

Next, we note that

$$\begin{aligned} +\Big[E_1 \oplus E_2\Big] &= \{+((e_1, e_2)) : (e_1, e_2) \in E_1 \oplus E_2\} \\ &= \{e_1 + e_2 : e_1 \in E_1, e_2 \in E_2\} \\ &= E_1 + E_2. \end{aligned}$$

So, to prove (2.4) is equivalent to proving:

$$+\Big[\operatorname{CL}_R(E_1) \oplus \operatorname{CL}_R(E_2)\Big] \subseteq \operatorname{CL}_R\Big[E_1 + E_2\Big]$$

or

$$\operatorname{CL}_R(E_1) + \operatorname{CL}_R(E_2) \subseteq \operatorname{CL}_R\Big(E_1 + E_2\Big).$$

This is what we proved in part (a) of Proposition 2.2.2.4. Hence addition is continuous.

(c) Next we show that the inversion map $- : R \to R$ that maps $a \in R$ to $-a$ is continuous. We use (c) of Theorem 1.3.4.2, to prove that for each closed set V in R, we have that $-^{-1}[V]$ is closed in R with respect to the spectral topology. i.e. $\operatorname{CL}_R(-^{-1}[V]) = -^{-1}[V]$. Notice that:

$$-^{-1}[V] = \{a \in R : -a \in V\} = -V.$$

To verify this, consider arbitrary $a \in -^{-1}[V]$. Then $-a \in V$ giving that $a \in -V$ hence $-^{-1}[V] \subseteq -V$. For the reverse inclusion, consider arbitrary $a \in -V$ then $-a \in V$ giving $a \in -^{-1}[V]$. Thus, the reverse inclusion $-V \subseteq -^{-1}[V]$, holds, giving equality, as desired.

Now, suppose that V is closed in R. We must show that $-V$ is closed in R. i.e $\operatorname{CL}(V) = V \implies \operatorname{CL}(-V) = -V$. Suppose that $\operatorname{CL}(V) = V$, then from part (b) of Lemma 2.2.2.5, we have that $\operatorname{CL}(-V) = -\operatorname{CL}(V) = -V$, giving that the inversion map is continuous.

(d) Suppose that $\langle R, \tau \rangle$ is a T_1 space. Then from part (b) of Proposition 1.3.3.5, we have that singletons are closed in R. This gives that $\operatorname{CL}(\{0\}) = \{0\}$. By Proposition 2.2.2.3, we know for all $a \in R$ that $\operatorname{CL}(\{a\}) = a + \operatorname{Rad} R$. Therefore $\operatorname{Rad} R = 0 + \operatorname{Rad} R = \operatorname{CL}(\{0\}) = \{0\}$. Hence R is semisimple.

Conversely, suppose that R is semisimple. Consider arbitrary, distinct $a, b \in R$. Since R is semisimple, we have that $\operatorname{CL}(\{a\}) = a + \operatorname{Rad} R = a + \{0\} = \{a\}$ and $\operatorname{CL}(\{b\}) = b + \operatorname{Rad} R = \{b\}$. So $\{a\}$ and $\{b\}$ are closed sets. Therefore $R \setminus \{b\}$ is an open set containing a and not b and $R \setminus \{a\}$ is an open set containing b and not containing a. Hence, $\langle R, \tau \rangle$ is a T_1 space. $\square$

Remark 2.3.0.3 Parts (a) to (c) of Theorem 2.3.0.2 means that the spectral topology is a ring topology as defined in Defintion 1.4.0.2.

Lemma 2.3.0.4 Let J is a subset of the ring $\mathbb{Z}$ containing more than two nonzero elements. Then the following implication holds:

$$1 - Jy \subseteq \mathbb{Z}^{-1} \implies y = 0.$$

Proof: Suppose that $\{a_1, a_2, a_3\} \subseteq J$ with $a_1 \neq a_2 \neq a_3 \neq a_1$ and assume that a_i is nonzero for each $i \in \{1, 2, 3\}$. The condition

$$1 - Jy \subseteq \{\pm 1\} \tag{2.5}$$

means that:

$$\begin{array}{lcccr} & 1 - a_1 y = 1 & \text{or} & 1 - a_1 y = -1 & \text{(a)} \\ \text{and} & 1 - a_2 y = 1 & \text{or} & 1 - a_2 y = -1 & \text{(b)} \\ \text{and} & 1 - a_3 y = 1 & \text{or} & 1 - a_3 y = -1 & \text{(c)} \end{array}$$

In determining the set of values for y that satisfy (2.5) we have the following options:

$$\begin{array}{lccccccr} & 1 - a_1 y = 1 & \text{and} & 1 - a_2 y = 1 & \text{and} & 1 - a_3 y = 1 & & (1) \\ \text{or} & 1 - a_1 y = -1 & \text{and} & 1 - a_2 y = -1 & \text{and} & 1 - a_3 y = -1 & & (2) \\ \text{or} & 1 - a_1 y = -1 & \text{and} & 1 - a_2 y = 1 & \text{and} & 1 - a_3 y = 1 & & (3) \\ \text{or} & 1 - a_1 y = 1 & \text{and} & 1 - a_2 y = -1 & \text{and} & 1 - a_3 y = 1 & & (4) \\ \text{or} & 1 - a_1 y = 1 & \text{and} & 1 - a_2 y = 1 & \text{and} & 1 - a_3 y = -1 & & (5) \\ \text{or} & 1 - a_1 y = 1 & \text{and} & 1 - a_2 y = -1 & \text{and} & 1 - a_3 y = -1 & & (6) \\ \text{or} & 1 - a_1 y = -1 & \text{and} & 1 - a_2 y = 1 & \text{and} & 1 - a_3 y = -1 & & (7) \\ \text{or} & 1 - a_1 y = -1 & \text{and} & 1 - a_2 y = -1 & \text{and} & 1 - a_3 y = 1 & & (8) \end{array}$$

Consider the set of equations (1). They imply that $-a_i y = 0$ which means that $a_i y = 0$ for each $i \in \{1, 2, 3\}$. Since we assumed that $a_i \neq 0$ for each $i \in \{1, 2, 3\}$ and since we know that $\mathbb{Z}$ is an integral domain (Example 1.2.0.8), we conclude that $y = 0$.

Next, consider the set of equations (2). They imply that $a_1 y = 2$ and $a_2 y = 2$ and $a_3 y = 2$. Our options for values of y are $y = 1$, $y = -1$, $y = 2$, $y = -2$. Suppose $y = 1$. Then we must have that $a_1 = a_2 = a_3 = 2$, contradicting our assumption that the a_i are distinct. Suppose $y = -1$. Then we get $a_1 = a_2 = a_3 = -2$, contradicting the same assumption. Each of the other two options for a value for y contradicts the same assumption. Hence the set of equations (2) has no solution.

Next, consider the set (3). They imply that $a_1 y = 2$ and $a_2 y = 0$ and $a_3 y = 0$. The first of these equations imply that $y \in \{-2, -1, 1, 2\}$. The other two imply that $y = 0$. Clearly there is no solution then.

The remaining sets of equations ((4) - (8)) all fail in a way similar to equations (3) to produce a solution for y. The result follows. $\square$

Example 2.3.0.5 ([9], p. 270) On $\mathbb{Z}$ the spectral closure generates the discrete topology.

Proof: Suppose $K \subseteq \mathbb{Z}$. First we show that $\mathrm{CL}(K) = K$. From K1 of Theorem 2.3.0.1 we have that $K \subseteq \mathrm{CL}(K)$. To see that $\mathrm{CL}(K) \subseteq K$, suppose that $a \in \mathrm{CL}(K)$. Let J be an arbitrary finite subset of $\mathbb{Z}$. Then there exists $a' \in K$ such that $1 - J(a - a') \subseteq \mathbb{Z}^{-1}$. Since this expression holds for all finite J, it must also hold when J contains more than two nonzero elements. By Lemma 2.3.0.4, this means that $a - a' = 0$ which means that $a = a'$, hence $a \in K$, so that $\mathrm{CL}(K) = K$. This shows that every subset of $\mathbb{Z}$ is closed.

Again, let $K \subseteq \mathbb{Z}$, K arbitrary. Then $\mathbb{Z} \setminus K$ is closed by the above reasoning, so K is open. We have shown that every subset of $\mathbb{Z}$ is open, i.e. the spectral closure generates the discrete topology on $\mathbb{Z}$. $\square$

Example 2.3.0.6 ([9], p. 270) For a Boolean ring R, the spectral closure generates the discrete topology.

Proof: Suppose R is a Boolean ring. As was noted in Remark 1.2.0.12, $R^{-1} = \{1\}$.

Let K be an arbitrary subset of R. Then from K1 of Theorem 2.3.0.1 we know that $K \subseteq \mathrm{CL}(K)$. It remains to show that $\mathrm{CL}(K) \subseteq K$. Suppose that $a \in \mathrm{CL}(K)$. Then there exists $a' \in K$ such that $1 - \{1\}(a - a') \subseteq \{1\}$, hence

$$1 - (a - a') = 1 \implies -(a - a') = 0 \implies a - a' = 0 \implies a' = a$$

Hence $a \in K$. The rest of the argument is the same as in Example 2.3.0.5. We conclude that the spectral topology on R is discrete. $\square$

Proposition 2.3.0.7 ([9], p. 270) If R is a division ring, the spectral closure gives the co-finite topology:

(a) if $K \subseteq R, K$ finite then $\mathrm{CL}(K) = K$,

(b) if $K \subseteq R, K$ infinite then $\mathrm{CL}(K) = R$.

Proof: From Definition 1.2.0.10, we can write $R = R^{-1} \cup \{0\}$.

(a) Assume that $K \subseteq R$, and that K is finite. Either R is finite, or R is infinite. Suppose R is finite. Then either $K = R$ or $K \neq R$.

So first, suppose that $K = R$. By definition of the $\mathrm{CL}(\cdot)$ operation, $\mathrm{CL}(R) = R$, so $\mathrm{CL}(K) = \mathrm{CL}(R) = R = K$ as required.

Next, suppose that $K \neq R$. We show that $a \notin K \implies a \notin \mathrm{CL}(K)$, as follows. Suppose $a \notin K$. This is possible since $K \subseteq R$ and $K \neq R$, hence $R \setminus K \neq \emptyset$. We will show that $a \notin \mathrm{CL}(K)$. So let $a' \in K$, a' arbitrary. Then $a \neq a'$ which means $a-a' \neq 0$. Since R is a division ring, we have that $a-a' \in R^{-1}$. Consider $J = \{(a-a')^{-1}\}$, a finite subset of R. Then the expression $1-J(a-a')$ simplifies to give $1 - \{(a-a')^{-1}\}(a-a') = 1 - \{1\} = \{0\} \nsubseteq R^{-1}$. Hence $a \notin \mathrm{CL}(K)$. Since a was arbitrary, we have that $\mathrm{CL}(K) \subseteq K$, giving equality.

(b) Next, we assume that K is infinite. We always have that $\mathrm{CL}(K) \subseteq R$. It remains to show that $R \subseteq \mathrm{CL}(K)$. Let $a \in R$ and suppose that $a \notin \mathrm{CL}(K)$. Then by definition of the spectral closure there exists J, a finite subset of R such that for all $a' \in K$, we have

$$1 - J(a - a') \nsubseteq R^{-1}. \tag{2.6}$$

This means that for each value of $a - a'$, there exists a value $j \in J$ such that

$$1 - j(a - a') \notin R^{-1}. \tag{2.7}$$

Using Remark 1.2.0.16, we have that (2.7) is equivalent to

$$1 - (a - a')j \notin R^{-1}. \tag{2.8}$$

Consider (2.7). Since R is a division ring it means that $1 - j(a - a') = 0$, which means that $j(a - a') = 1$. This means that each $a - a'$ has a left inverse in J. Similarly, (2.8) implies that j is also a right inverse for $a - a'$ in J. Hence

$(a - a')$ is invertible. J was assumed to be finite, and the set $\{a - a' : a' \in K\}$ is an infinite set (since K is infinite). This means we have a finite list of unique inverses for an infinite set, which is impossible, i.e. it contradicts the fact that inverses are unique (by (1.1)). The contradiction stems from our assumption that $a \notin \mathrm{CL}(K)$. Hence $a \in \mathrm{CL}(K)$, and part (b) is proved.

Example 2.3.0.8 The rings $\mathbb{R}$ and $\mathbb{C}$, under the usual addition and multiplication of real numbers and complex numbers respectively, are division rings, and so Proposition 2.3.0.7 describes what the spectral topology looks like for these rings.

Example 2.3.0.9 In this example we illustrate the fact that $\mathbb{R}(\mathbb{C})$, equipped with the spectral topology, is a T_1 space but not a T_2 space. To do so, we show that $\mathbb{R}$, equipped with the spectral topology, is a T_1 space but not a T_2 space. The argument for $\mathbb{C}$, equipped with the spectral topology, is exactly the same.

To show that $\mathbb{R}$ is a T_1 space, we note that $\mathbb{R}$ is a division ring, hence as discussed in Example 2.3.0.8, the spectral topology on $\mathbb{R}$ is the co-finite topology. Hence any finite subset of $\mathbb{R}$ is s closed set in the spectral topology. Now $\{0\}$ is a finite subset of $\mathbb{R}$, hence closed. Hence $\mathrm{CL}(\{0\}) = \{0\}$. From Proposition 2.2.2.3, we have that $\operatorname{Rad}\mathbb{R} = \mathrm{CL}(\{0\}) = \{0\}$. Hence $\mathbb{R}$ is semisimple, and so by Proposition 2.3.0.2, we know that $\mathbb{R}$, endowed with the spectral topology is a T_1 space.

Next, we prove that $\mathbb{R}$ equipped with the spectral topology, is not a T_2 space. To see this, we note that the spectral closure on $\mathbb{R}$ generates the co-finite topology, and hence by the second part of Remark 1.3.3.4 the topology cannot be T_2. $\square$

2.3.1 The spectral topology on R / Rad R

Lemma 2.3.1.1 ([3] - Theorem 3.1.5, p. 35) Let R be a ring. Then $[x] \in (R/\operatorname{Rad} R)^{-1}$ if and only if $x \in R^{-1}$. $\square$

Proposition 2.3.1.2 Let R be a ring, $K \subseteq R$. Then

$$a \in \mathrm{CL}_R(K) \iff [a] \in \mathrm{CL}_{R/\operatorname{Rad} R}(\pi(K)).$$

Proof: Suppose $a \in \mathrm{CL}_R(K)$. We will show that $[a] \in \mathrm{CL}_{R/\operatorname{Rad} R}(\pi(K))$. So let $\tilde{J}$ be any finite subset of $R/\operatorname{Rad} R$. By picking one representative from each equivalence class in $\tilde{J}$ we can construct a finite $J \subseteq R$ such that $\pi(J) = \tilde{J}$. Since $a \in \mathrm{CL}_R(K)$ there exists $a' \in K$ such that $1 - J(a - a') \subseteq R^{-1}$. By Lemma 2.3.1.1 we have

$$[1] - \pi(J)([a]-[a']) \subseteq (R/\operatorname{Rad} R)^{-1}$$

Next $\pi(J) = \tilde{J}$ and $[a'] \in \pi(K)$. Hence

$$[1] - \tilde{J}([a]-[a']) \subseteq (R/\operatorname{Rad} R)^{-1},$$

which means that $[a] \in \mathrm{CL}_{R/\operatorname{Rad} R}(\pi(K))$.

Conversely, let $[a] \in \mathrm{CL}_{R/\operatorname{Rad} R}(\pi(K))$, for $K \subseteq R$. To see that $a \in \mathrm{CL}_R(K)$, let $J \subseteq R$, J finite. Then $\pi(J)$ is a finite subset of $R/\operatorname{Rad} R$. Since $[a] \in \mathrm{CL}_{R/\operatorname{Rad} R}(\pi(K))$ there exists $b \in R$ such that $[b] \in \pi(K)$ and

$$[1] - \pi(J)([a]-[b]) \subseteq (R/\operatorname{Rad} R)^{-1}.$$

Since $[b] \in \pi(K)$ we know there exists $a' \in K$ such that $[b] = [a']$. Hence we have

$$[1] - \pi(J)([a]-[a']) \subseteq (R/\operatorname{Rad} R)^{-1}.$$

By Lemma 2.3.1.1 we have $1 - J(a-a') \subseteq R^{-1}$. Since $a' \in K$ we have $a \in \mathrm{CL}_R(K)$. □

Proposition 2.3.1.3 Let R be a ring, $K \subseteq R$. Then

$$K \text{ is closed in } R \implies \pi(K) \text{ is closed in } R/\operatorname{Rad} R.$$

Proof: Suppose that K is closed in R. We show that $\pi(K)$ is closed in $R/\operatorname{Rad} R$. From Theorem 2.3.0.1 we know that $\pi(K) \subseteq \mathrm{CL}_{R/\operatorname{Rad} R}(\pi(K))$. It remains to show that $\mathrm{CL}_{R/\operatorname{Rad} R}(\pi(K)) \subseteq \pi(K)$. So let $[a] \in \mathrm{CL}_{R/\operatorname{Rad} R}(\pi(K))$. By Proposition 2.3.1.2 we have that $a \in \mathrm{CL}_R(K)$. Since K is closed this means that $a \in K$. Hence $[a] \in \pi(K)$ as required. □

Let R be a ring. We will denote the spectral topology on R by τ_{CL_R}.

Proposition 2.3.1.4 Let R be a ring. The map

$$\begin{aligned} \pi : R &\to R/\operatorname{Rad} R \\ a &\mapsto [a] \end{aligned}$$

from $\langle R, \tau_{\mathrm{CL}_R} \rangle$ onto $\langle R/\operatorname{Rad} R, \tau_{\mathrm{CL}_{R/\operatorname{Rad} R}} \rangle$ is continuous.

Proof: We use the characterization of continuity - part (d) of Proposition 1.3.4.2. To see that π is continuous, let R be ring and let $E \subseteq R$. We show that

$$\pi(\mathrm{CL}_R(E)) \subseteq \mathrm{CL}_{R/\operatorname{Rad} R}(\pi(E)).$$

To see that this is the case, let $[a] \in \pi(\mathrm{CL}_R(E))$. Then there exists $b \in \mathrm{CL}_R(E)$ such that $[a] = [b]$. Let $\tilde{J} \subseteq R/\operatorname{Rad} R, \tilde{J}$ finite. Since $\tilde{J}$ is finite there exists $J \subseteq R, J$ finite such that $\pi(J) = \tilde{J}$. Since J is finite and $b \in \mathrm{CL}_R(E)$ there exists $c \in E$ such that $1 - J(b-c) \subseteq R^{-1}$. By Lemma 2.3.1.1, we know that

$$[1] - \pi(J)([b] - [c]) \subseteq (R/\operatorname{Rad} R)^{-1}.$$

Hence $[1] - \tilde{J}([a] - [c]) \subseteq (R/\operatorname{Rad} R)^{-1}$, and since $[c] \in \pi(E)$ we have that $[a] \in \mathrm{CL}_{R/\operatorname{Rad} R}(\pi(E))$, as required. □

Proposition 2.3.1.5 Let R be a ring. The spectral topology on $R/\operatorname{Rad} R, \tau_{\mathrm{CL}_{R/\operatorname{Rad} R}}$, equals the quotient topology generated by the natural homomorphism,

$$\pi : R \to R/\operatorname{Rad} R.$$

Proof: By Proposition 2.3.1.4 we know that π is continuous. By Proposition 2.3.1.3, π is closed. In view of Theorem 1.3.2.5, the result follows. □

We state the next result without proof.

Proposition 2.3.1.6 ([9], p. 270) Let R be a local ring, and $K \subset R$. Then K is closed in R if and only if it is finite modulo the radical.

□

We can now apply Proposition 2.3.1.6 to characterize the closed sets in $\mathbb{C}[[z]]$, as follows.

Theorem 2.3.1.7 ([9], p. 270) Let $R = \mathbb{C}[[z]]$ be the ring of all formal power series over the field $\mathbb{C}$ in the indeterminate z and let $K \subseteq R$. Then necessary and sufficient for K to be closed in R is that the set

$$K_0 = \{a_0 \in \mathbb{C} : a_0 + \textstyle\sum_{n\geq 1} a_n z^n = f(z) \in K\}$$

of all leading coefficients in K be finite.

□

✠

Chapter 3

Properties of the spectral topology

3.1 Introduction

In this chapter, we discuss some interesting properties of the spectral topology. We start by looking at notions of continuity for the multiplication map defined on a ring with a topology defined on it. Then we discuss the neighbourhood system around a point of the ring.

3.2 Joint and separate continuity

The multiplication operation on a ring R is essentially a map $\cdot : R \times R \to R$. This map from the product space can be continuous in more than one way. We look at the two notions of continuity that are relevant for us. We start with the most general definitions of the notions and then make the definitions specific for our needs.

Definition 3.2.0.1 ([29], p. 517) A map f from the product $X \times Y$ of topological spaces X and Y into a topological space Z is said to be *separately continuous* if, for each $(x_0, y_0) \in X \times Y$, the maps $x \mapsto f(x, y_0)$ from X to Z and $y \mapsto f(x_0, y)$ from Y to Z are continuous. When f is continuous at (x_0, y_0) relative to the product topology, we shall say that f is *jointly continuous* at (x_0, y_0).

To make our discussions below easier we will introduce some terms, used by the authors in [21].

Let R be a ring and let $\cdot : R \times R \to R$ be the multiplication operation on R. Fix $r_0 \in R$. Then we define the maps $\cdot_{(r_0,\cdot)}$ and $\cdot_{(\cdot,r_0)}$ from R to R as:

(a) $\cdot_{(r_0,\cdot)}(r) = r_0 \cdot r$, and

(b) $\cdot_{(\cdot,r_0)}(r) = r \cdot r_0$.

We will call $\cdot_{(r_0,\cdot)}$ the *vertical section of* $\cdot$ *by* r_0 and $\cdot_{(\cdot,r_0)}(r)$ the *horizontal section of* $\cdot$ *by* r_0.

It is now easy to see that Definition 3.2.0.1 above is equivalent to

Definition 3.2.0.2 Let R be a ring with a topology τ defined on it. The map

$$\cdot : R \times R \to R$$

is separately continuous w.r.t. τ if and only if for every $r_0 \in R$, the vertical and horizontal sections of $\cdot$ by r_0 are continuous w.r.t. τ.

Now we are ready to construct the working definition of separate continuity for our multiplication operation.

Lemma 3.2.0.3 Suppose R is a ring with a topology τ defined on it. Suppose that the multiplication operation $\cdot$ on R is separately continuous w.r.t. τ. Then for each element $a \in R$ and for each neighbourhood of zero, U, there exists a neighbourhood of zero, V, such that $a \cdot V \subseteq U$ and $V \cdot a \subseteq U$.

Proof: Let R, τ and $\cdot$ be as described and let $a \in R, a$ arbitrary. By assumption the map $\cdot_{(a,\cdot)} : R \to R$ is continuous at 0. Next, note that $\cdot_{(a,\cdot)}(0) = a \cdot 0 = 0$. So suppose that U is a neighbourhood of $0 = \cdot_{(a,\cdot)}(0)$. By the continuity of $\cdot_{(a,\cdot)}$ we have that there exists a neighbourhood of 0, V say, such that $\cdot_{(a,\cdot)}[V] \subseteq U$. But $\cdot_{(a,\cdot)}[V] = \{a \cdot v : v \in V\} = aV$. The second part of the statement is proved using the fact that $\cdot_{(\cdot,a)}$ is continuous at 0 as well. The result follows. $\square$

3.3 Comparison of closures

Proposition 3.3.0.1 ([9], p. 269) If A is a Banach algebra with identity 1 and invertible group A^{-1} then for $K \subseteq A$ we have $\mathrm{cl}_{\|\cdot\|}(K) \subseteq \mathrm{CL}(K) \subseteq \mathrm{cl}_{\mathrm{alg}}(K)$.

Proof: Let A be a Banach algebra and let $K \subseteq A$. To prove that $\mathrm{cl}_{\|\cdot\|}(K) \subseteq \mathrm{CL}(K)$ let $a \in \mathrm{cl}_{\|\cdot\|}(K)$. Then $a \in K \cup \mathrm{der}_{\|\cdot\|}(K)$. If $a \in K$ then from part (a) of Theorem 2.3.0.1 we have that $a \in \mathrm{CL}(K)$. If $a \in \mathrm{der}_{\|\cdot\|}(K)$ there exists a sequence of points (b_n) from K such that $b_n \xrightarrow[n]{\infty} a$. This means $\|a - b_n\| \xrightarrow[n]{\infty} 0$. Let J be a finite subset of A. Then $J = \{0\}$ or $J \neq \{0\}$. If $J = \{0\}$ then for every $k \in K$ we have that $1 - J(a-k) = \{1\} \subseteq A^{-1}$. Hence we have that $a \in \mathrm{CL}(K)$. If $J \neq \{0\}$ then $\max\limits_{j \in J} \|j\| \in \mathbb{R}^+$. From $\|a - b_n\| \xrightarrow[n]{\infty} 0$ we have that there exists $N \in \mathbb{N}$ such that $n \geqslant N \implies \|a - b_n\| < \frac{1}{\max_{j\in J}\|j\|}$ giving $\max\limits_{j\in J} \|j\|\|a - b_n\| < 1$. Let $j \in J$. Then $\|j(a - b_N)\| \leq \|j\|\|a - b_N\| \leq \max_{j\in J}\|j\|\|a - b_N\| < 1$. Hence by Theorem 1.5.0.3 we have that $1 - j(a - b_N) \in A^{-1}$. Since $j \in J$ was chosen arbitrarily, we have that $1 - J(a - b_N) \subseteq A^{-1}$, giving $a \in \mathrm{CL}(K)$. Thus $\mathrm{cl}_{\|\cdot\|}(K) \subseteq \mathrm{CL}(K)$ as required.

To show that $\mathrm{CL}(K) \subseteq \mathrm{cl}_{\mathrm{alg}}(K)$, let $a \in \mathrm{CL}(K)$. Then for all finite $J \subseteq A$ there exists $a' \in K$ such that $1 - J(a - a') \subseteq A^{-1}$. Let $b \in A$. Then $\{b\}$ is a finite subset of A and by assumption there exists $a' \in K$ such that $1 - b(a - a') \in A^{-1}$ from which the result follows. Hence $\mathrm{CL}(K) \subseteq \mathrm{cl}_{\mathrm{alg}}(K)$. □

Corollary 3.3.0.2 On a Banach algebra the spectral topology is coarser than the norm topology.

Proof: The proof follows as an application of Lemma 1.3.1.17 and Proposition 3.3.0.1. □

Theorem 3.3.0.3 ([9], p. 270) Let R be a ring. Then R^{-1} is open in the spectral topology.

Proof: We show that $\mathrm{CL}(R \setminus R^{-1}) = R \setminus R^{-1}$. From the fact that the spectral closure satisfies K1 from Theorem 2.3.0.1 we already have that $R \setminus R^{-1} \subseteq \mathrm{CL}(R \setminus R^{-1})$, so it remains to show that $\mathrm{CL}(R \setminus R^{-1}) \subseteq R \setminus R^{-1}$. We prove this by showing that $a \notin R \setminus R^{-1} \implies a \notin \mathrm{CL}(R \setminus R^{-1})$. So suppose that $a \notin R \setminus R^{-1}$. Then $a \in R^{-1}$. Let $J = \{a^{-1}\}$ and pick arbitrary $a' \in R \setminus R^{-1}$. Then $1 - J(a - a') = 1 - \{a^{-1}\}(a - a') = 1 - (1 - a^{-1}a') = a^{-1}a'$. Notice that with $a^{-1} \in R^{-1}$ and $a' \notin R^{-1}$, from part (d) of Lemma 1.2.0.4 we have that $a^{-1}a' \notin R^{-1}$ so it follows that $a \notin \mathrm{CL}(R \setminus R^{-1})$. □

The fact that in a ring R, the group of invertibles is open in the spectral topology enables us to construct neighbourhoods of 0.

Theorem 3.3.0.4 Let R be a ring and J a finite subset of R. The set

$$U_J = \{a : 1 - Ja \subseteq R^{-1}\}$$

is a neighbourhood of 0.

Proof: Let R and J be as described. Since $1 \in R^{-1}$ and R^{-1} is open in the spectral topology, there exists $V \in \mathcal{N}_1$ such that $V \subseteq R^{-1}$. By Theorem 1.4.0.4, since $V \in \mathcal{N}_1$ we have that $1 - V \in \mathcal{N}_0$. By the separate continuity of multiplication, for each $j \in J$ there exists $\widetilde{U}_j \in \mathcal{N}_0$ with the property that $j\widetilde{U}_j \subseteq 1 - V$. This means that

$$1 - j\widetilde{U}_j \subseteq 1 - (1 - V) = V \subseteq R^{-1}. \tag{3.1}$$

Let $\widetilde{U}_J = \bigcap\limits_{j \in J} \widetilde{U}_j$. Since $\widetilde{U}_J$ is a finite intersection of neighbourhoods of 0, it is also a neighbourhood of 0.

We show next that $\widetilde{U}_J \subseteq U_J$. To see this, let $a \in \widetilde{U}_J$. Then $a \in \widetilde{U}_j$ for each $j \in J$. By (3.1) we have that $1 - ja \in R^{-1}$ for each $j \in J$. Hence $1 - Ja \subseteq R^{-1}$ and so $a \in U_J$. Hence $\widetilde{U}_J \subseteq U_J$ and so $U_J \in \mathcal{N}_0$. $\square$

Theorem 3.3.0.5 ([9], Theorem 4 - p. 270) Let R be a ring with topology ρ for which the multiplication operation is separately continuous. For $K \subseteq R$, let $\mathrm{cl}_\rho(K)$ represent the closure of K with respect to ρ. Then

$$\mathrm{cl}_\rho(K) \subseteq \mathrm{CL}(K) \text{ for all } K \subseteq R \iff A^{-1} \in \rho.$$

Proof: Suppose that $A^{-1} \in \rho$. We show that $\mathrm{cl}_\rho(K) \subseteq \mathrm{CL}(K)$ for all $K \subseteq R$. Let $x \in \mathrm{cl}_\rho(K)$. Since A^{-1} is open and $1 \in A^{-1}$ there exists a $V \in \mathcal{N}_1$ with the property that $V \subseteq R^{-1}$. By Theorem 1.4.0.4, we have that $V - 1 \in \mathcal{N}_0$. Let $J \subseteq R$, J finite. Since multiplication is separately continuous, for every $j \in J$, there exists $U_j \in \mathcal{N}_0$ such that $jU_j \subseteq V - 1$. This gives us $1 + jU_j \subseteq V \subseteq A^{-1}$. Next, let $U_J = \bigcap\limits_{j \in J} U_j$. Then U_J is a finite intersection of neighbourhoods of zero, hence $U_J \in \mathcal{N}_0$. By Lemma 1.4.0.5 $-U_J \in \mathcal{N}_0$ also. Again by Theorem 1.4.0.4, $x - U_J \in \mathcal{N}_x$. Since $x \in \mathrm{cl}_\rho(K)$ there exists $x' \in K$ such that $x' \in x - U_J$. Hence $x' - x \in -U_J \implies x - x' \in U_J$. So $1 - J(x - x') \subseteq 1 - JU_J \subseteq A^{-1}$. Hence $x \in \mathrm{CL}(K)$.

Conversely, suppose that $\mathrm{cl}_\rho(K) \subseteq \mathrm{CL}(K)$ for every $K \subseteq R$. Then by Lemma 1.3.1.17 the spectral topology is weaker than ρ. Since A^{-1} is open in the spectral topology, we have that $A^{-1} \in \rho$. $\square$

Definition 3.3.0.6 ([9], p. 271) Let R be a ring. An element $x \in R$ is called:

(a) *Nearly left invertible* if $x \in \mathrm{CL}(R_l^{-1})$.

(b) *Nearly right invertible* if $x \in \mathrm{CL}(R_r^{-1})$.

(c) *Nearly invertible* if $x \in \mathrm{CL}(R^{-1})$.

In the spectral topology we have the following version of the well known property in the norm closure (Proposition 1.5.0.6).

Proposition 3.3.0.7 ([9], p. 271) Let R be a ring. Then

$$R_l^{-1} \cap \mathrm{CL}(R_r^{-1}) = R^{-1} = \mathrm{CL}(R_l^{-1}) \cap R_r^{-1}.$$

Proof: We show $R_l^{-1} \cap \mathrm{CL}(R_r^{-1}) \subseteq R^{-1}$ and $R^{-1} \subseteq R_l^{-1} \cap \mathrm{CL}(R_r^{-1})$

To see the first inclusion, let $a \in R_l^{-1} \cap \mathrm{CL}(R_r^{-1})$. Then $a \in R_l^{-1}$ and $a \in \mathrm{CL}(R_r^{-1})$. Since $a \in R_l^{-1}$ we know there exists $a' \in R_r^{-1}$ such that $a'a = 1$. Since $a \in \mathrm{CL}(R_r^{-1})$ we know that there exists $a'' \in R_r^{-1}$ such that $1 - a'(a - a'') \in R^{-1}$. Hence $a'a'' \in R^{-1}$. So there exists $b \in R$ with the property that $(ba')a'' = b(a'a'') = (a'a'')b = 1$ which means that $a'' \in R_l^{-1}$, hence $a'' \in R^{-1}$. Since $a'' \in R^{-1}$ there exists $c \in R$ such that $ca'' = a''c = 1$.

Next, we show that $a' \in R^{-1}$. Since we already have that $a' \in R_r^{-1}$ all we have to show is that $a' \in R_l^{-1}$. We know that $ba'a'' = 1$. Hence $ba'a''c = c$, which gives $ba' = c$ which gives us $a''ba' = a''c = 1$. Hence $a' \in R_l^{-1}$. So we have that $a'a = 1$ and $a''ba' = 1$, so that a' is both left invertible and right invertible. From the paragraph preceding (1.1) we must have that $a''b = a$, so that $a'a = aa' = 1$. Hence $a \in R^{-1}$. Hence $R_l^{-1} \cap \mathrm{CL}(R_r^{-1}) \subseteq R^{-1}$.

For the reverse inclusion, we use the fact that $R_r^{-1} \subseteq \mathrm{CL}(R_r^{-1})$ (K1 from Theorem 2.3.0.1) and reason as follows:

$$R^{-1} = R_l^{-1} \cap R_r^{-1} \subseteq R_l^{-1} \cap \mathrm{CL}(R_r^{-1})$$

and the containment follows. Hence we have proved that both containments hold, and so the result follows. $\square$

In words the above says that a nearly invertible element with one sided inverse has to be invertible.

The fact that the collection of sets $\{U_J : J \subseteq R, J \text{ finite}\}$ are all neighbourhoods of zero allows us to define a concept of convergence that applies to a general ring, as follows.

Definition 3.3.0.8 Let R be a ring and let (x_n) be a sequence with $x_n \in R$ for all $n \in \mathbb{N}$. Then we say (x_n) converges to 0 if there exists a family (N_J) of natural numbers, indexed by finite $J \subseteq R$, for which $n \geqslant N_J \implies 1 + Jx_n \subseteq R^{-1}$.

✠

Chapter 4

Quasinilpotents in general rings

4.1 Introduction

In this chapter, we introduce concepts generally associated with Banach algebras, suitably extended to general rings by means of the spectral closure and we explore some consequences.

4.2 Quasinilpotent elements in a ring

Proposition 4.2.0.1 ([20], p. 255) If A is a normed algebra, then a necessary and sufficient condition for $a \in A$ to be quasinilpotent is the following ($n \in \mathbb{N}$):

for all $m \in \mathbb{N}$ and for any $\{c, d\} \subseteq A^m$ the sequence $c_m^n \cdots c_2^n c_1^n a^n d_1^n d_2^n \cdots d_m^n \xrightarrow[n]{\infty} 0$. $\square$

The above proposition relaxes the requirement for the structure A to be a Banach algebra and merely stipulates that A should have a topology defined by a norm $\|\cdot\|$.

R. Harte and D. Cvetković-Ilić generalize this a step further, by defining a concept of quasinilpotence on a general ring equipped with the spectral topology, so we take the following as definition of a quasinilpotent element in a general ring:

Definition 4.2.0.2 ([9], p. 272) Let R be a ring and let $m, n \in \mathbb{N}$. Let $c = (c_1, c_2, ..., c_m) \in R^m$. We define

$$c^{(n)} = c_1^n c_2^n \cdots c_m^n \quad \text{and} \quad c_{(n)} = c_m^n \cdots c_2^n c_1^n$$

Definition 4.2.0.3 ([9], p. 272) An element a of a ring R is said to be quasinilpotent if for all $m \in \mathbb{N}$ and for any $\{c, d\} \subseteq R^m$ the sequence $c_{(n)} a^n d^{(n)} \xrightarrow[n]{\infty} 0$. We use $QN(R)$ to represent the set of quasinilpotent elements of the ring R.

It is easy to see that if R is a Banach (more generally, a normed) algebra, then we recover the concept of quisinilpotence via Proposition 4.2.0.1.

Theorem 4.2.0.4 ([9], p. 272) Let R be a ring, $a, b \in R$. The set $\mathrm{QN}(R)$ satisfies:

(a) $0 \in \{a^k : k \in \mathbb{N}\} \implies a \in \mathrm{QN}(R)$,

(b) $a \in \mathrm{QN}(R), ab = ba \implies ab \in \mathrm{QN}(R)$,

(c) $1 - \mathrm{QN}(R) \subseteq R^{-1}$,

(d) $\mathrm{Rad}\, R \subseteq \mathrm{QN}(R)$.

Proof: To prove that (a) holds, suppose a is nilpotent. Then $a^k = 0$ for some $k \in \mathbb{N}$. Let $J \subseteq R$, J finite. Let $N_J = k$ and let $\{c, d\} \subseteq R^m$. Then, if $n \geqslant N_J$ we have that $1 - J c_m^n \cdots c_2^n c_1^n a^n d_1^n d_2^n \cdots d_m^n = \{1\} \subseteq R^{-1}$. Hence $c_m^n \cdots c_2^n c_1^n a^n d_1^n d_2^n \cdots d_m^n \xrightarrow[n]{\infty} 0$, so that $a \in \mathrm{QN}(R)$.

To prove (b), suppose that $ab = ba$ and $a \in \mathrm{QN}(R)$. It is easy to see that if $ab = ba$ then $(ab)^n = a^n b^n$. Let $\{c, d\} \subseteq R^m$. Then

$$c_m^n \cdots c_2^n c_1^n (ab)^n d_1^n d_2^n \cdots d_m^n = c_m^n \cdots c_2^n c_1^n \cdot 1 \cdot a^n b^n d_1^n d_2^n \cdots d_m^n$$

We have that $\{(c_1, \cdots, c_m, 1), (b, d_1, \cdots, d_m)\} \subseteq R^{m+1}$, and since $a \in \mathrm{QN}(R)$ we have that $c_m^n \cdots c_2^n c_1^n \cdot 1 \cdot a^n b^n d_1^n d_2^n \cdots d_m^n \xrightarrow[n]{\infty} 0$, hence $c_m^n \cdots c_2^n c_1^n (ab)^n d_1^n d_2^n \cdots d_m^n \xrightarrow[n]{\infty} 0$, and so $ab \in \mathrm{QN}(R)$.

To prove (c), suppose that $a \in \mathrm{QN}(R)$, a arbitrary. We show that $1 - a \in R^{-1}$. We define $x_n = a^n$. Since $a \in \mathrm{QN}(R)$ and $(1, \cdots, 1) \in R^n$ we have that $a^n \xrightarrow[n]{\infty} 0$. Let $J = \{1\}$. By Definition 3.3.0.8, we know that for the set J there exists $N_J \in \mathbb{N}$ with the property that $n \geqslant N_J \implies 1 - J x_n \subseteq A^{-1}$. So let $m \in \mathbb{N}, m > N_J$. Then $1 - J x_m = 1 - \{1\} a^m \subseteq A^{-1}$, hence $1 - a^m \in A^{-1}$. Now we write:

$$(1 - a)(1 + a + a^2 + \cdots + a^{m-1}) = 1 - a^m = (1 + a + a^2 + \cdots + a^{m-1})(1 - a).$$

Since $1 - a^m \in R^{-1}$, we know there exists $b \in R$ with the property that $(1 - a^m)b = 1 = b(1 - a^m)$. Hence, we have $(1 - a)(1 + a + a^2 + \cdots + a^{m-1}) \cdot b = 1$, hence $1 - a \in R_r^{-1}$. Also, we have $b \cdot (1 + a + a^2 + \cdots + a^{m-1})(1 - a) = 1$, hence $1 - a \in R_l^{-1}$. Since $1 - a$ is then left and right invertible, it must be invertible, as discussed in the paragraph preceding equation (1.1).

To prove (d), we show that $\operatorname{Rad} R \subseteq \mathrm{QN}(R)$:

To see that $\operatorname{Rad} R \subseteq \mathrm{QN}(R)$ suppose that $a \in \operatorname{Rad} R$. We show that $a \in \mathrm{QN}(R)$. So let $m \in \mathbb{N}, m$ arbitrary, and let $\{c, d\} \subseteq R^m, c, d$ also arbitrary. We show that $c_{(n)} a^n d^{(n)} \xrightarrow[n]{\infty} 0$. To show this, let $J \subset R, J$ finite and arbitrary. Let $J' = -J$. Then J' is also finite. Let $j \in J', j$ also arbitrary. Then, since $a \in \operatorname{Rad} R$, for $n \in \mathbb{N} \setminus \{0\}$ we have that $1 - a[a^{n-1} d^{(n)} j c_{(n)}] \in R^{-1}$. From Lemma 1.2.0.15 this means that $1 - j c_{(n)} a [a^{n-1} d^{(n)}] \in R^{-1}$, which means that $1 - j c_{(n)} a^n d^{(n)} \in R^{-1}$. Hence $1 - J' c_{(n)} a^n d^{(n)} \subseteq R^{-1}$. Finally, this means that $1 + J c_{(n)} a^n d^{(n)} \subseteq R^{-1}$, hence $a \in \mathrm{QN}(R)$. □

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Chapter 5

Idempotents and generalized inverses

5.1 Introduction

In this chapter we discuss ring homomorphisms in connection with generalized inverses (relatively Fredholm & relatively Weyl) as well as indempotent elements.

5.2 Relatively Fredholm and relatively Weyl elements

Definition 5.2.0.1 ([20], p. 246 & [9], p. 272) Let R be a ring. The *relatively Fredholm (regular)* elements of R are the elements of R belonging to the set

$$R^{\cap} = \{a \in R : a \in aRa\}.$$

Definition 5.2.0.2 ([20], p. 246 & [9], p. 272) Let R be a ring. The *relatively Weyl (decomposably regular)* elements of R are those elements belonging to the set

$$R^{\cup} = \{a \in R : a \in aR^{-1}a\}.$$

Definition 5.2.0.3 ([20], p. 247 & [9], p. 272) Let R be a ring. The *idempotent* elements $R^{\bullet}$ of R are those elements of R belonging to the set

$$R^{\bullet} = \{p \in R : p^2 = p\}.$$

Lemma 5.2.0.4 ([20] - Theorem 7.3.4, p. 248) Let R be a ring. Then $R^{\cup} = R^{-1}R^{\bullet}$.

Proof: We prove that $R^{-1}R^{\bullet} \subseteq R^{\cup}$ and $R^{\cup} \subseteq R^{-1}R^{\bullet}$. To prove the first inclusion let $a \in R^{-1}R^{\bullet}$. Then there exists $b \in R^{-1}$ and $p \in R^{\bullet}$ such that $a = bp$. Now $a = bp = bpp = bpb^{-1}bp = ab^{-1}a \in R^{\cup}$. Hence $R^{-1}R^{\bullet} \subseteq R^{\cup}$. Next, suppose $a \in R^{\cup}$. Then there exists $b \in R^{-1}$ such that $a = aba$. Then $ba = baba$, so that $ba \in R^{\bullet}$. Then $a = b^{-1}ba \in R^{-1}R^{\bullet}$. Hence $R^{\cup} \subseteq R^{-1}R^{\bullet}$ and the proof is complete. □

Remark 5.2.0.5 Let R be a ring. We briefly discuss the following observations relating invertible, idempotent, regular and decomposably regular elements in R:

(a) $R^{-1} \subseteq R^{\cup}$,

(b) $R^{\cup} \subseteq R^{\cap}$,

(c) $R^{\bullet} \subseteq R^{\cup}$.

Proof: To prove that (a) holds, let $a \in R^{-1}$. Then there exists $b \in R^{-1}$ such that $ab = 1 = ba$. So from $1 = ba$ we get $a = aba$, hence $a \in R^{\cup}$.

To prove that (b) holds, let $a \in R^{\cup}$. Then there exists $b \in R^{-1}$ such that $a = aba$. Since $A^{-1} \subseteq A$, we have that $a \in R^{\cap}$, and the result follows.

To see that (c) holds, let $p \in R^{\bullet}$. Then $p = p \cdot 1 \in R^{\bullet}R^{-1} = R^{\cup}$ □

Theorem 5.2.0.6 ([9], p. 272) Let R be a ring. Nearly invertibles with generalized inverses in R have invertible generalized inverses:

(a) There is inclusion $R^{\cap} \cap \mathrm{CL}(R^{-1}) \subseteq R^{\cup}$.

(b) Necessary and sufficient for equality in (a) is that $R^{\bullet} \subseteq \mathrm{CL}(R^{-1})$.

Proof: Let $a \in R^{\cap} \cap \mathrm{CL}(R^{-1})$. We show that $a \in R^{\cup}$ by using Lemma 5.2.0.4. $a \in R^{\cap} \cap \mathrm{CL}(R^{-1})$ means that $a \in R^{\cap}$ and $a \in \mathrm{CL}(R^{-1})$. Since $a \in R^{\cap}$, there exists $a' \in R$ such that $a = aa'a$. Hence we have $a'a = a'aa'a = (a'a)^2$ hence $p = a'a \in R^{\bullet}$. Notice that $ap = aa'a = a$. Now, since $a \in \mathrm{CL}(R^{-1})$, there exists $b \in R^{-1}$ such that $1 - (a-b)a' \in R^{-1}$. Let $c^{-1} = 1 - (a-b)a'$. Then $1 = cc^{-1} = c + c(b-a)a'$, so left-multiplying by a gives $a = ca + c(b-a)a'a$. But $p = a'a$ so $a = ca + c(b-a)p$. Hence $a = ca + cbp - cap$. Since $ap = a$ we have that $a = ca + cbp - ca$, hence

$a = cbp$. Since $c, b \in R^{-1}$ we have that $cb \in R^{-1}$ (Lemma 1.2.0.4) and since $p \in R^\bullet$, from Lemma 5.2.0.4, we have that $a \in R^\cup$ thus proving (a).

To prove (b), suppose that $R^\cup \subseteq R^\cap \cap \mathrm{CL}(R^{-1})$ and let $a \in R^\bullet$. We show that $a \in \mathrm{CL}(R^{-1})$. From part 3 of Remark 5.2.0.5 we have that $R^\bullet \subseteq R^\cup$, so $a \in R^\bullet \implies a \in R^\cup \implies a \in \mathrm{CL}(R^{-1})$, by assumption.

Conversely, suppose that $R^\bullet \subseteq \mathrm{CL}(R^{-1})$. We show that $R^\cup \subseteq R^\cap \cap \mathrm{CL}(R^{-1})$. Let $a \in R^\cup$. From part b of Remark 5.2.0.5, we have that $a \in R^\cap$. We show that $a \in \mathrm{CL}(R^{-1})$, which will prove the statement. Since $a \in R^\cup$ there exists $b \in R^{-1}$ such that $a = aba$. This gives $ab = abab = (ab)^2$, hence $ab \in R^\bullet$. By assumption $R^\bullet \subseteq \mathrm{CL}(R^{-1})$ thus $ab \in \mathrm{CL}(R^{-1})$ which means that for all finite $J \subseteq R$, there exists $c \in R^{-1}$ such that $1 - J(ab - c) \subseteq R^{-1}$. Let $J = \{b^{-1}\}$. Then $1 - (ab - c)\{b^{-1}\} \subseteq R^{-1}$ equivalently $1 - (a - cb^{-1}) \in R^{-1}$. Since $c \in R^{-1}$ and $b \in R^{-1}$ we have (by part a, Lemma 1.2.0.4) that $cb^{-1} \in R^{-1}$. Thus $a \in \mathrm{CL}(R^{-1})$ proving part b. □

Theorem 5.2.0.7 ([9], p. 273) Let A be a Banach algebra. Then $0 \notin \mathrm{int}(\sigma(a)) \implies a \in \mathrm{cl}_{\|\cdot\|}(A^{-1}) \subseteq \mathrm{CL}(A^{-1})$.

Proof: If $0 \notin \mathrm{int}\,\sigma(a)$ then $0 \in A \setminus \sigma(a)$ or $0 \in \partial\sigma(a)$.

Suppose $0 \in A \setminus \sigma(a)$. Then $0 \notin \sigma(a) \implies -a \in A^{-1} \implies a \in A^{-1} \subseteq \mathrm{cl}_{\|\cdot\|}(A^{-1})$.

Suppose $0 \in \partial\sigma(a)$. From the fact that $\sigma(a)$ is compact (Theorem 1.5.1.3), hence closed and bounded, there exists a sequence (λ_n) with $\lambda_n \in \mathbb{C}$ for all $n \in \mathbb{N}$ with the properties that $\lambda_n \to 0$ and $a - \lambda_n \in A^{-1}$. Since $\lambda_n \to 0$ we have that $a - \lambda_n \to a$. Hence every neighbourhood of a (in the norm topology) will contain an element from A^{-1}, hence $a \in \mathrm{cl}_{\|\cdot\|}(A^{-1})$. The fact that $\mathrm{cl}_{\|\cdot\|}(A^{-1}) \subseteq \mathrm{CL}(A^{-1})$ follows from Proposition 3.3.0.1. The result follows.

□

✠

Chapter 6

Fredholm and Weyl elements

6.1 Introduction

Motivated by Atkinson's Theorem, Harte in [17], uses the concept of a Fredholm operator to define for a an element of a Banach algebra, what it means for a to be *Fredholm relative to a Banach algebra homomorphism*. He then develops a theory called Fredholm Theory relative to a Banach algebra homorphism. In this chapter we illustrate how the spectral closure interfaces with that theory.

6.2 T-Fredholm and T-Weyl elements

Definition 6.2.0.1 ([20], p. 261) Let R_1 and R_2 be rings and $T : R_1 \to R_2$ be a ring homomorphism. An element a of the ring R_1 is said to be *T-Fredholm* if $Ta \in R_2^{-1}$.

Definition 6.2.0.2 ([20], p. 261) Let R_1 and R_2 be rings and $T : R_1 \to R_2$ be a ring homomorphism. An element a of the ring R_1 is said to be *T-Weyl* if $a \in R_1^{-1} + T^{-1}(\{0\})$.

Definition 6.2.0.3 ([20], p. 356) Let R_1 and R_2 be rings.
A homomorphism $T : R_1 \to R_2$ is said to have the *Gelfand property* if for all $a \in R_1$,

$$Ta \in R_2^{-1} \implies a \in R_1^{-1}.$$

Definition 6.2.0.4 ([9], p. 273) Let R_1 and R_2 be rings.
A homomorphism $T : R_1 \to R_2$ is called *relatively open* if and only if

$$K \subseteq R_1 \implies T(R_1) \cap \mathrm{CL}_{R_2}(T(K)) \subseteq T(\mathrm{CL}_{R_1}(K)).$$

Definition 6.2.0.5 ([9], p. 273) Let R_1 and R_2 be rings. A homomorphism $T : R_1 \to R_2$ has *inverse closed range* if and only if $T(R_1) \cap R_2^{-1} \subseteq T(R_1)^{-1}$.

Remark 6.2.0.6 ([20] - Theorem 9.6.5, p. 358) Necessary and sufficient for the Gelfand homomorphism $T : R_1 \to R_2$ on R_1 to be one-to-one, is that R_1 is semisimple.

Theorem 6.2.0.7 ([9], p. 273) Let R_1, R_2 be rings and let $T : R_1 \to R_2$ be a ring homomorphism from R_1 onto R_2. Then we have:

$$R_1^{-1} \subseteq R_1^{-1} + T^{-1}(\{0\}) \subseteq T^{-1}(R_2^{-1}). \tag{6.1}$$

Proof: Suppose $a \in R_1^{-1}$. Then we can write $a = a + 0 \in R_1^{-1} + T^{-1}(\{0\})$. Since a was arbitrary we have that $R_1^{-1} \subseteq R_1^{-1} + T^{-1}(\{0\})$, i.e. invertible elements are T-Weyl.

Next, we show that T-Weyl elements are also T-Fredholm by showing $R_1^{-1}+T^{-1}(\{0\}) \subseteq T^{-1}(R_2^{-1})$. Suppose that $a \in R_1^{-1} + T^{-1}(\{0\})$. Then there exists $a_1 \in R_1^{-1}$ and $a_2 \in T^{-1}(0)$ such that $a = a_1 + a_2$. Since T is a ring homomorphism we have that

$$Ta = T(a_1 + a_2) = Ta_1 + Ta_2 = Ta_1 + 0 = Ta_1.$$

Hence $a \in T^{-1}(R_2^{-1}) \iff a_1 \in T^{-1}(R_2^{-1})$. Since $a_1 \in R_1^{-1}$, there exists $a_1^{-1} \in R_1$ with the property that $a_1a_1^{-1} = a_1^{-1}a_1 = 1$. Since T is onto we have (by Proposition 1.2.3.3) that

$$T(a_1)T(a_1^{-1}) = T(a_1a_1^{-1}) = T(1) = 1 \text{ and } T(a_1^{-1})T(a_1) = T(a_1^{-1}a_1) = T(1) = 1.$$

Hence $a_1 \in T^{-1}(R_2^{-1})$, which means that $a \in T^{-1}(R_2^{-1})$, and so a is T-Fredholm. □

Lemma 6.2.0.8 Let R_1, R_2 be rings. If $T : R_1 \to R_2$ is a homomorphism, then $T^{-1}(0) + T^{-1}(0) \subseteq T^{-1}(0)$.

Proof: Suppose that $a \in T^{-1}(\{0\}) + T^{-1}(\{0\})$. Then there exist $a_1, a_2 \in T^{-1}(\{0\})$ such that $a = a_1 + a_2$. Then

$$Ta = T(a_1 + a_2) = Ta_1 + Ta_2 = 0 + 0 = 0$$

Hence $a \in T^{-1}(\{0\})$, and the result follows. □

Lemma 6.2.0.9 *Let R_1 and R_2 be rings and let $T : R_1 \to R_2$ be a ring homomorphism onto R_2. Let $L \subseteq R_2, L$ finite. There exists $J \subseteq R_1, J$ finite such that $T(J) = L$.*

Proof: Let T, R_1, R_2 and L be as described, and let $l \in L$. Since T is onto the set $T^{-1}(\{l\})$ is not empty. The set J is constructed as follows. For each $l \in L$, pick one element from $T^{-1}(\{l\})$ to go into J. Then it is clear that $T(J) = L$ and that J is finite. □

Theorem 6.2.0.10 ([9], p. 273) *Let R_1 and R_2 be rings and $T : R_1 \to R_2$ a homomorphism from R_1 onto R_2. Then T is continuous with respect to the spectral topology.*

Proof: We use again part (d) of Proposition 1.3.4.2. So we will show that $K \subseteq R_1 \implies T(\mathrm{CL}_{R_1}(K)) \subseteq \mathrm{CL}_{R_2}(T(K))$. So let $y \in T(\mathrm{CL}_{R_1}(K))$ and let $L \subseteq R_2, L$ finite and arbitrary. Then, by Lemma 6.2.0.9 there exists $J \subseteq R_1, J$ finite such that $L = T(J)$. Since $y \in T(\mathrm{CL}_{R_1}(K))$ there exists $x \in \mathrm{CL}_{R_1}(K)$ such that $y = Tx$. Hence there exists $x' \in K$ with the property that $1_{R_1} - J(x - x') \subseteq R_1^{-1}$. Next we show that $a \in R_1^{-1} \implies Ta \in R_2^{-1}$. To see this, let $a \in R_1^{-1}$. Then

$$T(a)T(a^{-1}) = T(aa^{-1}) = T(1) = 1 \text{ and } T(a^{-1})T(a) = T(a^{-1}a) = T(1) = 1$$

prove the point. Hence $1_{R_1} - J(x - x') \subseteq R_1^{-1} \implies T(1_{R_1} - J(x - x')) \subseteq R_2^{-1}$. By the linearity of T, we have

$$T(1_{R_1} - J(x - x')) = T(1_{R_1}) - T(J)(Tx - Tx')) = 1_{R_2} - L(y - y') \subseteq R_2^{-1}$$

Since L was finite and arbitrary, the last line proves that $y \in \mathrm{CL}_{R_2}(T(K))$ as required. □

Theorem 6.2.0.11 ([9], p. 273) *Let R_1, R_2 be rings and let $T : R_1 \to R_2$ be a homomorphism. If T has the Gelfand property then:*

$$K \subseteq R_1 \implies T(R_1) \cap \mathrm{CL}_{R_2}(T(K)) \subseteq T(\mathrm{CL}_{R_1}(K)).$$

Proof: Suppose $K \subseteq R_1$. Let $y \in T(R_1) \cap \mathrm{CL}_{R_2}(T(K))$. We show that $y \in T(\mathrm{CL}_{R_1}(K))$, i.e. we show that there exists $x \in \mathrm{CL}_{R_1}(K)$ with the property that $y = Tx$. First, $y \in T(R_1)$ means there exists $x \in R_1$ such that $y = Tx$. We will show that $x \in \mathrm{CL}_{R_1}(K)$. To do so, let $J \subset R_1, J$ finite and arbitrary. Since J is finite, we have that $L = T(J)$ is also finite and $y \in \mathrm{CL}_{R_2}(T(K))$ means that there exists $y' \in T(K)$ such that $1_{R_2} - L(y - y') \subseteq R_2^{-1}$. Since $y' \in T(K)$ this means there exists $x' \in K$ with the property that $y' = T(x')$. So we have that

$$T(1_{R_1}) - T(J)(Tx - Tx') \subseteq R_2^{-1} \text{ or } T(1_{R_1} - J(x - x')) \subseteq R_2^{-1}.$$

Since T has the Gelfand property we have $1_{R_1} - J(x-x') \subseteq R_1^{-1}$ and so $x \in \mathrm{CL}_{R_1}(K)$ as required. □

Theorem 6.2.0.12 ([9], p. 273) Let R_1, R_2 be rings and $T : R_1 \to R_2$ a ring homomorphism onto R_2. If T has inverse closed range then:

$$K \subseteq R_1 \implies \mathrm{CL}_{R_1}(K) \cap T^{-1}(R_2^{-1}) \subseteq T^{-1}(\{0\}) + R_1^{-1}K.$$

Proof: Assume that $T : R_1 \to R_2$ is a ring homomorphism and suppose that T has inverse closed range. Let $K \subseteq R_1$ and let $a \in \mathrm{CL}_{R_1}(K) \cap T^{-1}(R_2^{-1})$. We show that $a \in T^{-1}(\{0\}) + R_1^{-1}K$.

We have that $a \in \mathrm{CL}_{R_1}(K) \cap T^{-1}(R_2^{-1})$ so $a \in \mathrm{CL}_{R_1}(K)$ and $a \in T^{-1}(R_2^{-1})$. Since $a \in T^{-1}(R_2^{-1})$ we know that $Ta \in R_2^{-1}$. Since T has inverse closed range we have that there exists $d \in R_1$ such that $(Ta)^{-1} = Td$. Since T is onto we have that

$$TaTd = 1 \implies T(ad) = T(1) \implies T(1 - ad) = 0$$

and

$$TdTa = 1 \implies T(da) = T(1) \implies T(1 - da) = 0.$$

Hence $\{1 - ad, 1 - da\} \subseteq T^{-1}(\{0\})$. Since $a \in \mathrm{CL}_{R_1}(K)$ there exists $c \in K$ such that $1 - d(a - c) \in R_1^{-1}$, or equivalently, by Lemma 1.2.0.15, $1 - (a - c)d \in R_1^{-1}$.

Let $e^{-1} = 1 - (a - c)d$. Then we have

$$\begin{aligned}
a = ee^{-1}a &= e(1 - (a - c)d)a \\
&= e(1 - ad + cd)a \\
&= e(1 - ad)a + ecda \\
&= e(1 - ad)a + ec(da - 1 + 1) \\
&= e(1 - ad)a + ec(da - 1) + ec \\
&\in T^{-1}(\{0\}) + T^{-1}(\{0\}) + R_1^{-1}K \\
&\subseteq T^{-1}(\{0\}) + R_1^{-1}K \qquad \text{(from Lemma 6.2.0.8)}
\end{aligned}$$

□

Example 6.2.0.13 Let R be a ring and let J be a two sided ideal in R. The canonical map $\pi_J : R \to R/J$ is a homomorphism onto. Hence Theorem 6.2.0.10 says that the spectral topology for R/J is weaker than or equal to the quotient of the spectral topology of R. □

Example 6.2.0.14 Let R_1 be a subring of a ring R_2 and $T : R_1 \to R_2$ be a homomorphism such that $T^{-1}(R_2^{-1}) \subseteq R_1^{-1}$. Then Theorem 6.2.0.11 tells us that the spectral topology on R_1 is weaker than or equal to the restriction of the spectral topology of R_2. □

6.3 Nearly invertible Fredholm, Weyl and weakly Riesz elements

Definition 6.3.0.1 Let R_1, R_2 be rings and $T : R_1 \to R_2$ be a homomorphism. Then $a \in R_1$ is a *nearly invertible Fredholm* element if $a \in \mathrm{CL}(R_1^{-1}) \cap T^{-1}(R_2^{-1})$.

Definition 6.3.0.2 ([11], p. 14) Let R be a ring and I a two sided ideal of R. Then I is *weakly Riesz* if $1 + I \subseteq \mathrm{CL}(R^{-1})$.

Lemma 6.3.0.3 Let R_1, R_2 be rings and $T : R_1 \to R_2$ be a ring homomorphism. Then

$$R_1^{-1} + T^{-1}(\{0\}) = R_1^{-1}(1 + T^{-1}(\{0\})).$$

Proof: Suppose that $a \in R_1^{-1} + T^{-1}(\{0\})$. Then $a = a_1 + a_2$ with $a_1 \in R_1^{-1}$ and $a_2 \in T^{-1}(\{0\})$. So $a = a_1(1 + a_1^{-1}a_2)$ showing, using Lemma 1.2.3.4, that $a \in R_1^{-1}(1 + T^{-1}(\{0\}))$. Hence $R_1^{-1} + T^{-1}(\{0\}) \subseteq R_1^{-1}(1 + T^{-1}(\{0\}))$

Conversely, suppose that $a \in R_1^{-1}(1+T^{-1}(\{0\}))$. Then $a = a_1(1+a_2)$ with $a_1 \in R_1^{-1}$ and $a_2 \in T^{-1}(\{0\})$. Then $a = a_1 + a_1a_2$. Since $a_2 \in T^{-1}(\{0\})$, using Lemma 1.2.3.4, we have that $a_1a_2 \in T^{-1}(\{0\})$ also. Hence we have that $a \in R_1^{-1} + T^{-1}(\{0\})$. Hence $R_1^{-1}(1 + T^{-1}(\{0\})) \subseteq R_1^{-1} + T^{-1}(\{0\})$, and the result follows. □

Theorem 6.3.0.4 ([9], p. 274) Let R_1, R_2 be rings and let $T : R_1 \to R_2$ be a ring homomorphism. Then

(a) If T has inverse closed range then

$$\mathrm{CL}(R_1^{-1}) \cap T^{-1}(R_2^{-1}) \subseteq R_1^{-1} + T^{-1}(\{0\}).$$

(b) $R_1^{-1} + T^{-1}(\{0\}) \subseteq \mathrm{CL}(R_1^{-1}) \cap T^{-1}(R_2^{-1})$ if and only if $T^{-1}(\{0\})$ is weakly Riesz.

Proof: To prove part (a), we apply Theorem 6.2.0.12 with $K = R_1^{-1}$ to get

$$\begin{aligned} R_1^{-1} \subseteq R_1 \implies \mathrm{CL}_{R_1}(R_1^{-1}) \cap T^{-1}(R_2^{-1}) &\subseteq T^{-1}(\{0\}) + R_1^{-1} R_1^{-1} \\ &\subseteq T^{-1}(\{0\}) + R_1^{-1}. \end{aligned}$$

The last inclusion above follows from part (a) of Lemma 1.2.0.4.

To prove part (b), suppose that

$$R_1^{-1} + T^{-1}(\{0\}) \subseteq \mathrm{CL}(R_1^{-1}) \cap T^{-1}(R_2^{-1}).$$

Then

$$\begin{aligned} 1 + T^{-1}(\{0\}) &\subseteq R_1^{-1} + T^{-1}(\{0\}) \\ &\subseteq \mathrm{CL}(R_1^{-1}) \cap T^{-1}(R_2^{-1}) \qquad \text{(by assumption)} \\ &\subseteq \mathrm{CL}(R_1^{-1}), \end{aligned}$$

hence $T^{-1}(\{0\})$ is weakly Riesz, as required.

Conversely, suppose that $T^{-1}(\{0\})$ is weakly Riesz. Then

$$\begin{aligned} R_1^{-1} + T^{-1}(\{0\}) &= R_1^{-1}(1 + T^{-1}(\{0\})) && \text{from lemma 7.2.0.4} \\ &\subseteq R_1^{-1}\,\mathrm{CL}(R_1^{-1}) && \text{by assumption} \\ &= \mathrm{CL}(R_1^{-1}) && \text{from Proposition 2.2.2.2.} \end{aligned}$$

From (6.1) we have that $R_1^{-1} + T^{-1}(\{0\}) \subseteq T^{-1}(R_2^{-1})$. Combined with the above containment, $R_1^{-1} + T^{-1}(\{0\}) \subseteq \mathrm{CL}(R_1^{-1})$ gives $R_1^{-1} + T^{-1}(\{0\}) \subseteq \mathrm{CL}(R_1^{-1}) \cap T^{-1}(R_2^{-1})$ □

✠

Chapter 7

Closure operations and Bass stable rank

7.1 Introduction

In this chapter we look at two additional operators. The first of these is just a simple extension of the spectral closure operation to n tuples. The second is a different idea proposed and studied by a different team of authors - Ara, Pedersen and Perera in [1, 2]. Both of these operations appear to be useful in the study of the concepts of Bass stable rank of a ring.

7.2 The spectral closure in n dimensions

We extend the notion of spectral closure to tuples as follows.

Definition 7.2.0.1 ([9], p. 275) Let R be a ring and $n \in \mathbb{N}$. Let $a, a' \in R^n$. We define $a' * a$ as

$$a' * a = \sum_{j=1}^{n} a'_j a_j \in R.$$

We use this dot product type operation to define the spectral closure of a subset of R^n.

Definition 7.2.0.2 ([9], p. 275) Let R be a ring and $n \in \mathbb{N}$. Let $K \subseteq R^n$. We define:

$$\mathrm{CL}^{(n)}_{\mathrm{left}}(K) = \{x \in R^n : \forall J \subseteq R^n \ \exists\, x' \in K, 1 - J * (x - x') \subseteq A^{-1}\},$$

and similarly

$$\mathrm{CL}^{(n)}_{\mathrm{right}}(K) = \{x \in R^n : \forall J \subseteq R^n \ \exists\, x' \in K, 1 - (x - x') * J \subseteq A^{-1}\}.$$

We also define left and right invertible tuples as follows.

Definition 7.2.0.3 ([9], p. 275) Let R be a ring, $n \in \mathbb{N}$. We define the set of *left invertible n-tuples* as

$$R^{-n}_{\mathrm{left}} = \{a \in R^n : 1 \in R^n * a = \sum_{j=1}^{n} Ra_j,$$

and the set of *right invertible n-tuples* as

$$R^{-n}_{\mathrm{right}} = \{a \in R^n : 1 \in a * R^n = \sum_{j=1}^{n} a_j R.$$

Definition 7.2.0.4 ([9], p. 276) Let R be a ring and $n \in \mathbb{N}$. We say that R has *left (Bass) stable rank* $\leqslant n$ provided that the following condition holds:

If $(a, b) \in R^n \times R$ and $(a, b) \in R^{-n-1}_{\mathrm{left}}$ then there exists $c \in R^n$ with the property that $a - cb \in R^{-n}_{\mathrm{left}}$.

Remark 7.2.0.5 The operations $\mathrm{CL}^n_{\mathrm{right}}$ and $\mathrm{CL}^n_{\mathrm{left}}$ are Kuratowski closure operations - see [9], p. 275. We do not include the proof that they are, because the arguments are largely a repeat of those in Theorem 2.3.0.1. □

Using the closure operation defined in Definition 7.2, Harte is able to construct a sufficient condition for a ring to have Bass stable rank $\leqslant n$. We state the result without proof.

Theorem 7.2.0.6 ([9], p. 276) Let R be a ring and suppose that

$$R^n \subseteq \mathrm{CL}^{(n)}_{\mathrm{right}}(R^{-1}_{\mathrm{right}}).$$

Then R has stable rank $\leqslant n$. □

7.3 A second operation

In [1, 2] the authors construct and analyse a different closure operator which is of interest to us. Below, we first define the operation and discuss and prove some of its interesting properties. Then we look at how it is connected to the main closure operator we have been discussing so far.

Definition 7.3.0.1 Let R be a unital ring, and let $K \subseteq R, a \in R$.
Then $a \in \mathrm{Cl}_1(K)$ if and only if the following condition holds:

If $xa + b = 1$ for some $x, b \in R$ then there exists $y \in R$ such that $a + yb \in K$.

As in the case of the spectral closure, we have an alternative definition, which we give next.

Definition 7.3.0.2 Let R be a unital ring, and let $K \subseteq R, a \in R$.
Then $a \in \mathrm{Cl}_2(K)$ if and only if the following condition holds:

$$Ra + Rb = R \implies (a + Rb) \cap K \neq \emptyset.$$

Next we show that the two definitions above are in fact equivalent.

Lemma 7.3.0.3 Let R be a ring and $K \subseteq R$. Then $\mathrm{Cl}_1(K) = \mathrm{Cl}_2(K)$.

Proof: We show that $\mathrm{Cl}_1(K) \subseteq \mathrm{Cl}_2(K)$ and $\mathrm{Cl}_2(K) \subseteq \mathrm{Cl}_1(K)$.

To see the first inclusion, suppose that $a \in \mathrm{Cl}_1(K)$, and that $b \in R$ is such that

$$Ra + Rb = R. \tag{7.1}$$

We show that $(a + Rb) \cap K \neq \emptyset$. From (7.1) we have that there exist $r_1, r_2 \in R$ such that $r_1 a + r_2 b = 1$. By assumption there exists $y \in R$ such that $a + yr_2 b \in K$. Hence $(a + Rb) \cap K \neq \emptyset$. Hence $a \in \mathrm{Cl}_2(K)$, which shows that $\mathrm{Cl}_1(K) \subseteq \mathrm{Cl}_2(K)$.

To see the second containment, let $a \in \mathrm{Cl}_2(K)$. Also, suppose that there exists $x, b \in R$ with the property that

$$xa + b = 1. \tag{7.2}$$

We will show that $Ra + Rb = R$. Since R is a ring we always have $Ra + Rb \subseteq R$. Let $r \in R$. Then from (7.2) we have that $rxa + rb = r$, hence $r \in Ra + Rb$. This means that $R \subseteq Ra + Rb$. By assumption this means that $(a + Rb) \cap K \neq \emptyset$. So there exists $y \in R$ such that $a + yb \in A$, as required. This means that $\mathrm{Cl}_2(K) \subseteq \mathrm{Cl}_1(K)$ and the result follows. □

Remark 7.3.0.4 Lemma 7.3.0.3 means we can now refer to the closure of a set with a single symbol. For $K \subseteq R$, R a ring, we will simply write $\mathrm{Cl}(K)$. In what follows we will alternate between the two definitions. $\square$

The operation $\mathrm{Cl}(\cdot)$ has some interesting properties, which we list and prove in the following proposition.

Proposition 7.3.0.5 Let R be a ring, $A, B \subseteq R$. Then

(a) $\mathrm{Cl}(\emptyset) = \emptyset$,

(b) $A \subseteq B \implies \mathrm{Cl}(A) \subseteq \mathrm{Cl}(B)$,

(c) $A \subseteq \mathrm{Cl}(A)$,

(d) $\mathrm{Cl}(R) = R$,

(e) $\mathrm{Cl}(A) = \mathrm{Cl}(\mathrm{Cl}(A))$,

(f) If $A \neq \emptyset$ then $\mathrm{Rad}\, R \subseteq \mathrm{Cl}(A)$,

(g) $\mathrm{Cl}(\{0\}) = \mathrm{Rad}\, R$.

Proof: Property (a) is trivial to see.

To see that (b) holds, suppose $A \subseteq B$ and that $a \in \mathrm{Cl}(A)$. To see that $a \in \mathrm{Cl}(B)$, suppose that $b \in R$ is such that $Ra + Rb = R$. Since $a \in \mathrm{Cl}(A)$ we have $(a + Rb) \cap A \neq \emptyset$. Since $A \subseteq B$, we also have $(a + Rb) \cap B \neq \emptyset$. Hence $a \in \mathrm{Cl}(B)$, and so $\mathrm{Cl}(A) \subseteq \mathrm{Cl}(B)$.

To see that (c) holds, suppose that $A \subseteq R$, and that $a \in A$. Suppose that $b \in R$ is such that $Ra + Rb = R$. Notice that $a = a + 0b \in (a + Rb) \cap A$. Hence $a \in \mathrm{Cl}(A)$. Hence $A \subseteq \mathrm{Cl}(A)$.

From (c) we have that $R \subset \mathrm{Cl}(R)$. By definition of $\mathrm{Cl}(\cdot)$ we have that $\mathrm{Cl}(R) \subseteq R$. Hence (d) holds.

To see that (e) holds, we first notice that from (b) and (c) we have that $\mathrm{Cl}(A) \subseteq \mathrm{Cl}(\mathrm{Cl}(A))$. To see the reverse inclusion, let $a \in \mathrm{Cl}(\mathrm{Cl}(A))$, and suppose that $xa + b = 1$ for some $x, b \in R$. By definition of $\mathrm{Cl}(\cdot)$ there exists $y \in R$ with the property that $a + yb \in \mathrm{Cl}(A)$. Next we have

$$\begin{aligned} x(a + yb) + (1 - xy)b &= xa + xyb + b - xyb \\ &= xa + b = 1 \end{aligned}$$

Hence $a + yb + z(1 - xy)b \in R$ for some $z \in R$. Hence $a + (y + z - zxy)b \in A$. This means that $a \in \mathrm{Cl}(A)$.

To see that (f) holds, suppose that $A \neq \emptyset$, and let $z \in \operatorname{Rad} R$. We show that $z \in \mathrm{Cl}(A)$. Suppose that $xz + b = 1$ for some $x, b \in R$. Then $xz = 1 - b \in \operatorname{Rad} R$. Hence $b \in R^{-1}$. Then $(a - z)b^{-1} \in R$. Let $a \in R$. Then

$$a = z + (a - z)b^{-1}b \in z + Rb.$$

Hence $z \in \mathrm{Cl}(A)$. Hence $\operatorname{Rad} R \subseteq \mathrm{Cl}(A)$ and (f) holds.

Finally, to see that (g) holds, notice first that from (f) we have that $\operatorname{Rad} R \subset \mathrm{Cl}(\{0\})$. To see that $\mathrm{Cl}(\{0\}) \subseteq \operatorname{Rad} R$, suppose that $a \notin \operatorname{Rad} R$. Then there exists a maximal ideal L such that $a \notin L$. Then $Ra + L$ is also a left ideal and $L \subseteq Ra + L$. Since L is maximal, we have that $Ra + L = R$. So there exists $x \in R$, $l \in L$ such that $xa + l = 1$. Suppose that $a \in \mathrm{Cl}(\{0\})$. Then $a + yl = 0$ for some $y \in R$, so that $a = -yl$. But this would mean that $a \in L$, contradicting our initial assumption. Hence we have proved that $a \notin \operatorname{Rad} R \implies a \notin \mathrm{Cl}(\{0\})$. This means that $\mathrm{Cl}(\{0\}) \subseteq \operatorname{Rad} R$. The result follows. $\square$

Remark 7.3.0.6 Let R be a ring, $E, F \subseteq R$. For non-commutative rings the equation $\mathrm{Cl}(E \cup F) = \mathrm{Cl}(E) \cup \mathrm{Cl}(F)$ is not true in general (see Example 1.10 in [2]), so that $\mathrm{Cl}(\cdot)$ is not a closure operation in the sense of Kuratowski, even though the conditions $(a), (c)$ and (e) strongly suggest that. $\square$

✠

Chapter 8

Conclusion

8.1 Introduction

In this final chapter we summarize what was discussed and briefly indicate some questions raised by the study.

8.2 Overview

Suppose A is a Banach algebra, and $a \in A$. A cornerstone of the theory of Banach algebras is the fact that

$$\|a\| < 1 \implies 1 - a \in A^{-1}.$$

This is a deep result in Banach algebra theory since it connects the algebraic and analytic foundations of the subject. A similarly deep result in spectral theory is the fact that if $a, b \in A$, then

$$1 - ab \in A^{-1} \iff 1 - ba \in A^{-1}.$$

This last fact holds in a general ring as well. Using mainly these two facts as motivation, the authors of [9], in the same article, define a set valued mapping that is to generate a topology on a ring.

In Chapter 2 we focused on proving that the set valued mapping is a Kuratowski closure operation. We discussed some basic properties of the closure operation. We also looked at how the closure of a product of sets is related to the product of the closure of the sets. These relationships were needed to prove that the closure operation is in fact a Kuratowski closure operation, hence generates a topology, called the spectral topology, on the ring. We also proved that a ring with the spectral topology is a topological ring. In this chapter we also looked at some examples of the spectral topology on different types of rings.

In Chapter 3 we compare the closure of a set in a Banach algebra relative to the spectral topology with its closure relative to the norm topology. From this we see that the spectral topology on a Banach algebra is coarser than the norm topology. In this chapter we also showed that the set of invertibles in the ring is open in the spectral topology. This fact enabled us to analyse what neighbourhoods of 0 look like in the spectral topology.

In Chapter 4 we look at how the spectral closure and its topology allows us to define a concept of quasinilpotent that applies to a general ring.

In Chapter 5 we looked at how the spectral closure intervenes in concepts of generalized invertibility.

In Chapter 6 we look at how the spectral closure intervenes with Fredholm Theory relative to a Banach algebra homomorphism.

Finally, in Chapter 7, we look at how the spectral closure intervenes in the concepts of Bass Stable Rank of a ring. In this chapter we discuss a variant of the main closure operator, specifically defined on n tuples of elements from a ring. We also discussed an alternative operation studied by a different team of researchers.

8.3 Open Quesions

We briefly list some questions that are either described in [9] as being unknown, or seem to us to be unknown facts, and could potentially lead to research questions or simply a deeper understanding of the spectral topology.

In Theorem 3.3.0.4 we proved that if R is a ring and J is a finite subset of R then the set

$$U_J = \{a \in R : 1 - Ja \in R^{-1}\}$$

is a neighbourhood of 0. We notice that each such set is a superset of $\operatorname{Rad} R$, ie $\operatorname{Rad} R \subset U_J$ for each J a finite subset of R. This raises the question as to whether there are rings R for which $\operatorname{Rad} R$ is an open set in the spectral topology. If such rings do exist, what would be the consequences of this fact? For example, if the ring was also semisimple, then the spectral topology would necessarily be discrete.

In Theorem 2.3.0.2, we see that the spectral topology on a ring R is T_1 if and only if the ring is semisimple. From Example 2.3.0.5 we have that on $\mathbb{Z}$ the spectral topology is discrete, hence T_2. This raises the question as to whether the following implication always holds. If R is a ring, let τ be the spectral topology. Is it the case that:

$$\langle R, \tau \rangle \text{ is } T_2 \implies \langle R, \tau \rangle \text{ is discrete?}$$

In Definition 4.2.0.3, we defined a general notion of quasinilpotent, one that can be applied to any ring with the spectral topology. In [9], the authors mention that it is not known whether the following implication holds, for R a ring:

$$a, b \in \mathrm{QN}(R), ab = ba \implies a + b \in \mathrm{QN}(R).$$

As is mentioned in section 7.3 above, the operation defined by Ara, Pedersen and Perera in Definitions 7.3.0.1 and 7.3.0.2, do not generate a topology on a ring. In [9], p. 277, the authors make the following suggested modification to the concept of Ara, Pedersen and Perera:

The concept by Ara, Pedersen and Perera is essentially:

Let R be a ring and $K \subseteq R$. Then $a \in \mathrm{cl}^{\bullet}_{\text{left}}(K)$ if and only if for arbitrary $b \in R$ there is implication:

$$(a, b) \in R^{-2}_{\text{left}} \subseteq R^2 \implies (a - Rb) \cap K \neq \emptyset.$$

As mentioned in Remark 7.3.0.6, the operation does not satisfy all of the Kuratowski closure conditions. Consider the following definition of an operator:

Definition 8.3.0.1 Let R be a ring, $K \subseteq R^n$. Then $a \in \mathrm{Cl}^{\bullet}_{\mathrm{left}}(K) \subseteq R^n$ if and only if for every finite $J \subseteq R$

$$(a, J) \subseteq R^{-n-1}_{\mathrm{left}} \implies \bigcap_{b \in J} (a - R^n b) \cap K \neq \emptyset.$$

Harte et al in [9] say that Definition 8.3.0.1 makes a 'cosmetic' change to the concept in Definition 7.2.0.4 which may or may not actually alter it. The first question here is whether the change does alter the definition. Harte et al also mention that they believe that the modified operator will satisfy all the properties of a Kuratowski closure operator. The second question is whether the new operator does satisfy the conditions and of course, how to prove that it does, or does not.

✠

List of Symbols

$\emptyset$	empty set
$\wp(A)$	power set of A
$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set of rationals
$\mathbb{R}$	real number field
$\mathbb{C}$	complex number field
$\mathbb{R}^+$	positive real numbers
$[n]$, I	(finite) index set
G, R, A	group/ring/algebra
$\prod_{i \in I} R_i$	direct sum (product) of rings
R^{-1}, (R_l^{-1}/R_r^{-1})	set of (left/right) invertibles
$\mathcal{M}_l$, $\mathcal{M}_r$	set of all left/right maximal ideals
R/J	quotient ring modulo ideal J
$\operatorname{Rad} R$	Jacobson radical of R
(z)	principal ideal generated by z
$R[[z]]$	formla power series ring
K_0	set of leading coefficients of polynomials
a^{-1}, a	(invertible) element
$[a]$	equivalence class of a

$\neq$, $=$	(non-)equality relation
$\notin$, $\in$	(non-)membership relation
$<$, $>$	less/greater-than relation
$\leq$, $\geq$	less/greater-than-or-equal-to relation
$\subset$, $\subseteq$	subset relation
$\supset$, $\supseteq$	superset relation
$\sim$	equivalent/equinumerous
$\#$	cardinality
$\cap$	intersection
$\cup$	union
$\forall$	universal quantifier
$\exists$	existential quantifier
$\Longrightarrow$	(one-way) implication
$\Longleftrightarrow$	bi-implication (if and only if)
$\overset{?}{\underset{\text{¿}}{\Longrightarrow}}$	unverified implication
∞	absolutus infinitus
ϵ, δ	infinitesimally small numbers
0	additive identity
1	multiplicative identity
λ, α, β	scalars / eigenvalues

$G + H$	sum set	$\mapsto$	function definition
$G \cdot H$	product set	$f^{-1}(\cdot),\ f(\cdot)$	(inverse) function/functional
K^{-1}	inverses of elements in K	$f(\cdot,\cdot)$	multivariable (two-argument) function
τ, σ	topology	$\circ$	infix function/operator composition
$\tau^{\times}$	product topology	$+$	infix binary addition operation
$\langle X, \tau\rangle$	topological space	$\cdot$	infix binary multiplication operation
$\langle X, \tau^{\times}\rangle$	product (topological) space	$*$	general infix binary operation
$\langle X, d\rangle$	metric space	$\alpha(\cdot,\cdot)$	prefix binary addition operation
$\langle X, \\|\cdot\\|\rangle$	metric space	$\mu(\cdot,\cdot)$	prefix binay multiplication operation
$X \setminus A$	complement of a set	$\beta(\cdot,\cdot)$	general prefix binary relation
$\mathcal{N}_x$	neighbourhood system at x	$p_j(\cdot)$	j^{th} projection map
$\mathscr{D}$, $\mathscr{B}$	(sub)base for a given topology	$I(\cdot)$	identity map
$B(x_0, \epsilon)$	ϵ-neighbourhood about x_0	$\mathbf{0}(\cdot)$	zero map
$\overline{A}$	(familiar) closure of a set	$\Gamma(\cdot)$	gamma function
∂A	boundary of a set	$d(\cdot,\cdot)$	metric
$\sigma(a)$	spectrum of Banach algebra element a	$\\|\cdot\\|$	norm
$\rho(a)$	Gelfand's spectral radius	$\operatorname{dist}(\cdot, K)$	distance from a point to a set K
T	linear operator	$\operatorname{int}(\cdot)$	interior operation
$T\|_A$	restriction of a linear operator	$\operatorname{der}(\cdot)$	derived set operation
$\tilde{T}$	extension of a linear operator	$\operatorname{cl}(\cdot)$	topological closure operation
$T^{\times}$	adjoint of a linear operator	$\operatorname{cl}_{\tau}(\cdot)$	closure operation w.r.t. τ
$\mathscr{D}(T)$, $\mathscr{R}(T)$	domain/range of a linear operator	$\operatorname{cl}_X(\cdot)$	closure operation w.r.t. X
$\mathcal{G}(T)$	graph of a linear operator	$\operatorname{cl}_{\text{alg}}(\cdot)$	algebraic closure operation
$\frac{\partial}{\partial x}$, $\frac{d}{dx}$	(partial) derivative operator w.r.t. x	$\operatorname{cl}_{\\|\cdot\\|}(\cdot)$	norm closure operation
∇	nabla/del operator	$\operatorname{CL}(\cdot)$	spectral closure operation
$\int$	(Lebesgue) integral operator	$\operatorname{CL}_{\tau}(\cdot)$	spectral closure operation w.r.t. τ
V, V^{α}	Voltera/Riemann-Liouville operator	$\operatorname{CL}_A(\cdot)$	spectral closure operation w.r.t. A

$\mathrm{CL}(A^{-1})$	set of nearly invertibles in A	$\mathrm{bsr}(R)$	bass stable rank of R
$\mathrm{QN}_{\|\cdot\|}(A)$	norm quasinilpotent elements of A	(x_n)	sequence
$\mathrm{QN}(A)$	ring quasinilpotent elements of A	$\lim\limits_{n\to\infty}$, $\xrightarrow[\infty]{n}$	limit (shorthand)
		inf, sup	infimum/supremum of a set
		max	maximum value

$\dim(V)$	dimension of a vector space
$T(\cdot)$, $\phi(\cdot)$	ring/vectorspace homo/epi/isomorphism
$T^{-1}(0)$, $\ker(\phi)$	kernel/null space of a vector space homomorphism
$\mathrm{coker}(\phi)$	cokernel of a vector space homomorphism
$\alpha(T)$, $\alpha(\phi)$	kernel index of a vector space homomorphism
$\beta(T)$, $\beta(\phi)$	deficiency index of a vector space homomorphism
$\iota(T)$, $\iota(\phi)$	index of a vector space homomorphism
$\mathcal{BL}(X,Y)$	set of bounded linear operators from X to Y
$\Phi(X,Y)$	set of Fredholm elements from X to Y
$\Omega(X,Y)$	set of Weyl operators from X to Y
$R^{\bullet}$	set of indempotent elements in R
$R^{\cap}$	regular (relatively Fredholm) elements in R
$R^{\cup}$	decomposably regular (relatively Weyl) elements in R

Bibliography

[1] P. Ara, G. Pedersen, F. Perera *An infinite analogue of rings with stable rank one*, J. Algebra 230 (2000), 608 - 655.

[2] P. Ara, G. Pedersen, F. Perera *A closure operation in rings*, Int. J. Mat. 12 (2001), 791 - 812.

[3] B. Aupetit, *A Primer on spectral theory*, Springer Science & Business Media, 2012.

[4] B. Barnes, G. Murphy, M. Smyth and T. West, *Riesz and Fredholm theory in Banach algebras*, Pitman Publishing Company, 1982.

[5] H. Bass, *K - Theory and stable algebra*, Publ. IHES 22 (1964), 5 - 60.

[6] A.-L. Cauchy, *Sur l'équation à l'aide de laquelle on détermine les inégalités séculaires des mouvements des planétes*, Exercises de mathématiques 4, in Œuvres complètes d'Augustin Cauchy, Paris: Gauthier-Villars et fils, 2 No. 9, 174-95

[7] D. Cvetkovic-Ilić, and R. E. Harte, *On Jacobson's Lemma and Drazin Invertibility*, Applied mathematics letters 23 (4) (2010), 417-420.

[8] D. Cvetkovic-Ilić, and R. E. Harte, *On the Algebraic Closure in Rings*, Proceedings of the American Mathematical Society, 135 (11) (2007), 3547-3552.

[9] D. Cvetkovic-Ilić, and R. E. Harte, *The Spectral Topology in Rings*, Studia Mathematica 200 (3) (2010), 267-278.

[10] H. A. M. Dzinotyiweyi, *The analogue of the group algebra for topological semigroups*, pitman (1984).

[11] D. S. Djordjević, R. E. Harte, C. M. Stack, *On left-right consistency in Rings*, Math. Proc. Roy. Irish Acad. 106A (2006), 11 - 17.

[12] K. J. Engel & R. Nagel, *A short course on operator semigroups*, Springer 256s (2006).

[13] J. B. Fraleigh, *A First Course in Abstract Algebra*, Addison-Wesley Publishing Company Inc. 1989.

[14] J. A. Gallian, *Contemporary Abstract Algebra*, 6th Edition, Houghton Mifflin Company, 2006.

[15] L. J. Goldstein, *Abstract Algebra - a first course*, Prentice-Hall, New Jersey, 1973.

[16] S. Goldberg, *Unbounded Linear Operators - Theory and Applications*, McGraw-Hill Book Company. 1966.

[17] R. E. Harte, *Fredholm theory relative to a Banach algebra homomorphism*, Mathematische Zeitschrift 179, (1982), 431 - 436.

[18] R. E. Harte, *Regular Boundary Elements*, Proceedings of the American Mathematical Society 99 (2) (1987), 328 - 330.

[19] R. E. Harte, *Arens-Royden and the Spectral Landscape*, Filomat (Niš) 16 (2002), 31-42.

[20] R. E. Harte, *Invertibility and Singularity for Bounded Linear Operators*, Courier Dover Publications, 2016.

[21] M. Henriksen, R. G. Woods, *Separate versus Joint Continuity*, Topology and its Applications, 97 (1999), 175 - 205.

[22] D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Nachrichten von d. Königl. Ges. d. Wissensch. zu Göttingen (Math.-physik. Kl.), (1904), 49-91.

[23] P. W. Jones, D. Marshall and T. Wolff, *Stable rank of the disc algebra*, Proceedings of the American Mathematical Society, 96 (4) (1986), 603-604.

[24] E. Kreyszig, *Introductory Functional Analysis with Applications*, Vol. 1. Wiley & Sons. Inc. New York, 1978.

[25] K. Kuratowski, *Sur l'opération $\bar{A}$ de l'Analysis Situs*, Fundamenta Mathematicae (3) (1922), 182–199.

[26] T. Y. Lam, *A First Course in Noncummutative Rings - Second Edition*, Springer, New York, 2001.

[27] S. Lipschutz, *Theory and Problems of General Topology*, Schaum's Outline Series (1965).

[28] V. Müller, *Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras*, Operator Theory Advances and Applications, Vol. 139, Birkhäuser, (2003).

[29] I. Namioka, *Separate continuity and joint continuity*, Pacific Journal of Mathematics, Vol. 51, No. 2, 1974.

[30] Z. Piotrowski, *The Genesis of Separate versus Joint Continuity*, Tatra Mountains Mathematical Publications, (8) (1996), 113-126.

[31] W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1973.

[32] J. Rotman, *Advanced Modern Algebra*, Prentice Halls, (2003).

[33] R. Y. Sharpe, *Steps in Commutative Algebra*, Cambridge University Press, 1990.

[34] A. E. Taylor and D. C. Lay, *Introduction to functional analysis*, Robert, E. Krieger Publishing Company Inc. 1980.

[35] S. Warner, *Topological Rings*, Elsevier Science Publishers B.V. Amsterdam, 1993.

[36] S. Willard, *General Topology*, Addison-Wesley Publishing Company Inc, 1970.

[37] Y. Xue, *A note about a theorem of R. Harte*, Filomat, 22 (2) (2008), 95-98.