Greedy Algorithms: Main Ideas

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Higher School of Economics

Algorithmic Toolbox
Data Structures and Algorithms

Outline

Learning objectives

- Come up with a greedy algorithm yourself

Largest Number

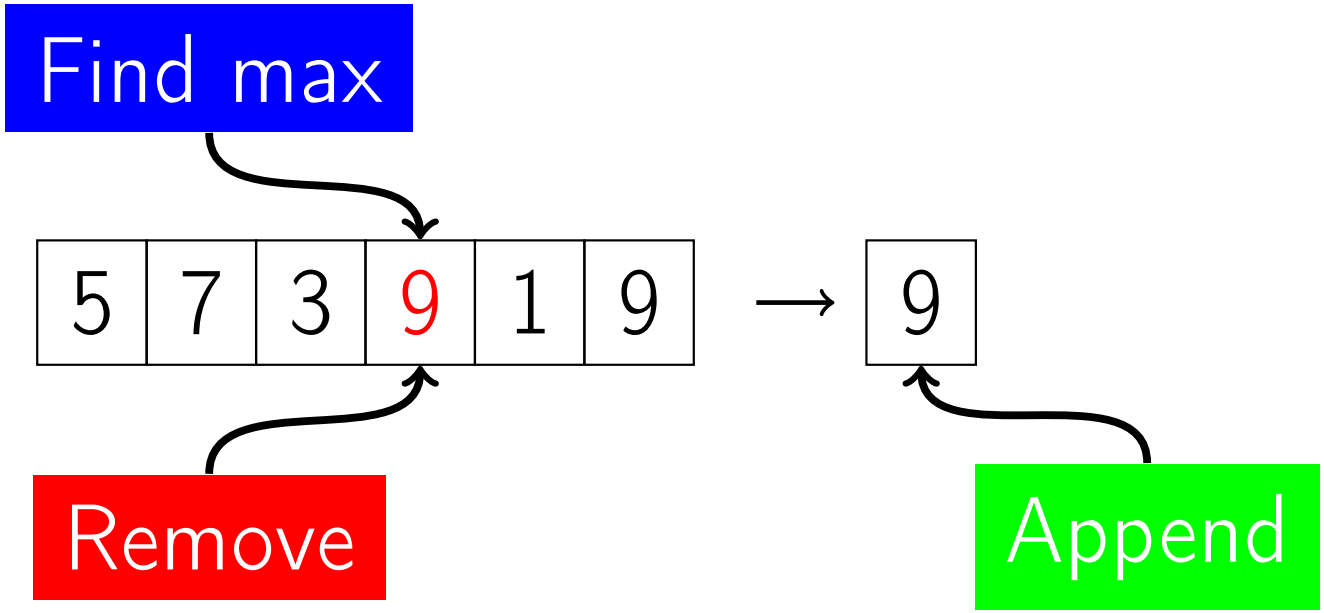
Toy problem

What is the largest number that consists of digits 3, 9, 5, 9, 7, 1? Use all the digits.

Examples

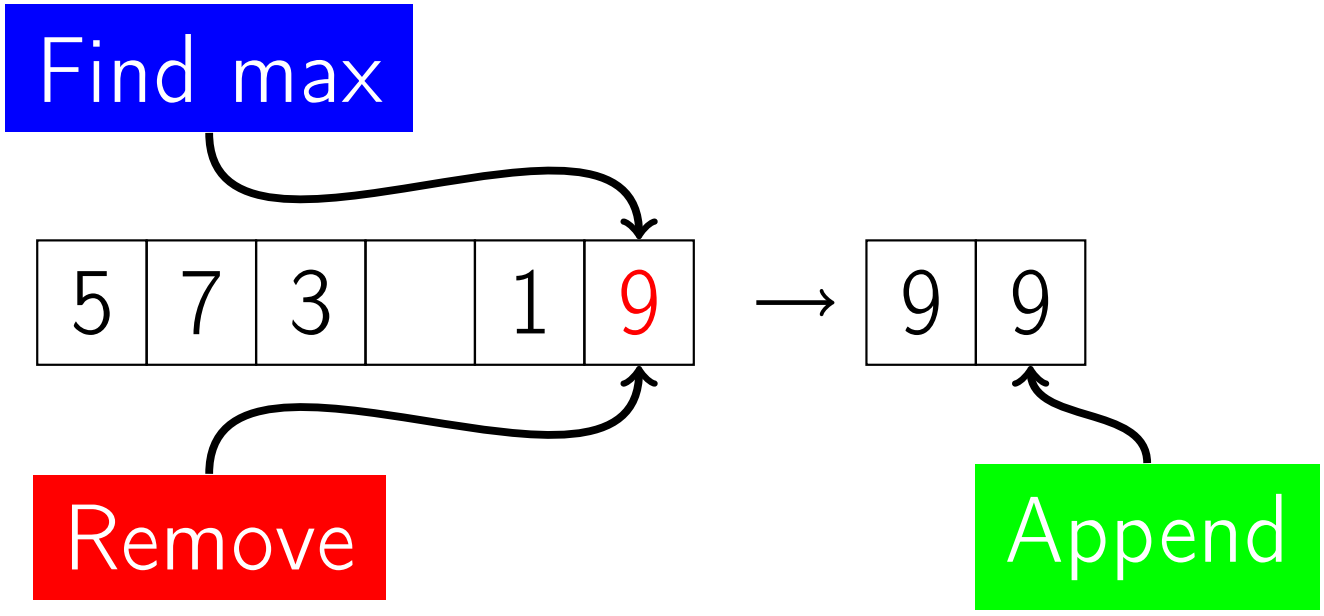
$$359179, 537991, 913579, \ldots$$

Greedy Strategy



- Find max digit
- Append it to the number
- Remove it from the list of digits

Greedy Strategy



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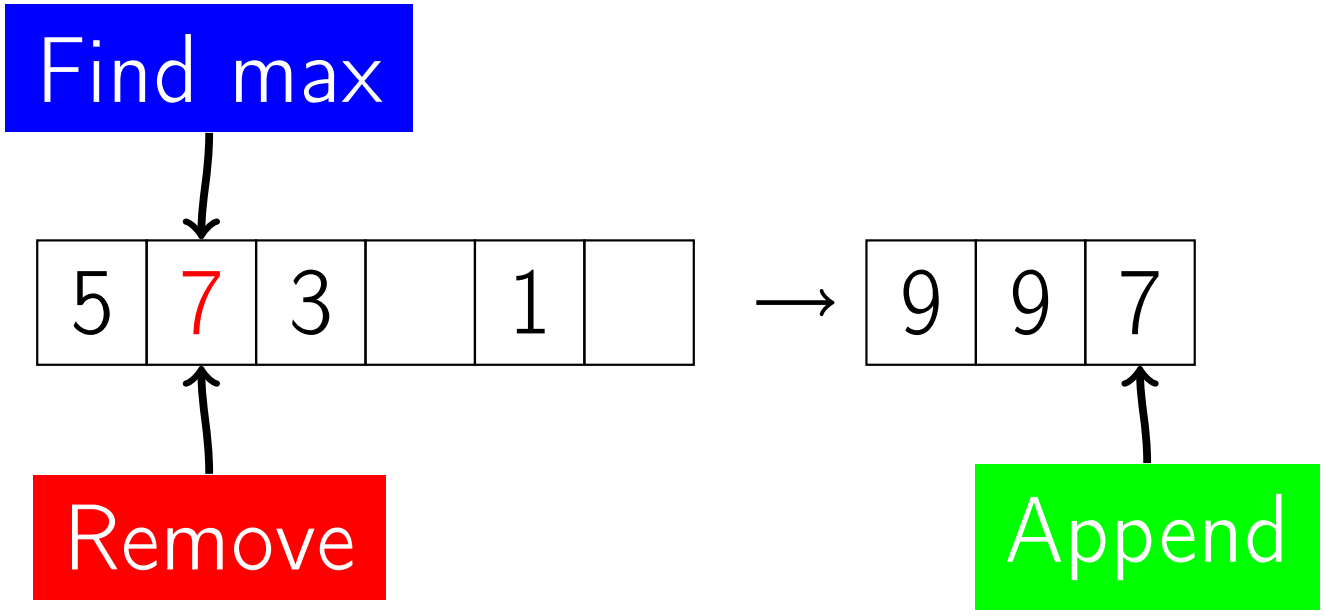

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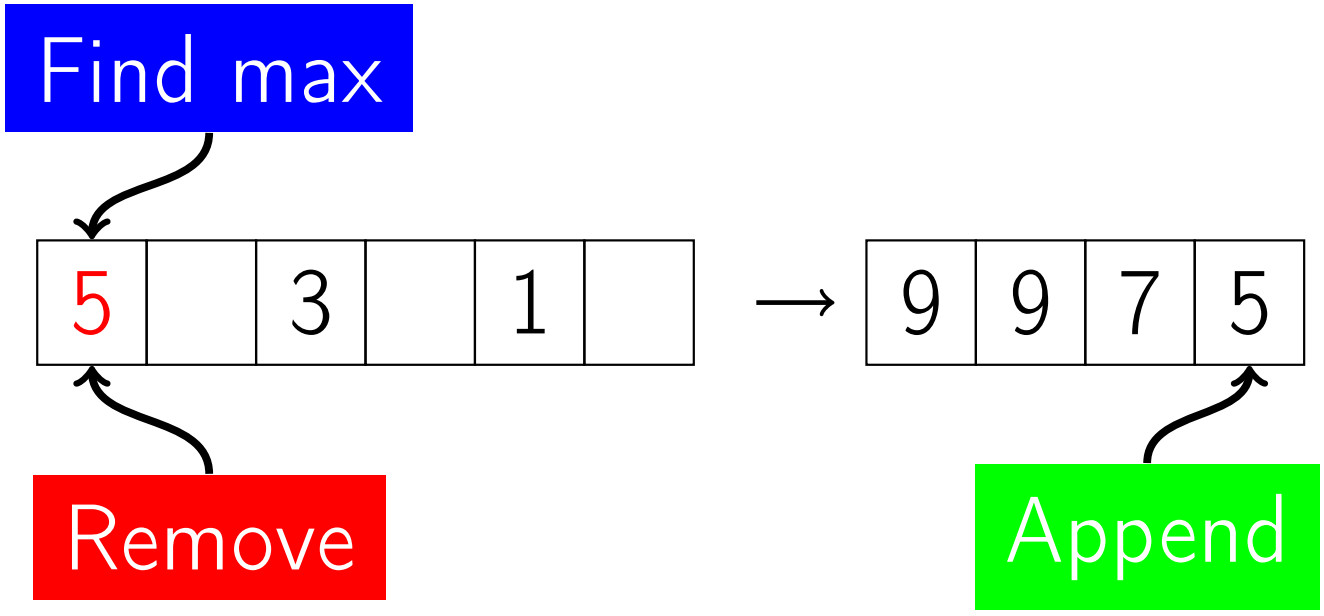

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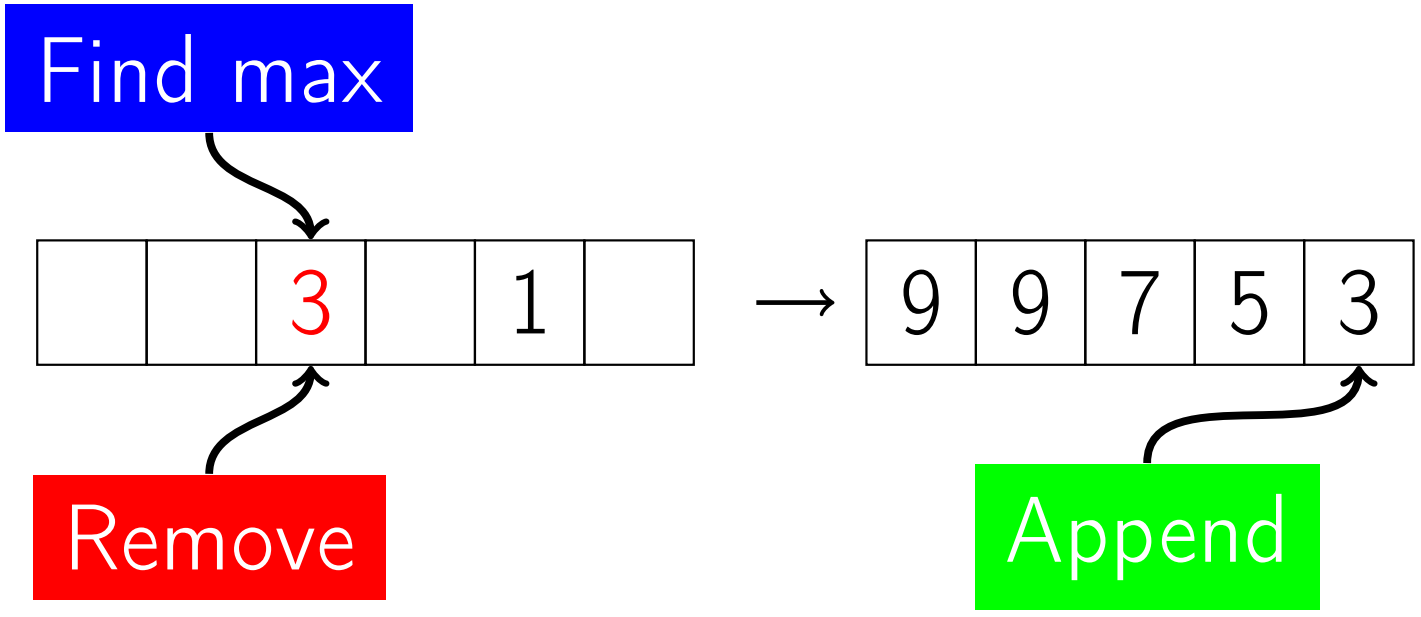

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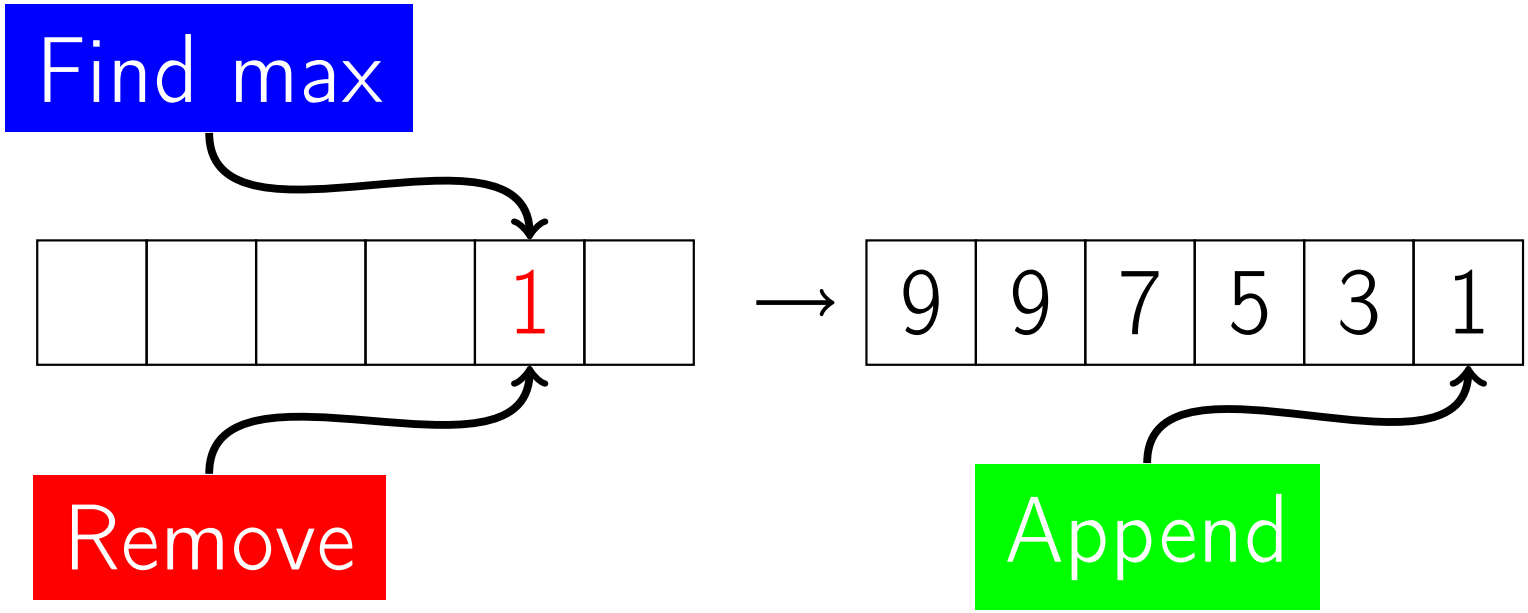

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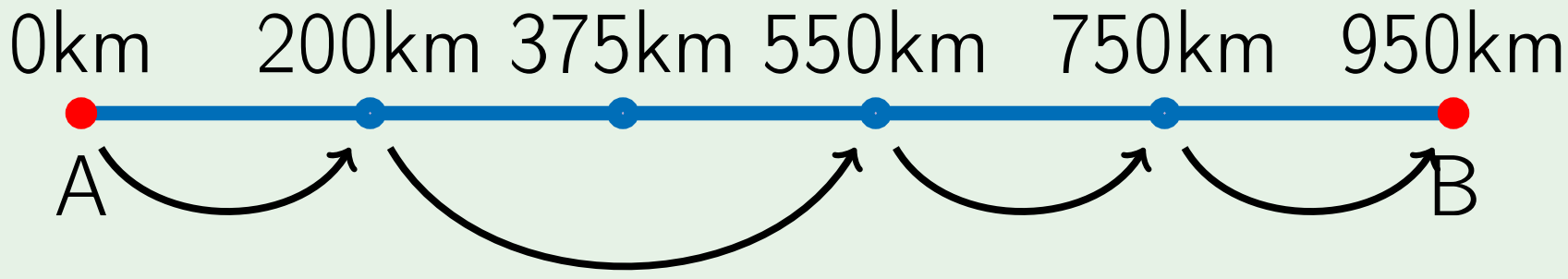
Outline

Car Fueling

Distance with full tank = 400km



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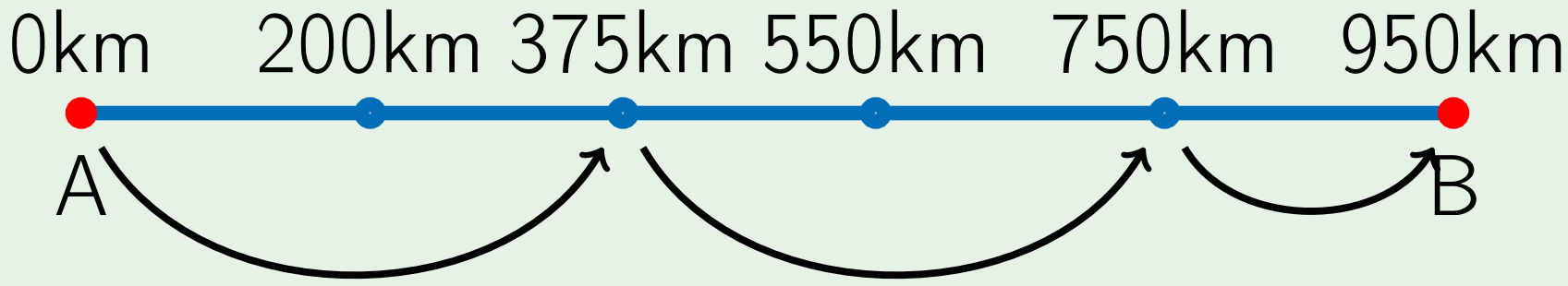

Car Fueling

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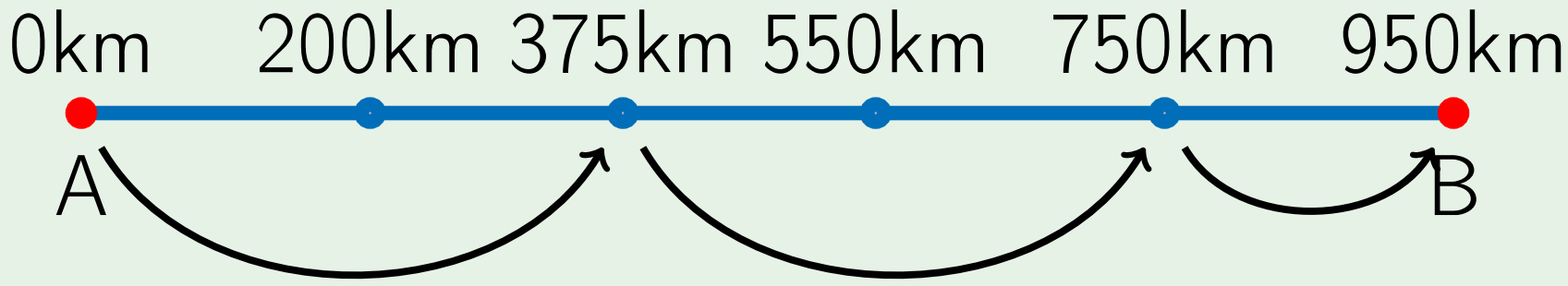


Minimum number of refills = 2

Car Fueling

Input: A car which can travel at most L kilometers with full tank, a source point A, a destination point B and n gas stations at distances $x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n$ in kilometers from A along the path from A to B.

Output: The minimum number of refills to get from A to B, besides refill at A.

Greedy Strategy

- Make some greedy choice
- Reduce to a smaller problem
- Iterate

Greedy Choice

- Refill at the the closest gas station
- Refill at the farthest reachable gas station
- Go until there is no fuel

Greedy Algorithm

- Start at A
- Refill at the farthest reachable gas station G
- Make G the new A
- Get from new A to B with minimum number of refills

Definition

Subproblem is a similar problem of smaller size.

Subproblem

Examples

- $\texttt{LargestNumber}(3, 9, 5, 9, 7, 1) =$ “9” $+ \texttt{LargestNumber}(3, 5, 9, 7, 1)$
- Min number of refills from A to $B =$ first refill at G + min number of refills from G to B

Safe Move

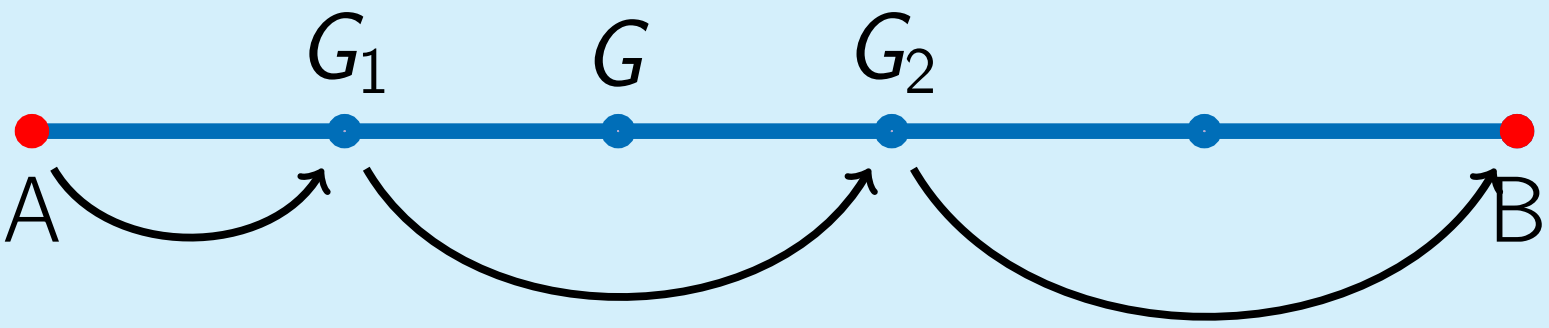
Definition

A greedy choice is called safe move if there is an optimal solution consistent with this first move.

Lemma

To refill at the farthest reachable gas station is a safe move.

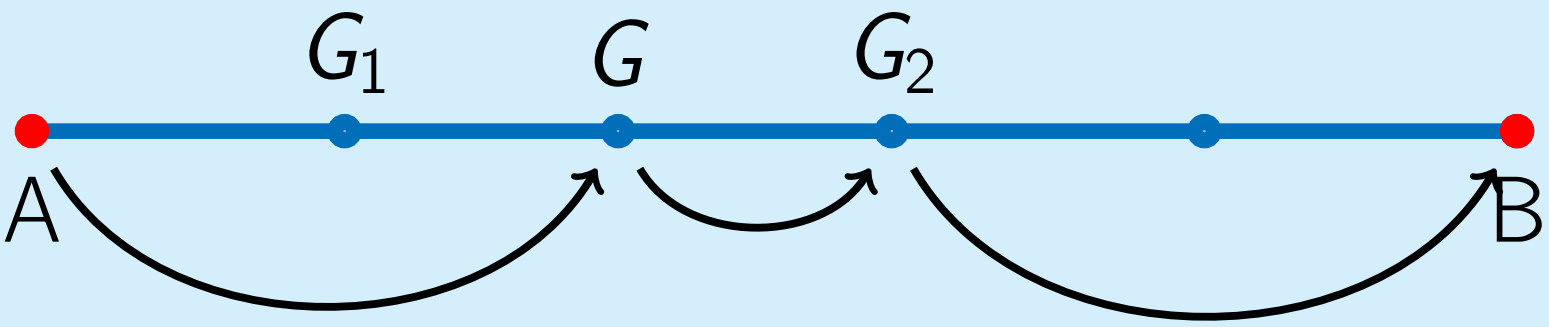
Proof



First case: G is closer than G_2

Refill at G instead of G_1

Proof



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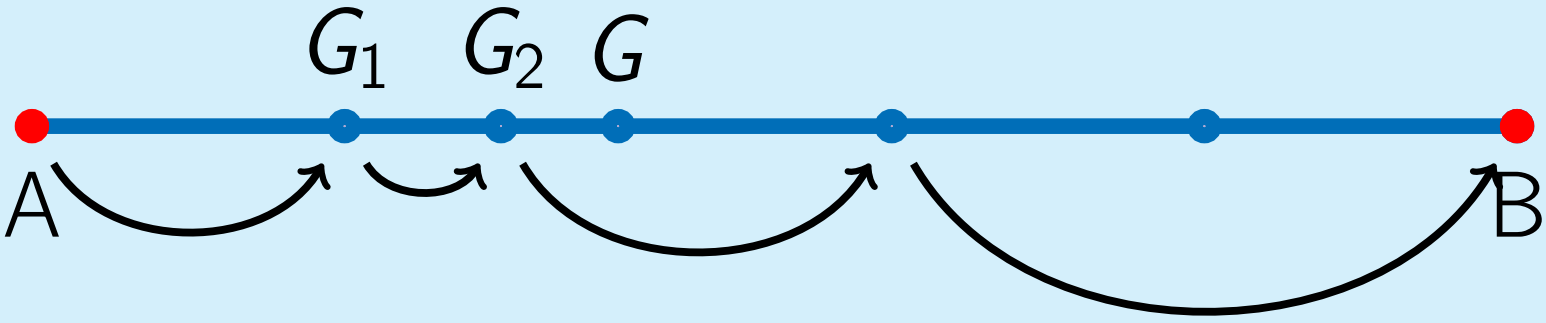
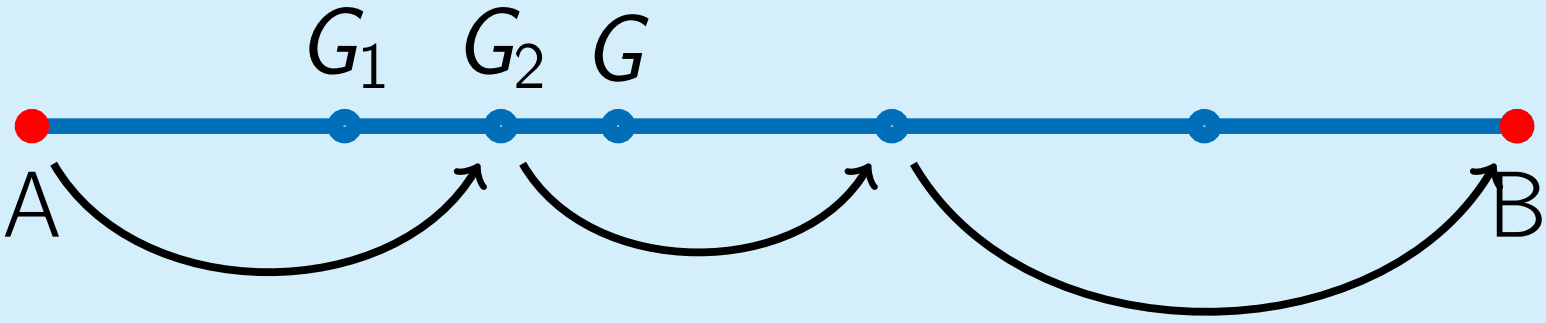


Second case: G_2 is closer than G

Avoid refill at G_1

Proof



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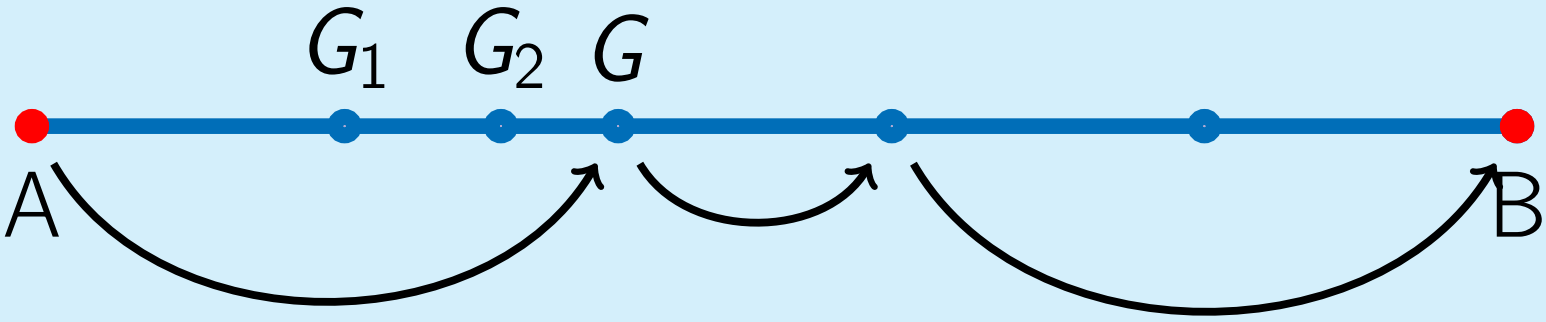


Second case: G_2 is closer than G

Avoid refill at G_1

Proof

- Route R with the minimum number of refills
- G_1 — position of first refill in R
- G_2 — next stop in R (refill or B)
- G — farthest refill reachable from A
- If G is closer than G_2, refill at G instead of G_1
- Otherwise, avoid refill at G_1 □

Outline

$A = x_0 \le x_1 \le x_2 \le \cdots \le x_n \le x_{n+1} = B$

MinRefills(x, n, L)

```
numRefills ← 0,  currentRefill ← 0
while currentRefill ≤ n:
  lastRefill ← currentRefill
  while (currentRefill ≤ n and
         x[currentRefill + 1] − x[lastRefill] ≤ L):
    currentRefill ← currentRefill + 1
  if currentRefill == lastRefill:
    return IMPOSSIBLE
  if currentRefill ≤ n:
    numRefills ← numRefills + 1
return numRefills
```

Lemma

The running time of `MinRefills`(x, n, L) is $O(n)$.

Proof

- *currentRefill* changes from 0 to $n + 1$, one-by-one
- *numRefills* changes from 0 to at most n, one-by-one
- Thus, $O(n)$ iterations □

Outline

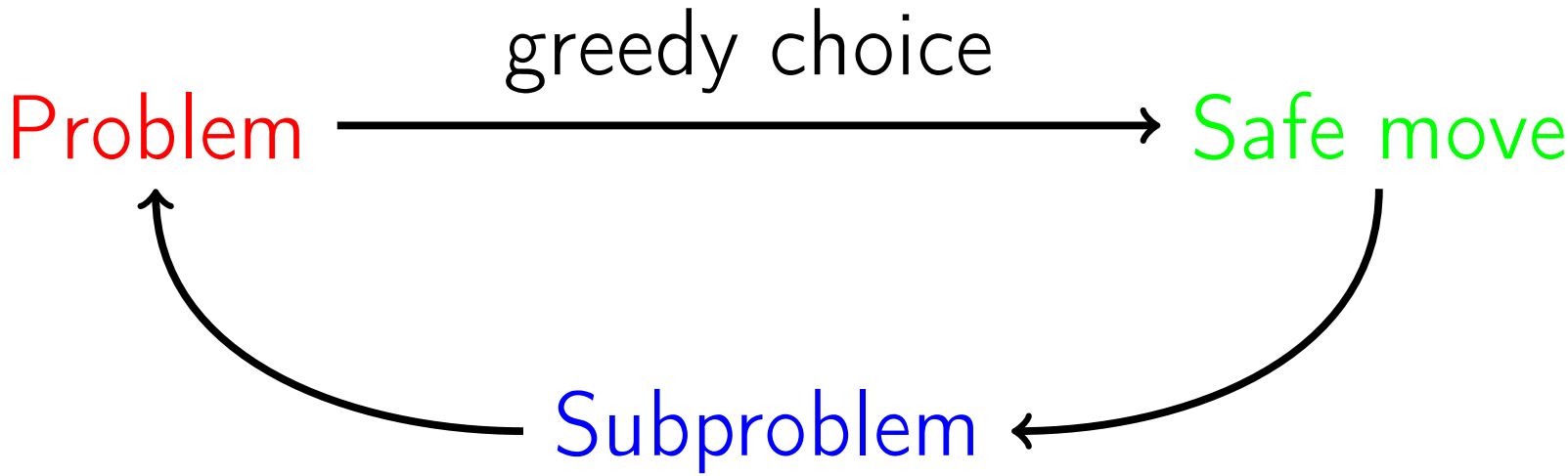
Reduction to Subproblem

- Make a first move
- Then solve a problem of the same kind
- Smaller: fewer digits, fewer fuel stations
- This is called a "subproblem"

Safe move

- A move is called safe if there is an optimal solution consistent with this first move
- Not all first moves are safe
- Often greedy moves are not safe

General Strategy



- Make a greedy choice
- Prove that it is a safe move
- Reduce to a subproblem
- Solve the subproblem