Dynamic Programming: Change Problem

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Data Structures and Algorithms
Algorithmic Toolbox

Outline

1. Greedy Change
2. Recursive Change
3. Dynamic Programming

Change problem

Find the minimum number of coins needed to make change.

Formally

Change problem

Input: An integer *money* and positive integers $coin_1, \ldots, coin_d$.

Output: The minimum number of coins with denominations $coin_1, \ldots, coin_d$ that changes *money*.

Greedy Way

```
GreedyChange(money)
Change ← empty collection of coins
while money > 0:
    coin ← largest denomination
           that does not exceed money
    add coin to Change
    money ← money − coin
return Change
```

Changing Money

in the US

40 cents $= 25 + 10 + 5$

Greedy

Changing Money

in Tanzania

40 cents $=$ $25 + 10 + 5$ $=$ $20 + 20$

Greedy is not Optimal

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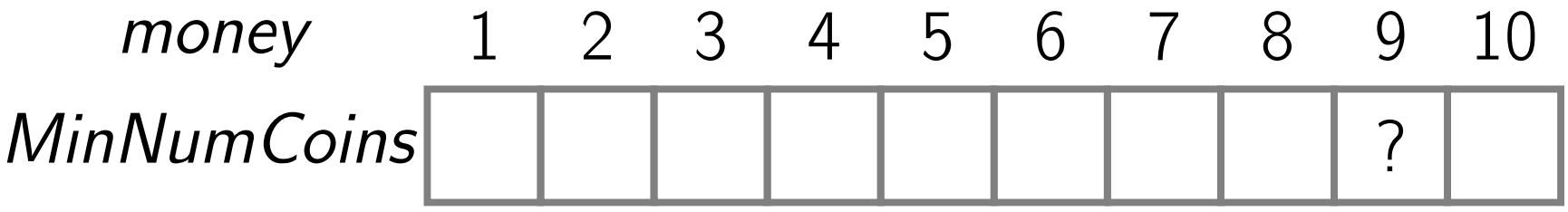
Recursive Change

Given the denominations 6, 5, and 1, what is the minimum number of coins needed to change 9 cents?

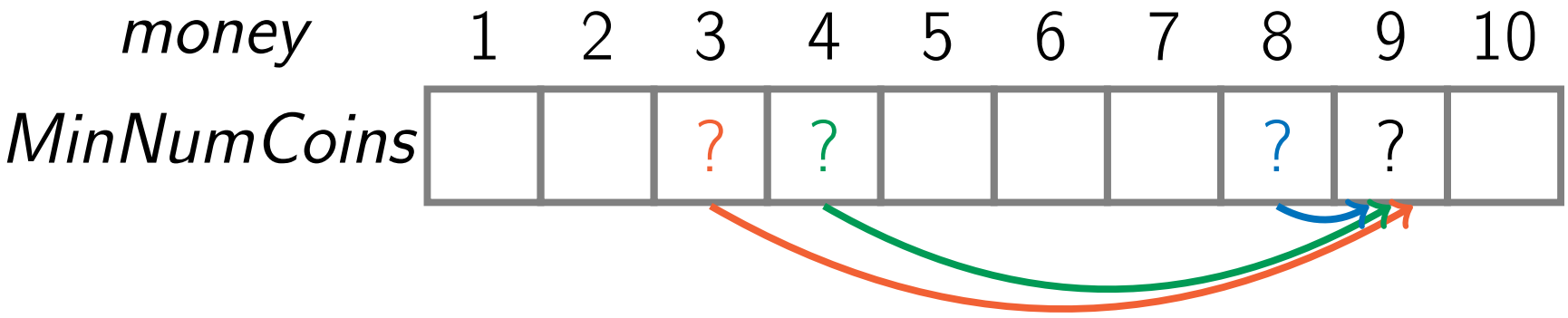
$$MinNumCoins(9) = \min \begin{cases} MinNumCoins(9-6)+1 \\ MinNumCoins(9-5)+1 \\ MinNumCoins(9-1)+1 \end{cases}$$



Recursive Change

Given the denominations 6, 5, and 1, what is the minimum number of coins needed to change 9 cents?

$$MinNumCoins(9) = \min \begin{cases} MinNumCoins(3) + 1 \\ MinNumCoins(4) + 1 \\ MinNumCoins(8) + 1 \end{cases}$$



Recurrence for Change Problem

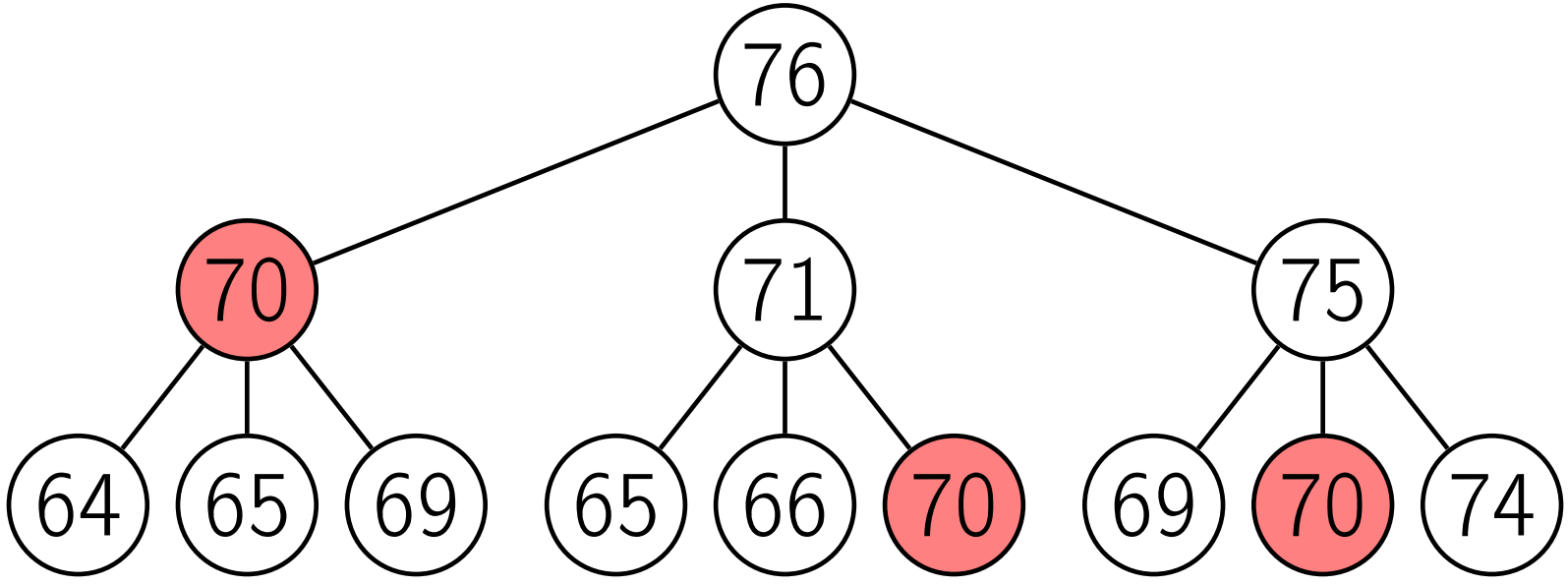
$$MinNumCoins(money) =$$

$$\min \begin{cases} MinNumCoins(money - coin_1) + 1 \\ MinNumCoins(money - coin_2) + 1 \\ \cdots \\ MinNumCoins(money - coin_d) + 1 \end{cases}$$

RecursiveChange($money$, $coins$)

```
if money = 0:
  return 0
MinNumCoins ← ∞
for i from 1 to |coins|:
  if money ≥ coin_i:
    NumCoins ← RecursiveChange(money − coin_i, coins)
    if NumCoins + 1 < MinNumCoins:
      MinNumCoins ← NumCoins + 1
return MinNumCoins
```

How Fast is `RecursiveChange`?



the optimal coin combination for 70 cents is computed at least **three** times!

the optimal coin combination for 30 cents is computed **trillions** of times!

Hint

Wouldn't it be nice to know all the answers for changing $money - coin_i$ by the time we need to compute an optimal way of changing *money*?

Instead of the time-consuming calls to

`RecursiveChange`$(money - coin_i, coins)$

we would simply look up these values!

Outline

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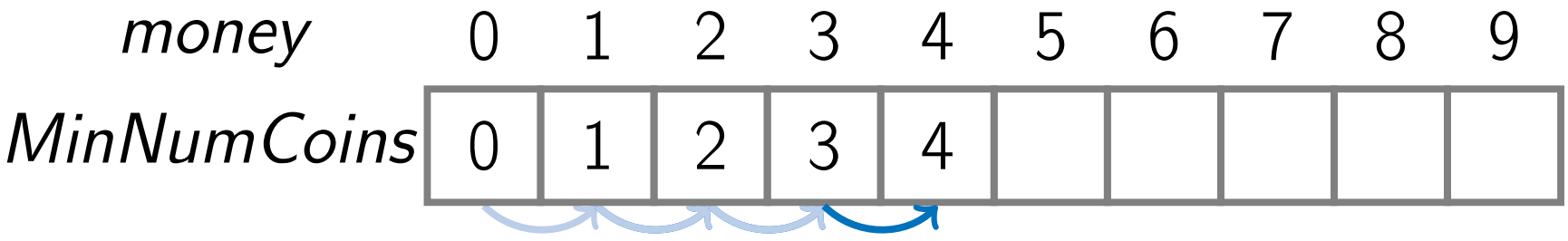
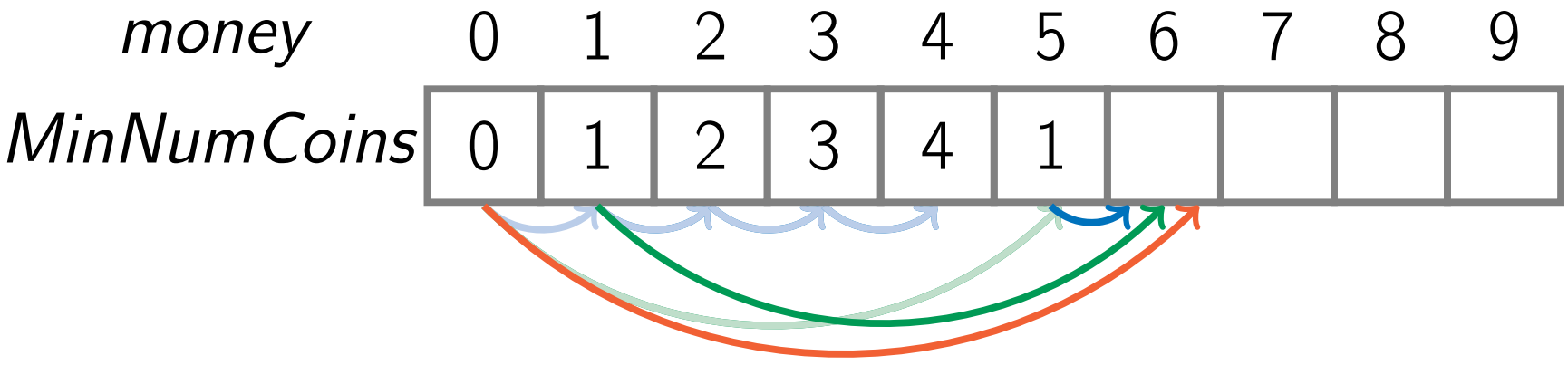
Dynamic Programming

What is the minimum number of coins needed to change 4 cents for denominations 6, 5, and 1?



Dynamic Programming

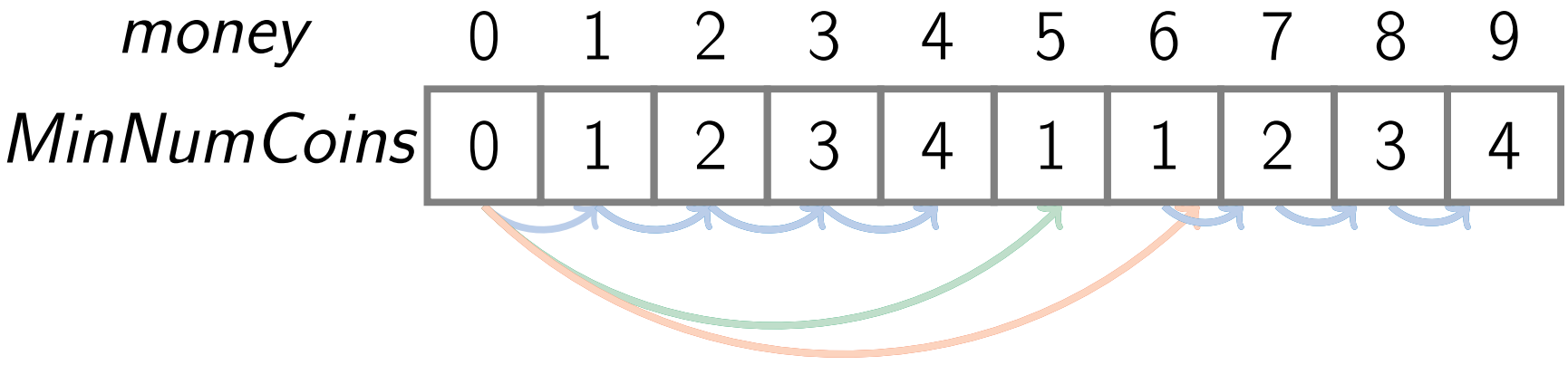
What is the minimum number of coins needed to change 6 cents for denominations 6, 5, and 1?



$$\min \begin{cases} MinNumCoins(0) + 1 \\ MinNumCoins(1) + 1 \\ MinNumCoins(5) + 1 \end{cases}$$

Dynamic Programming

What is the minimum number of coins needed to change 9 cents for denominations 6, 5, and 1?



DPChange($money$, $coins$)

```
MinNumCoins(0) ← 0
for m from 1 to money:
    MinNumCoins(m) ← ∞
    for i from 1 to |coins|:
        if m ≥ coin_i:
            NumCoins ← MinNumCoins(m − coin_i) + 1
            if NumCoins < MinNumCoins(m):
                MinNumCoins(m) ← NumCoins
return MinNumCoins(money)
```

"Programming" in "Dynamic Programming" Has Nothing to Do with Programming!

Richard Bellman developed this idea in 1950s working on an Air Force project.

At that time, his approach seemed completely impractical.

He wanted to hide that he is really doing math from the Secretary of Defense.

Richard Bellman

...What name could I choose? I was interested in planning but planning, is not a good word for various reasons. I decided therefore to use the word, "programming" and I wanted to get across the idea that this was dynamic. **It was something not even a Congressman could object to.** So I used it as an umbrella for my activities.