Greedy Algorithms: Grouping Children

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Algorithmic Toolbox
Data Structures and Algorithms

Outline

1. The Problem
2. Naive Algorithm
3. Efficient Algorithm

Many children came to a celebration. Organize them into the minimum possible number of groups such that the age of any two children in the same group differ by at most one year.

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MinGroups(C)

```
m ← len(C)
for each partition into groups
C = G_1 ∪ G_2 ∪ ··· ∪ G_k:
  good ← true
  for i from 1 to k:
    if max(G_i) − min(G_i) > 1:
      good ← false
  if good:
    m ← min(m, k)
return m
```

Running time

Lemma

The number of operations in `MinGroups`(C) is at least 2^n, where n is the number of children in C.

Proof

- Consider just partitions in two groups
- $C = G_1 \cup G_2$
- For each $G_1 \subset C$, $G_2 = C \setminus G_1$
- Size of C is n
- Each item can be included or excluded from G_1
- There are 2^n different G_1
- Thus, at least 2^n operations $\square$

Asymptotics

- Naive algorithm works in time $\Omega(2^n)$
- For $n = 50$ it is at least

$$2^{50} = 1125899906842624$$

operations!
- We will improve this significantly

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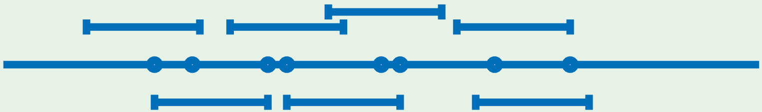
Covering points by segments

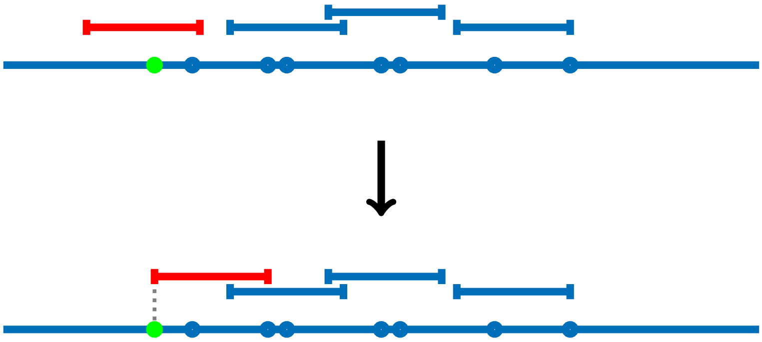
Input: A set of n points $x_1, \ldots, x_n \in \mathbb{R}$.

Output: The minimum number of segments of unit length needed to cover all the points.

Example



Safe move: cover the leftmost point with a unit segment which starts in this point.



Assume $x_1 \le x_2 \le \ldots \le x_n$

```
PointsCoverSorted(x_1, ..., x_n)

R ← {}, i ← 1
while i ≤ n:
   [ℓ, r] ← [x_i, x_i + 1]
   R ← R ⋃ {[ℓ, r]}
   i ← i + 1
   while i ≤ n and x_i ≤ r:
      i ← i + 1
return R
```

Lemma

The running time of `PointsCoverSorted` is $O(n)$.

Proof

- i changes from 1 to n
- For each i, at most 1 new segment
- Overall, running time is $O(n)$ $\square$

Total Running Time

- `PointsCoverSorted` works in $O(n)$ time
- Sort $\{x_1, x_2, \ldots, x_n\}$, then call `PointsCoverSorted`
- Soon you'll learn to sort in $O(n \log n)$
- Sort + `PointsCoverSorted` is $O(n \log n)$

Asymptotics

- Straightforward solution is $\Omega(2^n)$
- Very long for $n = 50$
- Sort + greedy is $O(n \log n)$
- Fast for $n = 10\,000\,000$
- Huge improvement!

Conclusion

- Straightforward solution is exponential
- Important to reformulate the problem in mathematical terms
- Safe move is to cover leftmost point
- Sort in $O(n \log n)$ + greedy in $O(n)$