

# Divide-and-Conquer: Polynomial Multiplication

Neil Rhodes

Department of Computer Science and Engineering  
University of California, San Diego

Data Structures and Algorithms  
Algorithmic Toolbox

# Outline

- 1 Problem Overview
- 2 Naïve Algorithm
- 3 Naïve Divide and Conquer Algorithm
- 4 Faster Divide and Conquer

# Uses of multiplying polynomials

- Error-correcting codes
- Large-integer multiplication
- Generating functions
- Convolution in signal processing

# Multiplying Polynomials

## Example

$$A(x) = 3x^2 + 2x + 5$$

$$B(x) = 5x^2 + x + 2$$

$$A(x)B(x) = 15x^4 + 13x^3 + 33x^2 + 9x + 10$$

# Multiplying polynomials

Input: Two  $n - 1$  degree polynomials:

$$a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$$

$$b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$$

Output: The product polynomial:

$$c_{2n-2}x^{2n-2} + c_{2n-3}x^{2n-3} + \cdots + c_1x + c_0$$

where:

$$c_{2n-2} = a_{n-1}b_{n-1}$$

$$c_{2n-3} = a_{n-1}b_{n-2} + a_{n-2}b_{n-1}$$

...

$$c_2 = a_2b_0 + a_1b_1 + a_0b_2$$

$$c_1 = a_1b_0 + a_0b_1$$

$$c_0 = a_0b_0$$

# Multiplying Polynomials

## Example

Input:  $n = 3$ ,  $A = (3, 2, 5)$ ,  $B = (5, 1, 2)$

$$A(x) = 3x^2 + 2x + 5$$

$$B(x) = 5x^2 + x + 2$$

$$A(x)B(x) = 15x^4 + 13x^3 + 33x^2 + 9x + 10$$

Output:  $C = (15, 13, 33, 9, 10)$

# Outline

- 1 Problem Overview
- 2 Naïve Algorithm
- 3 Naïve Divide and Conquer Algorithm
- 4 Faster Divide and Conquer

## MultPoly( $A, B, n$ )

```
pair  $\leftarrow$  Array[ $n$ ][ $n$ ]  
for  $i$  from 0 to  $n - 1$ :  
    for  $j$  from 0 to  $n - 1$ :  
        pair[ $i$ ][ $j$ ]  $\leftarrow$   $A[i] * B[j]$   
product  $\leftarrow$  Array[ $2n - 1$ ]  
for  $i$  from 0 to  $2n - 1$ :  
    product[ $i$ ]  $\leftarrow$  0  
for  $i$  from 0 to  $n - 1$ :  
    for  $j$  from 0 to  $n - 1$ :  
        product[ $i + j$ ]  $\leftarrow$  product[ $i + j$ ] + pair[ $i$ ][ $j$ ]  
return product
```



# Multiplying Polynomials

Naïve Solution:  $O(n^2)$

- Multiply all  $a_i b_j$  pairs ( $n^2$  multiplications)
- Sum needed pairs ( $n^2$  additions)

# Outline

- 1 Problem Overview
- 2 Naïve Algorithm
- 3 Naïve Divide and Conquer Algorithm
- 4 Faster Divide and Conquer

# Multiplying Polynomials

- Let  $A(x) = D_1(x)x^{\frac{n}{2}} + D_0(x)$  where
$$D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + \dots + a_{\frac{n}{2}}$$
$$D_0(x) = a_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + a_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + \dots + a_0$$
- Let  $B(x) = E_1(x)x^{\frac{n}{2}} + E_0(x)$  where
$$E_1(x) = b_{n-1}x^{\frac{n}{2}-1} + b_{n-2}x^{\frac{n}{2}-2} + \dots + b_{\frac{n}{2}}$$
$$E_0(x) = b_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + b_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + \dots + b_0$$
- $AB = (D_1x^{\frac{n}{2}} + D_0)(E_1x^{\frac{n}{2}} + E_0)$ 
$$= (D_1E_1)x^n + (D_1E_0 + D_0E_1)x^{\frac{n}{2}} + D_0E_0$$
- Calculate  $D_1E_1$ ,  $D_1E_0$ ,  $D_0E_1$ , and  $D_0E_0$

Recurrence:  $T(n) = 4T(\frac{n}{2}) + kn$ .

## Polynomial Mult: Divide & Conquer

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_0(x) = 2x + 1$$

$$E_1(x) = x + 2$$

$$E_0(x) = 3x + 4$$

$$D_1E_1 = 4x^2 + 11x + 6$$

$$D_1E_0 = 12x^2 + 25x + 12$$

$$D_0E_1 = 2x^2 + 5x + 2$$

$$D_0E_0 = 6x^2 + 11x + 4$$

$$AB = (4x^2 + 11x + 6)x^4 +$$

$$(12x^2 + 25x + 12 + 2x^2 + 5x + 2)x^2 +$$

$$6x^2 + 11x + 4$$

$$= 4x^6 + 11x^5 + 20x^4 + 30x^3 + 20x^2 + 11x + 4$$

## Function Mult2( $A, B, n, a_l, b_l$ )

$R = \text{array}[0..2n - 2]$

if  $n = 1$ :

$R[0] = A[a_l] * B[b_l]$  ; return  $R$

$R[0..n - 2] = \text{Mult2}(A, B, \frac{n}{2}, a_l, b_l)$

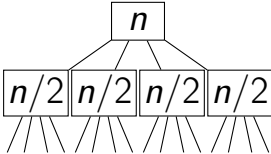

$R[n..2n - 2] = \text{Mult2}(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l + \frac{n}{2})$

$D_0E_1 = \text{Mult2}(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$

$D_1E_0 = \text{Mult2}(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l)$

$R[\frac{n}{2} \dots n + \frac{n}{2} - 2] += D_1E_0 + D_0E_1$

return  $R$

level		#	work
0		1	$kn$
1	$n/2$ $n/2$ $n/2$ $n/2$	4	$4k\frac{n}{2} = k2n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$i$	$\dots$ $n/2^i$ $\dots$	$4^i$	$4^i k \frac{n}{2^i} = k2^i n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\log_2 n$		$4^{\log_2 n}$	$k4^{\log_2 n} = kn^2$

---

Total:  $\sum_{i=0}^{\log_2 n} 4^i k \frac{n}{2^i} = \Theta(n^2)$

# Outline

- 1 Problem Overview
- 2 Naïve Algorithm
- 3 Naïve Divide and Conquer Algorithm
- 4 Faster Divide and Conquer

## Karatsuba approach

$$A(x) = a_1x + a_0$$

$$B(x) = b_1x + b_0$$

$$C(x) = a_1b_1x^2 + (a_1b_0 + a_0b_1)x + a_0b_0$$

Needs 4 multiplications

Rewrite as:

$$C(x) = a_1b_1x^2 + ((a_1 + a_0)(b_1 + b_0) - a_1b_1 - a_0b_0)x + a_0b_0$$

Needs 3 multiplications



## Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_0(x) = 2x + 1$$

$$E_1(x) = x + 2$$

$$E_0(x) = 3x + 4$$

$$D_1E_1 = 4x^2 + 11x + 6$$

$$D_0E_0 = 6x^2 + 11x + 4$$

$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$$

$$= 24x^2 + 52x + 24$$


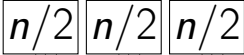



$$AB = (4x^2 + 11x + 6)x^4 +$$

$$(24x^2 + 52x + 24 - (4x^2 + 11x + 6)$$

$$- (6x^2 + 11x + 4))x^2 +$$

$$6x^2 + 11x + 4$$

$$= 4x^6 + 11x^5 + 20x^4 + 30x^3 + 20x^2 + 11x + 4$$

level		#	work
0		1	$kn$
1		3	$3k\frac{n}{2} = k\frac{3}{2}n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$i$	$\dots$  $\dots$	$3^i$	$3^i k\frac{n}{2^i} = k(\frac{3}{2})^i n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\log_2 n$	 $\dots$ 	$3^{\log_2 n}$	$k3^{\log_2 n} = kn^{\log_2 3}$

$$\begin{aligned}
 \text{Total: } \sum_{i=0}^{\log_2 n} 3^i k \frac{n}{2^i} &= \Theta(n^{\log_2 3}) \\
 &= \Theta(n^{1.58})
 \end{aligned}$$