Exercises from Chapter 5

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Problem 1. Suppose $(X, \mathcal{A})$ is a measurable space, f is a real-valued function, and $\{x : f(x) > r\} \in \mathcal{A}$ for each rational number r. Prove that f is measurable.

Proof. We need to show that $\{x : f(x) > q\} \in \mathcal{A}$ holds for any irrational q. Let q be any irrational number. Define the sequence of rational numbers $a_0, a_1, a_2, ...$ such that $a_i > a_j$ for $i > j$ and $\lim a_i = q$. Then, $\cup_0^\infty \{x : f(x) > a_i\}$ is equivalent to $\{x : f(x) > q\}$ and is the countable union of elements in $\mathcal{A}$, hence is an element of $\mathcal{A}$. Therefore, $\{x : f(x) > r\} \in \mathcal{A}$ for all real r, meaning f is measurable. $\square$

Problem 3. Suppose f is a measurable function and $f(x) > 0$ for all x. Let $g(x) = 1/f(x)$. Prove that g is a measurable function.

Proof. By proposition 5.5, f being measurable implies that $\{x : f(x) \leq r\} \in \mathcal{A}$ for all real r. But each set $\{x : f(x) \leq r\}$ is equal to $\{x : g(x) > r\}$. Hence, $\{x : g(x) > r\} \in \mathcal{A}$ for all r and g is measurable. $\square$

Problem 5. If $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, prove that there exists a Borel measurable function g such that $f = g$ a.e.

Proof. Since the Lebesgue algebra is the closure of the Borel algebra, any non-Borel measurable Lebesgue set will have measure 0. Hence, we can construct a function

$$g(x) = \begin{cases} 0 & x \in N \\ f(x) & \text{else} \end{cases}$$

Where N is the union of non Borel measurable Lebesgue sets. This function differs from f on a domain of measure 0. Hence, $f = g$ a.e. //TODO make this more rigorous; pretty sure it doesn't work right now $\square$

Problem S. uppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at each point. Prove that f and f' are Borel measurable.

Proof. Differentiability implies continuity, hence f is Borel measurable. Recall the definition of the derivative is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) + f(x)}{h}$$

Which is the limit of the sum of measurable functions times a scalar, hence is a measurable function. $\square$