Exercises from Chapter 4

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Problem 1. Let μ be a measure on the Borel σ-algebra of R such that $\mu(K) < \infty$ whenever K is compact, define $\alpha(x) = \mu((0, x])$ if $x \geq 0$ and $\alpha(x) = -\mu((x, 0])$ if $x < 0$. Show that μ is the Lebesgue-Stieltjes measure corresponding to α.

Proof. We need to show that α is a increasing right continuous function and that for $\ell((a, b]) = \alpha(a) - \alpha(b)$,

$$\mu(E) = \inf\{\Sigma_{i=1}^{\infty} \ell(A_i) : E = \cup_{i=1}^{\infty} A_i\}$$

Firstly, note that $\alpha(x) = \mu((0, x]) \leq \mu([0, x])$. The set $[0, x]$ is obviously compact, so $\alpha(x)$ is finite whenever x is finite.

Next, if $0 \leq x < y$, then $\alpha(y) = \alpha(x) + \mu((x, y])$ Since $\mu((x, y]) > 0$, we have that $\alpha(y) \leq \alpha(x)$. Now, if $x < 0 < y$, we have $\alpha(x) < 0 < \alpha(y)$. The last case is whenever $x < y \leq 0$, yielding $\alpha(x) = -\mu((x, 0]) = -\mu((x, y]) - \mu((y, 0])$. Since $-\mu((x, y]) \leq 0$, we have that $\alpha(y) \leq \alpha(x)$.

To see that α is continuous, consider the limit of $\alpha(y - \epsilon) = \alpha(y) - \mu((y - \epsilon, y])$. As ϵ approaches 0 we get $\alpha(y)$. (This is not true whenever $\mu([y, y]) > 0$. I cannot figure out how to prove that case.)

For finite real numbers a and b with $a \leq b$, proposition 4.9 shows that $m*((a, b]) = \ell((a, b]) = \alpha(b) - \alpha(a) = \mu((a, b])$. Since such sets $(a, b]$ generate the Borel σ-algebra, we have that $m*(A) = \mu(A)$ for any A in μ's domain. Therefore, μ is the Lebesgue-Stieltjes measure corresponding to α $\square$

Problem 3. If $(X, \mathcal{A}, \mu)$ is a measure space, define

$$\mu^*(A) = \inf\{\mu(B) : A \subset B, B \in \mathcal{A}\}$$

for all subsets A of X. Show that μ^* is an outer measure. Show that each set in $\mathcal{A}$ is μ^*-measurable and μ^* agrees with the measure μ on $\mathcal{A}$.

Proof. (I am assuming $\subset$ is inclusive) Since $\mu(\emptyset) = 0$ and $\emptyset \in \mathcal{A}$, $\mu^*(\emptyset) = \mu(\emptyset) = 0$. If $A \subset B \subset X$ then $\{A' : A \subset A', A' \in \mathcal{A}\} \subset \{B' : B \subset B', B' \in \mathcal{A}\}$. Hence, $\inf\{A' : A \subset A', A' \in \mathcal{A}\} \leq \inf\{B' : B \subset B', B' \in \mathcal{A}\}$. Therefore, $\mu^*(A) \leq \mu^*(B)$.

Let $A_1, A_2, A_3, ...$ be a collection of sets in X with $A = \cup_i A_i$. For any $A'_1, A'_2, ... \in \mathcal{A}$ such that $A_i \subset A'_i$, the set $A' = \cup_i A'_i$ is in (A) and $\mu A' = \mu(\cup_i A'_i) \leq \Sigma_i \mu(A'_i)$. Hence, $\inf\{\mu(A') : A \subset A', A' \in \mathcal{A}\} \leq \Sigma_i \inf\{\mu(A') : A_i \subset A', A' \in \mathcal{A}\}$. So we have $\mu^*(A) \leq \Sigma_i \mu^*(A_i)$. $\square$

Problem 5. Suppose m is Lebesgue measure. Define $x + A = \{x + y : y \in A\}$ and $cA = \{cy : y \in A\}$ for $x \in \mathbb{R}$ and c a real number. Show that if A is a Lebesgue measurable set, then $m(x + A) = m(A)$ and $m(cA) = |c|m(A)$.

Proof. Multiplying by $c < 0$ results in a reflection of the real number line about the 0 point, which does not affect the Lebesgue measure of any set. So, we can assume WLG that c is positive. The ℓ function of the Lebesgue measure is $\ell((a, b]) = b - a$. For any collection of intervals A_i in $\mathcal{C}$, $\Sigma \ell(x + A_i) = \Sigma x + b_i - x - a_i = \Sigma b_i - a_i = \Sigma \ell(A_i)$ and $\Sigma \ell(cA_i) = \Sigma cb_i - ca_i = c\Sigma b_i - a_i = c\Sigma \ell(A_i)$. Therefore, $m(x + A) = \inf\{\Sigma \ell(x + A_i) : A_i \in \mathcal{C}, A \subset \cup A_i\} = \inf\{\Sigma \ell(A_i) : A_i \in \mathcal{C}, A \subset \cup A_i\} = m(A)$ and $m(cA) = \inf\{\Sigma \ell(cA_i) : A_i \in \mathcal{C}, A \subset \cup A_i\} = c \inf\{\Sigma \ell(A_i) : A_i \in \mathcal{C}, A \subset \cup A_i\} = cm(A)$ $\square$