# Exercises from Chapter 5

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**Problem 1.** Suppose $(X, \mathcal{A})$ is a measurable space, $f$ is a real-valued function, and $\{x : f(x) > r\} \in \mathcal{A}$ for each rational number $r$. Prove that $f$ is measurable.

*Proof.* We need to show that $\{x : f(x) > q\} \in \mathcal{A}$ holds for any irrational $q$. Let $q$ be any irrational number. Define the sequence of rational numbers $a_0, a_1, a_2, ...$ such that $a_i > a_j$ for $i > j$ and $\lim a_i = q$. Then, $\cup_0^\infty \{x : f(x) > a_i\}$ is equivalent to $\{x : f(x) > q\}$ and is the countable union of elements in $\mathcal{A}$, hence is an element of $\mathcal{A}$. Therefore, $\{x : f(x) > r\} \in \mathcal{A}$ for all real $r$, meaning $f$ is measurable. $\square$

**Problem 3.** Suppose $f$ is a measurable function and $f(x) > 0$ for all $x$. Let $g(x) = 1/f(x)$. Prove that $g$ is a measurable function.

*Proof.* By proposition 5.5, $f$ being measurable implies that $\{x : f(x) \leq r\} \in \mathcal{A}$ for all real $r$. But each set $\{x : f(x) \leq r\}$ is equal to $\{x : g(x) > r\}$. Hence, $\{x : g(x) > r\} \in \mathcal{A}$ for all $r$ and $g$ is measurable. $\square$

**Problem 5.** If $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, prove that there exists a Borel measurable function $g$ such that $f = g$ a.e.

*Proof.* Since the Lebesgue algebra is the closure of the Borel algebra, $\square$