# Exercises from Chapter 3

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**Problem 1.** Suppose $(X, \mathcal{A})$ is a measurable space and $\mu$ is a non-negative set function that is finitely additive and such that $\mu(\emptyset) = 0$ and $\mu(B)$ is finite for some non-empty $B \in \mathcal{A}$. Suppose that whenever $A_i$ is an increasing sequence of sets in $\mathcal{A}$, then $\mu(\cup_i A_i) = lim_i \to \infty \mu(A_i)$. Show that $\mu$ is a measure.

*Proof.* We are given the first requirement for $\mu$ to be a measure, namely that $\mu(\emptyset) = 0$. So, we need only show countable additivity. Let $B = \{B_0, B_1, ...\}$ be a countably infinite collection of pairwise disjoint sets in $\mathcal{A}$. Define $A = \{A_0, A_1, ...\}$ to be the set where $A_n = \cup_{i=0}^n B_i$. Clearly, $A_i$ is an increasing sequence of sets.

Since $\mu$ is finitely additive, we have that $\mu(\cup_{i=0}^n B_i) = \Sigma_{i=0}^n \mu(B_i)$. And, since $\cup_{i=0}^n B_i = A_n$, we can conclude

$$\mu(\cup_i B_i) = \mu(\cup_i A_i) = lim_i \to \infty \mu(A_i) = lim_n \to \infty \Sigma_{i=0}^n \mu(B_i) = \Sigma_{i=0}^\infty \mu(B_i)$$

□

**Problem 3.** Let $X$ be an uncountable set and let $\mathcal{A}$ be the collection of subsets $A$ of $X$ such that either $A$ or $A^c$ is countable. Define $\mu(A) = 0$ if $A$ is countable and $\mu(A) = 1$ if $A$ is uncountable. Prove that $\mu$ is a measure.

*Proof.* Although this would need to be proven, we take it as given that $\mathcal{A}$ is a $\sigma$-algebra on $X$ because this was demonstrated in the book and we are lazy.

Since $\emptyset^c = X$ is uncountable, $\emptyset$ is countable and $\mu(\emptyset) = 0$. Hence, we have the first part of the definition covered.

For countable additivity, let $A = \{A_0, A_1, ...\}$ be any collection of pairwise disjoint elements of $\mathcal{A}$. If some $A_i$ in $A$ is uncountable, say $A_0$, then $A_0^c$ must be countable. And, because $A$ is pairwise disjoint, $\cup_{i=1}^\infty A_i \subseteq A_0^c$. So must $\{A_1, A_2 ...\}$ also be countable. Therefore, at most one of the sets in $A$ can be uncountable.

Thus, if $A$ contains only countable sets, $\mu(\cup_i A_i) = 0 = \Sigma_{i=0}^\infty \mu(A_i)$. Otherwise, $\mu(\cup_i A_i) = 1 = \Sigma_{i=0}^\infty \mu(A_i)$. Therefore, $\mu$ is countably additive and is a measure. □

**Problem 5.** Prove that if $\mu_1, \mu_2, ...$ are measures on a measurable space and $a_1, a_2, ... \in [0, \infty)$, then $\Sigma_{n=1}^\infty a_n \mu_n$ is also a measure.

*Proof.* First, we show the measure of the empty set is zero $\Sigma_{n=1}^\infty a_n \mu_n(\emptyset) = \Sigma_{n=1}^\infty 0 = 0$.

Next, let $A_i$ be a countable collection of sets, in the given measurable space. Then,

$$\Sigma_{n=1}^\infty a_n \mu_n(\cup_i A_i) = \Sigma_{n=1}^\infty \Sigma_{i=0}^\infty a_n \mu_n(A_i) = \Sigma_{i=1}^\infty \Sigma_{n=0}^\infty a_n \mu_n(A_i)$$

Therefore, the given function has countable additivity and is thus a measure. □

**Problem 5.** Suppose $\mu_1, \mu_2, ...$ are measures on a measurable space $(X, \mathcal{A})$ and $\mu_n(A) \uparrow$ for each $A \in \mathcal{A}$. Define $\mu(A) = \lim \mu_n(A)$. Is $\mu$ necessarily a measure? If not, give a counterexample. What if $\mu_n(A) \downarrow$ for each $A \in \mathcal{A}$ and $\mu_1(X) < \infty$?