Exercises from Chapter 3

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Problem 1. Suppose $(X, \mathcal{A})$ is a measurable space and μ is a non-negative set function that is finitely additive and such that $\mu(\emptyset) = 0$ and $\mu(B)$ is finite for some non-empty $B \in \mathcal{A}$. Suppose that whenever A_i is an increasing sequence of sets in $\mathcal{A}$, then $\mu(\cup_i A_i) = lim_i \to \infty \mu(A_i)$. Show that μ is a measure.

Proof. We are given the first requirement for μ to be a measure, namely that $\mu(\emptyset) = 0$. So, we need only show countable additivity. Let $B = \{B_0, B_1, ...\}$ be a countably infinite collection of pairwise disjoint sets in $\mathcal{A}$. Define $A = \{A_0, A_1, ...\}$ to be the set where $A_n = \cup_{i=0}^{n} B_i$. Clearly, A_i is an increasing sequence of sets.

Since μ is finitely additive, we have that $\mu(\cup_{i=0}^{n} B_i) = \Sigma_{i=0}^{n} \mu(B_i)$. And, since $\cup_{i=0}^{n} B_i = A_n$, we can conclude

$$\mu(\cup_i B_i) = \mu(\cup_i A_i) = lim_i \to \infty \mu(A_i) = lim_n \to \infty \Sigma_{i=0}^{n} \mu(B_i) = \Sigma_{i=0}^{\infty} \mu(B_i)$$

□

Problem 3. Let X be an uncountable set and let $\mathcal{A}$ be the collection of subsets A of X such that either A or A^c is countable. Define $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ if A is uncountable. Prove that μ is a measure.

Proof. Although this would need to be proven, we take it as given that $\mathcal{A}$ is a σ-algebra on X because this was demonstrated in the book and we are lazy.

Since $\emptyset^c = X$ is uncountable, $\emptyset$ is countable and $\mu(\emptyset) = 0$. Hence, we have the first part of the definition covered.

For countable additivity, let $A = \{A_0, A_1, ...\}$ be any collection of pairwise disjoint elements of $\mathcal{A}$. If some A_i in A is uncountable, say A_0, then A_0^c must be countable. And, because A is pairwise disjoint, $\cup_{i=1}^{\infty} A_i \subseteq A_0^c$. So must $\{A_1, A_2 ...\}$ also be countable. Therefore, at most one of the sets in A can be uncountable.

Thus, if A contains only countable sets, $\mu(\cup_i A_i) = 0 = \Sigma_{i=0}^{\infty} \mu(A_i)$. Otherwise, $\mu(\cup_i A_i) = 1 = \Sigma_{i=0}^{\infty} \mu(A_i)$. Therefore, μ is countably additive and is a measure. □

Problem 5. Prove that if $\mu_1, \mu_2, ...$ are measures on a measurable space and $a_1, a_2, ... \in [0, \infty)$, then $\Sigma_{n=1}^{\infty} a_n \mu_n$ is also a measure.

Proof. First, we show the measure of the empty set is zero $\Sigma_{n=1}^{\infty} a_n \mu_n(\emptyset) = \Sigma_{n=1}^{\infty} 0 = 0$.

Next, let A_i be a countable collection of sets, in the given measurable space. Then,

$$\Sigma_{n=1}^{\infty} a_n \mu_n(\cup_i A_i) = \Sigma_{n=1}^{\infty} \Sigma_{i=0}^{\infty} a_n \mu_n(A_i) = \Sigma_{i=1}^{\infty} \Sigma_{n=0}^{\infty} a_n \mu_n(A_i)$$

Therefore, the given function has countable additivity and is thus a measure. □

Problem 7. Suppose $\mu_1, \mu_2, ...$ are measures on a measurable space $(X, \mathcal{A})$ and $\mu_n(A) \uparrow$ for each $A \in \mathcal{A}$. Define

$$\mu(A) = \lim \mu_n(A).$$

Is μ necessarily a measure? If not, give a counterexample. What if $\mu_n(A) \downarrow$ for each $A \in \mathcal{A}$ and $\mu_1(X) < \infty$?

Proof. In both cases, $\mu(\emptyset) = 0$. So we just need countable additivity for μ to be a measure. In both cases, it is also obvious that $\lim_n \mu_n(\cup_i A_i) = \mu(\cup_i A_i)$. So it suffices to show that $\lim_n \Sigma_i \mu_n(A_i) = \Sigma_i \mu(A_i)$.

In the first case, we begin by noting that because $\mu_n(A) \uparrow$, $\Sigma_i \mu(A_i) \geq \Sigma_i \mu_n(A_i)$ for all n.

Let $\epsilon > 0$ and let $\{n_1, n_2, ...\}$ be the set of natural numbers such that $\mu_{n_i}(A_i) \geq \mu(A_i) - \frac{\epsilon}{2^i}$. We know such a set exists because of the limit condition on the μ_n. Now we have that $\Sigma_i \mu_{n_i}(A_i) \geq \Sigma_i \mu(A_i) - \frac{\epsilon}{2^i} = \Sigma_i \mu(A_i) - \epsilon$. Hence, we have

$$\Sigma_i \mu(A_i) \geq \Sigma_i \mu_{n_i}(A_i) \geq \Sigma_i \mu(A_i) - \epsilon$$

Taking the limit as ϵ aproaches 0 proves that $\Sigma_i \mu(A_i) \geq \lim_n \Sigma_i \mu_n(A_i) \geq \Sigma_i \mu(A_i)$. Therefore, $\lim_n \Sigma_i \mu_n(A_i) = \Sigma_i \mu(A_i)$ $\square$

Problem 9. Suppose X is the set of real numbers, $\mathcal{B}$ is the Borel σ-algebra, and m and n are two measures on $(X, \mathcal{B})$ such that $m((a,b)) = n((a,b)) < \infty$ whenever $\infty < a < b < \infty$. Prove that $m(A) = n(A)$ whenever $A \in \mathcal{B}$.

Proof. Proposition 2.1 states that the Borel σ-algebra on the real numbers is generated by (a,b) sets. Hence, any set in the Borel σ-algebra can be generated by sets on which m and n agree. $\square$