Exercises from Chapter 4

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Problem 1. Let μ be a measure on the Borel σ-algebra of R such that $\mu(K) < \infty$ whenever K is compact, define $\alpha(x) = \mu((0, x])$ if $x \geq 0$ and $\alpha(x) = -\mu((x, 0])$ if $x < 0$. Show that μ is the Lebesgue-Stieltjes measure corresponding to α.

Proof. We need to show that α is a increasing right continuous function and that for $\ell((a, b]) = \alpha(a) - \alpha(b)$,

$$\mu(E) = \inf\{\Sigma_{i=1}^{\infty} \ell(A_i) : E = \cup_{i=1}^{\infty} A_i\}$$

Firstly, note that $\alpha(x) = \mu((0, x]) \leq \mu([0, x])$. The set $[0, x]$ is obviously compact, so $\alpha(x)$ is finite whenever x is finite.

Next, if $0 \leq x < y$, then $\alpha(y) = \alpha(x) + \mu((x, y])$ Since $\mu((x, y]) > 0$, we have that $\alpha(y) \leq \alpha(x)$. Now, if $x < 0 < y$, we have $\alpha(x) < 0 < \alpha(y)$. The last case is whenever $x < y \leq 0$, yielding $\alpha(x) = -\mu((x, 0]) = -\mu((x, y]) - \mu((y, 0])$. Since $-\mu((x, y]) \leq 0$, we have that $\alpha(y) \leq \alpha(x)$.

To see that α is continuous, consider the limit of $\alpha(y - \epsilon) = \alpha(y) - \mu((y - \epsilon, y])$. As ϵ approaches 0 we get $\alpha(y)$. (This is not true whenever $\mu([y, y]) > 0$. I cannot figure out how to prove that case.)

For finite real numbers a and b with $a \leq b$, proposition 4.9 shows that $m*((a, b]) = \ell((a, b]) = \alpha(b) - \alpha(a) = \mu((a, b])$. Since such sets $(a, b]$ generate the Borel σ-algebra, we have that $m*(A) = \mu(A)$ for any A in μ's domain. Therefore, μ is the Lebesgue-Stieltjes measure corresponding to α $\square$

Problem 3. If $(X, \mathcal{A}, \mu)$ is a measure space, define

$$\mu^*(A) = \inf\{\mu(B) : A \subset B, B \in \mathcal{A}\}$$

for all subsets A of X. Show that μ^* is an outer measure. Show that each set in $\mathcal{A}$ is μ^*-measurable and μ^* agrees with the measure μ on $\mathcal{A}$.

Proof. (I am assuming $\subset$ is inclusive) Since $\mu(\emptyset) = 0$ and $\emptyset \in \mathcal{A}$, $\mu^*(\emptyset) = \mu(\emptyset) = 0$. If $A \subset B \subset X$ then $\{A' : A \subset A', A' \in \mathcal{A}\} \subset \{B' : B \subset B', B' \in \mathcal{A}\}$. Hence, $\inf\{A' : A \subset A', A' \in \mathcal{A}\} \leq \inf\{B' : B \subset B', B' \in \mathcal{A}\}$. Therefore, $\mu^*(A) \leq \mu^*(B)$.

Let $A_1, A_2, A_3, ...$ be a collection of sets in X with $A = \cup_i A_i$. For any $A'_1, A'_2, ... \in \mathcal{A}$ such that $A_i \subset A'_i$, the set $A' = \cup_i A'_i$ is in (A) and $\mu A' = \mu(\cup_i A'_i) \leq \Sigma_i \mu(A'_i)$. Hence, $\inf\{\mu(A') : A \subset A', A' \in \mathcal{A}\} \leq \Sigma_i \inf\{\mu(A') : A_i \subset A', A' \in \mathcal{A}\}$. So we have $\mu^*(A) \leq \Sigma_i \mu^*(A_i)$. $\square$

Problem 5. Suppose m is Lebesgue measure. Define $x + A = \{x + y : y \in A\}$ and $cA = \{cy : y \in A\}$ for $x \in \mathbb{R}$ and c a real number. Show that if A is a Lebesgue measurable set, then $m(x + A) = m(A)$ and $m(cA) = |c|m(A)$.

Proof. Multiplying by $c < 0$ results in a reflection of the real number line about the 0 point, which does not affect the Lebesgue measure of any set. So, we can assume WLG that c is positive. The ℓ function of the Lebesgue measure is $\ell((a, b]) = b - a$. For any collection of intervals A_i in $\mathcal{C}$, $\Sigma\ell(x + A_i) = \Sigma x + b_i - x - a_i = \Sigma b_i - a_i = \Sigma\ell(A_i)$ and $\Sigma\ell(cA_i) = \Sigma cb_i - ca_i = c\Sigma b_i - a_i = c\Sigma\ell(A_i)$. Therefore, $m(x + A) = \inf\{\Sigma\ell(x + A_i) : A_i \in \mathcal{C}, A \subset \cup A_i\} = \inf\{\Sigma\ell(A_i) : A_i \in \mathcal{C}, A \subset \cup A_i\} = m(A)$ and $m(cA) = \inf\{\Sigma\ell(cA_i) : A_i \in \mathcal{C}, A \subset \cup A_i\} = c \inf\{\Sigma\ell(A_i) : A_i \in \mathcal{C}, A \subset \cup A_i\} = cm(A)$ $\square$

Problem 7. Suppose $\epsilon \in (0, 1)$ and m is Lebesgue measure. Find a measurable set $E \subset [0, 1]$ such that the closure of E is $[0, 1]$ and $m(E) = \epsilon$.

Proof. Consider the set $[\mathbb{Q} \cap (\epsilon, 1)] \cup (0, \epsilon)$. The measure of this set is $m([\mathbb{Q} \cap (\epsilon, 1)]) + m((0, \epsilon)) = 0 + \epsilon = \epsilon$. Since this set contains every rational number in $(0, 1)$, its closure is obviously $[0, 1]$. $\square$

Problem 9. Let m be Lebesgue measure. Find an example of Lebesgue measurable subsets $A_1, A_2, ...$ of $[0, 1]$ such that $m(A_n) > 0$ for each n, $m(A_n \triangle A_m) > 0$ if $n \neq m$, and $m(A_n \cap A_m) = m(A_n)m(A_m)$ if $n \neq m$.

Proof. Let A_1 be the set of measure 0.5 centered on 0.5 (ie. $A_1 = (1/4, 3/4)$). For any A_k we define A_{k+1} as the collecting of sets which: 1 contain the endpoints of A_k, 2 all have equal measure summing to $\frac{m(A_k)}{2}$, and 3 are each shifted from centering on the endpoints of A_k in such a way that $\frac{m(A_k \cap A_{k+1})}{m(A_{k+1})} = m(A_k)$.

Let $m \neq n$ be such that $m > n$. $m(A_n \triangle A_m)$ is not zero, since A_n contains elements outside of A_m. We also have $m(A_n \cap A_)$ //TODO $\square$

Problem 11. Suppose m is Lebesgue measure and A is a Borel measurable subset of $\mathbb{R}$ with $m(A) > 0$. Prove that if

$$B = \{x - y : x, y \in A\},$$

then B contains a non-empty open interval centered at the origin. This is known as the Steinhaus theorem.

Proof. Given that A is Borel, it can be represented as the union of open intervals. Let $(a, b) \subset A$ be any such interval. With the operations defining B, $b - a$ and $a - b$ are the inf and sup elements respectively. Hence, the set in B corresponding to (a, b) is $(a - b, b - a)$, which is obviously centered at 0 $\square$

Problem 13. Let N be the non-measurable set defined in Section 4.4. Prove that if $A \subset N$ and A is Lebesgue measurable, then $m(A) = 0$.

Proof. Suppose A is any measurable subset of N. Then, as in the proof, we have:

$$\cup_{q \in [-1,1] \cap \mathbb{Q}} (A + q) \subset [-1, 2]$$

Hence,

$$3 \geq \Sigma_{q \in [-1,1], q \in \mathbb{Q}} m(A + q)$$

Therefore, $m(A) = 0$. $\square$

Problem 15. Let X be a set and $\mathcal{A}$ a collection of subsets of X that form an algebra of sets. Suppose ℓ is a measure on A such that $\ell(X) < \infty$. Define μ^* using ℓ as in (4.1). Prove that a set A is μ^*-measurable if and only if

$$\mu^*(A) = \ell(X) - \mu^*(A^c).$$

Proof. First, consider $m^*(X) = \inf\{\Sigma \ell(A_i) | \cup A_i = X\}$. Setting $A_1, A_2, ...$ to $X, \emptyset, \emptyset$ clearly yields the inf. Hence, $\ell(X) = m^*(X)$.

We can rewrite the given equation as $\ell(X) = \mu^*(A) + \mu^*(A^c)$. Let E be any subset of X. Then, subtracting both sides by $\mu^*(E^c)$ yields $\ell(X) - m^*(E^c) = m^*(X) - \mu^*(E^c)$ on the left, which can be expanded to

$$\inf\{\Sigma \ell(A_i) | \cup A_i = X\} - \inf\{\Sigma \ell(B_i) | \cup B_i = E^c\}$$

Subtracting this by $m^*(E)$ yields

$$\inf\{\Sigma \ell(A_i) | \cup A_i = X\} - \inf\{\Sigma \ell(B_i) | \cup B_i = E^c\} - \inf\{\Sigma \ell(C_i) | \cup C_i = E\}$$

Which can be combined to

$$\inf\{\Sigma \ell(A_i) | \cup A_i = X\} - \inf\{\Sigma \ell(B_i) + \ell(C_i) | \cup B_i = E^c, \cup C_i = E\}$$

Which is clearly $\inf\{\Sigma\ell(A_i)|\cup A_i = X\} - \inf\{\Sigma\ell(A_i)|\cup A_i = X\} = 0$. Hence, we have $[m^*(X) - m^*(E^c)] - m^*(E) = 0$. This clearly implies that $m^*(X) - m^*(E^c) = m^*(E)$.

On the right, we have $\mu * (A) + \mu * (A^c) - \mu * (E^c)$, which similarly can be written as

$$\inf\{\Sigma\ell(A_i)|\cup A_i = A\} + \inf\{\Sigma\ell(A_i)|\cup A_i = A^c\} - \inf\{\Sigma\ell(B_i)|\cup B_i = E^c\}$$

Splitting $\inf\{\Sigma\ell(B_i)|\cup B_i = E^c\}$ into two yields

$$\inf\{\Sigma\ell(A_i)|\cup A_i = A\} + \inf\{\Sigma\ell(A_i)|\cup A_i = A^c\} - \inf\{\Sigma\ell(B_i)|\cup B_i = E^c \cap A\} - \inf\{\Sigma\ell(B_i)|\cup B_i = E^c \cap A^c\}$$

Rearranging gives us

$$[\inf\{\Sigma\ell(A_i)|\cup A_i = A\} - \inf\{\Sigma\ell(B_i)|\cup B_i = E^c \cap A\}] + [\inf\{\Sigma\ell(A_i)|\cup A_i = A^c\} - \inf\{\Sigma\ell(B_i)|\cup B_i = E^c \cap A^c\}]$$

Combining yields

$$[\inf\{\Sigma\ell(A_i) - \ell(B_i)|\cup A_i = A, \cup B_i = E^c \cap A\}] + [\inf\{\Sigma\ell(A_i) - \ell(B_i)|\cup A_i = A^c, \cup B_i = E^c \cap A^c\}]$$

Which can be rewritten as

$$[\inf\{\Sigma\ell(A_i)|\cup A_i = E \cap A\}] + [\inf\{\Sigma\ell(A_i)|\cup A_i = E^c \cap A^c\}]$$

Which is of course the definition of $m^*(A \cap E) + m^*(A \cap E^c)$. Thus, we have that, $\mu^*(A) = \ell(X) - \mu^*(A^c)$ and for all $E \subset X$, $m^*(E) = m^*(A \cap E) + m^*(A \cap E^c)$, are equivalent statements. The second statement is the definition of m^* measurability. Therefore, A is m^*-measurable if and only if $m^*(A) = \ell(X) - m^*(A^c)$. $\square$

Problem 17. Suppose A is a Lebesgue measurable subset of $\mathbb{R}$ and

$$B = \cup_{x \in A}[x - 1, x + 1].$$

Prove that B is Lebesgue measurable.

Proof. By definition, $m(A) = \inf\{\Sigma(b_i - a_i)|\cup(a_i, b_i) = A\}$ exists. The set B can be defined as //TODO $\square$