Exercises from Dummit and Foote Chapter 15 on Commutative Rings and Algebraic Geometry

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Problem 1.1. Prove the converse to Hilbert's Basis Theorem: if the polynomial ring $R[x]$ is Noetherian,then R is Noetherian.

Proof. Suppose $R[x]$ is Noetherian. Then any ideal in $I \subseteq R$ is also in $R[x]$. So I must have finite generators $f_1, f_2, ..., f_n$ in $R[x]$. It remains to be seen that these generators can be strictly contained in R.

Let $\alpha = a_0 + a_1x + ... + a_nx^n$ be any element of I. Then, α can be expressed by $\alpha = g_1f_1 + ... + g_nf_n$. If α is strictly in R, then setting all variables to 0 is the identity. Hence $\alpha(0) = \alpha$, which lends $g_1(0)f_1(0) + ... + g_n(0)f_n(0) = \alpha$. Since $f_1(0), g_1(0), ..., f_n(0), g_n(0) \in R$, we have shown that any element of $I \cap R$ can be expressed by finite generators. Hence, I is finitely generated in R. □

Problem 1.2. Show that each of the following rings are not Noetherian by exhibiting an explicit infinite increasing chain of ideals:

(a) the ring of continuous real valued functions on $[0, 1]$,

(b) the ring of all functions from any infinite set X to $\mathbb{Z}/2\mathbb{Z}$.

Proof. (a) Let I_n be the ideal generated by functions which are 0 on the interval $[1/n, 1]$. Clearly $I_1 \subseteq I_2 \subseteq I_3 \subseteq ...$ is an infinite chain. Since I_n contains functions which are nonzero on $[1/(n+1), 1/n]$, $I_n \neq I_{n+1}$. So each inclusion id proper an the chain is infinitely increasing. □

(b). Using the AOC, let $x_1, x_2, x_3, ...$ be an ordered infinite subset of X. Let I_n be the ideal generated by all elements $\sigma(x_i) = 0$ for $i \leq n$. Then, $I_1 \subseteq I_2 \subseteq I_3 \subseteq ...$ is an infinite chain. Since I_n contains functions which send x_{n+1} to 1, the inclusions are again proper and the chain is increasing. □

Problem 1.3. Prove that the field $k(x)$ of rational functions over k in the variable x is not a finitely generated k-algebra. (Recall that $k(x)$ is the field of fractions of the polynomial ring $k[x]$. Note that $k(x)$ is a finitely generated field extension over k.)

Proof. content... □