Exercises from Chapter 10

Wesley Basener

June 6, 2025

I am skipping the first few exercises of each section in this chapter because they didn't look challenging enough to be helpful.

Problem 1.15. If M is a finite abelian group then M is naturally a $\mathbb{Z}$-module. Can this action be extended to make M into a $\mathbb{Q}$-module?

Proof. No, consider an order two element $m \in M$. Supposing such a $\mathbb{Q}$-module were possible, multiplying m by $\frac{2}{2} = (\frac{1}{2})(2)$ yields

$$\frac{2}{2}m = 1m = m \neq e = \frac{1}{2}e = \frac{1}{2}2m$$

Which of course contradicts part 2-b of the definition of modules. $\square$

Problem 1.17. Let T be the shift operator on the vector space V and let $e_1, ..., e_n$ be the usual basis vectors described in the example of $F[x]$-modules. If $m \geq n$ find $(a_m x^m + a_{m-1}x^{m-1} + ... + a_0)e_n$.

Proof. Each x^i shifts the 1 in the basis over i spaces left. Every x^i where $i > n$ will leave $e_i = 0$. Hence, we have,

$$(a_m x^m + a_{m-1}x^{m-1} + ... + a_0)e_n = (a_n, a_{n-1}, ..., a_0, 0, ..., 0)$$

where a_0 is in the nth spot. $\square$

Problem 1.19. Let $F = \mathbb{R}$, let $V = \mathbb{R}^2$ and let T be the linear transformation from V to V which is projection onto the y-axis. Show that V, 0, the x-axis and the y-axis are the only $F[x]$ submodules for this T.

Proof. Suppose for the sake of contradiction that this is not true, then there must exist a submodule N with an element (a, b) not in either of the nontrivial submodules mentioned. (ie. $a, b, \neq 0$).

Let (c, d) be any other element of M not in N. Multiplying (a, b) by $(\frac{d}{b} - \frac{c}{a})x + \frac{c}{a}$ yields

$$((\frac{d}{b} - \frac{c}{a})x + \frac{c}{a})(a, b) = (0, d - \frac{cb}{a}) + (c, \frac{cb}{a}) = (c, d)$$

Sending (a, b) outside of its submodule N is obviously a contradiction. Hence, there are no other non trivial submodules of M. In other words, V, 0, the x-axis, and the y-axis are the only submodules of M on this T. $\square$

Problem 1.21. Let $n \in \mathbb{Z}^+, n > 1$ and let R be the ring of $n \times n$ matrices with entries from a field F. Let M be the set of $n \times n$ matrices with arbitrary elements of F in the first column and zeros elsewhere. Show that M is a submodule of R when R is considered as a left module over itself, but M is not a submodule of R when R is considered as a right R-module.

Proof. Consider two elements of M: a and b. Let r be any element of R. If R is a right module, we have

$$a + rb = \begin{pmatrix} a_{1,1} & \dots & 0 \\ \vdots & & \vdots \\ a_{n,1} & \dots & 0 \end{pmatrix} + \begin{pmatrix} r_{1,1} & \dots & r_{1,n} \\ \vdots & & \vdots \\ r_{n,1} & \dots & r_{n,n} \end{pmatrix} \begin{pmatrix} b_{1,1} & \dots & 0 \\ \vdots & & \vdots \\ b_{n,1} & \dots & 0 \end{pmatrix} = \begin{pmatrix} a_{1,1} + r_{1,*} \cdot b_{*,1} & \dots & 0 \\ \vdots & & \vdots \\ a_{1,n} + r_{1,*} \cdot b_{*,1} & \dots & 0 \end{pmatrix}$$

which is in M. Hence M is a submodule of R, when R is a left module over itself.

If we consider R as a right module over itself, multiplication of any elememt a in M by any element r not in M yields.

$$ar = \begin{pmatrix} a_{1,1} & \dots & 0 \\ \vdots & & \vdots \\ a_{n,1} & \dots & 0 \end{pmatrix} \begin{pmatrix} r_{1,1} & \dots & r_{1,n} \\ \vdots & & \vdots \\ r_{n,1} & \dots & r_{n,n} \end{pmatrix} = \begin{pmatrix} a_{1,1}r_{1,1} & \dots & a_{1,1}r_{1,n} \\ \vdots & & \vdots \\ a_{n,1}r_{1,1} & \dots & a_{n,1}r_{1,n} \end{pmatrix}$$

which is not necessarily in M. Therefore, M is not a submodule of R, when R is a right module over itself. □

Problem 3.1. Prove that if A and B are sets of the same cardinality, then the free modules $F(A)$ and $F(B)$ are isomorphic.

Proof. If A and B have the same cardinality, then there is a bijective map $\phi : A \to B$. We can extend this map to the function $\psi : F(A) \to F(B)$, with the definition $\psi(r_1a_1 + r_2a_2 + ... + r_na_n) = r_1\phi(a_1) + r_2\phi(a_2) + ... + r_n\phi(a_n)$.

This map is a homomorphis since $\psi(r_1a_1 + ... + r_na_n + r(r'_1a'_1 + ... + r'_ma'_m)) = r_1\phi(a_1) + ... + r_n\phi(a_n) + r(r'_1\phi(a'_1) + ... + r'_n\phi(a'_n))$.

Furthermore, for unique a_i, $\psi(r_1a_1 + ... + r_na_n) = 0$ would necessitate $\phi(a_i) = \phi(a_j)$ for some $j \neq i$, which is impossible since ϕ is a bijection. Hence, the kernel of the homomorphism is 0 and it is thus an isomorphism. Therefore, $F(A)$ is isomorphic to $F(B)$. □

Problem 3.3. Show that the $F[x]$-modules in Exercises 18 and 19 of Section I are both cyclic.

Proof. In exercise 18, the $F[x]$-module is $\mathbb{R}^2$ with T being the rotation about the origin by $\pi/2$ radians. The element $(1, 0)$ can generate any other element (a, b) with the polynomial $bx + a$.

In exercise 19, the $F[x]$-module is again $\mathbb{R}^2$ but this time with T being the projection onto the y-axis. We showed in this exercise that any element $(a, b) \in \mathbb{R}^2$ not on the y or x axis can be mapped to any other element in $(c, d) \in \mathbb{R}^2$ with the polynomial $(\frac{d}{b} - \frac{c}{a})x + \frac{c}{a}$. Thus, any such element can generate the entire module. □

Problem 3.5. Let R be an integral domain. Prove that every finitely generated torsion R-module has a nonzero annihilator i.e., there is a nonzero element $r \in R$ such that $rm = 0$ for all $m \in M$ here r does not depend on m (the annihilator of a module was defined in Exercise 9 of Section 1). Give an example of a torsion R-module whose annihilator is the zero ideal.

Proof. Let $\{m_1, ..., m_n\}$ be the generators for M and let $t_1, ..., t_n \in R$ be the elements in R such that $r_im_i = 0$ for all i. Multiplying the product of the t_i by any element in M yields $(t_1...t_n)(r_1m_1 + ... + r_nm_n = (t_2...t_mr_1)t_1m_1 + ... + (t_1...t_{n-1}r_n)t_nm_n = 0$. Hence, $(t_1...t_n)$ is the desired element. □

Problem 3.7. Let N be a submodule of M. Prove that if both M/N and N are finitely generated then so is M.

Proof. We have that the set of elements $x_1, x_2, ...$ such that $x_1 + N, x_2 + N, ...$ are unique elements in M/N is finitely generated. Say, that this set is generated by $\{g_1, g_2, ...g_n\}$. Also say that N is generated by $\{n_1, ..., n_m\}$.

Every element in $m \in M$ maps to a unique element in some set $x + N \in M/N$, for some $x \in M$. Say for example, $m = x' + n'$. Both x' and n' have unique generations $x' = \Sigma r_ig_i$ and $n' = \Sigma r_jn_j$. Hence, $m = \Sigma r_ig_i + \Sigma r_jn_j$ is a unique generation of m. □

Problem 3.9. An R-module M is called irreducible if $M \neq 0$ and if 0 and M are the only submodules of M. Show that M is irreducible if and only if $M \neq 0$ and M is a cyclic module with any nonzero element as generator. Determine all the irreducible Z-modules.

Proof. Suppose M is irreducible. Then for any two elements $m, n \in M$, with $m \neq 0$ there must be an $r \in R$ such that $rm = n$. If this were not so, m would generate a non trivial submodule of M. Hence, any nonzeoro element generates M. We also have that $M \neq 0$.

Now, suppose $M \neq 0$ and that M is cyclic, being generated by any nonzero element. Then for any elements $m, n \in M$ with $m \neq 0$ there is an $r \in R$ such that $rm = n$. In other words, we can send any nonzero $m \in M$ to any other element in M. Hence, no elements in M are a part of a non trivial submodule.

If M is a irreducible Z-module, then any non zero m in M must generate M. This means there must be an integer n such that $nm = m$..(n times)..$m = e$. Thus, the irreducible Z modules are those modules whose base group is cyclically generated by any non element item. □

Problem 3.11. Show that if M_1 and M_2 are irreducible R-modules, then any nonzero R-module homomorphism from M_1 to $M2$ is an isomorphism. Deduce that if M is irreducible then $\mathrm{End}_R(M)$ is a division ring (this result is called Schur's Lemma). [Consider the kernel and the image.]

Proof. Let $\phi : M_1 \to M_2$ be a homomorphism from M_1 to M_2. Suppose that $m \in \ker(\phi)$ and $m \neq 0$. From exercise 9, we know that any nonzero element of M_1 generates the entire module. So there must be an $r \in R$ such that $rm \notin \ker(\phi)$. But this is a contradiction since $0 \neq \phi(rm) = r\phi(m) = r0 = 0$. Hence the kernel of ϕ is precisely the element 0 and ϕ is therefore an isomorphism.

It has been established that $\mathrm{End}_R(M)$ is a ring, with multiplication being defined as function composition. If $\psi \in \mathrm{End}_R(M)$, then by the prior proof, ψ is an isomorphism. Thus, it has an inverse ψ^{-1}. Multiplying these elements in $\mathrm{End}_R(M)$ lends $\psi^{-1} \circ \psi = \psi \circ \psi^{-1} = 1$ the identity element on $\mathrm{End}_R(M)$. Therefore, every element of $\mathrm{End}_R(M)$ has a multiplicative inverse and it is a division ring. □

Problem 3.13. Let R be a commutative ring and let F be a free R-module of finite rank. Prove the following isomorphism of R-modules: $\mathrm{Hom}_R(F, R) \cong F$.

Proof. Define $\{a_0, ..., a_n\}$ as the finite generators of F. Let $\phi \in \mathrm{Hom}_R(F, R)$, and define a map π between ϕ and F as $\pi(\phi) = \phi(a_1)a_1 + ... + \phi(a_n)a_n$. This function is a homomorphism since

$$\pi(\psi + r\phi) = \psi(a_1) + ... + \psi(a_n) + r(\phi(a_1) + ... + \phi(a_n)) = \pi(\psi) + r\pi(\phi)$$

The kernel of π is also only the zero function of $\mathrm{Hom}_R(F, R)$, since any other function would send some a_i to a nonzero r. Therefore, π is an isomorphism. □

Problem 3.15. An element $e \in R$ is called a central idempotent if $e^2 = e$ and $er = re$ for all $r \in R$. If e is a central idempotent in R, prove that $M = eM \oplus (1 - e)M$. [Recall Exercise 14 in Section 1.]

Proof. Exercise 14 asked us to prove that zM is a submodule of M for any z in the center of R. This is done by noticing that $zm_1 + r(zm_2) = e(m_1 + rm_2)$.

We have that both e and 1 (so by extent $1 - e$) are in the center of M. So eM and $(e - 1)M$ are both submodules of M.

For any r in R, either e factors out of r, in which case $r(1-e)M = r'(e-e^2)M = r'(e-e)M = r'0M = 0$. Or, e does not factor out of r, in which case $r(1 - e)M \cap eM = \emptyset$. So, $(1 - e)M$ and eM only overlap at 0.

We can get any $m \in M$ from the direct sum $eM \oplus (1-e)M$ with the element $em + (1-e)m = m$. Hence, $M = eM \oplus (1 - e)M$ spans M. □

Problem 3.17. In the notation of the preceding exercise, assume further that the ideals $A_t, ..., A_k$ are pairwise comaximal (i.e., $A_i + A_j = R$ for all $i \neq j$). Prove that $M/(A_1...A_k)M \cong M/A_1M \times ... \times M/A_kM$. [See the proof of the Chinese Remainder Theorem for rings in Section 7.6.]

Proof. The previous exercise proved that the map $M \to M/A_1M \times ... \times M/A_kM$ defined by $m \to (m + A_1M, ..., m + A_kM)$ is a homomorphism with kernel $A_1M \cap ... \cap A_kM$.

We can extend this map as one between $M/(A_1...A_k)M$ and $M/A_1M \times ... \times M/A_kM$. Doing so preserves the same kernel. And, we find that the kernel $A_1M \cap ... \cap A_kM$ is percisly the 0 element of $M/(A_1...A_k)M$. Therefore, the homomorphism becomes an isomorphism between these two modules. □

Problem 3.19. Show that if M is a finite abelian group of order $a = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ then, considered as a $\mathbb{Z}$-module, M is annihilated by (a), the p_i-primary component of M is the unique Sylow p_i-subgroup of M and M is isomorphic to the direct product of its Sylow subgroups.

Proof. Any element of M must have order dividing a. Say, $m \in M$ has order p and $p = ap'$. Then p is in (a) and $m^p = m^{p'} m^a = m^{p'} 0 = 0$. Thus, (a) annihilated by (a).

If m is in the p_i-primary component of M, then $m^{p_i^{\alpha_i}} = 0$. Thus, m has order $p_i^{\alpha_i}$ and is in the Sylow p_i subgroup of M.

The Sylow subgroups of M are pairwise disjoint. So, by proposition 5, M is isomorphic to the direct product of its Sylow subgroups. □

Problem 3.21. Let I be a nonempty index set and for each $i \in I$ let N_i be a submodule of M. Prove that the following are equivalent:

(i) the submodule of M generated by all the N_i's is isomorphic to the direct sum of the $N_i's$.

(ii) if $\{i_1, i_2, ..., i_k\}$ is any finite subset of I then $N_{i_1} \cap (N_{i_2} + \cdots + N_{i_k}) = 0$

(iii) if $\{i_1, i_2, ..., i_k\}$ is any finite subset of I then $N_1 + \cdots + N_k = N_1 \oplus \cdots \oplus N_k$

(iv) for every element x of the submodule of M generated by the N_i's there are unique elements $a_i \in N_i$ for all $i \in I$ such that all but a finite number of the a_i are zero and x is the (finite) sum of the a_i.

Proof. This is just proposition 5 with the added necessity of the axiom of choice. □

Problem 3.23. Show that any direct sum of free R-modules is free.

Proof. Let $F_1, ..., F_k$ all be free R-modules over the respective sets $A_1, ..., A_k$. Any element of their direct product can be decomposed into $x_1 + ... + x_k$ where $x_i \in F_i$. We can further decompose this into $r_{1,1}a_{1,1} + ... + r_{1,m}a_{1,m} + ... + r_{k,1}a_{k,1} + ... + r_{k,n}a_{k,n}$, where $r_{i,j} \in R$ and $a_{i,j} \in A_i$. Hence, any element of the direct product of the free groups is free over the set $A_1 \cup ... \cup A_k$. □

Problem 3.25. In the construction of direct limits, Exercise 8 of Section 7.6, show that if all A_i are R-modules and the maps ρ_{ij} are R-module homomorphisms, then the direct limit $A = \lim A_i$ may be given the structure of an R-module in a natural way such that the maps $\rho_i : A_i \to A$ are all R-module homomorphisms. Verify the corresponding universal property (part (e)) for R-module homomorphisms $\phi_i : A_i \to C$ commuting with the ρ_{ij}.

Proof. Let a and b be two elements of A. Define ra as $\rho_{ij}(ra') = r\rho_{ij}(a')$ for all $a' \in a$ and define $a + b$ as $\rho_{ik}(a') + \rho_{ik}(b')$ for all $a' \in a$ and $b' \in b$ where $i, j \leq k$. The R-module structure of the A_i ensures that we have defined a R-module on A.

For any element a in A, define the map $\phi : A \to C$ as $\phi(a) = \{a' | \phi_j \circ \rho_{ij}(a') = a\}$. This map is a homomorphism since

$$\phi(a+rb) = \{x | \phi_j \circ \rho_{ij}(x) = a + rb\} = \{a' + rb' | \phi_j \circ \rho_{ij}(a' + rb') = a + rb\} = \{a' | \phi_j \circ \rho_{ij}(a') = a\} + r\{b' | \phi_j \circ \rho_{kj}(b') = b\}$$

for $r \in R$ and $a' \in a, b' \in b$. (//TODO make this more formal and fleshed out; prove uniqueness.) □

Problem 3.27. *(Free modules over noncommutative rings need not have a unique rank)* Let M be the $\mathbb{Z}$-module $Z \times Z \times ...$ of Exercise 24 and let R be its endomorphism ring, $R = \mathrm{End}_{\mathbb{Z}}(M)$ (cf. Exercises 29 and 30 in Section 7.1). Define $\phi_1, \phi_2 \in R$ by

$$\phi_1(a_1, a_2, a_3, ...) = (a_1, a_3, a_5, ...)$$

$$\phi_1(a_1, a_2, a_3, ...) = (a_2, a_4, a_6, ...)$$

(a) Prove that $\{\phi_1, \phi_2\}$ is a free basis of the left R-module R. [Define the maps ψ_1 and ψ_2 bY $\psi_1(a_1, a_2, ...) = (a_1, 0, a_2, 0, ...)$ and $\psi_2(a_1, a_2, ...) = (O, a_1, 0, a_2, ...)$. Verify that $\phi_i\psi_i = 1, \phi_1\psi_2 = 0 = \phi_2\psi_1$ and $\psi_1\phi_1 + \psi_2\phi_2 = 1$. Use these relations to prove that ϕ_1, ϕ_2 are independent and generate R as a left R-module.]

(b) Use (a) to prove that $R \cong R^2$ and deduce that $R \cong R^n$ for all $n \in \mathbb{Z}^+$.

Definition 1. *Tensor Products Let S be a ring and N be a left R-module. In the free group $F(S \times N)$, define the subgroup H of the free group as all elements of the form $(s_1+s_2, n)-(s_1, n)-(s_2, n), (s, n_1+n_2)-(s, n_1)-(s, n_2), (sr, n)-(s, rn)$ where $s, s_1, s_2 \in S, n, n_1, n_2 \in N, r \in R$. Then the tensor product $S \otimes_R N$ is defined as $F(S \times N)/H$*

Theorem 8. Let R be a subring of S, let N be a left R-module and let $\iota : N \to S \otimes_R N$ be the R-module homomorphism defined by $\iota(n) = 1 \oplus n$. Suppose that L is any left S module (hence also an R-module) and that $\varphi : N \to L$ is an R-module homomorphism from N to L. Then there is a unique S-module homomorphism $\phi : S \otimes_R N \to L$ such that ϕ factors through ϕ, i.e., $\phi = \varphi \circ \iota$ and the diagram

$$\begin{tikzcd} N \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & S \otimes_R N \arrow[d, "\phi"] \\ & L \end{tikzcd}$$

commutes. Conversely, if $\phi : S \otimes_R N \to L$ is an S-module homomorphism then $\phi = \iota \circ \varphi$ is an R-module homomorphism from N to L.

Proof. □

Problem 4.1. Let $f : R \to S$ be a ring homomorphism from the ring R to the ring S with $f(1_R) = 1_S$. Verify the details that $sr = sf(r)$ defines a right R-action on S under which S is an (S, R)-bimodule.

Proof. In virtue of S being a ring, we already have that $sr = sf(s) \in S$ is a left S-module. So we only need to show that it is also a right R-module by proving part 2 of the definition of modules on page 337.

For a, we have that $s(r_1 + r_2) = sf(r_1 + r_2) = sf(r_1) + sf(r_2) = sr_1 + sr_2$. For part b, we have $s(r_1r_2) = sf(r_1r_2) = sf(r_1)f(r_2) = s(r_1)(r_2)$. For c, we have $(s_1 + s_2)r = (s_1 + s_2)f(r) = s_1f(r) + s_2f(r)$. Lastly, for part d, we have $s1_R = sf(1_R) = s1_S = s$. Together, these conditions imply that the set is a right R-module and is hence an (S, R)-bimodule. □

Problem 4.3. Show that $\mathbb{C} \otimes_\mathbb{R} \mathbb{C}$ and $\mathbb{C} \otimes_\mathbb{C} \mathbb{C}$ are both left $\mathbb{R}$-modules but are not isomorphic as $\mathbb{R}$-modules.

Proof. In virtue of the fact that $\mathbb{C}$ is a left $\mathbb{R}$-module, both tensor products are left $\mathbb{R}$-modules.

Suppose ϕ is an isomorphism between $\mathbb{C} \otimes_\mathbb{R} \mathbb{C}$ and $\mathbb{C} \otimes_\mathbb{C} \mathbb{C}$. Then $\phi((1, i) - (i, 1)) = (1, i) - (i, 1) = (1, i) - (1, i) = 0$. Which is a contradiction because on $\mathbb{C} \otimes_\mathbb{R} \mathbb{C}$, $(1, i) - (i, 1) \neq 0$. So, there is no such isomorphism. □

Problem 4.5. Let A be a finite abelian group of order n and let p^k be the largest power of the prime p dividing n. Prove that $\mathbb{Z}/p^k\mathbb{Z} \otimes_\mathbb{Z} A$ is isomorphic to the Sylow p-subgroup of A.

Proof. Let ϕ be the homomorphism from A to $\mathbb{Z}/p^k\mathbb{Z} \otimes_\mathbb{Z} A$ defined by $\phi(a) = 1 \otimes a$.

Let a be an element of A and $|a| = m$. If a is not in the Sylow p^k-subgroup of A, then and $(p^k, m) = 1$. So, by Bezout's lemma, there are numbers x and y such that $mx + p^ky = 1$. Therefore, $\phi(a) = 1 \otimes a = mx \otimes a = 1 \otimes xma = 1 \otimes 0$.

On the other hand, if a is in the Sylow p^k-subgroup of A, then $(p^k, m) = m$, then $1 \otimes a \neq 0$.

Thus, a is in the kernel of the homomorphism, if and only if a is not in the Sylow p^k-subgroup of A. So we have $\mathbb{Z}/p^k\mathbb{Z} \otimes_\mathbb{Z} A \cong A/\ker(\phi)$, which is of course isomorphic to the Sylow p^k-subgroup of A. □

Problem 4.7. If R is any integral domain with quotient field Q and N is a left R-module, prove that every element of the tensor product $Q \otimes_R N$ can be written as a simple tensor of the form $(1/d) \otimes n$ for some nonzero $d \in R$ and some $n \in N$.

Proof. Let $a/d \otimes b$ be any element of $Q \otimes_R N$, then we can use the distributive property of tensor products to yield

$$a/d \otimes b = 1/d \otimes ab$$

Which is obviously the desired format. Since every element of $Q \otimes_R N$ has the form $a/d \otimes b$, we are done. $\square$