# Exercises from Dummit and Foote Chapter 14 on Galois Theory

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**Problem 1.1.** **(a)** Show that if the field $K$ is generated over $F$ by the elements $\alpha_1, ..., \alpha_n$ then an automorphism $\sigma$ of $K$ fixing $F$ is uniquely determined by $\sigma(\alpha_1), ..., \sigma(\alpha_n)$. In particular show that an automorphism fixes $K$ if and only if it fixes a set of generators for $K$.

**(b)** Let $G \leq \mathrm{Gal}(K/F)$ b e a subgroup of the Galois group of the extension $K/F$ and suppose $\sigma_1, ..., \sigma_k$ are generators for $G$. Show that the subfield $E/F$ is fixed by $G$ if and only if it is fixed by the generators $\sigma_1, ..., \sigma_k$.

*Proof.* (a) Let $\sigma$ be any automorphism on $K$ fixing $F$. Then, for any $k = a_0 + a_1\alpha_1 + ... + a_n\alpha_n$ in $K$, $\sigma(k) = \sigma(a_0)+\sigma(a_1)\sigma(\alpha_1)+...+\sigma(a_n)\sigma(\alpha_n)$. Using the fact that $\sigma$ fixes $F$, we have $\sigma(k) = a_0+a_1\sigma(\alpha_1)+...+a_n\sigma(\alpha_n)$. Hence the image of any $k \in K$ on $\sigma$ is uniquely determined by $\sigma(\alpha_1), ..., \sigma(\alpha_n)$. From this, it is obvious that $\sigma$ fixes $K$ if it fixes the generators for $K$. □

*(b).* Denote the generators of $E$ over $F$ by $\alpha_1, ..., \alpha_m$. Suppose $G$ fixes $E/F$. From part (a), this is true if and only if $\sigma_i(\alpha_j) = \alpha_j$ for all $i \in [1, k], j \in [1, m]$. Hence, any element of a $E/F$ is fixed by any element of $G$. □

**Problem 1.3.** Determine the fixed field of complex conjugation on $\mathbb{C}$.

*Proof.* Complex conjugation is the function $\sigma : a + bi \mapsto a - bi$, which obviously fixes $a$. Hence, the fixed field of complex conjugation is $\mathbb{R}$ the real numbers. □

**Problem 1.5.** Determine the automorphisms of the extension $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}(\sqrt{2})$ explicitly.

*Proof.* There is only one basis element to this extension, namely $\sqrt[4]{2}$. Since $-\sqrt[4]{2} \neq \sqrt[4]{2}$, the automorphism $\sigma : a + b\sqrt[4]{2} \mapsto a - b\sqrt[4]{2}$ is not the identity. Hence, the automorphisms of this extension are $\{1, \sigma\}$. □

**Problem 1.7.** This problem determines $\mathrm{Aut}(\mathbb{R}/\mathbb{Q})$.

**(a)** Prove that any $\sigma \in \mathrm{Aut}(\mathbb{R}/\mathbb{Q})$ takes squares to squares and takes positive reals to positive reals. Conclude that $a < b$ implies $\sigma a < \sigma b$ for every $a, b \in \mathbb{R}$ Conclude that $a < b$ implies $\sigma a < \sigma b$ for every $a, b \in \mathbb{R}$.

**(b)** Prove that $-\frac{1}{m} < a - b < \frac{1}{m}$ implies $-\frac{1}{m} < \sigma a - \sigma b < \frac{1}{m}$ for every positive integer $m$. Conclude that $\sigma$ is a continuous map on $\mathbb{R}$.

**(c)** Prove that any continuous map on $\mathbb{R}$ which is the identity on $\mathbb{Q}$ is the identity map, hence $\mathrm{Aut}(\mathbb{R}/\mathbb{Q}) = 1$.

*Proof.* (a) Let $\sigma$ be an automorphism on $\mathbb{R}/\mathbb{Q}$. Suppose $x$ is a real square. Then, $x = p^2$ for real number $p$. Hence, we have $\sigma(x) = \sigma(p^2) = \sigma(p)\sigma(p)$. Thus, $\sigma$ sends squares to squares.

Let $y$ be any positive real number. Since $y$ is positive, $\sqrt{y}$ is real. From the first part of this proof, we know that $\sigma(y) = \sigma(\sqrt{y}\sqrt{y}) = q^2$ for some real number $q$. Since we are limited to the real numbers, $q^2$ is positive. Hence, $\sigma(y)$ is positive.

For any $a, b \in \mathbb{R}$, $a < b$ implies $0 < b - a$. Hence, from the prior paragraph, $0 < \sigma(b) - \sigma(a)$. Adding $\sigma(a)$ to both sides yields $\sigma(a) < \sigma(b)$. Note that setting $b = 0$ proves that $\sigma$ sends negatives to negatives. □

*(b).* Suppose $a, b$ are real numbers such that $-\frac{1}{m} < a - b < \frac{1}{m}$ for some positive integer $m$.

Since $m$ is an integer, it can be rewritten as $\Sigma_{i=0}^{m} 1$. Hence, $\sigma(m) = \Sigma_{i=0}^{m}\sigma(1) = \Sigma_{i=0}^{m} 1 = m$.

We can rewrite the above inequality as $-1 < m(a-b) < 1$. Which is the same as having $m(a-b)-1$ is negative and $m(a-b)+1$ is positive. From part (a), we know $\sigma$ sends positives to positives and negatives to negatives. Hence, $\sigma(m(a-b)-1) = m(\sigma(a)-\sigma(b))-1$ is negative and $\sigma(m(a-b)-1) = m(\sigma(a)-\sigma(b))+1$ is positive. Which of course implies $-\frac{1}{m} < \sigma(a) - \sigma(b) < \frac{1}{m}$

To show that $\sigma$ is continuous, let $x$ be any real number and let $\epsilon > 0$. We can find a natural number $N$ such that $\frac{1}{N} < \epsilon$. Then, for any $x_0$ such that $|x - x_0| < \frac{1}{N}$, we have $|\sigma(x) - \sigma(x_0)| < \frac{1}{N} < \epsilon$. Hence, $\sigma$ is continuous. $\square$

*(c).* Suppose $\sigma \in \mathrm{Aut}(\mathbb{R}/\mathbb{Q})$ fixes $\mathbb{Q}$. Let $x$ be any real number. Then by the density of the rationals in $\mathbb{R}$, for any $\epsilon > 0$, there exists some $q \in \mathbb{Q}$ such that $|x - q| < \epsilon$ Hence, $|\sigma(x-q)| = |\sigma(x) - q| < \epsilon$ which is only possible if $\sigma(x) = x$. Thus, any such $\sigma$ mus be the identity function. Therefore, $\mathrm{Aut}(\mathbb{R}/\mathbb{Q}) = 1$. $\square$

**Problem 1.9.** Determine the fixed field of the automorphism $t \mapsto t+1$ of $k(t)$.

*Proof.* Any element of $k(t)$ will have the form $\frac{\Sigma a_i t^i}{\Sigma b_i t^i}$ with $\gcd(\Sigma a_i x^i, \Sigma b_i x^i) = 1$. Suppose we have an element such that $\frac{\Sigma a_i (t+1)^i}{\Sigma b_i (t+1)^i} = \frac{\Sigma a_i t^i}{\Sigma b_i t^i}$. Then, $\frac{\Sigma a_i (t+1)^i}{\Sigma b_i (t+1)^i} - \frac{\Sigma a_i t^i}{\Sigma b_i t^i} = 0$ and since both fractions remain irreducible, we would have $\Sigma b_i (t+1)^i = \Sigma b_i t^i$. Thus, we would also have $\Sigma a_i (t+1)^i = \Sigma a_i (t)^i$. Hence, the fixed field of $k(t)$ is precisely the set of rational functions whose numerators and denominators are both fixed by the automorphism. $\square$