Exercises from Chapter 9

Wesley Basener

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Problem 1.1. Let $p(x, y, z) = 2x^2y - 3xy^3z + 4y^2z^5$ and $q(x, y, z) = 7x^2 + 5x^2y^3z^4 - 3x^2z^3$ be polynomials in $\mathbb{Z}[x, y, z]$.

(a) Write each p and q as a polynomial in x with coefficients in $\mathbb{Z}[y, z]$.

(b) Find the degree of each of p and q.

(c) Find the degree of p and q in each of the three variables x, y, and z.

(d) Compute pq and find the degree of pq in each of the three variables x, y, and z.

(e) Write pq as a polynomial in the variable z with coefficients in $\mathbb{Z}[x, y]$

Proof. For part a, $p = (2y)x^2 - (3y^3z)x + (4y^2z^5)x^0$ and $q = (7 + 5y^3z^4 - 3z^3)x^2$. For part b, the degree of p is the degree of the last term $2 + 5 = 7$ and the degree of q is the degree of the center second term $2 + 3 + 4 = 9$. For part c, x, y, z degrees of p are 2, 3,and 5 respectively and for q they are $2, 3$, and 4 respectively. For part d,

$$pq = (2x^2y - 3xy^3z + 4y^2z^5)(7x^2 + 5x^2y^3z^4 - 3x^2z^3)$$

$$= 14x^4y - 21x^3y^3z - 6x^4yz^3 + 9x^3y^3z^4 + 10x^4y^4z^4 + 28x^2y^2z^5 - 15x^3y^6z^5 - 12x^2y^2z^8 + 20x^2y^5z^9$$

The degrees of x, y, and z are $4, 6$, and 9 respectively. Lastly, for part e, we have

$$(20x^2y^5)z^9 - (12x^2y^2)z^8 + (28x^2y^2 - 15x^3y^6)z^5 + (9x^3y^3 + 10x^4y^4)z^4 - (6x^4y)z^3 - (21x^3y^3)z + (14x^4y)z^0$$

□

Problem 1.2. Repeat the preceding exercise under the assumption that the coefficients are of p and q are in $\mathbb{Z}/3\mathbb{Z}$.

Proof. We can start by rewriting p and q's coefficients in $\mathbb{Z}/3\mathbb{Z}$.

$$p = 2x^2y + y^2z^5 \text{ and } q = x^2 + 2x^2y^3z^4$$

For part a, we have $p = (2y)x^2 + (y^2z^5)x^0$ and $q = (1 + 3y^3z^4)x^2$. For part b, the degree of p is 7 and the degree of q is 9. For part c, the degree of p in x, y, and z is $2, 2$, and 5 respectively and for q it is $2, 3$, and 4 respectively. For part d,

$$pq = 2x^4y + x^4y^4z^4 + x^2y^2z^5 + 2x^2y^5z^9$$

The degrees of pq in x, y, and z are $4, 6$, and 9 respectively. Finally, for part e,

$$pq = (2x^2y^5)z^9 + (x^2y^2)z^5 + (x^4y^4)z^4 + (2x^4y)z^0$$

□

Problem 1.3. If R is a commutative ring and and $x_1, x_2, ..., x_n$ are independent variables over R, prove that $R[x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)}]$ is isomorphic to $R[x_1, x_2, ..., x_n]$ for any permutation of $\{1, 2, ..., n\}$.

Proof. Any element of $R[x_1, x_2, ..., x_n]$ will have the form

$$\alpha = (\Sigma_{i_1=0}^{m}(\Sigma_{i_2=0}^{m}...(\Sigma_{i_n=0}^{m}\alpha_{i_1,i_2,...,i_n}x_1^{i_1}x_2^{i_2}...x_n^{i_n})...))$$

Let α and β be two such polynomials and let $\phi : R[x_1, x_2, ..., x_n] \to R[x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)}]$ be the variable permutation function. Then, for addition,

$$\phi(\alpha) + \phi(\beta) =$$

$$\phi((\Sigma_{i_1=0}^{m}(\Sigma_{i_2=0}^{m}...(\Sigma_{i_n=0}^{m}\alpha_{i_1,i_2,...,i_n}x_1^{i_1}x_2^{i_2}...x_n^{i_n})...))) + \phi((\Sigma_{i_1=0}^{m}(\Sigma_{i_2=0}^{m}...(\Sigma_{i_n=0}^{m}\beta i_1, i_2, ..., i_n x_1^{i_1}x_2^{i_2}...x_n^{i_n})...)) =$$

$$\phi((\Sigma_{i_1=0}^{m}(\Sigma_{i_2=0}^{m}...(\Sigma_{i_n=0}^{m}\alpha_{i_1,i_2,...,i_n}x_{\pi(1)}^{i_1}x_{\pi(2)}^{i_2}...x_{\pi(n)}^{i_n})...))) + \phi((\Sigma_{i_1=0}^{m}(\Sigma_{i_2=0}^{m}...(\Sigma_{i_n=0}^{m}\beta i_1, i_2, ..., i_n x_{\pi(1)}^{i_1}x_{\pi(2)}^{i_2}...x_{\pi(n)}^{i_n})...))) =$$

$$\phi((\Sigma_{i_1=0}^{m}(\Sigma_{i_2=0}^{m}...(\Sigma_{i_n=0}^{m}(\alpha_{i_1,i_2,...,i_n} + \beta i_1, i_2, ..., i_n)x_{\pi(1)}^{i_1}x_{\pi(2)}^{i_2}...x_{\pi(n)}^{i_n})...))) =$$

$$\phi(\alpha + \beta)$$

Hence, the function satisfies the homomorphism condition on addition.

For multiplication, the coefficient of the term $x_1^{i_1}x_2^{i_2}...x_n^{i_n}$ in $\phi(\alpha \cdot \beta)$ will be the sum of all $\alpha_{k_1,k_2,...,k_n} \cdot \beta_{j_1,j_2,...,j_n}$ where $k_1 + j_1 = i_{\pi^{-1}(1)}, k_2 + j_2 = i_{\pi^{-1}(2)}, ..., k_n + j_n = i_{\pi^{-1}(n)}$ are all true. The coefficients of the $x_1^{i_1}x_2^{i_2}...x_n^{i_n}$ term in $\phi(\alpha) \cdot \phi(\beta)$ will be the sum of all $\alpha_{k_1,k_2,...,k_n} \cdot \beta_{j_1,j_2,...,j_n}$ where $k_{\pi(1)} + j_{\pi(1)} = i_1, k_{\pi(2)} + j_{\pi(2)} = i_2, ..., k_{\pi(n)} + j_{\pi(n)} = i_n$ are all true. But $k_{\pi(l)} + j_{\pi(l)} = i_l$ is true for all l in $[n]$ if and only if $k_l + j_l = i_{\pi^{-1}(l)}$ is also true for all l in $[n]$. So $\phi(\alpha \cdot \beta)$ and $\phi(\alpha) \cdot \phi(\beta)$ have precisely the same coefficients.

Since the set of units in $R[x_1, x_2, ..., x_n]$ are the set of units in R, and ϕ is the identity element on R, $\phi(1_{R[x_1,x_2,...,x_n]}) = 1_{R[x_1,x_2,...,x_n]}$.

These three cases prove that ϕ is a homomorphism on $R[x_1, x_2, ..., x_n]$. To show that it is an isomorphism, notice first that for r in R, $\phi(rx_1^{i_1}x_2^{i_2}...x_n^{i_n}) = rx_{\pi(1)}^{i_1}x_{\pi(2)}^{i_2}...x_{\pi(n)}^{i_n}$ is nonzero if and only if r is nonzero. Second, recall that ϕ is the identity function on R. Hence, the inverse of the kernal of ϕ is the 0 element of R. Therefore, ϕ is an isomorphism. □

Problem 1.4. Prove that the ideals (x) and (x, y) are prime ideals in $\mathbb{Q}[x, y]$ but only the later ideal is maximal.

Proof. If the second ideal were not prime, then there would exist a, b in R such that $ab = cx^ny^m$ for nonzero c in R. But this would violate the closure of R under multiplication, because cx^ny^m is not in R. Thus, (x, y) is prime. The same result follows for the first ideal (x) by letting a, b, and c be in $R[y]$.

Since $(x) \subset (x, y)$, (x) is not maximal. For any $\alpha = \Sigma_{i=0,j=0}^{n,m}\alpha_{i,j}x^iy^j$, α is not in (x, y) if and only if $\alpha_{0,0} \neq 0$. For every such α, $(x, y) + (\alpha) = (\alpha_{0,0}) = (1) = \mathbb{Q}[x, y]$. So the only ideal containing (x, y) is $\mathbb{Q}[x, y]$. Hence, (x, y) is maximal.

□

Problem 1.5. Prove that (x, y) and $(2, x, y)$ are prime ideals in $\mathbb{Z}[x, y]$ but only the latter ideal is a maximal ideal.

Proof. There are no elements not equal to x, y, or 2 in $\mathbb{Z}[x,y]$ that multiply to equal x, y, or 2 respectively. So the ideals generated by these elements are prime.

Since $(x,y) \subset (2,x,y) \subset \mathbb{Z}[x,y]$, where all containment is proper, (x,y) is not maximal. For any polynomial $\alpha = \Sigma_{i=0,j=0}^{m,n} \alpha_{i,j} x^i y^j$ is not in $(2,x,y)$ if and only if $\alpha_{0,0}$ is not zero and is not divisible by 2. This means that there are a,b in $\mathbb{Z}$ such that $a\alpha_{0,0} + b2 = 1$. Hence, $(2,x,y) + (\alpha) = \mathbb{Z}[x,y]$. So there are no ideals containing $(2,x,y)$ aside from $\mathbb{Z}[x,y]$. Therefore, $(2,x,y)$ is a maximal ideal. $\square$

Problem 1.6. Prove that (x,y) is not principle in $\mathbb{Q}[x,y]$.

Proof. The gcd of x and y is 1. So, the only element that could generate (x,y) is 1, which would also generate the rest of $\mathbb{Q}[x,y]$. $\square$

Problem 1.7. Let R be a commutative ring with 1. Prove that a polynomial ring in more than one variable over R is not a PID.

Proof. For any $R[x,y]$, the ideal (x,y) is not principle. By induction, this is true for any polynomial ring with more than one variable. $\square$

Problem 1.8. Let F be a field and $R = F[x, x^2y, x^3y^2, ..., x^ny^{n-1}, ...]$ be a subring of the polynomial ring $F[x,y]$.

(a) Prove that the fields of fractions of R and $F[x,y]$ are the same.

(b) Prove that R contains an ideal that is not finitely generated.

Proof. For any f in the field F, $fx \cdot 1/x = f$ is in the fraction field of R and so are $x^2y \cdot 1/x^2 = y$ and x. Therfore, the fraction field is $(F,x,y) = F[x,y]$.

For part b, consider the ideal of elements fx^my^n, where $m > n+1$. Suppose $x^{m+2}y^m$, which is indeed part of the ideal, is not a generator of the ideal. Then this element must be divisible by an element of the ideal. This is impossible because there is no partition a_1, b_1 and $a_2, b2$ of the integers $m+2, m$ such that $a_1 > b_1 + 1$ and $a_2 > b_2$. Therefore, the ideal is not finitely generated. $\square$

Problem 1.9. Prove that a polynomial ring in infinitely many variables with coefficients in any commutative ring contains ideals that are not finitely generated.

Proof. In $R[x_1, x_2, ...]$, the ideal of all elements containing variables is not finitely generated, every variable x_i by itself is in the ideal, but is not divisible by other variables. $\square$

Problem 1.10. Prove that the ring $\mathbb{Z}[x_1, x_2, x_3, ...]/(x_1x_2, x_3x_4, x_5x_6, ...)$ contains infinitely many minimal prime ideals.

Proof. Consider the ideal $(x_{\beta_1}, x_{\beta_2}, ...)$, where β_1 is either 1 or 2, β_2 is either 3 or 4 and so on. This is a prime ideal, since each of its generators are prime. It contains the 0 ideal of the quotient $(x_1x_2, x_3x_4, x_5x_6, ...)$. And, removing any of its generators would keep it from containing the 0 ideal, so it is minimal. Since there are infinite possibilities of the combinations of βs, there are infinite such minimal ideals in the ring. $\square$

Problem 1.11. Show that the radical of the ideal $I = (x, y^2)$ in $\mathbb{Q}[x,y]$ is (x,y). Deduce that I is a primary ideal that is not a power of a prime ideal.

Proof. The radical of I is the ideal formed by the square root of all elements in I, if the square root exists. The only elements that have a square root in I are of the form q^2x^{2n}, q^2y^{2n}, and $q^2x^{2m}y^{2n}$. The square root of these elements is qx^n, qy^n, and qx^my^n, which is obviously generated by the elements x, y. Hence, $\text{rad}(I) = (x,y)$

The above works to show (x,y) is any root of I. Simply replace 2 with any other natural number > 0. Therefore, I, which is obviously primary, is not the power of any prime ideal. $\square$

Problem 1.12. Let $R = \mathbb{Q}[x,y,z]$ and let bars denote passage to $\mathbb{Q}[x,y,z]/(xy - z^2)$. Prove that $\overline{P} = (\overline{x}, \overline{y})$ is a prime ideal. Show that $\overline{xy} \in \overline{P}^2$ but that no power (This shows that $\overline{P}$ is a prime ideal whose square is not a primary ideal).

Proof. Since (x, y) is prime in $\mathbb{Q}[x, y, z]$, it is prime in $\mathbb{Q}[x, y, z]/(xy - z^2)$. Hence, $\overline{P}$ is a prime ideal.

In $\mathbb{Q}[x, y, z]/(xy - z^2)$, xy is equivalent to z^2. This can be seen by noticing that in $\mathbb{Q}[x, y, z]/(xy - z^2)$, $xy = xy + (-1) \cdot (xy - z^2) = xy - xy + z^2 = z^2$. Hence, $\overline{xy}$ is in $\overline{P}^2$.

To see that no power of y is in $\overline{P}^2$, we consider the quotient ring definition of $\overline{y}^n$, which is $y^n + (xy - z^2)$. We want to show that no element of the ideal $(xy - z^2)$ adds to y^n to create an element of $\overline{P}^2$. Suppose this is not true, then there must be some a in $\mathbb{Q}[x, y, z]/(xy - z^2)$ where $y^n + a(xy - z^2) \in \overline{P}^2 = (x^2, z^2)$. This would require either axy or az^2 to cancel y^n, which is not possible. So, there is no such a. Hence, no power of y is in $\overline{P}^2$. □

Problem 1.13. Prove that the rings $F[x, y]/(y^2 - x)$ and $F[x, y]/(y^2 - x^2)$ are not isomorphic for any field F.

Proof. For brevity, denote $F[x, y]/(y^2 - x)$ as A and $F[x, y]/(y^2 - x^2)$ as B.

Notice that $y^2 - x^2 = (y - x) \cdot (y + x)$. Since neither $(y - x)$ nor $(y + x)$ are in $(y^2 - x^2)$, $(y^2 - x^2)$ is not prime. So, B has zero divisors and is not an integral domain. $(y^2 - x)$ however is prime (see exercise 1.14 below). So A has no zero divisors and is an integral domain.

Suppose there is an isomorphism $\phi : B \to A$. Then since $(y - x) \cdot (y + x)$ is in the kernel of B, $\phi((y - x) \cdot (y + x))$ must be in the kernel of A. However $\phi((y - x) \cdot (y + x)) = \phi(y - x) \cdot \phi(x + y)$ is also in the kernel of A.

But since ϕ is an isomorphism and neither $(y - x)$ nor $(y + x)$ are in the kernel of B, neither $\phi((y - x))$ nor $\phi((y + x))$ can be in the kernel of A. This implies that A is not an integral domain, contradicting our earlier results. Therefore, such an isomorphism cannot exist, regardless of the field F. □

Problem 1.14. Let R be an integral domain and let i, j be relatively prime integers. Prove that the ideal $(x^i - y^j)$ is a prime ideal in $R[x, y]$. [Consider the ring homomorphism ϕ from $R[x, y]$ to $R[t]$ defined by mapping x to t^j and mapping y to t^i. Show that an element of $R[x, y]$ differs from an element in $(x^i - y^j)$ by a polynomial $f(x)$ of degree at most $j - 1$ in y and observe that the exponents of $\phi(x^r y^s)$ are distinct for $0 \leq s < j$.]

Proof. □