Exercises from Chapter 1

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Problem 1.1. Beginning with the formula for the sum of a geometric series, use differentiation to obtain the identity.

$$\sum_{n=0}^{\infty} ne^{-An} = \frac{e^{-A}}{(1-e^{-A})^2}$$

Solution. First, we integrate the left side to get the expression into geometric series form.

$$\int \sum_{n=0}^{\infty} ne^{-An} dA = \sum_{n=0}^{\infty} -e^{-An}$$

Next, recall that the sum of a geometric series is $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$. Using this fact, the previous result can be rewritten,

$$\frac{-1}{1-e^{A}}$$

Finally, we take the derivative with respect to A, to undo our previous integration.

$$\frac{d}{dA}\frac{-1}{1-e^{A}} = \frac{e^{-A}}{(1-e^{-A})^2}$$

□

Problem 1.2. In Planck's model of blackbody radiation, the energy in a given frequency ω of electromagnetic radiation is distributed randomly over all numbers of the form $n\hbar\omega$, where $n = 0, 1, 2, ...$ Specifically, the likelihood of finding energy $n\hbar\omega$ is postulated to be

$$p(E = n\hbar\omega) = \frac{1}{Z}e^{-\beta n\hbar\omega},$$
$$Z = \frac{1}{1-e^{-\beta n\hbar\omega}}$$

Where Z is a normalization constant, which is chosen so that the sum over n of the probabilities is 1. Here $\beta = \frac{1}{k_B T}$, where T is the temperature and k_B is Boltzman's constant. The expected value of the energy, denoted $< E >$, is defined to be

$$< E >= \frac{1}{Z}\sum_{n=0}^{\infty}(n\hbar\omega)e^{-\beta n\hbar\omega}.$$

(a) using exercise 1, show that

$$< E >= \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}.$$

(b) Show that $< E >$ behaves like $\frac{1}{\beta} = k_B T$ for small ω, but that $< E >$ decays exponentially as ω tends to infinity.

Proof. (a) From exercise 1, we can rewrite the sum term as

$$\frac{\hbar\omega e^{-\hbar\omega\beta}}{(1-e^{-\hbar\omega\beta})^2}$$

Multiply this by $\frac{1}{Z} = 1 - e^{-\hbar\omega\beta}$.

$$\begin{aligned} 1 - e^{-\hbar\omega\beta}\frac{\hbar\omega e^{-\hbar\omega\beta}}{(1-e^{-\hbar\omega\beta})^2} \\ = \frac{\hbar\omega e^{-\hbar\omega\beta}}{1-e^{-\hbar\omega\beta}} \\ = \frac{\hbar\omega}{e^{\hbar\omega\beta}-1} \end{aligned}$$

(b) Using the Taylor series expansion of e^x, we can rewrite $< E >$

$$\begin{aligned} \frac{\hbar\omega}{e^{\hbar\omega\beta}-1} \\ = \frac{\hbar\omega}{(1+\hbar\omega\beta+\sum_{n=2}^{\infty}\frac{(\hbar\omega\beta)^n}{n})-1} \\ = \frac{1}{\beta+\sum_{n=2}^{\infty}\frac{(\hbar\omega)^{n-1}\beta^n}{n}} \end{aligned}$$

as ω approaches 0, the sum disappears and the fraction approaches $\frac{1}{\beta}$.
It is easy to find the limit of $< E >$ as ω approaches ∞ using L'Hospital's rule

$$\frac{\hbar\omega}{e^{\hbar\omega\beta}-1} \overset{\text{H}}{=} \frac{\hbar}{\hbar\beta e^{\hbar\omega\beta}}$$

$\square$