

# SI231b: Matrix Computations

## Lecture 12: Eigenvalue Revealing Decomposition

Yue Qiu

[qiuyue@shanghaitech.edu.cn](mailto:qiuyue@shanghaitech.edu.cn)

School of Information Science and Technology  
ShanghaiTech University

Oct. 22, 2020

## Algebraic Multiplicity

- ▶ Characteristic polynomial  $p(\lambda) = \prod_{i=1}^n (\lambda - \lambda_i)$
- ▶ denote  $\mu_i$  as the number of repeated eigenvalues of  $\lambda_i$  ( $i = 1, \dots, k$ )

$$p(\lambda) = (\lambda - \lambda_1)^{\mu_1} (\lambda - \lambda_2)^{\mu_2} \cdots (\lambda - \lambda_k)^{\mu_k},$$

with  $\mu_1 + \mu_2 + \cdots + \mu_k = n$  and  $\lambda_i$  is distinct with  $\lambda_j$ .

- ▶  $\mu_i$  is called the **algebraic multiplicity** of the eigenvalue  $\lambda_i$

## Geometric Multiplicity

- ▶ every  $\lambda_i$  can have more than one eigenvector (scaling not counted)
- ▶ eigenspace  $\mathcal{E}_{\lambda_i}$  associated with  $\lambda_i$ ,  $\mathcal{E}_{\lambda_i} = \mathcal{N}(A - \lambda_i I)$
- ▶  $\gamma_i = \dim(\mathcal{E}_{\lambda_i})$  is called the **geometric multiplicity** of the eigenvalue  $\lambda_i$

**Fact:**  $\mu_i \geq \gamma_i$  for each  $\lambda_i$ .

- ▶ Diagonalization
- ▶ Similarity Transformation
- ▶ Schur Decomposition
- ▶ Eigenvalues of Hermitian Matrices

**Theorem 1:** An  $n \times n$  matrix  $A$  is nondefective if and only if it has an eigenvalue decomposition

$$A = V\Lambda V^{-1},$$

with  $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$  and the  $k$ -th column of  $V$  being the eigenvector  $v_k$  associated with  $\lambda_k$ .

*Hint:* you need the following lemma to prove the theorem

**Lemma 1:** Let  $A \in \mathbb{C}^{n \times n}$ , and suppose that the eigenvalues  $\lambda_1, \dots, \lambda_n$  are ordered such that  $\{\lambda_1, \dots, \lambda_k\}$ ,  $k \leq n$ , is the set of all distinct eigenvalues of  $A$ . Also, let  $v_i$  be *any* eigenvector associated with  $\lambda_i$ . Then  $v_1, \dots, v_k$  must be linearly independent.

From the theorem 1, another term for nondefective is diagonalizable.

# Properties of Eigenvalue Decomposition

If  $A$  admits an eigenvalue decomposition, the following properties can be shown (easily):

- ▶  $\det(A) = \prod_{i=1}^n \lambda_i$
- ▶  $\operatorname{tr}(A) = \sum_{i=1}^n \lambda_i$
- ▶ the eigenvalues of  $A^k$  are  $\lambda_1^k, \dots, \lambda_n^k$
- ▶  $A$  is nonsingular if and only if  $A$  does not have zero eigenvalues
- ▶ suppose that  $A$  is also nonsingular. Then,  $A^{-1} = V\Lambda^{-1}V^{-1}$

**Note:** the first three properties does not require the eigenvalue decomposition to prove.

# Similarity Transformation

For  $A \in \mathbb{R}^{n \times n}$  (or  $\mathbb{C}^{n \times n}$ ), if  $T \in \mathbb{R}^{n \times n}$  (or  $\mathbb{C}^{n \times n}$ ) is nonsingular, the map  $A \mapsto T^{-1}AT$  is called a similarity transformation of  $A$ .

**Theorem 2** If  $T$  is nonsingular, then  $A$  and  $T^{-1}AT$  have the same

- ▶ characteristic polynomial
- ▶ eigenvalues
- ▶ algebraic multiplicity
- ▶ geometric multiplicity

*Hint:* using characteristic polynomial to show.

Let  $A \in \mathbb{C}^{n \times n}$ , the Schur decomposition of  $A$  is given by

$$A = QTQ^H,$$

where  $Q$  is unitary ( $Q^H Q = I$ ), and  $T$  is upper-triangular.

**Property:** Since  $A$  and  $T$  are similar, the eigenvalues of  $A$  appear on the diagonal of  $T$ .

**Theorem 3:** Every square matrix  $A$  has a Schur decomposition.

**Hint:** applying induction to prove.

## Real Eigenvalues

Let  $A \in \mathbb{C}^{n \times n}$  be Hermitian ( $A = A^H$ ), then

1. the eigenvalues  $\lambda_1, \dots, \lambda_n$  of  $A$  are real
2. suppose that  $\lambda_i$ 's are ordered such that  $\{\lambda_1, \dots, \lambda_k\}$  is the set of all distinct eigenvalues of  $A$ . Also, let  $v_i$  be *any* eigenvector associated with  $\lambda_i$ . Then  $v_1, \dots, v_k$  must be orthonormal.

## Remark:

- ▶ the above results apply to real symmetric matrices, recall that

$$A = A^T \Rightarrow A = A^H.$$

## Corollary:

- ▶ for a real symmetric matrix, all eigenvectors  $v_1, \dots, v_n$  can be chosen as real



# Diagonalization of Hermitian Matrices

**Theorem 4:** Every Hermitian matrix  $A \in \mathbb{C}^{n \times n}$  has an eigenvalue decomposition given by

$$A = V\Lambda V^H,$$

where  $V \in \mathbb{C}^{n \times n}$  is unitary ( $V^H V = I$ ),  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$  with  $\lambda_i \in \mathbb{R}$  for all  $i$ . Also, if  $A \in \mathbb{R}^{n \times n}$  is symmetric,  $V$  is an orthogonal matrix.

*Hint:* can you use Schur decomposition to prove this?

**Remark:**

- ▶ does not require the assumption of  $\mu_i = \gamma_i$  for all  $\lambda_i$

**Corollary:**

- ▶ If  $A$  is Hermitian or real symmetric,  $\mu_i = \gamma_i$  for all  $\lambda_i$  (no. of repeated eigenvalues = no. of linearly independent eigenvectors)

# Eigenvalue Revealing Decomposition

Factorize a matrix to a form in which eigenvalues are explicitly displayed

- ▶ **Diagonalization**,  $A = V\Lambda V^{-1}$ , exists if and only if  $A$  is nondefective.
- ▶ **Schur decomposition**,  $A = QTQ^H$  always exists.
- ▶ **Jordan canonical form**,  $A = SJS^{-1}$  always exists (**will not be introduced in our lecture**), where

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{bmatrix}$$

with

$$J_i = \begin{bmatrix} \lambda_i & & & \\ & \lambda_i & & \\ & & \ddots & \\ & & & \lambda_i \end{bmatrix}, \quad \text{or} \quad J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

You are supposed to read

- ▶ Gene H. Golub and Charles F. Van Loan. *Matrix Computations*, Johns Hopkins University Press, 2013.

Chapter 7.1, 8.1