## Necessary And Sufficient Conditions For Existence of the LU Factorization of an Arbitrary Matrix.

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July 31, 1997

#### Abstract

If A is an n-by-n matrix over a field F ( $A \in M_n(F)$ ), then A is said to "have an LU factorization" if there exists a lower triangular matrix  $L \in M_n(F)$  and an upper triangular matrix  $U \in M_n(F)$  such that

$$A = LU$$
.

We give necessary and sufficient conditions for LU factorability of a matrix. Also simple algorithm for computing an LU factorization is given. It is an extension of the Gaussian elimination algorithm to the case of not necessarily invertible matrices. We consider possibilities

<sup>\*</sup>This research was partially supported by NSF REU grant DMS-96-19577 and by the College of William and Mary.

to factors a matrix that does not have an LU factorization as the product of an "almost lower triangular" matrix and an "almost upper triangular" matrix. There are many ways to formalize what almost means. We consider some of them and derive necessary and sufficient conditions. Also simple algorithms for computing of an "almost LU factorization" are given.

If A is an n-by-n matrix over a field F  $(A \in M_n(F))$ , then A is said to "have an LU factorization" if there exists a lower triangular matrix  $L \in M_n(F)$  and an upper triangular matrix  $U \in M_n(F)$  such that

$$A = LU$$
.

Historically, such factorizations have been extensively used in the solution of the linear system Ax = b, especially when a series of right hand sides b is presented. For, if A = LU, then each solution x is a solution to

$$Ux = y$$

for some solution y to

$$Ly = b$$
,

and if L is invertible, y is uniquely determined; and if U is invertible, any solution y to Ly = b will give a solution x to Ax = b via Ux = y. Thus, much of the discussion of LU factorization has occurred in the computational literature. However, LU factorization has also emerged as a useful theoretical tool.

Though an LU factorization is quite useful when it exists, unfortunately not every matrix has one. The simplest example is

$$A = \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right].$$

If  $A = LU = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$ , then L and U would have to be invertible because A is (i.e.  $l_{11}, u_{11}, l_{22}, u_{22} \neq 0$ ), but this would contradict  $l_{11}u_{11} = a_{11} = 0$ .

Which  $A \in M_n(F)$ , then, do have an LU factorization? We first observe a necessary condition on A by writing a supposed factorization in partitioned

form. Suppose

$$A = \left[ \begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right]$$

with  $A_{11}$  k-by-k, and that

$$A = \left[ \begin{array}{cc} L_{11} & 0 \\ L_{21} & L_{22} \end{array} \right] \left[ \begin{array}{cc} U_{11} & U_{12} \\ 0 & U_{22} \end{array} \right]$$

partitioned conformally. We then have

$$[A_{11}A_{12}] = L_{11}[U_{11}U_{12}]$$

and

$$\left[\begin{array}{c} A_{11} \\ A_{21} \end{array}\right] = \left[\begin{array}{c} L_{11} \\ L_{21} \end{array}\right] U_{11}$$

and

$$A_{11} = L_{11}U_{11}$$

We use well known inequalities involving ranks of products. If Z = XY with X having k columns and Y having k rows,

$$rankX + rankY - k \le rankZ \le min\{rankX, rankY\}$$

First applying the right hand inequality we have

$$rank[A_{11}A_{12}] \le rankL_{11}$$

and

$$rank \left[ \begin{array}{c} A_{11} \\ A_{21} \end{array} \right] \le rank U_{11}$$

Then applying the left hand inequality we get

$$rankL_{11} + rankU_{11} - k \leq rankA_{11}$$

Combining, we obtain

$$rank[A_{11}A_{12}] + rank \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} - k \le rankA_{11}$$

which we may write as

$$rankA_{11} + k \ge rank[A_{11}A_{12}] + rank\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}$$

this must hold for each k = 1, ..., n - 1. We can also write these conditions as

$$rankA[\{1...k\}] + k \ge rankA[\{1...k\}, \{1...n\}] + rankA[\{1...n\}, \{1...k\}]$$
 (1) for all  $k = 1, ..., n$ .

Our main result is that these conditions are also sufficient.

**Theorem 1** The matrix  $A = (a_{ij}) \in M_n(F)$  has an LU factorization iff it satisfies conditions (1).

*Proof.* Necessity was proved above. We will prove sufficiency by induction on n. In the case n=1 the theorem is trivially true because any 1-by-1 matrix has an LU factorization. In particular, if A=[0] we let L=[0] and U=[0]. Though there are many other LU factorizations of the zero matrix we choose this particular one for the purposes of theorem 5.

Suppose that conditions (1) are sufficient for  $n \leq m$ . We want to prove that conditions (1) are also sufficient for n = m + 1.

The idea of the proof is as follows. We want to find a lower triangular matrix L and an upper triangular matrix U such that A = LU. Suppose

$$A = \left[ \begin{array}{cc} a_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right]$$

in which  $a_{11}$  is the (1,1) entry of A, and that

$$A = LU = \begin{bmatrix} l_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} u_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}$$

partitioned conformally.

Block multiplication gives us

$$a_{11} = l_{11}u_{11} (2)$$

$$A_{21} = L_{21}u_{11} (3)$$

$$A_{12} = l_{11}U_{12} (4)$$

$$A_{22} = L_{21}U_{12} + L_{22}U_{22}. (5)$$

We will prove that it is possible to choose the first column of L and the first row of U in such a way that equations 2–4 are satisfied and additionally that the (n-1)-by-(n-1) matrix

$$B = A_{22} - L_{21}U_{12}$$

satisfies the conditions (1). We call B "a factor complement" of A. Then by the inductive hypothesis matrix B has an LU factorization or  $B = L_0U_0$  with  $L_0$  lower triangular and  $U_0$  upper triangular. We let

$$L_{22} = L_0$$

and

$$U_{22} = U_0.$$

Notice that a matrix L defined this way is lower triangular, and a matrix U defined this way is upper triangular. Also equations 2–5 are satisfied which implies that A = LU.

We will now describe how to choose the first column of L and the first row of U. We need to consider several cases.

Case 1:  $a_{11} \neq 0$ . In this case we let

$$L[\{1 \dots n\}, \{1\}] = A[\{1 \dots n\}, \{1\}]$$

and

$$U[\{1\}, \{1 \dots n\}] = a_{11}^{-1} A[\{1\}, \{1 \dots n\}]$$

which gives us

$$l_{11} = a_{11}$$

$$L_{21} = A_{21}$$

$$u_{11} = 1$$

$$U_{12} = a_{11}^{-1} A_{12}.$$

Equations (2)–(4) are obviously satisfied by this choice of the first column of L and the first row of U. It remains to prove that the matrix  $B = A_{22} - L_{21}U_{12}$  satisfies conditions (1).

Let us consider the matrix

$$C = A - \begin{bmatrix} a_{11} \\ L_{21} \end{bmatrix} \begin{bmatrix} 0 \ U_{12} \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ A_{21} & A_{22} - L_{21}U_{12} \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ A_{21} & B \end{bmatrix}.$$
 (6)

Matrix C satisfies conditions (1). Indeed, let us fix k  $(1 \le k \le n)$ . Then matrix  $C[\{1...k\}]$  can be obtained from matrix  $A[\{1...k\}]$  by applying several type-3 elementary column operations. Therefore

$$rankC[\{1 \dots k\}] = rankA[\{1 \dots k\}].$$

In particular,

$$rankC = rankA$$

Similarly,

$$rankC[\{1...k\}, \{1...n\}] = rankA[\{1...k\}, \{1...n\}]$$

and

$$rankC[\{1...n\}, \{1...k\}] = rankA[\{1...n\}, \{1...k\}].$$

Since A satisfies conditions (1) we conclude that C also satisfies conditions (1).

From the formula (6) it now follows that matrix B satisfies conditions (1). Indeed, let us fix k  $(1 \le k \le n-1)$ . Because  $c_{11} = a_{11} \ne 0$  and  $c_{12} = \cdots = c_{1,k+1} = 0$  we have

$$rankC[\{1 \dots n\}, \{2 \dots k+1\}] = rankC[\{1 \dots n\}, \{1 \dots k+1\}] - 1$$

Also because  $c_{12} = \cdots = c_{1,k+1} = 0$  we see that

$$rankC[\{1...n\}, \{2...k+1\}]] = rankC[\{2...n\}, \{2...k+1\}],$$

but 
$$C[\{2...n\}, \{2...k+1\}] = B[\{1...n-1\}, \{1...k\}]$$
 and therefore

$$rankB[\{1\dots(n-1)\},\{1\dots k\}] = rankC[\{1\dots n\},\{1\dots(k+1)\}] - 1 \quad (7)$$

Similarly,

$$rankB[\{1...k\}, \{1...(n-1)\}] = rankC[\{1...(k+1)\}, \{1...n\}] - 1(8)$$
$$rankB[\{1...k\}] = rankC[\{1...(k+1)\}] - 1$$
(9)

In particular,

$$rankB = rankC - 1 = rankA - 1$$

Also since C satisfies conditions (1) we have

$$rankC[\{1 \dots (k+1)\}] + (k+1) \ge rankC[\{1 \dots (k+1)\}, \{1 \dots n\}] + rankC[\{1 \dots n\}, \{1 \dots (k+1)\}]. (10)$$

From (7)–(10) we conclude that

$$rankB[\{1...k\}] + k \ge rankB[\{1...k\}, \{1...(n-1)\}] + rankB[\{1...(n-1)\}, \{1...k\}].$$

In other words B satisfies conditions (1).

Case 2:  $a_{11} = 0$ . If  $a_{11} = 0$ , then conditions (1) for k = 1 imply that either the first row or the first column of A or both are zero. We first consider the case when the first row of A is equal to zero but the first column is not zero. Let i be the smallest integer such that  $a_{1i} \neq 0$ . We let

$$L[\{1\dots n\}, \{1\}] = A[\{1\dots n\}, \{1\}]$$

and

$$U[\{1\}, \{1 \dots n\}] = a_{1i}^{-1} A[\{i\}, \{1 \dots n\}]$$

which gives us

$$l_{11} = 0$$

$$L_{21} = A_{21}$$

$$u_{11} = 1$$

$$U_{12} = a_{1i}^{-1} A[\{i\}, \{2...n\}]$$

Again equations (2)–(4) are obviously satisfied by this choice of the first column of L and the first row of U. It remains to prove that the matrix  $B = A_{22} - L_{21}U_{12}$  satisfies conditions (1).

Let us consider the matrix

$$C = A - \begin{bmatrix} 0 \\ L_{21} \end{bmatrix} \begin{bmatrix} 0 & U_{12} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} - L_{21}U_{12} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ A_{21} & B \end{bmatrix}.$$
 (11)

Exactly as in case 1 we prove that matrix C satisfies conditions (1), and

$$rankC = rankA$$

From the formula (11) it now follows that matrix B satisfies conditions (1). Indeed, let us fix k  $(1 \le k \le n-1)$ . Because  $c_{i1} = a_{1i} \ne 0$  and  $c_{i2} = \cdots = c_{i,k+1} = 0$  we have

$$rankC[\{1...n\}, \{2...k+1\}] = rankC[\{1...n\}, \{1...k+1\}] - 1$$

Also because  $c_{12} = \ldots = c_{1,k+1} = 0$  we see that

$$rankC[\{1...n\}, \{2...k+1\}] = rankC[\{2...n\}, \{2...k+1\}],$$

but 
$$C[\{2...n\}, \{2...k+1\}] = B[\{1...n-1\}, \{1...k\}]$$
 and therefore

$$rankB[\{1...n-1\},\{1...k\}] = rankC[\{1...n\},\{1...k+1\}] - 1$$
 (12)

If  $k+1 \ge i$  similar reasoning gives

$$rankB[\{1 \dots k\}, \{1 \dots (n-1)\}] = rankC[\{1 \dots (k+1)\}, \{1 \dots n\}] - 1(13)$$
$$rankB[\{1 \dots k\}] = rankC[\{1 \dots (k+1)\}] - 1$$
(14)

In particular,

$$rankB = rankC - 1 = rankA - 1$$

But if k + 1 < i we have

$$rankB[\{1...k\}, \{1...(n-1)\}] = rankC[\{1...(k+1)\}, \{1...n\}]$$
(15)  
$$rankB[\{1...k\}] = rankC[\{1...(k+1)\}]$$
(16)

Also since C satisfies conditions (1) we have

$$rankC[\{1...(k+1)\}] + (k+1) \ge rankC[\{1...(k+1)\}, \{1...n\}] + rankC[\{1...n\}, \{1...(k+1)\}]. (17)$$

From (12)–(17) we conclude that

$$rankB[\{1...k\}] + k \ge rankB[\{1...k\}, \{1...(n-1)\}] + rankB[\{1...(n-1)\}, \{1...k\}].$$

In other words B satisfies conditions (1).

We now consider the case when the first column of A is zero, but the first row is not zero. We let

$$C = A^T$$

Notice that C satisfies conditions (1) and also the first row of C is zero, but the first column of C is not zero. It was shown above that there are scalars  $m_{11}$ , and  $v_{11}$  and a column vector  $M_{21}$  and a row vector  $V_{12}$  such that

$$c_{11} = a_{11} = m_{11}v_{11}$$

$$C_{21} = A_{12}^{T} = M_{21}v_{11}$$

$$C_{12} = A_{21}^{T} = m_{11}V_{12}$$

and additionally that matrix

$$C_{22} - M_{21}V_{12}$$

satisfies conditions (1) and

$$rank(C_{22} - M_{21}V_{12}) = rankC - 1 = rankA - 1$$

We now let

$$l_{11} = v_{11}$$

$$L_{21} = V_{12}^{T}$$

$$u_{11} = m_{11}$$

$$U_{12} = M_{21}^{T}$$

Equations (2)–(4) are obviously satisfied by this choice of the first column of L and the first row of U. Additionally a factor complement of A

$$B = A_{22} - L_{21}U_{12} = (C_{22} - M_{21}V_{12})^T$$

satisfies conditions (1) and also

$$rankB = rank(C_{22} - M_{21}V_{12}) = rankA - 1$$

The last case we have to consider is when both the first row and the first column of A are equal to zero. If A is the zero matrix we let the first column of L and the first row of U to be equal zero. Equations (2)–(4) are obviously satisfied by this choice of the first column of L and the first row of U. Also matrix

$$B = A_{22} - L_{21}U_{12}$$

is the zero matrix and therefore obviously satisfies conditions (1). In this case

$$rankB = rankA = 0$$

If A is not the zero matrix let i be the smallest integer such that the ith column of A is not zero. Let us consider the matrix C such that the first column of C is equal to the ith column of A and columns 2 through n of C are equal to the corresponding columns of A. We want to prove that C satisfies conditions (1). Fix k  $(1 \le k \le n)$ . If k is less than i then because the first row of C is zero we have

$$rankC[\{1\dots k\},\{1\dots n\}] \le k-1$$

and because columns 2 through k of C are zero, but the first column of C is not zero we have

$$rankC[\{1\dots n\},\{1\dots k\}]=1$$

We conclude that

$$rankC[\{1...k\}] + k \ge rankC[\{1...k\}, \{1...n\}] + rankC[\{1...n\}, \{1...k\}]$$

which means that C satisfies condition (1) for index k.

If k is greater or equal than i, then because columns 2 through n of C are equal to the corresponding columns of A and the first column of C is equal to the ith column of A and also because the first column of A is zero we have

$$rankC[\{1\dots k\}] = rankA[\{1\dots k\}]$$

In particular

$$rankC = rankA$$

Also

$$rankC[\{1\ldots k\},\{1\ldots n\}] = rankA[\{1\ldots k\},\{1\ldots n\}]$$

and

$$rankC[\{1\dots n\},\{1\dots k\}] = rankA[\{1\dots n\},\{1\dots k\}]$$

Since A satisfies conditions (1) for index k, we conclude that C satisfies conditions (1) for index k. We already know that because C satisfies conditions (1) and the first row of C is zero, but the first column of C is not zero there

are scalars  $m_{11}$ , and  $v_{11}$  and a column vector  $M_{21}$  and a row vector  $V_{12}$  such that

$$c_{11} = 0 = m_{11}v_{11}$$

$$C_{21} = M_{21}v_{11}$$

$$C_{12} = A_{12} = m_{11}V_{12}$$

and additionally that matrix

$$C_{22} - M_{21}V_{12}$$

satisfies conditions (1) and

$$rank(C_{22} - M_{21}V_{12}) = rankC - 1 = rankA - 1$$

We now let

$$l_{11} = m_{11}$$

$$L_{21} = M_{21}$$

$$u_{11} = 0$$

$$U_{12} = V_{12}$$

Equations (2)–(4) are obviously satisfied by this choice of the first column of L and the first row of U. Additionally a factor complement of A

$$B = A_{22} - L_{21}U_{12} = C_{22} - M_{21}V_{12}$$

satisfies conditions (1) and also

$$rankB = rank(C_{22} - M_{21}V_{12}) = rankA - 1$$

Note that this case can also be treated by considering matrix C such that the first row of C is equal to the first nonzero row of A and such that rows 2 through n of C are equal to the corresponding rows of A.  $\Box$ 

We now prove the following well known result

Corollary 1 Let A be an n-by-n invertible matrix then A has an LU factorization iff all principal leading submatrices of A have full rank.

*Proof.* Since A is invertible we must have

$$rankA[\{1...k\}, \{1...n\}] = rankA[\{1...n\}, \{1...k\}] = k$$

for all k = 1, ..., n. Then by theorem (1) A has an LU factorization iff

$$rankA[\{1\dots k\}] = k$$

for  $k = 1, \ldots, n \square$ 

We now give pseudo-code description of the algorithm derived in theorem (1) for finding L and U.

```
Algorithm 1
Input n-by-n matrix A
for k from 1 to n
for i from 1 to n
for j from i to n
if a_{i,j} \neq 0
kth column of L:= jth column of A
kth row of U:=a_{i,j}^{-1} multiplied by ith row of A
GOTO 10
elseif a_{j,i} \neq 0
kth column of L:= ith column of A
kth row of U:=a_{j,i}^{-1} multiplied by jth row of A
GOTO 10
endif
endif
next j
next i
kth column of L:= the zero vector
kth row of U:=the zero vector
10 A=A-(kth column of L multiplied by kth row of U)
next k
```

We also give a verbal description of the algorithm:

1) Assign priorities to all possible positions in n-by-n matrix.

For all k from 1 to n repeat steps 2 through 4

- 2) Find a nonzero element of matrix A which has the smallest possible integer assigned to its position. If there are no nonzero elements (i.e if A is the zero matrix) then let kth column of L and kth row of U equal to zero and skip to step 4.
- 3) Suppose the element chosen in step 2 is in position (i,j) then let kth column of L equal to jth column of A and kth row of U equal to ith row of A divided by  $a_{i,j}$ .
  - 4) Let

$$A = A - L[\{1 \dots n\}, \{k\}] U[\{k\}, \{1 \dots n\}]$$

How to assign priorities?

We give a simple pseudo-code description and a 4-by-4 example

let counter=0
for i from 1 to n
for j from i to n
counter=counter+1
assign value of counter to (i,j) and (j,i) positions

next j next i

Here is how we would assign priorities to positions in 4-by-4 matrix

$$\left[\begin{array}{ccccc}
1 & 2 & 3 & 4 \\
2 & 5 & 6 & 7 \\
3 & 6 & 8 & 9 \\
4 & 7 & 9 & 10
\end{array}\right]$$

We may now state

**Theorem 2**  $A \in M_n(F)$  has an LU factorization iff algorithm (1) produces LU factorization of A.

*Proof. Necessity.* If A has an LU factorization then by theorem (1) it satisfies conditions (1). Then the algorithm will produce an LU factorization of A. Sufficiency is trivial.  $\square$ 

#### Note 1

Suppose that an n-by-n matrix A satisfies conditions (1) and additionally the first m rows of A are equal to zero and the first m columns of A are zero. If we apply the algorithm to obtain an LU factorization of A, then the first m rows of L will be equal to zero and the first m columns of U will be zero.

#### Note 2

It was shown in the proof of the theorem (1) that when we perform one step of the algorithm (i.e. go from k=p to k=p+1) the rank of modified A is less by one than the rank of A before modification, unless the rank of A before modification was already zero. So if we start with an n-by-n matrix A satisfying conditions (1) such that rank deficiency of A is m, then after (n-m) steps modified A will be the zero matrix. Therefore the last m columns of L and the last m row of U will be equal to zero (which means that the algorithm produces "full rank" factorization of A). In particular, if A has m rows of zeros or m columns of zeros, then the last m columns of L and rows of L are zero.

#### Note 3

By previous note the ranks of L and U are less or equal to (n-m). But they also must be greater or equal than the rank of A which is equal to (n-m). Therefore rankL = rankU = n - m.

**Definition 1** An n-by-n matrix K is said to be an almost lower triangular with m extra diagonals if a(i, j) = 0 whenever j > i + m. An almost upper triangular matrix with m extra diagonals is defined similarly.

**Definition 2** We say that an n-by-n matrix A fails conditions(1) by at most (no more than) m if

$$rankA[\{1...k\}]+k+m \ge rankA[\{1...k\},\{1...n\}]+rankA[\{1...n\},\{1...k\}]$$
  
for all  $k = 1,...,n$ .

Given an n-by-n matrix A we obtain (n+m)-by-(n+m) matrix C by bordering A with m columns of zeros on the left and m rows of zeros on the top.

We state a trivial but useful lemma

**Lemma 1** If A and C are as above then C satisfies conditions (1) iff A fails conditions (1) by no more than m.

We now prove

**Theorem 3**  $A \in M_n(F)$  can be written as

$$A = KW$$

where K is an almost lower triangular matrix with m extra diagonals and W is an almost upper triangular matrix with m extra diagonals iff A fails conditions (1) at most by m.

*Proof.* Sufficiency. Given an n-by-n matrix A that fails conditions (1) by no more than m we obtain (n+m)-by-(n+m) matrix C by bordering A with m columns of zeros on the left and m rows of zeros on the top. Matrix C satisfies conditions (1). We apply the algorithm to obtain LU factorization of C. As was noted above the first m rows and the last m columns of L are zero. Also the first m columns and the last m rows of U are zero. We let

$$K = L[\{m+1...n+m\}, \{1...n\}]$$

and

$$W = U[\{1 \dots n\}, \{m+1 \dots n+m\}]$$

Matrices K and W defined this way have the desired form. Also block multiplication shows that

$$A = KW$$

Necessity. Suppose A can be written as

$$A = KW$$

with K and W such as above. We define C, L and U as above. Block multiplication shows that

$$C = LU$$

Therefore by theorem (1) C satisfies conditions (1). It follows that A fails conditions (1) by no more than m.  $\square$ 

#### Note 4

If matrix A has rank deficiency p then matrix C defined as above has rank deficiency m+p and by note (2) the last m+p columns of L and rows of U are equal to zero. Therefore the last p columns of K are zero and the last p rows of W are zero. In particular our algorithm produces a full rank factorization of A.

#### Note 5

Notice that algorithm (1) when applied to matrix C such as above simply "ignores" first m zero columns and first m zero rows. Therefore matrices K and W can be obtained by applying algorithm (1) to matrix A directly and then letting

$$K = L$$

and

$$W = U$$

We now state

**Theorem 4** An n-by-n matrix A can be written as

$$A = KW$$

where K is an almost lower triangular matrix with m extra diagonals and W is an almost upper triangular matrix with m extra diagonals iff the algorithm described in note (5) produces such a factorization.

*Proof.* Suppose A has such a factorization then by theorem (3) it fails conditions (1) by no more than m. Then the algorithm produces a factorization of desired form. Sufficiency is obvious.

**Definition 3** An n-by-(n+m) matrix H is said to be an almost lower triangular with m extra columns if the matrix  $H[\{1...n\}, \{m+1...n+m\}]$  is lower triangular. An (n+m)-by-n matrix W is said to be an almost upper triangular with m extra rows if the matrix  $W[\{m+1...n+m\}, \{1...n\}]$  is upper triangular.

**Theorem 5** An n-by-n matrix A can be written as product of an almost lower triangular matrix H with m extra columns and an almost upper triangular matrix V with m extra rows iff A fails conditions (1) by no more than m.

Proof.

*Necessity.* Suppose A can be written as

$$A = HV$$

with H and V such as above. We obtain an (n+m)-by-(n+m) matrix C by bordering A with m rows of zeros on the top and m columns of zeros on the left. We obtain a lower triangular matrix L by bordering H with m rows of zeros on the top, and an upper triangular matrix U by bordering V with m columns of zeros on the left. Block multiplication shows that

$$C = LU$$

Therefore by theorem (1) C satisfies conditions (1). It follows that A fails conditions (1) by no more than m.

Sufficiency. Suppose A fails conditions (1) by no more than m. We define matrix C as above. Matrix C satisfies conditions (1). We now apply algorithm (1) to obtain LU factorization of A. By note (1) the first m rows of L are zeros and the first m columns of U are zeros. Block multiplication shows that A can be written as a product of two matrices of the desired form

$$A = HV$$

with

$$H = L[\{m+1...n+m\}, \{1...n+m\}]$$

and

$$V = U[\{1 \dots n + m\}, \{m + 1 \dots n + m\}]$$

#### Note 6

Additionally by note (2) last m columns of H are zero and last m rows of V are zero.

#### Note 7

Note 4 shows that factorization of A obtained in this way is a full rank factorization.

# Notes about computation of LU factorization in floating point arithmetic .

Every invertible matrix which has an LU factorization has a neighborhood in which every matrix is invertible and has LU factorization. Moreover, if we require that  $u_{ii} = 1$  then such a factorization is unique. Therefore we can define function  $A \mapsto [L, U]$ . Such a function can be defined in some neighborhood of any invertible matrix that has an LU factorization. The function will be invertible and continuous in the neighborhood. The inverse will also be 1-1 and continuous. These facts make it possible to compute LU factorization of an invertible matrix satisfying conditions (1) using floating point arithmetic.

However in general a matrix that has an LU factorization does not have a neighborhood in which every matrix has an LU factorization. Also if a factorization exists it does not have to be unique. Further LU factorization does not have to depend continuously on the entries of A. Thus, because of the possibility of the rounding error it is not generally possible to compute an LU factorization in the general case in floating point arithmetic using algorithm (1).

### Some Applications.

**Theorem 6** Any n-by-n matrix A can be written as

$$A = U_1 L U_2$$

with  $U_1$  and  $U_2$  upper triangular matrices and L lower triangular matrix.

*Proof.* By a series of type-3 elementary row operations matrix A can be transformed into a matrix C such that for all  $k = 1, \ldots, n$  we have

$$rankC[\{1\dots k\}] = rankC[\{1\dots n\}, \{1\dots k\}]$$

In particular, since

$$rankC[\{1\dots k\},\{1\dots n\}] \leq k$$

we see that C satisfies conditions (1). In fact one can transform A into C using only elementary type-3 row operations that add multiples of ith row

to the jth row with j < i. Any series of such operations can be realized by multiplying A by an invertible upper triangular matrix U on the left. We have

$$C = UA$$

but by theorem (1)

$$C = LU_2$$

with L lower triangular and  $U_2$  upper triangular. Since U is invertible we can let

$$U_1 = U^{-1}$$

That gives us

$$U_1UA = A = U_1LU_2$$

#### Note 8

Notice that  $U_1$  can be taken invertible. Also because

$$rankC[\{1\dots k\}] = rankC[\{1\dots n\}, \{1\dots k\}]$$

it follows that L can be taken invertible [ see LM ].

Corollary 2 Any n-by-n matrix A can be written as

$$A = L_1 U L_2$$

with  $L_1$  and  $L_2$  lower triangular matrices and U an upper triangular matrix.

*Proof.* It follows from the fact that  $A^T$  can be written as

$$A^T = U_1 L U_2$$

with  $U_1$  and  $U_2$  upper triangular matrices and L lower triangular matrix.  $\square$ 

#### Note 9

Notice that U and  $L_2$  can be taken invertible. We prove the following well known result **Theorem 7** Any n-by-n matrix A can be written as

$$A = PLU$$

with U an upper triangular matrix, L a lower triangular matrix and P permutation matrix.

*Proof.* Any matrix A can be multiplied on the left by such permutation matrix  $P_0$  that matrix

$$C = P_0 A$$

satisfies the following equality for all k = 1, ..., n

$$rankC[\{1 \dots k\}] = rankC[\{1 \dots n\}, \{1 \dots k\}]$$

In particular, since

$$rankC[\{1\dots k\},\{1\dots n\}] \le k$$

we see that C satisfies conditions (1). therefore by theorem (1) C can be written as

$$C = LU$$

with a lower triangular L and an upper triangular U. We let

$$P = P_0^{-1}$$

That gives us

$$A = PP_0A = PC = PLU$$

 $\Box$ .

Corollary 3 Any n-by-n matrix A can be written as

$$A = LUP$$

with U an upper triangular matrix, L a lower triangular matrix and P permutation matrix.

*Proof.* It follows from the fact that

$$A^T = P_0 L_0 U_0$$

with  $U_0$  an upper triangular matrix,  $L_0$  a lower triangular matrix and  $P_0$  permutation matrix. Indeed, we can write

$$A = LUP$$

with

$$L = U_0^T$$

$$U = L_0^T$$

and

$$P = P_0^T$$

## References

- [G] F.R. Gantmacher, *The Theory of Matrices*, Vol II, Chelsea, New York, 1959.
- [HJ1] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, 1985.
- [HJ2] R. A. Horn and C. R. Johnson, Topics in Matrix Analysis, Cambridge University Press, New York, 1991.
- [LM] C.M. Lau and T.M. Markham, *LU factorizations*, Czechoslovak Mathematical Journal, 29 (104) 1979, Praha