SI231b: Matrix Computations

Lecture 8: Orthogonal Projection and QR Factorization

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Oct. 10, 2020

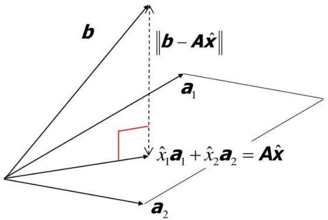
Recap

Overdetermined System: $\mathrm{Ax} = \mathrm{b}$, $\mathrm{A} \in \mathbb{R}^{m \times n}$ $(m > n)$, the least square (LS) solution x_{LS},

$$\mathrm{x}_{LS} = \arg\min \|\mathrm{b} - \mathrm{Ax}\|_2^2,$$

where $\|\cdot\|_2$ represents the vector 2-norm and A is full rank.

1. find $\tilde{\mathrm{b}} \in \mathcal{R}(\mathrm{A})$ such that $\|\mathrm{b} - \tilde{\mathrm{b}}\|_2$ is minimized
2. solve $\mathrm{Ax}_{LS} = \tilde{\mathrm{b}}$ to obtain x_{LS}



Key: orthogonal projection on $\mathcal{R}(\mathrm{A})$

Outline

- Orthogonal Projection
- Projection with Orthogonal Basis
- Projection with Arbitrary Basis
- Computing Orthogonal Basis

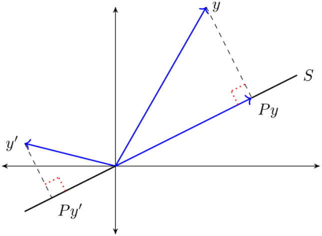


Figure 1: orthogonal projection

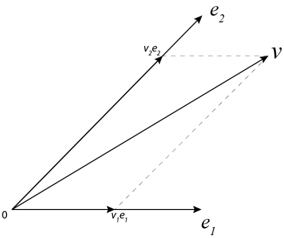


Figure 2: oblique projection

Projection onto Subspaces

Projection onto subspaces

Suppose $\mathcal{V} = \mathcal{U} \oplus \mathcal{W}$, then there is a projector P such that $\mathcal{R}(\mathrm{P}) = \mathcal{U}$ and $\mathcal{N}(\mathrm{P}) = \mathcal{W}$, we say that P is a projector onto $\mathcal{U}$ along $\mathcal{W}$.

Orthogonal projector

An orthogonal projector P is the one that projects onto a subspace $\mathcal{U}$ along a subspace $\mathcal{W}$ when $\mathcal{U}$ and $\mathcal{W}$ are orthogonal.

Warning: orthogonal projectors are not orthogonal matrices.

Previous analysis show that $\mathsf{P} \in \mathbb{R}^{m\times m}$ seperates $\mathbb{R}^m$ into two subspaces

- $\mathcal{R}(\mathsf{P})$
- $\mathcal{N}(\mathsf{P})$

and

$$\mathbb{R}^m = \mathcal{R}(\mathsf{P}) \oplus \mathcal{N}(\mathsf{P}) \quad \text{can you prove this?}$$

P projects $\mathbb{R}^m$ onto $\mathcal{R}(\mathsf{P})$ along $\mathcal{N}(\mathsf{P})$.

Theorem

A projector P is orthogonal if and only if $\mathsf{P} = \mathsf{P}^T$.

Proof ?

Projection with Orthogonal Basis

When $\{q_1, \ q_2, \ \cdots, \ q_n\}$ form an orthogonal basis of $\mathcal{R}(\mathrm{P})$, then the orthogonal projector is given by

$$\mathrm{P} = \mathrm{Q}\mathrm{Q}^T,$$

where $\mathrm{Q} = [q_1, \ q_2, \ \cdots, \ q_n]$

Can you explain why?

Projection with Arbitrary Basis

When $\{a_1, \ a_2, \ \cdots, \ a_n\}$ form a basis of $\mathcal{R}(\mathrm{P})$, then the orthogonal projector is given by

$$\mathrm{P} = \mathrm{A}(\mathrm{A}\mathrm{A}^T)^{-1}\mathrm{A}^T,$$

where $\mathrm{A} = [a_1, \ a_2, \ \cdots, \ a_n]$

How to obtain?

Computing Orthogonal Basis

Given a basis $\{a_1, \ a_2, \ \cdots, \ a_n\}$ of a subspace $\mathcal{S}$, how to compute its orthogonal basis?

Gram-Schmidt orthogonalization.

Key: orthogonal projection of vector a onto vector b

$$\mathrm{proj_b(a)} = \frac{<\mathrm{a}, \mathrm{b}>}{<\mathrm{b}, \mathrm{b}>}\mathrm{b},$$

where $<>$ represents the inner product of two vectors.