Solution 23.20

First, we will use forward expansions. The Taylor series expansion about $a = x_i$ and $x = x_{i+2}$ ($2\Delta x$ steps forward) can be written as:

$$f(x_{i+2}) = f(x_i) + f'(x_i)2\Delta x + \frac{1}{2}f''(x_i)(2\Delta x)^2 + \frac{1}{6}f'''(x_i)(2\Delta x)^3 + \frac{1}{24}f^{(4)}(x_i)(2\Delta x)^4 + \frac{1}{120}f^{(5)}(x_i)(2\Delta x)^5 + \cdots$$

$$f(x_{i+2}) = f(x_i) + 2f'(x_i)\Delta x + 2f''(x_i)\Delta x^2 + \frac{8}{6}f'''(x_i)\Delta x^3 + \frac{16}{24}f^{(4)}(x_i)\Delta x^4 + \frac{32}{120}f^{(5)}(x_i)\Delta x^5 + \cdots \quad \textbf{(1)}$$

Taylor series expansion about $a = x_i$ and $x = x_{i+1}$ (Δx steps forward):

$$f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x + \frac{1}{2}f''(x_i)\Delta x^2 + \frac{1}{6}f'''(x_i)\Delta x^3 + \frac{1}{24}f^{(4)}(x_i)\Delta x^4 + \frac{1}{120}f^{(5)}(x_i)\Delta x^5 + \cdots \quad \textbf{(2)}$$

Multiply Eq. 2 by 2 and subtract the result from Eq. 1 to yield

$$f(x_{i+2}) - 2f(x_{i+1}) = -f(x_i) + f''(x_i)\Delta x^2 + \frac{6}{6}f'''(x_i)\Delta x^3 + \frac{14}{24}f^{(4)}(x_i)\Delta x^4 + \frac{30}{120}f^{(5)}(x_i)\Delta x^5 + \cdots \quad \textbf{(3)}$$

Next, we will use backward expansions. The Taylor series expansion about $a = x_i$ and $x = x_{i-2}$ ($2\Delta x$ steps backward) can be written as:

$$f(x_{i-2}) = f(x_i) + f'(x_i)(-2\Delta x) + \frac{1}{2}f''(x_i)(-2\Delta x)^2 + \frac{1}{6}f'''(x_i)(-2\Delta x)^3 + \frac{1}{24}f^{(4)}(x_i)(-2\Delta x)^4 + \frac{1}{120}f^{(5)}(x_i)(-2\Delta x)^5 + \cdots$$

$$f(x_{i-2}) = f(x_i) - 2f'(x_i)\Delta x + 2f''(x_i)\Delta x^2 - \frac{8}{6}f'''(x_i)\Delta x^3 + \frac{16}{24}f^{(4)}(x_i)\Delta x^4 - \frac{32}{120}f^{(5)}(x_i)\Delta x^5 + \cdots \quad \textbf{(4)}$$

Solution continued on the next page...

Taylor series expansion about $a = x_i$ and $x = x_{i-1}$ (Δx steps backward):

$$f(x_{i-1}) = f(x_i) - f'(x_i)\Delta x + \frac{1}{2} f''(x_i)\Delta x^2 - \frac{1}{6} f'''(x_i)\Delta x^3 + \frac{1}{24} f^{(4)}(x_i)\Delta x^4 - \frac{1}{120} f^{(5)}(x_i)\Delta x^5 + \cdots \quad \textbf{(5)}$$

Multiply Eq. 5 by 2 and subtract the result from Eq. 4 to yield

$$2f(x_{i-1}) - f(x_{i-2}) = f(x_i) - f''(x_i)\Delta x^2 + \frac{6}{6} f'''(x_i)\Delta x^3 - \frac{14}{24} f^{(4)}(x_i)\Delta x^4 + \frac{30}{120} f^{(5)}(x_i)\Delta x^5 + \cdots \quad \textbf{(6)}$$

Add Eqs (3) and (6)

$$f(x_{i+2}) - 2f(x_{i+1}) + 2f(x_{i-1}) - f(x_{i-2}) = 2f'''(x_i)\Delta x^3 + \frac{60}{120} f^{(5)}(x_i)\Delta x^5 + \cdots \quad \textbf{(7)}$$

Equation 7 can be solved for

$$f'''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + 2f(x_{i-1}) - f(x_{i-2})}{2\Delta x^3} - \frac{\frac{60}{120} f^{(5)}(x_i)\Delta x^5}{2\Delta x^3} + \cdots$$

$$f'''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + 2f(x_{i-1}) - f(x_{i-2})}{2\Delta x^3} - \frac{1}{4} f^{(5)}(x_i)\Delta x^2 + \cdots$$