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Avaliação 03 de Álgebra Linear

32. [M] Construct a pair $\mathbf{u}$, $\mathbf{v}$ of random vectors in $\mathbb{R}^4$, and let

$$A = \begin{bmatrix} .5 & .5 & .5 & .5 \\ .5 & .5 & -.5 & -.5 \\ .5 & -.5 & .5 & -.5 \\ .5 & -.5 & -.5 & .5 \end{bmatrix}$$

a. Denote the columns of A by $\mathbf{a}_1, \dots, \mathbf{a}_4$. Compute the length of each column, and compute $\mathbf{a}_1 \cdot \mathbf{a}_2$, $\mathbf{a}_1 \cdot \mathbf{a}_3$, $\mathbf{a}_1 \cdot \mathbf{a}_4$, $\mathbf{a}_2 \cdot \mathbf{a}_3$, $\mathbf{a}_2 \cdot \mathbf{a}_4$, and $\mathbf{a}_3 \cdot \mathbf{a}_4$.

b. Compute and compare the lengths of $\mathbf{u}$, $A\mathbf{u}$, $\mathbf{v}$, and $A\mathbf{v}$.

c. Use equation (2) in this section to compute the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$. Compare this with the cosine of the angle between $A\mathbf{u}$ and $A\mathbf{v}$.

d. Repeat parts (b) and (c) for two other pairs of random vectors. What do you conjecture about the effect of A on vectors?

33. [M] Generate random vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{v}$ in $\mathbb{R}^4$ with integer entries (and $\mathbf{v} \neq \mathbf{0}$), and compute the quantities

$$\left(\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v},\ \left(\frac{\mathbf{y}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v},\ \frac{(\mathbf{x}+\mathbf{y})\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v},\ \frac{(10\mathbf{x})\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}$$

Repeat the computations with new random vectors $\mathbf{x}$ and $\mathbf{y}$. What do you conjecture about the mapping $\mathbf{x} \mapsto T(\mathbf{x}) = \left(\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$ (for $\mathbf{v} \neq \mathbf{0}$)? Verify your conjecture algebraically.

34. [M] Let $A = \begin{bmatrix} -6 & 3 & -27 & -33 & -13 \\ 6 & -5 & 25 & 28 & 14 \\ 8 & -6 & 34 & 38 & 18 \\ 12 & -10 & 50 & 41 & 23 \\ 14 & -21 & 49 & 29 & 33 \end{bmatrix}$. Construct a matrix N whose columns form a basis for Nul A, and construct a matrix R whose *rows* form a basis for Row A (see Section 4.6 for details). Perform a matrix computation with N and R that illustrates a fact from Theorem 3.

35. [M] Show that the columns of the matrix A are orthogonal by making an appropriate matrix calculation. State the calculation you use.

$$A = \begin{bmatrix} -6 & -3 & 6 & 1 \\ -1 & 2 & 1 & -6 \\ 3 & 6 & 3 & -2 \\ 6 & -3 & 6 & -1 \\ 2 & -1 & 2 & 3 \\ -3 & 6 & 3 & 2 \\ -2 & -1 & 2 & -3 \\ 1 & 2 & 1 & 6 \end{bmatrix}$$

36. [M] In parts (a)–(d), let U be the matrix formed by normalizing each column of the matrix A in Exercise 35.

a. Compute $U^T U$ and $U U^T$. How do they differ?

b. Generate a random vector $\mathbf{y}$ in $\mathbb{R}^8$, and compute $\mathbf{p} = U U^T \mathbf{y}$ and $\mathbf{z} = \mathbf{y} - \mathbf{p}$. Explain why $\mathbf{p}$ is in Col A. Verify that $\mathbf{z}$ is orthogonal to $\mathbf{p}$.

c. Verify that $\mathbf{z}$ is orthogonal to each column of U.

d. Notice that $\mathbf{y} = \mathbf{p} + \mathbf{z}$, with $\mathbf{p}$ in Col A. Explain why $\mathbf{z}$ is in $(\text{Col } A)^\perp$. (The significance of this decomposition of $\mathbf{y}$ will be explained in the next section.)