Mathematical Analysis of the Q Factor in a Driven Oscillation

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1 Maximum Amplitude of a Driven Oscillation

The equation of motion of a driven oscillation system is

$$m\ddot{x} + b\dot{x} + kx = F(t) = F_0 \cos(\omega t) \tag{1.1}$$

where $b\dot{x}$ is the drag term, kx is the force model of an oscillating object, and $F(t)$ is a periodic external driven force. We rewrite the problem as

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega t) \tag{1.2}$$

Where $2\beta \equiv b/m$; $\omega_0^2 \equiv k/m$; $f_0 = F_0/m$, and the driven term ω means the driven frequency. After solving the second-order differential equation, we obtained the solution

$$x(t) = \frac{f_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2}} \cos(\omega t - \delta) + e^{-\beta t}\left(C_1 e^{\sqrt{\beta^2-\omega_0^2}\, t} + C_2 e^{-\sqrt{\beta^2-\omega_0^2}\, t}\right) \tag{1.3}$$

Here, δ denotes the phase difference between the oscillatory solution $x(t)$ and the initial periodic driving force $F_0 \cos(\omega t)$. The phase difference satisfies

$$\delta = \tan^{-1}\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right) \tag{1.4}$$

The long-term behavior $x_l(t)$ depends on the externally driven term, that is,

$$x_l(t) = \frac{f_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2}} \cos(\omega t - \delta) = A\cos(\omega t - \delta) \tag{1.5}$$

The amplitude A of the oscillating system is

$$A = \frac{f_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2}} \tag{1.6}$$

When the natural frequency ω_0 is fixed and the driving frequency ω is varied, the maximum amplitude occurs at the point where the first derivative of A with respect to ω equals zero, that is

$$\left.\frac{dA}{d\omega}\right|_{\omega=\omega_R} = 0 \tag{1.7}$$

where ω_R represents the resonance frequency. Then we can compute that

$$\left.\frac{dA}{d\omega}\right|_{\omega=\omega_R} = \left.\frac{-f_0 \cdot \frac{d}{d\omega}[\left(\omega_0^2-\omega^2\right)^2+4\omega^2\beta^2]^{1/2}}{\left(\omega_0^2-\omega^2\right)^2+4\omega^2\beta^2}\right|_{\omega=\omega_R} = \left.\frac{-f_0(4\omega^3-4\omega\omega_0^2+8\omega\beta^2)}{2[\left(\omega_0^2-\omega^2\right)^2+4\omega^2\beta^2]^{3/2}}\right|_{\omega=\omega_R}$$

$$= \frac{-f_0(4\omega_R^3-4\omega_R\omega_0^2+8\omega_R\beta^2)}{2[\left(\omega_0^2-\omega_R^2\right)^2+4\omega_R^2\beta^2]^{3/2}} = 0$$

Then

$$4\omega_R^3-4\omega_R\omega_0^2+8\omega_R\beta^2 = 0 \Rightarrow \omega_R^2 = \omega_0^2 - 2\beta^2$$

So, the resonance frequency is

$$\omega_R = \sqrt{\omega_0^2-2\beta^2} \tag{1.8}$$

Substituting this into (1.6), we have

$$A_{\max} = \frac{f_0}{2\beta\sqrt{\omega_0^2-\beta^2}} \approx \left.\frac{f_0}{2\beta\omega_0}\right|_{\beta\ll\omega_0} \tag{1.9}$$

This indicates that when $\beta \ll \omega_0$, a resonance phenomenon occurs as the driving frequency approaches the natural frequency. At this point, the amplitude reaches its maximum. The corresponding graph will be discussed in Section 3.

2 Total Energy and Average Power of a Driven Oscillation

When the time $t \gg 1/\beta$, the complementary solution decays and no longer contributes to the motion. Therefore, we only need to focus on the contribution of the particular solution, also known as long-term behavior, to the total energy E. That is,

$$E = T(t) + V(t) = \frac{1}{2}m[\dot{x}_l(t)]^2 + \frac{1}{2}kx_l(t) \tag{2.1}$$

Substituting (1.5) into (2.1), we have

$$\begin{aligned} E &= \frac{1}{2}m\omega^2A^2[-\sin(\omega t-\delta)]^2 + \frac{1}{2}kA^2[\cos(\omega-\delta)]^2 \\ &= \frac{1}{2}m\omega^2A^2\sin^2(\omega t-\delta) + \frac{1}{2}kA^2\cos^2(\omega t-\delta) \end{aligned} \tag{2.2}$$

Note that this is the instantaneous mechanical energy. When the kinetic energy reaches its maximum, i.e., when $\sin^2(\omega t-\delta) = 1$, the kinetic energy represents the total mechanical energy of the system, that is

$$E = \frac{1}{2}m\omega^2A^2 \tag{2.3}$$

If we want to describe the whole behavior of the system, then we need to get the average energy of the system, denoted by $\langle E \rangle$.

$$\begin{aligned}\langle E \rangle &= \left\langle \frac{1}{2} m\omega^2 A^2 \sin^2(\omega t - \delta) \right\rangle + \left\langle \frac{1}{2} k A^2 \cos^2(\omega t - \delta) \right\rangle \\ &= \frac{1}{2} m\omega^2 A^2 \left\langle \sin^2(\omega t - \delta) \right\rangle + \frac{1}{2} k A^2 \left\langle \cos^2(\omega t - \delta) \right\rangle \\ &= \frac{1}{2} m\omega^2 A^2 \left\langle \frac{1 - \cos(2\omega t - \delta)}{2} \right\rangle + \frac{1}{2} k A^2 \left\langle \frac{1 + \cos(2\omega t - \delta)}{2} \right\rangle \end{aligned} \tag{2.4}$$

By the formula of the average of a periodic function

$$\langle f(t) \rangle = \frac{1}{T} \int_0^T f(t)\, dt \tag{2.5}$$

Then the averages of the two trigonometric functions are

$$\begin{cases} \left\langle \dfrac{1 - \cos(2\omega t - \delta)}{2} \right\rangle = \dfrac{\omega}{\pi} \displaystyle\int_0^{\pi/\omega} \dfrac{1 - \cos(2\omega t - \delta)}{2} dt = \dfrac{1}{2} \\ \left\langle \dfrac{1 + \cos(2\omega t - \delta)}{2} \right\rangle = \dfrac{\omega}{\pi} \displaystyle\int_0^{\pi/\omega} \dfrac{1 + \cos(2\omega t - \delta)}{2} dt = \dfrac{1}{2} \end{cases} \tag{2.6}$$

Substituting (2.6) into (2.4), we can finally get the average energy

$$\langle E \rangle = \frac{1}{4} m\omega^2 A^2 + \frac{1}{4} k A^2 = \frac{1}{4} m A^2 (\omega^2 + \omega_0^2) \tag{2.7}$$

In addition, we can apply a similar method to the system's average power $\langle P \rangle$. The instantaneous power P is

$$P = F(t) \cdot \dot{x}_l(t) = F_0 \cos(\omega t) \cdot [-A\omega \sin(\omega t - \delta)] \tag{2.8}$$

We expand the equation to

$$\begin{aligned} P &= -m f_0 A\omega \cos(\omega t) \left[\sin(\omega t) \frac{A(\omega_0^2 - \omega^2)}{f_0} - \cos(\omega t) \frac{2A\omega\beta}{f_0} \right] \\ &= -\frac{mA^2\omega(\omega_0^2 - \omega^2)}{2} \sin(2\omega t) + 2mA^2\omega^2\beta \cos^2(\omega t) \\ &= -\frac{mA^2\omega(\omega_0^2 - \omega^2)}{2} \sin(2\omega t) + mA^2\omega^2\beta + mA^2\omega^2\beta \cos(2\omega t) \end{aligned}$$

By (2.4) and (2.5), we obtain

$$
\begin{aligned}
\langle P \rangle &= -\frac{mA^2\omega(\omega_0^2-\omega^2)}{2}\langle \sin(2\omega t)\rangle + mA^2\omega^2\beta + mA^2\omega^2\beta\langle\cos(2\omega t)\rangle \\
&= -\frac{mA^2\omega(\omega_0^2-\omega^2)}{2}\cdot\frac{\omega}{\pi}\int_0^{\pi/\omega}\sin(2\omega t)dt + mA^2\omega^2\beta \\
&+ mA^2\omega^2\beta\cdot\frac{\omega}{\pi}\int_0^{\pi/\omega}\cos(2\omega t)dt = mA^2\omega^2\beta
\end{aligned}
\tag{2.9}
$$

Thus, we can claim that

$$\langle P \rangle = mA^2\omega^2\beta \propto A^2 \tag{2.10}$$

3 Bandwidth of the Driven Oscillation

To identify a reasonable range of resonance strength, also known as the response interval, we define this interval as the region where the system maintains at least half of its average power. The rationale is that, from an engineering perspective, oscillations below half the average power decay rapidly, rendering the resonance effect insignificant. In other words, this interval is confined to the region above half the average power, and its length is referred to as the bandwidth, denoted by $\Delta\omega$. This means the effective resonance region is only as wide as the bandwidth.
Let the boundaries of the bandwidth be $\omega_0 \pm \delta\omega$, as shown below.

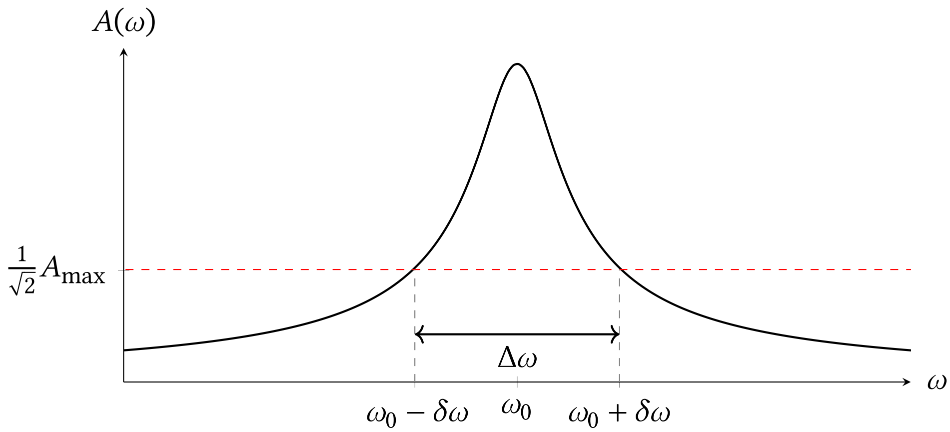

Figure 1: Resonance curve of a driven oscillator. The shaded region represents the frequencies over which the amplitude exceeds $\frac{1}{\sqrt{2}}A_{\max}$, indicating the bandwidth.

By (2.10), we know that the average power $\langle P\rangle \propto A^2$, then half the average power satisfies

$$\langle P\rangle = mA^2\omega^2\beta \Longleftrightarrow \frac{1}{2}\langle P\rangle = m\left(\frac{1}{\sqrt{2}}A\right)^2\omega^2\beta \tag{3.1}$$

Then we can further assert that

$$\frac{1}{2}\langle P\rangle_{\text{max}} \longleftrightarrow \frac{1}{\sqrt{2}}A_{\text{max}} \tag{3.2}$$

In other words, to determine the response interval, we can directly refer to the amplitude–frequency function. The effective resonance range must lie above the level of $\frac{1}{\sqrt{2}}A_{\text{max}}$. Let $\omega = \omega_0 + \delta\omega$, then

$$\omega^2 = \omega_0^2 + 2\omega_0\omega + O(\delta\omega^2) \tag{3.3}$$

since $\delta\omega \ll \omega_0$. Substituting this into (1.6) and comparing with (1.9), we obtain

$$\begin{aligned}\frac{1}{\sqrt{2}}A_{\text{max}} &= \frac{f_0}{\sqrt{4\omega_0^2\delta\omega^2 + 4(\omega_0^2 + 2\omega_0\delta\omega)\beta^2}} \\ &= \frac{f_0}{\sqrt{4\omega_0^2\delta\omega^2 + 4\omega_0^2\beta^2 + 8\omega_0\delta\omega\beta^2}} \\ &= \frac{f_0}{2\sqrt{2}\beta\sqrt{\omega_0^2 - \beta^2}}\end{aligned} \tag{3.4}$$

When $\delta\omega,\ \beta \ll \omega_0$, we can neglect the term $8\omega_0\delta\omega\beta^2$. Then (3.4) becomes

$$\frac{f_0}{2\sqrt{2}\beta\omega_0} = \frac{f_0}{\sqrt{4\omega_0^2\delta\omega^2 + 4\omega_0^2\beta^2}} \Rightarrow 8\beta^2\omega_0^2 = 4\omega_0^2\delta\omega^2 + 4\beta^2\omega_0^2 \tag{3.5}$$

Therefore,

$$\delta\omega = \pm\beta \tag{3.6}$$

So, the bandwidth $\Delta\omega$ is

$$\Delta\omega = |(\omega_0 \pm \delta\omega) - (\omega_0 \mp \delta\omega)| = 2|\delta\omega| = 2\beta \tag{3.7}$$

4 Definition of the Q Factor

The quality factor (Q factor) quantifies how underdamped an oscillatory system is and describes the ratio of energy stored to energy dissipated per cycle. It's defined as

$$Q \equiv 2\pi\frac{\text{Energy stored in the oscillator}}{\text{Average energy dissipated in a period}} \tag{4.1}$$

Namely,

$$Q = 2\pi \frac{E}{\langle \Delta E \rangle_{\text{per period}}} \tag{4.2}$$

We've discussed the total energy in (2.3), that is, $E = \frac{1}{2}m\omega^2 A^2$. Furthermore, the average energy dissipated in a period is related to the average power, that is

$$\langle \Delta E \rangle_{\text{per period}} = \langle P \rangle \cdot \frac{2\pi}{\omega} = 2\pi m A^2 \omega \beta \tag{4.3}$$

Substituting this into (4.2), the Q factor becomes

$$Q = 2\pi \frac{\frac{1}{2}m\omega^2 A^2}{2\pi m A^2 \omega \beta} = \frac{\omega}{2\beta} \tag{4.4}$$

When $\beta \ll \omega_0$, by (3.7), we can replace 2β with $\Delta\omega$. So, the Q factor becomes the ratio of the frequency ω to the bandwidth $\Delta\omega$. We obtain

$$Q = \frac{\omega}{2\beta} = \left.\frac{\omega}{\Delta\omega}\right|_{\beta \ll \omega_0} \tag{4.5}$$

As previously mentioned, the parameter β represents the damping coefficient. Therefore, when the quality factor Q increases, β becomes smaller, indicating that the oscillatory system experiences less resistance and dissipates energy at a lower rate. As a result, the maximum amplitude at resonance becomes larger. At the same time, the bandwidth becomes narrower since $\Delta\omega = 2\beta$.

Physically, this means that a system with high Q exhibits a sharp frequency response and significant resonance only when the driving frequency is very close to the natural frequency.

Conversely, when Q is small, β increases, indicating greater resistance and higher energy dissipation in the system. In this case, the system decays more quickly, responds more weakly and broadly across a wider frequency range, and the resonance peak becomes broader and flatter, as shown in the following figure.

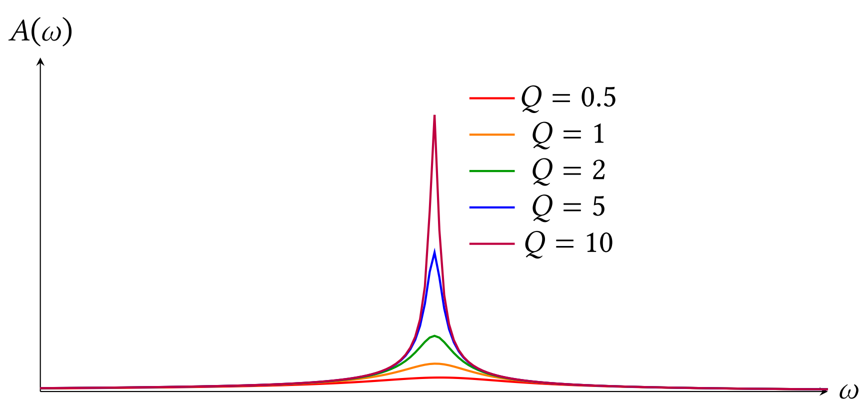

Figure 2: Resonance curves $A(\omega)$ for different quality factors Q.