Mathematical Methods for Taylor's Mechanics

National Central University

Teaching Material for Incoming Physics Students

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Preface

This book explains and introduces the mathematical methods used in *Classical Mechanics* by John R. Taylor, methods that are not typically taught in high school. Unlike many other universities, we are not required to start with "General Physics." Instead, we begin directly with one of the four fundamental branches of mechanics: classical mechanics. This book is written for students who are just beginning their new academic journey in college, especially at NCU. :(

Classical mechanics is definitely not the same thing as high school physics. Sure, it still talks about motion and force, but trust me, it's a whole different world. High school physics uses the Newtonian system to describe motion, but only with simple tools: triangles, basic vector addition and subtraction, and maybe the occasional dot or cross product. And let's be honest, the goal back then was just to get "high scores" to get into a "good college," right? But here, in this course and this book, it's no longer about scores or grinding through endless problem sets. Now, it's about something deeper, we're aiming for the **essence of the world**. So inevitably, we can't just stay in the comfort zone of simple equations and expect to explain a world this complex. We need a deeper mathematical language, one that speaks in calculus, vectors, and differential equations, if we want to get even a little closer to how this world truly works.

To avoid saying "What the hell is that?" during the course, I put together this booklet to introduce nearly all the math tools(almost 90 percent) you'll need while reading Taylor's Classical Mechanics. When I first encountered these topics, I was genuinely shocked by how different they were from high school math. I struggled a lot, feeling frustrated and totally lost because I didn't have the right background or theoretical intuition. So, to fix that, I spent some time organizing all the math I believe is essential for this course, and tailored it specifically to match the structure of Taylor's textbook. That way, you'll have a reference that walks alongside the physics, not behind it. After all, you really don't want to be solving differential equations and taking gradients in mechanics class while your calculus course is still teaching you how to take limits.

Unlike a typical textbook, this booklet aims to explain the math and classical systems analysis more rigorously and thoroughly. Most textbooks skip over the elegant details(mainly due to page limits), and since Professor Kuo only has one semester to cover so much, things inevitably get rushed or skipped. That's why I decided to create something to complement both the book and the class: a companion guide that fills in the mathematical gaps. Pair it with our professor's iconic voice, and you're in for a top-tier classroom experience. Also, unless something unexpected happens, you'll probably need to write a lot of reports. This booklet can be your go-to math background reference, so you don't get lost in the mathematical jungle, wondering where to even start. Anyway, enough rambling. I hope this semester gives you a smooth, meaningful journey into mechanics and that you'll experience, for yourself, just how beautiful the structure of physics and mathematics can be. Trust me, you won't regret it.

Some grammar and derivations in this booklet were refined with the help of OpenAI ChatGPT (GPT-4), which also assisted in verifying equations and providing ideas during the writing process.

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Chapter 1. Basic Calculus for Physics

This chapter serves as the foundational prerequisite for entering theoretical physics. Whether in classical mechanics, electrodynamics, or any other discipline in the physics curriculum, calculus is regarded as a basic tool, akin to addition, subtraction, multiplication, and division in the eyes of high school students. Therefore, before formally beginning Taylor's *Classical Mechanics*, this book will provide a concise summary to help readers review the key concepts of differentiation, integration, and partial differentiation. The chapter will also include some common formula derivations used in classical mechanics. Unlike other chapters, this section will not focus on rigorous mathematical proofs or formal derivations. Instead, it will serve as a transitional bridge to more advanced content. Readers who are interested in more detailed mathematical justifications are encouraged to consult their own calculus textbooks.

1.1 Differentiation

▷ The Chain Rule

If $y = f(u)$ and $u = g(x)$ are both differentiable functions, then the composite function $y = (f \circ g)(x) = f(g(x))$ is also differentiable, and its derivative at the point $x = a$ is

$$(f \circ g)'(a) = f'[g(a)] \cdot g'(a) \tag{1.1.1}$$

Which can also be denoted by

$$\left.\frac{df[g(x)]}{dx}\right|_{x=a} = \left.\frac{df(y)}{dy}\right|_{y=g(a)} \cdot \left.\frac{dg(x)}{dx}\right|_{x=a} \tag{1.1.2}$$

This can also be intuitively understood using Leibniz notation as:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{1.1.3}$$

Although this notation is not a rigorous mathematical proof, it is very commonly seen and extremely useful in physics and engineering.

Example. *Find the derivative of the following function*

1. $f(x) = (2x^2 + 3x)^5$
2. $f(x) = \sqrt{\dfrac{1}{x^2 - 1}}$

1. Let $f(x) = g(u)$, where $g(x) = x^5$ and $u = 2x^2 + 3x$, by chain rule

$$\frac{df}{dx} = \frac{dg}{du} \cdot \frac{du}{dx} = g'(u) \cdot u'$$

So the final answer is

$$f'(x) = g'(u) \cdot u' = 5(2x^2 + 3x)^4 \cdot (4x + 3)$$

2. Let $f(x) = h(u)$, where $h(x) = \sqrt{x} = x^{1/2}$ and $u = 1/(x^2 - 1)$, by chain rule

$$\frac{df}{dx} = \frac{dh}{du} \cdot \frac{du}{dx} = h'(u) \cdot u'$$

Note that the derivative of u is

$$u' = \frac{-2x}{(x^2 - 1)^2}$$

So the final answer is

$$f'(x) = h'(u) \cdot u' = \frac{1}{2} x^{-\frac{1}{2}} \cdot \frac{-2x}{(x^2 - 1)^2} = \frac{-x}{\sqrt{x}(x^2 - 1)^2}$$

▷ Trigonometric Differentiation

Let us first review the definitions of the basic trigonometric ratios and the fundamental trigonometric identities.

$$\begin{cases} \tan x = \dfrac{\sin x}{\cos x} \\ \cot x = \dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x} \\ \sec x = \dfrac{1}{\cos x} \\ \csc x = \dfrac{1}{\sin x} \end{cases} \qquad \begin{cases} \sin^2 x + \cos^2 x = 1 \\ \tan^2 x + 1 = \sec^2 x \\ \cot^2 x + 1 = \csc^2 x \end{cases} \tag{1.1.4}$$

The basic trigonometric differentiations are sin and cos. Which is

$$\begin{cases} \dfrac{d}{dx} \sin x = \cos x \\ \dfrac{d}{dx} \cos x = -\sin x \end{cases} \tag{1.1.5}$$

Note that when the x term become $u = g(x)$, which is $\sin(u)$, we should apply the chain rule on it. For example,

$$\frac{d}{dx} \sin(g(x)) = \cos(g(x)) \cdot g'(x) \tag{1.1.6}$$

So is the cosine term. After discussing the basic trigonometric differentiations, I will prove the other four differentiations. Starting with $\tan x$, because $\tan x = \sin x / \cos x$, by the differentiation rule, we get

$$\frac{d}{dx} \tan x = \frac{d}{dx} \left(\frac{\sin x}{\cos x} \right) = \frac{\frac{d}{dx}(\sin x) \cos x - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} = \frac{1}{\cos^2 x}$$

So, the derivative of $\tan x$ is

$$\boxed{\frac{d}{dx} \tan x = \sec^2 x} \tag{1.1.7}$$

Apply the same logic, we can calculate the other three derivatives

$$\boxed{\begin{cases} \dfrac{d}{dx}\cot x = -\csc^2 x \\ \dfrac{d}{dx}\sec x = \sec x \tan x \\ \dfrac{d}{dx}\csc x = -\csc x \cot x \end{cases}} \tag{1.1.8}$$

Problem. *Find the derivative of the following function*

1. $f(t) = A\cos(\omega t - \delta) + B\sin(\omega_0 t - \delta)$
2. $f(x) = \cos(3x^2 + 4x - 1)$
3. $f(t) = \sin(u)$, *where* $u = \theta(t^2 - 3t)$

1. The differentiation rule

$$\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$$

So the derivative of $f(t)$ is

$$f'(t) = \frac{d}{dt}A\cos(\omega t - \delta) + \frac{d}{dt}B\sin(\omega_0 t - \delta)$$

Remember the chain rule,

$$\frac{d}{dt}A\cos(\omega t - \delta) + \frac{d}{dt}B\sin(\omega_0 t - \delta) = -A\omega\sin(\omega t - \delta) + B\omega_0\cos(\omega_0 t - \delta)$$

So the final answer is

$$f'(t) = -A\omega\sin(\omega t - \delta) + B\omega_0\cos(\omega_0 t - \delta)$$

2. Let $f(x) = g(u)$, where $g(x) = \cos x$ and $u = 3x^2 + 4x - 1$, then

$$\frac{d}{dx}f(x) = \frac{d}{du}g(u) \cdot \frac{d}{dx}u$$

So the final answer is

$$f'(x) = -\sin(3x^2 + 4x - 1) \cdot (6x + 4)$$

3. In this problem, there are three layers of composition, $\sin t, \theta(t)$ and $u = t^2 - 3t$, so we need to apply the chain rule repeatedly.

$$\frac{d}{dx}(f(g(u))) = f'(g(u)) \cdot g'(u) \cdot u'$$

So

$$\frac{d}{dt}f(t) = \dot{f}(t) = \frac{d}{d\theta}\sin(\theta(t^2 - 3t)) \cdot \frac{d}{du}\theta(u) \times \frac{d}{dt}u$$

The final answer is

$$\dot{f}(t) = (2t - 3)\dot{\theta}\cos(\theta)$$

▷ Derivative of Exponential Function

An exponential function is defined as

$$f(x) = b^x \tag{1.1.9}$$

Where $b > 0$, and x can be a rational or irrational number. If $b > 0, b \neq 1$, the exponential function is continuous, which is

$$\lim_{x \to r} b^x = b^r \tag{1.1.10}$$

To differentiate the exponential function $f(x) = b^x$, we need to do it by the definition of differentiation that

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} = \lim_{h \to 0} \frac{b^x(b^h - 1)}{h} \\ &= b^x \lim_{h \to 0} \frac{b^h - 1}{h} \end{aligned} \tag{1.1.11}$$

Note that we can rewrite the limit term as

$$\lim_{h \to 0} \frac{b^h - 1}{h} = \lim_{h \to 0} \frac{b^{(0+h)} - 1}{h} = f'(0) \tag{1.1.12}$$

So we can obtain

$$f'(x) = b^x \cdot f'(0) \tag{1.1.13}$$

For example, for $b = 2$, $f'(0) = \lim_{h \to 0}(2^h - 1)/h \approx 0.693$, while $b = 3$, $f'(0) = \lim_{h \to 0}(3^h - 1)/h \approx 1.099$. By these two examples, it is obvious that there exists a number Ξ $(2 < \Xi < 3)$, such that $f'(0) = 1$. We can define this number as $\Xi = e$(called "exponential"), which is

$$f'(0) = \lim_{h \to 0} \frac{e^h - 1}{h} = 1 \tag{1.1.14}$$

So the derivative of the function $f(x) = e^x$ is

$$\boxed{f'(x) = e^x \cdot f'(0) = e^x} \tag{1.1.15}$$

We will not provide a detailed proof or explanation of how to approximate the value of e here. If you are really, really curious, please refer to *Stewart's Calculus*. In this book, we simply adopt the standard approximation: $e \approx 2.7182818$. When there is a composite function of exponential, for example $f(u) = e^u$, $u = \sin x$, we need to apply the chain rule

$$f'(u) = e^u \cdot u' = e^{\sin x} \cdot \cos x$$

After discussing the exponential function, we can continue to the logarithmic function

$$f(x) = \log_b a \tag{1.1.16}$$

where $b > 0, b \neq 1$ and $a > 0$. The logarithmic function and the exponential function are the inverse functions of each other, which means

$$f(x) = b^x, \ f^{-1}(b^x) = \log_b b^x = x$$

When we replace the base of the logarithm with the number $\Xi = e$, we call it **natural log** and rewrite it as

$$\boxed{\log_e x \coloneqq \ln x} \tag{1.1.17}$$

After understanding the differentiation rules for exponential functions, we now turn to the differentiation rules for logarithmic functions. Let $f(x) = \ln x$, we need to find $f'(x)$. By the definition of the natural log, we obtain

$$e^{f(x)} = x \Rightarrow \frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x) = 1 \tag{1.1.18}$$

Dividing both sides by $e^{f(x)}$, then we can get

$$f'(x) = \frac{1}{e^{f(x)}} = \frac{1}{x} \tag{1.1.19}$$

Note that we substituted $e^{f(x)} = x$ into the result, so

$$\boxed{\frac{d}{dx}\ln x = \frac{1}{x}} \tag{1.1.20}$$

By the way, sometimes, when the exponent of e becomes too long, we tend to write it as $\exp(f(x))$ for convenience.
Now, let's return to the differentiation of logarithmic functions with bases other than e. We can let $f(x) = b^x = e^{\ln b^x} = e^{x\ln b}$, then

$$f'(x) = \frac{d}{dx}e^{x\ln b} = e^{x\ln b}\cdot\frac{d}{dx}(x\ln b) = e^{x\ln b}\cdot\ln b \tag{1.1.21}$$

Substituting the equation $e^{x\ln b} = b$ into it, we can verify that

$$\boxed{\frac{d}{dx}b^x = b^x\cdot\ln b} \tag{1.1.22}$$

In addition, we can differentiate the logarithmic function $f(x) = \log_a x$ by the change of base formula, which is

$$\log_a x = \frac{\log_e x}{\log_e a} = \frac{\ln x}{\ln a} \tag{1.1.23}$$

So

$$\boxed{\frac{d}{dx}\log_a x = \frac{1}{\ln a}\frac{d}{dx}\ln x = \frac{1}{x\ln a}} \tag{1.1.24}$$

There is another type of the exponential function, which is the "variable exponent function". It's a function in which the variable appears both as the base and as the exponent. Consider the form

$$f(x) = x^x \tag{1.1.25}$$

In such a problem, if we want to get $f'(x)$, we should take the natural logarithm of both sides of the equation. That is

$$\ln f(x) = \ln x^x = x\ln x \tag{1.1.26}$$

In this way, we change the variable exponent function into a simple composite function. Then, differentiate both sides,

$$\frac{d}{dx}\ln f(x) = \frac{1}{f(x)}\cdot f'(x) = \frac{d}{dx}(x\ln x) = \ln x + x\cdot\frac{1}{x}$$

So it becomes

$$f'(x) = f(x)(\ln x + 1)$$

Substituting $f(x) = x^x$ into the result, we can get

$$\boxed{\frac{d}{dx}x^x = x^x(\ln x + 1)} \tag{1.1.27}$$

Example. *Find the derivative of the following function*

1. $f(x) = 2^{3x^2-x}$
2. $f(x) = e^{x^{3x}}$
3. $f(x) = e^x x^{\sin x}$

1. We can directly differentiate this function by applying the chain rule to it, that is

$$\frac{d}{dx}2^{3x^2-x} = 2^{3x^2-x}\cdot\ln(3x^2-x)\cdot\frac{d}{dx}(3x^2-x)$$

So the final answer is

$$f'(x) = (6x-1)\cdot\ln(3x^2-x)\cdot 2^{3x^2-x}$$

2. Let $f(x) = e^u$, and $u = x^{3x}$, the differentiation is

$$f'(x) = e^u\cdot u'$$

By $u = x^{3x}$, it's a variable exponent function, we need to take natural logarithm of both sides of the function, that is

$$\ln u = \ln(x^{3x}) = 3x\ln x \Rightarrow \frac{d}{dx}\ln u = \frac{1}{u}u' = 3\ln x + 3$$

So $u' = u(3\ln x + 3) = x^{3x}(3\ln x + 3)$. Substituting this into $f'(x) = e^u\cdot u'$, we can finally get

$$f'(x) = e^{x^{3x}}\cdot x^{3x}(3\ln x + 3)$$

3. the function consists of the function $g(x) = e^x$ and the function $h(x) = x^{\sin x}$, so the differentiation

$$f'(x) = g'(x)h(x) + g(x)h'(x) = \left(\frac{d}{dx}e^x\right)x^{\sin x} + e^x\left(\frac{d}{dx}x^{\sin x}\right)$$

The function $h(x)$ is a variable exponent function, so let $u = x^{\sin x}$ and take the natural log of both sides $\ln u = \ln x^{\sin x} = \sin x\cdot\ln x$. Then

$$\frac{d}{dx}\ln u = \frac{1}{u}u' = \frac{d}{dx}(\sin x\cdot\ln x) = \cos x\cdot\ln x + \frac{\sin x}{x}$$

So we get $u' = u(\cos x\cdot\ln x + \frac{\sin x}{x}) = x^{\sin x}(\cos x\cdot\ln x + \frac{\sin x}{x})$. Substituting this into $f'(x)$

$$f'(x) = e^x\sin x + e^x x^{\sin x}\left(\cos x\cdot\ln x + \frac{\sin x}{x}\right)$$

▷ Implicit Differentiation

An implicit function is a non-standard form of a function on the two-dimensional plane. It is expressed as an equation involving both variables x and y, and we cannot easily determine which variable is a function of the other. This is why it is called an "implicit" function. A classic example is the equation of a circle or an ellipse, both of which are implicit functions, such as

$$x^2 + y^2 = r^2 \quad \text{or} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

In this part, I'll show you how to get the derivative of y with respect to x. The key point is the **chain rule**. Consider the form

$$F(x, y) = f(x)g(y) = C \tag{1.1.28}$$

Where $F(x, y) = 0$ is the standard form of an implicit function. Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(f(x)g(y)) = \left(\frac{d}{dx}f(x)\right)g(y) + f(x)\left(\frac{d}{dx}g(y)\right) = 0 \tag{1.1.29}$$

Note that $g(y)$ is a function with respect to y, so we need to apply the chain rule to it. That is

$$\frac{d}{dx}g(y) = \frac{dg(y)}{dy} \cdot \frac{dy}{dx} \tag{1.1.30}$$

Substituting (1.1.30) into (1.1.29), we obtain

$$\left(\frac{d}{dx}f(x)\right)g(y) + f(x)\left(\frac{dg(y)}{dy} \cdot \frac{dy}{dx}\right) = 0 \tag{1.1.31}$$

So the derivative of y is

$$\frac{dy}{dx} = \frac{-f'(x)g(y)}{f(x)g'(y)} \tag{1.1.32}$$

For example, we can use implicit differentiation to solve the problem of finding the tangent line to a circle, which is commonly encountered in high school mathematics. There is an equation of the circle $(x-1)^2 + (y-2)^2 = 4$, and we need to get dy/dx of it. First,

$$\frac{d}{dx}(x-1)^2 + \frac{d}{dx}(y-2)^2 = 0 \Rightarrow (2x-2) + \left(\frac{d}{dy}(y-2)^2 \cdot \frac{dy}{dx}\right) = 0$$

The differentiation $d/dy\,(y-2)^2$ is $2y-4$. Substituting it into the original equation,

$$(2x-2) + (2y-4)\frac{dy}{dx} = 0$$

So the final answer is

$$\frac{dy}{dx} = \frac{-2x+2}{2y-4}$$

The slope of the tangent line passing through the point $(2, \sqrt{3}+2)$ is $dy/dx = (-2\times 2 + 2)/(2\times(\sqrt{3}+2)-4) = -1/\sqrt{3}$. There is another classic example. Consider an implicit function

$$y^7 - y^3 + x^3y^3 - 6x = 8$$

If we want to find dy/dx, we need to differentiate both sides

$$\frac{d}{dx}y^7 - \frac{d}{dx}y^3 + \frac{d}{dx}(x^3y^3) - \frac{d}{dx}6x = 0$$

$$\Rightarrow 7y^6\frac{dy}{dx} - 3y^2\frac{dy}{dx} + 3x^2y^3 + 2x^3y^2\frac{dy}{dx} - 6 = 0$$

$$\Rightarrow \frac{dy}{dx}(7y^6 - 3y^2 + 2x^3y^2) = 6 - 3x^2y^3$$

$$\Rightarrow \frac{dy}{dx} = \frac{6 - 3x^2y^3}{7y^6 - 3y^2 + 2x^3y^2}$$

This brings us to the final answer. Below, I've created two challenging problems that integrate all the differentiation techniques covered in this book. If you can solve both of them independently, it means you've truly mastered the concepts, and it's also a great way to test whether you've been paying attention in class or reading this book carefully. :)

Example. *Find the derivative dy/dx of the following implicit functions.*

1. $y = 2^{x^y}$
2. $y^{\exp(\sin x)} = x^{\sin x}$

1. This implicit function is an variable exponent function, and it must be differentiated using logarithmic differentiation. That is

$$\ln y = \ln\left(2^{x^y}\right) = x^y \cdot \ln 2$$

Differentiating both sides, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(x^y)\ln 2 + x^y\frac{d}{dx}\ln 2 = \frac{d}{dx}(x^y)\ln 2$$

Let $x^y = \eta$ then

$$\frac{d}{dx}\ln\eta = \frac{1}{\eta}\left(\frac{d}{dx}\eta\right) = \frac{d}{dx}(y\ln x) = \frac{dy}{dx}\ln x + \frac{y}{x}$$

Substituting this into $x^y = \eta$, we get

$$\frac{d}{dx}\eta = \frac{d}{dx}(x^y) = x^y\left(\frac{dy}{dx}\ln x + \frac{y}{x}\right)$$

Put this result back to the original equation

$$\frac{dy}{dx} = y\left(\frac{d}{dx}(x^y)\ln 2\right) = y\left[x^y\left(\frac{dy}{dx}\ln x + \frac{y}{x}\right)\ln 2\right]$$

By rearranging the equation, we obtain the final result that

$$\frac{dy}{dx} = \frac{\ln 2\cdot\ y^2}{x - x\ln 2\cdot\ y\cdot\ x^y\cdot\ln x}$$

2. This implicit function is a variable exponent function, too. So, it must be differentiated using logarithmic differentiation.

$$e^{\sin x} \ln y = \sin x \ln x$$

Differentiating both sides, we get

$$\begin{cases} \dfrac{d}{dx}(e^{\sin x} \ln y) = \cos x e^{\sin x} \ln y + e^{\sin x} \cdot \dfrac{1}{y} \dfrac{dy}{dx} \\ \dfrac{d}{dx}(\sin x \ln x) = \cos x \ln x + \sin x \cdot \dfrac{1}{x} \end{cases}$$

Combining them to get dy/dx

$$\frac{dy}{dx} = \frac{y \cos x (x \ln x - x \cdot e^{\sin x} \ln y) + y \sin x}{x e^{\sin x}}$$

1.2 Fundamental Techniques of Integration

▷ Fundamental Theorem of Calculus

Consider the functions $F(x),\ f(x)$, where $f, F : [a, b] \to \mathbb{R}$ and $a, b \in \mathbb{R}$, and they satisfy

$$F'(x) = f(x) \tag{1.2.1}$$

Then, we call $F(x)$ the antiderivative of $f(x)$. To find the antiderivative, we create a notation $\int$, which is

$$\int f(x)\ dx = F(x) + C \tag{1.2.2}$$

(1.2.2) means that if we know a function $f(x)$, its antiderivative is not unique since any two antiderivatives can differ by a constant. You can check this in the equation (1.2.1). By (1.2.1) and (1.2.2), we know that differentiation and integration are inverse operations. This book does not cover the definition of Riemann sums in detail. If you have forgotten it, please review your high school mathematics textbook.

To rigorously establish the connection between differentiation and integration, we introduce the **Fundamental Theorem of Calculus**(FTC). This theorem consists of two parts. Let's begin with the first part(FTC I). The antiderivative $F(x)$ of $f(x)$ is defined to be

$$\boxed{F(x) = \int_a^x f(t)\ dt} \tag{1.2.3}$$

By the definition, the derivative of $F(x)$ is

$$\boxed{F'(x) = \frac{d}{dx}\int_a^x f(t)\ dt = f(x)} \tag{1.2.4}$$

We can extend it to a more general definition. When (1.2.3) becomes

$$F(x) = \int_a^{g(x)} f(t)\, dt \tag{1.2.5}$$

Then the derivative of $F(x)$ must be applied to the chain rule

$$F'(x) = \frac{d}{du}\frac{du}{dx}\int_a^u f(t)\, dt = f(u) \cdot \frac{du}{dx}$$

where $u = g(x)$, so the general form, it satisfy

$$F'(x) = \frac{d}{dx}\int_a^{g(x)} f(t)\ dt = f(g(x)) \cdot g'(x) \tag{1.2.6}$$

In short, FTC I tells us that integrating a function gives an antiderivative, and differentiating it brings us back to the original function.
We now turn to the Second Fundamental Theorem of Calculus (FTC II), which is based on the assumption that FTC I holds. If $x \in [a, b]$, and $F'(x) = f(x)$, with $f(x)$ integrable

over $[a, b]$, then we can formally link the operations of definite integration, differentiation, and antiderivatives. That is,

$$\boxed{\int_a^b f(x)\ dx = F(x)\big|_a^b = F(b) - F(a)} \tag{1.2.7}$$

Simply put, the indefinite integral is a kind of "inverse operation" that represents the set of all antiderivatives of a function $f(x)$. The First Fundamental Theorem of Calculus (FTC I) describes the differentiability of the accumulation function $F(x)$, and shows that differentiating it recovers the original function $f(x)$. Finally, the Second Fundamental Theorem of Calculus (FTC II) states that the difference of the accumulation function at the endpoints a and b is equal to the area under its derivative $f(x)$ over the interval $[a, b]$, thereby connecting the concepts of differentiation, derivatives, antiderivatives, and Riemann sums.

Although both parts of the Fundamental Theorem of Calculus are deeply related, they have a logical order: FTC I must be established first. This theorem shows that the integral of a continuous function defines a differentiable function whose derivative recovers the integrand. Only after FTC I is proven can we then use this result to define and compute definite integrals using antiderivatives, which is what FTC II asserts. Therefore, FTC II depends logically on FTC I, not the other way around.

▷ **Integration by Substitution**

The function $F(u)$ is the antiderivative of the function $f(u)$, which is

$$F'(u) = f(u) \tag{1.2.8}$$

Consider a composite function $F(g(x))$, by the chain rule, we obtain

$$\frac{d}{dx}\left[F(g(x))\right] = F'(g(x)) \cdot g'(x) \tag{1.2.9}$$

Apply FTC and let the function $g(x)$ be u, we can thus verify

$$\int F'(g(x)) \cdot g'(x)dx = \int \frac{d}{dx}\left[F(g(x))\right] dx = F(g(x)) + C = F(u) + C \tag{1.2.10}$$

Note that we can rewrite the $F(u) + C$ as an indefinite integral and by (1.2.8), we obtain

$$F(u) + C = \int F'(u)du = \int f(u)du \tag{1.2.11}$$

By (1.2.10) and (1.2.11), we can finally verify that

$$\boxed{\int f(g(x)) \cdot g'(x)\ dx = \int f(u)\ du} \tag{1.2.12}$$

This is the **substitution rule** in integration. Substitution requires that the substitution function $g(x)$ is differentiable and the resulting integral is well-defined. Improper substitutions can lead to incorrect results. So that you can quickly write down the equation

$$du = g'(x)\ dx$$

Note that if you are dealing with a definite integral, you must change the limits of integration to match the new variable u. For example

$$\int_a^b f(g(x)) \cdot g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du \tag{1.2.13}$$

Since $u = g(x)$, $a \to g(a)$, $b \to g(b)$. However, if you're feeling lazy (like I usually am), you can first compute the antiderivative in terms of u, and then substitute $u = g(x)$ back into the result. That way, you don't have to change the limits. For example, we want to evaluate the integral

$$\int \frac{1}{x^2}\sqrt{1+\frac{1}{x}}\ dx$$

Let $u = 1 + x^{-1}$, then $du = -x^{-2}dx$, so the integral becomes

$$\int \sqrt{1+\frac{1}{x}}\left(\frac{1}{x^2}\ dx\right) = \int \sqrt{u}(-du) = -\int \sqrt{u}\ du = -2u^{3/2} + C$$

substituting $u = 1 + x^{-1}$ back into the result, we can obtain

$$\int \frac{1}{x^2}\sqrt{1+\frac{1}{x}}\ dx = \left(1+\frac{1}{x}\right)^{-3/2} + C$$

Example. *Evaluate the following integrals.*

1. $\displaystyle\int \frac{(\ln x)^2}{3x}\ dx$
2. $\displaystyle\int_1^2 (x-1)e^{(x-1)^2}\ dx$

1. Let $\ln x = u$, so $du = d/dx(\ln x)\ dx = 1/x\ dx$, the integral becomes

$$\int \frac{(\ln x)^2}{3x}\ dx = \frac{1}{3}\int (\ln x)^2\left(\frac{1}{x}dx\right) = \frac{1}{3}\int u^2\ du = \frac{1}{9}u^3 + C$$

substituting $u = \ln x$ back into the result, we can obtain the answer

$$\int \frac{(\ln x)^2}{3x}\ dx = \frac{1}{9}(\ln x)^3 + C$$

2. Let $(x-1)^2 = u$, then $du = 2(x-1)$, the integral becomes

$$\int_1^2 (x-1)e^{(x-1)^2}\ dx = \int_1^2 e^{(x-1)^2}\left[\frac{1}{2}\cdot 2(x-1)dx\right] = \frac{1}{2}\int_0^1 e^u du = \frac{1}{2}e^u\Big|_0^1 = \frac{e-1}{2}$$

▷ Integration by Parts

Next, we introduce what I consider to be one of the most important techniques in integration, **integration by parts**. When dealing with a complicated function, this

method allows you to eliminate the interference of polynomial terms (see Chapter 4). Moreover, in Chapter 6 of Taylor's *Classical Mechanics*, the variational method (see Chapter 5) is also built upon this technique.
Consider two functions $u,\ v$ with respect to x, which satisfy the rule of differentiation

$$\frac{d}{dx}(uv) = \left(\frac{d}{dx}u\right)v + u\left(\frac{d}{dx}v\right) \tag{1.2.14}$$

Which can be simplified to

$$\frac{d}{dx}(uv) = u'v + uv' \tag{1.2.15}$$

Then it becomes

$$d(uv) = u'v\ dx + uv'\ dx \tag{1.2.16}$$

Integrate both sides over the interval $[a, b]$, we obtain

$$\int_a^b d(uv) = uv|_a^b = \int_a^b u'v\ dx + \int_a^b uv'\ dx \tag{1.2.17}$$

Bring any term from the right side of the equation to the left, we can finally get the formula

$$\boxed{\int_a^b uv'\ dx = uv|_a^b - \int_a^b u'v\ dx} \tag{1.2.18}$$

Sometimes, we may write

$$\int_a^b u\ dv = uv|_a^b - \int_a^b v\ du \tag{1.2.19}$$

Since $u'dx = du,\ v'dx = dv$.
When you encounter a product of several functions, first check whether it can be solved using substitution. If that doesn't work, try expanding or rewriting the expression to see if substitution becomes possible. If all else fails, it's time to consider using integration by parts. *"If once is not enough, do it twice."*. That's my guiding principle when dealing with such cases.

Example. *Evaluate the following integral*

$$\int x \ln x\, dx$$

Let $x = u',\ \ln x = v$, by integration by parts, we obtain

$$\begin{aligned}\int x \ln x\ dx &= \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2 \cdot \frac{1}{x} dx \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\ dx\end{aligned}$$

So the final answer is

$$\int x \ln x\ dx = \frac{1}{4}x^2(2\ln x - 1) + C$$

▷ Exponential and Logarithmic Integral

In the section on the Fundamental Theorem of Calculus, we have already discussed that integration and differentiation are inverse operations. So, the first one

$$\frac{d}{dx}b^x = b^x \cdot \ln b \Leftrightarrow \int b^x \, dx = \frac{b^x}{\ln b} + C \tag{1.2.20}$$

The second term is

$$\frac{d}{dx}e^{ax} = ae^{ax} \Leftrightarrow \int e^{ax}dx = \frac{1}{a}e^{ax} + C \tag{1.2.21}$$

The most important one is the integral of $1/x$, which is

$$\frac{d}{dx}\ln x = \frac{1}{x} \Leftrightarrow \int \frac{1}{x} \, dx = \ln x + C \tag{1.2.22}$$

The equation (1.2.22) often appears in the integration by substitution. Consider an integral

$$\int \frac{g'(x)}{g(x)}dx \tag{1.2.23}$$

Let $u = g(x)$, then $du = g'(x)dx$, so the integral becomes

$$\int \frac{g'(x)}{g(x)}dx = \int \frac{1}{u}du = \ln u + C = \ln g(x) + C \tag{1.2.24}$$

For example,

$$\int \frac{x}{x^2 - 1}dx$$

Let $x^2 - 1 = u, \ du = 2xdx$, so

$$\int \frac{x}{x^2 - 1}dx = \int \frac{1}{2u}du = \frac{1}{2}\ln u + C = \frac{1}{2}\ln(x^2 - 1) + C$$

After introducing the three most fundamental types of integrals, we will now discuss the integration of logarithmic functions, beginning with the natural logarithm. Note that this requires the technique of integration by parts. If you've reached this point and still don't understand how integration by parts works, please turn back to the previous page and review it first.

The integral

$$\int \ln x \, dx \tag{1.2.25}$$

Let $u = x, \ v = \ln x$, so

$$\int \ln x \, dx = \int u'vdx = x\ln x - \int x \cdot \frac{1}{x} \, dx = x\ln x - x + C \tag{1.2.26}$$

Next, we will deal with the general logarithmic function, where the base is not e. In such cases, we must use the change of base formula to convert all logarithmic expressions into natural logarithms (ln).

The integral

$$\int \log_a g(x)dx = \int \frac{\ln g(x)}{\ln a}dx = \frac{1}{\ln a}\int \ln g(x)dx \tag{1.2.27}$$

Let $u = x,\ v = \ln g(x)$. Note that the derivative of v is

$$v' = \frac{1}{g(x)} \cdot g'(x) \tag{1.2.28}$$

So we apply the integration by parts to $\int \ln g(x)dx$, we can obtain

$$\int \ln g(x)dx = x \ln g(x) - \int x \cdot \frac{g'(x)}{g(x)}dx + C \tag{1.2.29}$$

So, the final result of the integral (1.2.27) becomes

$$\int \log_a g(x)dx = \frac{x \ln g(x)}{\ln a} - \int \frac{x}{\ln a} \cdot \frac{g'(x)}{g(x)}dx + C \tag{1.2.30}$$

The integral on the right-hand side of the equation depends on the form of $g(x)$.

Example. *Evaluate the following integrals.*

1. $\displaystyle\int \left(xe^x + \frac{\ln x}{x} + x2^x\right) dx$

2. $\displaystyle\int e^{-\beta x} \sin(bx)\, dx$

1. The integral can be decomposed to

$$\int x \cdot e^x\ dx + \int \frac{\ln x}{x}\ dx + \int x \cdot 2^x\ dx$$

We can let $x = u,\ e^x = v'$ of the first term, we obtain

$$\int x \cdot e^x\ dx = xe^x - \int e^x\ dx = e^x(x-1) + C$$

In the second term, let $\ln x = u,\ v' = 1/x \Rightarrow v = \ln x$, we obtain

$$\int \frac{\ln x}{x}\ dx = (\ln x)^2 - \int \frac{\ln x}{x}\ dx \Rightarrow 2\int \frac{\ln x}{x}\ dx = (\ln x)^2$$

So the final result of the second term is

$$\int \frac{\ln x}{x}\ dx = \frac{(\ln x)^2}{2}$$

In the final term, let $x = u,\ 2^x = v' \Rightarrow v = 2^x/\ln 2$, we get

$$\int x \cdot 2^x\ dx = \frac{x \cdot 2^x}{\ln 2} - \int \frac{2^x}{\ln 2}\ dx = \frac{x \cdot 2^x}{\ln 2} - \frac{2^x}{(\ln 2)^2} + C$$

Combine these results together, we can thus get the answer to the problem

$$\int \left(xe^x + \frac{\ln x}{x} + x2^x\right) dx = e^x(x-1) + \frac{(\ln x)^2}{2} + \frac{x \cdot 2^x}{\ln 2} - \frac{2^x}{(\ln 2)^2} + C$$

2. Let $e^{-\beta x} = u'$, $\sin(bx) = v$, so

$$\int e^{-\beta x}\sin(bx)\,dx = \frac{-1}{\beta}e^{-\beta x}\sin(bx) + \frac{b}{\beta}\int e^{-\beta x}\cos(bx)\,dx$$

Note that we can get $\sin(bx)$ when we apply integration by parts again, which is

$$\frac{b}{\beta}\int e^{-\beta x}\cos(bx)\,dx = -\frac{b}{\beta^2}e^{-\beta x}\cos(bx) - \frac{b^2}{\beta^2}\int e^{-\beta x}\sin(bx)dx$$

So we obtain

$$\int e^{-\beta x}\sin(bx)\,dx = \frac{-1}{\beta}e^{-\beta x}\sin(bx) - \frac{b}{\beta^2}e^{-\beta x}\cos(bx) - \frac{b^2}{\beta^2}\int e^{-\beta x}\sin(bx)dx$$

$$\Rightarrow \frac{\beta^2+b^2}{\beta^2}\int e^{-\beta x}\sin(bx)\,dx = \frac{-1}{\beta}e^{-\beta x}\sin(bx) - \frac{b}{\beta^2}e^{-\beta x}\cos(bx)$$

The final answer is

$$\int e^{-\beta x}\sin(bx)\,dx = -\frac{\beta}{\beta^2+b^2}e^{-\beta x}\sin(bx) - \frac{b}{\beta^2+b^2}e^{-\beta x}\cos(bx) + C$$

▷ Trigonometric Substitution

Trigonometric integrals and trigonometric substitutions are among the most important techniques in integral calculus. In the analysis of physical systems, the appearance of trigonometric functions often implies information about the system's periodicity, trajectories, and underlying symmetries. Thus, understanding a physical system often begins with mastering these types of integrals. We will start with basic trigonometric integrals, which can be viewed as the reverse process of differentiation.
The basic trigonometric differentiations are sin and cos. Which is

$$\boxed{\begin{cases}\int \sin\theta\,d\theta = -\cos\theta + C\\ \int \cos\theta\,d\theta = \sin\theta + C\end{cases}} \tag{1.2.31}$$

Other trigonometric integrals require certain techniques to evaluate. We begin with those involving $\tan\theta$,

$$\int \tan\theta\,d\theta = \int \frac{\sin\theta}{\cos\theta}d\theta$$

Let $\cos\theta = u$, $du = -\sin\theta\,d\theta$, so

$$\int \frac{\sin\theta}{\cos\theta}d\theta = -\int \frac{1}{u}du = -\ln u \Rightarrow \boxed{\int \tan\theta\,d\theta = -\ln\cos\theta + C} \tag{1.2.32}$$

And the next one is $\cot\theta$, which is

$$\int \cot\theta\,d\theta = \int \frac{\cos\theta}{\sin\theta}d\theta$$

Let $\sin\theta = u$, $du = \cos\theta\, d\theta$, so

$$\int \frac{\cos\theta}{\sin\theta} d\theta = \int \frac{1}{u} du = \ln u \Rightarrow \boxed{\int \cot\theta\, d\theta = \ln\sin\theta + C} \tag{1.2.33}$$

The third one, $\sec\theta$, is the most skillful one in the trigonometric integral.

$$\int \sec\theta\, d\theta = \int \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta} d\theta$$

Let $\sec\theta + \tan\theta = u$, $du = (\tan\theta\sec\theta + \sec^2\theta)d\theta$, so the integral can be written as

$$\begin{aligned} &\int \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta} d\theta = \int \frac{1}{u} du = \ln u \\ &\Rightarrow \boxed{\int \sec\theta\, d\theta = \ln(\sec\theta + \tan\theta) + C} \end{aligned} \tag{1.2.34}$$

The final one is $\csc\theta$, we can use the former results to calculate the integral,

$$\begin{aligned} \int \csc\theta\, d\theta = \int \frac{1}{\sin\theta} d\theta &= \int \frac{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} d\theta \\ &= \frac{1}{2}\left(\int \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} d\theta + \int \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} d\theta\right) \end{aligned}$$

Note that when $u = \cos(\theta/2)$, $du = -(1/2)\sin(\theta/2)$, so the coefficient $1/2$ disappeared. By the results (1.2.33) and (1.2.32), we can obtain

$$\boxed{\int \csc\theta\, d\theta = \int \frac{1}{\sin\theta} d\theta = \ln\tan\left(\frac{\theta}{2}\right) + C} \tag{1.2.35}$$

Trigonometric integrals do not have a unique form. You can express the answer using various trigonometric identities. Likewise, there is no single correct method for evaluating these integrals, as long as you choose an appropriate substitution or transformation, even involving special functions, your approach is valid. The method provided here is just one reference path. So if you see different solutions or final expressions elsewhere, don't panic!

After these trigonometric integrals, we can continue to deal with the trigonometric substitution. There are six forms of integrals, and every type of integral has its corresponding substitution method.

$$\begin{cases} \displaystyle\int \frac{b}{\sqrt{a^2 - x^2}} dx \Rightarrow \text{Let } x = a\sin\theta \\ \displaystyle\int \frac{b}{\sqrt{a^2 + x^2}} dx \Rightarrow \text{Let } x = a\tan\theta \\ \displaystyle\int \frac{b}{\sqrt{x^2 - a^2}} dx \Rightarrow \text{Let } x = a\sec\theta \end{cases} \qquad \begin{cases} \displaystyle\int \frac{b}{a^2 - x^2} dx \Rightarrow \text{Let } x = a\sin\theta \\ \displaystyle\int \frac{b}{a^2 + x^2} dx \Rightarrow \text{Let } x = a\tan\theta \\ \displaystyle\int \frac{b}{x^2 - a^2} dx \Rightarrow \text{Let } x = a\sec\theta \end{cases} \tag{1.2.36}$$

1. The integral
$$\int \frac{b}{\sqrt{a^2 - x^2}} dx$$
Let $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$, so the integral becomes
$$\int \frac{b}{\sqrt{a^2 - x^2}} dx = b\int \frac{a\cos\theta\, d\theta}{a\cos\theta} = b\theta + C$$
We're not done yet, recall that our original variable was x, so we need to express the result of the integral in terms of x. Since we made the substitution $x = a\sin\theta$, we can invert it using the inverse trigonometric function
$$x = a\sin\theta \quad \Rightarrow \quad \theta = \sin^{-1}\left(\frac{x}{a}\right) = \arcsin\left(\frac{x}{a}\right) \tag{1.2.37}$$
Substituting this back into our earlier result, we obtain the final answer
$$\int \frac{b}{\sqrt{a^2 - x^2}}\, dx = \frac{b}{a}\theta + C = b\arcsin\left(\frac{x}{a}\right) + C \tag{1.2.38}$$

2. The integral
$$\int \frac{b}{\sqrt{a^2 + x^2}} dx$$
Let $x = a\tan\theta$, $dx = a\sec^2\theta\, d\theta$, so the integral becomes
$$\int \frac{b}{\sqrt{a^2 + x^2}} dx = b\int \frac{a\sec^2\theta\, d\theta}{a\sec\theta} = b\int \sec\theta\, d\theta = b\ln(\sec\theta + \tan\theta) + C$$
Because $x = a\tan\theta$, so $\tan\theta = x/a$; $\sec\theta = \sqrt{a^2 + x^2}/a$, so we substitute this back into our earlier result, we obtain the final answer
$$\int \frac{b}{\sqrt{a^2 + x^2}} dx = b\ln(\sec\theta + \tan\theta) + C = b\ln(x + \sqrt{a^2 + x^2}) + C \tag{1.2.39}$$
The term $-b\ln a$ is absorbed into the constant of integration.

3. The integral
$$\int \frac{b}{\sqrt{x^2 - a^2}} dx$$
Remember the identity $\sec\theta - 1 = \tan\theta$, so we let $x = a\sec\theta$, $dx = a\sec\theta\tan\theta\, d\theta$, the integral becomes
$$\int \frac{b}{\sqrt{x^2 - a^2}} dx = b\int \frac{a\sec\theta\tan\theta\, d\theta}{a\tan\theta} = b\int \sec\theta\, d\theta = b\ln(\sec\theta + \tan\theta) + C$$
Because $x = a\sec\theta$, so $\sec\theta = x/a$; $\tan\theta = \sqrt{x^2 - a^2}/a$, so the final answer is familiar with the last one
$$\int \frac{b}{\sqrt{x^2 - a^2}} dx = b\ln(\sec\theta + \tan\theta) + C = b\ln(x + \sqrt{x^2 - a^2}) + C \tag{1.2.40}$$

4. The integral

$$\int \frac{b}{a^2 - x^2} dx$$

Let $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$, so

$$\int \frac{b}{a^2 - x^2} dx = b\int \frac{a\cos\theta\, d\theta}{a^2\cos^2\theta} = \frac{b}{a}\int \sec\theta\, d\theta = \frac{b}{a}\ln(\sec\theta + \tan\theta) + C$$

Because $x = a\sin\theta$, so $\sec\theta = a/\sqrt{a^2 - x^2}$; $\tan\theta = x/\sqrt{a^2 - x^2}$, we can obtain

$$\int \frac{b}{a^2 - x^2} dx = \frac{b}{a}\ln(\sec\theta + \tan\theta) + C = \frac{b}{a}\ln\left(\frac{a + x}{\sqrt{a^2 - x^2}}\right) + C \tag{1.2.41}$$

5. The integral

$$\int \frac{b}{a^2 + x^2} dx$$

Let $x = a\tan\theta$, $dx = a\sec^2\theta\, d\theta$, so

$$\int \frac{b}{a^2 + x^2} dx = b\int \frac{a\sec^2\theta\, d\theta}{a^2\sec^2\theta} = \frac{b}{a}\int d\theta = \frac{b}{a}\theta + C$$

Where $\theta = \arctan(x/a)$, and we can obtain the final answer

$$\int \frac{b}{a^2 + x^2} dx = \frac{b}{a}\theta + C = \frac{b}{a}\arctan\left(\frac{x}{a}\right) + C \tag{1.2.42}$$

6. We have finally reached the final stage of trigonometric substitution. The integral

$$\int \frac{b}{x^2 - a^2} dx$$

Let $x = a\sec\theta$, $dx = a\tan\theta\sec\theta\, d\theta$, so

$$\int \frac{b}{x^2 - a^2} dx = b\int \frac{a\sec\theta\tan\theta\, d\theta}{a^2\tan\theta} = \frac{b}{a}\int \sec\theta\ d\theta = \frac{b}{a}\ln(\sec\theta + \tan\theta) + C$$

Because $\sec\theta = x/a$, we get $\tan\theta = \sqrt{x^2 - a^2}/a$. The final answer

$$\int \frac{b}{x^2 - a^2} dx = \frac{b}{a}\ln(\sec\theta + \tan\theta) + C = \frac{b}{a}\ln(x + \sqrt{x^2 - a^2}) + C \tag{1.2.43}$$

The six forms above serve as standard templates. More complex expressions are often transformations or combinations of these cases. In later examples, you are encouraged to identify how they relate to these core types.

Example. *Evaluate the following integrals.*

1. $\displaystyle\int \frac{x^2}{(a^2 + x^2)^{3/2}} dx$
2. $\displaystyle\int \frac{1}{x^2\sqrt{x^2 - a^2}} dx$

1. The integral
$$\int \frac{x^2}{(a^2+x^2)^{3/2}}dx$$
Let $x = a\tan\theta$, $dx = a\sec^2\theta\, d\theta$, so
$$\int \frac{x^2}{(a^2+x^2)^{3/2}}dx = \int \frac{a^2\tan^2\theta \cdot a\sec^2\theta\, d\theta}{a^3\sec^3\theta} = \int \frac{\tan^2\theta\, d\theta}{\sec\theta} = \int \frac{\sin^2\theta}{\cos\theta}d\theta$$
The identity $\sin^2\theta = 1-\cos^2\theta$, we substitute this into the integral
$$\int \frac{\sin^2\theta}{\cos\theta}d\theta = \int \frac{1-\cos^2\theta}{\cos\theta}d\theta = \int \sec\theta\, d\theta - \int \cos\theta\ d\theta = \ln(\sec\theta+\tan\theta) - \sin\theta + C$$
Because $x = a\tan\theta$, we can get $\sec\theta = \sqrt{x^2+a^2}/a$; $\sin\theta = x/\sqrt{x^2+a^2}$. We obtain
$$\int \frac{x^2}{(a^2+x^2)^{3/2}}dx = \ln(\sqrt{x^2+a^2}+x) - \frac{x}{\sqrt{x^2+a^2}} + C$$

2. The integral
$$\int \frac{1}{x^2\sqrt{x^2-a^2}}dx$$
Let $x = a\sec\theta\, d\theta$, $dx = a\tan\theta\sec\theta\, d\theta$, so
$$\int \frac{1}{x^2\sqrt{x^2-a^2}}dx = \int \frac{a\tan\theta\sec\theta\, d\theta}{a^2\sec^2\theta\cdot a\tan\theta} = \frac{1}{a^2}\int \frac{1}{\sec\theta}d\theta = \frac{1}{a^2}\int \cos\theta\, d\theta = \frac{1}{a^2}\sin\theta + C$$
By the equation $x = a\sec\theta$ we get $\sin\theta = \sqrt{x^2-a^2}/x$. So, the final answer is
$$\int \frac{1}{x^2\sqrt{x^2-a^2}}dx = \frac{1}{a^2}\sin\theta + C = \frac{1}{a^2}\cdot\frac{\sqrt{x^2-a^2}}{x} + C$$

▷ **Hyperbolic Functions**

Back in high school, we learned about the standard form of a hyperbola
$$x^2 - y^2 = 1 \tag{1.2.44}$$
In calculus, we extend this equation to define a new class of functions called the **hyperbolic functions**. If we look closely at the structure of this equation, it resembles that of the unit circle:
$$x^2 + y^2 = 1$$
The unit circle can be parametrized using angles and radii to describe points on the circle. This leads to the identity:
$$\cos^2\theta + \sin^2\theta = 1$$
Here, the angle θ is the key parameter in the circular parametrization. Now, we ask a natural question: Is there a similar kind of function and parameter that can describe points on the hyperbola? This motivates the definition of the hyperbolic functions, which satisfy the identity
$$\cosh^2 a - \sinh^2 a = 1 \tag{1.2.45}$$

This mirrors the structure of the circular identity and provides a way to parametrize the hyperbola. However, unlike the circular case where the parameter is an angle, the parameter a here is defined differently: it is **twice the area** enclosed by the ray from the origin, the hyperbola, and the x-axis, as illustrated in the figure below.

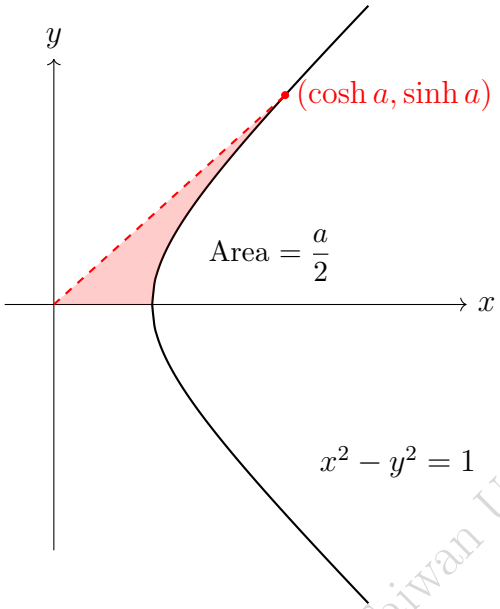

Figure 1: The illustration of the hyperbolic function and its definition of parameter

Interestingly, the hyperbolic functions $\sinh x$ and $\cosh x$ were not originally defined using exponentials. They were first introduced to describe geometric properties of the hyperbola, analogous to how trigonometric functions describe the circle. Later, it was discovered that these functions could be neatly expressed in terms of exponentials

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2} \tag{1.2.46}$$

This remarkable identity unites the geometric and algebraic aspects of the hyperbolic functions. Then, we can define other hyperbolic functions with similar forms as the trigonometric functions, that is

$$\begin{cases} \tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} \\ \coth x = \dfrac{1}{\tanh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}} \\ \operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{1}{e^x - e^{-x}} \\ \operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{1}{e^x + e^{-x}} \end{cases} \tag{1.2.47}$$

And the identities

$$\begin{cases} \tanh^2 x = 1 - \operatorname{sech}^2 x \\ \coth^2 x = 1 + \operatorname{csch}^2 x \end{cases} \tag{1.2.48}$$

At the same time, we can verify the differential relationships of the hyperbolic functions using their exponential definitions. Let's begin with the most fundamental ones

$$\begin{cases} \dfrac{d}{dx}\sinh x = \cosh x \\ \dfrac{d}{dx}\cosh x = \sinh x \end{cases} \tag{1.2.49}$$

Next, we consider the derivatives of the remaining hyperbolic functions, starting with $\tanh x$. We know that

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

To differentiate this expression, we apply the quotient rule

$$\begin{aligned} \frac{d}{dx}\tanh x &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) \\ &= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} \\ &= 1 - \tanh^2 x = \operatorname{sech}^2 x \end{aligned} \tag{1.2.50}$$

Using the same method, we can derive the derivatives of the other hyperbolic functions as follows:

$$\begin{cases} \dfrac{d}{dx}\coth x = -\operatorname{csch}^2 x \\ \dfrac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x \\ \dfrac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x \end{cases} \tag{1.2.51}$$

However, the most important aspect in this section is to explore the inverse relationships among these functions. In most integration problems, we don't necessarily need to use hyperbolic substitution, in fact, trigonometric substitution is often more than sufficient.

But in certain physical problems, such as the classic catenary problem, where hyperbolic functions were first discovered, the result of the integration may initially appear messy. Yet by expressing the solution in terms of hyperbolic functions, the form becomes clean and elegant, and the overall behavior of the function is immediately clear at a glance.

The inverse function of $\sinh x$ is defined as $\operatorname{arsinh} x$. Note! it is not "arc" but "ar", where "ar" means area. So, we have

$$y = \sinh x = \frac{e^x - e^{-x}}{2} \Leftrightarrow x = \sinh y = \frac{e^y - e^{-y}}{2} \tag{1.2.52}$$

So, by (1.2.52), we can construct a quadratic equation in one variable

$$\begin{aligned} \frac{e^y - e^{-y}}{2} = x &\Rightarrow e^{2y} - 1 = 2xe^y \\ &\Rightarrow (e^y)^2 - 2x(e^y) - 1 = 0 \\ &\Rightarrow e^y = \frac{2x + \sqrt{4x^2 + 4}}{2} = x + \sqrt{x^2 + 1} \end{aligned}$$

Note that we should take the positive term of the quadratic equation because $e^y > 0$. So the inverse function

$$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right) \tag{1.2.53}$$

Using the same method, we can derive the inverse function

$$\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2-1}\right) \tag{1.2.54}$$

Do these two functions look familiar to you? That's right, they already appeared as solutions to the trigonometric substitution integrals (1.2.39) and (1.2.40). In fact, these two integrals can be re-expressed in terms of inverse hyperbolic functions

$$\boxed{\begin{cases} \displaystyle\int \frac{b}{\sqrt{x^2+a^2}}\,dx = b\operatorname{arsinh}(ax) + C \\ \displaystyle\int \frac{b}{\sqrt{x^2-a^2}}\,dx = b\operatorname{arcosh}(ax) + C \end{cases}} \tag{1.2.55}$$

It's amazing that we can unify these two strikingly similar integrals using hyperbolic functions, it's so cool! In addition, we can get the same results by letting $x = a\cosh u$ or $x = a\sinh u$. I'll show you this in the case $\operatorname{artanh} x$.
Now, let's turn to the last and most commonly encountered inverse hyperbolic function: $\operatorname{artanh} x$.

$$y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \Leftrightarrow x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}} \tag{1.2.56}$$

Simplify the expression and rewrite the inverse function in terms of x.

$$\begin{aligned} \frac{e^y - e^{-y}}{e^y + e^{-y}} = x &\Rightarrow e^y - e^{-y} = x(e^y + e^{-y}) \\ &\Rightarrow (1-x)e^y = (1+x)e^{-y} \\ &\Rightarrow e^{2y} = \frac{1+x}{1-x} \\ &\Rightarrow y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \end{aligned}$$

So the inverse function

$$\operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \tag{1.2.57}$$

This result is, in fact, equivalent to equation (1.2.41). We will verify this in two parts. First, let's rewrite the form of equation (1.2.41). That is

$$\begin{aligned} \frac{b}{a}\ln\left(\frac{a+x}{\sqrt{a^2-x^2}}\right) &= \frac{b}{a}\left[\ln(a+x) - \frac{1}{2}\ln(a-x) - \frac{1}{2}\ln(a+x)\right] \\ &= \frac{b}{a}\cdot\frac{1}{2}\ln\left(\frac{a+x}{a-x}\right) \\ &= \frac{b}{a}\cdot\frac{1}{2}\ln\left(\frac{1+\frac{x}{a}}{1-\frac{x}{a}}\right) = \frac{b}{a}\operatorname{artanh}\left(\frac{x}{a}\right) \end{aligned}$$

The second method is the substitution, consider the integral

$$\int \frac{b}{a^2 - x^2} dx$$

Let $x = a \tanh u$, $dx = a \operatorname{sech}^2 u \, du$. We can rewrite the integral

$$\int \frac{b}{a^2 - x^2} dx = b \int \frac{a \operatorname{sech}^2 u \, du}{a^2 \operatorname{sech}^2 u} = \frac{b}{a} \int du = \frac{b}{a} u + C$$

By the equation $x = a \tanh u$, we obtain

$$u = \operatorname{artanh} \left(\frac{x}{a} \right)$$

So, we can verify this important result, which is

$$\boxed{\int \frac{b}{a^2 - x^2} dx = \frac{b}{a} \operatorname{artanh} \left(\frac{x}{a} \right) + C} \tag{1.2.58}$$

For example, we have an integral

$$\int \frac{dx}{1 - x^2/\omega^2} = \omega^2 \int \frac{dx}{\omega^2 - x^2}$$

Thus, it returns to the familiar integral form given in equation (1.2.41). By (1.2.58), we obtain the answer to the integral

$$\int \frac{dx}{1 - x^2/\omega^2} = \omega \operatorname{artanh} \left(\frac{x}{\omega} \right) + C$$

▷ **The Method of Partial Fraction**

Shapes like $\frac{N(x)}{D(x)}$, where $\deg(N) > \deg(D)$, can all be denoted by

$$\frac{N(x)}{D(x)} = \left(\sum_i C_i x^i \right) + \sum_i \frac{A_i}{(px + q)^i} + \sum_i \frac{A_i x + A_{i+1}}{(ax^2 + bx + c)^i} \tag{1.2.59}$$

Where $N(x)$, $D(x)$ are polynomials, and A_j, $C_j \in \mathbb{R}$. While we will not prove here why such a decomposition is always possible, this result comes from a fundamental theorem in algebra. Our focus will be on how to apply it effectively to compute integrals. If both the numerator and the denominator of a function are polynomials, we can decompose it into several simpler fractions and integrate them individually. This makes the original problem much easier to solve, this technique is known as the **method of partial fractions**.
For example, the fraction can be decomposed to partial fractions as shown below.

$$\frac{3x + 4}{(x - 1)(x^2 + 2x + 9)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 2x + 9}$$

Where A, B, $C \in \mathbb{R}$. After decomposing the expression into partial fractions, the most important step is to determine the constants in each term. Of course, you can expand

both sides and compare coefficients, but here, I will use the *limit method* to eliminate unnecessary terms, making it easier to identify the constants directly. Consider a fraction

$$\frac{N(x)}{D(x)} = \frac{A}{ax+b} + \frac{B}{x+c} + \frac{C}{(x+c)^2} + \frac{Dx+E}{dx^2+ex+f} \tag{1.2.60}$$

Where $D(x) = (ax+b)(x+c)^2(dx^2+ex+f)$. To solve for A, we multiply both sides of the equation by $ax+b$, that is

$$(ax+b)\frac{N(x)}{D(x)} = \frac{N(x)}{(x+c)^2(dx^2+ex+f)} = A+(ax+b)\left[\frac{B}{x+c} + \frac{C}{(x+c)^2} + \frac{Dx+E}{dx^2+ex+f}\right]$$

The limit

$$\lim_{x\to\frac{-b}{a}} A + \lim_{x\to\frac{-b}{a}} (ax+b)\left[\frac{B}{x+c} + \frac{C}{(x+c)^2} + \frac{Dx+E}{dx^2+ex+f}\right] = A$$

The limit of the left-hand side is totally the same as the right-hand side, so we can obtain the constant A, which is

$$\lim_{x\to\frac{-b}{a}} \frac{N(x)}{(x+c)^2(dx^2+ex+f)} = A \tag{1.2.61}$$

However, this method cannot be applied to solve for b because taking the limit would cause the term $C/(x+c)$ on the right-hand side to diverge. Instead, we must multiply both sides by $(x+c)^2$ before taking the limit in order to isolate C. Following the same logic as before, we can verify that

$$\lim_{x\to -c} \frac{N(x)}{(ax+b)(dx^2+ex+f)} = C \tag{1.2.62}$$

Suppose the factorization of dx^2+ex+f is $(\lambda x - \lambda' i)(\omega x - \omega' i)$, and we select one of the factors to take the limit. We apply the same method, we can get

$$\lim_{x\to\frac{\lambda'}{\lambda}i} \frac{N(x)}{(ax+b)(x+c)^2} = \left(\frac{D\lambda' i}{\lambda} + E\right) \tag{1.2.63}$$

If there are any remaining unknown constants, such as B, we can group them with the known constants and solve a system of equations. This allows us to determine all the constants involved.

Example. *Evaluate the following integrals.*

1. $\displaystyle\int \frac{x}{(x-1)(x-2)^2}dx$
2. $\displaystyle\int \frac{1}{x^3+1}dx$

1. First, apply the method of partial fraction

$$\int \frac{x}{(x-1)(x-2)^2}dx = \int \frac{A}{x-1}dx + \int \frac{B}{x-2}dx + \int \frac{C}{(x-2)^2}dx$$

Then we use the limit method to get $A,\ C$, that is

$$\begin{cases} A = \lim\limits_{x\to 1} \dfrac{x}{(x-2)^2} = 1 \\ C = \lim\limits_{x\to 2} \dfrac{x}{x-1} = 2 \end{cases}$$

By $A = 1$, $C = 2$, we can obtain the remaining constant $B = -1$. So we can start to calculate the integrals.

$$\int \frac{x}{(x-1)(x-2)^2}dx = \int \frac{1}{x-1}dx + \int \frac{-1}{x-2}dx + \int \frac{2}{(x-2)^2}dx$$

The final answer is

$$\int \frac{x}{(x-1)(x-2)^2}dx = \ln\left(\frac{x-1}{x-2}\right) + \frac{2}{x-2} + C$$

2. This problem is a classic example of the partial fraction. It looks simple, but it is actually complicated. Although this problem is difficult, it is also beautiful. It brings together all the integration techniques you've learned so far, which is why it is an important one.
We factorize the fraction to be

$$\int \frac{1}{x^3+1}dx = \int \frac{1}{(x+1)(x^2-x+1)}dx = \int \frac{A}{x+1}dx + \int \frac{Bx+C}{x^2-x+1}dx$$

Then we use the limit method to get $A,\ C$, that is

$$\begin{cases} A = \lim\limits_{x\to 1} \dfrac{1}{x^2-x+1} = \dfrac{1}{3} \\ C = \lim\limits_{x\to 0} \dfrac{1}{(x+1)(x^2-x+1)} - A = \dfrac{2}{3} \end{cases}$$

So the constant $B = -1/3$. Then, the integral becomes

$$\int \frac{A}{x+1}dx + \int \frac{Bx+C}{x^2-x+1}dx = \frac{1}{3}\int \frac{1}{x+1}dx - \frac{1}{3}\int \frac{x-2}{x^2-x+1}dx$$

We can start to calculate the integral

$$\frac{1}{3}\int \frac{1}{x+1}dx - \frac{1}{3}\int \frac{x-2}{x^2-x+1}dx = \frac{1}{3}\ln(x+1) - \frac{1}{6}\int \frac{2x-4}{x^2-x+1}dx$$

$$= \frac{1}{3}\ln(x+1) - \frac{1}{6}\int \frac{(2x-1)-3}{x^2-x+1}dx$$

$$= \frac{1}{3}\ln(x+1) - \frac{1}{6}\ln(x^2-x+1) - \frac{1}{6}\int \frac{-3}{x^2-x+1}dx$$

$$= \frac{1}{3}\ln(x+1) - \frac{1}{6}\ln(x^2-x+1) + \frac{1}{2}\int \frac{1}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}dx$$

Let $x - 1/2 = \sqrt{3}/2 \tan\theta$, $dx = \sqrt{3}/2 \sec^2\theta \, d\theta$, and the remaining integral becomes

$$\int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx = \int \frac{\frac{\sqrt{3}}{2} \sec^2\theta \, d\theta}{\frac{3}{4}\sec^2\theta} = \frac{2}{\sqrt{3}}\theta + C$$

Because $x - 1/2 = \sqrt{3}/2 \tan\theta$, then

$$\theta = \arctan\left(\frac{2}{\sqrt{3}}x - \frac{1}{\sqrt{3}}\right)$$

So, the final answer is

$$\int \frac{1}{x^3+1} dx = \frac{1}{3}\ln(x+1) - \frac{1}{6}\ln(x^2 - x + 1) + \frac{1}{\sqrt{3}} \arctan\left(\frac{2}{\sqrt{3}}x - \frac{1}{\sqrt{3}}\right) + C$$

1.3 Taylor Expansion

▷ General Form of Taylor Expansion

Taylor expansion is one of the most important mathematical tools in physics. It allows us to represent all types of functional relationships as an infinite series of polynomials. Moreover, this method provides an explanation for linear approximation behavior in small neighborhoods. Many modern mathematical techniques have been developed based on this fundamental property.
As introduced in *Classical Dynamics* by Thornton and Marion, we define a definite integral as

$$I \coloneqq \int_{x_0}^{x_0+h} f'(x)\, dx = f(x_0+h) - f(x_0) \tag{1.3.1}$$

let $x = x_0 + h - t$, where t is a variable, so $dx = -dt$ (This is the integration by substitution). We rewrite the integral as

$$I = \int_h^0 f'(x_0+h-t)\,(-dt) = \int_0^h f'(x_0+h-t)\, dt \tag{1.3.2}$$

We need to apply the integration by parts to decompose the integral

$$\int_0^h f'(x_0+h-t)\, dt = t\cdot f'(x_0+h-t)|_0^h + \int_0^h t\cdot f''(x_0+h-t)\, dt \tag{1.3.3}$$

Note that we need to apply the chain rule to get the derivative of $f'(x_0+h-t)$, which is

$$\frac{d}{dt}f'(x_0+h-t) = f''(x_0+h-t)\cdot(-1)$$

Therefore, the integration by parts in (1.3.3) results in a positive term. We will continue to use integration by parts to break down the integral expression.

$$\begin{aligned} t\cdot f'(x_0+h-t)|_0^h + \int_0^h t\cdot f''(x_0+h-t)\, dt &= hf'(x_0) + \frac{1}{2}t^2 f''(x_0+h-t)|_0^h + \cdots \\ &= hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \cdots + \frac{h^n}{n!}f^{(n)}(x_0) \end{aligned}$$

Note that the notation $f^{(n)}(x)$ represents the nth derivative of the function. So, the original integral (1.3.1)

$$\begin{aligned} I = f(x_0+h) - f(x_0) &= hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \cdots + \frac{h^n}{n!}f^{(n)}(x_0) \\ \Rightarrow f(x_0+h) &= f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \cdots + \frac{h^n}{n!}f^{(n)}(x_0) \end{aligned}$$

Here, x_0 represents the expansion point, and h denotes a small displacement from that point. This formulation makes it easier to introduce the concept of limits and the definition of derivatives (by considering $h \to 0$). Such an expansion is a **local approximation**, meaning that it describes the behavior of the function near x_0, including its value, slope, and curvature. As more terms are included, the approximation becomes more accurate,

and the closer x is to x_0, the smaller the error.
Let $x_0 = 0$, $h = x - 0$, then I describes the behavior of the function near the point $x = 0$, it's called the **Maclaurin Expansion**.

$$\boxed{f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots} \tag{1.3.4}$$

We can also write this as a series; it's called the Maclaurin series.

$$f(x) = \sum_i \frac{x^i f^{(i)}(0)}{i!} \tag{1.3.5}$$

In general, we can let $h = x - x_0$, then I describes the behavior of the function near the point x_0, where $x_0 \in f$, it's called the **Taylor Expansion**. It's denoted by

$$\boxed{f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)}{2!}f''(x_0) + \cdots + \frac{(x - x_0)}{n!}f^{(n)}(x_0) + \cdots} \tag{1.3.6}$$

We can also write this as a series; it's called the Taylor series.

$$f(x) = \sum_i \frac{(x - x_0)^i f^{(i)}(x_0)}{i!} \tag{1.3.7}$$

There are three important expansions,

$$\begin{cases} \sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots \\ \cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots \\ e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots \end{cases} \tag{1.3.8}$$

In addition, we can expand any differentiable, continuous, and smooth function $f(x)$, including functions such as the natural logarithm. For example,

$$\ln(x_0 + (x - x_0)) = \ln(x_0) + (x - x_0)\frac{d}{dx}\ln(x)|_{x=x_0} + \frac{(x - x_0)^2}{2!}\left.\frac{d^2\ln(x)}{dx^2}\right|_{x=x_0} + \cdots$$

$$= \ln(x_0) + \frac{x - x_0}{x_0} - \frac{(x - x_0)^2}{2x_0^2} + \frac{(x - x_0)^3}{3x_0^3} + \cdots$$

Note that $|x - x_0| < x_0$ since $\ln x, x > 0$. So, the Taylor series of $\ln(x)$ can be denoted by

$$\ln(x) = \ln(x_0) + \sum_i (-1)^{i+1}\frac{(x - x_0)^i}{i \cdot x_0^i} \tag{1.3.9}$$

Similarly, when $x_0 = 1,\ h = -\epsilon$, the function $\ln(1 - \epsilon)$ can be expanded into

$$\ln(1 - \epsilon) = -\epsilon - \frac{\epsilon^2}{2} - \frac{\epsilon^3}{3} - \cdots = -\sum_i \frac{\epsilon^i}{i} \tag{1.3.10}$$

▷ Linear Approximation

Taylor expansion allows us to approximate a function locally based on the choice of an expansion point x_0. This property enables us to describe how the function behaves under small variations in the input variable. If we take $x_0 = x,\ h = \Delta x$, which means that in a more general sense, the variation of a function can be described according to its domain. That is

$$f(x_0 + h) = f(x + \Delta x) = f(x) + \Delta x f'(x) + \frac{(\Delta x)^2}{2!} f''(x) + \cdots \tag{1.3.11}$$

The function increment Δf is defined as

$$\Delta f = f(x + \Delta x) - f(x) = \Delta x f'(x) + \frac{(\Delta x)^2}{2!} f''(x) + \cdots \tag{1.3.12}$$

Note that if we take $\|\Delta\| \to 0$, in the case where two points are very close to each other, the behavior of the function resembles motion along the tangent line. This is why we call it a **Linear Approximation**. That is, the ratio of change $\Delta y/\Delta x$ remains nearly constant, which corresponds to the slope you learned in high school. Since this approximates the tangent, the Taylor expansion only needs to retain the first-order derivative term (i.e., the slope of the tangent), and the higher-order terms can be neglected because the variation becomes extremely small. So, it becomes

$$\Delta f|_{\|\Delta\|\to 0} \coloneqq df = f(x + dx) - f(x) = f'(x)\ dx + O((dx)^2) \tag{1.3.13}$$

We keep only the first-order term and ignore all higher-order terms, which are collectively denoted as $\mathcal{O}((dx)^2)$. So the linear approximation is

$$\boxed{df \approx f'(x) \cdot dx} \tag{1.3.14}$$

We can also generalize the Taylor expansion to n dimensions, which can be denoted by

$$f(\mathbf{x} + \mathbf{a}) = f(\mathbf{x}) + \sum_i \frac{\partial f}{\partial x_i} a_i + \frac{1}{2!} \sum_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} a_i a_j + \frac{1}{3!} \sum_{ijk} \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k} a_i a_j a_k + \cdots$$

It can be written in a general form

$$f(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + \sum_{|\alpha|=1} \frac{1}{\alpha!} \partial^\alpha f(\mathbf{x}_0) \mathbf{h}^\alpha + R_n \tag{1.3.15}$$

Where $\mathbf{x}_0 = (x_1, x_2, \cdots, x_n)$, $\mathbf{h} = (h_1, \cdots, h_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and R_n is the remainder term of the expansion. The term $\partial^\alpha f$ is defined as

$$\partial^\alpha f \coloneqq \frac{\partial^{|\alpha|} f}{\prod_i \partial^{\alpha_i} x_i}$$

In this book, we will not delve deeply into the Taylor expansion of multivariable functions, as it involves extensive mathematical background that cannot be fully covered at once. Those who are interested may refer to real analysis and advanced calculus for a more rigorous treatment. Multivariable functions will be introduced on the next page, and the Taylor expansion for multivariable cases will appear later in the context of total differentials.

1.4 Partial Differentiation

▷ Functions of Several Variables

In real-world physics problems, very few quantities depend on a single variable. For instance, the temperature in a room varies with both position and time: $T(x, y, z, t)$. Similarly, the potential energy of a charged particle depends on its spatial coordinates: $V(x, y, z)$. These are all multivariable functions. To analyze these systems mathematically, we need to generalize our understanding of functions from a single variable to multiple variables.

We start with two variables, there's a function $z = f(x, y)$. The domain has extended from the one-dimensional real line to the two-dimensional real plane, meaning that for every input from the domain, the function f maps it to a unique real number. The mapping relationship is illustrated as follows.

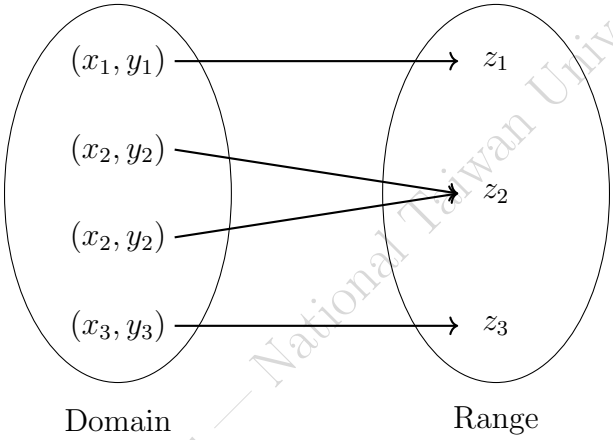

Figure 2: The correspondence between the domain and the range of a multivariable function

In mathematics and physics, a function of two variables is typically defined as a mapping from $\mathbb{R}^2$ to $\mathbb{R}$, that is, $f : \mathbb{R}^2 \to \mathbb{R}$. In other words, each input pair (x, y) is assigned a unique real number $z = f(x, y)$. Geometrically, this defines a surface in three-dimensional space, determined by the coordinates (x, y, z). When the function value is held constant as $f(x, y) = c$, the resulting set of points forms a *level curve* or *contour line*, which represents the intersection between the surface and the horizontal plane $z = c$. These contour lines are widely used to illustrate scalar fields in two dimensions.

Extending this idea, a function of three variables is a mapping $f : \mathbb{R}^3 \to \mathbb{R}$, written as $\Phi = f(x, y, z)$. This defines a scalar field in three-dimensional space, where each point (x, y, z) is associated with a real-valued quantity Φ. Since we cannot visualize four-dimensional graphs, such functions are usually represented through *level surfaces*, that is, surfaces of constant function value $\Phi = c$. These level surfaces describe regions in space where the function maintains the same value and are frequently used in physics to represent quantities such as temperature, pressure, potential, and density fields.

More generally, we can extend the concept of functions to n-dimensional space. Suppose we have a function of n variables, $f(x_1, x_2, \cdots, x_n)$. In physics, when the number of variables exceeds three, it is common to denote the input variables using indexed notation

such as x_i for $i = 1, 2, \cdots, n$. This notation will become especially important in later physical applications. Such a function is formally defined as a mapping

$$f : \mathbb{R}^n \to \mathbb{R},$$

which assigns a real number to each point in n-dimensional space. Due to the limitation of human visualization, it is no longer possible to represent these functions geometrically in a simple way when $n > 3$. Nevertheless, these high-dimensional functions play a crucial role in various branches of physics, particularly in the calculus of variations discussed in Chapter 5, where functionals often depend on multiple variables and their behavior in high-dimensional domains.

▷ Partial Derivatives

In single-variable calculus, we define the derivative and differential using the limit of a small change in a single variable:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

As previously mentioned, in physics, we often need multivariable functions to accurately and rigorously describe physical phenomena. Therefore, we must find a way to generalize the idea of differentiation to three, four, or even n dimensions. This leads us to the concept of **partial derivatives**, which describe the rate of change of a function in the direction of a specific variable.
For a two-variable function $f(x, y)$, the partial derivatives are defined as:

$$\begin{cases} \dfrac{\partial f}{\partial x} = f_x(x, y) := \lim\limits_{h \to 0} \dfrac{f(x+h, y) - f(x, y)}{h} \\ \dfrac{\partial f}{\partial y} = f_y(x, y) := \lim\limits_{h \to 0} \dfrac{f(x, y+h) - f(x, y)}{h} \end{cases} \tag{1.4.1}$$

This can be interpreted geometrically. In three-dimensional space, taking the partial derivative with respect to x means that we fix the value of y, and study the behavior of f as x varies, yielding a result analogous to the slope of the tangent in the x-direction. This process isolates the rate of change along a single variable, treating all others as constants. The same logic applies for the partial derivative with respect to y.
Additionally, when performing partial differentiation multiple times with respect to different variables, we define:

$$\begin{cases} f_{xy}(x, y) = \dfrac{\partial^2 f}{\partial x \partial y} \\ f_{yx}(x, y) = \dfrac{\partial^2 f}{\partial y \partial x} \end{cases} \tag{1.4.2}$$

For a function with three variables, say $f(x, y, z)$, the third-order mixed partial derivative is written as:

$$f_{xyz}(x, y, z) = \frac{\partial^3 f}{\partial x \partial y \partial z} \tag{1.4.3}$$

More generally, for an n-variable function $f(x_1, x_2, \ldots, x_n)$, the partial derivative with respect to a specific variable x_i is defined as:

$$\frac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h} \tag{1.4.4}$$

To illustrate the computation, consider the function $f(x, y) = xy^2 + y^5 + x^4y + x^6$. If we take the partial derivative with respect to x, we treat y as a constant and differentiate only with respect to x. This is an important idea: when computing a partial derivative, all other variables are treated as constants. Thus,

$$f_x = \frac{\partial f}{\partial x} = y^2 + 4x^3y + 6x^5$$

Note that the term y^5 is eliminated because it is treated as a constant during differentiation with respect to x. Similarly, the partial derivative with respect to y is

$$f_y = \frac{\partial f}{\partial y} = 2xy + 5y^4 + x^4$$

This procedure can be extended to functions of more variables. This is the essence of partial differentiation. In classical mechanics, electromagnetism, quantum mechanics, and beyond, we will no longer be satisfied with single-variable functions. We will frequently encounter multivariable systems that require partial derivatives. Therefore, mastering this concept early is essential. Since partial differentiation is not usually taught in high school, this section of the book focuses more on the theoretical understanding rather than problem-solving. If you're interested, you may consult a calculus textbook to find practice problems, none will be found here :3

To find the extrema of a multivariable function, such as $f(x, y)$, we follow the same logic as in the single-variable case: the slope of the tangent at an extremum must be zero. Therefore, for a function of two variables, the critical points where extrema may occur are those that satisfy

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \tag{1.4.5}$$

▷ **Total Derivative**

Consider a set $\mathbf{x} = (x_1, x_2, \cdots), \mathbf{a} = (a_1, a_2, \cdots)$ $f : \mathbb{R}^n \to \mathbb{R}$, and we've known that the Taylor series for functions of several variables

$$f(\mathbf{x} + \mathbf{a}) = f(\mathbf{x}) + \sum_i \frac{\partial f}{\partial x_i} a_i + \frac{1}{2!} \sum_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} a_i a_j + \frac{1}{3!} \sum_{ijk} \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k} a_i a_j a_k + \cdots$$

When we consider $\mathbf{a} = \Delta\mathbf{x} = (\Delta x_1, \Delta x_2, \cdots)$ a small variation in x, then the Taylor series becomes

$$f(\mathbf{x} + \Delta\mathbf{x}) = f(\mathbf{x}) + \sum_i \frac{\partial f}{\partial x_i} \Delta x_i + \frac{1}{2!} \sum_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \Delta x_i \Delta x_j + \cdots \tag{1.4.6}$$

The function increment is

$$\Delta f = f(\mathbf{x} + \Delta\mathbf{x}) - f(\mathbf{x}) = \sum_i \frac{\partial f}{\partial x_i} \Delta x_i + \frac{1}{2!} \sum_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \Delta x_i \Delta x_j + .. \tag{1.4.7}$$

While we take $\|\Delta\| \to 0 \coloneqq d$ like what we did in 1-3, the higher-order terms can be neglected because the variation becomes extremely small. So, we only need to retain the first-order partial derivative term

$$df = f(\mathbf{x} + d\mathbf{x}) - f(\mathbf{x}) = \sum_i \frac{\partial f}{\partial x_i} dx_i + O(dx_1 dx_2 \cdots) \tag{1.4.8}$$

So we define this approximation as the **Total Derivatives**, which can be denoted by

$$\boxed{df = \sum_i \frac{\partial f}{\partial x_i} dx_i} \tag{1.4.9}$$

In other words, the total differential is the linear approximation given by the first-order partial derivatives in the multivariable Taylor expansion. From a physical point of view, each variable of the function f undergoes a small variation, and the variation of each variable can be approximated as a movement along the tangent line with all other variables held constant. The total contribution df is then the accumulation of all these infinitesimal changes across each direction.
For example, there is a 3-dimensional function $f(x, y, z)$, which is what you'll often meet in Taylor's *Classical Mechanics*, then the total derivatives of the function f is

$$df = f(x + dx, y + dy, z + dz) - f(x, y, z) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \tag{1.4.10}$$

▷ The Chain Rule of Several Variables

We have discussed the chain rule in single-variable function $f(x)$. When the variable x becomes a function $g(x)$, we need to apply the chain rule to it because f depends on g; in the meantime, g depends on x, so we should consider both functions. Similarly, consider a function $f(x, y, z)$, and x, y, z depend on the variable t, which can be denoted by

$$f(x(t), y(t), z(t)) \tag{1.4.11}$$

If we want to know the derivative of f with respect to t, then we need to "dig" the variable t out from each variable, that is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \tag{1.4.12}$$

However, when the elements of f depend on two (or more) variables $u,\ v$, which is

$$f(x(u, v), y(u, v), z(u, v)) \tag{1.4.13}$$

Then the chain rule becomes

$$\begin{cases} \dfrac{\partial f}{\partial u} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial u} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial u} \\ \dfrac{\partial f}{\partial v} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial v} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial v} \end{cases} \tag{1.4.14}$$

Then the total derivatives of f can be denoted by

$$df = \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}\right) du + \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}\right) dv \tag{1.4.15}$$

That is to say, we must take into account how the variables u and v contribute to x, y, z. Since x, y, z in turn influence the overall behavior of the function f, we can say that u and v indirectly affect the function.

In general, consider a function $f(x_1(u_1, u_2, \cdots, u_n), \cdots, x_n(u_1, u_2, \cdots, u_n))$, the total derivative of f can be generalized to

$$df = \sum_i \boldsymbol{\nabla} f \cdot \frac{\partial \mathbf{r}}{\partial u_i} du_i \tag{1.4.16}$$

Where $\boldsymbol{\nabla} = (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})$ and $\mathbf{r} = (x_1, \cdots, x_n)$, both of them are vector. This will be discussed in detail in the next chapter *introduction to vector analysis.*

▷ Multiple Integrals

We've learned that if we want to calculate the area under a curve $f(x)$, we have

$$A = \int_a^b f(x)\ dx$$

But, in three dimensions, when we want to calculate the area of a specific region, and the boundary of that region is no longer constant but varies with functions, we can no longer use single-variable integration as we did in two-dimensional problems. In such cases, we must introduce **double integrals**, also known as **area integrals**. Its general form is:

$$\iint_A f(x, y)\, dA \tag{1.4.17}$$

The concept is analogous to the Riemann sum you are familiar with, the domain is infinitely partitioned into smaller rectangles, and the total area is obtained by summing the contributions of $f(x, y)$ over each small patch using a limiting process.

Similarly, we also have **triple integrals**, also known as **volume integrals**, which cut the volume into infinitesimal cubes and then sum them. The form is:

$$\iiint_V f(x, y, z)\, dV \tag{1.4.18}$$

Note that in some physics textbooks, the notation for multiple integrals is simplified as:

$$\iint_A f(x, y)\, dA \coloneqq \int_A f(x, y)\, dA, \quad \iiint_V f(x, y, z)\, dV \coloneqq \int_V f(x, y, z)\, dV$$

However, the meaning remains the same. In this book, we will not use this shorthand notation to avoid confusion.

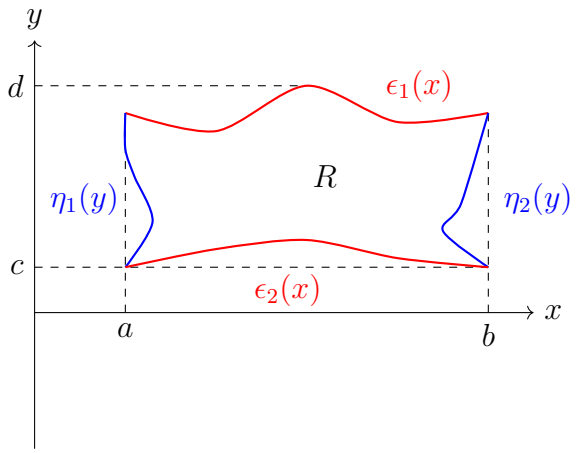

Figure 3: Illustration of the integration region R with variable limits. The region can be described either by fixing x and integrating over y between $\epsilon_2(x)$ and $\epsilon_1(x)$, or by fixing y and integrating over x between $\eta_1(y)$ and $\eta_2(y)$.

To compute an area integral, let us take a closer look at how to actually perform the calculation. The figure above shows a region R in the xy-plane bounded by curved edges, rather than straight lines.
Let us start with the more familiar case: integrating with respect to x first. In this case, the integration bounds can no longer be expressed simply as constants a and b, since the left and right boundaries are given by curves $\eta_1(y)$ and $\eta_2(y)$. Therefore, the inner integral becomes

$$\int_{\eta_1(y)}^{\eta_2(y)} f(x,y)\,dx \tag{1.4.19}$$

We can interpret this as follows: at a fixed value of y, the function $f(x,y)$ becomes a single-variable function in x, and integrating it gives the contribution from a "vertical strip" at that specific y. This is the inner integral.

Then, by allowing y to vary from c to d, we add up all of these vertical strips to obtain the total area over the region R. This gives the full double integral:

$$\int_c^d \int_{\eta_1(y)}^{\eta_2(y)} f(x,y)\,dx\,dy \tag{1.4.20}$$

By the same logic, we can also reverse the order of integration. That is, we can first fix x and integrate with respect to y, then integrate with respect to x. The integral becomes:

$$\int_a^b \int_{\epsilon_1(x)}^{\epsilon_2(x)} f(x,y)\,dy\,dx \tag{1.4.21}$$

This naturally leads to the introduction of **Fubini's Theorem**. Let $D \subset \mathbb{R}^2$ be a bounded, measurable domain, and $f : \mathbb{R}^2 \to \mathbb{R}$ be integrable on D. Then we have

$$\iint_D f(x,y)\,dx\,dy = \int_{x\in\pi_x(D)} \left(\int_{y\in D_x} f(x,y)\,dy \right) dx \tag{1.4.22}$$

where $D_x = \{y \mid (x,y) \in D\}$ represents the vertical slice of D at fixed x. In simpler terms, you are free to choose the order of integration, as long as you correctly determine

the limits based on the geometry of the domain. Similarly, for volume integrals, we must identify three bounding surfaces, and the integral is written as:

$$\int_a^b \int_{\epsilon_1(x)}^{\epsilon_2(x)} \int_{\xi_1(x,y)}^{\xi_2(x,y)} f(x,y,z)\, dz\, dy\, dx \tag{1.4.23}$$

This also satisfies the property that the order of integration can be freely rearranged, provided the limits are properly adjusted. In terms of computation, the idea is very similar to how you handle partial derivatives. When integrating with respect to one variable, you treat the others as constants and perform integration accordingly. For instance, to evaluate the following volume integral:

$$\int_{-1}^{1} \int_{0}^{2} \int_{0}^{1-x^2} dz\, dy\, dx$$

you first integrate with respect to z, treating both x and y as constants, then proceed with y, and finally x. That is

$$\begin{aligned}
\int_{-1}^{1} \int_{0}^{2} \int_{0}^{1-x^2} dz\, dy\, dx &= \int_{-1}^{1} \int_{0}^{2} \left(z|_0^{1-x^2} \right) dydx \\
&= \int_{-1}^{1} \int_{0}^{2} (1-x^2) dydx \\
&= \int_{-1}^{1} y(1-x^2)|_0^2 \, dx \\
&= \int_{-1}^{1} 2 - 2x^2 \, dx = \frac{8}{3}
\end{aligned}$$

▷ **Lagrange multipliers**

We begin with the case of a two-variable function. In this section, we deal with the problem of a **constraint function**. A constraint function $g(x,y) = c$ (which is an *iso-contour*) restricts the domain of a function $f(x,y)$ whose extremum d we want to find. That is, we are seeking the value $f(x,y) = d$ *subject to the condition* $g(x,y) = c$. In other words, $f(x,y) = d \in g(x,y)$, and d must be an extremum of f.

You can think of this like filtering college admission scores: we're looking for the highest score among applicants, but only among those who had perfect scores in math and science. This is the basic idea behind a constraint.

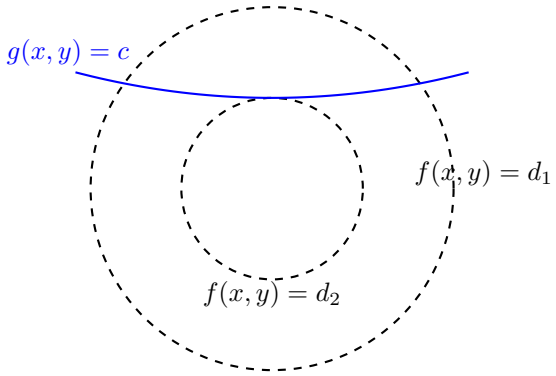

Figure 4: A constraint curve $g(x,y)=c$ (blue) tangent to a level curve $f(x,y)=d$ (black dashed)

As illustrated above, we have contour lines $f(x,y)=d_1$, $f(x,y)=d_2$, etc. For the extremum to satisfy the constraint $g(x,y)=c$, the point must lie on both f and g, and more importantly, the contours must be **tangent** at the extremum point. As we mentioned earlier, the extremum of a multivariable function must satisfy

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0$$

So this tangency condition is natural. Since f and g are tangent, their normal vectors must be **parallel** or anti-parallel. Here, we subtly introduce the concept of a normal vector and the gradient (see Chapter 2). The normal vector is given by

$$\boldsymbol{\nabla} f=\mathbf{n} \tag{1.4.24}$$

where $\boldsymbol{\nabla}=\left(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\right)$. Returning to our earlier condition, since the gradients of f and g are parallel at the point of tangency, they differ by a scalar multiple. We define this scalar as λ, leading to

$$\boldsymbol{\nabla} f=-\lambda\boldsymbol{\nabla} g \tag{1.4.25}$$

This constant λ is called the **Lagrange multiplier**. In the two-variable case, this becomes

$$\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)=-\lambda\left(\frac{\partial g}{\partial x},\frac{\partial g}{\partial y}\right) \tag{1.4.26}$$

From this, we obtain the following system

$$\begin{cases}\dfrac{\partial f}{\partial x}+\lambda\dfrac{\partial g}{\partial x}=0\\[2ex] \dfrac{\partial f}{\partial y}+\lambda\dfrac{\partial g}{\partial y}=0\end{cases} \tag{1.4.27}$$

We now multiply the first equation by dx and the second by dy

$$\begin{cases}\dfrac{\partial f}{\partial x}dx+\lambda\dfrac{\partial g}{\partial x}dx=0\\[2ex] \dfrac{\partial f}{\partial y}dy+\lambda\dfrac{\partial g}{\partial y}dy=0\end{cases} \tag{1.4.28}$$

Integrating both and summing them yields

$$\left(\int \frac{\partial f}{\partial x}dx\right) + \lambda\left(\int \frac{\partial g}{\partial x}dx\right) + \left(\int \frac{\partial f}{\partial y}dy\right) + \lambda\left(\int \frac{\partial g}{\partial y}dy\right) = C$$

$$\Rightarrow \int \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) + \lambda \int \left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy\right)$$

Both are total differentials (df, dg), so we can write

$$f(x,y) + \lambda g(x,y) = C \tag{1.4.29}$$

This motivates the definition of a new function, the **Lagrangian**:

$$\mathcal{L}(x,y,\lambda) \coloneqq f(x,y) + \lambda g(x,y) \tag{1.4.30}$$

Thus, the original problem of finding an extremum of $f(x,y)$ under the constraint $g(x,y) = c$ is transformed into the unconstrained extremum of the Lagrangian:

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \mathcal{L}}{\partial y} = 0 \tag{1.4.31}$$

This method, **the method of Lagrange multipliers**, is crucial and will appear frequently in Lagrangian mechanics later.
The method can also be generalized to n-dimensional functions:

$$\mathcal{L}(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = f(x_1, \ldots, x_n) + \sum_i \lambda_i g_i(x_1, \ldots, x_n) \tag{1.4.32}$$

For example, if we must find the extremum of $f(x,y,z) = xyz > 0$ under the constraint $g(x,y,z) = 6x + 4y + 3z - 24 = 0$. Let the Lagrange multiplier $\lambda \in \mathbb{R}$, and define a function which depends on $x,\ y,\ z$ and λ, that is

$$\mathcal{L}(x,y,z,\lambda) = xyz + \lambda(6x + 4y + 3z - 24)$$

So, to find the extremum of $\mathcal{L}$, we have

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x} = yz + 6\lambda = 0 \\ \dfrac{\partial \mathcal{L}}{\partial y} = xz + 4\lambda = 0 \\ \dfrac{\partial \mathcal{L}}{\partial z} = xy + 3\lambda = 0 \\ \dfrac{\partial \mathcal{L}}{\partial \lambda} = 6x + 4y + 3z - 24 = 0 \end{cases}$$

After calculating, we have $x = 4/3$, $y = 2$, $z = 8/3$, so the extremum of f under the constraint g is

$$xyz = \frac{64}{9}$$

▷ Jacobian Matrix and Coordinate Transformation

Let $\Phi(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ denote the original coordinates in the reference coordinate system. We define a smooth transformation $F : \mathbb{R}^n \to \mathbb{R}^n$, such that

$$\Phi(x_1, x_2, \cdots, x_n) \xrightarrow{F} \Phi'(x_1', x_2', \cdots, x_n') \tag{1.4.33}$$

This means that each point of Φ in the original system is mapped to a new point of Φ' via the transformation F, which defines the new coordinate frame. We define matrices (see Chapter 6.) that

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad X' = \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} \tag{1.4.34}$$

then we can use the matrices to describe the coordinates $\Phi(X)$ and $\Phi'(X')$. Rewrite (1.4.33) as

$$X \xrightarrow{F} X' \tag{1.4.35}$$

So, we can write down the relationship of function between the new coordinate and the old one, that is

$$X' = F(X) \Leftrightarrow X = F^{-1}(X') \tag{1.4.36}$$

Based on the relationship above, we can express the following equality

$$\Phi(X) = \Phi\left(F^{-1}(X)\right) \tag{1.4.37}$$

In other words, we can construct a new one-to-one function using the functional relationship between the old and new coordinates. This function takes the new coordinates as its domain, meaning that for each input in the new coordinate system, we obtain a unique output value. This mapping is entirely equivalent to the correspondence established by the new coordinates themselves. That is,

$$\Phi(X) = \Phi'(X') \tag{1.4.38}$$

From a physical standpoint, this implies that the physical quantity remains unchanged regardless of whether we express it in the old or the new coordinate system. Coordinates merely serve as a reference framework for describing physical quantities, they do not alter the physical essence of the quantity itself.
The transformation F defines a mapping from the original coordinate system to the new one. It consists of a collection of scalar-valued component functions, such that,

$$F(X) = \begin{pmatrix} f_1(X) \\ f_2(X) \\ \vdots \\ f_n(X) \end{pmatrix} \tag{1.4.39}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$. So, in detail

$$\begin{aligned} (x_1, x_2, \cdots, x_n) &\xrightarrow{f_1} x_1' = f_1(x_1, x_2, \cdots, x_n) \\ &\xrightarrow{f_2} x_2' = f_2(x_1, x_2, \cdots, x_n) \\ &\vdots \end{aligned} \tag{1.4.40}$$

Under the inverse transformation, this can be written as

$$\begin{aligned}(x'_1, x'_2, \cdots, x'_n) &\xrightarrow{f_1^{-1}} x_1 = f_1^{-1}(x'_1, x'_2, \cdots, x'_n) \\ &\xrightarrow{f_2^{-1}} x_2 = f_2^{-1}(x'_1, x'_2, \cdots, x'_n) \\ &\vdots\end{aligned} \tag{1.4.41}$$

That is, according to equations (1.4.8) and (1.4.9), the new coordinates are functions of the original coordinates; conversely, the original coordinates can also be expressed as functions of the new ones. By (1.4.6), it can be denoted by

$$\Phi'(x'_1, x'_2, \cdots, x'_n) = \Phi(x_1(x'_1, x'_2, \cdots, x'_n), x_2(x'_1, x'_2, \cdots, x'_n), \cdots, x_n(x'_1, x'_2, \cdots, x'_n)) \tag{1.4.42}$$

Φ depends on $x_1,\ x_2,\ \cdots$, and Φ' depends on $x'_1,\ x'_2,\ \cdots$, so Φ also depends on $x'_1,\ x'_2,\ \cdots$. We can naturally obtain the chain rule of the partial differentiation under the coordinate transformation,

$$\boxed{\frac{\partial \Phi'}{\partial x'_i} = \sum_j \frac{\partial \Phi}{\partial x_j}\frac{\partial x_j}{\partial x'_i}} \tag{1.4.43}$$

After discussing the chain rule under the coordinate transformation, we can further define a crucial matrix in such a transformation, the **Jacobian**. Consider a first-order infinitesimal variation in a Taylor expansion dX', then

$$X' + dX' = \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix} + \begin{pmatrix} dx'_1 \\ dx'_2 \\ \vdots \\ dx'_n \end{pmatrix} \tag{1.4.44}$$

By (1.4.8), the latter term can be rewritten as the total differential of the function f_i, that is

$$dX' = \begin{pmatrix} dx'_1 \\ dx'_2 \\ \vdots \\ dx'_n \end{pmatrix} = \begin{pmatrix} df_1 \\ df_2 \\ \vdots \\ df_n \end{pmatrix} = \begin{pmatrix} \sum_i \frac{\partial f_1}{\partial x_i} dx_i \\ \sum_i \frac{\partial f_2}{\partial x_i} dx_i \\ \vdots \\ \sum_i \frac{\partial f_n}{\partial x_i} dx_i \end{pmatrix} \tag{1.4.45}$$

We can decompose the matrix into the product of two matrices.

$$\begin{pmatrix} \sum_i \frac{\partial f_1}{\partial x_i} dx_i \\ \sum_i \frac{\partial f_2}{\partial x_i} dx_i \\ \vdots \\ \sum_i \frac{\partial f_n}{\partial x_i} dx_i \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}_{n\times n} \begin{pmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{pmatrix}_{n\times 1}$$

Then, we define the $n \times n$ matrix as

$$J_F \coloneqq \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \Rightarrow \quad \det(J_F) \coloneqq \frac{\partial(f_1, f_2, \cdots, f_n)}{\partial(x_1, x_2, \cdots, x_n)} \tag{1.4.46}$$

The Jacobian matrix defined here represents how an infinitesimal variation in the new coordinates, under the transformation F, corresponds to variations in the original coordinates. It describes the relative scaling and directional contribution of each element across the coordinate systems. We can write down

$$\boxed{dX' = J_F \; dX} \tag{1.4.47}$$

In other words, this matrix connects the rate of change between the two coordinate systems. We will explore the meaning and formal definition of this matrix in vector spaces more thoroughly and rigorously in Chapter 6.

In high school, we learned that if an area or volume is transformed by a matrix, then the resulting area or volume must be multiplied by the determinant of that matrix. For example, an area A is transformed by λ, then the resulting area A' becomes

$$A' = |\det(\lambda)| A \tag{1.4.48}$$

So do the Jacobian and the coordinate transformation. Under the coordinate transformation in two dimensions, if we wish to describe how variables in a different coordinate system contribute to the original Cartesian coordinate system, we must use the Jacobian matrix to connect the two systems.

Suppose we have a coordinate system $\Phi'(u, v)$, and changes in u and v correspond to changes in the actual Cartesian system $\Phi(x, y)$. To quantify how much change in x and y results from a variation in u and v, we need to map Φ' onto Φ, that is, we write $\Phi(x, y)$ as $\Phi(x(u, v), y(u, v))$. In this case, both u and v contribute to x and y, consistent with the relationship described on the previous page

$$\Phi = J \cdot \Phi' \tag{1.4.49}$$

At this point, a small area element $du\,dv$ in the Φ' system is mapped through the Jacobian transformation to the small area element $dx\,dy$ in the Φ system. As you learned in high school, this results in

$$dx\,dy = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv = |\det(J)|\,du\,dv = \left|\left|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix}\right|\right| dudv \tag{1.4.50}$$

We call $dxdy$, $dvdu$ the area element. So the area integrals become

$$\iint_A dxdy = \iint_\Omega \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv = \iint_\Omega |\det(J)|\,du\,dv \tag{1.4.51}$$

For example, the polar coordinate $\Phi'(r,\theta)$ and the Cartesian system $\Phi(x,y)$. We've known the relationship between them

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

The relationship between the area elements of them is

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta = \left|\left|\begin{matrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{matrix}\right|\right| dr\,d\theta = r\cdot dr\,d\theta$$

So the integral must be rewritten as

$$\int_{y_1}^{y_2}\int_{\epsilon_1(x)}^{\epsilon_2(x)} dx\,dy = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} r\cdot dr\,d\theta \tag{1.4.52}$$

We can also apply the same logic to the volume elements. Take the spherical coordinates for example. The relationship between spherical coordinates $\Phi'(r,\theta,\phi)$ and Cartesian coordinate $\Phi(x,y,z)$ is

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\cos\phi \\ z = r\cos\theta \end{cases}$$

Where θ is the polar angle and ϕ is the azimuthal angle (see Chapter 1-6). The relationship between the volume elements of them is

$$dx\,dy\,dz = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\right| dr\,d\theta\;d\phi = \left|\left|\begin{matrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{matrix}\right|\right| dr\,d\theta\,d\phi = r^2\sin\theta\cdot dr\,d\theta\,d\phi$$

The volume integral becomes

$$\iiint_V dx\,dy\,dz = \iiint r^2\sin\theta\cdot dr\,d\theta\,d\phi \tag{1.4.53}$$

1.5 Elliptic Integrals in the Pendulum Problem(Optional)

When dealing with complex physical behaviors, such as large-angle pendulum motion, celestial mechanics, and orbital curvature, it is often impossible to represent the solutions using elementary functions. Instead, we must introduce special functions and evaluate them using numerical methods or series expansions to approximate real-world values. One such integral is called the **elliptic integral**, which was originally developed in the context of computing the arc length of an ellipse. There are three types of elliptic integrals: the first kind, second kind, and third kind.

The elliptic integral of the first kind is commonly written as $F(k, \phi)$, where k is a parameter determined by the initial condition of the system, and ϕ is the upper limit of integration, also known as the amplitude. Its definition is:

$$F(k, \phi) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}, \quad k^2 < 1 \tag{1.5.1}$$

The elliptic integral of the second kind, denoted $E(k, \phi)$, corresponds to the original arc length formula of an ellipse and is defined as:

$$E(k, \phi) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta, \quad k^2 < 1 \tag{1.5.2}$$

The elliptic integral of the third kind, written as $\Pi(n, k, \phi)$, is the most complex among the three and often appears in advanced simulations such as space mission trajectory modeling. Its definition is:

$$\Pi(n, k, \phi) = \int_0^{\phi} \frac{d\theta}{(1 + n \sin^2 \theta)\sqrt{1 - k^2 \sin^2 \theta}} \tag{1.5.3}$$

In this chapter, we will focus solely on the first kind of elliptic integral to address the large-angle pendulum problem.

▷ **Gaussian Integral**

The Gaussian integral I is

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx \tag{1.5.4}$$

It is an **improper integral** over an infinite domain. In this book, we will not go into the detailed theory of improper integrals. At this point, it is sufficient for us to know whether the integrand is convergent and integrable over the given interval. We observe that the function $\exp(-x^2)$ decays faster than any rational function as $|x| \to \infty$, so the integral converges.

If we want to calculate this integral, we need to use some methods. First, we have

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx = \int_{-\infty}^{\infty} e^{-y^2}\, dy \tag{1.5.5}$$

So, we multiply both integrals, and then we get

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dxdy \tag{1.5.6}$$

We transform the Cartesian coordinate to the polar coordinates, that is

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

By (1.4.52), the area element $dx\,dy$ becomes $r \cdot drd\theta$. Substituting these into (1.5.3), we can obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dxdy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,rdrd\theta \tag{1.5.7}$$

Let $u = r^2$, $du = 2rdr$, so the integral becomes

$$\int_0^{2\pi}\int_0^{\infty} \frac{1}{2}e^{-u}\,dud\theta = \int_0^{2\pi} -\frac{1}{2}e^{-u}\Big|_0^{\infty} d\theta = \frac{1}{2}\int_0^{2\pi} d\theta = \pi$$

Substituting this into (1.5.3), $I^2 = \pi$, so we can finally obtain the Gaussian integral

$$\boxed{I = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}} \tag{1.5.8}$$

▷ Gamma Function

The Gamma function is introduced to extend the factorial function to noninteger values in a continuous manner. It's defined as

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\,dt,\ x > 0 \tag{1.5.9}$$

When $x = n, n \in \mathbb{N}$, it is

$$\Gamma(n) = \int_0^{\infty} t^{n-1}e^{-t}\,dt \tag{1.5.10}$$

The formula $\Gamma(n) = (n-1)!$ is important, where we define $0! = 1$. We can use integration by parts and the mathematical induction to analyze it. When $x = 1$ and $x = 2$, we get

$$\Gamma(1) = \int_0^{\infty} e^{-t}\,dt = -\,e^{-t}\big|_0^{\infty} = 1 = 1!$$

We assume that the formula holds when $x = n,\ n \in \mathbb{N}$, which is

$$\Gamma(n) = \int_0^{\infty} t^{n-1}e^{-t}\,dt = (n-1)!$$

When $x = n + 1$, it becomes

$$\begin{aligned}\Gamma(n+1) &= \int_0^{\infty} t^n e^{-t}\,dt \\ &= -t^n e^{-t}|_0^{\infty} + n\int_0^{\infty} t^{n-1}e^{-t}\,dt\end{aligned}$$

Note that the limit $\lim_{t\to\infty} -t^n e^{-t} = 0$ because the growth rate of exponential functions is faster than that of polynomial functions. So

$$\Gamma(n+1) = n\int_0^{\infty} t^{n-1}e^{-t}\,dt = n(n-1)! = n!$$

By mathematical induction, we can verify that for all $x \in \mathbb{N}$, the formula $\Gamma(n) = (n-1)!$ holds. More specifically, you can generalize it to $\Gamma(x)$ in the same way. After that, we have two crucial properties of the Gamma function

$$\boxed{\begin{cases} \Gamma(n) = (n-1)!, \ n \in \mathbb{N} \\ \Gamma(x+1) = x\Gamma(x), \ \forall x \in \mathbb{R}^+ \end{cases}} \tag{1.5.11}$$

However, when x is a noninteger value, such as $x = 1/2, \ x = 1/3 \cdots$, it may become the familiar form that we have discussed, the Gaussian integral. For example, $x = 1/2$, the Gamma function

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}} e^{-t} \, dt \tag{1.5.12}$$

Let $u = \sqrt{t}$, $du = 1/2 \cdot t^{-1/2}$, so (1.5.12) becomes

$$\int_0^\infty t^{-\frac{1}{2}} e^{-t} \, dt = 2 \int_0^\infty e^{-u^2} \, du \tag{1.5.13}$$

(1.5.13) is the form of the Gaussian integral. Since the function e^{-x^2} is symmetric at $x = 0$, so

$$\int_0^\infty e^{-u^2} \, du = \frac{1}{2} \int_{-\infty}^\infty e^{-u^2} \, du = \frac{\sqrt{\pi}}{2}$$

Substituting it into (1.5.13), we can finally get

$$\Gamma\left(\frac{1}{2}\right) = 2 \int_0^\infty e^{-u^2} \, du = \sqrt{\pi} \tag{1.5.14}$$

When $x = 3/2$, by the formula (1.5.11), we obtain

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2} + 1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \tag{1.5.15}$$

However, except for $x = 1/2$, which can be evaluated using the Gaussian integral, most other cases, such as $x = 1/3, \ 2/3, \cdots$ can only be computed numerically.
We have learned binomial expansion in high school, which is

$$(1+x)^r = \sum_i \binom{r}{i} x^i y^{r-i}$$

We can use the Gamma function to generalize the binomial expansion. Which can be denoted by

$$\boxed{(x+y)^r = \sum_i \frac{\Gamma(r+1)}{\Gamma(i+1)\Gamma(r-i+1)} x^i y^{r-i}} \tag{1.5.16}$$

Although we have just introduced the generalized binomial theorem, note that if the value of $r - i + 1$ becomes less than zero, the Gamma function involved will diverge and become undefined. That is, $(1-x)^{-r}$, if we differentiate it with respect to x, we get

$$\frac{d^n}{dx^n}(1-x)^{-r} = \prod_{i=0} (r+i)(1-x)^{-r-n}$$

Let $f(x) = (1-x)^{-r}$, we've known that

$$f(x) = \sum_{i=0} \frac{f^{(i)}(0)}{i!} x^i$$

So the Taylor series of $f(x) = (1-x)^{-r}$ is

$$(1-x)^{-r} = \sum_{i=0} \frac{\prod_{j=0}^{i-1}(r+j)}{i!}\, x^i = \sum_i \binom{r+i-1}{i} x^i = \sum_i \frac{(r+i-1)!}{i!(r-1)!} = \sum_i \frac{\Gamma(r+i)}{\Gamma(i+1)\Gamma(r)} x^i$$

Take $r = -1/2$ for example, we obtain

$$(1-x)^{-1/2} = \sum_i \frac{\Gamma\left(i+\frac{1}{2}\right)}{\Gamma(i+1)\Gamma\left(\frac{1}{2}\right)} x^i \tag{1.5.17}$$

Where $\Gamma(i+1) = i!$, $\Gamma(1/2) = \sqrt{\pi}$. To express these Gamma functions in terms of factorials, we can repeatedly apply the identity $\Gamma(x+1) = x\Gamma(x)$ to break them down. Let's deal with $\Gamma\left(i+\frac{1}{2}\right)$, we can obtain

$$\begin{aligned}
\Gamma\left(i+\frac{1}{2}\right) &= \Gamma\left(i-\frac{1}{2}+1\right) \\
&= \left(i-\frac{1}{2}\right)\Gamma\left(i-\frac{1}{2}\right) \\
&= \left(i-\frac{1}{2}\right)\left(i-\frac{3}{2}\right)\Gamma\left(i-\frac{3}{2}\right) \\
&= \left(i-\frac{1}{2}\right)\left(i-\frac{3}{2}\right)\left(i-\frac{5}{2}\right)\cdots\frac{1}{2}\Gamma\left(\frac{1}{2}\right)
\end{aligned}$$

By factoring out i terms of $1/2$, the remaining product becomes a sequence of odd integers starting from $2i-1$ and decreasing by 2. That is,

$$\frac{1}{2^i}(2i-1)(2i-1)\cdots 3\cdot 2$$

We can perform a common-denominator transformation to introduce the missing even-numbered terms in the denominator

$$\frac{(2i)!}{2^i \times i! \times 2^i} = \frac{(2i)!}{i!4^i}$$

So, $\Gamma\left(i+\frac{1}{2}\right)$ can be rewritten as

$$\Gamma\left(i+\frac{1}{2}\right) = \frac{(2i)!}{i!4^i}\sqrt{\pi} \tag{1.5.18}$$

We substitute this result (1.5.18) into (1.5.17), and we get the final expansion

$$(1-x)^{-1/2} = \sum_i \frac{\frac{(2i)!}{i!4^i}\sqrt{\pi}}{i!\sqrt{\pi}} x^i = \sum_i \frac{(2i)!}{(i!)^2 \times 4^i} x^i = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots \tag{1.5.19}$$

▷ Beta Function

To better understand the relationship between the integral and the factorial, I must introduce another special function, the Beta function. You may be surprised, or even shocked, by the beauty of this structure later.
Define the Beta function $B(p,q)$

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt \tag{1.5.20}$$

We can use this definition to combine the integrals and the Gamma function in complicated problems, such as the elliptic integral. To find the relationship between the Gamma function and the Beta function, we construct

$$\begin{aligned}\Gamma(p)\Gamma(q) &= \left(\int_0^\infty x^{p-1}e^{-x}dx\right)\left(\int_0^\infty y^{q-1}e^{-y}dy\right) \\ &= \int_0^\infty\int_0^\infty x^{p-1}y^{q-1}e^{-(x+y)}dxdy\end{aligned} \tag{1.5.21}$$

Let $x = ru,\ y = r(1-u)$, where $r \in [0,\infty),\ u \in [0,1]$. By the coordinate transformation equation

$$dx\ dy = \left|\frac{\partial(x,y)}{\partial(r,u)}\right| dr\,du = r\ dr\ du \tag{1.5.22}$$

So the integral (1.5.21) becomes

$$\int_0^\infty\int_0^\infty x^{p-1}y^{q-1}e^{-(x+y)}dxdy = \int_0^1\int_0^\infty (ru)^{p-1}(r(1-u))^{q-1}e^{-r}r\,dr\,du \tag{1.5.23}$$

Rewriting the integrand and calculating this integral

$$\begin{aligned}\int_0^1\int_0^\infty (ru)^{p-1}(r(1-u))^{q-1}e^{-r}r\,dr\,du &= \int_0^1\int_0^\infty u^{p-1}(1-u)^{q-1}r^{p+q-1}e^{-r}dr\,du \\ &= \int_0^1\left(\int_0^\infty r^{p+q-1}e^{-r}dr\right)u^{p-1}(1-u)^{q-1}\,du\end{aligned}$$

Note that the inner integral is the same as the definition of the Gamma function, and the outer integral is the same as the definition of the Beta function, so

$$\begin{aligned}\int_0^1\left(\int_0^\infty r^{p+q-1}e^{-r}dr\right)u^{p-1}(1-u)^{q-1}\,du &= \int_0^1\Gamma(p+q)u^{p-1}(1-u)^{q-1}\,du \\ &= \Gamma(p+q)\int_0^1 u^{p-1}(1-u)^{q-1}\,du \\ &= \Gamma(p+q)\cdot B(p,q)\end{aligned}$$

Substituting this into (1.5.21), we can obtain a crucial formula for the relationship between the Gamma and Beta functions. That is

$$\boxed{B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}} \tag{1.5.24}$$

▷ Elliptic Integral and Pendulum

The motion of a simple pendulum is one of the most fundamental problems in classical mechanics, as it helps us understand the behavior of nonlinear physical systems. In high school, we typically focus on pendulum motion with small initial angles. However, as the angle becomes larger, the same method of analysis is no longer valid.

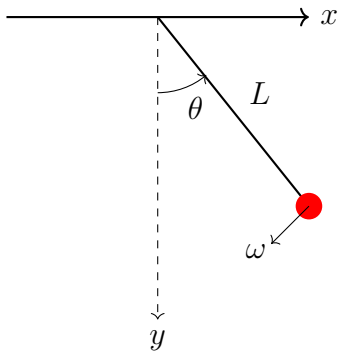

Figure 5: Sample of a system constrained by rod

As shown in the above figure, the bob is released with the initial angle θ_0 and $v = L\dot{\theta} = 0$, so the kinetic energy of the bob is 0 at $\theta = \theta_0$, and the total energy and the potential energy are

$$\begin{cases} V = mgL(1-\cos\theta) = 2mgL\sin^2\left(\dfrac{\theta}{2}\right) \\ E = mgL(1-\cos\theta_0) = 2mgL\sin^2\left(\dfrac{\theta_0}{2}\right) \end{cases} \tag{1.5.25}$$

We use the difference between the potential energy and the total energy to express the kinetic energy

$$T = \frac{1}{2}mL^2\dot{\theta}^2 = E - V = 2mgL\left(\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right)$$

So

$$\dot{\theta} = \frac{d\theta}{dt} = 2\sqrt{\frac{g}{L}}\sqrt{\left(\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right)} \tag{1.5.26}$$

Here, we separate the variable (see Chapter 4-1), and we can easily solve the differential equation

$$\frac{1}{2}\sqrt{\frac{L}{g}}\left[\left(\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right)\right]^{-1/2} d\theta = dt$$

By integrating both sides, we obtain the time t it takes for the pendulum to swing from $\theta = \theta_0$ to $\theta = 0$, which corresponds to one-quarter of the total period T. That is,

$$T = 4t = 4\int dt = 2\sqrt{\frac{L}{g}}\int_0^{\theta_0} \frac{d\theta}{\left[\left(\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right)\right]^{1/2}} \tag{1.5.27}$$

This is a typical elliptic integral of the first kind (1.5.1). Factor $\sin\theta_0/2$ out of the square root, it becomes

$$2\sqrt{\frac{L}{g}}\int_0^{\theta_0}\frac{d\theta}{\left[\left(\sin^2\left(\frac{\theta_0}{2}\right)-\sin^2\left(\frac{\theta}{2}\right)\right)\right]^{1/2}}=2\sqrt{\frac{L}{g}}\int_0^{\theta_0}\frac{d\theta}{\sin\left(\frac{\theta_0}{2}\right)\left[1-\left(\frac{\sin(\theta/2)}{\sin(\theta_0/2)}\right)^2\right]^{1/2}} \quad (1.5.28)$$

Apply the integration by substitution, let

$$z=\frac{\sin(\theta/2)}{\sin(\theta_0/2)};\ k=\sin(\theta_0/2)$$

Differentiating this to get

$$dz=\frac{\cos(\theta/2)}{2\sin(\theta_0/2)}d\theta=\frac{\sqrt{1-\sin^2(\theta/2)}}{2\sin(\theta_0/2)}d\theta=\frac{\sqrt{1-k^2z^2}}{2k}d\theta\Rightarrow d\theta=\frac{2k\ dz}{\sqrt{1-k^2z^2}}$$

substituting this into (1.5.20), we have

$$2\sqrt{\frac{L}{g}}\int_0^{\theta_0}\frac{d\theta}{\sin\left(\frac{\theta_0}{2}\right)\left[1-\left(\frac{\sin(\theta/2)}{\sin(\theta_0/2)}\right)^2\right]^{1/2}}=2\sqrt{\frac{L}{g}}\int_0^1\frac{2k\,dz}{k\left(1-z^2\right)^{1/2}\left(1-k^2z^2\right)^{1/2}}$$

Remember to change the limits of integration after the substitution, $z(\theta_0)=1$. So, the total period of the pendulum is

$$T=4\sqrt{\frac{L}{g}}\int_0^1\left[\left(1-z^2\right)\left(1-k^2z^2\right)\right]^{-1/2}dz \quad (1.5.29)$$

We can expand the term $\left(1-k^2z^2\right)^{-1/2}$ by (1.5.19)

$$\left(1-k^2z^2\right)^{-1/2}=\sum_i\frac{(2i)!}{(i!)^2\times4^i}(k^2z^2)^i=1+\frac{k^2z^2}{2}+\frac{3k^4z^4}{8}+\frac{5k^6z^6}{16}+\cdots \quad (1.5.30)$$

Substituting (1.5.30) into (1.5.29), we can start to calculate the integral

$$T=4\sqrt{\frac{L}{g}}\int_0^1\frac{1}{\sqrt{(1-z^2)}}\left(1+\frac{k^2z^2}{2}+\frac{3k^4z^4}{8}+\frac{5k^6z^6}{16}+\cdots\right)dz$$

$$=4\sqrt{\frac{L}{g}}\left(\int_0^1\frac{dz}{\sqrt{(1-z^2)}}+\frac{k^2}{2}\int_0^1\frac{z^2dz}{\sqrt{(1-z^2)}}+\cdots\right)$$

Let $z=\sin\phi$, $dz=\cos\theta\,d\theta$, then the integral can be described as

$$4\sqrt{\frac{L}{g}}\left(\int_0^{\frac{\pi}{2}}d\theta+\frac{k^2}{2}\int_0^{\frac{\pi}{2}}\sin^2\theta\,d\theta+\frac{3k^4}{8}\int_0^{\frac{\pi}{2}}\sin^4\theta\,d\theta+\frac{5k^6}{16}\int_0^{\frac{\pi}{2}}\sin^6\theta\,d\theta\cdots\right)$$

We can observe that each term of the integral contains the integral of $\sin^{2n}\theta$ from 0 to $\pi/2$. Let $u = \sin^2\theta$, $du = 2\sin\theta\cos\theta\, d\theta$, so $d\theta = 1/2 \times u^{-1/2} \times (1-u)^{-1/2}\ du$. Substituting this into the integral, it becomes

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^{2n}\theta\ d\theta &= \frac{1}{2}\int_0^1 u^n \times u^{-1/2} \times (1-u)^{-1/2} du \\ &= \frac{1}{2}\int_0^1 u^{n-1/2}(1-u)^{-1/2} du \\ &= \frac{1}{2}\int_0^1 u^{(n+1/2)-1}(1-u)^{1/2-1} du \\ &= \frac{1}{2}B\left(n+\frac{1}{2},\frac{1}{2}\right) = \frac{1}{2}\frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(n+1)}\end{aligned}$$

By (1.5.18), we get

$$\int_0^{\frac{\pi}{2}} \sin^{2n}\theta\ d\theta = \frac{1}{2}\frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(n+1)} = \frac{1}{2}\frac{\left(\frac{(2n)!}{n!4^n}\sqrt{\pi}\right)\sqrt{\pi}}{n!} = \frac{(2n)!}{(n!)^2 4^n}\cdot\frac{\pi}{2} \tag{1.5.31}$$

Each integral of the form $\int_0^{\pi/2}\sin^{2n}\theta\, d\theta$ yields a result of the form $c\times\pi/2$, where c is a constant. By comparing with equation (1.5.30), we observe that the coefficient of each $\frac{\pi}{2}$ exactly matches the coefficient of k^{2n} in the expansion of (1.5.30). Therefore,

$$\begin{aligned}T &= 4\sqrt{\frac{L}{g}}\left(\frac{\pi}{2}+\frac{k^2}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2}+\frac{3k^4}{8}\cdot\frac{3}{8}\cdot\frac{\pi}{2}+\frac{5k^6}{16}\cdot\frac{5}{16}\cdot\frac{\pi}{2}+\cdots\right) \\ &= 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{k^2}{4}+\frac{9k^4}{64}+\frac{25k^6}{512}+\cdots\right) = 2\pi\sqrt{\frac{L}{g}}\sum_i\left(\frac{(2i)!}{(i!)^2 4^i}\right)^2 k^{2i}\end{aligned} \tag{1.5.32}$$

By the Taylor expansion of $\sin x$ function, we can approximate the term $k = \sin(\theta_0/2)$, which is

$$k = \sin\left(\frac{\theta_0}{2}\right) = \frac{\theta_0}{2} - \frac{\theta_0^3}{48} + \frac{\theta_0^5}{3840} - \frac{\theta_0^7}{645120} + \cdots \tag{1.5.33}$$

So, the final result of the period of the simple pendulum is

$$T = 2\pi\sqrt{\frac{L}{g}}\sum_i\left(\frac{(2i)!}{(i!)^2 4^i}\right)^2\left(\frac{\theta_0}{2} - \frac{\theta_0^3}{48} + \frac{\theta_0^5}{3840} - \frac{\theta_0^7}{645120} + \cdots\right)^{2i} \tag{1.5.34}$$

From this equation, we can clearly see that when the pendulum undergoes small-angle motion, the contribution of the angular terms approaches zero, resulting in an approximately linear periodic behavior.

$$\lim_{\theta_0\to 0}\sum_i\left(\frac{(2i)!}{(i!)^2 4^i}\right)^2\left(\frac{\theta_0}{2} - \frac{\theta_0^3}{48} + \frac{\theta_0^5}{3840} - \frac{\theta_0^7}{645120} + \cdots\right)^{2i} = 1 \Rightarrow T = 2\pi\sqrt{\frac{L}{g}}\bigg|_{\theta_0\to 0}$$

1.6 Acceleration in Three Coordinates

In mathematics and physics, the most commonly used coordinate systems can be categorized into four types: Cartesian, polar, cylindrical, and spherical coordinates. In this chapter, I will start with the unit vectors and their differential relationships in the Cartesian coordinates and use them to transform the unit vectors of the other three systems. By doing so, we will derive the relationships between the unit vectors in each coordinate system and further obtain the expressions for velocity and acceleration in these various coordinate frameworks.

▷ Polar Coordinates

We begin with the polar coordinate system, which can be decomposed into the radial unit vector $\mathbf{e}_r$ and the angular (or tangential) unit vector $\mathbf{e}_\phi$, as shown in the figure below

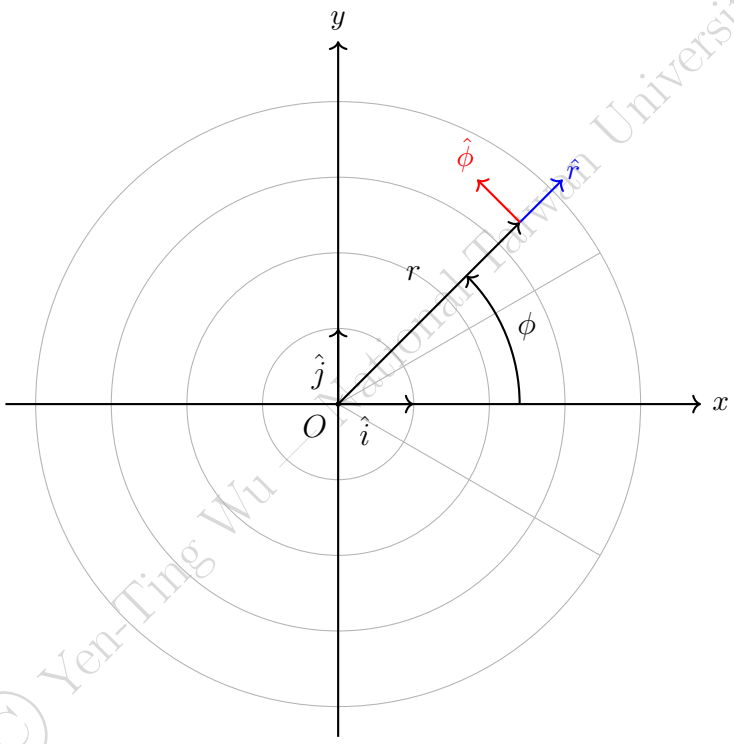

Figure 6: Polar coordinate system with unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\phi$, along with the Cartesian basis vectors $\hat{i}$ and $\hat{j}$.

We can first write down the relationship between the two coordinates, that is

$$\begin{cases} \mathbf{e}_r = \cos\phi\hat{i} + \sin\phi\hat{j} \\ \mathbf{e}_\phi = -\sin\phi\hat{i} + \cos\phi\hat{j} \end{cases} \tag{1.6.1}$$

The two unit vectors in polar coordinates depend only on the angle ϕ. Differentiating both unit vectors with respect to ϕ, we get

$$\begin{cases} \dfrac{d\mathbf{e}_r}{d\phi} = -\sin\phi\hat{i} + \cos\phi\hat{j} = \mathbf{e}_\phi \\ \dfrac{d\mathbf{e}_\phi}{d\phi} = -\cos\phi\hat{i} - \sin\phi\hat{j} = -\mathbf{e}_r \end{cases} \tag{1.6.2}$$

We can express the above equation as a second-order Taylor expansion approximation.

$$\begin{cases} d\mathbf{e}_r = d\phi\ \mathbf{e}_\phi \Rightarrow \dot{\mathbf{e}}_r = \dot{\phi}\ \mathbf{e}_\phi \\ d\mathbf{e}_\phi = -d\phi\ \mathbf{e}_r \Rightarrow \dot{\mathbf{e}}_\phi = -\dot{\phi}\ \mathbf{e}_r \end{cases} \tag{1.6.3}$$

then the position vector can be written as

$$\mathbf{r}(t) = r(t)\mathbf{e}_r \tag{1.6.4}$$

Differentiating the position vector, we can obtain

$$\dot{\mathbf{r}}(t) = \frac{d}{dt}(r(t)\mathbf{e}_r) = \dot{r}(t)\mathbf{e}_r + r(t)\dot{\mathbf{e}}_r$$

So the velocity is

$$\mathbf{v}(t) = \dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\phi}\ \mathbf{e}_\phi \tag{1.6.5}$$

Next, by differentiating the velocity vector once more, we obtain the acceleration

$$\begin{aligned} \mathbf{a}(t) = \dot{\mathbf{v}} = \ddot{\mathbf{r}} &= \frac{d}{dt}(\dot{r}\mathbf{e}_r + r\dot{\phi}\ \mathbf{e}_\phi) \\ &= \ddot{r}\mathbf{e}_r + \dot{r}\dot{\mathbf{e}}_r + \dot{r}\dot{\phi}\mathbf{e}_\phi + r\ddot{\phi}\mathbf{e}_\phi + r\dot{\phi}\dot{\mathbf{e}}_\phi \end{aligned}$$

Substituting (1.6.3) into it, then

$$\ddot{\mathbf{r}} = \ddot{r}\mathbf{e}_r + \dot{r}\dot{\phi}\ \mathbf{e}_\phi + \dot{r}\dot{\phi}\mathbf{e}_\phi + r\ddot{\phi}\mathbf{e}_\phi - r\dot{\phi}^2\ \mathbf{e}_r = (\ddot{r} - r\dot{\phi}^2)\mathbf{e}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\mathbf{e}_\phi$$

So, the final result of the acceleration in polar coordinates is

$$\boxed{\mathbf{a} = \begin{cases} \text{radial} : (\ddot{r} - r\dot{\phi}^2)\mathbf{e}_r \\ \text{angular} : (r\ddot{\phi} + 2\dot{r}\dot{\phi})\mathbf{e}_\phi \end{cases}} \tag{1.6.6}$$

When an object undergoes uniform circular motion, its radius(r) of motion remains constant, and its angular velocity($\dot{\phi}$) stays uniform. That is to say, $\dot{r}, \ddot{r} = 0$, and $\ddot{\phi} = 0$. The final result is familiar in high school, which is

$$\mathbf{a} = -r\dot{\phi}^2\ \mathbf{e}_r = -r\omega^2\mathbf{e}_r \tag{1.6.7}$$

The negative sign here indicates that the direction of acceleration always points opposite to the radial direction, that is, toward the center of the circle.

▷ Cylindrical Coordinates

The second coordinate, the cylindrical system, can be decomposed into the radial unit vector $\mathbf{e}_r$, an angular (azimuthal) unit vector $\mathbf{e}_\phi$, and a vertical unit vector $\mathbf{e}_z$. As shown in figure

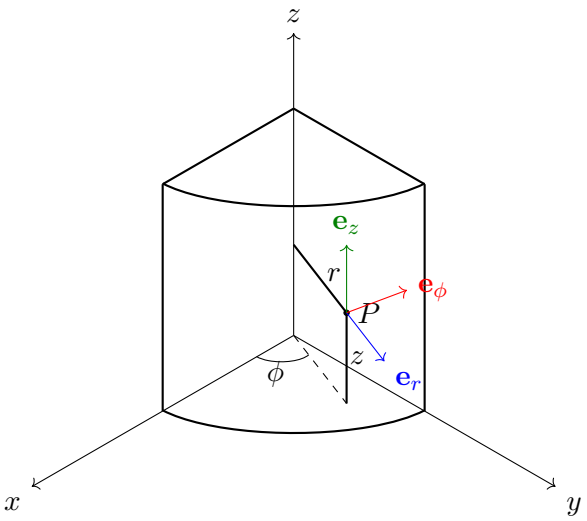

Figure 7: Cylindrical coordinate system with unit vectors $\mathbf{e}_r$, $\mathbf{e}_\phi$, and $\mathbf{e}_z$.

We begin by rewriting these unit vectors in terms of the Cartesian basis vectors, which is

$$\begin{cases} \mathbf{e}_r = \cos\phi\hat{i} + \sin\phi\hat{j} \\ \mathbf{e}_\phi = -\sin\phi\hat{i} + \cos\phi\hat{j} \\ \mathbf{e}_z = \hat{k} \end{cases} \tag{1.6.8}$$

We can also express the above equation as a second-order Taylor expansion approximation, as we did in the last analysis.

$$\begin{cases} d\mathbf{e}_r = \dfrac{d\mathbf{e}_r}{d\phi}d\phi = d\phi\,\mathbf{e}_\phi \Rightarrow \dot{\mathbf{e}}_r = \dot{\phi}\,\mathbf{e}_\phi \\ d\mathbf{e}_\phi = \dfrac{d\mathbf{e}_\phi}{d\phi}d\phi = -d\phi\,\mathbf{e}_r \Rightarrow \dot{\mathbf{e}}_\phi = -\dot{\phi}\,\mathbf{e}_r \\ d\mathbf{e}_z = \dfrac{d\mathbf{e}_z}{d\phi}d\phi = 0 \end{cases} \tag{1.6.9}$$

The result indicates that the unit vector $\mathbf{e}_z$ does not change with respect to the angle or the position; it remains $(0, 0, 1)$. The position vector is given by $\mathbf{r} = r\mathbf{e}_r + z\mathbf{e}_z$, since $r\mathbf{e}_r$ can only describe vectors in the xy-plane. So, the velocity is

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d}{dt}(r\mathbf{e}_r + z\mathbf{e}_z) = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r + \dot{z}\mathbf{e}_z + z\dot{\mathbf{e}}_z \tag{1.6.10}$$

Substituting (1.6.9) into (1.6.10), we get

$$\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\phi}\mathbf{e}_\phi + \dot{z}\mathbf{e}_z \tag{1.6.11}$$

Next, by differentiating the velocity vector once more, we obtain the acceleration

$$\begin{aligned}\mathbf{a} = \ddot{\mathbf{r}} &= \frac{d}{dt}(\dot{r}\mathbf{e}_r + r\dot{\phi}\mathbf{e}_\phi + \dot{z}\mathbf{e}_z) \\ &= \ddot{r}\mathbf{e}_r + \dot{r}\dot{\mathbf{e}}_r + \dot{r}\dot{\phi}\mathbf{e}_\phi + r\ddot{\phi}\mathbf{e}_\phi + r\dot{\phi}\dot{\mathbf{e}}_\phi + \ddot{z}\mathbf{e}_z + \dot{z}\dot{\mathbf{e}}_z \\ &= \ddot{r}\mathbf{e}_r + \dot{r}\dot{\phi}\mathbf{e}_\phi + \dot{r}\dot{\phi}\mathbf{e}_\phi + r\ddot{\phi}\mathbf{e}_\phi - r\dot{\phi}^2\mathbf{e}_r + \ddot{z}\mathbf{e}_z \\ &= (\ddot{r} - r\dot{\phi}^2)\mathbf{e}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\mathbf{e}_\phi + \ddot{z}\mathbf{e}_z\end{aligned} \tag{1.6.12}$$

So, the final result of the acceleration in cylindrical coordinates is

$$\boxed{\mathbf{a} = \begin{cases} \text{radial} : (\ddot{r} - r\dot{\phi}^2)\mathbf{e}_r \\ \text{angular} : (r\ddot{\phi} + 2\dot{r}\dot{\phi})\mathbf{e}_\phi \\ \text{vertical} : \ddot{z}\mathbf{e}_z \end{cases}} \tag{1.6.13}$$

▷ Spherical Coordinates

The third coordinate system, the spherical coordinate system, can be decomposed into the radial unit vector $\mathbf{e}_r$, the polar (or colatitudinal) unit vector $\mathbf{e}_\theta$, and the azimuthal unit vector $\mathbf{e}_\phi$. As shown in the figure, these unit vectors form an orthonormal basis that changes direction depending on the position.

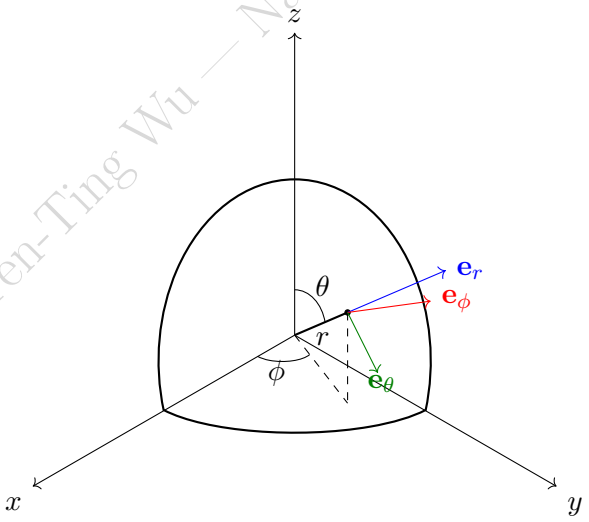

Figure 8: Spherical coordinate system with unit vectors $\mathbf{e}_r$, $\mathbf{e}_\phi$, and $\mathbf{e}_\theta$.

Similarly, we begin by rewriting these unit vectors in terms of the Cartesian basis vectors, which is

$$\begin{cases} \mathbf{e}_r = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k} \\ \mathbf{e}_\phi = -\sin\phi\hat{i} + \cos\phi\hat{j} \\ \mathbf{e}_\theta = \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k} \end{cases} \tag{1.6.14}$$

Since this coordinate system involves both θ and ϕ as independent variables, we can consider the total differentials of the three unit vectors, which are given by

$$\begin{cases} d\mathbf{e}_r = \dfrac{\partial \mathbf{e}_r}{\partial \theta} d\theta + \dfrac{\partial \mathbf{e}_r}{\partial \phi} d\phi = d\theta\, \mathbf{e}_\theta + d\phi \sin\theta\, \mathbf{e}_\phi \\ d\mathbf{e}_\phi = \dfrac{\partial \mathbf{e}_\phi}{\partial \theta} d\theta + \dfrac{\partial \mathbf{e}_\phi}{\partial \phi} d\phi = (-\sin\theta\, \mathbf{e}_r - \cos\theta\, \mathbf{e}_\theta) d\phi \\ d\mathbf{e}_\theta = \dfrac{\partial \mathbf{e}_\theta}{\partial \theta} d\theta + \dfrac{\partial \mathbf{e}_\theta}{\partial \phi} d\phi = -d\theta\, \mathbf{e}_r + d\phi \cos\theta\, \mathbf{e}_\phi \end{cases} \tag{1.6.15}$$

The position vector is given by $\mathbf{r} = r\mathbf{e}_r$, so the velocity is

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d}{dt}(r\mathbf{e}_r) = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r \tag{1.6.16}$$

Substituting (1.4.15) into (1.4.16), we get

$$\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + r\dot{\phi}\sin\theta\mathbf{e}_\phi \tag{1.6.17}$$

Next, by differentiating the velocity vector once more, we obtain the acceleration

$$\begin{aligned} \mathbf{a} = \ddot{\mathbf{r}} &= \frac{d}{dt}(\dot{r}\mathbf{e}_r) + \frac{d}{dt}(r\dot{\theta}\mathbf{e}_\theta) + \frac{d}{dt}(r\dot{\phi}\sin\theta\mathbf{e}_\phi) \\ &= \ddot{r}\mathbf{e}_r + \dot{r}\dot{\mathbf{e}}_r + \dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta + r\dot{\theta}\dot{\mathbf{e}}_\theta + \dot{r}\dot{\phi}\sin\theta\mathbf{e}_\phi \\ &\quad + r\ddot{\phi}\sin\theta\mathbf{e}_\phi + r\dot{\phi}\left(\frac{d}{dt}\sin\theta\right)\mathbf{e}_\phi + r\dot{\phi}\sin\theta\dot{\mathbf{e}}_\phi \end{aligned} \tag{1.6.18}$$

Note that $\sin\theta$ is the function of θ, while θ is the function of t. By chain rule, we get

$$\left(\frac{d}{dt}\sin\theta\right) = \dot{\theta}\cos\theta \tag{1.6.19}$$

So, (1.4.18) becomes

$$\begin{aligned} \ddot{\mathbf{r}} &= \ddot{r}\mathbf{e}_r + (\dot{r}\dot{\theta}\mathbf{e}_\theta + \dot{r}\dot{\phi}\sin\theta\mathbf{e}_\phi) + \dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta + (-r\dot{\theta}^2\mathbf{e}_r + r\dot{\theta}\dot{\phi}\cos\theta\mathbf{e}_\phi) \\ &\quad + \dot{r}\dot{\phi}\sin\theta\mathbf{e}_\phi + r\ddot{\phi}\sin\theta\mathbf{e}_\phi + r\dot{\theta}\dot{\phi}\cos\theta\mathbf{e}_\phi + (-r\dot{\phi}^2\sin^2\theta\mathbf{e}_r - r\dot{\phi}^2\sin\theta\cos\theta\mathbf{e}_\theta) \\ &= (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\mathbf{e}_\theta + (2\dot{r}\dot{\phi}\sin\theta + \\ &\quad 2r\dot{\theta}\dot{\phi}\cos\theta + r\ddot{\phi}\sin\theta)\mathbf{e}_\phi \end{aligned}$$

The final result of the acceleration in spherical coordinates is

$$\boxed{\mathbf{a} = \begin{cases} \text{radial} : (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\mathbf{e}_r \\ \text{colatitudinal} : (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\mathbf{e}_\theta \\ \text{azimuthal} : (2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\ddot{\phi}\sin\theta)\mathbf{e}_\phi \end{cases}} \tag{1.6.20}$$

Chapter 2. Introduction to Vector Analysis

They appear extensively in mechanics, electromagnetism, and many other branches of physics. Unlike the simple introduction to vectors and scalars in high school, this chapter will redefine and reconstruct our understanding of vectors through a more abstract yet rigorous algebraic approach. In addition, we will introduce powerful new operators and apply calculus directly to vector quantities. This is not just a technical extension, it is the key to formulating physical laws in a generalized way that is valid under all coordinate systems and boundary conditions, unlike the limited scenarios often seen in high school problems. This chapter might be one of the most challenging parts of this book, but believe me, if you push through, you'll find yourself in one of the most beautiful and exciting areas in mathematical physics...Maybe.

2.1 Basic Vector Operations

▷ Algebra of Vector

Before moving on to the next concept, I'd like to introduce two notations you may not have seen before, the Levi-Civita symbol ε and the Kronecker delta δ. In this book, I'll not introduce these two notations in detail because they are related to tensor analysis and are far beyond the scope of first-year classical mechanics. Interested students are encouraged to explore it on their own. These notations are fundamental in expressing vector calculus identities and linear algebra, especially in index notation.
The Levi-Civita symbol ε represents the sign of a permutation. It equals $+1$ for even permutations, -1 for odd permutations, and 0 if any two indices are repeated, which can be denoted by

$$\varepsilon_{ijk} = \begin{cases} +1 \text{ even permutations} \\ -1 \text{ odd permutations} \\ 0 \text{ repeated indices} \end{cases} \tag{2.1.1}$$

For example, starting from the number set $(1, 2, 3)$: to reach $(2, 1, 3)$, , you need an odd permutation (one swap); to reach $(3, 1, 2)$, , you need an even permutation (two swaps). That is

$$\varepsilon_{213} = -1; \quad \varepsilon_{312} = 1 \tag{2.1.2}$$

The other notation δ, called **Kronecker delta symbol**, can be denoted by

$$\delta_{ij} = \begin{cases} +1, & \text{When } i = j \\ 0, & \text{When } i \neq j \end{cases} \tag{2.1.3}$$

In three dimensions, the relationship between the two notations is given by

$$\varepsilon_{ijk}\varepsilon_{lmm} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \tag{2.1.4}$$

This determinant form arises from the identity involving Levi-Civita and Kronecker delta symbols. The determinant indicates that there are 3×3 types of combinations of non-zero indices. We can generalize to an N-dimensional system, which is

$$\varepsilon_{a_1a_2\cdots}\varepsilon_{b_1b_2\cdots} = \begin{vmatrix} \delta_{a_1b_1} & \delta_{a_1b_2} & \cdots & \delta_{a_1b_n} \\ \delta_{a_2b_1} & \delta_{a_2b_2} & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{a_nb_1} & \delta_{a_nb_2} & \cdots & \delta_{a_nb_n} \end{vmatrix} \tag{2.1.5}$$

Under this relationship, we will focus on discussing one specific case.

$$\boxed{\sum_k \varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}} \tag{2.1.6}$$

We will use this equation shortly. It won't be proved here, but if you're sharp, you might find it pretty obvious.
We have a N-dimensional coordinate $\phi(x_1, x_2, \cdots, x_n)$, it becomes $\phi'(x_1', x_2', \cdots, x_n')$ after an orthogonal rotation(see Chapter 6.5). We define the cosine between the original coordinate and the new one as

$$\lambda_{ij} = \cos(x_i', x_j) \tag{2.1.7}$$

Where (x_i', x_j) is the angle of the intersection. So, the equation of the coordinate transformation is

$$x_i' = \sum_j \lambda_{ij} x_j \tag{2.1.8}$$

This means that the new coordinates depend on the projection of the original ones. There are two sets in this n-dimensional system $\Phi_1 = (x_1, x_2, \cdots, x_n)$ and $\Phi_2 = (x_1, x_2, \cdots, x_n)$. If Φ_1 is absolutely the same as the original one after coordinate transformation, then Φ_1 is a **Scalar**, while Φ_2 changes into $\Phi_2' = (x_1', x_2', \cdots, x_n')$, then Φ_2 is a **Vector** and defined as $\phi_2(x_1, x_2, \cdots, x_n) \coloneqq \mathbf{\Phi_2}$. The length of a vector is denoted by

$$|\mathbf{\Phi}| = \Phi = \sqrt{\sum_i x_i^2} \tag{2.1.9}$$

The unit vector $\mathbf{u}$ is defined to be a vector with the length $|\mathbf{u}| = 1$. We can also make all vectors into a unit vector by dividing its length, that is

$$\mathbf{u} = \frac{\mathbf{\Phi}}{\Phi} \tag{2.1.10}$$

Under the different coordinates, the unit vectors have different kinds of forms, they can be denoted by the following forms

$$\begin{cases} \mathbf{i},\ \mathbf{j},\ \mathbf{k} \\ \\ \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3} \end{cases} \tag{2.1.11}$$

▷ Vector Products

I'm going to introduce the most important products in vector analysis, the **Dot product** and the **Cross product.** First, the dot product of the two vectors **A**, **B** is defined as

$$\boxed{\mathbf{A}\cdot\mathbf{B}\coloneqq\sum_i A_iB_i} \tag{2.1.12}$$

A_i is the element of **A**, while B_i is the element of **B**. We can use this equation to define the cosine between the two by using both of their **direction cosine**(Thornton 2003), that is

$$\frac{\mathbf{A}\cdot\mathbf{B}}{A\cdot B}=\sum_i\frac{A_i}{A}\frac{B_i}{B}\coloneqq\sum_i\Lambda_i^A\Lambda_i^B=\cos\theta \tag{2.1.13}$$

Where Λ is the direction cosine. The meaning of direction cosine is to use the cosine values to illustrate the **unit vector**, for instance, $\mathbf{A}/A=(\Lambda_1^A,\Lambda_2^A,\cdots,\Lambda_n^A)$. The result is obvious. If we transform the coordinate from $\phi(x_1,\cdot,x_n)$ to $\phi'(x_1',\cdots,x_n')$, then the elements of the vector **A**, **B** become

$$\begin{cases}A_i'\to\sum_i\lambda_{ij}A_j\\ B_i'\to\sum_k\lambda_{ik}B_k\end{cases} \tag{2.1.14}$$

So, the dot product becomes

$$\begin{aligned}\mathbf{A}'\cdot\mathbf{B}'&=\sum_i A_i'B_i'\\ &=\sum_i\left(\sum_j\lambda_{ij}A_j\right)\left(\sum_k\lambda_{ik}B_k\right)\\ &=\sum_{jk}\left(\sum_i\lambda_{ij}\lambda_{ik}\right)A_jB_k\\ &=\sum_{jk}\delta_{jk}A_jB_k=\sum_j A_jB_j\\ &=\mathbf{A}\cdot\mathbf{B}\end{aligned}$$

It's independent of the coordinate transformation, so we can verify that the dot product is a **scalar**.
The other vector product, the cross product of **A** with **B** is defined as

$$\boxed{(\mathbf{A}\times\mathbf{B})_i\coloneqq\sum_{jk}\varepsilon_{ijk}A_jB_k} \tag{2.1.15}$$

In three-dimensional coordinates, it's also denoted by (see chapter 6)

$$\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\mathbf{e_1}&\mathbf{e_2}&\mathbf{e_3}\\ A_1&A_2&A_3\\ B_1&B_2&B_3\end{vmatrix} \tag{2.1.16}$$

If we transform the coordinate, like what we did in the dot product, the cross product becomes

$$\begin{aligned}(\mathbf{A}' \times \mathbf{B}')_i &= \sum_{jk} \varepsilon_{ijk} A'_j B'_k \\ &= \sum_{jk} \varepsilon_{ijk} \left(\sum_l \lambda_{jl} A_l \right) \left(\sum_m \lambda_{km} B_m \right) \\ &= \sum_{jklm} \varepsilon_{ijk} \lambda_{jl} \lambda_{km} A_l B_m \neq (\mathbf{A} \times \mathbf{B})_i\end{aligned}$$

It depends on the coordinate transformation, so the cross product is a **vector.** Because it's a vector, the cross product must have length denoted by $|\mathbf{A} \times \mathbf{B}|$. I'll focus on the three-dimensional coordinates.
We can expand the result of (2.1.16), that is

$$\mathbf{A} \times \mathbf{B} = (A_2B_3 - A_3B_2)\mathbf{e_1} + (A_3B_1 - A_1B_3)\mathbf{e_2} + (A_1B_2 - A_2B_1)\mathbf{e_3} \tag{2.1.17}$$

So the length is

$$\begin{aligned}|\mathbf{A} \times \mathbf{B}|^2 &= (A_2B_3 - A_3B_2)^2 + (A_3B_1 - A_1B_3)^2 + (A_1B_2 - A_2B_1)^2 \\ &= A_2^2B_3^2 + A_3^2B_2^2 + A_3^2B_1^2 + A_1^2B_3^2 + A_1^2B_2^2 + A_2^2B_1^2 \\ &- 2(A_2A_3B_2B_3 + A_1A_3B_1B_3 + A_1A_2B_1B_2) \\ &+ (A_1^2B_1^2 - A_1^2B_1^2) + (A_2^2B_2^2 - A_2^2B_2^2) + (A_3^2B_3^2 - A_3^2B_3^2) \\ &= (A_1^2 + A_2^2 + A_3^2)(B_1^2 + B_2^2 + B_3^2) - (A_1B_1 + A_2B_2 + A_3B_3)^2\end{aligned}$$

Note that $(A_1^2 + A_2^2 + A_3^2)(B_1^2 + B_2^2 + B_3^2)$ is A^2B^2, and $(A_1B_1 + A_2B_2 + A_3B_3)^2 = (\mathbf{A} \cdot \mathbf{B})^2$. So, the final result is

$$\begin{aligned}|\mathbf{A} \times \mathbf{B}| &= \sqrt{A^2B^2 - (\mathbf{A} \cdot \mathbf{B})^2} \\ &= AB\sqrt{1 - \left(\sum_i \Lambda_i^A \Lambda_i^B \right)^2} \\ &= AB \sin\theta\end{aligned}$$

That's the familiar form of the length of cross product, namely

$$\boxed{|\mathbf{A} \times \mathbf{B}| = AB \sin\theta} \tag{2.1.18}$$

▷ Vector Identities

There are four fundamental vector identities presented below. In this section, I will verify each of them using algebraic methods rather than geometric intuition.

1. $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{D}) = \mathbf{D} \cdot (\mathbf{A} \times \mathbf{B})$
   The definition of the cross product is

$$(\mathbf{B} \times \mathbf{D})_i = \sum_{jk} \varepsilon_{ijk} B_j D_k \tag{2.1.19}$$

Because both $\mathbf{B}\times\mathbf{D}$ and $\mathbf{A}$ are vectors, the dot product of the two is

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}\times\mathbf{D}) &= \sum_i A_i(\mathbf{B}\times\mathbf{D})_i \\ &= \sum_i A_i\left(\sum_{jk}\varepsilon_{ijk}B_jD_k\right) \\ &= \sum_{ijk}\varepsilon_{ijk}A_iB_jD_k\end{aligned} \tag{2.1.20}$$

At the same time, we can use the same way to get

$$\mathbf{D}\cdot(\mathbf{A}\times\mathbf{B}) = \sum_{ijk}\varepsilon_{ijk}D_iA_jB_k \tag{2.1.21}$$

Note that the Levi-Civita symbol equals $+1$ for even permutations and -1 for odd permutations. The permutations of the indices also satisfy this rule, for example, $(i,j,k)\to(j,i,k)$ is an odd permutation, while $(i,j,k)\to(k,i,j)$ is an even permutation. So, we can get that

$$\sum_{ijk}\varepsilon_{ijk}D_iA_jB_k = \sum_{ijk}\varepsilon_{jki}A_jB_kD_i = \sum_{ijk}\varepsilon_{ijk}A_iB_jD_k = \mathbf{A}\cdot(\mathbf{B}\times\mathbf{D})$$

Thus, we can obtain the first identity

$$\boxed{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{D}) = \mathbf{D}\cdot(\mathbf{A}\times\mathbf{B})} \tag{2.1.22}$$

This is what you've learned in high school; it represents the volume of a parallelepiped in three-dimensional space. We can also define this identity as

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{D}) = \mathbf{D}\cdot(\mathbf{A}\times\mathbf{B}) \coloneqq \mathbf{ABD} \tag{2.1.23}$$

2. $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$
First, the cross product of $\mathbf{B}$ with $\mathbf{C}$ is

$$(\mathbf{B}\times\mathbf{C})_i = \sum_{jk}\varepsilon_{ijk}B_jC_k \tag{2.1.24}$$

After that, the cross product of $\mathbf{A}$ with $\mathbf{B}\times\mathbf{C}$ is

$$\begin{aligned}\left(\mathbf{A}\times(\mathbf{B}\times\mathbf{C})\right)_i &= \sum_{lm}\varepsilon_{ilm}A_l(\mathbf{B}\times\mathbf{C})_m \\ &= \sum_{lm}\varepsilon_{ilm}A_l\left(\sum_{jk}\varepsilon_{mjk}B_jC_k\right) \\ &= \sum_{jklm}\varepsilon_{ilm}\varepsilon_{jkm}A_lB_jC_k\end{aligned} \tag{2.1.25}$$

We need to use (2.1.6), so (2.1.25) becomes

$$\sum_{jklm}\varepsilon_{ilm}\varepsilon_{jkm}A_lB_jC_k = \sum_{jkl}(\delta_{ij}\delta_{lk}-\delta_{ik}\delta_{lj})A_lB_jC_k \tag{2.1.26}$$

When $i = j,\ l = k$, $(\delta_{ij}\delta_{lk} - \delta_{ik}\delta_{lj})A_lB_jC_k = A_lC_lB_i$; when $i = k,\ l = j$, $(\delta_{ij}\delta_{lk} - \delta_{ik}\delta_{lj})A_lB_jC_k = -A_lB_lC_i$. Substituting this result into (2.1.26), we get

$$\sum_{jkl}(\delta_{ij}\delta_{lk} - \delta_{ik}\delta_{lj})A_lB_jC_k = \left(\sum_l A_lC_l\right)B_i - \left(\sum_l A_lB_l\right)C_i \tag{2.1.27}$$

Thus, we verify that

$$(\mathbf{A}\times(\mathbf{B}\times\mathbf{C}))_i = \left(\sum_l A_lC_l\right)B_i - \left(\sum_l A_lB_l\right)C_i \tag{2.1.28}$$

Note that the elements in the brackets are the dot products, so we can finally claim that

$$\boxed{\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}} \tag{2.1.29}$$

3. $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$
First, the cross products are

$$\begin{cases}(\mathbf{A}\times\mathbf{B})_i = \sum_{jk}\varepsilon_{ijk}A_jB_k \\ (\mathbf{C}\times\mathbf{D})_i = \sum_{lm}\varepsilon_{ilm}C_lD_m\end{cases} \tag{2.1.30}$$

Then the dot product of $\mathbf{A}\times\mathbf{B}$ with $\mathbf{C}\times\mathbf{D}$ is

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) &= \sum_i\left(\sum_{jk}\varepsilon_{ijk}A_jB_k\right)\left(\sum_{lm}\varepsilon_{ilm}C_lD_m\right) \\ &= \sum_{jklm}\left(\sum_i\varepsilon_{ijk}\varepsilon_{ilm}\right)A_jB_kC_lD_m \\ &= \sum_{jklm}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})A_jB_kC_lD_m\end{aligned} \tag{2.1.31}$$

When $j = l,\ k = m$, $(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})A_jB_kC_lD_m = A_jC_jB_kD_k$; when $j = m,\ k = l$, $(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})A_jB_kC_lD_m = -A_jD_jB_kC_k$. Substituting this result into (2.1.31), we get

$$\begin{aligned}\sum_{jklm}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})A_jB_kC_lD_m &= \left(\sum_{jk}A_jC_jB_kD_k\right) - \left(\sum_{jk}A_jD_jB_kC_k\right) \\ &= \left(\sum_j A_jC_j\right)\left(\sum_k B_kD_k\right) - \left(\sum_j A_jD_k\right)\left(\sum_k B_kC_k\right) \\ &= (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})\end{aligned}$$

So we can finally verify the identity

$$\boxed{(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})} \tag{2.1.32}$$

4. $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})=(\mathbf{ABD})\mathbf{C}-(\mathbf{ABC})\mathbf{D}$
We can directly use the result of 2. That is

$$(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})=[(\mathbf{A}\times\mathbf{B})\cdot\mathbf{D}]\mathbf{C}-[(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}]\mathbf{D} \tag{2.1.33}$$

So, the final result is undeniable that

$$\boxed{(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})=(\mathbf{ABD})\mathbf{C}-(\mathbf{ABC})\mathbf{D}} \tag{2.1.34}$$

This is probably the easiest proof in this book. :) You need to cherish it!

2.2 Definition of Vector Differentiation

▷ **Vector Field and Curve**

A vector function $\mathbf{\Phi}$ is a function that assigns a unique vector to each point in its domain D, which lies in a plane or in space. The collection of all such output vectors forms the image of the vector function, which in physics is often called a **field**.
More formally, a vector function is a function from a domain $D\subset\mathbb{R}^n$ to $\mathbb{R}^m$, where each point $\mathbf{r}\in D$ corresponds to a unique vector $\mathbf{F}(\mathbf{r})\in\mathbb{R}^m$.
In a three-dimensional system, a vector function $\mathbf{\Phi}$ consists of three scalar functions along the three coordinate directions. That is

$$\mathbf{\Phi}(x_1,x_2,x_3)=\Phi_1(x_1,x_2,x_3)\mathbf{e_1}+\Phi_2(x_1,x_2,x_3)\mathbf{e_2}+\Phi_3(x_1,x_2,x_3)\mathbf{e_3} \tag{2.2.1}$$

Where $\Phi_1,\ \Phi_2\ ,\Phi_3$ are scalar functions. So, in an N-dimensional system, it becomes

$$\mathbf{\Phi}(x_1,x_2,\cdots,x_n)=\sum_i\Phi_i(x_1,\cdots,x_n)\mathbf{e_i} \tag{2.2.2}$$

For example, a vector function $\mathbf{F}(x,y,z)=xyz^2\mathbf{e_1}+x^3z\ \mathbf{e_2}+(2y+3x)\mathbf{e_3}$, the vector of the point $(2,1,-1)$ is

$$\mathbf{F}(2,1,-1)=2\mathbf{e_1}-8\mathbf{e_2}+8\mathbf{e_3}$$

In addition, we can use the vector and parameter t to describe the graph in space. A parameter t is a controlling quantity used to label or describe the position or variation of a geometric object, such as a point, a curve, or a surface. While a parameter does not necessarily carry physical meaning by itself, varying the parameter allows us to trace out the entire shape step by step.

We start with a curve C. We can construct a vector function using a parameter as its domain. By assigning a unique vector to each value of the parameter and collecting all such vectors, we generate a curve in space. This method of describing a curve is called parametrization. It can be denoted as

$$C:\{\mathbf{r}(t)=x(t)\mathbf{e_1}+y(t)\mathbf{e_2}+z(t)\mathbf{e_3},\quad t\in[a,b]\} \tag{2.2.3}$$

where $\mathbf{r}(t)$ is the parametrization of the curve. In mathematics, it is defined as

$$\mathbf{r}(t):\mathbb{R}\supset I\to\mathbb{R}^n \tag{2.2.4}$$

More specifically, the idea of parametrization is that each parameter value corresponds to a point in space, and each point is associated with a direction. By smoothly connecting these directional changes, we form a winding curve, one that can be seen as composed of a sequence of varying tangent vectors along its path.

▷ The Definition of Limit and Differentiation

After discussing the vector function, I will introduce the differentiation of a vector function in space. Starting with the single variable function $\mathbf{\Phi}(t)$, the limit of it is defined as

$$\lim_{t\to t_0} \mathbf{\Phi}(t) = \mathbf{A} \Longleftrightarrow |\mathbf{\Phi}(t) - \mathbf{A}| = 0, \quad t \to t_0 \tag{2.2.5}$$

It means that when t approaches t_0, the angle of intersection of vector $\mathbf{\Phi}$ and $\mathbf{A}$, in other words, the components of $\mathbf{\Phi}$ approach those of $\mathbf{A}$. That is

$$\lim_{t\to t_0} \Phi_i = A_i \tag{2.2.6}$$

So that the limit of a vector function is

$$\boxed{\lim_{t\to t_0} \mathbf{\Phi}(t) = \left(\lim_{t\to t_0} \Phi_1(t), \lim_{t\to t_0} \Phi_2(t), \lim_{t\to t_0} \Phi_3(t)\right) = (A_1, A_2, A_3)} \tag{2.2.7}$$

Assume that $\mathbf{\Phi}(t)$ is continuous at $t = t_0$, when the limit exists, we can define the derivative of vector function as

$$\left.\frac{d\mathbf{\Phi}}{dt}\right|_{t=t_0} = \lim_{\|\Delta\|\to 0} \frac{\mathbf{\Phi}(t_0 + \Delta t) - \mathbf{\Phi}(t_0)}{\Delta t} \tag{2.2.8}$$

Geometrically, the derivative of a vector function represents the tangent vector to the curve traced out by $\mathbf{\Phi}$. Expanding the components of $\mathbf{\Phi}$, (2.2.8) becomes

$$\lim_{\|\Delta\|\to 0} \frac{(\Phi_1(t_0 + \Delta t) - \Phi_1(t), \Phi_2(t_0 + \Delta t) - \Phi_2(t), \Phi_3(t_0 + \Delta t) - \Phi_3(t))}{\Delta t} \tag{2.2.9}$$

So, the derivative of the vector function $\mathbf{\Phi}$ is

$$\boxed{\left.\frac{d\mathbf{\Phi}}{dt}\right|_{t=t_0} = \left(\left.\frac{d\Phi_1}{dt}\right|_{t=t_0}, \left.\frac{d\Phi_2}{dt}\right|_{t=t_0}, \left.\frac{d\Phi_3}{dt}\right|_{t=t_0}\right)} \tag{2.2.10}$$

For example, a position vector function $\mathbf{r}(t)$ [1] Then velocity and the acceleration of it is

$$\begin{cases} \mathbf{v}(t) = \dfrac{d\mathbf{r}(t)}{dt} \coloneqq \dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z}) \\ \mathbf{a}(t) = \dfrac{d^2\mathbf{r}(t)}{dt^2} \coloneqq \ddot{\mathbf{r}} = (\ddot{x}, \ddot{y}, \ddot{z}) \end{cases}$$

We can apply the same logic to multiple variables and vector functions. Consider a vector function $\mathbf{\Phi}(x_1, x_2, \cdots, x_n)$, the partial differentiation is

$$\frac{\partial \mathbf{\Phi}}{\partial x_j} = \sum_i \frac{\partial \Phi_i}{\partial x_j} \mathbf{e_i} \tag{2.2.11}$$

[1] position vector function represents $\mathbf{r}(t) = (x(t), y(t), z(t))$, it's a fixed usage.

Similarly, the total differential is denoted by

$$\begin{aligned} d\mathbf{\Phi} &= \left(\sum_i \frac{\partial \Phi_1}{\partial x_i} dx_i, \sum_i \frac{\partial \Phi_2}{\partial x_i} dx_i, \cdots, \sum_i \frac{\partial \Phi_n}{\partial x_i} dx_i \right) \\ &= \sum_{ij} \frac{\partial \Phi_j}{\partial x_i} dx_i \mathbf{e_j} \end{aligned} \tag{2.2.12}$$

The meaning of this is that a small change in a vector field is the sum of small directional changes in each component. Note that the total differential of a vector field is still a vector.

In addition, let $\mathbf{r}(t)$ be a differentiable vector function defined with respect to time t. We can expand it around the point t_0 using a Taylor series, that is

$$\mathbf{r}(t_0 + dt) = \mathbf{r}(t_0) + \left.\frac{d\mathbf{r}}{dt}\right|_{t_0} dt + \frac{1}{2!} \left.\frac{d^2\mathbf{r}}{dt^2}\right|_{t_0} (dt)^2 + \cdots \tag{2.2.13}$$

We take the approximation up to the second term. In fact, the second term can be regarded as the infinitesimal displacement $d\mathbf{r}$ along the curve, which is essentially the core idea of the total differential. To simplify the following proof, we can rewrite the second term as

$$\left.\frac{d\mathbf{r}}{dt}\right|_{t_0} dt \coloneqq \Delta\mathbf{r}|_{\|\Delta\| \to 0} \tag{2.2.14}$$

Below are six important identities. I'll show you the proofs of (3) to (6) since (1), (2) are simple and obvious.

---

1. $\boxed{\frac{d}{dt}(\mathbf{A}(t) \pm \mathbf{B}(t)) = \frac{d\mathbf{A}}{dt} \pm \frac{d\mathbf{B}}{dt}}$

2. $\boxed{\frac{d}{dt}(f(t)\mathbf{A}(t)) = \frac{df}{dt}\mathbf{A} + f\frac{d\mathbf{A}}{dt}}$

3. $\boxed{\frac{d}{dt}(\mathbf{A}(t) \cdot \mathbf{B}(t)) = \frac{d\mathbf{A}}{dt} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{d\mathbf{B}}{dt}}$

   We expand the dot product of $\mathbf{A}$ with $\mathbf{B}$

$$\begin{aligned} \frac{d}{dt}(\mathbf{A}(t) \cdot \mathbf{B}(t)) &= \lim_{\|\Delta\| \to 0} \frac{(\mathbf{A} + \Delta\mathbf{A}) \cdot (\mathbf{B} + \Delta\mathbf{B}) - \mathbf{A} \cdot \mathbf{B}}{\Delta t} \\ &= \lim_{\|\Delta\| \to 0} \frac{\mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \Delta\mathbf{B} + \mathbf{B} \cdot \Delta\mathbf{A} + \Delta\mathbf{A} \cdot \Delta\mathbf{B} - \mathbf{A} \cdot \mathbf{B}}{\Delta t} \\ &= \lim_{\|\Delta\| \to 0} \left(\mathbf{A} \cdot \frac{\Delta\mathbf{B}}{\Delta t} + \mathbf{B} \cdot \frac{\Delta\mathbf{A}}{\Delta t} + \frac{\Delta\mathbf{A} \cdot \Delta\mathbf{B}}{\Delta t} \right) \end{aligned}$$

   Since the third term is sufficiently small, we neglect it in the mathematical approximation, and thus obtain the final result that

$$\frac{d}{dt}(\mathbf{A}(t) \cdot \mathbf{B}(t)) = \frac{d\mathbf{A}}{dt} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{d\mathbf{B}}{dt} \tag{2.2.15}$$

4. $\boxed{\dfrac{d}{dt}(\mathbf{A}(t)\times\mathbf{B}(t))=\dfrac{d\mathbf{A}}{dt}\times\mathbf{B}+\mathbf{A}\times\dfrac{d\mathbf{B}}{dt}}$

Apply the same method to prove this, so we need to expand the cross product

$$\begin{aligned}\frac{d}{dt}(\mathbf{A}(t)\times\mathbf{B}(t)) &= \lim_{\|\Delta\|\to 0}\frac{(\mathbf{A}+\Delta\mathbf{A})\times(\mathbf{B}+\Delta\mathbf{B})-\mathbf{A}\times\mathbf{B}}{\Delta t}\\ &= \lim_{\|\Delta\|\to 0}\frac{\mathbf{A}\times\mathbf{B}+\mathbf{A}\times\Delta\mathbf{B}+\mathbf{B}\times\Delta\mathbf{A}+\Delta\mathbf{A}\times\Delta\mathbf{B}-\mathbf{A}\times\mathbf{B}}{\Delta t}\\ &= \lim_{\|\Delta\|\to 0}\left(\mathbf{A}\times\frac{\Delta\mathbf{B}}{\Delta t}+\mathbf{B}\times\frac{\Delta\mathbf{A}}{\Delta t}+\frac{\Delta\mathbf{A}\times\Delta\mathbf{B}}{\Delta t}\right)\end{aligned}$$

Since the third term is sufficiently small, we neglect it in the mathematical approximation, and thus obtain the final result that

$$\frac{d}{dt}(\mathbf{A}(t)\times\mathbf{B}(t))=\frac{d\mathbf{A}}{dt}\times\mathbf{B}+\mathbf{A}\times\frac{d\mathbf{B}}{dt} \tag{2.2.16}$$

5. $\boxed{\dfrac{d}{dt}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})=\mathbf{A}\cdot\mathbf{B}\times\dfrac{d\mathbf{C}}{dt}+\mathbf{A}\cdot\dfrac{d\mathbf{B}}{dt}\times\mathbf{C}+\dfrac{d\mathbf{A}}{dt}\cdot\mathbf{B}\times\mathbf{C}}$

Let the cross product $\mathbf{B}\times\mathbf{C}$ be the vector $\mathbf{r}$, use the result of (3), we can get

$$\frac{d}{dt}(\mathbf{A}\cdot\mathbf{r})=\frac{d\mathbf{A}}{dt}\cdot\mathbf{r}+\mathbf{A}\cdot\frac{d\mathbf{r}}{dt} \tag{2.2.17}$$

Substituting $\mathbf{B}\times\mathbf{C}=\mathbf{r}$ into (2.2.17), we can finally get

$$\frac{d\mathbf{A}}{dt}\cdot\mathbf{B}\times\mathbf{C}+\mathbf{A}\cdot\frac{d(\mathbf{B}\times\mathbf{C})}{dt}=\frac{d\mathbf{A}}{dt}\cdot\mathbf{B}\times\mathbf{C}+\mathbf{A}\cdot\left(\frac{d\mathbf{B}}{dt}\times\mathbf{C}+\mathbf{B}\times\frac{d\mathbf{C}}{dt}\right)$$

So

$$\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})=\mathbf{A}\cdot\mathbf{B}\times\frac{d\mathbf{C}}{dt}+\mathbf{A}\cdot\frac{d\mathbf{B}}{dt}\times\mathbf{C}+\frac{d\mathbf{A}}{dt}\cdot\mathbf{B}\times\mathbf{C} \tag{2.2.18}$$

6. $\boxed{\dfrac{d}{dt}(\mathbf{A}\times(\mathbf{B}\times\mathbf{C}))=\mathbf{A}\times\left(\mathbf{B}\times\dfrac{d\mathbf{C}}{dt}\right)+\mathbf{A}\times\left(\dfrac{d\mathbf{B}}{dt}\times\mathbf{C}\right)+\dfrac{d\mathbf{A}}{dt}\times(\mathbf{B}\times\mathbf{C})}$

Similarly, let the cross product $\mathbf{B}\times\mathbf{C}$ be the vector $\mathbf{r}$, and use the result of (4), we can get

$$\frac{d}{dt}(\mathbf{A}\times\mathbf{r})=\frac{d\mathbf{A}}{dt}\times\mathbf{r}+\mathbf{A}\times\frac{d\mathbf{r}}{dt} \tag{2.2.19}$$

Substituting $\mathbf{B}\times\mathbf{C}=\mathbf{r}$ into (2.2.19), we can finally get

$$\frac{d\mathbf{A}}{dt}\times(\mathbf{B}\times\mathbf{C})+\mathbf{A}\times\frac{d(\mathbf{B}\times\mathbf{C})}{dt}=\frac{d\mathbf{A}}{dt}\times(\mathbf{B}\times\mathbf{C})+\mathbf{A}\times\left(\frac{d\mathbf{B}}{dt}\times\mathbf{C}+\mathbf{B}\times\frac{d\mathbf{C}}{dt}\right)$$

So the final result is solved

$$\frac{d}{dt}(\mathbf{A}\times(\mathbf{B}\times\mathbf{C}))=\mathbf{A}\times\left(\mathbf{B}\times\frac{d\mathbf{C}}{dt}\right)+\mathbf{A}\times\left(\frac{d\mathbf{B}}{dt}\times\mathbf{C}\right)+\frac{d\mathbf{A}}{dt}\times(\mathbf{B}\times\mathbf{C}) \tag{2.2.20}$$

▷ **Directional Derivative**

Consider an infinitesimal variation at $z = f(x, y) \in \mathbb{R}^3$ shown below,

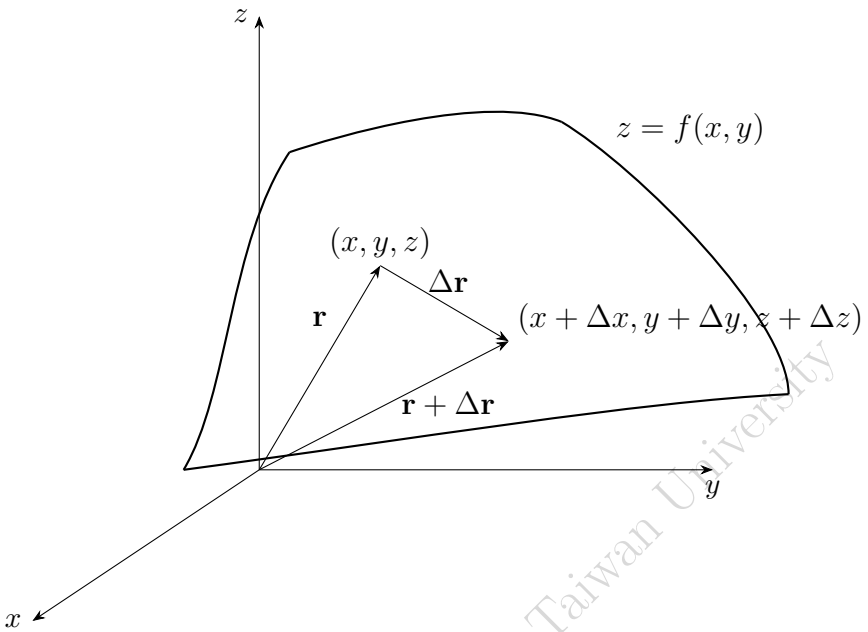

Figure 9: Infinitesimal variation on the surface $z = f(x, y)$

In this figure, the original position vector $\mathbf{r} = (x, y, z)$ undergoes a small variation in the direction of $\Delta\mathbf{r}$. We regard this variation as a progression in a certain direction $\mathbf{u}$. Along this direction, the change in the function's value is equal to

$$\Delta f = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z) \tag{2.2.21}$$

By the linear approximation of a multivariable function via Taylor expansion(see Chapter 1), we get

$$\Delta f \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z \tag{2.2.22}$$

Dividing both sides by $\Delta r = |\Delta\mathbf{r}|$, that is

$$\frac{\Delta f}{\Delta r} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta r} + \frac{\partial f}{\partial y}\frac{\Delta x}{\Delta r} + \frac{\partial f}{\partial z}\frac{\Delta z}{\Delta r} \tag{2.2.23}$$

As the point approaches infinitely close, the direction of variation becomes the tangent vector at the initial point. So, if we take $\|\Delta\| \to 0$, it becomes

$$\frac{df}{dr} = \lim_{\|\Delta\|\to 0} \frac{\Delta f}{\Delta r} \tag{2.2.24}$$

Then we can define df/ds as the **Directional Derivative** along the tangent vector $d\mathbf{r}/dr$. The directional derivative of f at a point in a direction $\mathbf{u}$ measures how fast the function changes in that direction, where $\mathbf{u} = d\mathbf{r}/dr$. More formally and rigorously, the directional derivative is defined as

$$\boxed{D_u f(\mathbf{r}) = \lim_{\alpha\to 0} \frac{f(\mathbf{r} + \alpha\mathbf{u}) - f(\mathbf{r})}{\alpha}} \tag{2.2.25}$$

Where $\mathbf{r} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and $\mathbf{u} = (dx_1/dr, dx_2/dr, \cdots, dx_n/dr)$ is the vector along the tangent direction. By the linear approximation of a multivariable function via Taylor expansion, we get

$$\begin{aligned} D_u f(\mathbf{r}) &= \lim_{\alpha \to 0} \frac{f(\mathbf{r}) + \left(\frac{\partial f}{\partial x_1} \alpha \frac{dx_1}{dr} + \cdots + \frac{\partial f}{\partial x_n} \alpha \frac{dx_n}{dr} \right) - f(\mathbf{r})}{\alpha} \\ &= \lim_{\alpha \to 0} \frac{\frac{\partial f}{\partial x_1} \alpha \frac{dx_1}{dr} + \cdots + \frac{\partial f}{\partial x_n} \alpha \frac{dx_n}{dr}}{\alpha} \\ &= \frac{\partial f}{\partial x_1} \frac{dx_1}{dr} + \cdots + \frac{\partial f}{\partial x_n} \frac{dx_n}{dr} \end{aligned}$$

So, this result can be written as the dot product of two vectors, that is

$$\begin{aligned} \frac{\partial f}{\partial x_1} \frac{dx_1}{dr} + \cdots + \frac{\partial f}{\partial x_n} \frac{dx_n}{dr} &= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n} \right) \cdot \left(\frac{dx_1}{dr}, \frac{dx_2}{dr}, \cdots, \frac{dx_n}{dr} \right) \\ &= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n} \right) \cdot \mathbf{u} \end{aligned}$$

At first glance, this may seem like a "strange" vector because it consists entirely of partial derivatives of f. However, this vector is of fundamental importance and is known as the **Gradient**. That is,

$$\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n} \right) = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n} \right) f \coloneqq \boldsymbol{\nabla} f \tag{2.2.26}$$

The notation ∇ is a differential operator; it is

$$\boldsymbol{\nabla} \coloneqq \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n} \right) \tag{2.2.27}$$

We can use this operator to redefine the directional derivative as

$$D_u f(\mathbf{r}) = \boldsymbol{\nabla} f \cdot \mathbf{u} \tag{2.2.28}$$

It is given by the projection of the gradient $\boldsymbol{\nabla}$ onto the unit direction u. This formula arises naturally from the first-order Taylor expansion of f, and geometrically, it tells us how much of the gradient's influence is felt along direction $\mathbf{u}$.

Note that the vector $\mathbf{u}$ here refers to an arbitrary direction vector. When we previously referred to it as a "tangent vector," we did not mean the tangent vector to the level curve $f(x, y, z) = C$ specifically. Rather, $\mathbf{u}$ lies in the tangent plane of the surface $z = f(x, y)$, and represents a general direction along which one may move tangentially to the surface. The tangent vector to the level curve is simply one special case of such a direction.

▷ Gradient

After introducing the directional derivatives, I'll show you the meaning. We know that the gradient can be denoted by (2.2.27), but we don't know why it could be described as

a vector. We have discussed the coordinate transformation in Chapter 1, which is

$$\frac{\partial \Phi'}{\partial x_i'} = \sum_j \frac{\partial \Phi}{\partial x_j} \frac{\partial x_j}{\partial x_i'}$$

In such a transformation, we can write down the inverse transformation

$$x_j = \sum_k \lambda_{kj} x_k' \tag{2.2.29}$$

So, the partial differentiation of x_j can be denoted by

$$\frac{\partial x_j}{\partial x_i'} = \frac{\partial}{\partial x_i'} \left(\sum_k \lambda_{kj} x_k' \right) = \sum_k \lambda_{kj} \frac{\partial x_k'}{\partial x_i'} = \sum_k \lambda_{kj} \delta_{ki} \tag{2.2.30}$$

The result is now evident. Based on the property of the Kronecker delta, only the term with $i = k$ remains, while all others vanish. That is

$$\frac{\partial x_j}{\partial x_i'} = \lambda_{ij} \tag{2.2.31}$$

Therefore, we can substitute this result into the coordinate transformation equation to obtain

$$\frac{\partial \Phi'}{\partial x_i'} = \sum_j \lambda_{ij} \frac{\partial \Phi}{\partial x_j} \tag{2.2.32}$$

It follows the rule of vectors(see Chapter 2-1), the term $\partial\Phi/\partial x_j$ is the j-th component of the vector. Since the coordinate Φ is a scalar function, the term, $\partial/\partial x_j$ must be a vector. In this way, we can verify that the gradient $\boldsymbol{\nabla}$ is truly a vector, and it's defined as

$$\boxed{\text{grad} = \boldsymbol{\nabla} \coloneqq \sum_i \frac{\partial}{\partial x_i} \mathbf{e_i}} \tag{2.2.33}$$

Now, let us return to the concept of the total differential for a function of n variables $\Phi(x_1, \cdot, x_n)$. The total differential is

$$d\Phi = \sum_i \frac{\partial \Phi}{\partial x_i} dx_i$$

Substituting the gradient into the total differential, we can rewrite the infinitesimal displacement of an n-variable function as

$$d\Phi = \boldsymbol{\nabla}\Phi \cdot d\mathbf{r} = |\boldsymbol{\nabla}\Phi||d\mathbf{r}| \cos\theta \tag{2.2.34}$$

where $d\mathbf{r} = (dx_1, dx_2, \cdots, dx_n)$. The meaning of the vector $d\mathbf{r}$ shares a similarity with the direction vector $\mathbf{u}$ in directional derivatives: both describe a certain direction of variation. However, $d\mathbf{r}$ is not a unit vector, but rather a collection of infinitesimal displacements along each coordinate direction. Therefore, the dot product of the gradient and $d\mathbf{r}$ yields the total amount of change in the function. On the other hand, $\mathbf{u}$ is a unit vector, so the dot product with the gradient yields the rate of change in that specific direction.

The direction of $\boldsymbol{\nabla}\Phi$ is fixed, and the vector $d\mathbf{r}$ is arbitrary. So, the infinitesimal displacements $d\Phi$ have different values with different directions of $d\mathbf{r}$. If we choose $d\mathbf{r} \parallel \boldsymbol{\nabla}\Phi$, the angle $\theta = 0$, so

$$\max(d\Phi) = |\boldsymbol{\nabla}\Phi||d\mathbf{r}| \tag{2.2.35}$$

Equation (2.2.35) implies that the gradient of a function points in the direction of the greatest change at a given point. Therefore, by choosing a small displacement in this direction, one can ensure movement in the direction of the most rapid increase of the function. While we choose $d\mathbf{r} \perp \boldsymbol{\nabla}\Phi$,

$$d\Phi = |\boldsymbol{\nabla}\Phi||d\mathbf{r}| \cos\left(\frac{\pi}{2}\right) = 0$$

In other words, if we move along the direction tangent to the level curve at a given point, the function value clearly remains unchanged since we are staying on the same contour. This is similar to hiking along a trail that maintains a constant elevation; you will never reach the summit or descend to the base, because you are moving horizontally along the same height.
On the other hand, the gradient direction is naturally perpendicular to the tangent direction of the level curve. In the hiking analogy, this corresponds to walking directly uphill or downhill, moving in the direction of steepest ascent or descent, which brings you to the summit or base as quickly as possible.

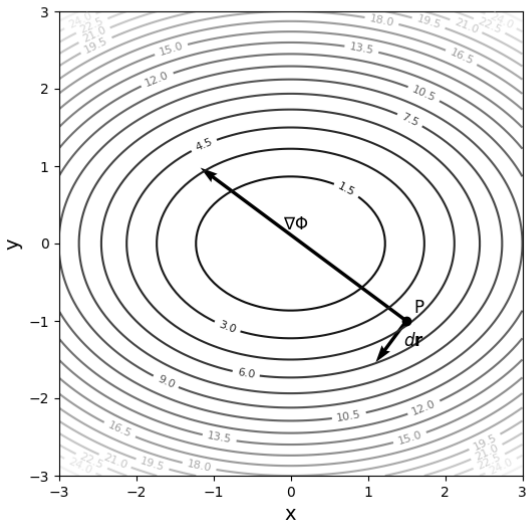

Figure 10: Gradient Points in the Direction of Steepest Increase

There is another operator I should introduce, the **Laplacian**. It is defined as

$$\boxed{\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \boldsymbol{\nabla}^2 \coloneqq \sum_i \frac{\partial^2}{\partial x_i^2}} \tag{2.2.36}$$

When the Laplacian operates on a scalar, we have(Thornton, 2003)

$$\boldsymbol{\nabla}^2 \psi = \sum_i \frac{\partial^2 \psi}{\partial x_i^2} \tag{2.2.37}$$

However, within the scope of undergraduate classical mechanics, this operator does not typically appear. Therefore, it will not be discussed in detail in this book.

▷ Definition of Divergence and Curl

Gradient is an operator that describes the variation of a scalar function. To describe the vector field, we need to introduce another operator, **divergence** and **curl**.
First, consider a vector field $\mathbf{\Phi} = x_1\mathbf{e_1} + x_2\mathbf{e_2} + \cdots$, then its divergence $\text{div}\mathbf{\Phi} = \boldsymbol{\nabla} \cdot \mathbf{\Phi}$ is defined as

$$\boxed{\text{div}\mathbf{\Phi} = \boldsymbol{\nabla} \cdot \mathbf{\Phi} = \sum_i \frac{\partial \Phi_i}{\partial x_i}} \tag{2.2.38}$$

For example, $\mathbf{\Phi} = (x^2 - y^2)\mathbf{e_1} + (x^2 - z^2)\mathbf{e_2} + (y^2 - z^2)\mathbf{e_3}$, the the divergence of this vector function is

$$\boldsymbol{\nabla} \cdot \mathbf{\Phi} = \frac{\partial}{\partial x}(x^2 - y^2) + \frac{\partial}{\partial y}(x^2 - z^2) + \frac{\partial}{\partial z}(y^2 - z^2) = 2x - 2z$$

Second, consider a 3-dimensional vector field $\mathbf{\Phi} = x_1\mathbf{e_1} + x_2\mathbf{e_2} + x_3\mathbf{e_3}$, then its curl $\text{curl}\mathbf{\Phi} = \boldsymbol{\nabla} \times \mathbf{\Phi}$ is defined as

$$\boxed{\text{curl}\mathbf{\Phi} = \boldsymbol{\nabla} \times \mathbf{\Phi} = \sum_{ijk} \varepsilon_{ijk} \frac{\partial \Phi_k}{\partial x_j} \mathbf{e_i}} \tag{2.2.39}$$

At this point, I will not yet introduce the geometric meanings of divergence and curl, because they require vector integration in order to be properly explained. If you're too curious to wait, feel free to skip ahead to Gauss' Law and Stokes' Theorem!

Problem. *There is a vector function* $\mathbf{F} = F_x\mathbf{e_1} + F_y\mathbf{e_2} + F_z\mathbf{e_3}$*, prove that*

$$\begin{cases} \boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = 0 \\ \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{F}) = 0 \end{cases} \tag{2.2.40}$$

1. The first equation is really **important**, we will check it later. By the definition of the curl and the gradient, we can get

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix}$$

   So, it becomes

$$\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{e_1} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\mathbf{e_2} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{e_3} = 0$$

2. it is obvious that

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{F}) = \frac{\partial}{\partial x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) = 0$$

▷ **Identities of Vector Operators**

Before proving the multiple identities of gradient, divergence, and curl, I must first introduce a completely new rule of operation that you have likely never encountered before to make the proof clearer: the **Einstein summation convention.**
The core idea of this convention is to perform a summation over any index that appears twice in a single term. So, the summation must be redefined as

$$\sum_i A_i B_i \coloneqq A_i B_i \tag{2.2.41}$$

and the partial differentiation is redefined as

$$\sum_i \frac{\partial}{\partial x_i} f(x_1, x_2, \cdots, x_n) \coloneqq \partial_i f(x_1, x_2, \cdots, x_n) \tag{2.2.42}$$

Here, ∂_i is shorthand for $\partial/\partial x_i$. From now on, we will use this simplified notation. The dot product and cross product become

$$\begin{cases} \mathbf{A} \cdot \mathbf{B} = A_i B_i \\ (\mathbf{A} \times \mathbf{B})_k = \varepsilon_{kij} A_i B_j \end{cases} \tag{2.2.43}$$

Now that we have defined the Einstein summation rule and simplified notation, we can re-express all our familiar vector operators in this unified index form

$$\begin{cases} (\boldsymbol{\nabla} f)_i = \partial_i f \\ \boldsymbol{\nabla} \cdot \mathbf{A} = \partial_i A_i \\ (\boldsymbol{\nabla} \times \mathbf{A})_k = \varepsilon_{kij} \partial_i A_j \end{cases} \tag{2.2.44}$$

After the introduction, I can show you 15 identities and prove them all.

1. $\boxed{\boldsymbol{\nabla} c = 0,\ c \in \mathbb{R}}$
   It is obvious that the differentiation of constant c equals 0, namely

$$\boldsymbol{\nabla} c = \partial_i c \mathbf{e}_i = 0 \tag{2.2.45}$$

2. $\boxed{\boldsymbol{\nabla}(cf) = c\boldsymbol{\nabla} f}$ where f is a function.
   We directly expand the gradient that

$$\boldsymbol{\nabla}(cf) = \partial_i c f \mathbf{e_i} = c \partial_i f \mathbf{e_i} = c \boldsymbol{\nabla} f \tag{2.2.46}$$

3. $\boxed{\boldsymbol{\nabla}(f \pm g) = \boldsymbol{\nabla} f \pm \boldsymbol{\nabla} g}$

$$\boldsymbol{\nabla}(f \pm g) = \partial_i (f \pm g) \mathbf{e_i} = \partial f\ \mathbf{e_i} \pm \partial_i g\ \mathbf{e_2} = \boldsymbol{\nabla} f \pm \boldsymbol{\nabla} g \tag{2.2.47}$$

4. $\boxed{\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla}g + g\boldsymbol{\nabla}f}$

$$\begin{aligned}\boldsymbol{\nabla}(fg) &= \partial_i(fg)\mathbf{e_i} \\ &= (f\partial_i g + g\partial_i f)\mathbf{e_i} \\ &= f\partial_i g\ \mathbf{e_i} + g\partial_i f\ \mathbf{e_i} = f\boldsymbol{\nabla}g + g\boldsymbol{\nabla}f\end{aligned} \tag{2.2.48}$$

5. $\boxed{\boldsymbol{\nabla}(f/g) = \frac{1}{g^2}(g\boldsymbol{\nabla}f - f\boldsymbol{\nabla}g)}$

$$\begin{aligned}\boldsymbol{\nabla}\left(\frac{f}{g}\right) &= \frac{1}{g}\boldsymbol{\nabla}f + f\boldsymbol{\nabla}\left(\frac{1}{g}\right) \\ &= \frac{1}{g}\partial_i f\ \mathbf{e_i} + f\partial_i g^{-1}\ \mathbf{e_i} \\ &= \frac{1}{g}\partial_i f\ \mathbf{e_i} - f\frac{1}{g^2}\partial_i g\ \mathbf{e_i} \\ &= \frac{1}{g}\boldsymbol{\nabla}f - \frac{f}{g^2}\boldsymbol{\nabla}g = \frac{1}{g^2}(g\boldsymbol{\nabla}f - f\boldsymbol{\nabla}g)\end{aligned} \tag{2.2.49}$$

6. $\boxed{\boldsymbol{\nabla}(f(u)) = f'(u)\boldsymbol{\nabla}u}$

$$\begin{aligned}\boldsymbol{\nabla}(f(u)) &= \partial_i f(u)\mathbf{e_i} \\ &= f'(u)\partial_i u\ \mathbf{e_i} = f'(u)\boldsymbol{\nabla}u\end{aligned} \tag{2.2.50}$$

The proofs of the above identities are similar to what you did in differentiation, so I don't want to stay here so long. The next proof is the climax of this section.

7. $\boxed{\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A}) + \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B})}$
First, we need to write down the expansion of the gradient term, which is

$$[\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B})]_i = \partial_i(A_jB_j) = B_j\partial_i A_j + A_j\partial_i B_j \tag{2.2.51}$$

We have to use some skills to deal with terms $\partial_i A_j$ and $\partial_i B_j$. So,

$$\partial_i A_j = \frac{1}{2}(\partial_i A_j + \partial_j A_i) + \frac{1}{2}(\partial_i A_j - \partial_j A_i) \tag{2.2.52}$$

The term $\partial_i A_j - \partial_j A_i$ can be written as the combination of Kronecker delta symbols, that is

$$\begin{aligned}\partial_i A_j - \partial_j A_i &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_l A_m \\ &= (\varepsilon_{ijk}\varepsilon_{lmk})\partial_l A_m \\ &= \varepsilon_{ijk}(\varepsilon_{lmk}\partial_l A_m)\end{aligned} \tag{2.2.53}$$

Note that the curl $(\boldsymbol{\nabla}\times\mathbf{A})_k = \varepsilon_{kij}\partial_i A_j$, so (2.2.53) can be rewritten as

$$\partial_i A_j - \partial_j A_i = \varepsilon_{ijk}(\boldsymbol{\nabla}\times\mathbf{A})_k \tag{2.2.54}$$

In the same way, we obtain

$$\partial_i B_j - \partial_j B_i = \varepsilon_{ijk}(\boldsymbol{\nabla} \times \mathbf{B})_k \tag{2.2.55}$$

Substituting (2.2.54) and (2.2.55) into (2.2.51), we get

$$\begin{aligned} B_j\partial_i A_j + A_j\partial_i B_j = &\frac{1}{2}(\partial_i A_j + \partial_j A_i)B_j + \frac{1}{2}(\partial_i B_j + \partial_j B_i)A_j \\ &+ \frac{1}{2}[\varepsilon_{ijk}B_j(\boldsymbol{\nabla} \times \mathbf{A})_k] + \frac{1}{2}[\varepsilon_{ijk}A_j(\boldsymbol{\nabla} \times \mathbf{B})_k] \end{aligned} \tag{2.2.56}$$

And we separate the term $B_j\partial_i A_j$ with $A_j\partial_i B_j$, we get the equations

$$\begin{aligned} B_j\partial_i A_j &= \frac{1}{2}B_j\partial_i A_j + \frac{1}{2}B_j\partial_j A_i + \frac{1}{2}\varepsilon_{ijk}B_j(\boldsymbol{\nabla} \times \mathbf{A})_k \\ \Rightarrow B_j\partial_i A_j &= B_j\partial_j A_i + \varepsilon_{ijk}B_j(\boldsymbol{\nabla} \times \mathbf{A})_k \end{aligned} \tag{2.2.57}$$

Similarly,

$$A_j\partial_i B_j = A_j\partial_j B_i + \varepsilon_{ijk}A_j(\boldsymbol{\nabla} \times \mathbf{B})_k \tag{2.2.58}$$

Substituting both into the beginning equation (2.2.51), we can get

$$[\boldsymbol{\nabla}(\mathbf{A} \cdot \mathbf{B})]_i = (B_j\partial_j)A_i + (A_j\partial_j)B_i + \varepsilon_{ijk}B_j(\boldsymbol{\nabla} \times \mathbf{A})_k + \varepsilon_{ijk}A_j(\boldsymbol{\nabla} \times \mathbf{B})_k \tag{2.2.59}$$

We can finally get this "beautiful" result, and that is what we want

$$\boxed{\boldsymbol{\nabla}(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A} + (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} + \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{A}) + \mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{B})} \tag{2.2.60}$$

This is the first major identity in this book that fully integrates the core principles of index notation, partial derivatives, symmetry and antisymmetry, and the Levi-Civita structure. By proving it purely algebraically, we cross the threshold from vector intuition into rigorous mathematical formulation.

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1. $\boxed{\boldsymbol{\nabla} \cdot (c\mathbf{A}) = c\boldsymbol{\nabla} \cdot \mathbf{A}}$

$$\boldsymbol{\nabla} \cdot (c\mathbf{A}) = \partial_i cA_i = c\partial_i A_i = c\boldsymbol{\nabla} \cdot \mathbf{A} \tag{2.2.61}$$

2. $\boxed{\boldsymbol{\nabla} \cdot (\mathbf{A} \pm \mathbf{B}) = \boldsymbol{\nabla} \cdot \mathbf{A} \pm \boldsymbol{\nabla} \cdot \mathbf{B}}$

$$\boldsymbol{\nabla} \cdot (\mathbf{A} \pm \mathbf{B}) = \partial_i(A_i \pm B_i) = \partial_i A_i \pm \partial_i B_i = \boldsymbol{\nabla} \cdot \mathbf{A} \pm \boldsymbol{\nabla} \cdot \mathbf{B} \tag{2.2.62}$$

3. $\boxed{\boldsymbol{\nabla} \cdot (f\mathbf{A}) = f\boldsymbol{\nabla} \cdot \mathbf{A} + \mathbf{A} \cdot \boldsymbol{\nabla} f}$

$$\boldsymbol{\nabla} \cdot (f\mathbf{A}) = \partial_i(fA_i) = f\partial_i A_i + A_i\partial_i f = f\boldsymbol{\nabla} \cdot \mathbf{A} + \mathbf{A} \cdot \boldsymbol{\nabla} f \tag{2.2.63}$$

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1. $\boxed{\boldsymbol{\nabla} \times (c\mathbf{A}) = c\boldsymbol{\nabla} \times \mathbf{A}}$

$$\boldsymbol{\nabla} \times (c\mathbf{A}) = \varepsilon_{ijk}\partial_j cA_k = c\varepsilon_{ijk}\partial_j A_k = c\boldsymbol{\nabla} \times \mathbf{A} \tag{2.2.64}$$

2. $\boxed{\boldsymbol{\nabla} \times (\mathbf{A} \pm \mathbf{B}) = \boldsymbol{\nabla} \times \mathbf{A} \pm \boldsymbol{\nabla} \times \mathbf{B}}$

$$\boldsymbol{\nabla} \times (\mathbf{A} \pm \mathbf{B}) = \varepsilon_{ijk}\partial_j(A_k \pm B_k) = \varepsilon_{ijk}\partial_j A_k \pm \varepsilon_{ijk}\partial_j B_k = \boldsymbol{\nabla} \times \mathbf{A} \pm \boldsymbol{\nabla} \times \mathbf{B} \quad (2.2.65)$$

3. $\boxed{\boldsymbol{\nabla} \times (f\mathbf{A}) = f\boldsymbol{\nabla} \times \mathbf{A} + \boldsymbol{\nabla} f \times \mathbf{A}}$

$$\begin{aligned} \boldsymbol{\nabla} \times (f\mathbf{A}) &= \varepsilon_{ijk}\partial_j(fA_k)\mathbf{e_i} \\ &= \varepsilon_{ijk}(f\partial_j A_k + A_k\partial_j f)\mathbf{e_i} \\ &= f\varepsilon_{ijk}\partial_j A_k\mathbf{e_i} + (\varepsilon_{ijk}\partial_j f)A_k\mathbf{e_i} \\ &= f\boldsymbol{\nabla} \times \mathbf{A} + \boldsymbol{\nabla} f \times \mathbf{A} \end{aligned} \quad (2.2.66)$$

4. $\boxed{\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \boldsymbol{\nabla} \times \mathbf{A} - \mathbf{A} \cdot \boldsymbol{\nabla} \times \mathbf{B}}$

$$\begin{aligned} \boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) &= \partial_i\varepsilon_{ijk}A_j B_k \\ &= \varepsilon_{ijk}\partial(A_j B_k) \\ &= \varepsilon_{ijk}B_k\partial_i A_j + \varepsilon_{ijk}A_j\partial_i B_k \end{aligned} \quad (2.2.67)$$

We need to change the order of the indices of ε. That is, $\varepsilon_{ijk} = \varepsilon_{kij}$ because of the even permutation; $\varepsilon_{ijk} = -\varepsilon_{jik}$ because of the odd permutation. So (2.2.) becomes

$$\varepsilon_{ijk}B_k\partial_i A_j + \varepsilon_{ijk}A_j\partial_i B_k = B_k(\varepsilon_{kij}\partial_i A_j) - A_j(\varepsilon_{jik}\partial_i B_k) \quad (2.2.68)$$

Note that the reason why we change the order of the indices is to get the curl term satisfying the expansion of it, so

$$\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \boldsymbol{\nabla} \times \mathbf{A} - \mathbf{A} \cdot \boldsymbol{\nabla} \times \mathbf{B} \quad (2.2.69)$$

5. $\boxed{\mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \frac{1}{2}\boldsymbol{\nabla}A^2 - (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{A}}$

We first expand the cross product term that

$$[\mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{A})]_i = \varepsilon_{ijk}A_j(\boldsymbol{\nabla} \times \mathbf{A})_k = \varepsilon_{ijk}A_j(\varepsilon_{klm}\partial_l A_m) \quad (2.2.70)$$

Note that we have another structure of $\varepsilon_{ijk}\varepsilon_{klm}$, and it equals to $\varepsilon_{ijk}\varepsilon_{lmk}$, so it can be written as

$$\begin{aligned} \varepsilon_{ijk}A_j(\varepsilon_{klm}\partial_l A_m) &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})A_j\partial_l A_m \\ &= A_m\partial_i A_m - (A_l\partial_l)A_i \\ &= \frac{1}{2}\partial_i(A_m^2) - (A_l\partial_l)A_i \end{aligned} \quad (2.2.71)$$

That is a detail, $\partial_i(A_m \times A_m) = A_m\partial_i A_m + A_m\partial_i A_m$, so the term $A_m\partial_i A_m = 1/2\partial_i(A_m \times A_m)$. After all, the result is

$$\mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \frac{1}{2}\boldsymbol{\nabla}A^2 - (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{A} \quad (2.2.72)$$

Where A is the length of the vector $\mathbf{A}$.

2.3 Vector Integral

▷ Definition of Vector Integration

After becoming familiar with vector differentiation and vector operators, we are now ready to introduce one of the most important mathematical tools in vector analysis: vector integration. We begin with the fundamental theorem of calculus for vector fields, whose definition parallels that of scalar differentiation. The indefinite integral of the vector field $\mathbf{\Phi}(t) = (\Phi_1(t), \Phi_2(t), \Phi_3(t))$ is

$$\int \mathbf{\Phi}(t)dt = \left(\int \Phi_1(t)dt, \int \Phi_2(t)dt, \int \Phi_3(t)dt \right) \tag{2.3.1}$$

And the definite integral is defined as

$$\int_a^b \mathbf{\Phi}(t)dt = \left(\int_a^b \Phi_1(t)dt, \int_a^b \Phi_2(t)dt, \int_a^b \Phi_3(t)dt \right) \tag{2.3.2}$$

Let $\mathbf{F} = d/dt\, \mathbf{\Phi}(t)$, so that $\mathbf{\Phi}(t)$ is the antiderivative vector function of $\mathbf{F}$. By (2.3.1) and (2.3.2), we can apply the fundamental theorem of calculus to the functions, namely

$$\begin{cases} \int \mathbf{F}(t)dt = \int \dfrac{d\mathbf{\Phi}(t)}{dt}dt = \mathbf{\Phi}(t) + C \\ \displaystyle\int_a^b \mathbf{F}(t)dt = \mathbf{\Phi}(b) - \mathbf{\Phi}(a) \end{cases} \tag{2.3.3}$$

Vector integration also possesses integral identities analogous to the differential identities discussed earlier. In fact, one can derive similar results by simply integrating the differential identities on pages 66 and 67 in reverse. Therefore, we will not repeat those derivations here.

However, there is one important property that deserves special attention: integration by parts for vector fields. There are two vector functions $\mathbf{A}(t)$ and $\mathbf{B}(t)$, each of which depends on the parameter t. By (2.2.15), we have

$$\frac{d}{dt}(\mathbf{A}(t) \cdot \mathbf{B}(t)) = \frac{d\mathbf{A}(t)}{dt} \cdot \mathbf{B}(t) + \mathbf{A}(t) \cdot \frac{d\mathbf{B}(t)}{dt}$$

Multiply both sides by dt and integrate both sides with respect to t,

$$\int d(\mathbf{A}(t) \cdot \mathbf{B}(t)) = \mathbf{A}(t) \cdot \mathbf{B}(t) = \int \dot{\mathbf{A}}(t) \cdot \mathbf{B}(t)dt + \int \mathbf{A}(t) \cdot \dot{\mathbf{B}}(t)dt$$

Thus, we obtain

$$\int \dot{\mathbf{A}}(t) \cdot \mathbf{B}(t)dt = \mathbf{A}(t) \cdot \mathbf{B}(t) - \int \mathbf{A}(t) \cdot \dot{\mathbf{B}}(t)dt \tag{2.3.4}$$

Similarly, the integration by parts formula involving the cross product also exhibits analogous properties

$$\int \dot{\mathbf{A}}(t) \times \mathbf{B}(t)dt = \mathbf{A}(t) \times \mathbf{B}(t) - \int \mathbf{A}(t) \times \dot{\mathbf{B}}(t)dt \tag{2.3.5}$$

For example, there is an integral

$$\int \mathbf{A} \times \ddot{\mathbf{A}} dt$$

Apply the integration by parts, we can evaluate the integral

$$\int \mathbf{A} \times \ddot{\mathbf{A}} dt = \mathbf{A} \times \dot{\mathbf{A}} - \int \dot{\mathbf{A}} \times \dot{\mathbf{A}} dt = \mathbf{A} \times \dot{\mathbf{A}} + \mathbf{C}$$

Where $\mathbf{C}$ is a constant vector.

▷ **Line Integral and Kinetic Energy**

Back in calculus, we already encountered the equation of the length$L$ of a curve $S : \{(x, y)|y = f(x),\ x \in \mathbb{R}\}$ in the two-dimensional plane

$$L = \int_a^b \sqrt{(dx)^2 + (dy)^2} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

Now, we extend this concept to three-dimensional space. Since a curve is a one-dimensional object, it can still be described using a single parameter t, which determines the position of a point $(x(t), y(t), z(t))$ along the curve. Thus, a space curve can be represented as a set of points in the following form

$$C = \{\, (x, y, z) \mid x = x(t),\ y = y(t),\ z = z(t),\ t \in [a, b] \,\} \tag{2.3.6}$$

Or

$$C = \{\, \mathbf{r}(t) = x(t)\mathbf{e}_1 + y(t)\mathbf{e}_2 + z(t)\mathbf{e}_3,\ a \le t \le b\} \tag{2.3.7}$$

So, the **arc length**$(s(t))$ of the curve C is similar to the curve in the two-dimensional plane, which is

$$s(t) = \int_0^t \sqrt{(dx)^2 + (dz)^2 + (dz)^2} = \int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

Note that $\mathbf{r}'(t) = (dx/dt,\, dy/dt,\, dz/dt)$, so $\sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2} = |d\mathbf{r}/dt|$. Then the curve length in the three-dimensional plane can be denoted by

$$\boxed{s(t) = \int_0^t \left|\frac{d\mathbf{r}(t)}{dt}\right| dt = \int_0^t |\mathbf{r}'(t)| dt} \tag{2.3.8}$$

In the physical perspective, the derivative of $\mathbf{r}(t)$ with respect to t means the velocity of a particle, and dt means infinitesimal time difference, so $|\mathbf{r}'(t)|dt$ represents the infinitesimal distance. The integral adds all the infinitesimal distances together, so we obtain the arc length. By FTC I, we have

$$\frac{ds(t)}{dt} = \frac{d}{dt}\int_0^t |\mathbf{r}'(t)| dt = |\mathbf{r}'(t)| \Rightarrow ds = |\mathbf{r}'(t)| dt \tag{2.3.9}$$

We call the notation ds as **arc length element**.
After understanding the concept of the arc length element, we can proceed to introduce one of the most important types of integrals in mechanics and electromagnetism - the

line integral.
We begin with the first type of line integral, namely, the line integral of a scalar function. Let C be a piecewise smooth curve defined on the interval $[a, b]$, and let f be a continuous function defined on the curve C, that is, the domain of f satisfies $D_f \subseteq C$. Then the line integral of f along the curve C is given by

$$\boxed{\int_C f(x, y, z)ds = \int_C f(\mathbf{r}(t))|\mathbf{r}'(t)|dt} \tag{2.3.10}$$

In other words, the line integral of a scalar function represents the total accumulation of function values along the curve. From a physical perspective, if $f(x, y, z)$ denotes the linear density of a rope, then the line integral of f over the curve gives the total mass of the rope.
For example, C is the upper semicircle of radius C, which is

$$C = \{(x, y)|x = a\cos t,\ y = a\sin t,\quad 0 \le t \le \pi\}$$

Then evaluate the line integral

$$\int_C x^2 y\, ds$$

We need to compute the arc length element ds

$$ds = |\mathbf{r}'(t)|dt = a\, dt$$

So the line integral becomes

$$\int_C x^2 y\, ds = \int_0^{\pi} \left(a^2\cos^2 t\right)(a\sin t)\, a\, dt = \frac{2}{3}a^4$$

Now it is time for vector fields to take the stage! After discussing line integrals of scalar functions, we have seen that the key idea behind a line integral is to map the domain of the function onto a curve and then perform the integration *along* the curve C. Similarly, we can apply the same idea to vector-valued functions.

Vector fields were originally developed to address problems in physics, particularly the computation of work. In earlier physics courses, we encountered work calculations restricted to special cases, such as constant force along straight-line paths. However, such formulas could not be applied to compute the work done along arbitrary paths in space. To overcome this limitation, we developed the concept of the **line integral of a vector field**. There are a vector function $\mathbf{F} = (F_1, F_2, F_3)$ and a infinitesimal tangent vector $d\mathbf{r} = (dx, dy, dz)$ of the curve C. The work done by force, or called line integral of a vector field is defined as

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C F_1 dx + F_2 dy + F_3 dz \tag{2.3.11}$$

In other words, we sum the dot product between the vector field and each infinitesimal displacement along the curve, according to the parameter domain of the curve. From a physical perspective, this corresponds to adding up the infinitesimal amounts of work

done at each moment, resulting in the total work performed along the path.
We can rewrite (2.3.11) as

$$\int_C F_1dx+F_2dy+F_3dz = \int_C \left(F_1\frac{dx}{dy} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\right)dt = \int_C (F_1,F_2,F_3)\cdot\left(\frac{dx}{dy},\frac{dy}{dt},\frac{dz}{dt}\right)dt$$

So, the generalization of the line integral of the vector function can be denoted by

$$\boxed{\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{r}'(t)dt} \tag{2.3.12}$$

After understanding these theorems and rules, we now proceed to generalize the expression for kinetic energy along an arbitrary path. Consider a force given by $\mathbf{F} = m\mathbf{a} = m\ddot{\mathbf{r}}$, and let C be an arbitrary path parameterized with respect to time t. Assume that the curve C is smooth on its domain, and it is given by

$$C = \{\mathbf{r}(t) = x(t), y(t), z(t),\ t_1 \le t \le t_2\} \tag{2.3.13}$$

Then the work done by the force $\mathbf{F}$ is the line integral (2.3.12), we substitute $\mathbf{F} = m\mathbf{a} = m\ddot{\mathbf{r}}$ into it, it becomes

$$W = \int_{t_1}^{t_2} m\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}}\,dt \tag{2.3.14}$$

We need to apply the integration by parts such as (2.3.4), we obtain

$$\begin{aligned} W &= \int_{t_1}^{t_2} m\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}}\,dt \\ &= m(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} m\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}}\,dt \end{aligned}$$

Since the integral $\int_{t_1}^{t_2} m\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}}\,dt$ and $\int_{t_1}^{t_2} m\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}}\,dt$ are the same, we let the integrals be I. Therefore

$$2I = m(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})\Big|_{t_1}^{t_2} \Rightarrow W = I = \frac{1}{2}\,m|\dot{\mathbf{r}}|^2\Big|_{t_1}^{t_2} = \frac{1}{2}m[v(t_2)]^2 - \frac{1}{2}m[v(t_1)]^2$$

So, we can finally get the **The work–energy theorem** in a rigorous way, that is

$$\boxed{W_F = \int_C \mathbf{F}\cdot d\mathbf{r} = \frac{1}{2}m[v(t_2)]^2 - \frac{1}{2}m[v(t_1)]^2} \tag{2.3.15}$$

Before going to the next part, I must introduce another integrating notation – the closed integral $\oint$. As long as the starting point and the ending point of the curve are the same, then we can rewrite the line integral (2.3.13) as

$$\oint_C \mathbf{F}\cdot d\mathbf{r} \tag{2.3.16}$$

Moreover, when the line integral is taken along the boundary of a region, it can also be expressed as a closed line integral. Consider a continuous surface with boundary S in $\mathbb{R}^3$, its boundary can be denoted by ∂S, the line integral becomes

$$\oint_{\partial S} \mathbf{F}\cdot d\mathbf{r} \tag{2.3.17}$$

In a closed line integral, the curve C must have a specified direction, that is, we must define whether the path is traversed in a clockwise or counterclockwise manner. This orientation is usually indicated by the symbol $\oint$, which denotes integration along a *closed* curve with a given direction. For example, in the plane, a **positively oriented** curve typically means counterclockwise, while a negatively oriented one is clockwise. Although the exact orientation becomes very important in later theorems (such as Green's and Stokes' Theorem), for now, you only need to recognize this symbol and understand that direction matters when computing line integrals over closed curves.

▷ Conservative Field and Potential Functions

There exists a continuous scalar function $\Phi(x, y, z)$, by the properties of the gradient $\boldsymbol{\nabla}$, we obtain the line integral of the gradient function of Φ, which is

$$\begin{aligned}\int_A^B \boldsymbol{\nabla}\Phi \cdot d\mathbf{r} &= \int_A^B \left(\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}, \frac{\partial\Phi}{\partial z}\right) \cdot d\mathbf{r} \\ &= \int_A^B \left(\frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz\right)\end{aligned}$$

Where A and B represent the position. Note that the term $\frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz$ is the total derivatives of the scalar function $d\Phi$, that is

$$d\Phi = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz$$

So, the integral can be written as

$$\int_A^B \boldsymbol{\nabla}\Phi \cdot d\mathbf{r} = \int_A^B d\Phi = \Phi(B) - \Phi(A) \tag{2.3.18}$$

By (2.3.18), we can observe that the line integral $\boldsymbol{\nabla}\Phi$ is independent of the path it traverses through. The function only depends on the initial and the final positions A and B. That is to say, when a vector function $\mathbf{F}$ can be written in the form $\boldsymbol{\nabla}\Phi$, the vector function $\mathbf{F}$ is called a **Conservative Vector Field**, and the scalar function Φ is the **Potential Function** of $\mathbf{F}$. It is similar to potential energy in textbooks.

There are some methods to tell whether the vector functions are conservative or not. By the identities (2.2.40), we've known that

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} f = \mathbf{0}$$

Where f is a scalar function. So, when a vector function $\mathbf{F}$ satisfy that

$$\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0} \tag{2.3.19}$$

Then the vector function $\mathbf{F}$ is a conservative function since there exists a potential function Φ satisfying $\boldsymbol{\nabla}\Phi = \mathbf{F}$, so we obtain

$$\boldsymbol{\nabla} \times \mathbf{F} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla}\Phi) = \mathbf{0} \tag{2.3.20}$$

On the other hand, when a line integral of a conservative field $\mathbf{F}$ along a closed curve, then the line integral is 0. It is given by

$$\oint_C \mathbf{F} \cdot d\mathbf{r} \coloneqq \int_{x_0}^{x_0} \mathbf{F} \cdot d\mathbf{r} = \int_{x_0}^{x_0} \boldsymbol{\nabla}\Phi \cdot d\mathbf{r} = \Phi(x_0) - \Phi(x_0) = 0 \tag{2.3.21}$$

This integral may be mentioned in Chapter 2.6. Back to the physical perspective, the work W done by force $\mathbf{F}$ is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

We find that if the work done by a force $\mathbf{F}$ depends only on the initial and final positions, and not on the path taken, then the force is called a **conservative force**. In such cases, we can define a type of energy, denoted by V, which allows us to express the work in terms of the force itself. This quantity is known as the **potential energy**. That is

$$\int_{x_1}^{x_2} \mathbf{F} \cdot d\mathbf{r} \coloneqq V(x_1) - V(x_2) \Leftrightarrow \boxed{\mathbf{F} = -\boldsymbol{\nabla} V} \tag{2.3.22}$$

Note that, since potential energy is defined based on the work done by a conservative force, its value corresponds to the initial potential energy minus the final potential energy. Therefore, there is a negative sign difference between this definition and equation (2.3.18). In other words, the negative sign between conservative force and potential energy arises naturally from the definition.

So, we can thus verify the law of conservation of energy by this definition in vector analysis. Suppose an object is subjected only to conservative forces $\mathbf{F}$, by (2.3.15), we can first write down the work-energy theorem

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2} m[v(t_2)]^2 - \frac{1}{2} m[v(t_1)]^2 \tag{2.3.23}$$

Then, by (2.3.22), we obtain

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} = -\int_{\mathbf{r}_1}^{\mathbf{r}_2} \boldsymbol{\nabla} V \cdot d\mathbf{r} = V(\mathbf{r}_1) - V(\mathbf{r}_2) \tag{2.3.24}$$

The work value (2.3.23) is equal to (2.3.24), we have

$$\begin{aligned} &\frac{1}{2} m[v(t_2)]^2 - \frac{1}{2} m[v(t_1)]^2 = V(\mathbf{r}(t_1)) - V(\mathbf{r}(t_2)) \\ &\Rightarrow \frac{1}{2} m[v(t_1)]^2 + V(\mathbf{r}(t_1)) = \frac{1}{2} m[v(t_2)]^2 + V(\mathbf{r}(t_2)) \end{aligned} \tag{2.3.25}$$

When the potential energy depends on time t, which is written as $V(\mathbf{r}, t)$, and $\mathbf{r} = (x_1, x_2, \cdots, x_n)$, so the total derivative of the potential energy becomes

$$dV = \sum_i \frac{\partial V}{\partial x_i} dx_i + \frac{\partial V}{\partial t} dt = \boldsymbol{\nabla} V \cdot d\mathbf{r} + \frac{\partial V}{\partial t} dt = \boldsymbol{\nabla} V \cdot \dot{\mathbf{r}} dt + \frac{\partial V}{\partial t} dt \tag{2.3.26}$$

Then the work done by the conservative force becomes

$$W = -\int dV = -\int_{t_1}^{t_2} \boldsymbol{\nabla} V \cdot \dot{\mathbf{r}} dt - \int_{t_1}^{t_2} \frac{\partial V}{\partial t} dt = V(\mathbf{r_1}) - V(\mathbf{r_2}) + V(\mathbf{r}, t_1) - V(\mathbf{r}, t_2) \tag{2.3.27}$$

So the conservation of energy must be rewritten as

$$T_{\mathbf{r_1}} + V(\mathbf{r_1}) + V(\mathbf{r}, t_1) = T_{\mathbf{r}_2} + V(\mathbf{r_2}) + V(\mathbf{r}, t_2) \tag{2.3.28}$$

If you differentiate the total energy $E = T + V$ with respect to t, you may get the result of the power p

$$\mathrm{p} = \frac{\partial V}{\partial t} \tag{2.3.29}$$

This part only focuses on the mathematical principle of the relationship between energy and line integrals, I'll not discuss the physical meaning in detail. Those who are interested may refer to your own classical mechanics textbook.

▷ Surface integral

Unlike the line integral, the surface integral is used to calculate the flux and the surface area in three dimensions. In this section, we focus on discussing surfaces in three-dimensional coordinates. In Cartesian coordinates $\mathbb{R}^3$, the surface is often written as $z = f(x, y)$, so the surface can be denoted by a set consisting of the elements of the function, which is

$$S = \big\{(x, y, z) | z = f(x, y), f : \mathbb{R}^2 \to \mathbb{R}\big\} \tag{2.3.30}$$

Just like curves in space, surfaces can also be parameterized. However, we need two parameters to describe a surface since it is a two-dimensional structure. So, the surfaces can be rewritten as

$$S = \{(x, y, z) | x = x(u, v),\ y = y(u, v),\ z = z(u, v)\} \tag{2.3.31}$$

To calculate the surface area or the surface integral of scalar functions, we need to understand the relationship between the area element after parametrization and the original surface element dS.
To begin, let us first consider the case of a non-parameterized surface $z = f(x, y)$. There is a point $(x, y, f(x, y))$ on the surface, and we can obtain the tangent vector by taking the partial derivative $\mathbf{r}_x$, $\mathbf{r}_y$ of x and y.

$$\begin{aligned} \mathbf{r}_x &= (x + dx,\ y,\ f(x + dx,\ y)) - (x, y, f(x, y)) \\ &= (dx, 0, df) = \left(dx, 0, \frac{\partial f}{\partial x} dx\right) = \left(1, 0, \frac{\partial f}{\partial x}\right) dx \end{aligned} \tag{2.3.32}$$

Similarly, tangent vector $\mathbf{r}_y$ is

$$\mathbf{r}_y = \left(0, 1, \frac{\partial f}{\partial y}\right) dy \tag{2.3.33}$$

The area, which is also known as the surface element dS, enclosed by these two vectors $\mathbf{r}_x$ and $\mathbf{r}_y$ is

$$dS = |\mathbf{r}_x \times \mathbf{r}_y| = \left| \left(1, 0, \frac{\partial f}{\partial x}\right) \times \left(0, 1, \frac{\partial f}{\partial y}\right) \right| dxdy$$

So

$$dS = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dxdy \tag{2.3.34}$$

If we consider a parameterized surface $S : \mathbf{r}(u,v) = (x(u,v), y(u.v), z(u,v))$, then the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_y$ of the point $(x(u,v), y(u.v), z(u,v))$ satisfy

$$\begin{aligned}\mathbf{R}_u &= (x(u+du,v),\ y(u+du,v),\ z(u+du,v)) - ((x(u,v), y(u.v), z(u,v))) \\ &= (dx, dy, dz) = \left(\frac{\partial x}{\partial u}du, \frac{\partial y}{\partial u}du, \frac{\partial z}{\partial u}du\right) = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) du\end{aligned} \tag{2.3.35}$$

Then we define the tangent vector $\mathbf{r}_u$ as

$$\mathbf{r}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$$

Similarly, tangent vector $\mathbf{r}_v$ is

$$\mathbf{r}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) \tag{2.3.36}$$

The surface element dS enclosed by these two vectors $\mathbf{r}_u$ and $\mathbf{r}_u$ is

$$\boxed{dS = |\mathbf{R}_u \times \mathbf{R}_v| = |\mathbf{r}_u \times \mathbf{r}_v| dudv = \left|\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \times \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)\right| dudv} \tag{2.3.37}$$

In general, we can also use Jacobian to describe a surface when we meet a higher-dimensional coordinate. While the full Jacobian determinant does not directly apply to a mapping from the parameterization $\mathbf{r}(u,v) : \mathbb{R}^2 \to \mathbb{R}^3$ and $\Omega \to S$, we can still use the Jacobian-type structure to compute the surface area element. We define a Jacobian and its transpose matrix

$$J \coloneqq \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}_{3\times 2} \Rightarrow J^T = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{pmatrix}_{2\times 3} \tag{2.3.38}$$

So, the matrix product of J^T and J is

$$\begin{aligned}\det\left(J^T J\right) &= \left| \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \right| = \begin{vmatrix} \frac{|\mathbf{r}_u|^2}{(du)^2} & \frac{\mathbf{r}_u \cdot \mathbf{r}_v}{du\, dv} \\ \frac{\mathbf{r}_v \cdot \mathbf{r}_u}{du\, dv} & \frac{|\mathbf{r}_v|^2}{(dv)^2} \end{vmatrix} \\ &= \left[|\mathbf{r}_u|^2 |\mathbf{r}_u|^2 - (\mathbf{r}_u \cdot \mathbf{r}_v)^2\right] (du\, dv)^2 = |\mathbf{r}_u \times \mathbf{r}_v|^2 (du\, dv)^2\end{aligned}$$

Here, we illustrate only the case of a parametrized surface in three-dimensional space, so the surface element in space can be expressed using the Jacobian as

$$\boxed{dS = \sqrt{\det\left(J^T J\right)} du\, dv} \tag{2.3.39}$$

In fact, this approach offers greater generality and consistency with the underlying geometric structure. This method of defining surface elements naturally extends to higher-dimensional volume integrals and manifold theory. In higher-dimensional spaces, it is

no longer feasible to rely on geometric visualization to interpret the relationship between the original coordinates and their parametrized counterparts, and therefore, the Jacobian becomes an essential tool in capturing the local distortion of volume or area under the transformation.
After discussing the surface element in space, we can further apply this to integrals. Just as line integrals are used to compute the length of a curve, we can apply a similar method to compute the area of a surface. The only difference is that, since the surface is described by two variables, a double integral is required. So, the area of the surface S is

$$\boxed{S = \iint_S dS = \iint_\Omega |\mathbf{r}_u \times \mathbf{r}_v| du\, dv = \iint_\Omega \sqrt{\det(J^T J)} du\, dv} \tag{2.3.40}$$

We can now further extend the concept to the surface integral of a scalar function. Suppose a scalar function f is a continuous real-valued function defined on a surface S, then the surface integral of f over S is given by

$$\boxed{\begin{aligned} \iint_S f(x,y,z) dS &= \iint_\Omega f(\mathbf{r}(u,v)) |\mathbf{r}_u \times \mathbf{r}_v| du\, dv \\ &= \iint_\Omega f(\mathbf{r}(u,v)) \sqrt{\det(J^T J)} du\, dv \end{aligned}} \tag{2.3.41}$$

The final part involves the surface integral of a vector field. In this context, the surface element is treated as a vector-valued area element $d\mathbf{S}$, defined as the unit normal vector to the surface patch S, which means $d\mathbf{S} = \mathbf{n} dS$. The unit normal vector $\mathbf{n}$ is

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \tag{2.3.42}$$

By (2.3.42), the vector-valued area element $d\mathbf{S}$ becomes

$$d\mathbf{S} = \mathbf{n} dS = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \times |\mathbf{r}_u \times \mathbf{r}_v| du\, dv = \mathbf{r}_u \times \mathbf{r}_v\, du\, dv \tag{2.3.43}$$

Over this surface, suppose a vector field $\mathbf{F} = P\mathbf{e_1} + Q\mathbf{e_2} + R\mathbf{e_3}$ passes through the region. Then, the surface integral of $\mathbf{F}$ over S is

$$\boxed{\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} dS &= \iint_\Omega \mathbf{F} \cdot \mathbf{r}_u \times \mathbf{r}_v du dv \\ &:= \iint_\Omega [\mathbf{F}, \mathbf{r}_u, \mathbf{r}_v] du dv \end{aligned}} \tag{2.3.44}$$

We can observe that the meaning of the surface integral of the vector function is to project the vector function $\mathbf{F}$ onto the direction of the vector-valued area element $d\mathbf{S}$. It describes the component $\mathbf{F}\cos\theta$ of the vector field passing through each surface element, then the integral means to add all the dot products of $\mathbf{F}$ and $d\mathbf{S}$. Overall, this integral represents one of the most important physical quantities in fluid mechanics and electromagnetism, namely, the **flux**. As illustrated in the figure below.

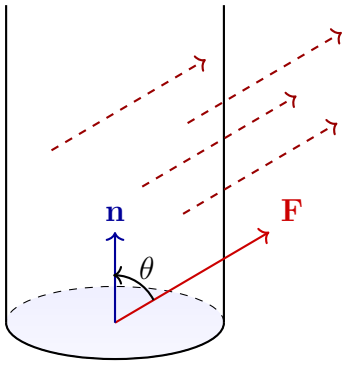

Figure 11: Geometric illustration of the field through curve integrals (fluxes)

We take a classic physical problem for example. Calculate the flux(Φ) of the vector field $\mathbf{F} = \frac{q}{r^3}\mathbf{r}$ through the spherical surface $S : x^2 + y^2 + z^2 = a^2$, where $\mathbf{r}$ denotes the position vector (x, y, z). Since the outward unit normal vector on the sphere coincides with the radial direction of the position vector, we observe that $\mathbf{n} = \frac{\mathbf{r}}{r}$. So, the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_S \frac{q}{r^2} dS = \frac{q}{r^2} \iint_S dS$$

Because the surface area of a sphere $S = 4\pi r^2$, so the flux(Φ) of the vector field through the spherical surface is

$$\frac{q}{r^2} \iint_S dS = 4\pi q$$

2.4 Green's Theorem

Green's Theorem is one of the most fundamental and important components in all of vector analysis. It describes how the line integral of a vector field $\mathbf{F}$ over the boundary $\partial\Omega$ of a bounded region $\Omega \subset \mathbb{R}^2$ can be transformed into a double integral over the region Ω itself. Specifically, it generalizes the concept of the line integral introduced in equation (2.3.17) to a closed path surrounding a two-dimensional region and expresses the result in a remarkably elegant and simplified form.

Consider a region Ω in the plane defined by:

$$\Omega = \{\, (x, y) \mid x \in [a, b],\ y \in [g_1(x), g_2(x)] \,\} \tag{2.4.1}$$

as illustrated in the figure below.

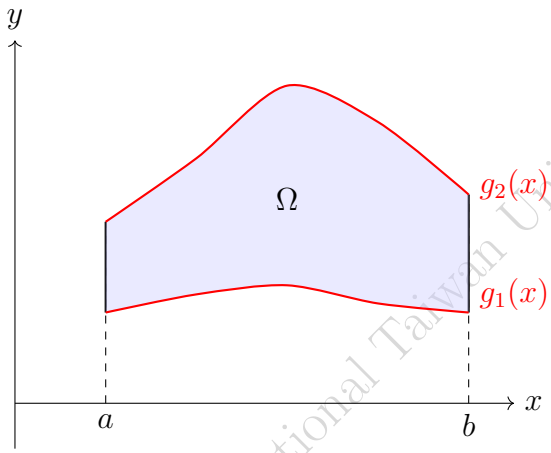

Figure 12: Region Ω bounded by the curves $g_1(x)$ and $g_2(x)$, for use in Green's Theorem

We now consider a vector field $\mathbf{F} = (P(x, y), Q(x, y))$ passing through the region Ω, then the closed line integral can be written as

$$\oint_{\partial\Omega} \mathbf{F} \cdot d\mathbf{r} = \oint_{\partial\Omega} P(x, y)dx + Q(x, y)dy \tag{2.4.2}$$

We can divide this line integral into four parts, corresponding to the four segments of the boundary. These are $(a, g_1(x)) \to (b, g_1(x))$; $(b, g_1(x)) \to (b, g_2(x))$; $(b, g_2(x)) \to (a, g_2(x))$; $(a, g_2(x)) \to (a, g_1(x))$. So, we can extend (2.4.2) to

$$\begin{aligned}\oint_{\partial\Omega} P(x, y)dx + Q(x, y)dy = &\int_a^b P(x, g_1(x))dx + \int_{g_1(x)}^{g_2(x)} Q(b, y)dy \\ &+ \int_b^a P(x, g_2(x))dx + \int_{g_2(x)}^{g_1(x)} Q(a, y)dy\end{aligned}$$

By rewriting the integrals of the P and Q terms with matching limits of integration, we can combine them into a single expression.

$$\int_a^b P(x, g_1(x))dx + \int_{g_1(x)}^{g_2(x)} Q(b, y)dy + \int_b^a P(x, g_2(x))dx + \int_{g_2(x)}^{g_1(x)} Q(a, y)dy$$

$$= \int_{g_1(x)}^{g_2(x)} [Q(b, y) - Q(a, y)]dy - \int_a^b [P(x, g_2(x)) - P(x, g_1(x))]dx$$

Note that the terms $[Q(b,y)-Q(a,y)]$ and $[P(x,g_2(x))-P(x,g_1(x))]$ can be rewritten as the integrals that

$$\begin{cases} Q(b,y)-Q(a,y)=\displaystyle\int_a^b \frac{\partial Q(x,y)}{\partial x}dx \\ P(x,g_2(x))-P(x,g_1(x))=\displaystyle\int_{g_1(x)}^{g_2(x)} \frac{\partial P(x,y)}{\partial y}dy \end{cases} \tag{2.4.3}$$

substituting (2.4.3) into the original integrals, we obtain

$$\int_{g_1(x)}^{g_2(x)}[Q(b,y)-Q(a,y)]dy-\int_a^b[P(x,g_2(x))-P(x,g_1(x))]dx$$

$$=\int_{g_1(x)}^{g_2(x)}\left(\int_a^b \frac{\partial Q(x,y)}{\partial x}dx\right)dy-\int_a^b\left(\int_{g_1(x)}^{g_2(x)}\frac{\partial P(x,y)}{\partial y}dy\right)dx$$

$$=\int_{g_1(x)}^{g_2(x)}\int_a^b\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$$

Here, we rewrite $dx\,dy$ as the area element dA. Thus, combining the previous derivations and concepts, we have derived the most well-known Green's Theorem.

$$\boxed{\oint_{\partial\Omega}\mathbf{F}\cdot d\mathbf{r}=\iint_\Omega\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA=\iint_\Omega\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ P & Q\end{vmatrix}dA} \tag{2.4.4}$$

For example, let C be the boundary of the triangle enclosed by the x-axis, the y-axis, and the line $x+y=1$. Then, evaluate the integral

$$\oint_C xy^2dx+(x^2+y)dy$$

First, let $xy^2=P$, $x^2+y=Q$, so the integral becomes

$$\oint_C Pdx+Qdy=\iint_\Omega\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA=\iint_\Omega(2x-2xy)dxdy$$

Next, we need to determine the range of the two variables, $0\le x\le 1$ and $0\le y\le 1-x$. substituting it into the integral, we obtain

$$\iint_\Omega(2x-2xy)dxdy=\int_0^1\int_0^{1-x}(2x-2xy)dydx$$

$$=\int_0^1 2x(1-x)-x(1-x)^2dx=\frac{1}{4}$$

Moreover, this theorem is closely related to the concept of area. The area of Ω is

$$\Omega=\iint_\Omega dA \tag{2.4.5}$$

If we take

$$\begin{cases} Q = x,\ P = 0 \\ Q = 0,\ P = -y \end{cases} \Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 \tag{2.4.6}$$

substituting (2.4.6) into (2.4.5) and combine it with Green's Theorem, and we have

$$\boxed{\Omega = \iint_{\Omega} dA = \oint_{\partial\Omega} x dy = \oint_{\partial\Omega} -y\, dx} \tag{2.4.7}$$

We can use this property to evaluate the area of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

First, parameterize the ellipse

$$\mathbf{r}(\theta) = \{(x, y) |\, x = a\cos\theta,\ y = b\sin\theta,\ 0 \le \theta \le 2\pi\}$$

We take $\mathbf{F} = (x, 0)$, and $dy = b\cos\theta$ then we have

$$\begin{aligned} \Omega &= \oint_{\partial\Omega} \mathbf{F} \cdot d\mathbf{r} \\ &= \oint_{\partial\Omega} x dy = \int_0^{2\pi} ab\cos^2\theta\, d\theta = \pi ab \end{aligned}$$

2.5 Gauss's Divergence Theorem

▷ The Derivation of the Theorem

The Divergence Theorem is a reformulation of Green's Theorem. In this book, I will only discuss the two-dimensional and three-dimensional cases. In Green's Theorem, we integrate a vector field along the direction of the curve, meaning we compute the line integral with respect to the tangent vector at each point. However, we now take a different perspective: instead of integrating along the direction of the curve$d\mathbf{r}$, we perform the integration using the outward-pointing normal vector $\mathbf{n}$ at each point on the curve. See the illustration below.

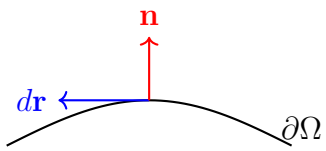

Figure 13: A curve with the tangent vector $d\mathbf{r}$ and the outward normal vector $\mathbf{n}$.

The vector

$$d\mathbf{r} = \left(\frac{dx}{dr}, \frac{dy}{dr}\right), d\mathbf{s} = \mathbf{n} dr \tag{2.5.1}$$

Note that the normal vector we use here is always taken to point outward. In order to maintain this outward orientation consistently, we apply a rotation matrix that you have

already encountered in high school: a 90° rotation to the right of the tangent vector $d\mathbf{r}$. This ensures that the resulting vector always represents the outward normal in the positive (counterclockwise) orientation. That is

$$\begin{pmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} \\ -\sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} \begin{pmatrix} \frac{dx}{dr} \\ \frac{dy}{dr} \end{pmatrix} = \begin{pmatrix} \frac{dy}{dr} \\ -\frac{dx}{dr} \end{pmatrix} \tag{2.5.2}$$

So, the closed integral of the vector function $\mathbf{F} = (P, Q)$ with respect to the normal vector $\mathbf{n}$ is

$$\oint_{\partial\Omega} \mathbf{F} \cdot d\mathbf{s} = \oint_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} dr = \oint_{\partial\Omega} (P, Q) \cdot \left(\frac{dy}{dr}, -\frac{dx}{dr} \right) dr = \oint_{\partial\Omega} P dy - Q dx \tag{2.5.3}$$

Substituting (2.5.3) into Green's theorem, and we can obtain

$$\oint_{\partial\Omega} P dy - Q dx = \iint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA = \iint_{\Omega} \boldsymbol{\nabla} \cdot \mathbf{F} \, dA \tag{2.5.4}$$

The meaning of (2.5.11) is that the line integral of a vector field around a closed curve is equal to the area integral of the divergence of the vector field over the enclosed region. We can extend this idea to the three-dimensional case using the same logic.

If V is a three-dimensional region (volume), then its boundary ∂V is a two-dimensional manifold, which is the surface enclosing the volume. There's a volume $V \subset \mathbb{R}^3$, and ∂V is positive. Consider $V = [a, b] \times [c, d] \times [e, f]$ and the vector function $\mathbf{F}(P, Q, R)$, we start from the $x-$direction, the integral

$$\iint \left(\int_a^b \frac{\partial P}{\partial x} dx \right) dy \, dz = \iint [P(b, y, z) - P(a, y, z)] dy dz \tag{2.5.5}$$

On the other hand, the surface integral of $\mathbf{F}$ with respect to $S_{x=a}$ is

$$\begin{aligned} \iint_{S_{x=a}} \mathbf{F} \cdot d\mathbf{S} &= \iint_{S_{x=a}} \mathbf{F} \cdot \mathbf{n} dS \\ &= \iint_{S_{x=a}} (P, Q, R) \cdot (-1, 0, 0) dS \\ &= \iint_{S_{x=a}} -P(a, y, z) dS \end{aligned} \tag{2.5.6}$$

Note that we always take the normal vector outside, so the normal vector of the surface $S_{x=a}$ is $(-1, 0, 0)$, and the normal vector of $S_{x=b}$ is $(1, 0, 0)$. So,

$$\iint_{S_{x=b}} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_{x=a}} (P, Q, R) \cdot (1, 0, 0) dS = \iint_{S_{x=b}} P(b, y, z) dS \tag{2.5.7}$$

At this moment, the normal vector of the surface S_x points in the same direction as the normal vector of the area element $dy \, dz$, so we rewrite the area element dS of S_x as $dy \, dz$. So the surface integral over S_x is

$$\begin{aligned} \iint_{S_x} \mathbf{F} \cdot \mathbf{n} \, dS &= \iint_{S_{x=b}} P(b, y, z) dy \, dz - \iint_{S_{x=a}} P(a, y, z) dy \, dz \\ &= \iint [P(b, y, z) - P(a, y, z)] dy \, dz \end{aligned} \tag{2.5.8}$$

substituting (2.5.8) into (2.5.5), we can finally get

$$\iiint_V \frac{\partial P}{\partial x} dV = \iint_{S_x} \mathbf{F} \cdot \mathbf{n} dS \tag{2.5.9}$$

Where $dV = dx\, dy\, dz$, apply the same method, we can also obtain

$$\begin{cases} \iiint_V \dfrac{\partial Q}{\partial y} dV = \iint_{S_y} \mathbf{F} \cdot \mathbf{n} dS \\ \iiint_V \dfrac{\partial R}{\partial z} dV = \iint_{S_z} \mathbf{F} \cdot \mathbf{n} dS \end{cases} \tag{2.5.10}$$

Combine (2.5.9) with (2.5.10), we have

$$\begin{cases} \iiint_V \dfrac{\partial P}{\partial x} dV + \iiint_V \dfrac{\partial Q}{\partial y} dV + \iiint_V \dfrac{\partial R}{\partial z} dV = \iiint_V \left(\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} \right) dV \\ \iint_{S_x} \mathbf{F} \cdot \mathbf{n} dS + \iint_{S_y} \mathbf{F} \cdot \mathbf{n} dS + \iint_{S_z} \mathbf{F} \cdot \mathbf{n} dS = \iint_{\partial V} \mathbf{F} \cdot \mathbf{n} dS \end{cases}$$

Note that we can rewrite the term

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \boldsymbol{\nabla} \cdot \mathbf{F}$$

Finally, based on the derivation above, we can generalize Gauss's divergence theorem to three dimensions, leading to a significant result

$$\boxed{\iint_{\partial V} \mathbf{F} \cdot \mathbf{n} dS = \iiint_V \boldsymbol{\nabla} \cdot \mathbf{F}\, dV} \tag{2.5.11}$$

As we discussed on page 86, the surface integral of a vector field represents the physical concept of flux. In the case of a closed volume V, the surface integral corresponds to the total flux across all the faces enclosing the volume. This is equivalent to the volume integral of the divergence of the vector field over V. In other words, the physical interpretation of divergence is that, when applied to a vector field passing through a closed volume, the total accumulation of divergence within the volume is equal to the net flux through the entire surface of the volume.

Put differently, the divergence represents the total flux density of a system. More precisely, the divergence theorem states that

$$\iiint_V \boldsymbol{\nabla} \cdot \mathbf{F}\, dV = \lim_{\|\Delta\| \to 0} \sum \boldsymbol{\nabla} \cdot \mathbf{F} \times \Delta V = \iint_{\partial V} \mathbf{F} \cdot \mathbf{n} dS$$

So

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \lim_{\|\Delta\| \to 0} \frac{1}{\Delta V} \iint_{\partial V} \mathbf{F} \cdot \mathbf{n} dS \tag{2.5.12}$$

In summary, the divergence represents the **rate of flux change per unit volume**. Therefore, we can determine the type of physical system simply by examining the value of the divergence.

1. When $\nabla \cdot \mathbf{F} > 0$, it is called **source point**.
   Because the equation

$$\nabla \cdot \mathbf{F} = \lim_{\|\Delta\| \to 0} \frac{1}{\Delta V} \iint_{\partial V} \mathbf{F} \cdot \mathbf{n} dS > 0$$

   Therefore, we can further verify that

$$\forall \mathbf{r} \text{ such that } \mathbf{r} \cdot \mathbf{n} > 0 \Rightarrow \frac{d}{d\alpha} \mathbf{F}(\mathbf{r_0} + \alpha \mathbf{r}) \cdot \mathbf{n} \Big|_{\alpha=0} > 0 \tag{2.5.13}$$

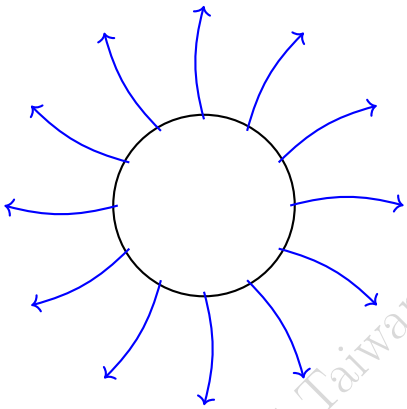

Figure 14: A region with outward curved flux arrows representing positive divergence

2. When $\nabla \cdot \mathbf{F} < 0$, it is called **sink point**.
   Because the equation

$$\nabla \cdot \mathbf{F} = \lim_{\|\Delta\| \to 0} \frac{1}{\Delta V} \iint_{\partial V} \mathbf{F} \cdot \mathbf{n} dS < 0$$

   Therefore, we can further verify that

$$\forall \mathbf{r} \text{ such that } \mathbf{r} \cdot \mathbf{n} > 0 \Rightarrow \frac{d}{d\alpha} \mathbf{F}(\mathbf{r_0} + \alpha \mathbf{r}) \cdot \mathbf{n} \Big|_{\alpha=0} < 0 \tag{2.5.14}$$

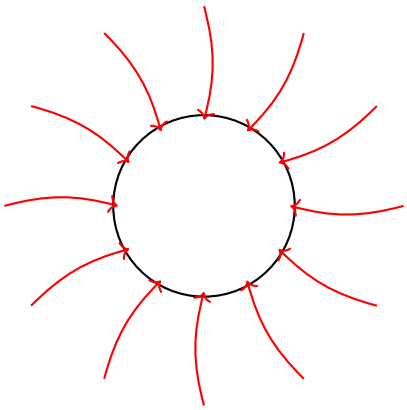

Figure 15: A region with inward curved flux arrows representing negative divergence

3. When $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$, it is called **source-free field**. Because the equation

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \lim_{\|\Delta\| \to 0} \frac{1}{\Delta V} \iint_{\partial V} \mathbf{F} \cdot \mathbf{n} dS = 0$$

Therefore, we can further verify that

$$\forall \mathbf{r} \text{ such that } \mathbf{r} \cdot \mathbf{n} > 0 \Rightarrow \frac{d}{d\alpha} \mathbf{F}(\mathbf{r_0} + \alpha \mathbf{r}) \cdot \mathbf{n} \Big|_{\alpha=0} = 0 \tag{2.5.15}$$

This illustrates that, regardless of the direction of displacement within this region, the flux variation rate of the vector field remains constant. Therefore, it is also referred to as a source-free field.

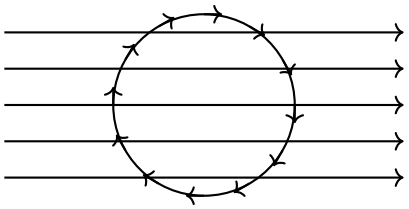

Figure 16: A divergence-free field with both circular and uniform flow showing zero net flux

▷ **Gauss's Law**

Let S is an arbitrary closed surface, and $\mathbf{r} = (x, y, z)$, $r = |\mathbf{r}|$. We have to evaluate the following integral

$$\iint_S \frac{\mathbf{r}}{r^3} \cdot \mathbf{n} dS = \iint_{\partial V} \frac{\mathbf{r}}{r^3} \cdot \mathbf{n} dS \tag{2.5.16}$$

Note that $(0, 0, 0)$ is the singularity [2]of the integral, so we need to consider three separate cases depending on the location of $(0, 0, 0)$.

1. When $(0, 0, 0)$ lies outside the surface S, the integrand is smooth over the domain, and the integral can be evaluated directly. In addition, we can directly apply Gauss's divergence theorem

$$\iint_S \frac{\mathbf{r}}{r^3} \cdot \mathbf{n} dS = \iiint_V \boldsymbol{\nabla} \cdot \left(\frac{\mathbf{r}}{r^3} \right) dV$$

Note that the divergence

$$\boldsymbol{\nabla} \cdot \left(\frac{\mathbf{r}}{r^3} \right) = r^{-3} \boldsymbol{\nabla} \cdot \mathbf{r} + \mathbf{r} \cdot \boldsymbol{\nabla} r^{-3} = 3r^{-3} + \mathbf{r} \cdot (-3r^{-5} \mathbf{r}) = 0 \tag{2.5.17}$$

So, when $(0, 0, 0)$ lies outside the surface S, the integral

$$\boxed{\iint_S \frac{\mathbf{r}}{r^3} \cdot \mathbf{n} dS = 0} \tag{2.5.18}$$

[2]Singularity refers to a point where a function, equation, or geometric object behaves abnormally, such as becoming infinite, undefined, discontinuous, or non-differentiable.

2. When $(0,0,0)$ lies inside the surface S, the integrated $\mathbf{r}/r^3$ is differentiable at the point $(0,0,0)$. We can't evaluate the integral directly. We must remove a spherical region B_ϵ of radius ϵ centered at the origin. The remaining volume V_ϵ and the original volume V satisfy

$$V_\epsilon = V - B_\epsilon \Rightarrow \partial V_\epsilon = \partial V + \partial B_\epsilon \tag{2.5.19}$$

Note that since we have removed the small spherical region at the center, the total surface now includes an additional spherical boundary. So the integral

$$\iint_{\partial V_\epsilon} \frac{\mathbf{r}}{r^3} \cdot \mathbf{n} dS = \iint_{\partial V} \frac{\mathbf{r}}{r^3} \cdot \mathbf{n} dS + \iint_{\partial B_\epsilon} \frac{\mathbf{r}}{r^3} \cdot \mathbf{n} dS \tag{2.5.20}$$

By removing the spherical region, the point $(0,0,0)$ doesn't lie inside the volume V_ϵ, so we can apply the divergence theorem to it

$$\iint_{\partial V_\epsilon} \frac{\mathbf{r}}{r^3} \cdot \mathbf{n} dS = 0 \tag{2.5.21}$$

We must take the outward-pointing normal vector with respect to V_ϵ; that is, for the spherical surface B_ϵ, the corresponding normal vector $\mathbf{n} = -\mathbf{r}/r$ points inward. So the surface integral of the sphere

$$\iint_{\partial B_\epsilon} \frac{\mathbf{r}}{r^3} \cdot \mathbf{n} dS = \iint_{\partial B_\epsilon} \frac{-r^2}{r^4} \cdot \mathbf{n} dS = -\frac{1}{\epsilon^2} \iint_{\partial B_\epsilon} dS \tag{2.5.22}$$

Because the surface integral $\iint_{\partial B_\epsilon} dS$ is the surface area of a sphere $4\pi r^2$, so

$$\iint_{\partial B_\epsilon} \frac{\mathbf{r}}{r^3} \cdot \mathbf{n} dS = -\frac{1}{\epsilon^2} \times 4\pi\epsilon^2 = -4\pi \tag{2.5.23}$$

Substituting (2.5.21) and (2.5.23) into (2.5.20), we obtain

$$0 = \iint_{\partial V} \frac{\mathbf{r}}{r^3} \cdot \mathbf{n} dS - 4\pi$$

So, when $(0,0,0)$ lies inside the surface S, the integral

$$\boxed{\iint_{\partial V} \frac{\mathbf{r}}{r^3} \cdot \mathbf{n} dS = 4\pi} \tag{2.5.24}$$

3. When $(0,0,0)$ lies on the surface S, we only need to remove a hemispherical region B'_ϵ and apply the same logic to the problem

$$\iint_{\partial V} \frac{\mathbf{r}}{r^3} \cdot \mathbf{n} dS + \iint_{\partial B'_\epsilon} \frac{\mathbf{r}}{r^3} \cdot \mathbf{n} dS = 0 \tag{2.5.25}$$

Because it is a hemisphere, the area of the hemisphere is only $2\pi r^2$. The integral

$$\iint_{\partial B'_\epsilon} \frac{\mathbf{r}}{r^3} \cdot \mathbf{n} dS = -\frac{1}{\epsilon^2} \times 2\pi\epsilon^2 = -2\pi \tag{2.5.26}$$

Substituting (2.5.26) into (2.5.25), we can get the final result

$$\boxed{\iint_{\partial V} \frac{\mathbf{r}}{r^3} \cdot \mathbf{n} dS = 2\pi} \tag{2.5.27}$$

2.6 Stokes' Theorem

For the sake of notational simplicity in the following proof, we rewrite the Cartesian coordinates (x, y, z) as the physicist's favorite form (x_1, x_2, x_3), so that we can conveniently apply the Einstein summation convention later. To ensure clarity in the subsequent derivations, let us first review the relationship between the curl $\boldsymbol{\nabla}\times$ and a vector field $\mathbf{F}(x_1, x_2, x_3) = (F_1, F_2, F_3)$, which is

$$\boldsymbol{\nabla} \times \mathbf{F} = \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}\right)\mathbf{e}_1 + \left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}\right)\mathbf{e}_2 + \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right)\mathbf{e}_3$$

So, the Green's theorem can be written as

$$\oint_C F_1 dx_1 + F_2 dx_2 = \iint_S \boldsymbol{\nabla} \times \mathbf{F} \cdot \mathbf{e}_3 \, dA \tag{2.6.1}$$

This is the form of Green's theorem on the $x-y$ plane. Next, we aim to extend this theorem to a three-dimensional surface. To do so, we must first establish a way to describe the surface S and the properties of its boundary ∂S.

The parametric form $\mathbf{r} : \Omega \to S \subset \mathbb{R}^3$ of the surface S is

$$S = \{\mathbf{r}(u, v) | x = x_1(u, v),\ x_2 = x_2(u, v),\ x_3 = x_3(u, v)\}$$

In addition, just as we did for curves, we can also parametrize $\mathbf{R} : C \to \Omega \subset \mathbb{R}^2$ the boundary ∂S. That is

$$\partial S = \{\mathbf{R}(t) = \mathbf{r}(u(t), v(t)) |\ t \in [a, b]\}$$

Where $\mathbf{r}$ is the surface parametrization defined above. By the chain rule of several variables, we have

$$\frac{d\mathbf{R}}{dt} = \frac{\partial \mathbf{r}}{\partial u}\frac{du}{dt} + \frac{\partial \mathbf{r}}{\partial v}\frac{dv}{dt} \Rightarrow d\mathbf{R} = \frac{\partial \mathbf{r}}{\partial u}du + \frac{\partial \mathbf{r}}{\partial v}dv \tag{2.6.2}$$

Next, we evaluate the line integral of the vector field $\mathbf{F}$ along the surface boundary S. In this case, the projection of the vector field is taken along the parameterized boundary ∂S, that is, along $d\mathbf{R}$. The closed integral

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{R} = \oint_C \mathbf{F}(\mathbf{r}(u, v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u}du + \frac{\partial \mathbf{r}}{\partial v}dv\right) = \oint_C \left(\mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial u}\right) du + \left(\mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial v}\right) dv \tag{2.6.3}$$

By (2.4.4), we can rewrite the integral

$$\oint_C \left(\mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial u}\right) du + \left(\mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial v}\right) dv = \iint_\Omega \left[\frac{\partial}{\partial u}\left(\mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial v}\right) - \frac{\partial}{\partial v}\left(\mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial u}\right)\right] du\, dv \tag{2.6.4}$$

According to the properties of differentiation and partial differentiation, we obtain

$$\begin{cases} \dfrac{\partial}{\partial u}\left(\mathbf{F} \cdot \dfrac{\partial \mathbf{r}}{\partial v}\right) = \dfrac{\partial \mathbf{F}}{\partial u} \cdot \dfrac{\partial \mathbf{r}}{\partial v} + \mathbf{F} \cdot \dfrac{\partial^2 \mathbf{r}}{\partial v \partial u} = \dfrac{\partial \mathbf{F}}{\partial u} \cdot \dfrac{\partial \mathbf{r}}{\partial v} + \mathbf{F} \cdot \mathbf{r}_{vu} \\ \dfrac{\partial}{\partial v}\left(\mathbf{F} \cdot \dfrac{\partial \mathbf{r}}{\partial u}\right) = \dfrac{\partial \mathbf{F}}{\partial v} \cdot \dfrac{\partial \mathbf{r}}{\partial u} + \mathbf{F} \cdot \dfrac{\partial^2 \mathbf{r}}{\partial u \partial v} = \dfrac{\partial \mathbf{F}}{\partial v} \cdot \dfrac{\partial \mathbf{r}}{\partial u} + \mathbf{F} \cdot \mathbf{r}_{uv} \end{cases}$$

Where $\mathbf{F}\cdot\mathbf{r}_{vu}=\mathbf{F}\cdot\mathbf{r}_{uv}$. Substituting this into (2.6.4), it becomes

$$\iint_\Omega\left[\frac{\partial}{\partial u}\left(\mathbf{F}\cdot\frac{\partial\mathbf{r}}{\partial v}\right)-\frac{\partial}{\partial v}\left(\mathbf{F}\cdot\frac{\partial\mathbf{r}}{\partial u}\right)\right]du\,dv=\iint_\Omega\left(\frac{\partial\mathbf{F}}{\partial u}\cdot\frac{\partial\mathbf{r}}{\partial v}-\frac{\partial\mathbf{F}}{\partial v}\cdot\frac{\partial\mathbf{r}}{\partial u}\right)du\,dv \quad (2.6.5)$$

Applying the Einstein summation convention, we can rewrite $\frac{\partial\mathbf{F}}{\partial u}\cdot\frac{\partial\mathbf{r}}{\partial v}-\frac{\partial\mathbf{F}}{\partial v}\cdot\frac{\partial\mathbf{r}}{\partial u}$ as

$$\begin{cases}\dfrac{\partial\mathbf{F}}{\partial u}\cdot\dfrac{\partial\mathbf{r}}{\partial v}=\partial_jF_i\dfrac{\partial x_j}{\partial u}\dfrac{\partial x_i}{\partial v}\\[2ex]\dfrac{\partial\mathbf{F}}{\partial v}\cdot\dfrac{\partial\mathbf{r}}{\partial u}=\partial_jF_i\dfrac{\partial x_j}{\partial v}\dfrac{\partial x_i}{\partial u}\end{cases}\Rightarrow\frac{\partial\mathbf{F}}{\partial u}\cdot\frac{\partial\mathbf{r}}{\partial v}-\frac{\partial\mathbf{F}}{\partial v}\cdot\frac{\partial\mathbf{r}}{\partial u}=\partial_jF_i\left(\frac{\partial x_j}{\partial u}\frac{\partial x_i}{\partial v}-\frac{\partial x_j}{\partial v}\frac{\partial x_i}{\partial u}\right)$$

Note that we should use the Kronecker delta and the Levi-Civita symbol again to unravel the structure of this; we have

$$\begin{aligned}\partial_jF_i\left(\frac{\partial x_j}{\partial u}\frac{\partial x_i}{\partial v}-\frac{\partial x_j}{\partial v}\frac{\partial x_i}{\partial u}\right)&=\partial_jF_i\left(\delta_{im}\delta_{jl}-\delta_{il}\delta_{jm}\right)\frac{\partial x_l}{\partial u}\frac{\partial x_m}{\partial v}\\&=\partial_jF_i\left(\varepsilon_{ijk}\varepsilon_{mlk}\right)\frac{\partial x_l}{\partial u}\frac{\partial x_m}{\partial v}\\&=\varepsilon_{ijk}\partial_jF_i\left(\varepsilon_{mlk}\frac{\partial x_l}{\partial u}\frac{\partial x_m}{\partial v}\right)\\&=\left(-\varepsilon_{kji}\partial_jF_i\right)\left(-\varepsilon_{klm}\frac{\partial x_l}{\partial u}\frac{\partial x_m}{\partial v}\right)\\&=\left(\boldsymbol{\nabla}\times\mathbf{F}\right)_k\left(\mathbf{r}_u\times\mathbf{r}_v\right)_k=\boldsymbol{\nabla}\times\mathbf{F}\cdot\mathbf{r}_u\times\mathbf{r}_v\end{aligned}$$

Substituting this into (2.6.5), you can obtain a familiar integral

$$\iint_\Omega\left(\frac{\partial\mathbf{F}}{\partial u}\cdot\frac{\partial\mathbf{r}}{\partial v}-\frac{\partial\mathbf{F}}{\partial v}\cdot\frac{\partial\mathbf{r}}{\partial u}\right)du\,dv=\iint_\Omega\boldsymbol{\nabla}\times\mathbf{F}\cdot\mathbf{r}_u\times\mathbf{r}_v\,du\,dv \quad (2.6.6)$$

Where $\mathbf{r}_u\times\mathbf{r}_v\,du\,dv=\mathbf{n}dS=d\mathbf{S}$. So (2.6.6) becomes

$$\iint_\Omega\boldsymbol{\nabla}\times\mathbf{F}\cdot\mathbf{r}_u\times\mathbf{r}_v\,du\,dv=\iint_S\boldsymbol{\nabla}\times\mathbf{F}\cdot\mathbf{n}dS=\iint_S\boldsymbol{\nabla}\times\mathbf{F}\cdot d\mathbf{S} \quad (2.6.7)$$

Combining (2.6.7) with (2.6.3), we obtain the final result, the **Stokes' Theorem.**

$$\boxed{\oint_{\partial S}\mathbf{F}\cdot d\mathbf{R}=\iint_S\boldsymbol{\nabla}\times\mathbf{F}\cdot d\mathbf{S}} \quad (2.6.8)$$

$\oint_{\partial S}\mathbf{F}\cdot d\mathbf{R}$ represents the line integral of the vector field along the boundary of the surface. The result reflects the total "amount of rotation" along the boundary of the surface. In other words, we can use the curl and the line integral over the boundary to describe the rotational behavior of the field, that is

$$\left|\boldsymbol{\nabla}\times\mathbf{F}\right|=\lim_{\|\Delta\|\to0}\frac{1}{S}\oint_{\partial S}\mathbf{F}\cdot d\mathbf{R} \quad (2.6.9)$$

The line integral represents the circulation of the vector field along the boundary of the surface. Its magnitude corresponds to the total projection of the curl of the vector field $\mathbf{F}$ onto the normal direction of each infinitesimal surface element $d\mathbf{S}$ over the surface.

We define the direction of $\nabla \times \mathbf{F}$ on $d\mathbf{S}$ to follow the **right-hand rule**, where the curled fingers represent the direction of rotation. We always define the counterclockwise direction as the positive orientation. This convention allows us to conveniently handle various problems related to electric current, magnetic fields, and other topics in electromagnetism. The direction of the curl is illustrated in the following diagram.

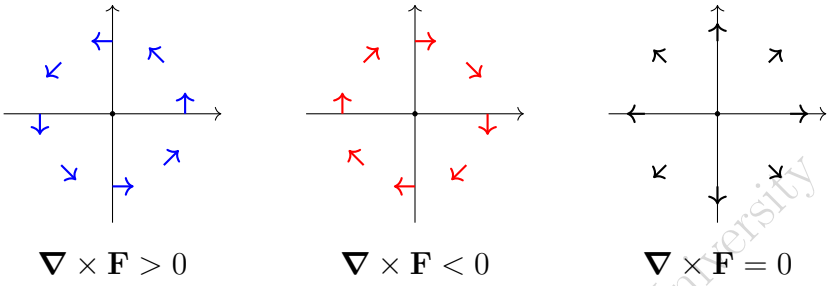

Figure 17: Examples of vector fields with positive, negative, and zero curl

In fact, every infinitesimal surface element on the surface satisfies Stokes' theorem. The projection of the curl onto the normal vector of each element represents the local direction and magnitude of rotation in that small region. However, the rotational effects in the interior cancel each other out, and after integrating over the entire surface, only the net circulation along the boundary remains, as illustrated below.

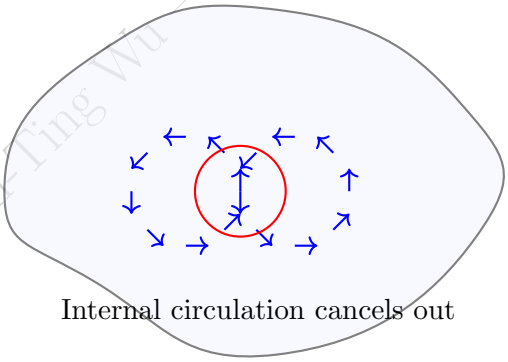

Figure 18: Internal rotational effects cancel out, leaving only boundary circulation (Stokes' Theorem)

Chapter 3. Complex Numbers in Physics

This chapter introduces one of the most important tools in both physics and engineering: the complex number system. Unlike real numbers, complex numbers cannot directly represent concrete physical quantities or magnitudes. However, their algebraic structure greatly simplifies many calculations in the real domain. In other words, complex numbers allow us to express various properties of a physical quantity, such as amplitude, phase, and periodicity, within a unified symbolic framework.
In electrical engineering, for example, complex numbers are used to simultaneously account for magnitude and phase in AC circuits. In mechanics, the use of complex numbers is generally more straightforward, often even simpler than problems encountered in high school physics. Therefore, this chapter will provide only a brief introduction to the essential tools and calculations involving complex numbers that are not typically covered in high school. The main focus will instead be placed on Fourier analysis.
You may consider this section a moment to catch your breath, a lighter, transitional chapter before moving on to more intense mathematical structures.

3.1 Euler's Formula and Complex Exponentials

▷ **Euler's Formula**

Review the Taylor expansion of the exponential function e^x, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

We can observe that the expansion of e^x is related to the expansion of $\sin x$ and $\cos x$, that is

$$\begin{cases} \sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots \\ \cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots \end{cases}$$

In the series expansions of these two functions, we observe that their terms are structurally identical to those of e^x, with only differences in sign. This observation allows us to introduce the imaginary unit i as a unifying tool, enabling us to combine both functions into a single exponential expression. Let $f(x) = e^{ix}$, where $i = \sqrt{-1}$, we can obtain

$$\begin{aligned} f(x) = e^{ix} &= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots \\ &= 1 + ix + \frac{x^2}{2!} \times i^2 + \frac{x^3}{3!} \times i^3 + \cdots \\ &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \cdots \end{aligned}$$

We can factor i out of the terms, which is

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \tag{3.1.1}$$

Substituting the expansion of $\cos x$ and $\sin x$ into (3.1.1), we can thus obtain the most important formula, the **Euler's formula**.

$$\boxed{e^{ix} = \cos x + i \sin x} \tag{3.1.2}$$

When $x = \theta$, we can also denote

$$e^{i\theta} = \cos\theta + i \sin\theta \tag{3.1.3}$$

When $\theta = \pi$, then the exponential function becomes $e^{i\pi}$, by (3.1.3), we can obtain

$$e^{i\pi} + 1 = (\cos\pi + i\sin\pi) + 1 = -1 + 1 = 0 \tag{3.1.4}$$

Previously, in Equation (1.2.46), we used the exponential function e^x to define the hyperbolic functions sinh and cosh. Now, we also observe that the ordinary trigonometric functions $\sin x$ and $\cos x$ share a similar connection with the exponential function. Thus, we can likewise express $\sin x$ and $\cos x$ in terms of e^{ix} using its properties. There are two functions e^{ix} and e^{-ix} =, we have

$$\begin{cases} e^{ix} = \cos x + i \sin x \\ e^{-ix} = e^{i(-x)} = \cos(-x) + i\sin(-x) = \cos x - i \sin x \end{cases}$$

Subtracting the two equations

$$e^{ix} - e^{-ix} = 2i \sin x$$

So, the trigonometric function $\sin x$ is denoted by

$$\boxed{\sin x = \frac{e^{ix} - e^{-ix}}{2i}} \tag{3.1.5}$$

Similarly, the function $\cos x$ is

$$\boxed{\cos x = \frac{e^{ix} + e^{-ix}}{2}} \tag{3.1.6}$$

Dividing (3.1.5) by (3.1.6), we can also get the function $\tan x$ in terms of e^{ix}

$$\boxed{\tan x = -\frac{i(e^{ix} - e^{-ix})}{e^{ix} + e^{-ix}}} \tag{3.1.7}$$

After introducing the basic structure, we now arrive at an interesting mathematical phenomenon. In the real domain, we all know that the range of the sine function is bounded by

$$-1 \leq \sin x \leq 1 \quad \text{for } x \in \mathbb{R} \tag{3.1.8}$$

However, once we extend our domain beyond the real numbers, this limitation no longer holds. For example, in the complex domain, it is entirely possible for

$$\sin x = 2 \quad \text{for some } x = z \in \mathbb{C} \tag{3.1.9}$$

Substituting $\sin x = \sin z$ into (3.1.5), we have

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = 2 \Rightarrow e^{iz} - e^{-iz} = 4i \tag{3.1.10}$$

We multiply both sides of (3.1.10) by e^{iz}, and we can obtain a quadratic equation in terms of e^{iz}.

$$\left(e^{iz}\right)^2 - 4i\left(e^{iz}\right) - 1 = 0$$

$$\Rightarrow e^{iz} = \frac{4i \pm \sqrt{-16+4}}{2} = i\left(2 \pm \sqrt{3}\right)$$

Taking the natural logarithm of both sides of the equation, we get

$$\ln e^{iz} = iz = \ln\left[i\left(2 \pm \sqrt{3}\right)\right] = \ln i + \ln\left(2 \pm \sqrt{3}\right)$$

Note that the term $\ln i$ is weird; we now consider this in the complex plane. Back in high school, we learned that the complex plane can be viewed as the real coordinate plane, with the y-axis replaced by the imaginary axis i. Therefore, we can also define an angular coordinate θ in the complex plane, ranging from $0 \leq \theta \leq 2\pi$. In the complex plane, any nonzero complex number z can be expressed in polar coordinates as

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta} \tag{3.1.11}$$

where $r = |z|$ is the modulus and $\theta = \arg(z)$ is the argument of the complex number. The illustration is shown below,

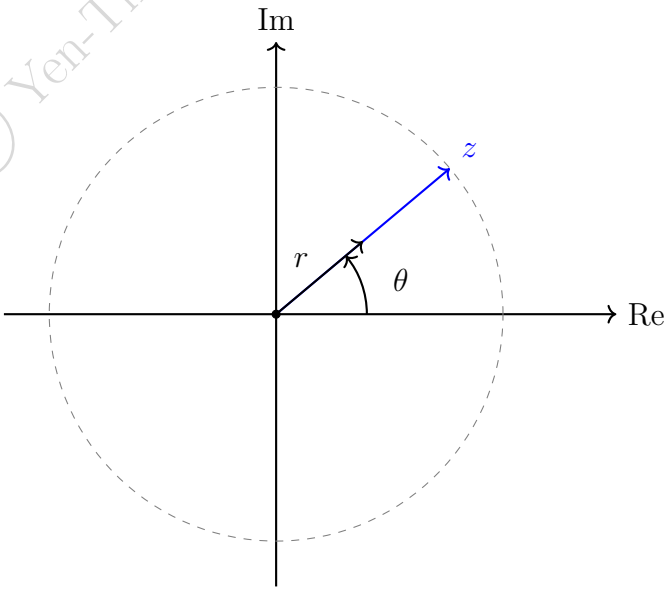

Figure 19: Polar Representation of a Complex Number

So, by (3.1.11), we have

$$i = 1 \times \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = |i|e^{i\pi/2} = e^{i\pi/2} \tag{3.1.12}$$

Substituting (3.1.12) into $\ln i$, we obtain

$$iz = \ln i + \ln\left(2 \pm \sqrt{3}\right) = \ln e^{i\pi/2} + \ln\left(2 \pm \sqrt{3}\right) = \frac{\pi}{2}i + \ln\left(2 \pm \sqrt{3}\right)$$

Dividing both sides by i, we can finally obtain the complex number making $\sin z = 2$, that is

$$z = \frac{\pi}{2} - i\ln\left(2 \pm \sqrt{3}\right) \tag{3.1.13}$$

So,

$$\sin\left(\frac{\pi}{2} - i\ln\left(2 \pm \sqrt{3}\right)\right) = 2 \tag{3.1.14}$$

In summary, the introduction of complex numbers allows us to resolve many problems that seem undefined or unreasonable within the real number system. Moreover, by working with the compact expression e^{ix}, we can significantly streamline calculations. In cases involving logarithmic functions, for example, the exponential can often be canceled, leaving only the exponent behind. In the next chapter, we will explore how complex numbers are meaningfully applied in physics and why they play such a crucial role.

▷ De Moivre's Theorem

Consider a complex number $z = r(\cos\theta + i\sin\theta)$, De Moivre's Theorem in high school can be denoted by

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)) \tag{3.1.15}$$

In the previous section, we discussed the exponential form of complex numbers in polar coordinates. We are now ready to prove De Moivre's Theorem in a remarkably elegant and concise way. Based on Equation (3.1.11), we rewrite the complex number z as:

$$z = re^{i\theta}$$

So, the nth power of z is

$$z^n = \left(re^{i\theta}\right)^n = r^n\left(e^{i\theta}\right)^n = r^n e^{i(n\theta)} \tag{3.1.16}$$

Substituting this into Euler's formula, we can obtain

$$\boxed{z^n = r^n e^{i(n\theta)} = r^n(\cos(n\theta) + i\sin(n\theta))} \tag{3.1.17}$$

We can apply the same logic to the $1/n$th power of z, that is

$$z^{1/n} = r^{1/n}e^{i\theta/n} = \sqrt[n]{r}\left(\cos\frac{\theta}{n} + \sin\frac{\theta}{n}\right) \tag{3.1.18}$$

Building on Equation (3.1.18), we can now use the complex exponential form to solve problems involving algebraic roots. Consider the equation

$$x^n = r$$

where we can rewrite $r \in \mathbb{C}$ as

$$r = |r|e^{i(\theta+2k\pi)} \quad k = 0, 1, \cdots \tag{3.1.19}$$

since $e^{i(\theta+2k\pi)} = e^{i\theta} = e^{i(\theta+2\pi)} = \cdots$, according to the fundamental theorem of algebra, the equation $x^n = r$ must have n roots, so

$$x = r^{1/n} = \left(|r|e^{i(\theta+2k\pi)}\right)^{1/n} = \sqrt[n]{|r|}\, e^{i\left(\frac{\theta+2k\pi}{n}\right)} \quad 0 \le k \le n-1 \tag{3.1.20}$$

For example, to compute the equation $x^3 = -8i$, we have

$$x^3 = -8i = 8e^{i(-\frac{\pi}{2}+2k\pi)}$$

So,

$$x = \sqrt[3]{-8} = \sqrt[3]{8}\, e^{i\left(-\frac{\pi}{6}+\frac{2k\pi}{3}\right)} \quad k = 0, 1, 2$$

That is,

$$x = \sqrt[3]{-8} = \sqrt[3]{8}\, e^{-i\frac{\pi}{6}},\ \sqrt[3]{8}\, e^{i\frac{\pi}{2}},\ \sqrt[3]{8}\, e^{i\frac{7\pi}{6}}$$

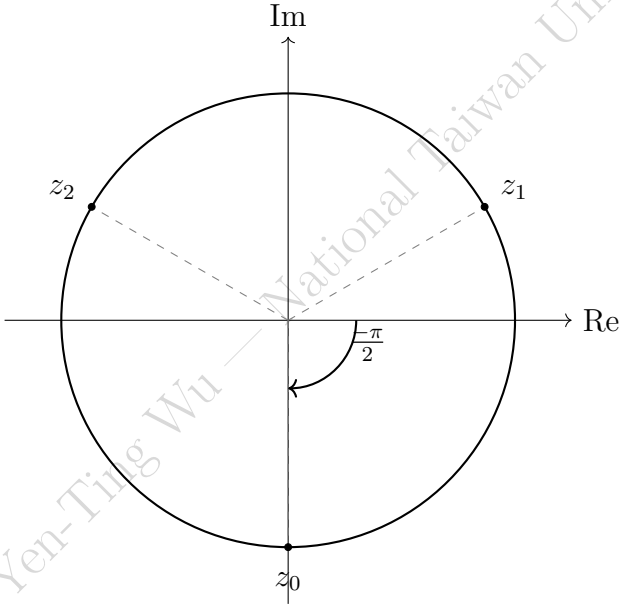

Figure 20: The three complex roots of the equation $x^3 = -8i$ are evenly distributed on a circle of radius 2 in the complex plane, starting at an angle of $-\frac{\pi}{2}$.

Furthermore, consider two complex numbers $z_1 = r_1 e^{i\alpha} = r_1(\cos\alpha + i\sin\alpha)$ and $z_2 = r_2 e^{i\beta} = r_2(\cos\beta + i\sin\beta)$, then $z_1^n z_2^m$ and z_1^n / z_2^m is

$$z_1^n z_2^m = \left(r_1 e^{i\alpha}\right)^n \left(r_2 e^{i\beta}\right)^m = r_1^n r_2^m e^{i(n\alpha+m\beta)} \quad \frac{z_1^n}{z_2^m} = \frac{\left(r_1 e^{i\alpha}\right)^n}{\left(r_2 e^{i\beta}\right)^m} = \frac{r_1^n}{r_2^n} e^{i(n\alpha-m\beta)}$$

Therefore,

$$\begin{cases} z_1^n z_2^m = r_1^n r_2^m e^{i(n\alpha+m\beta)} = r_1^n r_2^m(\cos(n\alpha+m\beta) + i\sin(n\alpha+m\beta)) \\ \dfrac{z_1^n}{z_2^m} = \dfrac{r_1^n}{r_2^m} e^{i(n\alpha-m\beta)} = \dfrac{r_1^n}{r_2^m}(\cos(n\alpha-m\beta) + i\sin(n\alpha-m\beta)) \end{cases} \tag{3.1.21}$$

▷ **Trigonometric Identities**

Back in high school, we learned many trigonometric identities, such as the double-angle, triple-angle, and sum and difference formulas. However, these identities were often used without formal proof. Now, with Euler's formula in hand, we are finally able to derive these identities in a much more elegant and straightforward way. We've known that Euler's formula satisfies

$$e^{ix} = \cos x + i \sin x$$

The sum of angles in terms of e^{ix} can be denoted by

$$e^{i(\alpha+\beta)} = \left(e^{i\alpha}\right)\left(e^{i\beta}\right) \tag{3.1.22}$$

Substituting $e^{i\alpha} = \cos\alpha + i\sin\alpha$ and $e^{i\beta} = \cos\beta + i\sin\beta$ into (3.1.22)

$$\begin{aligned}\left(e^{i\alpha}\right)\left(e^{i\beta}\right) &= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\ &= \cos\alpha\cos\beta + i\sin\beta\cos\alpha + i\sin\alpha\cos\beta - \sin\alpha\sin\beta \\ &= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\beta\cos\alpha + \sin\alpha\cos\beta)\end{aligned} \tag{3.1.23}$$

By (3.1.22),

$$e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta) \tag{3.1.24}$$

Comparing (3.1.23) with (3.1.24), we obtain

$$\begin{cases} i\sin(\alpha+\beta) = i(\sin\beta\cos\alpha + \sin\alpha\cos\beta) \\ \cos(\alpha+\beta) = (\cos\alpha\cos\beta - \sin\alpha\sin\beta) \end{cases}$$

So, we have proved that

$$\boxed{\begin{cases} \sin(\alpha+\beta) = (\sin\beta\cos\alpha + \sin\alpha\cos\beta) \\ \cos(\alpha+\beta) = (\cos\alpha\cos\beta - \sin\alpha\sin\beta) \end{cases}} \tag{3.1.25}$$

Similarly, let $\beta = -\gamma$, then (3.1.25) becomes

$$\begin{cases} \sin(\alpha-\gamma) = (\sin(-\gamma)\cos\alpha + \sin\alpha\cos(-\gamma)) \\ \cos(\alpha-\gamma) = (\cos\alpha\cos(-\gamma) - \sin\alpha\sin(-\gamma)) \end{cases} \tag{3.1.26}$$

since $\sin(-\gamma) = -\sin(\gamma)$ and $\cos(-\gamma) = \cos(\gamma)$, we obtain

$$\boxed{\begin{cases} \sin(\alpha-\gamma) = (\sin\alpha\cos\gamma - \sin\gamma\cos\alpha) \\ \cos(\alpha-\gamma) = (\cos\alpha\cos\gamma + \sin\alpha\sin\gamma) \end{cases}} \tag{3.1.27}$$

Furthermore, since our method of proof relies on comparing the real and imaginary parts of a complex expression, the operation of extracting a specific part from a complex number is called **taking the real part** or **taking the imaginary part**. We denote this as follows

$$\begin{cases} \sin(\alpha\pm\beta) = \operatorname{Im} e^{i(\alpha\pm\beta)} \\ \cos(\alpha\pm\beta) = \operatorname{Re} e^{i(\alpha\pm\beta)} \end{cases} \tag{3.1.28}$$

3.2 Basic Complex Calculus

The advantage of rewriting the polar form of complex numbers into its exponential form becomes especially clear in this chapter. Trigonometric functions are inherently periodic, and when analyzing a physical system involving differentiation and integration, we often desire an expression that both preserves this periodicity and is convenient to manipulate mathematically. This is where the exponential function e^x becomes our most powerful tool. Because e^x remains unchanged in form under both differentiation and integration, we are free to repeatedly apply calculus operations without the expression collapsing or becoming overly complicated.

Consider a function $f(z) = x_0(\cos(\omega z + \theta) + i\sin(\omega z + \theta))$, where $f : \mathbb{C} \to \mathbb{C}$, then we differentiate it and integrate it

$$\begin{aligned} \frac{d}{dz}f(z) &= z_0(-\omega\sin(\omega z + \theta) + i\omega\cos(\omega z + \theta)) \\ &= i\omega z_0((\cos(\omega z + \theta) + i\sin(\omega z + \theta)) \\ &= i\omega f(z) \end{aligned} \tag{3.2.1}$$

Similarly,

$$\frac{d^2}{dz^2}f(z) = -\omega^2 f(z) \tag{3.2.2}$$

Applying Euler's formula, the function $f(z)$ can be rewritten as

$$f(z) = z_0 e^{i(\omega z+\theta)} \tag{3.2.3}$$

By Chapter 1, we can directly obtain the relationship between $f'(z)$ and $f(z)$, which is

$$\begin{cases} \dfrac{d}{dz}f(z) = i\omega z_0 e^{i(\omega z+\theta)} = i\omega z_0 e^{i(\omega z+\theta)} = i\omega f(z) \\ \dfrac{d^2}{dz^2}f(z) = \dfrac{d}{dz}i\omega z_0 e^{i(\omega z+\theta)} = (i\omega)^2 z_0 e^{i(\omega z+\theta)} = -\omega^2 f(z) \end{cases} \tag{3.2.4}$$

We verify this fact by differentiating $\sin(\omega z - \theta)$. Let $f(z) = e^{i(\omega z-\theta)}$ By (3.1.28), we can denote it as

$$\sin(\omega z - \theta) = \mathrm{Im} f(z) = \mathrm{Im}\, e^{i(\omega z-\theta)} \tag{3.2.5}$$

Differentiating both sides

$$\frac{d}{dz}\sin(\omega z - \theta) = \frac{d}{dz}\mathrm{Im} f(z) = \mathrm{Im}\frac{d}{dz}e^{i(\omega z-\theta)} = \mathrm{Im}\left(i\omega e^{i(\omega z-\theta)}\right)$$

since

$$\mathrm{Im}\left(i\omega e^{i(\omega z-\theta)}\right) = \mathrm{Im}\left(i\omega\cos(\omega z - \theta) - \omega\sin(\omega z - \theta)\right) = \omega\cos(\omega z - \theta)$$

Hence, we have verified that

$$\frac{d}{dz}\sin(\omega z - \theta) = \omega\cos(\omega z - \theta) \tag{3.2.6}$$

as expected. In addition, we can express $\sin(\omega z - \theta)$ as the form of (3.1.5), that is

$$\sin(\omega z - \theta) = \frac{e^{i(\omega z-\theta)} - e^{i(\omega z-\theta)}}{2i} \tag{3.2.6}$$

Differentiating this with respect to z,

$$\begin{aligned}\frac{d}{dz}\frac{e^{i(\omega z-\theta)}-e^{-i(\omega z-\theta)}}{2i} &= \frac{i\omega e^{i(\omega z-\theta)}+i\omega e^{-i(\omega z-\theta)}}{2i}\\ &= \frac{\omega e^{i(\omega z-\theta)}+\omega e^{-i(\omega z-\theta)}}{2}\\ &= \omega\left(\frac{e^{i(\omega z-\theta)}+e^{-i(\omega z-\theta)}}{2}\right)\\ &= \omega\cos(\omega z-\theta)\end{aligned}$$

As before. The same logic applies to integration. Let $f(z)=z_0e^{i\omega z}$, we obtain

$$\int f(z)\,dz=\int z_0e^{i\omega z}\,dz=z_0\int e^{i\omega z}\,dz=\frac{z_0}{i\omega}e^{i\omega z}+C=\frac{-z_0i}{\omega}e^{i\omega z}+C \tag{3.2.7}$$

So we can verify that

$$\int f(z)\,dz=\frac{-i}{\omega}f(z)+C \tag{3.2.8}$$

This shows that even after integration, the exponential term $e^{i\omega z}$ remains, which allows us to easily retain information related to periodicity. In other words, by transforming the problem into an integral involving e^x, the entire computation becomes significantly simplified. This shows that even after integration, the exponential term $e^{i\omega z}$ remains, which allows us to easily retain information related to periodicity. In other words, by transforming the problem into an integral involving e^x, the entire computation becomes significantly simplified.

For example, the function z_1 and z_2 satisfy

$$\begin{cases}\dfrac{d}{dz}z_1=\zeta z_2\\ \dfrac{d}{dz}z_2=-\zeta z_1\end{cases} \tag{3.2.9}$$

Note that when the derivatives of two functions differ only by a minus sign and result in each other's form, we can safely assume a complex representation

$$\mathcal{Z}=z_1+iz_2 \tag{3.2.10}$$

Then the differentiation of $\mathcal{Z}$ can be denoted by

$$\frac{d}{dz}\mathcal{Z}=\frac{d}{dz}(z_1+iz_2)=\frac{d}{dz}z_1+i\frac{d}{dz}z_2$$

Substituting (3.2.9) into (3.2.10), we get

$$\frac{d}{dz}\mathcal{Z}=\zeta z_2-i\zeta z_1=-i\zeta(z_1+iz_2)=-i\zeta\mathcal{Z}$$

It becomes a first-order differential equation (See Chapter 4).

$$\begin{aligned} \frac{d\mathcal{Z}}{\mathcal{Z}} = -i\zeta\, dz \Rightarrow \int \frac{d\mathcal{Z}}{\mathcal{Z}} &= -i\zeta \int dz \\ \Rightarrow \ln \mathcal{Z} &= -i\zeta z + C \\ \Rightarrow \mathcal{Z} &= Ae^{-i\zeta z} \end{aligned} \tag{3.2.11}$$

where $A = e^C$. We can obtain A by the *initial condition* (See Chapter 4). So (3.2.11) can be rewritten as

$$z_1 + iz_2 = Ae^{-i\zeta z} = A\cos(\zeta z) - Ai\sin(\zeta z) \tag{3.2.12}$$

So, the functions z_1 and z_2 are

$$\begin{cases} z_1 = \mathrm{Re}(Ae^{-i\zeta z}) = A\cos(\zeta z) \\ z_2 = \mathrm{Im}(Ae^{-i\zeta z}) = -A\sin(\zeta z) \end{cases} \tag{3.2.13}$$

In summary, the introduction of Euler's formula allows us to handle differentiation, integration, and even differential equations in a more compact and elegant way. By working with the exponential form, we can extract the full behavior and periodicity of the function, and then isolate the real or imaginary part as needed. However, in physical systems, both the real and imaginary parts do not always correspond to physical quantities. In the next section, I will illustrate this with two classic examples in physics.

3.3 Real and Imaginary Parts: Physical Interpretation

▷ **Simple Harmonic Motion**

This part involves second-order differential equations, which are covered in detail in Chapter 4.2. Therefore, we will not go into the full solution here. Instead, we will present only the final result and focus on explaining its physical meaning. Students who are interested in the full derivation are encouraged to study the differential equations in Chapter 4 first, and then return to this section to work through the solution independently.
The equation of motion of SHM system is

$$m\ddot{x} = -kx \Rightarrow m\ddot{x} + kx = 0 \tag{3.3.1}$$

Let $\omega^2 \coloneqq k/m$, and this second-order differential equation becomes

$$\ddot{x} + \omega^2 x = 0 \tag{3.3.2}$$

The solution to the ODE is

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \tag{3.3.3}$$

(3.3.3) can be expanded to

$$\begin{aligned} x(t) &= [C_1\cos(\omega t) + iC_1\sin(\omega t)] + [C_2\cos(\omega t) - iC_2\sin(\omega t)] \\ &= (C_1 + C_2)\cos(\omega t) + i(C_1 - C_2)\sin(\omega t) \end{aligned} \tag{3.3.4}$$

Because we don't let the function $x(t)$ be the complex number on purpose, we can deduce that C_1, $C_2 \in \mathbb{C}$. The initial condition $x(0) = x_0$ and $\dot{x}(0) = v_0$ can help us to determine C_1 and C_2, and we substitute these into (3.3.3)

$$\begin{cases} x(0) = C_1 e^0 + C_2 e^0 = C_1 + C_2 = x_0 \\ \dot{x}(0) = i\omega C_1 e^0 - i\omega C_2 e^0 = i\omega(C_1 - C_2) = v_0 \end{cases}$$

So we can compute that

$$\begin{cases} C_1 = \dfrac{x_0}{2} - i\dfrac{v_0}{2\omega} \\ C_2 = \dfrac{x_0}{2} + i\dfrac{v_0}{2\omega} \end{cases} \tag{3.3.5}$$

Substituting (3.3.5) into (3.3.4), we obtain

$$x(t) = x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t) \tag{3.3.6}$$

The angle relationship is shown below

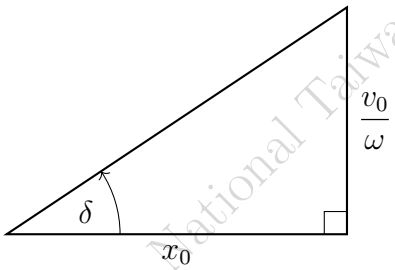

Figure 21: A right triangle representing the initial condition decomposition of the solution $x(t) = x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t)$, where x_0 and $\frac{v_0}{\omega}$ are the projections onto cosine and sine components, respectively.

By the figure, we can rewrite (3.3.6) as

$$\begin{aligned} x(t) &= \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2} \left(\frac{x_0}{\sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}} \cos(\omega t) + \frac{v_0/\omega}{\sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}} \sin(\omega t) \right) \\ &= \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2} \cos(\omega t - \delta) = \sqrt{x_0^2 + \frac{m v_0^2}{k}} \cos(\omega t - \delta) \end{aligned} \tag{3.3.7}$$

The angle δ is

$$\delta = \arctan\left(\frac{v_0}{\omega x_0}\right) = \arctan\left(\frac{v_0}{x_0}\sqrt{\frac{m}{k}}\right) \tag{3.3.8}$$

It determines the initial phase diagram by the initial condition, including the initial position x_0 and the initial velocity v_0. And the period of the system is given by the trigonometric function $\cos(\omega t - \delta)$, which is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \tag{3.3.9}$$

The key point here is that we did not deliberately choose to use complex numbers. What we've demonstrated is a result that naturally emerges from solving the differential equation itself. Therefore, we don't need to force ourselves to pick out the real or imaginary part, the solution already carries physical meaning as it is. Most importantly, from the very beginning, the exponential term $e^{i\omega t}$ directly reveals the periodic nature of the system's motion. In the next example, we will see a different situation, one where we intentionally introduce complex numbers in order to simplify the computation.

▷ **AC Circuit**

In the AC system, both voltage and current are quantities that oscillate with time. The voltage can be denoted by

$$V(t) = V_0 \cos(\omega t) \tag{3.3.10}$$

Let $\mathcal{V}(t)$ be the complex number, which is

$$\mathcal{V}(t) = V_0 e^{i\omega t} \tag{3.3.11}$$

Under the series RLC circuit, the impedance Z of the system is

$$Z = R + i\omega L + \frac{1}{i\omega C} = R + i\left(\omega L - \frac{1}{\omega C}\right) \tag{3.3.12}$$

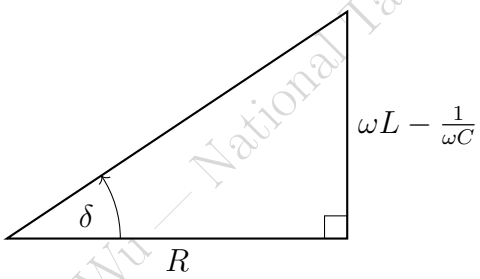

Figure 22: A right triangle representing the impedance triangle in a series RLC circuit, where the angle δ corresponds to the phase angle between voltage and current.

As shown in the figure, the modulus of the complex number $|Z|$ is

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{3.3.13}$$

So, the impedance can be rewritten as

$$\begin{aligned} Z &= |Z| \left(\frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} + i \frac{\omega L - \frac{1}{\omega C}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \right) \\ &\coloneqq |Z|(\cos\delta + i\sin\delta) = |Z| e^{i\delta} \end{aligned} \tag{3.3.14}$$

We can now write Ohm's law in the complex domain. Let the current be the function $\mathcal{I}(t)$, which satisfies

$$\mathcal{I}(t) = \frac{\mathcal{V}(t)}{Z} = \frac{V_0 e^{i\omega t}}{|Z| e^{i\delta}} = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} e^{i(\omega t - \delta)} \tag{3.3.15}$$

Where δ means the **phase difference** between the voltage and the current. It is given by

$$\delta = \arctan\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right) = \arctan\left(\frac{\omega^2 CL - 1}{R\omega C}\right) \tag{3.3.16}$$

Through the above derivations, we can extract several key pieces of information: the magnitude and frequency of the voltage, the magnitude and frequency of the current, and the phase difference between them. However, it is important to note that in the complex representation, the quantities we can actually measure using instruments such as oscilloscopes are given by the real parts of these complex expressions. That is,

$$\begin{cases} V(t) = \mathrm{Re}\left(\mathcal{V}(t)\right) = V_0 \cos(\omega t) \\ I(t) = \mathrm{Re}\left(\mathcal{I}(t)\right) = \dfrac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \cos(\omega t - \delta) \end{cases} \tag{3.3.17}$$

In this circuit system, we move all real-world quantities and physical values into the complex domain. As I mentioned earlier, in the complex domain, we can represent everything using exponential terms, which greatly simplifies the computation. (Trust me, you really don't want to abandon $e^{i\omega t}$ and go back to brute-force real-number trigonometry.)

Although the imaginary part cannot be directly measured in experiments, its presence is crucial for preserving phase information in the system. It's like a wedding, where the real part is the bride and groom, and the imaginary part is the best man. The best man is there to help the ceremony go smoothly, but in the end, no one really remembers who he was. And honestly, we don't need to. (Apologies to anyone who has ever been a best man.)

3.4 Introduction to Fourier Series and Transform

The Fourier series expresses a periodic function as an infinite sum of sine and cosine terms. In contrast, the Fourier transform generalizes this idea to non-periodic functions by replacing the discrete sum with a continuous integral, effectively decomposing the signal into a continuous spectrum of frequency components.

▷ Average Value of Sines and Cosines

Consider a function $f(x)$, $x \in [a, b]$, and we divide this interval into n subintervals. Then the average of $f(x)$ on $[a, b]$ can be denoted by

$$\langle f(x) \rangle \coloneqq \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$$

Let x_1, x_2 be Δx apart, then we have

$$\frac{[f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x}{n\Delta x}$$

When $\|\Delta\| \to 0$, the numerator can be written as definite integral, which is

$$\langle f(x) \rangle = \left.\frac{[f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x}{n\Delta x}\right|_{\|\Delta\| \to 0} = \frac{1}{b-a}\int_a^b f(x)\,dx \tag{3.4.1}$$

After introducing the average of a function, I'm going to introduce the periodic function next, such as $f(x) = \sin x$ and $f(x) = \cos x$, since this chapter focuses on the average over a period. A periodic function is defined as

$$f(x + T) = f(x) \tag{3.4.2}$$

Then the period of the function is T, or sometimes a multiple such as nT, depending on the frequencies involved. We know that $\sin x$ and $\cos x$ are periodic functions with period 2π. Therefore, combinations such as $\sin(nx)\sin(mx)$, $\cos(nx)\cos(mx)$, or $\sin(nx)\cos(mx)$ are also periodic, because the argument of each trigonometric function is shifted by integer multiples of 2π, which do not affect the function value. To illustrate

$$\sin[n(x + 2\pi)]\sin[m(x + 2\pi)] = \sin(nx + 2n\pi)\sin(mx + 2m\pi) = \sin(nx)\sin(mx)$$

since $\sin(\theta + 2\pi k) = \sin(\theta)$ for any integer k. Similarly,

$$\begin{cases} \cos[n(x + 2\pi)]\cos[m(x + 2\pi)] = \cos(nx)\cos(mx) \\ \sin[n(x + 2\pi)]\cos[m(x + 2\pi)] = \sin(nx)\cos(mx) \end{cases}$$

Although the combination $\sin(nx)\sin(mx)$, $\cos(nx)\cos(mx)$ or $\sin(nx)\cos(mx)$ may has a shorter period than 2π, in Fourier series we still integrate over the original period of the function, typically $[-\pi, \pi]$ or $[0, 2\pi]$. This is because the Fourier coefficients are defined relative to the period of the function being expanded, not the individual basis functions. We'll discuss this concept next part. So, we discuss the average values of them over a

period 2π. that is

$$\langle f(x)\rangle = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx \\ \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx \\ \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx \end{cases} \tag{3.4.3}$$

By (3.1.5) and (3.1.6), we express all trigonometric functions in terms of complex exponentials via Euler's formula, so

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{e^{inx}-e^{-inx}}{2i}\right)\left(\frac{e^{imx}-e^{-imx}}{2i}\right)\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{e^{i(n+m)x}-e^{i(n-m)x}-e^{i(m-n)x}+e^{-i(n+m)x}}{-4}\right)\,dx \end{aligned}$$

Then it becomes

$$\frac{-1}{8\pi}\left[\int_{-\pi}^{\pi} e^{i(n+m)x}dx - \int_{-\pi}^{\pi} e^{i(n-m)x}dx - \int_{-\pi}^{\pi} e^{i(m-n)x}dx + \int_{-\pi}^{\pi} e^{-i(n+m)x}dx\right] \tag{3.4.4}$$

When $m \neq n$, we have

$$\begin{aligned} &= \frac{-1}{8i(n+m)}\, e^{i(n+m)x}\Big|_{-\pi}^{\pi} + \frac{1}{8i(n-m)}\, e^{i(n-m)x}\Big|_{-\pi}^{\pi} + \frac{1}{8i(m-n)}\, e^{i(m-n)x}\Big|_{-\pi}^{\pi} \\ &+ \frac{1}{8i(n+m)}\, e^{-i(n+m)x}\Big|_{-\pi}^{\pi} \qquad (3.4.5) \\ &= \frac{-\sin[(n+m)\pi]}{4} + \frac{\sin[(n-m)\pi]}{4} + \frac{\sin[(m-n)\pi]}{4} = 0 \end{aligned}$$

Since $m, n \in \mathbb{Z}$, $\sin(n \pm m)\pi = \sin k\pi = 0$. When $m = n \neq 0$, (3.4.4) becomes

$$\begin{aligned} &\frac{-1}{8\pi}\left[\int_{-\pi}^{\pi} e^{i2nx}dx - \int_{-\pi}^{\pi} dx - \int_{-\pi}^{\pi} dx + \int_{-\pi}^{\pi} e^{-i2nx}dx\right] \\ &= \frac{-\sin(2n\pi)}{8\pi} + \frac{2\pi}{8\pi} + \frac{2\pi}{8\pi} = \frac{1}{2} \end{aligned} \tag{3.4.6}$$

When $m = n = 0$, (3.4.4) becomes

$$\frac{-1}{8\pi}\left[\int_{-\pi}^{\pi} dx - \int_{-\pi}^{\pi} dx - \int_{-\pi}^{\pi} dx + \int_{-\pi}^{\pi} dx\right] = 0 \tag{3.4.7}$$

So, we can obtain the average value of sines over a period 2π in three different situations, which is

$$\boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(nx)\sin(mx)\,dx=\begin{cases}0 & m\neq n\\ \dfrac{1}{2} & m=n\neq 0\\ 0 & m=n=0\end{cases}} \tag{3.4.8}$$

Apply the same logic to the cosines $\cos(nx)\cos(mx)$, we have

$$\boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(nx)\cos(mx)\,dx=\begin{cases}0 & m\neq n\\ \dfrac{1}{2} & m=n\neq 0\\ 1 & m=n=0\end{cases}} \tag{3.4.9}$$

Similarly, for the product of sine and cosine functions, regardless of the values of m or n, the result will be a combination of terms similar to equation (3.4.5), involving complex exponentials like $e^{i(kx)}$ and $e^{-i(kx)}$ with nonzero frequencies. Since the average value of such oscillating exponential terms over a full period is zero, the average of the entire product function is also zero. That is,

$$\boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(nx)\cos(mx)\,dx=0} \tag{3.4.10}$$

▷ Fourier Series in the Real Domain

Given a function $f(x)$ of period 2π, we can express the function as a combination of sine and cosine terms, since these are all periodic functions. By adjusting the size of each coefficient, we determine how much each periodic component contributes to the overall shape of $f(x)$. In this way, we aim to approximate the function as closely as possible using these fundamental building blocks. Therefore, our goal is to write

$$\begin{aligned}f(x)=A&+a_1\cos x+a_2\cos(2x)+a_3\cos(3x)+\cdots\\&+b_1\sin x+b_2\sin(2x)+b_3\sin(3x)+\cdots\end{aligned} \tag{3.4.11}$$

Where $A=a_0/2$, I'll show you the reason later. Let's start with the cosine term. When we want to get a_n, we multiply both sides by $\cos(nx)$, for example

$$\begin{aligned}f(x)\cos x=A\cos x+a_1\cos^2 x+a_2\cos(2x)\cos x+\cdots\\+b_1\sin x\cos x+b_2\sin(2x)\cos x+\cdots\end{aligned}$$

Then the average value over the period of both sides is given by

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\cos x dx=A\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos x dx+a_1\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^2 x\,dx+\cdots\\+b_1\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin x\cos x\,dx+\cdots\end{aligned}$$

In the cosine terms, according to (3.4.9), the average value of the function is zero whenever $m \neq n$. Similarly, based on (3.4.10), the integral of the product of sine and cosine functions is always zero. This means that all terms vanish except the a_1 term (you can verify that the A term integrates to zero as well). Thus, we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos x dx = a_1\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^2 x\, dx = \frac{a_1}{2} \Rightarrow a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x dx \tag{3.4.12}$$

Similarly, we can verify

$$\begin{cases} a_1 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos x dx \\ a_2 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos(2x) dx \\ \vdots \\ a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos(nx) dx \end{cases} \qquad \begin{cases} b_1 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin x dx \\ b_2 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin(2x) dx \\ \vdots \\ b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin(nx) dx \end{cases} \tag{3.4.13}$$

What about A? Because A can be seen as the coefficient of the term $\cos 0x$, we can integrate both sides directly

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = A\frac{1}{2\pi}\int_{-\pi}^{\pi} dx + a_1\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos x\, dx + \cdots$$
$$+ b_1\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin x\, dx + \cdots$$

According to (3.4.10), the integral of the products of sine and cosine functions is always zero($\sin x = \sin 1x \cos 0x$ and so on). So

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = A\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos 0x\, dx = A \tag{3.4.14}$$

As we can see from Equation (3.4.14), its integral does not follow the same pattern as that of Equation (3.4.13), and therefore cannot be written in a unified formula. To make it consistent with the pattern, we need to multiply the result by 2. That is

$$2A = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \tag{3.4.15}$$

We define $2A \coloneqq a_0$, then we can get $A = a_0/2$. Substituting $A = a_0/2$ into (3.4.11), we can thus write down the Fourier series as

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right)} \tag{3.4.16}$$

Where the coefficients a_n and b_n are denoted by

$$\boxed{\begin{cases} a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos(nx) dx \\ b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin(nx) dx \end{cases}} \tag{3.4.17}$$

Example. *Express the following functions using the Fourier series.*

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$$

Where $f(x + 2\pi) = f(x)$.

The coefficient a_n satisfies

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\cos(nx)\,dx + \int_{0}^{\pi}\cos(nx)\,dx\right]$$

We need to carefully examine the graph of the periodic function $f(x)$. If the function is not globally continuous, we must analyze it piecewise. For this example, we divide the interval into two parts, $[-\pi, 0]$ and $[0, \pi]$. Starting from a_n, when $n \neq 0$, we have

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\cos(nx)\,dx + \int_{0}^{\pi}\cos(nx)\,dx\right]$$

$$= \frac{1}{\pi}\int_{0}^{\pi}\cos(nx)\,dx = \frac{1}{n\pi}\sin(nx)\Big|_{0}^{\pi} = 0$$

When $n = 0$, we have

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\cos(0x)\,dx + \int_{0}^{\pi}\cos(0x)\,dx\right]$$

$$= \frac{1}{\pi}\int_{0}^{\pi} dx = 1$$

And the coefficient b_n satisfies

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\sin(nx)\,dx + \int_{0}^{\pi}\sin(nx)\,dx\right] = \frac{1}{\pi}\int_{0}^{\pi}\sin(nx)\,dx$$

$$= \frac{-1}{n\pi}\cos(nx)\Big|_{0}^{\pi} = \frac{-1}{n\pi}[(-1)^n - 1] = \begin{cases} 0 & n = 2k,\ k \in \mathbb{N} \\ \dfrac{2}{n\pi} & n = 2k+1 \end{cases}$$

Substituting a_n, b_n into (3.4.16), we can finally get

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots\right)$$

Next, we can generalize the Fourier series to functions that have periods other than 2π. Given a function $f(x)$ of period $2L$, where $L \in \mathbb{R}$. That is

$$f(x + 2L) = f(x)$$

When analyzing this function, we need to change the interval for computing the average value of the periodic function from $[-\pi, \pi]$ to $[-L, L]$. That is

$$\langle f(x) \rangle = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

To expand $f(x)$, we use functions that have the same periodicity, namely a period of $2L$. This can be done by rewriting the trigonometric functions $\sin(nx)$ and $\cos(nx)$ as $\sin(n\pi x/L)$ and $\cos(n\pi x/L)$, respectively, which ensures that the resulting functions have period $2L$ since

$$\begin{cases} \sin\left[\dfrac{n\pi}{L}(x+2L)\right] = \sin\left(\dfrac{n\pi x}{L} + 2n\pi\right) = \sin\left(\dfrac{n\pi x}{L}\right) \\ \cos\left[\dfrac{n\pi}{L}(x+2L)\right] = \cos\left(\dfrac{n\pi x}{L} + 2n\pi\right) = \cos\left(\dfrac{n\pi x}{L}\right) \end{cases} \tag{3.4.18}$$

As a reminder, in high school we learned that the period of $\sin(nx)$ is $2\pi/n$. However, for the sake of convenience in calculation and analysis, we usually treat them as if they have period 2π, as long as their behavior repeats correctly within that interval.
So, the Fourier series becomes

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]} \tag{3.4.19}$$

Where the coefficients a_n and b_n are changed into

$$\boxed{\begin{cases} a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x) \cos\left(\dfrac{n\pi x}{L}\right) dx \\ b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x) \sin\left(\dfrac{n\pi x}{L}\right) dx \end{cases}} \tag{3.4.20}$$

Similarly, these coefficients a_n and b_n determine how much of each sine or cosine wave contributes to the shape of $f(x)$. If we want to express

$$f(x) = \begin{cases} 0 & -L < x < 0 \\ 1 & 0 < x < L \end{cases}$$

Then the answer becomes

$$f(x) = \frac{1}{2} + \frac{2}{L}\left[\sin\left(\frac{\pi x}{L}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{L}\right) + \frac{1}{5}\sin\left(\frac{5\pi x}{L}\right) + \cdots\right]$$

▷ Complex Form of Fourier Series

Since we can express a function with period 2π as a sum of sine and cosine functions, it naturally follows from Euler's formula that we can also rewrite these sine and cosine terms using exponential functions of the form e^{ix}. This substitution not only simplifies the expression but also helps maintain the compactness and elegance of the overall formula. Given a function $f(x)$ of period 2π, we can express the function as

$$f(x) = c_0 + c_1 e^{ix} + c_{-1} e^{-ix} + c_2 e^{i2x} + c_{-2} e^{-i2x} + \cdots \tag{3.4.21}$$

To determine c_0, we need to compute the average value of $f(x)$ over the period

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = c_0 \frac{1}{2\pi}\int_{-\pi}^{\pi} dx + c_1 \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix}\,dx + \cdots \tag{3.4.22}$$

Note that when we integrate e^{ikx} over the interval from $-\pi$ to π, the result is always zero as long as $k \neq 0$. Because

$$\int_{-\pi}^{\pi} e^{ikx}\,dx = \frac{1}{ik} e^{ikx}\Big|_{-\pi}^{\pi} = \frac{1}{ik}\left(e^{ik\pi} - e^{-ik\pi}\right) = \frac{2\sin(k\pi)}{ik} = 0 \tag{3.4.23}$$

So, (3.4.22) becomes

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = c_0 \frac{1}{2\pi}\int_{-\pi}^{\pi} dx + 0 = c_0 \frac{2\pi}{2\pi} = c_0 \tag{3.4.24}$$

Similarly, when we want to determine c_n, then we should multiply both sides by e^{-inx} since all terms other than e^{inx} multiplied by e^{-inx} can be rewritten in the form e^{ikx} with $k \neq 0$, their integrals from $-\pi$ to π will vanish. As a result, only the term c_n survives in the integral, that is

$$\begin{aligned} & f(x)e^{-inx} = c_0 + c_1 e^{ix} e^{-inx} + \cdots + c_n e^{inx} e^{-inx} + \cdots \\ \Rightarrow\ & \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = 0 + \cdots + c_n \frac{1}{2\pi}\int_{-\pi}^{\pi} dx + 0 + \cdots = c_n \end{aligned} \tag{3.4.25}$$

Substituting these results into (3.4.21), we can express the periodic function using the Fourier series in the complex form, which is

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}} \tag{3.4.26}$$

Where the coefficient c_n is denoted by

$$\boxed{c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx} \tag{3.4.27}$$

Similarly, if the periodic function $f(x)$ now has period $2L$, we can follow the structure of Equation (3.4.19) and rewrite e^{inx} as $e^{in\pi x/L}$. In this case, the Fourier series becomes

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,dx} \tag{3.4.28}$$

▷ Convergence and the Gibbs Phenomenon(Optional)

Consider the partial sums of a general Fourier trigonometric series $S_N(x)$, which means we take the sum from $n = 0$ to $n = N$ of a function $f(x)$ defined on $[-L, L]$. That is

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \tag{3.4.29}$$

We substitute (3.4.20) into (3.4.29), we have

$$\begin{aligned} S_N(x) &= \frac{1}{2L}\int_{-L}^{L} f(t)\,dt + \sum_{n=1}^{N}\left[\left(\frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt\right)\cos\left(\frac{n\pi x}{L}\right)\right. \\ &\quad \left. + \left(\frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt\right)\sin\left(\frac{n\pi x}{L}\right)\right] \\ &= \frac{1}{L}\int_{-L}^{L}\left[\frac{1}{2} + \sum_{n=1}^{N}\left(\cos\left(\frac{n\pi t}{L}\right)\cos\left(\frac{n\pi x}{L}\right) + \sin\left(\frac{n\pi t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\right)\right] f(t)dt \\ &= \frac{1}{L}\int_{-L}^{L}\left[\frac{1}{2} + \sum_{n=1}^{N}\left(\cos\frac{n\pi(t-x)}{L}\right)\right] f(t)dt \end{aligned}$$

Then we have

$$S_N(x) = \frac{1}{L}\int_{-L}^{L} D_N(t-x)f(t)dt \tag{3.4.30}$$

Where we define

$$D_N(x) = \frac{1}{2} + \sum_{n=1}^{N}\left(\cos\frac{n\pi x}{L}\right) \tag{3.4.31}$$

Then we called the function D_N **Dirichlet kernel**. This representation shows that the partial sum $S_N(x)$ is the convolution of the Dirichlet kernel with the function $f(x)$.
Let $\pi x/L = \phi$, and we multiply $D_N(x)$ by $2\sin(\phi/2)$, we have

$$\begin{aligned} 2\sin\left(\frac{\phi}{2}\right) D_N(x) &= 2\sin\left(\frac{\phi}{2}\right)\left[\frac{1}{2} + \sum_{n=1}^{N}(\cos\, n\phi)\right] \\ &= 2\sin\left(\frac{\phi}{2}\right)\left[\frac{1}{2} + \cos\phi + \cos 2\phi + \cdots\right] \\ &= \sin\left(\frac{\phi}{2}\right) + 2\cos\phi\sin\left(\frac{\phi}{2}\right) + 2\cos 2\phi\sin\left(\frac{\phi}{2}\right) + \cdots \end{aligned} \tag{3.4.32}$$

We need to use trigonometric identities that we learned in high school, that is

$$\sin\left(N\phi+\frac{\phi}{2}\right)-\sin\left(N\phi-\frac{\phi}{2}\right)=\sin\left(N\phi\right)\cos\left(\frac{\phi}{2}\right)+\sin\left(\frac{\phi}{2}\right)\cos\left(N\phi\right)$$
$$-\sin\left(N\phi\right)\cos\left(\frac{\phi}{2}\right)+\sin\left(\frac{\phi}{2}\right)\cos\left(N\phi\right)$$
$$=2\sin\left(\frac{\phi}{2}\right)\cos\left(N\phi\right)$$

Substituting this into (3.4.32), we can write

$$2\sin\left(\frac{\phi}{2}\right)D_N(x)=\sin\left(\frac{\phi}{2}\right)+\left(\sin\frac{3\phi}{2}-\sin\frac{\phi}{2}\right)+\left(\sin\frac{5\phi}{2}-\sin\frac{3\phi}{2}\right)+\cdots$$
$$+\left[\sin\left(N\phi+\frac{\phi}{2}\right)-\sin\left(N\phi-\frac{\phi}{2}\right)\right]$$
$$=\sin\left(N\phi+\frac{\phi}{2}\right)=\sin\left(N+\frac{1}{2}\right)\phi$$

If $2\sin\left(\frac{\phi}{2}\right)\neq 0$, we can obtain

$$D_N(x)=\frac{\sin\left(N+\frac{1}{2}\right)\phi}{2\sin\left(\frac{\phi}{2}\right)}\quad \phi=\frac{\pi x}{L} \tag{3.4.33}$$

In the following, we will use the Dirichlet kernel to explain a notable phenomenon. We know that the more terms we include in the Fourier series, the more closely it approximates the target function. However, if you look carefully, you will Note that near points of discontinuity, the approximation visibly overshoots the actual function value. This overshoot is referred to as an **overshoot error**. As we increase the number of terms, the oscillations become more concentrated near the discontinuity, but the magnitude of the overshoot does not vanish. This persistent behavior is known as the **Gibbs phenomenon**. Ah shown below

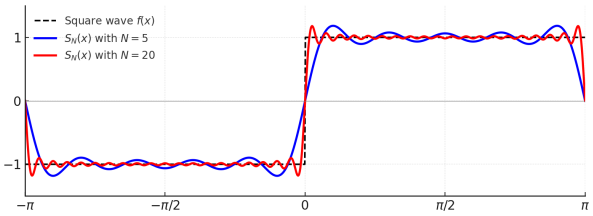

Figure 23: Comparison of the Fourier series approximations of a square wave using $N=5$ (blue) and $N=20$ (red).

Consider a square wave function $f(x)$ of period $2L$, which satisfies

$$f(x)=\begin{cases}-1 & x\in(-L,0)\\ 1 & x\in(0,L)\end{cases}$$

We apply the substitution rule to (3.4.30), let $\zeta = t - x$, then (3.4.30) becomes

$$S_N(x) = \frac{1}{L}\int_{-L-x}^{L-x} D_N(\zeta) f(\zeta + x) d\zeta \tag{3.4.34}$$

Because the function $f(x)$ has the same behavior over a period $2L$, we can shift the interval back to $[-L, L]$, that is

$$S_N(x) = \frac{1}{L}\int_{-L}^{L} D_N(\zeta) f(\zeta + x) d\zeta \tag{3.4.35}$$

We divide the interval into two parts, $[-L, 0]$ and $[0, L]$, then

$$S_N(x) = \frac{1}{L}\int_{-L}^{0} D_N(\zeta) f(\zeta + x) d\zeta + \int_{0}^{L} D_N(\zeta) f(\zeta + x) d\zeta$$

Let the first integral $\zeta \to -\zeta$, then $D_N(\zeta)f(\zeta+x)d\zeta$ becomes $D_N(-\zeta)f(x-\zeta)(-d\zeta)$. We can observe (3.4.31) that the Dirichlet kernel is an even function, so $D_N(-\zeta) = D_N(\zeta)$. We obtain

$$\begin{aligned}
\frac{1}{L}\int_{-L}^{0} D_N(\zeta) f(\zeta + x) d\zeta &= \frac{1}{L}\int_{L}^{0} D_N(-\zeta) f(x - \zeta)(-d\zeta) \\
&= \frac{1}{L}\int_{0}^{L} -D_N(\zeta) f(x - \zeta)(-d\zeta) \\
&= \frac{1}{L}\int_{0}^{L} D_N(\zeta) f(x - \zeta)\, d\zeta
\end{aligned}$$

Substituting this into $S_N(x)$, we can write down

$$S_N(x) = \frac{1}{L}\int_{0}^{L} [f(x - \zeta) + f(x + \zeta)] D_N(\zeta)\, d\zeta \tag{3.4.36}$$

To illustrate the Gibbs phenomenon of this square wave, we can narrow down the scope of analysis by focusing on the interval $x \in (0, L/2)$ and $\zeta \in [0, L]$. This allows us to examine only the amplitude extrema of the oscillations within this region. Due to the symmetry of the periodic function, if we can demonstrate the behavior in this interval, it implies that the same conclusion applies to the entire function.

Then we have three different values of the function $f(x - \zeta) + f(x + \zeta)$. When $f(x - \zeta) + f(x + \zeta) = 2$, it means that

$$x - \zeta > 0 \quad x + \zeta < L$$

Where $\zeta \geq 0$. Since $x \in (0, L/2)$, we can deduce that $x < L - x$, then we have

$$f(x - \zeta) + f(x + \zeta) = 2 \quad 0 \leq \zeta < x \tag{3.4.37}$$

Similarly,

$$\begin{cases}
f(x - \zeta) + f(x + \zeta) = 2 & 0 \leq \zeta < x \\
f(x - \zeta) + f(x + \zeta) = 0 & x < \zeta < L - x \\
f(x - \zeta) + f(x + \zeta) = -2 & L - x < \zeta < L
\end{cases} \tag{3.4.38}$$

Substituting the conditions of (3.4.38) into (3.4.36), we obtain

$$S_N(x) = \frac{2}{L}\int_0^x D_N(\zeta)\,d\zeta - \frac{2}{L}\int_{L-x}^{L} D_N(\zeta)\,d\zeta \tag{3.4.39}$$

Let $u = L - \zeta$, $d\zeta = -du$, and (3.4.39) becomes

$$\begin{aligned} S_N(x) &= \frac{2}{L}\int_0^x D_N(u)\,du - \frac{2}{L}\int_x^0 D_N(L-u)\,(-du) \\ &= \frac{2}{L}\int_0^x D_N(u)\,du - \frac{2}{L}\int_0^x D_N(L-u)\;du \end{aligned} \tag{3.4.40}$$

Since we are focusing on the region $x \in (0, L/2)$, the primary contribution to the partial sum $S_N(x)$ arises from the integration of $D_N(u)$ over the neighborhood of the origin, where the Dirichlet kernel is sharply peaked. The term $D_N(L-u)$, centered near $u = L$, lies far from the singularity and contributes only small oscillations. Owing to the localization property of D_N and its decay away from the origin, the second term becomes asymptotically negligible as $N \to \infty$, for any fixed $x \ll L$. And we have

$$D_N(x) \approx \left.\frac{\sin\left[N\left(\frac{\pi x}{L}\right)\right]}{2\left(\frac{\pi x}{2L}\right)}\right|_{N\to\infty,\,x\to 0} = \frac{\sin\left[N\left(\frac{\pi x}{L}\right)\right]}{\left(\frac{\pi x}{L}\right)} \tag{3.4.41}$$

So (3.4.40) can be written as

$$S_N(x) = \frac{2}{L}\int_0^x \frac{\sin\left[N\left(\frac{\pi u}{L}\right)\right]}{\left(\frac{\pi u}{L}\right)}\,du \tag{3.4.42}$$

Let $\pi u/L = \xi$, then $du = L/\pi \cdot d\xi$, and substitute this into (3.4.42), we have

$$S_N(x) = \frac{2}{L}\int_0^{\pi x/L} \frac{\sin N\xi}{\xi}\,\frac{L}{\pi}\,d\xi = \frac{2}{\pi}\int_0^{\pi x/L} \frac{\sin N\xi}{\xi}\,d\xi \tag{3.4.43}$$

Differentiating both sides with respect to x, we write down

$$\frac{d}{dx}S_N(x) = \frac{2}{\pi}\frac{d}{dx}\int_0^{\pi x/L} \frac{\sin N\xi}{\xi}\,d\xi = \frac{2}{\pi}\frac{\sin N\left(\frac{\pi x}{L}\right)}{\left(\frac{\pi x}{L}\right)}\cdot\frac{\pi}{L} = \frac{2}{L}\frac{\sin N\left(\frac{\pi x}{L}\right)}{\left(\frac{\pi x}{L}\right)} = 0 \tag{3.4.44}$$

When the extremum occurs, $\sin(N\pi x/L) = 0$, so $N\pi x/L = m\pi$, $m \in \mathbb{Z}$. Take $m = 1$, which means that $x = L/N$. Substituting x into (3.4.43), we obtain

$$S_N\left(\frac{L}{N}\right) = \frac{2}{\pi}\int_0^{\pi/N} \frac{\sin N\xi}{\xi}\,d\xi \tag{3.4.45}$$

Let $t = N\xi$, so $d\xi = dt/N$. (3.4.45) becomes

$$\frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{t/N}\,\frac{dt}{N} = \frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{t}\,dt \coloneqq \frac{2}{\pi}\mathrm{Si}(\pi) \tag{3.4.46}$$

So, we can verify that

$$\boxed{\max S_N(x)|_{x\to 0} = \frac{2}{\pi}\mathrm{Si}(\pi) \approx 1.17897} \tag{3.4.47}$$

In mathematics, even if we take a large number of terms to approximate the original function, oscillatory behavior will still appear near the discontinuities. The maximum overshoot caused by this oscillation converges to a value approximately 17.9% higher than the actual function value at the jump.

▷ **Fourier Transform of a Non-periodic Function**

In (3.4.28), we know that

$$f_L(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \quad c_n = \frac{1}{2L}\int_{-L}^{L} f_L(x)e^{-in\pi x/L}\,dx$$

Where $f_L(x)$ is a function of period $2L$. We define the frequency $\omega_n \coloneqq n\pi/L$, then c_n can be considered to be the function of ω_n, that is

$$c(\omega_n) = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\omega_n x}\,dx \tag{3.4.48}$$

As previously mentioned, the coefficient c_n represents the strength of contribution of each trigonometric function with different frequencies or periods. In other words, the function $c(\omega_n)$, defined over frequency ω_n, outputs values that correspond to the strength or contribution at each frequency. Therefore, $c(\omega_n)$ can be regarded as the amplitude spectra of $f_L(x)$ (Felix Lee, 2009).
For periodic functions with a finite period, the frequencies are discrete, and the spacing between them is given by

$$\Delta\omega = \omega_n - \omega_{n-1} = \frac{\pi}{L} \tag{3.4.49}$$

As $L \to \infty$, amplitudes $c(\omega_n)$ will approach a continuous function on ω axis. When $L \to \infty$, we say that $f_L(x) \to f(x)$ is a non-periodic function, that is

$$\lim_{L\to\infty} f_L(x) = f(x) \tag{3.4.50}$$

Furthermore, the spacing between the frequencies becomes

$$\Delta\omega = \frac{\pi}{L}\Big|_{L\to\infty} = d\omega \to 0 \tag{3.4.51}$$

Assume that $f(x)$ exists and the integral

$$\int_{-\infty}^{\infty} |f(x)|dx < M < \infty \tag{3.4.52}$$

Where W is a finite constant. By (3.4.52), we can ensure that this improper integral is convergent and integrable. Then we can rewrite the Fourier series as

$$f(x) = \lim_{L\to\infty} \sum_{n=-\infty}^{\infty} \left(\frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\omega x}\,dx\right) e^{i\omega x}$$

$$= \lim_{L\to\infty} \sum_{n=-\infty}^{\infty} \frac{1}{2L} e^{i\omega x}\left(\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx\right)$$

$$= \lim_{L\to\infty} \sum_{n=-\infty}^{\infty} \frac{1}{2L\Delta\omega} e^{i\omega x}\Delta\omega\left(\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx\right)$$

Substituting $\Delta\omega = \pi/L$, we have

$$\lim_{L\to\infty}\sum_{n=-\infty}^{\infty}\frac{1}{2L\Delta\omega}e^{i\omega x}\Delta\omega\left(\int_{-\infty}^{\infty}f(x)e^{-i\omega x}\,dx\right)=\frac{1}{2\pi}\lim_{\|\Delta\|\to 0}\sum_{n=-\infty}^{\infty}e^{i\omega x}\Delta\omega\left(\int_{-\infty}^{\infty}f(x)e^{-i\omega x}\,dx\right)$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega x}\,d\omega\int_{-\infty}^{\infty}f(x)e^{-i\omega x}\,dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\omega x}\,d\omega\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}\,dx$$

So, we can obtain

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\omega x}\,d\omega\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}\,dx \tag{3.4.53}$$

Then we define the conventional Fourier transform pair as

$$\boxed{\begin{cases}\mathcal{F}[f(x)]=F(\omega):=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty}f(x)e^{-i\omega x}\,dx\\[2ex]\mathcal{F}^{-1}[F(\omega)]=f(x)=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty}F(\omega)e^{i\omega x}\,d\omega\end{cases}} \tag{3.4.54}$$

Here, $f(x)$ is referred to as a function in the **time domain**, which represents the actual shape of the function as it varies with the input variable x. On the other hand, $F(\omega)$ lies in the **frequency domain**, representing the transformation of the original aperiodic function into its frequency components as a function of the continuous frequency variable ω. Since the frequency domain is continuous, we cannot precisely state which specific frequencies constitute the original function because all $\omega \in (-\infty, \infty)$ potentially contribute. However, the value of $F(\omega)$ at each ω tells us the strength (or contribution) of that frequency to the construction of $f(x)$. By examining $F(\omega)$, we can determine which frequencies contribute the most to $f(x)$. For example, in the case of sound waves, this allows us to identify the dominant frequency components of a signal, helping us analyze the pitch or tonal quality of a sound. The following is a graph of the real experimental data.

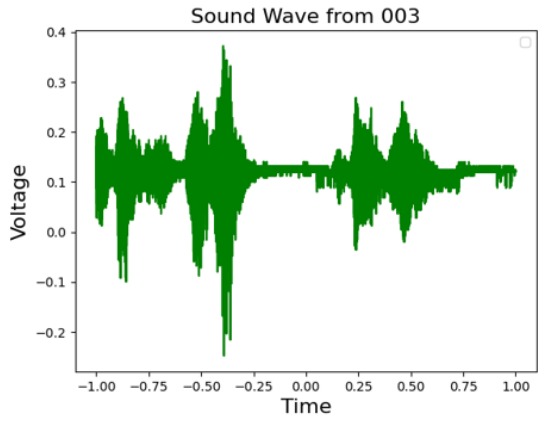

Figure 24: Waveform of the sound signal detected from 003 (Godtone's Wife), converted via oscilloscope

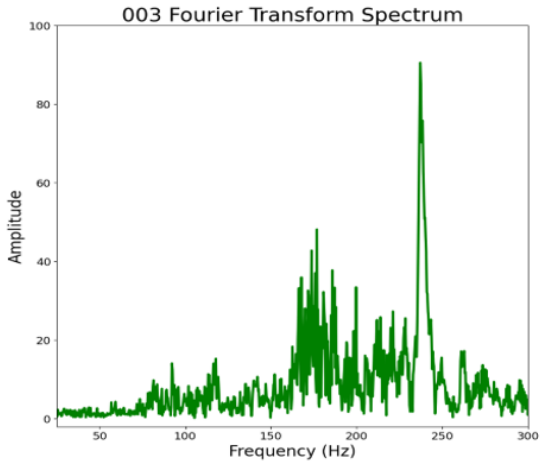

Figure 25: Frequency domain amplitude spectrum of the sound saveform of 003 (Godtone's Wife) after Fourier transform

Consider an odd function $f(x)$, which means that $f(-x) = -f(x)$, and we expand Euler's formula in Equation (3.4.54).

$$\begin{aligned} F(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)[\cos \omega x - i \sin \omega x]\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos \omega x\, dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -if(x) \sin \omega x\, dx \end{aligned} \tag{3.4.55}$$

Since the function $\cos \omega x$ is an even function, the product $f(x) \cos \omega x$ is odd, that is

$$f(-x) \cos(-\omega x) = -f(x) \cos \omega x \tag{3.4.56}$$

Similarly, the product $f(x) \sin \omega x$ is odd since

$$f(-x) \sin(-\omega x) = -f(x)(-\sin \omega x) = f(x) \sin \omega x \tag{3.4.57}$$

Note that the integral of an odd function over a symmetric interval about the origin $x = 0$ is 0, which is

$$\int_{-\infty}^{\infty} f(x) \cos \omega x\, dx = 0 \tag{3.4.58}$$

Conversely, the integral of an even function over a symmetric interval about the origin $x = 0$ is twice the integral over $x > 0$, that is

$$\int_{-\infty}^{\infty} f(x) \sin \omega x\, dx = 2 \int_{0}^{\infty} f(x) \sin \omega x\, dx \tag{3.4.59}$$

Substituting this into (3.4.55), we have

$$F(\omega) = \frac{-2i}{\sqrt{2\pi}} \int_{0}^{\infty} f(x) \sin \omega x\, dx = -i \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin \omega x\, dx \tag{3.4.60}$$

Similarly, because $F(-\omega)$ changes the sign of $\sin \omega x$, we can verify $F(-\omega) = -F(\omega)$, which is an odd function. Then we have

$$\begin{aligned} f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x}\, d\omega &= \frac{2i}{\sqrt{2\pi}} \int_{0}^{\infty} F(\omega) \sin \omega x\, d\omega \\ &= i \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F(\omega) \sin \omega x\, d\omega \end{aligned} \tag{3.4.61}$$

Substituting (3.4.60) into (3.4.61), we obtain

$$f(x) = i \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \left(-i \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin \omega x\, dx \right) \sin \omega x\, d\omega$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \sin \omega x\, d\omega \left(\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin \omega x\, dx \right)$$

Then we can define a pair of **Fourier sine transforms** representing **odd** function $f(x)$, that is

$$\boxed{\begin{cases} \mathcal{F}_s[f(x)] = F_s(\omega) = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^\infty f(x) \sin \omega x \, dx \\ \mathcal{F}_s^{-1}[F_s(\omega)] = f(x) = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^\infty F_s(\omega) \sin \omega x \, d\omega \end{cases}} \tag{3.4.62}$$

We can apply the same way to analyze an even function $f(x)$, we have

$$\begin{cases} f(-x)\cos(-\omega x) = f(x)\cos \omega x & \text{even function} \\ f(-x)\sin(-\omega x) = -f(x)\sin \omega x & \text{odd function} \end{cases}$$

So, the integral

$$\int_{-\infty}^{\infty} f(x)\cos \omega x \, dx = 2\int_0^\infty f(x)\cos \omega x \, dx \tag{3.4.63}$$

Similarly,

$$\int_{-\infty}^{\infty} f(x)\sin \omega x \, dx = 0 \tag{3.4.64}$$

Substituting these into (3.4.55), we can obtain

$$F(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos \omega x \, dx \tag{3.4.65}$$

We can deduce that $F(-\omega) = F(\omega)$, so

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(\omega)\cos \omega x \, d\omega \tag{3.4.66}$$

Then we can define a pair of **Fourier cosine transforms** representing **even** function $f(x)$, that is

$$\boxed{\begin{cases} \mathcal{F}_c[f(x)] = F_c(\omega) = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^\infty f(x) \cos \omega x \, dx \\ \mathcal{F}_c^{-1}[F_c(\omega)] = f(x) = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^\infty F_c(\omega) \cos \omega x \, d\omega \end{cases}} \tag{3.4.67}$$

Chapter 4. Ordinary Differential Equation

This chapter is about learning how to handle the equations of motion of physical systems mathematically. And honestly, it might be the most important chapter you'll come across in college. We are going to learn how to eliminate derivatives in equations to bring out the underlying structure, something you will see over and over again, not just in physics but also in electrical and civil engineering. If you are planning to survive and thrive in NCU Physics, the tools in this chapter are your best friends. So, please pay close attention; this one is worth it.

4.1 First-Order ODEs

▷ Homogeneous Problems and Separable Equations

An ordinary differential equation(ODE) is an equation with several forms. For example

$$4xy'' + x^2y' + 3xy = 3e^{3x}\tan x \tag{4.1.1}$$

In physics, we are interested in the original relationship between each variable so that we can illustrate and predict the behavior of the system. In this book, I'll focus on the differential equation in your mechanics courses.
We can simply divide the equation into two groups; one is the **Homogeneous** ODE, and the other is **Nonhomogeneous.** We can define another form of ODE, which can be denoted by

$$L[y] = P(x) \tag{4.1.2}$$

We call the notation L **differentiating operator**. The meaning of the operator is to be used to encapsulate the entire differentiation process and act directly on the target function y. For example, if the operator $L \coloneqq (d^2/dx^2 + d/dx + \omega_0)$, then the ODE is

$$L[y] = (d^2/dx^2 + d/dx + \omega_0)y = y'' + y' + \omega_0 y = P(x) \tag{4.1.3}$$

In mathematics, a homogeneous ODE of y is defined as

$$L[y] = 0 \tag{4.1.4}$$

Which means the right-hand side function $P(x) = 0$. Consider an equation

$$y = \int f(x)\ dx \tag{4.1.5}$$

Differentiating both sides, namely

$$y' = \frac{dy}{dx} = f(x) \tag{4.1.6}$$

So it becomes

$$dy = f(x)dx \tag{4.1.7}$$

In this procedure, we rewrite the equation so that there are only y terms on one side of the equation and only x terms on the other side. It's called the **separable** equation, and the skill is called **separating the variables.**
There's a differential equation denoted by

$$y' + P(x)y = 0 \tag{4.1.8}$$

We put the second term of the equation into the right side and rewrite the ODE as

$$\frac{dy}{dx} = -P(x)y \tag{4.1.9}$$

Multiplying both sides by dx/y, (4.1.9) becomes

$$\begin{aligned} \frac{dy}{y} = -P(x)\ dx \Rightarrow \int \frac{dy}{y} &= \int -P(x)\ dx \\ \Rightarrow \ln y &= -\int P(x)\ dx + const \\ \Rightarrow y &= C\exp\left(-\int P(x)\ dx\right) \end{aligned} \tag{4.1.10}$$

So, the general solution to the homogeneous ODE is

$$\boxed{y = C\exp\left(-\int P(x)\ dx\right)} \tag{4.1.11}$$

Example. *Consider a rocket moving vertically upward in space (neglecting air resistance and gravity). Let $m(t)$ be the mass of the rocket at time t, and assume the rocket expels gas at a constant rate, so $m(t) = m_0 - kt$, where $k > 0$ is a constant. According to the rocket equation, the velocity $v(t)$ satisfies the equation*

$$m(t)\frac{dv}{dt} + v\frac{dm}{dt} = 0$$

Solve for $v(t)$ as a function of t, and find the constant using the initial condition $v(0) = v_0$.

By the function $m(t) = m_0 - kt$, so that

$$\frac{dm(t)}{dt} = -k$$

Substituting this to the original ODE, we can get

$$(m_0 - kt)\frac{dv}{dt} - kv = 0 \Rightarrow \frac{dv}{v} = \frac{k}{m_0 - kt}dt$$

Let $m_0 - kt = u,\ du = -k\ dt$, and substitute this into the equation and integrate both sides,

$$\int \frac{dv}{v} = -\int \frac{1}{u}dt$$

So we obtain

$$\ln v = -\ln u + const \Rightarrow v = \frac{C}{u} = \frac{C}{m_0 - kt}$$

By the initial condition, $v(0) = v_0$, the constant $C = m_0 v_0$. So, the final answer is

$$v(t) = \frac{m_0 v_0}{m_0 - kt}$$

▷ **Integrating Factor**

Consider an ODE $P(x_i)dx + Q(x_j)dy = 0$, we can reduce the equation to the form of

$$\mathbf{F}(P(x_i), Q(x_j)) \cdot d\mathbf{r} = 0 \tag{4.1.12}$$

Which implies the meaning of vector field in the ODE. So the geometric meaning of an ODE is the projection of a vector field $\mathbf{F}$ onto the tangent vector $d\mathbf{r}$ to derive the equipotential surfaces while the ODE can be solved.

If we can find a scalar function ϕ which satisfies that

$$d\phi = Pdx + Qdy \tag{4.1.13}$$

or the equation

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{4.1.14}$$

Then the original ODE $Pdx + Qdy = 0$ is called an **exact differential equation**. In other words, the vector field $\mathbf{F} = (P, Q)$ in the exact differential equation can be written as $\mathbf{F} = \boldsymbol{\nabla}\phi$ where ϕ is the potential function of $\mathbf{F}$, which is a conservative vector field then we can solve the ODE naturally because the line integral

$$\int_A^x \mathbf{F} \cdot d\mathbf{r} = \int_A^x \boldsymbol{\nabla}\phi \cdot d\mathbf{r} = \phi(x) - \phi(A) \tag{4.1.15}$$

where we often define the constant term $-\phi(A)$ to be C. But if the differential equation $Pdx + Qdy = 0$ is not an exact ODE, we cannot integrate it directly. However, if there exists a function μ such that $\mu Pdx + \mu Qdy = 0$ be an exact ODE, then the function μ is called **integrating factor**.
That is to say, as long as the integrating factor exists, the ODE can be solved or called "integrable." Let's verify the facts in detail. If the differential equation $\mathbf{F} \cdot d\mathbf{r} = 0$ is integrable, then the integrating factor can make it be

$$\mu\mathbf{F} = \boldsymbol{\nabla}\phi \tag{4.1.16}$$

We know that the curl of the gradient is 0, so

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = \boldsymbol{\nabla} \times (\mu\mathbf{F}) = 0 \tag{4.1.17}$$

(4.1.17) can be expanded to

$$\boldsymbol{\nabla} \times (\mu\mathbf{F}) = \mu\boldsymbol{\nabla} \times \mathbf{F} + \boldsymbol{\nabla}\mu \times \mathbf{F} = 0$$
$$\Rightarrow \boldsymbol{\nabla} \times \mathbf{F} = -\frac{\boldsymbol{\nabla}\mu \times \mathbf{F}}{\mu}, \mu \neq 0$$

After that, the dot product between the original vector field can be denoted by

$$\mathbf{F} \cdot \boldsymbol{\nabla} \times \mathbf{F} = -\frac{1}{\mu} \left[\mathbf{F} \cdot (\boldsymbol{\nabla}\mu \times \mathbf{F}) \right] = 0 \tag{4.1.18}$$

So, when (4.1.18) holds, it shows that $\exists \mu$ such that the differential equation $Pdx + Qdy = 0,\ \mathbf{F} = (P, Q)$ is integrable and solvable.
Now I am going to show an example of the integrating factor by solving the first-order linear ODE, which is the following "main character." Consider an ODE

$$y' + P(x)y = Q(x) \tag{4.1.19}$$

We need to separate the variable that

$$dy + P(x)ydx = Q(x)dx$$

We must deal with the homogeneous term $dy + P(x)ydx = 0$ first because we need to determine its total derivative to introduce the integrating factor. Which means that

$$\mu dy + \mu P(x)ydx = d(\mu y) = 0 \tag{4.1.20}$$

So we can get y by solving $dy + P(x)ydx = 0$, that is,

$$\frac{dy}{y} = -P(x)dx \Rightarrow y = C \exp\left(-\int P(x)dx \right)$$

Because $d(\mu y) = d\left(\mu \times C \exp\left(-\int P(x)dx \right) \right) = 0$, it is obvious that we can take

$$\mu = \exp\left(\int P(x)dx \right) \tag{4.1.21}$$

back to the original ODE, we multiplied a factor μ in the whole equation, so

$$\mu dy + \mu P(x)ydx = d(\mu y) = \mu Q(x)dx$$

So that

$$\int d(\mu y) = \int \mu Q(x)dx \Rightarrow \mu y = \int \mu Q(x)dx + C'$$

We can finally get the familiar solution

$$\boxed{y = \exp\left(-\int P(x)dx \right) \left(\int \left[\exp\left(\int P(x)dx \right) Q(x) \right] dx + C' \right)} \tag{4.1.22}$$

That is why we should solve the homogeneous term of an ODE, and we need to put it back into the original equation to get the final answer. The point is **INTEGRABLE**.

Example. *Consider a radioactive decay chain: Ra-226 decays into Rn-222, and Rn-222 itself further decays into other elements. Let $N_{Ra}(t)$ and $N_{Rn}(t)$ denote the number of Ra-226 and Rn-222 atoms at time t, respectively.*
Assume that Ra-226 decays at a constant rate λ_1, Rn-222 decays at a rate λ_2, Initially, $N_{Ra}(t) \approx N_0$ (since Ra-226 decays very slowly compared to Rn-222, its number remains nearly constant over the timescale considered). Solve for $N_{Rn}(t)$ with initial condition $N_{Rn}(0) = 0$.

The key to this problem lies in understanding the physical meaning behind the decay equation. Before solving for the function $N_{Rn}(t)$, we must first consider the sequence of decay. As R_a decays into R_n, and R_n further decays into another atom, the quantity $N_{Rn}(t)$ is simultaneously affected by both the decay of R_a (which produces R_n) and the decay of R_n itself (which reduces the amount of R_n). In other words, $N_{Rn}(t)$ is governed by a balance between production and loss. The differential equation must therefore include a source term from R_a and a sink term from the decay of R_n.

Let's start with the $N_{Ra}(t)$, the decay function is shown below

$$\frac{dN_{Ra}(t)}{dt} = -\lambda_1 N_{Ra}(t)$$

The solution is

$$N_{Ra}(t) = N_0 e^{-\lambda_1 t}$$

Since the initial condition $N_{Ra}(0) = N_0$. In the same time, the decay function of $N_{Rn}(t)$ is

$$\frac{dN_{Rn}(t)}{dt} = \lambda_1 N_{Ra}(t) - \lambda_2 N_{R_n}(t)$$

Note that we must be careful to put $\lambda_1 N_{Ra}(t)$. So it becomes a nonhomogeneous ODE. The integrating factor of the equation is

$$\mu = e^{\lambda_2 t}$$

multiply the factor at both sides, and substitute $N_{Ra}(t) = N_0 e^{-\lambda_1 t}$ in the equation, we can solve that

$$\begin{aligned}\int d(N_{Rn} e^{\lambda_2 t}) &= \int \lambda_1 N_0 e^{-\lambda_1 t} \cdot e^{\lambda_2 t}\, dt \\ &= \lambda_1 N_0 \int e^{(\lambda_2 - \lambda_1)t}\, dt \\ &= \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} e^{(\lambda_2 - \lambda_1)t} + C\end{aligned}$$

and we can calculate that

$$N_{Rn}(t) = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + C e^{-\lambda_2 t}$$

At time $t = 0$, $N_{Rn}(0) = 0$, so the constant $C = \frac{-\lambda_1 N_0}{\lambda_2 - \lambda_1}$, substituting C into the result, so the final solution to the N_{Rn} is

$$N_{Rn} = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} \left(e^{-\lambda_1 t} - e^{\lambda_2 t}\right)$$

▷ The Bernoulli Equation

The differential equation that satisfies

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{4.1.23}$$

Where P, Q are functions of x, is called **Bernoulli equation.** To solve this equation, we need to use some methods to make it a common form of nonhomogeneous ODE $y' + Py = Q$. So, define the new variable

$$u = y^{1-n},\ u' = (1-n)y^{-n}y' \tag{4.1.24}$$

Note that you need to apply the chain rule because the function y depends on x. We multiply both sides of the Bernoulli equation by $(1-n)y^{-n}$, than the original problem becomes

$$(1-n)y^{-n}y' + (1-n)y^{1-n}P(x) = (1-n)Q(x) \tag{4.1.25}$$

This result is really beautiful and skillful because we can thus change the ODE for y into the ODE for u, and the right-side y^n disappears. That is

$$u' + P(x)u = (1-n)Q(x) \tag{4.1.26}$$

Since the result is the same as (4.1.11), we must substitute it back to $u = y^{1-n}$. So, the final solution is

$$u = y^{1-n} = \exp\left(-\int P(x)dx\right)\left(\int\left[\exp\left(\int P(x)dx\right)(1-n)Q(x)\right]dx + C'\right) \tag{4.1.27}$$

Example. *Consider a horizontal pipe through which an incompressible fluid is flowing with constant velocity v_0. Due to nonlinear friction (such as turbulent effects and viscous resistance), the pressure $p(x)$ along the length of the pipe is not constant, and its spatial rate of change satisfies the differential equation:*

$$\frac{dp}{dx} = -kp^2$$

where $k > 0$ is a constant describing the resistance per unit length, and $p(x)$ is the pressure at position x. The pressure at the inlet of the pipe $x = 0$ is p_0. Solve the equation using the Bernoulli method and find the particular solution for $p(x)$.

To solve the problem, we need to let $u = p^{1-2} = p^{-1}$. The derivative of u is

$$u' = -p^{-2}\frac{dp}{dx}$$

Multiply both sides of the ODE by $-p^{-2}$, and substitute this in the original ODE, and it becomes

$$-p^{-2}\frac{dp}{dx} = -p^{-2} \times (-kp^2) \Rightarrow du = k\ dx$$

Integrate both sides, and we can get the solution to u

$$u = \frac{1}{p(x)} = \int k\ dx = kx + C$$

So, the pressure at position x is

$$p(x) = \frac{1}{kx + C}$$

We've known that $p(0) = p_0$, substitute it into $p(x)$, so the constant $C = 1/p_0$. The final solution is

$$p(x) = \frac{1}{kx + \frac{1}{p_0}} = \frac{p_0}{1 + kp_0x}$$

This result shows that the pressure decreases nonlinearly along the pipe due to nonlinear friction and becomes asymptotically small as $x \to \infty$.

▷ Homogeneous Equation

A homogeneous equation in polynomials is slightly different from a homogeneous differential equation in the context of mathematics. A polynomial equation is said to be **homogeneous** of degree n if every term in the polynomial has the same total degree. It can be written as

$$\sum_{j=0} a_j x^j y^{n-j} = 0 \tag{4.1.28}$$

In such equations, since all terms have the same degree n, we can factor the entire expression by either x^n or y^n, allowing us to rewrite the equation as a function of y/x or x/y. For instance:

$$\frac{1}{x^n} \sum_j x^j y^{n-j} = \sum_j \left(\frac{y}{x}\right)^{n-j} := f\left(\frac{y}{x}\right) \tag{4.1.29}$$

In general, as long as an expression can be written in the form $x^n f(y/x)$, it is called a **homogeneous** expression.
To simplify the equation, we may introduce the substitution

$$u := \frac{y}{x} \Rightarrow y = xu \tag{4.1.30}$$

With this substitution, the function becomes $f(u)$, and the differential equation can now be simplified and solved more directly.

Example. *Solve the ODE* $x^2dy + (y^2 - xy)dx = 0$

First, divide the equation by x^2, and it becomes

$$dy + \left[\left(\frac{y}{x}\right)^2 - \frac{y}{x}\right] dx = 0$$

let $y/x = u$, then $y = xu$. By the rule of differentiation, we have

$$dy = d(xu) = xdu + udx$$

So, the equation becomes

$$xdu + udx + (u^2 - u)dx = xdu + u^2dx = 0$$

We change the original equation into the separable form by substitution, so

$$-\frac{1}{u^2}du = \frac{1}{x}dx \Rightarrow \int -\frac{1}{u^2}du = \int \frac{1}{x}dx \Rightarrow \frac{1}{u} = \ln(x) + C$$

Substituting back $y/x = u$ back into the expression, we obtain the general solution to the original ODE

$$x = y(\ln x + C)$$

This example illustrates how a seemingly complicated nonlinear first-order equation can be reduced to a separable form by identifying the homogeneous structure and using an appropriate substitution.

4.2 Second-Order ODEs

▷ Homogeneous Problems

After calculating the first-order ODE, I'm going to show the second-order ODE next. We just begin with the homogeneous problem and focus on the method of solving it. The homogeneous second-order ODE means that the right-hand side (RHS) is 0, as shown in (4.2.1).
Consider the homogeneous equation

$$y'' + ay' + by = 0 \tag{4.2.1}$$

We can introduce a new operator D, it's defined as

$$D \coloneqq \frac{d}{dx};\ D^2 = \frac{d^2}{dx^2} \tag{4.2.2}$$

Then substitute this into the original equation to be

$$(D^2 + aD + b)y = 0 \tag{4.2.3}$$

Factor the polynomial, and we get

$$(D - \alpha)(D - \beta)y = 0 \tag{4.2.4}$$

Where $\alpha\beta = b,\ -(\alpha + \beta) = a$. There are two different situations about the relationship between the constant α and β.
When $\alpha \neq \beta$, we let $u = (D - \beta)y$ and substitute it into (4.2.4), it becomes

$$(D - \alpha)u = u' - \alpha u = 0 \tag{4.2.5}$$

Note that we have transformed the original second-order ODE into a first-order one. We will discuss this technique in more detail in the next section. So, separate the variable

$$\frac{du}{u} = \alpha dx \Rightarrow u = C_1 e^{\alpha x}$$

Put the result back to $u = (D - \beta)y$, it becomes a first-order inhomogeneous ODE, that is

$$y' - \beta y = Ce^{\alpha x} \tag{4.2.6}$$

The integrating factor $\mu = e^{-\beta x}$, then

$$d\left(e^{-\beta x}y\right) = Ce^{(\alpha-\beta)x}dx \Rightarrow e^{-\beta x}y = \frac{C}{\alpha - \beta}e^{(\alpha-\beta)x} + C^{(1)}$$

So the final solution to y is

$$\boxed{y = C_1 e^{\alpha x} + C_2 e^{\beta x},\ \alpha \neq \beta} \tag{4.2.7}$$

Where $C_1 = \frac{C}{\alpha - \beta}$, $C_2 = C^{(1)}$.
The other situation is $\alpha = \beta$. Let $(D - \alpha)y = u$, and we've had the solution to u in (4.2.6), then substituting u into $(D - \alpha)y = u$, we get

$$y' - \alpha y = Ce^{\alpha x} \tag{4.2.8}$$

The integrating factor $\mu = e^{-\alpha x}$, then

$$d\left(e^{-\alpha x} y\right) = Ce^{(\alpha - \alpha)x} dx = Cdx \Rightarrow e^{-\alpha x} y = Cx + C^{(1)}$$

So the final solution to y is

$$\boxed{y = (C_1 x + C_2)e^{\alpha x},\ \alpha = \beta} \tag{4.2.9}$$

Where $C_1 = C,\ C_2 = C^{(1)}$. These solutions to the ODEs are called the **general solution**, the complementary solution(y_c) to inhomogeneous ODEs, and sometimes the homogeneous solution(y_h).

▷ **Nonhomogeneous Problems and The Operator Descent Method**

Another form of second-order ODEs is the nonhomogeneous one. We can use similar ways to solve homogeneous problems as you did in the last part. Consider a second-other ODE shown below, with RHS not 0.

$$y'' + ay' + by = f(x) \tag{4.2.10}$$

We define the derivative notation $D \coloneqq \frac{d}{dx},\ D^2 = \frac{d^2}{dx^2}$, so (3.1) can be written as

$$(D - \alpha)(D - \beta)y = f(x) \tag{4.2.11}$$

where $\alpha\beta = b,\ -(\alpha + \beta) = a$. Next, let the arbitrary one of the $(D - \cdot)y$ be a new variable u, for example

$$(D - \beta)y \coloneqq u \tag{4.2.12}$$

Substituting it into (4.2.11), it becomes

$$(D - \alpha)u = u' - \alpha u = f(x) \tag{4.2.13}$$

The original problem becomes a first-order nonhomogeneous ODE problem, such a method is called **the operator descent method.**
By Section 2, we have known that the integrating factor

$$\mu = e^{-\alpha x} \Rightarrow d(e^{-\alpha x} u) = e^{-\alpha x} f(x) dx$$

Integrate both sides, and we can get

$$e^{-\alpha x} u = \int e^{-\alpha x} f(x) dx + C \Rightarrow u = e^{\alpha x} \int e^{-\alpha x} f(x) dx + Ce^{\alpha x}$$

Note that we haven't done it yet; this is just a solution to the new variable u, but what we want is y. So we need to put it back to (4.2.12), that is

$$(D-\beta)y = u = e^{\alpha x}\int e^{-\alpha x}f(x)dx + Ce^{\alpha x} \tag{4.2.14}$$

We can find that it becomes a first-order problem again, which is

$$y' - \beta y = e^{\alpha x}\int e^{-\alpha x}f(x)dx + Ce^{\alpha x}$$

The integrating factor of this is

$$\mu = e^{-\beta x} \Rightarrow d(e^{-\beta x}y) = e^{(\alpha-\beta)x}\int e^{-\alpha x}f(x)dx + Ce^{(\alpha-\beta)x}$$

Use the same way to get y

$$\begin{aligned} e^{-\beta x}y &= \int \left[e^{(\alpha-\beta)x}\int e^{-\alpha x}f(x)dx + Ce^{(\alpha-\beta)x}\right]dx \\ \Rightarrow y &= e^{\beta x}\int \left[e^{(\alpha-\beta)x}\int e^{-\alpha x}f(x)dx\right]dx + C_1e^{\alpha x} + C_2e^{\beta x} \end{aligned} \tag{4.2.15}$$

Where $C_1 = C/(\alpha-\beta)$ and C_2 is the constant of the integral of RHS. Note the different terms in (4.2.15), the first term of the right is called the **particular solution** y_p, and the result of the integral depends on the form of $f(x)$ in the origin, so it's "particular"; the second and third term of the right is called **complementary solution** y_c of the ODE, which is the general solution of the homogeneous equation. So

$$\begin{cases} y_p = e^{\beta x}\int \left[e^{(\alpha-\beta)x}\int e^{-\alpha x}f(x)dx\right]dx \\ y_c = C_1e^{\alpha x} + C_2e^{\beta x} \end{cases} \tag{4.2.16}$$

In this method, we no longer need to memorize the particular solution to different kinds of ODEs, we just need to solve the first-order problem twice. In this way, we won't be afraid of any other complicated RHS, and as long as it is integrable, we can undoubtedly use the same method. Unlike the method of undetermined coefficients, which requires case-by-case memory, this unified structure emphasizes transparency over memorization.

4.3 Generalization to Arbitrary Right-Hand Side Functions

We've learned the unified method for solving the ODEs, so I will deal with some of the classical and difficult problems of ODEs. For example, $f(x) = \sin ax$, $f(x) = e^{cx}$, $f(x) = \ln x$, and so on. Remember, **you don't need to memorize any particular solution in this book.**

▷ **The Exponential Term**

Problem. $y'' - (\alpha+\beta)y' + \alpha\beta y = ae^{\gamma x}$

This question can be written as

$$(D-\alpha)(D-\beta)y = ae^{\gamma x}$$

Let $u = (D-\beta)y$, then we need to solve the first ODE

$$u' - \alpha u = ae^{\gamma x}$$

The integrating factor of it is

$$\mu = e^{-\alpha x} \Rightarrow d(e^{-\alpha x}u) = ae^{(\gamma-\alpha)x}dx$$
$$\Rightarrow u = e^{\alpha x}\int ae^{(\gamma-\alpha)x}dx + Ce^{\alpha x}$$

substituting it into $u = (D-\beta)y$, so that

$$\mu = e^{-\beta x} \Rightarrow d(e^{-\beta x}y) = \left[e^{(\alpha-\beta)x}\int ae^{(\gamma-\alpha)x}dx + Ce^{(\alpha-\beta)x}\right]dx$$

$$\Rightarrow y = e^{\beta x}\int\left[e^{(\alpha-\beta)x}\int ae^{(\gamma-\alpha)x}dx\right]dx + e^{\beta x}\int Ce^{(\alpha-\beta)x}dx + C^{(1)}e^{\beta x}$$

So we have the solution

$$y = e^{\beta x}\int\left[e^{(\alpha-\beta)x}\int ae^{(\gamma-\alpha)x}dx\right]dx + e^{\beta x}\int Ce^{(\alpha-\beta)x}dx + C^{(1)}e^{\beta x} \tag{4.3.1}$$

Note that we can't deal with the term of the integral of exponential, once the parameter $\gamma = \alpha$ or $\alpha = \beta$, the exponential becomes 1, and the result of the integral will be a linear or a quadratic term. There are three different situations of the parameters.

The first is $\alpha \neq \beta \neq \gamma$, thus the exponential terms exist and we can integrate it directly. So

$$\begin{aligned}
y &= e^{\beta x}\int\left[e^{(\alpha-\beta)x}\int ae^{(\gamma-\alpha)x}dx\right]dx + e^{\beta x}\int Ce^{(\alpha-\beta)x}dx + C^{(1)}e^{\beta x} \\
&= e^{\beta x}\int\left[e^{(\alpha-\beta)x}\left(\frac{a}{\gamma-\alpha}e^{(\gamma-\alpha)} + C^{(2)}\right)\right]dx + e^{\beta x}\left(\frac{C}{\alpha-\beta}e^{(\alpha-\beta)x} + C^{(3)}\right) + C^{(1)}e^{\beta x} \\
&= e^{\beta x}\int\left[\frac{a}{\gamma-\alpha}e^{(\gamma-\beta)x} + C^{(2)}e^{(\alpha-\beta)x}\right]dx + \frac{C}{\alpha-\beta}e^{\alpha x} + (C^{(1)}+C^{(3)})e^{\beta x} \\
&= \frac{a}{(\gamma-\alpha)(\gamma-\beta)}e^{\gamma x} + \frac{C^{(2)}}{\alpha-\beta}e^{\alpha x} + C^{(4)}e^{\beta x} + \frac{C}{\alpha-\beta}e^{\alpha x} + (C^{(1)}+C^{(3)})e^{\beta x}
\end{aligned}$$

So the final solution is

$$\boxed{y = C_3e^{\gamma x} + C_1e^{\alpha x} + C_2e^{\beta x}} \tag{4.3.2}$$

where

$$C_1 = \frac{C^{(2)}+C}{\alpha-\beta};\quad C_2 = (C^{(1)}+C^{(3)}+C^{(4)});\quad C_3 = \frac{a}{(\gamma-\alpha)(\gamma-\beta)}$$

Note that the coefficients $C, C^{(1)}, C^{(2)}, \cdots$ and their redefinition are not important; I will skip the explanation in the following analysis. Please forgive me :)

The second is $\alpha = \gamma \neq \beta$, the $e^{(\gamma-\alpha)x}$ will become 1. So

$$\begin{aligned} y &= e^{\beta x}\int\left[e^{(\alpha-\beta)x}\int ae^{(\gamma-\alpha)x}dx\right]dx + e^{\beta x}\int Ce^{(\alpha-\beta)x}dx + C^{(1)}e^{\beta x} \\ &= e^{\beta x}\int\left[e^{(\alpha-\beta)x}\int adx\right]dx + e^{\beta x}\left(\frac{C}{\alpha-\beta}e^{(\alpha-\beta)x} + C^{(2)}\right) + C^{(1)}e^{\beta x} \\ &= e^{\beta x}\int\left(axe^{(\alpha-\beta)x} + C^{(3)}e^{(\alpha-\beta)x}\right)dx + \frac{C}{\alpha-\beta}e^{\alpha x} + (C^{(1)} + C^{(1)})e^{\beta x} \\ &= e^{\beta x}\int axe^{(\alpha-\beta)x} + \left(\frac{C^{(3)}+C}{\alpha-\beta}\right)e^{\alpha x} + (C^{(1)} + C^{(2)} + C^{(4)})e^{\beta x} \end{aligned}$$

The left integral $\int axe^{(\alpha-\beta)x}$ must use integral by parts, I have an additional discussion about it; it is

$$\begin{aligned} \int axe^{(\alpha-\beta)x} &= \frac{a}{\alpha-\beta}xe^{(\alpha-\beta)x} - \frac{a}{\alpha-\beta}\int e^{(\alpha-\beta)x}dx \\ &= \frac{a}{\alpha-\beta}xe^{(\alpha-\beta)x} - \frac{a}{(\alpha-\beta)^2}e^{(\alpha-\beta)x} + C^{(5)} \end{aligned}$$

substituting it into the original equation, then

$$\boxed{y = C_3xe^{\alpha x} + C_1e^{\alpha x} + C_2e^{\beta x} = (Ax+B)e^{\alpha x} + Ce^{\beta x}} \tag{4.3.3}$$

The last situation is $\alpha = \beta = \gamma$, the exponential $e^{(\gamma-\alpha)x}$, $e^{(\gamma-\beta)x}$, $e^{(\alpha-\beta)x}$ become 1, so

$$\begin{aligned} y &= e^{\beta x}\int\left[e^{(\alpha-\beta)x}\int ae^{(\gamma-\alpha)x}dx\right]dx + e^{\beta x}\int Ce^{(\alpha-\beta)x}dx + C^{(1)}e^{\beta x} \\ &= e^{\beta x}\int\left(\int adx\right)dx + e^{\beta x}\int Cdx + C^{(1)}e^{\beta x} \\ &= e^{\beta x}\int(ax + C^{(2)})dx + e^{\beta x}(Cx + C^{(3)}) + C^{(1)}e^{\beta x} \\ &= e^{\beta x}(ax^2 + C^{(2)}x + C^{(4)}) + e^{\beta x}(Cx + C^{(3)}) + C^{(1)}e^{\beta x} \end{aligned}$$

factor out the common term $e^{\beta x}$, the final answer is

$$\boxed{y = e^{\beta x}(Ax^2 + Bx + C)} \tag{4.3.4}$$

▷ **The Trigonometric Term**

Problem. $y'' - (\alpha+\beta)y' + \alpha\beta y = af(bx)$, *where* $f(x)$ *is a trigonometric function*

In such problems, I will discuss the method in two different situations, where $f(x) = \sin x$, $f(x) = \cos x$. But I will only show the $\sin x$ one because they both have the same logic.

The first one, $f(x) = \sin x$, the problem becomes $y'' - (\alpha + \beta)y' + \alpha\beta y = a\sin(bx)$, it can be written as

$$(D - \alpha)(D - \beta)y = a\sin(bx) \Rightarrow u' - \beta u = a\sin(bx)$$

Where $u = (D - \alpha)y$. The integrating factor is

$$\mu = e^{-\beta x} \Rightarrow d(e^{-\beta x}u) = e^{-\beta x}a\sin(bx)dx$$

$$\Rightarrow u = ae^{\beta x}\int e^{-\beta x}\sin(bx)\,dx + Ce^{\beta x}$$

To solve the integral, we need to use integral by parts twice, that is

$$\int e^{-\beta x}\sin(bx)\,dx = \frac{-1}{\beta}e^{-\beta x}\sin(bx) + \frac{b}{\beta}\int e^{-\beta x}\cos(bx)\,dx$$

$$= \frac{-1}{\beta}e^{-\beta x}\sin(bx) - \frac{b}{\beta^2}e^{-\beta x}\cos(bx) - \frac{b^2}{\beta^2}\int e^{-\beta x}\sin(bx)\,dx$$

so the integral becomes

$$\int e^{-\beta x}\sin(bx)\,dx = -\frac{\beta}{\beta^2 + b^2}e^{-\beta x}\sin(bx) - \frac{b}{\beta^2 + b^2}e^{-\beta x}\cos(bx) + C \tag{4.3.5}$$

In the same way, we can calculate that

$$\int e^{-\beta x}\cos(bx)\,dx = \frac{b}{\beta^2 + b^2}e^{-\beta x}\sin(bx) - \frac{\beta}{\beta^2 + b^2}e^{-\beta x}\cos(bx) + C \tag{4.3.6}$$

Substituting the result to u, we can now know that

$$u = -\frac{a\beta}{\beta^2 + b^2}\sin(bx) - \frac{ab}{\beta^2 + b^2}\cos(bx) + Ce^{\beta x}$$

By the equation $u = (D - \alpha)y$, the factor of this ODE is

$$\mu = e^{-\alpha x} \Rightarrow y = e^{\alpha x}\int\left[-\frac{a\beta}{\beta^2 + b^2}e^{-\alpha x}\sin(bx) - \frac{ab}{\beta^2 + b^2}e^{-\alpha x}\cos(bx) + Ce^{(\beta-\alpha)x}\right]dx$$

So, the unresolved solution of the ODE is

$$y = \frac{-a\beta e^{\alpha x}}{\beta^2 + b^2}\int e^{-\alpha x}\sin(bx)\,dx - \frac{abe^{\alpha x}}{\beta^2 + b^2}\int e^{-\alpha x}\cos(bx)\,dx + C^{(1)}e^{\alpha x} + \frac{C}{\beta - \alpha}e^{\beta x}$$

$$= \left(\frac{a(\alpha\beta - b^2)}{(\beta^2 + b^2)(\alpha^2 + b^2)}\right)\sin(bx) + \left(\frac{ab(\alpha + \beta)}{(\beta^2 + b^2)(\alpha^2 + b^2)}\right)\cos(bx)$$

$$+ C^{(1)}e^{\alpha x} + \frac{C}{\beta - \alpha}e^{\beta x} \tag{4.3.7}$$

Both integral terms are already solved in (4.3.5) and (4.3.6), so I will directly use the result after calculating to save time. Note that I will skip the following integral like that. The final solution to the second-order trigonometric ODE $y''-(\alpha+\beta)y'+\alpha\beta y = a\sin(bx)$ is

$$\boxed{y = A\sin(bx) + B\cos(bx) + C_1e^{\alpha x} + C_2e^{\beta x}} \tag{4.3.8}$$

Because the two integrals in (4.3.7) will produce a sin term and a cos term, it is obvious to get the two forms of the trigonometric function as the particular solution y_p to the ODE. We use the same logic to solve the second situation, which is $y''-(\alpha+\beta)y'+\alpha\beta y = a\cos(bx)$. The process and the solution to this ODE are really similar to the first one, so the final solution can also be written as (4.3.8).

Let us explain why the particular solution to a second-order linear ODE with $f(x) = \sin x,\ \cos x$ is always the form $y_p = A\sin x + B\cos x$. This is because the set $\{\sin x, \cos x\}$ is closed under differentiation, which is

$$\begin{cases} \dfrac{d}{dx}\sin x = \cos x,\ \dfrac{d^2}{dx^2}\sin x = -\sin x \\ \frac{d}{dx}\cos x = -\sin x,\ \frac{d^2}{dx^2}\cos x = -\cos x \end{cases}$$

So, when we apply the differential operator $(D-\alpha)(D-\beta)y$ to $A\sin x + B\cos x$, the result is still a linear combination of $\sin x$ and $\cos x$, which matches the right-hand side form $f(x)$. Therefore, we can always expect that a particular solution will lie in the same function space as the RHS, in this case, the space spanned by $\{\sin x, \cos x\}$. This property is called **closure under differentiation**.

In addition, we can use complex analysis to solve the problem. Since Euler's formula

$$e^{iax} = \cos(ax) + i\sin(ax) \tag{4.3.9}$$

we can change the form of any function of $\sin x$ and $\cos x$ into Euler's formula. Problem 2 can be rewritten as

$$(D-\alpha)(D-\beta)Y = ae^{ibx} \tag{4.3.10}$$

Where Y is a complex-valued function, consisting of the real part Y_R and the imaginary part Y_I. So (4.3.10) is

$$(D-\alpha)(D-\beta)Y = (D-\alpha)(D-\beta)(Y_R+iY_I) = \mathrm{Re}(ae^{ibx}) + i\,\mathrm{Im}(ae^{ibx}) \tag{4.3.11}$$

separate both terms, that is

$$\begin{cases} (D-\alpha)(D-\beta)Y_R = \mathrm{Re}(ae^{ibx}) \\ (D-\alpha)(D-\beta)Y_I = \mathrm{Im}(ae^{ibx}) \end{cases} \tag{4.3.12}$$

And the solution $Y = f(z)$, $f : \mathbb{C} \to \mathbb{C}$. Note that the solutions to both ODEs in (4.3.12) are functions of the real numbers. $Y_R = f_1(x),\ f_1 : \mathbb{R} \to \mathbb{R}$; $Y_I = f_2(x),\ f_2 : \mathbb{R} \to \mathbb{R}$. Because the RHS of Problem 2 is $a\sin bx$, it conveys the same idea as the solution to the ODE below (4.3.8). But $Y = Y_R + iY_I$, iY_I will become the only one complex function of Y. In other words, we may directly solve the complex-valued ODE in (4.3.10), and extract the imaginary part of the solution, since $\mathrm{Im}(Y) = \mathrm{Im}f(z) = Y_I$. If the problem becomes $a\cos(bx)$, then we must take the real part. The meaning of this method is to simplify the original ODE problem since the exponential form is much easier to solve

than the trigonometric one; you just need to remember to take the imaginary part at the end. This provides an alternative explanation for solving trigonometric integrals.
Mathematically speaking, since the operator $(D-\alpha)(D-\beta)$ is linear, it commutes with the real and imaginary part operators. That is,

$$\begin{cases} (D-\alpha)(D-\beta)\mathrm{Re}(Y) = \mathrm{Re}[(D-\alpha)(D-\beta)Y] \\ (D-\alpha)(D-\beta)\mathrm{Im}(Y) = \mathrm{Im}[(D-\alpha)(D-\beta)Y] \end{cases} \tag{4.3.13}$$

which guarantees that we can apply the operator to Y as a whole and then extract the part we need. The remaining process is the same as before. I will walk through a full example in Section 4.4.

In addition, if the RHS has two terms of a trigonometric function, for example

$$(D-\alpha)(D-\beta)y = C_1\sin(\omega_1 x) + C_2\cos(\omega_2 x) \tag{4.3.14}$$

Consider the two ODEs shown below,

$$\begin{cases} (D-\alpha)(D-\beta)y_1 = C_1\sin(\omega_1 x) \\ (D-\alpha)(D-\beta)y_2 = C_2\cos(\omega_2 x) \end{cases}$$

Add both ODEs, we'll get

$$(D-\alpha)(D-\beta)y_1+(D-\alpha)(D-\beta)y_2 = (D-\alpha)(D-\beta)(y_1+y_2) = C_1\sin(\omega_1 x)+C_2\cos(\omega_2 x)$$

Compared with (4.3.14), we can observe that

$$(D-\alpha)(D-\beta)(y_1+y_2) = (D-\alpha)(D-\beta)y \Rightarrow y = y_1+y_2 \tag{4.3.15}$$

Therefore, solving the original ODE for y is equivalent to breaking it into two separate ODEs and then simply adding the results together.
By separating the two ODEs and solving them individually, the resulting forms become much simpler and easier to understand. This is precisely the idea behind the **superposition principle**.

I chose to introduce this principle only at the end of the trigonometric section because most physical systems that require this method typically involve periodic motion or oscillatory behavior, where the right-hand side consists of several trigonometric terms.

This allows us to focus exclusively on such cases in this report, as more complicated forms would carry little physical meaning at this level.

▷ **The Polynomial Term**

Problem. $y''-(\alpha+\beta)y'+\alpha\beta y = ae^{\gamma x}P_n(x)$, *where* $P_n(x)$ *is a polynomial of degree* n. *And* $\alpha \neq \beta \neq \gamma$.

As the same method, we change Problem 3 into

$$(D-\alpha)(D-\beta)y = ae^{\gamma x}P_n(x)$$

Let $u = (D - \alpha)y$, so the solution to u is

$$u = ae^{\beta x} \int e^{(\gamma-\beta)x} P_n(x)dx + Ce^{\beta x}$$

$$= ae^{\beta x} \left[\frac{1}{\gamma-\beta} e^{(\gamma-\beta)x} P_n(x) - \int \frac{1}{\gamma-\beta} e^{(\gamma-\beta)x} P_n^{(1)}(x)dx \right] + Ce^{\beta x}$$

$$= ae^{\beta x} \left[\frac{1}{\gamma-\beta} e^{(\gamma-\beta)x} P_n(x) - \frac{1}{(\gamma-\beta)^2} e^{(\gamma-\beta)x} P_n^{(1)}(x) + \int \frac{1}{(\gamma-\beta)^2} e^{(\gamma-\beta)x} P_n^{(2)}(x)dx \right] + Ce^{\beta x}$$

$$= ae^{\beta x} \left[\frac{1}{\gamma-\beta} e^{(\gamma-\beta)x} P_n(x) - \frac{1}{(\gamma-\beta)^2} e^{(\gamma-\beta)x} P_n^{(1)}(x) + \frac{1}{(\gamma-\beta)^3} e^{(\gamma-\beta)x} P_n^{(2)}(x) - \cdots \right] + Ce^{\beta x}$$

$$= ae^{\gamma x} \sum_j \frac{(-1)^{j-1}}{(\gamma-\beta)^j} P_n^{(j-1)}(x) + Ce^{\beta x}, \quad 1 \le j \le n$$

Note that I use integration by parts for n times to delete the polynomial terms. substituting this into $u = (D - \alpha)y$, we can get

$$d\left(e^{-\alpha x} y\right) = \left[ae^{(\gamma-\alpha)x} \sum_j \frac{(-1)^{j-1}}{(\gamma-\beta)^j} P_n^{(j-1)}(x) + Ce^{(\beta-\alpha)x} \right] dx$$

Integrate both sides, and we'll get the solution to the ODE

$$y = ae^{\alpha x} \int \left[e^{(\gamma-\alpha)x} \sum_j \frac{(-1)^{j-1}}{(\gamma-\beta)^j} P_n^{(j-1)}(x) \right] dx + e^{\alpha x} \int Ce^{(\beta-\alpha)x} dx + C^{(1)} e^{\alpha x}$$

$$= ae^{\alpha x} \sum_j \int e^{(\gamma-\alpha)x} \frac{(-1)^{j-1}}{(\gamma-\beta)^j} P_n^{(j-1)}(x)\, dx + \frac{C}{\beta-\alpha} e^{\beta x} + C^{(2)} e^{\alpha x}$$

$$= ae^{\alpha x} \left[\sum_{jk} e^{(\gamma-\alpha)x} \frac{(-1)^{(j+k-2)}}{(\gamma-\alpha)^k (\gamma-\beta)^j} P_n^{(j+k-2)}(x) \right] + \frac{C}{\beta-\alpha} e^{\beta x} + C^{(3)} e^{\alpha x}$$

$$= ae^{\gamma x} \sum_{jk} \frac{(-1)^{(j+k-2)}}{(\gamma-\alpha)^k (\gamma-\beta)^j} P_n^{(j+k-2)}(x) + \frac{C}{\beta-\alpha} e^{\beta x} + C^{(3)} e^{\alpha x}$$

Where$1 \le j \le k \le n$. We redefine the function

$$\sum_{jk} \frac{(-1)^{(j+k-2)}}{(\gamma-\alpha)^k (\gamma-\beta)^j} P_n^{(j+k-2)}(x) \coloneqq Q(x) \tag{4.3.16}$$

Obviously, the highest degree of the new function $Q(x)$ equals $P_n(x)$ because $j + k - 2$ starts from 0, that is

$$\deg P_n(x) = \deg Q(x) = n \tag{4.3.17}$$

Thus, the final solution will be the familiar form that we've seen before, that is

$$\boxed{y = ae^{\gamma x} Q(x) + C_1 e^{\alpha x} + C_2 e^{\beta x};\ \deg Q(x) = n} \tag{4.3.18}$$

There are two other types of parameters, $\gamma = \alpha \neq \beta$ and $\gamma = \alpha = \beta$, similar to those in Section, the exponential term, and their solutions follow the same procedure. In this paper, I will focus on the most common type for demonstration and explanation, while the remaining cases will be omitted for brevity.

4.4 Other Methods for Second-Order Equation

▷ Differential Equations Involving Only y

1. When the differential equation satisfies

$$f(y, y', y'') = 0 \tag{4.4.1}$$

That is, the function is independent of x. Then we let

$$\boxed{y'(x) = p(y(x))} \tag{4.4.2}$$

By the chain rule, we can verify that

$$\boxed{y''(x) = \frac{d}{dy}p(y) \cdot y'(x) = p(y) \cdot \frac{dp(y)}{dy}} \tag{4.4.3}$$

Then (4.4.1) becomes

$$f\left(y, p, p\frac{dp}{dy}\right) = 0 \tag{4.4.4}$$

This method is effective in reducing a second-order differential equation that is independent of the variable x into a first-order linear or nonlinear differential equation, which can then be solved using techniques discussed earlier. Thus, it enables us to indirectly recover the function $y(x)$.
The most famous example is the equation of motion of a damped oscillation with a quadratic drag term($-bv^2$).

$$m\ddot{x} + b\left(\dot{x}\right)^2 + kx = 0 \tag{4.4.5}$$

Let $\dot{x} = p(x(t))$, by (4.4.3), $\ddot{x} = p \cdot \frac{dp}{dx}$. The equation of motion becomes

$$mp\frac{dp}{dx} + bp^2 + kx = 0 \Rightarrow \frac{dp}{dx} + 2\beta p + \omega_0^2 x p^{-1} = 0 \tag{4.4.6}$$

Where $2\beta \coloneqq b/m$ and $\omega_0^2 \coloneqq k/m$. After that, this differential equation becomes a classical Bernoulli equation.

$$\begin{cases} \dfrac{dp}{dx} + 2\beta p = -\omega_0^2 x p^{-1} \\ \dfrac{dy}{dx} + P(x)y = Q(x)y^n \quad \text{(Bernoulli equation)} \end{cases} \tag{4.4.7}$$

Then you can let $z = p^2$ to solve this differential equation.

Example. *A particle moves vertically under the influence of a force that is inversely proportional to the square of its distance from the origin. The force acting on the particle is given by*

$$F(y) = -\frac{k}{y^2}, \quad \textit{where } k > 0.$$

Assuming the particle starts at position $y_0 > 0$ and is initially moving downward, find the relationship between the particle's velocity v and its position y.

From Newton's second law, we obtain

$$ma = -\frac{k}{y^2} \quad \Rightarrow \quad a = \frac{dv}{dt} = -\frac{k}{my^2}$$

By the chain rule

$$\frac{dv}{dt} = \frac{dv}{dy} \cdot \frac{dy}{dt} = v \cdot \frac{dv}{dy} \Rightarrow v \cdot \frac{dv}{dy} = -\frac{k}{my^2}$$

Separating the variables

$$v\,dv = -\frac{k}{my^2}dy \Rightarrow \int v\,dv = -\frac{k}{m}\int \frac{1}{y^2}dy \Rightarrow \frac{1}{2}v^2 = \frac{k}{my} + C$$

Thus, the velocity-position relationship is

$$v(y) = -\sqrt{\frac{2k}{my} + C'}$$

Where the negative sign chosen shows that the motion is downward.

---

2. When the differential equation satisfies

$$y'' + f(y) = 0 \Rightarrow y'' = -f(y) \tag{4.4.8}$$

Then we must multiply both sides by y', that is

$$\boxed{y' \cdot y'' = y' \cdot f(y)} \tag{4.4.9}$$

The reason why we must do this is to solve the differential equation for y', which is

$$y'y'' = -f(y)y' \Rightarrow y'\frac{dy'}{dx} = -f(y)\frac{dy}{dx} \Rightarrow y'dy' = -f(y)dy$$

$$\Rightarrow \int y'dy' = \int -f(y)dy \Rightarrow \frac{1}{2}y'^2 = -\int f(y)dy + C$$

$$\Rightarrow y' = \sqrt{-2\int f(y)dy + C}$$

Such differential equations most commonly appear in mechanics when we need to find the velocity-position relationship. In these cases, there is no need to indirectly solve for $y(t)$; instead, the velocity term can be obtained directly. This makes it more convenient to analyze the phase diagram and observe the trajectory of the system's motion.

▷ **Cauchy Equation**

Differential equations of the form

$$a_2x^2y'' + a_1xy' + a_0y = f(x) \tag{4.4.10}$$

are called **Cauchy equation**. Being more convenient to identify such equations, we can verify that each term of the equation satisfies

$$a_nx^nD^n[y] \tag{4.4.11}$$

Where $D = d/dx$. In such problems, we can reduce the differential equation to a linear equation without x. That is, we should change the variable x to z where

$$x = e^z \Leftrightarrow z = \ln x \tag{4.4.12}$$

After changing the variable, the linear term $x\frac{dy}{dx}$ can be written as

$$x\frac{dy}{dx} = x \cdot \frac{dy}{dz} \cdot \frac{dz}{dx}$$

Note that $z = \ln x$, so the derivative of z with respect to x is

$$\frac{dz}{dx} = \frac{d\ln x}{dx} = \frac{1}{x}$$

Then we obtain

$$x \cdot \frac{dy}{dz} \cdot \frac{dz}{dx} = x\frac{dy}{dz} \cdot \frac{1}{x} = \frac{dy}{dz}$$

So, the first important conclusion is

$$\boxed{x\frac{dy}{dx} = \frac{dy}{dz}} \tag{4.4.13}$$

By (4.4.13), the equation can be also denoted by

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz} \tag{4.4.14}$$

We can use the above conclusion to change the quadratic term with respect to z, that is

$$\begin{aligned} x^2\frac{d^2y}{dx^2} &= x^2\frac{d}{dx}\left(\frac{dy}{dx}\right) = x^2\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) \\ &= x^2\left[-\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\left(\frac{d}{dx}\frac{dy}{dz}\right)\right] \\ &= x^2\left[-\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\left(\frac{d}{dz}\frac{dz}{dx}\frac{dy}{dz}\right)\right] \\ &= x^2\left[-\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}\right] = \frac{d^2y}{dz^2} - \frac{dy}{dz} \end{aligned}$$

So, the second important conclusion is

$$\boxed{x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}} \tag{4.4.15}$$

Substituting (4.4.13) and (4.4.15) into (4.4.10), we can obtain a normal second-order differential equation

$$\boxed{a_2\frac{d^2y}{dz^2} + (a_1 - a_2)\frac{dy}{dz} + a_0y = f(e^z)} \tag{4.4.16}$$

You can see the example in the next chapter.

▷ A Complex Differential Equation with One of Its Solutions

When you meet the differential equation of the form

$$y'' + f(x)y' + g(x)y = 0 \tag{4.4.17}$$

and given $y = u(x)$ is one of the solutions, then we can assume that

$$y = u(x)v(x) \tag{4.4.18}$$

Substituting (4.4.18) into (4.4.17), we obtain

$$\begin{aligned}
&\frac{d^2}{dx^2}(uv) + f \cdot \frac{d}{dx}(uv) + guv \\
&= u''v + 2u'v' + 2uv'' + f(u'v + uv') + guv \\
&= 2uv'' + (2u' + fu)v' + (u'' + fu' + gu)v = 0
\end{aligned}$$

Because we've known that $y = u$ is a solution to the ODE, we can verify that the coefficient of v, namely $u'' + fu' + gu$, is 0. Then it shows that

$$2uv'' + (2u' + fu)v' = 2u\frac{dv'}{dx} + (2u' + fu)v' = 0 \tag{4.4.19}$$

By separating the variables, we obtain

$$\int \frac{dv'}{v'} = \int \frac{-(2u' + fu)}{2u}dx \tag{4.4.20}$$

The solution to v' is

$$v' = \exp\left(\int \frac{-(2u' + fu)}{2u}dx\right) \tag{4.4.21}$$

Furthermore, we can obtain the function $v(x)$ is

$$v = \int \exp\left(\int \frac{-(2u' + fu)}{2u}dx\right)dx \tag{4.4.22}$$

Substituting the result of v into (4.4.18), we can finally get the complete solution to the original ODE, that is

$$y = uv = u\left[\int \exp\left(\int \frac{-(2u' + fu)}{2u}dx\right)dx\right] \tag{4.4.23}$$

Note that we assume that the inner integral is

$$\int \frac{-(2u' + fu)}{2u} dx \coloneqq F(x) + C \tag{4.4.24}$$

Then the outer integral can be written as

$$\int C_1 e^{F(x)} dx \coloneqq G(x) + C_2 \tag{4.4.25}$$

Substituting $G(x) + C_2$ into y, the final solution is rewritten as

$$y = u(x)G(x) + C_2 u(x) \tag{4.4.26}$$

"Such a solution also satisfies the superposition principle, that is

$$\begin{cases} y_1 = u(x)G(x) \\ y_2 = C_2 u(x) \end{cases} \Rightarrow y = y_1 + y_2 = u(x)G(x) + C_2 u(x) \tag{4.4.27}$$

This means we can use one particular solution from the general solution to infer the form of another, and ultimately combine them to construct the complete general solution. The reason this is possible is that, regardless of the form of the general solution, we can always factor out one part of it as a product of functions. For example, we have the general solution $y = C_1 f(x) + C_2 g(x)$, and then we factor one of the solutions out($g(x)$), it becomes

$$C_1 f(x) + C_2 g(x) = g(x) \left[C_1 \frac{f(x)}{g(x)} + C_2 \right]$$

So,

$$u = g(x), \quad v = C_1 \frac{f(x)}{g(x)} + C_2$$

That is how this method works.

4.5 The Examples

Example. *Solve the ODE* $y'' - 3y' + 2y = \sin x$

First, rewrite the problem as

$$(D - 1)(D - 2)y = \sin x$$

Let $u = (D - 2)y$, the equation becomes $u' - u = \sin x$, so the solution to u is

$$u = e^x \int e^{-x} \sin x \, dx = -\frac{1}{2} \sin x - \frac{1}{2} \cos x + Ce^x$$

substituting it into $u = (D - 2)y$, we get

$$y' - 2y = -\frac{1}{2} \sin x - \frac{1}{2} \cos x + Ce^x$$

$$\Rightarrow y = e^{2x} \int e^{-2x} \left(-\frac{1}{2} \sin x - \frac{1}{2} \cos x + Ce^x \right) dx$$

we can easily get the final solution, that is

$$y = \frac{1}{10} \sin x + \frac{3}{10} \cos x + C_1 e^x + C_2 e^{2x}$$

Example. *Solve the ODE* $y'' - 2y' + y = \sin 2x + \cos 3x$

In this problem, I'm going to use the superposition principle like what I did in Section 4.2. We break it into two separate ODEs, which is

$$\begin{cases} (D-1)(D-1)y_1 = \sin 2x \\ (D-1)(D-1)y_2 = \cos 3x \end{cases}$$

We solve the first ODE first, let $u = (D-1)y_1$, then

$$u' - u = \sin 2x \Rightarrow e^{-x}u = \int e^{-x} \sin 2x \; dx$$

So u equals

$$u = \frac{-1}{5}\sin 2x - \frac{2}{5}\cos 2x + Ce^x$$

substituting it into $u = (D-1)y_1$, we get

$$\begin{aligned} y_1 &= e^x \int e^{-x} \left(-\frac{1}{5}\sin 2x - \frac{2}{5}\cos 2x + Ce^x \right) dx \\ &= \frac{-3}{25}\sin 2x + \frac{4}{25}\cos 2x + (Cx + C_1)e^x \end{aligned}$$

Similarly,

$$y_2 = -\frac{3}{50}\sin 3x - \frac{2}{25}\cos 3x + (C_2 x + C_3)e^x$$

By equation (4.3.15), the solution to the original ODE is

$$y = y_1 + y_2 = \frac{-3}{25}\sin 2x + \frac{4}{25}\cos 2x - \frac{3}{50}\sin 3x - \frac{2}{25}\cos 3x + (Ax + B)e^x$$

Example. *Solve the ODE* $y'' - 4y' + 3y = e^{5x}\cos 2x$

Here, I want to use the complex method to solve the problem. The ODE can be rewritten as

$$(D-1)(D-3)Y = e^{5x} \times \; e^{2ix}$$

Because $e^{2ix} = \cos 2x + i \sin x$, we need to take the real part of Y, which is a complex-valued function. By (4.3.13), we got

$$(D-1)(D-3)\mathrm{Re}(Y) = \mathrm{Re}\left[(D-1)(D-3)Y\right]$$

So, we need to solve the ODE

$$(D-1)(D-3)Y = e^{(5+2i)x}$$

By the result of (4.3.2), the final solution to the ODE of Y is

$$Y = \frac{1}{(5+2i-1)(5+2i-3)}e^{(5+2i)x} + C_1 e^x + C_2 e^{3x}$$

The complex term $(4+2i)(2+2i) = 4+12i$, so

$$\frac{1}{4+12i} = \frac{4-12i}{(4+12i)(4-12i)} = \frac{1-3i}{40}$$

Expand Y to get

$$\begin{aligned} Y &= \frac{1-3i}{40}e^{(5+2i)x} + C_1e^x + C_2e^{3x} \\ &= e^{5x}\left[\frac{1-3i}{40}(\cos 2x + i\sin 2x)\right] + C_1e^x + C_2e^{3x} \\ &= e^{5x}\left[\frac{3\sin 2x + \cos 2x}{40} + \frac{i(\sin 2x - 3\cos 2x)}{40}\right] + C_1e^x + C_2e^{3x} \end{aligned}$$

To get the solution to the original ODE, we must remember to take the real part of Y, so we can get the answer

$$y = \text{Re }(Y) = \frac{3}{40}e^{5x}\sin 2x + \frac{1}{40}e^{5x}\cos 2x + C_1e^x + C_2e^{3x}$$

Example. *Solve the ODE* $xy'' = y' + (y')^3$

In such a problem, we must let $y' = p(x)$. Therefore, we can rewrite the equation into

$$x\frac{dp}{dx} = p + p^3 = p(p^2+1)$$

Separate the variables, and we obtain

$$\frac{dp}{p(p^2+1)} = \frac{dx}{x} \Leftrightarrow \int \frac{dp}{p(p^2+1)} = \int \frac{dx}{x}$$

By the method of partial fractions, the left-hand integral can be written as

$$\int \frac{1}{p}dp - \int \frac{p}{p^2+1}dp = \ln x + C$$

Then

$$\ln p - \frac{1}{2}\ln(p^2+1) = \ln\left(\frac{p}{\sqrt{p^2+1}}\right) = \ln x + C$$

$$\Rightarrow \frac{p}{\sqrt{p^2+1}} = kx, \quad k = e^C$$

$$\Rightarrow p^2 = \left(\frac{dy}{dx}\right)^2 = \frac{k^2x^2}{1-k^2x^2}$$

$$\Rightarrow \frac{dy}{dx} = \pm\frac{kx}{\sqrt{1-k^2x^2}}$$

$$\Rightarrow \int dy = \pm\int \frac{kx}{\sqrt{1-k^2x^2}}dx$$

Note that the original problem becomes a trigonometric substitution. Let $kx = \sin\theta$, then $dx = \frac{1}{k}\cos\theta d\theta$. Substituting this into the integral, we get

$$y = \mp\frac{1}{k}\cos\theta + a \Rightarrow y - a = \mp\frac{1}{k}\cos\theta, \quad a = const$$

Because we have let $x = \frac{1}{k}\sin\theta$, by squaring them and adding, we obtain

$$x^2 + (y-a)^2 = \left(\frac{1}{k}\right)^2 \sin^2\theta + \left(\frac{1}{k}\right)^2 \cos^2\theta = \frac{1}{k^2}$$

It's an equation of a circle. Note that when y is a constant, it's obvious that a constant is a solution to the differential equation. So, the final solutions to the differential equation are

$$x^2 + (y-a)^2 = \frac{1}{k^2} \quad \text{or} \quad y = const$$

Example. *Solve the ODE* $2yy'' = (y')^2$

Let $y' = p$, by chain rule, $y'' = p\frac{dp}{dy}$. Then we can rewrite the equation into

$$2yp\frac{dp}{dy} = p^2 \Rightarrow 2y\frac{dp}{dy} = p$$

By separating the variables, we can obtain

$$\frac{dp}{p} = \frac{1}{2}\frac{dy}{y} \Leftrightarrow \int\frac{dp}{p} = \int\frac{1}{2}\frac{dy}{y}$$

Then the solution to p is

$$p = Cy^{1/2}$$

Substituting $y' = p$ into the equation

$$\frac{dy}{dx} = Cy^{1/2} \Rightarrow \int y^{-1/2}dy = \int Cdx$$

Then the final solution to the ODE is

$$2\sqrt{y} = Cx + D$$

Where C, D are constants.

Example. *The ODE* $y'' = \dfrac{2}{y}$, *find* y' *depending on* y

Multiplying both sides by y', the equation becomes

$$y'y'' = \frac{2}{y}y'$$

It equals

$$y'\frac{dy'}{dx} = \frac{2}{y}\frac{dy}{dx} \Rightarrow \int y'dy' = 2\int\frac{dy}{y}$$

Therefore, we can obtain the solution to y', that is

$$\frac{1}{2}y'^2 = 2\ln y + C$$

$$\Rightarrow y' = \sqrt{4\ln y + 2C}$$

Example. *Solve the ODE* $x^2y'' - 3xy' + 4y = 6x^2\ln x$

This is one of the Cauchy equations, and we let $x = e^z$. By (4.4.), we obtain

$$\frac{d^2y}{dz^2} + (-3-1)\frac{dy}{dz} + 4y = 6(e^z)^2\ln e^z = 6ze^{2z}$$

Then we get a second-order differential equation with respect to z, that is

$$y'' - 4y' + 4y = (D-2)(D-2)y = 6ze^{2z}$$

where $D = d/dz$. Let $(D-2)y = u$, we can know that

$$u' - 2u = 6ze^{2z} \Rightarrow e^{-2z}u = 6\int e^{-2z}(ze^{2z})dz = 6\int zdz$$

The solution to u is

$$u = (D-2)y = 3z^2e^{2z} + C_1e^{2z}$$

Similarly, the solution to y is

$$y = e^{2z}\int 3z^2dz + e^{2z}\int C_1dz = z^3e^{2z} + (C_1z + C_2)e^{2z}$$

Substituting $z = \ln x$ into this solution, we can finally get

$$y = x^2\left[(\ln x)^3 + C_1\ln x + C_2\right]$$

Example. *Solve the ODE* $x^2(2-x)y'' + 2xy' - 2y = 0$*, given that* $u(x) = x$ *is a solution*

First, let the solution y be

$$y = u(x)v(x) = xv$$

Substituting $y = xv$ into the ODE, we obtain

$$\begin{aligned} x^2(2-x)\frac{d^2}{dx^2}(xv) + 2x\frac{d}{dx}(xv) - 2xv &= x^2(2-x)(2v' + xv'') + 2x(v + xv') - 2xv \\ &= (6x^2 - 2x^3)v' + (2x^3 - x^4)v'' \\ &= (6x^2 - 2x^3)v' + (2x^3 - x^4)\frac{dv'}{dx} = 0 \end{aligned}$$

Then we write down the differential equation for v'

$$(6x^2 - 2x^3)v' + (2x^3 - x^4)\frac{dv'}{dx} = 0$$

$$\Rightarrow \int \frac{dv'}{v'} = \int \frac{2x^3 - 6x^2}{2x^3 - x^4} dx$$

$$\Rightarrow \ln v' = \ln\left(\frac{2-x}{x^3}\right) + C$$

$$\Rightarrow v' = C_1\left(\frac{2-x}{x^3}\right)$$

Solving the ODE again, we obtain

$$\frac{dv}{dx} = C_1\left(\frac{2-x}{x^3}\right) \Rightarrow \int dv = v = C_1 \int \left(\frac{2-x}{x^3}\right) dx = C_1(x^{-1} - x^{-2}) + C_2$$

Substituting v into $y = xv$, we can finally obtain the solution to the question

$$y = C_1(1 - x^{-1}) + C_2 x$$

4.6 Application: Driven Oscillation and RLC Circuit

▷ Driven Oscillation

After such long explanations and demonstrations, we can finally begin to solve the most important second-order nonhomogeneous differential equation in a classical physical system, the driven oscillation.
The equation of motion of a driven oscillation system is

$$m\ddot{x} + b\dot{x} + kx = F(t) \tag{4.6.1}$$

where $b\dot{x}$ is the drag term, kx is the force model of an oscillating object, and $F(t)$ is a periodic external driven force. We rewrite the problem as

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega t) \tag{4.6.2}$$

Where $2\beta \coloneqq b/m$; $\omega_0^2 \coloneqq k/m$, and the driven term ω means the driven frequency. To simplify the calculation, I'll use the complex function to solve this ODE. Factor the operator to (5.2), and it becomes

$$\left[D - \left(-\beta + \sqrt{\beta^2 - \omega_0^2}\right)\right]\left[D - \left(-\beta - \sqrt{\beta^2 - \omega_0^2}\right)\right] X = f_0 e^{i\omega t} \tag{4.6.3}$$

Because $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$, we need to take the real part of X,

To simplify the equation, I define the term $\sqrt{\beta^2 - \omega_0^2} = \gamma$, let $[D - (-\beta - \gamma)]\, X = u$, then

$$[D - (-\beta + \gamma)]\, u = f_0 e^{i\omega t}$$

$$\Rightarrow e^{(\beta-\gamma)} u = f_0 \int e^{[(\beta-\gamma+i\omega)t]}\, dt$$

$$\Rightarrow u = \frac{f_0}{(\beta + i\omega - \gamma)} e^{i\omega t} + C e^{(-\beta+\gamma)t}$$

substituting the result to $(D - (-\beta - \gamma))X = u$, then

$$e^{(\beta+\gamma)t} X = \frac{f_0}{[i\omega + (\beta - \gamma)]} \int e^{(i\omega+\beta+\gamma)t} dt + C \int e^{2\gamma t} dt$$

So the solution X is

$$\begin{aligned} X(t) &= \frac{f_0}{[i\omega + (\beta - \gamma)]\,[i\omega + (\beta + \gamma)]} e^{i\omega t} + C_1 e^{(-\beta+\gamma)t} + C_2 e^{(-\beta-\gamma)t} \\ &= \frac{f_0}{(\omega_0^2 - \omega^2) + 2i\omega\beta} e^{i\omega t} + e^{-\beta t}\left(C_1 e^{\gamma t} + C_2 e^{-\gamma t}\right) \\ &= \frac{f_0\left[(w_0^2 - \omega^2) - 2i\omega\beta\right]}{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2} e^{i\omega t} + e^{-\beta t}\left(C_1 e^{\gamma t} + C_2 e^{-\gamma t}\right) \end{aligned} \tag{4.6.4}$$

The result of (4.6.4) has not been done yet, we must take the real part of the complex-valued function $X(t)$, that is

$$x(t) = \mathrm{Re}(X(t)) = \frac{f_0\left[(\omega_0^2 - \omega^2)\cos(\omega t) + 2\omega\beta\sin(\omega t)\right]}{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2} + e^{-\beta t}\left(C_1 e^{\gamma t} + C_2 e^{-\gamma t}\right) \tag{4.6.5}$$

To simplify (4.6.5), we apply a trigonometric combination, then we can obtain a really beautiful result

$$x(t) = \frac{f_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2}} \cos(\omega t - \delta) + e^{-\beta t}\left(C_1 e^{\gamma t} + C_2 e^{-\gamma t}\right) \tag{4.6.6}$$

where the angle δ is

$$\delta = \tan^{-1}\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right) \tag{4.6.7}$$

Note that the homogeneous solution $e^{-\beta t}\left(C_1 e^{\gamma t} + C_2 e^{-\gamma t}\right)$, which is the decay term of the damping system in physical meaning, will disappear when the time $t \gg 1/\beta$. In mathematics, it is

$$\lim_{t\to\infty} e^{-\beta t}\left(C_1 e^{\gamma t} + C_2 e^{-\gamma t}\right) = 0 \tag{4.6.8}$$

It means that the long-term behavior $x_l(t)$ depends on the external driven term, that is,

$$x_l(t) = \frac{f_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2}} \cos(\omega t - \delta) \tag{4.6.9}$$

The amplitude A of the oscillating system is

$$A = \frac{f_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2}} \tag{4.6.10}$$

In physics, ω_0 is called **natural frequency**, while the driven frequency ω approaches the specific frequency ω_R, the amplitude of the system reaches its highest point, and the phenomenon is called **resonance**. Once the resonance happens, the amplitude satisfies

$$\left.\frac{dA}{d\omega}\right|_{\omega=\omega_R} = 0 \tag{4.6.11}$$

Then we can compute that

$$\left.\frac{dA}{d\omega}\right|_{\omega=\omega_R} = \left.\frac{-f_0 \cdot \frac{d}{d\omega}[\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2]^{1/2}}{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2}\right|_{\omega=\omega_R}$$

$$= \left.\frac{-f_0(4\omega^3 - 4\omega\omega_0^2 + 8\omega\beta^2)}{2[\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2]^{3/2}}\right|_{\omega=\omega_R} = 0$$

So, the resonance frequency is

$$\omega_R = \sqrt{\omega_0^2 - 2\beta^2}$$

Substituting this into (4.6.10), we can compute the highest amplitude (A_{max}), that is

$$A_{max} = \frac{f_0}{2\beta\sqrt{\omega_0^2 - \beta^2}} \approx \left.\frac{f_0}{2\beta\omega_0}\right|_{\beta\ll\omega_0}$$

This report focuses primarily on the analytical form of the solution to the driven oscillation problem, which is one of the most important second-order nonhomogeneous

differential equations in classical physics. The main objective is to derive the general solution using the operator descent method and to understand how the solution behaves with respect to the natural frequency ω_0 and the driving frequency ω.

A more detailed analysis of the physical interpretation, such as energy transfer, resonance bandwidth, or quality factor, is beyond the scope of this report and will be left to future study or more advanced physics courses. In this work, I concentrate on solving the equation rigorously and extracting the mathematical structure behind the solution.

▷ RLC Circuit

Another well-known and classic example of a second-order nonhomogeneous differential equation is the RLC circuit. Consider a series RLC circuit driven by an external voltage source $V(t) = V_0 \sin(\omega t)$. The current flows through a resistor R, a capacitor C, and an inductor L connected in series.
In this case, we can use the relationship between electric charge and time to formulate a second-order differential equation. Let $q(t)$ be the charge on the capacitor. By applying Kirchhoff's voltage law (KVL), we obtain:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V_0 \sin(\omega t) \tag{4.6.12}$$

This equation describes the dynamic behavior of the charge $q(t)$ in the circuit under the influence of the time-varying driving voltage.
First, rewrite the problem as

$$\ddot{q} + 2\beta\dot{q} + \omega_0^2 q = \epsilon \sin(\omega t) \tag{4.6.13}$$

Where $2\beta = R/L;$, $\omega_0^2 = 1/(LC)$ and $\epsilon = V_0/L$. Factor the operator to this, and it becomes

$$\left[D - \left(-\beta + \sqrt{\beta^2 - \omega_0^2}\right)\right]\left[D - \left(-\beta - \sqrt{\beta^2 - \omega_0^2}\right)\right] Q = \epsilon e^{i\omega t} \tag{4.6.14}$$

In the same way, I use the complex function to solve this ODE, which is like the driven oscillating problem, as long as we remember to take the imaginary part of the solution.
Define $\sqrt{\beta^2 - \omega_0^2} = \gamma$ and $[D - (-\beta - \gamma)]Q = u$, then

$$[D - (-\beta + \gamma)]\, u = \epsilon e^{i\omega t}$$

$$\Rightarrow e^{(\beta-\gamma)}u = \epsilon \int e^{[(\beta-\gamma+i\omega)t]}\, dt$$

$$\Rightarrow u = \frac{\epsilon}{(\beta + i\omega - \gamma)} e^{i\omega t} + Ce^{(-\beta+\gamma)t}$$

substituting the result to $(D - (-\beta - \gamma))Q = u$, then

$$e^{(\beta+\gamma)t}Q = \frac{\epsilon}{[i\omega + (\beta - \gamma)]} \int e^{(i\omega+\beta+\gamma)t} dt + C \int e^{2\gamma t} dt$$

So the solution Q is

$$\begin{aligned} Q(t) &= \frac{\epsilon}{[i\omega + (\beta - \gamma)]\,[i\omega + (\beta + \gamma)]} e^{i\omega t} + C_1 e^{(-\beta+\gamma)t} + C_2 e^{(-\beta-\gamma)t} \\ &= \frac{\epsilon}{(\omega_0^2 - \omega^2) + 2i\omega\beta} e^{i\omega t} + e^{-\beta t}\left(C_1 e^{\gamma t} + C_2 e^{-\gamma t}\right) \\ &= \frac{\epsilon\,[(w_0^2 - \omega^2) - 2i\omega\beta]}{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2} e^{i\omega t} + e^{-\beta t}\left(C_1 e^{\gamma t} + C_2 e^{-\gamma t}\right) \end{aligned} \tag{4.6.15}$$

The result of (4.6.15) has not been done yet, we must take the imaginary part of the complex-valued function $Q(t)$, that is

$$q(t) = \mathrm{Im}(Q(t)) = \frac{\epsilon\left[(\omega_0^2 - \omega^2)\sin(\omega t) - 2\omega\beta\cos(\omega t)\right]}{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2} + e^{-\beta t}\left(C_1 e^{\gamma t} + C_2 e^{-\gamma t}\right) \quad (4.6.16)$$

To simplify (4.6.16), we apply a trigonometric combination, and then we can obtain a really beautiful result

$$q(t) = \frac{V_0/L}{\sqrt{\left(1/LC - \omega^2\right)^2 + (R\omega/L)^2}}\sin(\omega t - \delta) + e^{-\beta t}\left(C_1 e^{\gamma t} + C_2 e^{-\gamma t}\right) \quad (4.6.17)$$

where the angle δ is

$$\delta = \tan^{-1}\left(\frac{R\omega/L}{1/LC - \omega^2}\right) \quad (4.6.18)$$

This angle represents the phase difference between the voltage and the charge (or current). Similarly, in the time-dependent behavior of the current, as time progresses, the transient terms decay, and only the steady-state response remains.

$$q(t) = \frac{V_0/L}{\sqrt{\left(1/LC - \omega^2\right)^2 + (R\omega/L)^2}}\sin(\omega t - \delta) \quad (4.6.19)$$

Typically, in circuit experiments or analysis reports, we are more interested in the current as the main physical quantity. As a result, we need to differentiate the charge function once more to get the current as a function of time, that is

$$I = \dot{q}(t) = \frac{\omega V_0}{L\sqrt{\left(1/LC - \omega^2\right)^2 + (R\omega/L)^2}}\cos(\omega t - \delta) \quad (4.6.20)$$

If you are unsure whether the result is correct, you can check the dimensional consistency to avoid potentially damaging the circuit.
Although the RLC circuit and mechanical oscillation share the same mathematical form, I deliberately chose a different way to extract the solution, not for you to blindly mimic steps, but to help you realize this: mathematics is only a medium; what truly determines how we solve the equation is the physics behind it.

Chapter 5. Introduction to Calculus of Variations

In this chapter, I want to focus on the mathematical derivation and explanation of functional variational methods, which were not thoroughly explored in my previous studies. This method is crucial in analytical mechanics, quantum mechanics, and dynamical systems. Since I have the time, I plan to gather information from various sources and books to compile a series of easily understandable articles. In high school, we learn about functions and ask, "At which point does a function attain its extremum?" In university, we ask: "Which function minimizes the energy?" This is the essence of variational calculus.

5.1 Definition of a Functional

▷ **Functional**

A functional is composed of many functions that are part of the functional domain.

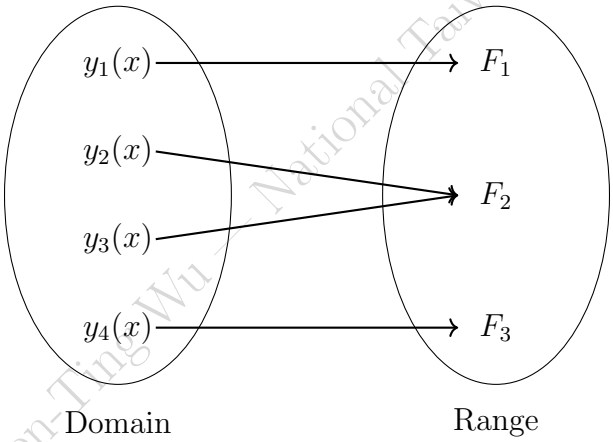

Figure 26: The correspondence between the domain and the range of a functional

For example, $F[y(x), y'(x), x] = \int_a^b dx f(y(x), y'(x), x)$ is a classical functional which is often used to deal with the analyti cal mechanics problems.

▷ **Extremum of Functional**

We aim to find the function $y(x)$ that makes the functional attain an extremum. This is done by evaluating how the functional changes under small variations of $y(x)$ and requiring that the first variation vanishes. The method for finding such an extremum is known as the calculus of variations.

5.2 Euler-Lagrange Equation and Lagrangian & Hamiltonian

▷ Euler-Lagrange Equation without Constraint Conditions

To find the extremum value, we need to add a minimal change at the functional, which is denoted by

$$F[y(x)] \Rightarrow F[y(x) + \delta y(x)] = F[y(x) + \alpha\eta(x)] \tag{5.2.1}$$

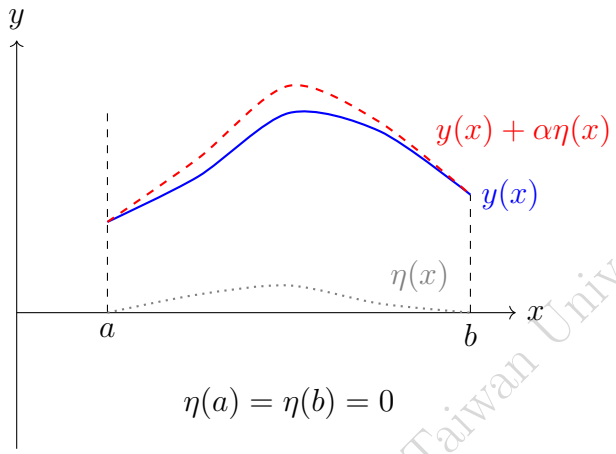

Figure 27: Illustration of Perturbation Function in Calculus of Variations

where α is a minimal value, and $\eta(x)$ is an arbitrary function that is differentiable to first order and $\eta(a) = \eta(b) = 0$. It's called the perturbation function.
When the extremum occurs at the point $y(x)$, then $\frac{\partial F}{\partial y} = 0$. Consider the bivariate function $f(y(x), y'(x))$ which depends on y, y', the functional F is defined to be the integral of f, that is

$$F[y(x) + \alpha\eta(x)] \coloneqq F(\alpha) = \int_a^b dx f(y(x) + \alpha\eta(x), y'(x) + \alpha\eta'(x)) \tag{5.2.2}$$

When $\alpha = 0$, it is obvious to see that $\frac{dF}{d\alpha} = 0$ because $\alpha = 0$ means $F(\alpha = 0) = F[y]$, which is the extremum value. However, we should need to go further to check what must happen and the correspondence between the original bivariate function and y.
To get this result, we need to get the Taylor Expansion of the functional $F(\alpha)$, we get

$$F(\alpha \to 0) = F(0) + \left.\frac{dF}{d\alpha}\right|_{\alpha=0} \cdot \alpha + O(\alpha^2) \tag{5.2.3}$$

Note that α is a variable that approaches 0. We have known that if we want to find the extremum value, the slope of the functional at the point α equals 0. We have discussed

this earlier. So,

$$\begin{aligned}\left.\frac{dF}{da}\right|_{\alpha=0} &= \frac{d}{d\alpha}\left[\int_a^b dx\, f\left(y(x)+\alpha\eta(x), y'(x)+\alpha\eta'(x)\right)\right]\Bigg|_{a=0} \\ &= \int_a^b dx\left\{\frac{\partial f}{\partial\left(y(x)+\alpha\eta(x)\right)}\cdot\frac{d}{d\alpha}\left(y(x)+\alpha\eta(x)\right)\right. \\ &\quad \left.+\frac{\partial f}{\partial\left(y'(x)+\alpha\eta'(x)\right)}\cdot\frac{d}{d\alpha}\left(y'(x)+\alpha\eta'(x)\right)\right\}\end{aligned} \tag{5.2.4}$$

The derivative of $\frac{d}{d\alpha}\left(y(x)+\alpha\eta(x)\right) = \eta(x)$; $\frac{d}{d\alpha}\left(y'(x)+\alpha\eta'(x)\right) = \eta'(x)$. In addition, $\alpha = 0$, the (5.2.4) can be written as

$$\left.\frac{dF}{da}\right|_{\alpha=0} = \int_a^b dx\left\{\frac{\partial f}{\partial\left(y(x)\right)}\cdot\eta(x)+\frac{\partial f}{\partial\left(y'(x)\right)}\cdot\eta'(x)\right\} \tag{5.2.5}$$

Use integration by parts, the integral

$$\int_a^b dx\frac{\partial f}{\partial y'(x)}\cdot\eta'(x) = \left.\frac{\partial f}{\partial y'(x)}\eta(x)\right|_a^b - \int_a^b dx\frac{d}{dx}\left[\frac{\partial f}{\partial y'(x)}\right]\cdot\eta(x) \tag{5.2.6}$$

Combining this with (5.2.5), we get

$$\begin{aligned}\left.\frac{dF}{da}\right|_{\alpha=0} &= \int_a^b dx\frac{\partial f}{\partial y(x)}\cdot\eta(x) - \int_a^b dx\frac{d}{dx}\left[\frac{\partial f}{\partial y'(x)}\right]\cdot\eta(x) \\ &= \int_a^b dx\left[\frac{\partial f}{\partial y(x)} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'(x)}\right)\right]\cdot\eta(x) = 0\end{aligned} \tag{5.2.7}$$

Because $\eta(x)$ is an arbitrary function that is differentiable to first order, the equation must satisfy

$$\boxed{\frac{\partial f}{\partial y(x)} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'(x)}\right) = 0} \tag{5.2.8}$$

Which is called **Euler-Lagrange equation**. So we finally get $y(x)$, which makes F have an extreme value, and satisfies the Euler-Lagrange equation for all $x \in [a, b]$.

Consider its another form, we have $F[y(x), y'(x), x] = \int_a^b dx f(y(x), y'(x), x)$, and differentiate f with respect to x, that is,

$$\frac{df}{dx} = \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} + \frac{\partial f}{\partial x}\frac{dx}{dx} \tag{5.2.9}$$

Because we've known $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$, so we can combine the first and the second terms that

$$\frac{df}{dx} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}y'\right) + \frac{\partial f}{\partial x} \Rightarrow \frac{d}{dx}\left(f - \frac{\partial f}{\partial y'}y'\right) = \frac{\partial f}{\partial x} \tag{5.2.10}$$

If the function f is independent of x, which means $\frac{\partial f}{\partial x} = 0$, we can finally get the second form of the Euler-Lagrange equation, denoted by

$$\boxed{f - \frac{\partial f}{\partial y'} y' = const} \tag{5.2.11}$$

where $f = f(y, y')$.
This result can be generalized into N-variable functions $F[y_1(x), y_1'(x), \cdots, y_N(x), y_N'(x)] = \int_a^b dx f(y_1(x), y_1'(x), \cdots, y_N(x), y_N'(x))$, there exists $y_i(x), i = 1, \cdots, N$ such that F has extreme value. We perform variations of them, we get

$$y_i(x) \to y_i(x) + \alpha\eta_i(x);\ y_i'(x) \to y_i'(x) + \alpha\eta_i'(x) \tag{5.2.12}$$

which means we can write F as $F(\alpha) = \int_a^b dx f(y_i + \alpha\eta_i, y_i' + \alpha\eta_i')$ where$i = 1, \cdots, N$. We have known that $\frac{dF}{d\alpha}\Big|_{\alpha=0} = 0$, so

$$\begin{aligned}
\frac{dF}{d\alpha}\bigg|_{\alpha=0} &= \frac{d}{d\alpha}\int_a^b dx f(y_i + \alpha\eta_i, y_i' + \alpha\eta_i')\bigg|_{\alpha=0} \\
&= \int_a^b dx \sum_i \left(\frac{\partial f}{\partial(y_i + \alpha\eta_i)}\frac{d(y_i + \alpha\eta_i)}{d\alpha} + \frac{\partial f}{\partial(y_i' + \alpha\eta_i')}\frac{d(y_i' + \alpha\eta_i')}{d\alpha}\right)\bigg|_{\alpha=0} \\
&= \int_a^b dx \sum_i \left(\frac{\partial f}{\partial y_i}\eta_i + \frac{\partial f}{\partial y_i'}\eta_i'\right) \\
&= \int_a^b dx \sum_i \left(\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'}\right)\eta_i = 0
\end{aligned} \tag{5.2.13}$$

Because η_i is arbitrary and independent of each other, we finally get

$$\boxed{\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} = 0, \quad i = 1, \cdots, N} \tag{5.2.14}$$

Before going to the constraint situation, we need to introduce a new notation δ, which reads "The variation of$\cdots$" As the same meaning in (5.2.2) and (5.2.3), the symbol δ can be considered to be a minimal change, which is the first order Taylor expansion at $\alpha = 0$, denoted by

$$F + \delta F = F(\alpha = 0) + \frac{dF}{d\alpha}\bigg|_{\alpha=0} d\alpha + O(\delta^2) \Rightarrow \delta F = \frac{dF}{d\alpha}\bigg|_{\alpha=0} d\alpha \tag{5.2.15}$$

We define $Y(x, \alpha) = y(x) + \alpha\eta(x);\ \ Y'(x, \alpha) = y'(x) + \alpha\eta'(x)$. Then the meaning of δy is

$$\delta y = \frac{\partial Y}{\partial \alpha}\bigg|_{\alpha=0} d\alpha = \eta(x) d\alpha \tag{5.2.16}$$

At the same time,

$$\delta y' = \frac{\partial Y'}{\partial \alpha}\bigg|_{\alpha=0} d\alpha = \eta'(x) d\alpha \tag{5.2.17}$$

So $\delta y' = \frac{d}{dx}\delta y$. Then the variation of $f[y, y', x]$ is

$$\delta f = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y' \tag{5.2.18}$$

This is just the total differential $df = (\partial f/\partial\alpha)_{\alpha=0}d\alpha$ of the function $f[Y(x,\alpha), Y'(x,\alpha), x]$ at $\alpha = 0$ with considered the only variable. Similarly, the boundary conditions of the variation functions are

$$\begin{cases} \delta y(a) = \delta y(b) = 0, \quad \delta y'(a) = \delta y'(b) = 0 \\ \eta(a) = \eta(b) = 0, \quad \eta'(a) = \eta'(b) = 0 \end{cases}$$

Then the variation of the functional F is

$$\begin{aligned} \delta F &= \delta \int_a^b f(y, y', x)dx \\ &= \int_a^b \delta f(y, y', x)dx = \int_a^b \left(\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y'\right)dx \\ &= \int_a^b \frac{\partial f}{\partial y}\delta y\, dx + \int_a^b \frac{\partial f}{\partial y'}\delta y'\, dx \\ &= \int_a^b \frac{\partial f}{\partial y}\delta y\, dx + \left(\frac{\partial f}{\partial y'}\delta y\Big|_a^b - \int_a^b \frac{d}{dx}\frac{\partial f}{\partial y'}\delta y\, dx\right) \\ &= \int_a^b \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\delta y\, dx \end{aligned} \tag{5.2.19}$$

Where the variation of the functional must be 0 ($\delta F = 0$), therefore the final result is as before. Note that the term $\frac{\partial f}{\partial y'}\delta y\Big|_b^a$ is 0 because of the boundary condition.

▷ **Euler-Lagrange Equation under Constraint Condition**

Consider a N-variable functional $F[y_1(x), y_1'(x), \cdots, y_N(x), y_N'(x)] = \int_a^b dx f(y_1(x), y_1'(x), \cdots, y_N(x), y_N'(x))$, but there are M constraint conditions between every coordinate where $1 \le M \le N-1$ such that the degree of freedom[3] becomes $N - M$.

We define the N constraint conditions to be the following functions that:

$$\begin{cases} g_1(y_1, y_1', \cdots, y_N, y_N') = 0 \\ g_2(y_1, y_1', \cdots, y_N, y_N') = 0 \\ \quad \vdots \\ g_M(y_1, y_1', \cdots, y_N, y_N') = 0 \end{cases} \tag{5.2.20}$$

[3]Degrees of freedom (DOF) refer to the number of independent parameters that define a system's state or motion. Constraints reduce the available DOF.

We have to perform variational calculus under the constraint of specific coordinates. We know that $g_k(y_j, y_j') = 0$, so it becomes $g_k(y_j + \alpha\eta_j, y_j' + \alpha\eta_j') = g(\alpha) = 0$
The functions are differentiable to first order and 0 when $\alpha = 0$, which is

$$\left.\frac{dg_k}{d\alpha}\right|_{\alpha=0} = \sum_j \left(\frac{\partial g_k}{\partial y_j}\eta_j + \frac{\partial g_k}{\partial y_j'}\eta_j'\right) \tag{5.2.21}$$

The function $\eta(x)$ is not arbitrary anymore, it should satisfy the differential equation shown above. But if we define a new function that

$$\tilde{f}(y_j, y_j') \coloneqq f(y_j, y_j') + \sum_k \lambda_k(x) g_k(\alpha) = \tilde{f}(y_j, y_j', \lambda_k(x)) \tag{5.2.22}$$

where $\lambda_k(x)$ is the Lagrange multiplier. The Lagrange multiplier projects the original function onto the constraint function in a certain way, so that the newly defined function automatically resides in the intersection of the original function and the constraint function. As a result, it transforms into an unconstrained variational problem within this intersection, somewhat analogous to solving for the intersection surface in three dimensions.
In formal terms, the Lagrange multiplier method transforms the problem in such a way that we can perform variation on a new unconstrained function, whose solution naturally satisfies the original constraint conditions.
So we can write down the performance of the variation on it

$$\left.\frac{d}{d\alpha}F(\alpha)\right|_{\alpha=0} = \left.\frac{d}{d\alpha}\int_a^b dx \tilde{f}(y_j + \alpha\xi_j(x), y_j' + \alpha\xi_j(x), \lambda_k(x))\right|_{\alpha=0} \tag{5.2.23}$$

Note that $\xi(x)$ is a structured subset of the arbitrary space $\eta(x)$, but it still inherently respects the constraint because the Lagrange multiplier ensures that the final equations account for it. Therefore, the reason why we choose $\xi(x)$ to be the arbitrary function is that it can be "freer" in the sense of being fully usable under constraints. The function $\xi : D_\xi \to R_\xi$, $\eta : D_\eta \to R_\eta$, so $D_\xi \subseteq D_\eta$ and $\mathrm{Im}(\xi) \subseteq \mathrm{Im}(\eta)$, where R represents the range of the function while "Im" represents the subset of R consisting all the images under the function of the variable x in the domain D, that is

$$\mathrm{Im}(\xi) : \{\xi(x) | \, x \in D_\xi\}$$

More generally, when a constraint is imposed, only variations that lie within the tangent space $T_y(C)$ are admissible, because these directions allow motion along the constraint surface without deviating from it. Where C is the constraint manifold[4], and $T_y(C)$ means that the tangent plane on the function $y(x)$ under the constraint manifold. It can be denoted by

$$\delta y \in T_y(C) \subseteq F; \quad \delta y' \in T_{y'}(C) \subseteq F$$

[4]You don't need to understand what it means, I was just saying it for fun. For those who are interested, you can refer to the book *Mathematical Methods of Classical Mechanics, Arnold.*

And the derivative will be

$$\begin{aligned}\left.\frac{d}{d\alpha}F(\alpha)\right|_{\alpha=0} &= \int_a^b dx \sum_j \left[\frac{\partial \tilde{f}}{\partial(y_j+\alpha\xi_j)}\frac{d(y_j+\alpha\xi_j)}{d\alpha}+\frac{\partial \tilde{f}}{\partial(y'_j+\alpha\xi'_j)}\frac{d(y'_j+\alpha\xi'_j)}{d\alpha}\right]\Bigg|_{\alpha=0} \\ &= \int_a^b dx \sum_j \left(\frac{\partial \tilde{f}}{\partial y_j}\xi_j+\frac{\partial \tilde{f}}{\partial y'_j}\xi'_j\right) = \int_a^b dx \sum_j \left(\frac{\partial \tilde{f}}{\partial y_j}-\frac{d}{dx}\frac{\partial \tilde{f}}{\partial y'_j}\right)\xi_j\end{aligned} \tag{5.2.24}$$

In detail, the index j can be separated into two different groups. One is under the M constraint conditions, there must exist $\lambda_k, 1 \le k \le M < N$ such that $\frac{\partial \tilde{f}}{\partial y_j}-\frac{d}{dx}\frac{\partial \tilde{f}}{\partial y'_j}=0$; while the other one is without constraint, where $1 \le j \le N-M$, so the perturbation function $\xi(x)$ is arbitrary. Thus, these two groups indicate that when a function is under constraint conditions, we can define a new function to get the extreme value of a functional, which satisfies

$$\boxed{\frac{\partial \tilde{f}}{\partial y_j}-\frac{d}{dx}\frac{\partial \tilde{f}}{\partial y'_j}=0} \tag{5.2.25}$$

where $\tilde{f}(y_j, y'_j) \coloneqq f(y_j, y'_j)+\sum_k \lambda_k(x)g_k(\alpha)$.
In another situation, we can also directly deal with the original function $f(y_j, y'_j)$ and $\lambda_k g_k(y_j, y'_j)$. In other words, we should calculate the variation of each one, δf and δg. We've solved the result of δf before, so by (5.2.21) the last variation δg_k can be written as

$$\delta g_k = \left.\frac{dg_k}{d\alpha}\right|_{\alpha=0} d\alpha = \sum_j \left(\frac{\partial g_k}{\partial y_j}\delta y_j+\frac{\partial g_k}{\partial y'_j}\delta y'_j\right)=0 \tag{5.2.26}$$

Because it is 0, so for every k, the variation of g_k are all 0, so

$$\sum_k \delta g_k = \sum_k \lambda_k \delta g_k = \sum_{jk} \lambda_k \left(\frac{\partial g_k}{\partial y_j}\delta y_j+\frac{\partial g_k}{\partial y'_j}\delta y'_j\right)=0 \tag{5.2.27}$$

So it should satisfy that

$$\begin{aligned}\int_a^b dx \sum_k \delta g_k &= \int_a^b dx \sum_k \lambda_k \delta g_k \\ &= \int_a^b dx \sum_{jk} \lambda_k \left(\frac{\partial g_k}{\partial y_j}\delta y_j+\frac{\partial g_k}{\partial y'_j}\delta y'_j\right) \\ &= \int_a^b dx \sum_k \lambda_k \sum_j \left[\frac{\partial g_k}{\partial y_j}-\left(\frac{d}{dx}\frac{\partial g_k}{\partial y'_j}\right)\right]\delta y_j\end{aligned} \tag{5.2.28}$$

So, to ensure the δF is under the constraint, we must combine them, which is equivalent

to (5.2.26).

$$\begin{aligned}
&\delta(F+\lambda g)=\delta F+\lambda\delta g=0\\
\Rightarrow&\int_a^b dx\sum_k\lambda_k\sum_j\left[\frac{\partial g_k}{\partial y_j}-\left(\frac{d}{dx}\frac{\partial g_k}{\partial y_j'}\right)\right]\delta y_j+\\
&\int_a^b dx\sum_j\left(\frac{\partial f}{\partial y_j}-\frac{d}{dx}\frac{\partial f}{\partial y_j'}\right)\delta y_j=0
\end{aligned}\tag{5.2.29}$$

For the jth parameter, the equation must be modified to

$$\begin{aligned}
&\left(\frac{\partial f}{\partial y_j}-\frac{d}{dx}\frac{\partial f}{\partial y_j'}\right)+\sum_k\lambda_k\left[\frac{\partial g_k}{\partial y_j}-\left(\frac{d}{dx}\frac{\partial g_k}{\partial y_j'}\right)\right]=0\\
\Rightarrow&\frac{\partial f}{\partial y_j}+\sum_k\lambda_k\frac{\partial g_k}{\partial y_j}=\frac{d}{dx}\left(\frac{\partial f}{\partial y_j'}+\sum_k\frac{\partial g_k}{\partial y_j'}\right)
\end{aligned}\tag{5.2.30}$$

Note that most of the physical or mathematical problems we've ever met initially focus on the constraint conditions, which are independent of the variable y_j', so the constraint functions become $g_k(y_j,x)$, so $\frac{\partial g_k}{\partial y_j'}=0$. After that, we can get the familiar Euler-Lagrange equation under the constraint situation

$$\boxed{\frac{\partial f}{\partial y_j}+\sum_k\lambda_k\frac{\partial g_k}{\partial y_j}=\frac{d}{dx}\frac{\partial f}{\partial y_j'}}\tag{5.2.31}$$

▷ **Legendre Transformation between Hamiltonian and Lagrangian**

We have had the Lagrangian, we can use it to define the general momentum p_i

$$\mathcal{L}(q_i,\dot{q}_i)=T(\dot{q}_i)-V(q_i)\Rightarrow p_i=\frac{\partial\mathcal{L}}{\partial\dot{q}_i}=p_i(q_i,\dot{q}_i)\tag{5.2.32}$$

Note that we call the general momentum a **conjugate momentum**[5], I'm going to discuss the relative mathematical principle in detail in the next chapter. So the general velocity $\dot{q}_i=\dot{q}_i(q_i,p_i)$. After this, we define the Hamiltonian as

$$\mathcal{H}\coloneqq\sum\dot{q}_ip_i-\mathcal{L}=\sum\dot{q}_i\frac{\partial\mathcal{L}}{\partial\dot{q}_i}-\mathcal{L}\tag{5.2.33}$$

More specifically, the velocity $\dot{q}_i$ and the momentum p_i conjugate, so the Hamiltonian $\mathcal{H}$ can be described as a function which depends on the parameters q_i and p_i, that is,

$$\mathcal{H}=\sum\dot{q}_i(p_i,q_i)p_i-\mathcal{L}(q_i,\dot{q}_i(p_i,q_i))\coloneqq\mathcal{H}(q_i,p_i)\tag{5.2.34}$$

The difference between Hamiltonian and Lagrangian is the variable. Lagrangian depends on the velocity $\dot{q}_i$ while Hamiltonian relies on the momentum of the particle, which is necessary to be induced because we can conveniently solve a dynamical system by the

[5]conjugate momentum means that we can identify two conjugate elements, and knowing one allows us to deduce the other.

particle's momentum. It is widely used in quantum physics. Before analyzing the relation between Hamiltonian and Lagrangian, we must introduce "Legendre Transformation" in its mathematical principle in detail.
In this report, we only focus on the **strictly convex function**.
Consider the interval $I \subset \mathbb{R}$, and there is a strictly convex function $f : I \to \mathbb{R}$.We assume that the transformed function $f^* : I^* \to \mathbb{R}$. The definition of Legendre transformation is

$$f^*(p) = \sup_{x \in I} (px - f(x)) \tag{5.2.35}$$

where $p \coloneqq \frac{df}{dx}$, in addition, the domain of f^* becomes

$$I^* : \left\{ p \in \mathbb{R} | \sup_{x \in I} (px - f(x)) < \infty \right\} \tag{5.2.36}$$

Understand it more straightforwardly. The definition of Legendre transformation can be interpreted into

$$f^*(p) = pu - f(u)|_{\frac{d[pu-f(u)]}{du}=0}, u \in I \tag{5.2.37}$$

Which means it represents the unique function in the Legendre transformation because we need to find the parameter $(u, f(u))$ to make the value of $f^*(p)$ have the maximum. More intuitively, when we find the slope of the tangent at the point $x = x_0$. All the curves passing through the parameter $(u, f(u))$ will compose a family of lines $y = f'(x_0)(x - u) + f(u)$ which is not a unique solution. That means we can't just use the slope to deduce the original function unless we set another condition.
The family of lines is

$$y = f'(x_0)x - (f'(x_0)u - f(u)) \tag{5.2.38}$$

When $u = x_0$, y reaches its maximum value, which means the function relationship $f'(x_0)u - f(u)$ has its minimum value. This statement precisely validates the previously mentioned goal of finding u that maximizes $f^*(p)$, and this extreme value $f^* = y$ corresponds to the y-intercept of the tangent line equation at the point $(x_0, f(x_0))$.
As a result, the standard solution of the Legendre transformation is strictly confined to the tangent point of the original function, ensuring a one-to-one relationship between the two. So x and $p(x)$ has a one-to-one inverse function relationship

$$p = f'(x) \Rightarrow x = (f')^{-1}(p) \tag{5.2.39}$$

The Legendre transformation thus becomes

$$f^*(p) = p\left[(f')^{-1}(p)\right] - f\left((f')^{-1}(p)\right) \tag{5.2.40}$$

For example, the original function $f(x) = e^x$, the slope of the function is

$$p = \frac{d}{dx}e^x = e^x \Rightarrow x = \ln p$$

So the Legendre transformation of $f(x) = e^x$ will be

$$f^*(p) = px - f(x) = p \ln p - e^{\ln p} = p(\ln p - 1)$$

After these, it's obvious that Hamiltonian $\mathcal{H}$ is the Legendre transformation of Lagrangian $\mathcal{L}$ as $\mathcal{L}$ is the original function depending on $(q_i, \dot{q}_i)$ because $\mathcal{H}(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}, q_i) = \sum \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L}$. So

the velocity $\dot{q}_i$ and the momentum p_i are called conjugated.
Let the Hamiltonian $\mathcal{H} = \mathcal{H}(q_i, p_i, t)$ and the Lagrangian $\mathcal{L} = \mathcal{L}(q_i, p_i, t)$. Then the total derivative of $\mathcal{H} = \mathcal{H}(q_i, p_i, t)$ is

$$d\mathcal{H} = \sum \frac{\partial \mathcal{H}}{\partial q_i} dq_i + \sum \frac{\partial \mathcal{H}}{\partial p_i} dp_i + \frac{\partial \mathcal{H}}{\partial t} dt \tag{5.2.41}$$

By the Legendre transformation of $\mathcal{L}$, $\mathcal{H}(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}, q_i) = \sum \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L}$, so

$$\begin{aligned} d\mathcal{H} &= d\left(\sum \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L}\right) \\ &= \sum p_i d\dot{q}_i + \sum \dot{q}_i dp_i - d\mathcal{L} \\ &= \sum p_i d\dot{q}_i + \sum \dot{q}_i dp_i - \sum \frac{\partial \mathcal{L}}{\partial q_i} dq_i - \sum \frac{\partial \mathcal{L}}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial \mathcal{L}}{\partial t} dt \end{aligned} \tag{5.2.42}$$

By $\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = p_i$, and the system satisfies Euler-Lagrange equation $\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$, so

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt} p_i = \dot{p}_i \Rightarrow \sum \frac{\partial \mathcal{L}}{\partial q_i} dq_i = \sum \dot{p}_i dq_i \tag{5.2.43}$$

The (5.2.42) becomes

$$d\mathcal{H} = \sum \dot{q}_i dp_i - \sum \dot{p}_i dq_i - \frac{\partial \mathcal{L}}{\partial t} dt \tag{5.2.44}$$

By coefficient comparison with (5.2.41), we can get the most important equation, known as the Hamilton-Jacobi equation

$$\boxed{\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}; \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}; \quad \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}} \tag{5.2.45}$$

With these results, we can ensure that $\mathcal{H}$ is the Legendre transformation of $\mathcal{L}$.

▷ Calculus of Variations in Hamiltonian function

We can also use the calculus of variations to get the Hamilton-Jacobi equation in the same way as in Lagrangian analysis.
We substitute it into the stationary action principle, which is

$$\delta S = \int_{t_1}^{t_2} dt\ \delta \mathcal{L} = \int_{t_1}^{t_2} dt\ \delta \left[\sum \dot{q}_i p_i - \mathcal{H}(q_i, p_i)\right] = 0 \tag{5.2.46}$$

And the variations of these parameters are

$$\begin{cases} q_i(t) \to q_i(t) + \alpha \eta_i(t) \Rightarrow \dot{q}_i(t) \to \dot{q}_i(t) + \alpha \dot{\eta}_i(t) \\ p_i(t) \to p_i(t) + \alpha \xi_i(t) \Rightarrow \dot{p}_i(t) \to \dot{p}_i(t) + \alpha \dot{\xi}_i(t) \end{cases} \tag{5.2.47}$$

So the variation of the Lagrangian is

$$\begin{aligned} \delta\left[\sum \dot{q}_i p_i - \mathcal{H}(q_i, p_i)\right] &= \sum \delta(\dot{q}_i p_i) - \delta \mathcal{H}(q_i, p_i) \\ &= \sum \frac{\partial(\dot{q}_i p_i)}{\partial \dot{q}_i} \delta \dot{q}_i + \sum \frac{\partial(\dot{q}_i p_i)}{\partial p_i} \delta p_i - \sum \frac{\partial \mathcal{H}}{\partial q_i} \delta q_i - \sum \frac{\partial \mathcal{H}}{\partial p_i} \delta p_i \\ &= \sum \left(p_i \delta \dot{q}_i + \dot{q}_i \delta p_i - \frac{\partial \mathcal{H}}{\partial q_i} \delta q_i - \frac{\partial \mathcal{H}}{\partial p_i} \delta p_i\right) \end{aligned} \tag{5.2.48}$$

Substituting it into (5.2.46), we can get

$$\begin{aligned}\delta S &= \int_{t_1}^{t_2} dt \sum \left(p_i \delta \dot{q}_i + \dot{q}_i \delta p_i - \frac{\partial \mathcal{H}}{\partial q_i} \delta q_i - \frac{\partial \mathcal{H}}{\partial p_i} \delta p_i \right) \\ &= \int_{t_1}^{t_2} dt \sum \left(p_i \delta \dot{q}_i - \frac{\partial \mathcal{H}}{\partial q_i} \delta q_i \right) + \int_{t_1}^{t_2} dt \sum \left(\dot{q}_i - \frac{\partial \mathcal{H}}{\partial p_i} \right) \delta p_i\end{aligned} \tag{5.2.49}$$

Note that the first term on the right-hand side of the second equality is

$$\int_{t_1}^{t_2} dt \sum \left(p_i \delta \dot{q}_i - \frac{\partial \mathcal{H}}{\partial q_i} \delta q_i \right) = \sum p_i \delta q_i \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt \sum \left(-\dot{p}_i - \frac{\partial \mathcal{H}}{\partial q_i} \right) \delta q_i$$

So, for the ith parameter, we can claim that

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}; \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} \tag{5.2.50}$$

This is the same result as (5.2.45).
In another situation, when the system is under the constraint conditions $g_j(q_l)$. We can use the same way to add Lagrange multipliers in the variation of the constraint conditions, which is

$$\begin{aligned}\delta S = \delta(S + \lambda g) &= \int_{t_1}^{t_2} dt \ \delta \left[\sum_l \dot{q}_l p_l - \mathcal{H}(q_l, p_l) \right] + \lambda \delta g = 0 \\ &= \int_{t_1}^{t_2} dt \ \delta \left[\sum_l \dot{q}_l p_l - \mathcal{H}(q_l, p_l) \right] + \int_{t_1}^{t_2} dt \ \delta \sum_j \lambda_j(t) g_j(q_l)\end{aligned} \tag{5.2.51}$$

In this way, we can see a similar result of the constraint condition that

$$\int_{t_1}^{t_2} dt \sum_l \left(-\dot{p}_l - \frac{\partial \mathcal{H}}{\partial q_l} + \sum_j \lambda_j(t) \frac{\partial g_j}{\partial q_l} \right) \delta q_l + \int_{t_1}^{t_2} dt \sum_l \left(\dot{q}_l - \frac{\partial \mathcal{H}}{\partial p_l} \right) \delta p_l = 0 \tag{5.2.52}$$

Note that the perturbation functions $\eta_l(t)$ are arbitrary functions because of the introduction of the Lagrange multipliers. On the other hand, the constraint conditions g_j are independent of p_l, so the perturbation functions $\xi_l(t)$ are arbitrary functions originally. Then we must change the lth form of the Euler-Lagrange function to be

$$\boxed{\begin{cases} \dot{p}_l - \sum_j \lambda_j(t) \dfrac{\partial g_j}{\partial q_l} = -\dfrac{\partial \mathcal{H}}{\partial q_l} \\ \dot{q}_l = \dfrac{\partial \mathcal{H}}{\partial p_l} \end{cases}} \tag{5.2.53}$$

5.3 Applications of the Calculus of Variations

▷ No One Has Told Me Why a Straight Line is the Shortest Distance!

From elementary to high school, every math teacher tells us that the shortest distance between two points is a straight line. We've always believed it and taken it for granted. However, no one ever questioned this claim, nor did any teacher actually prove it to us. In mathematics and physics, such intuitive reasoning is not acceptable. So, fellow physics students, let's uncover the truth behind this statement together.

Here, our proof and explanation will focus solely on the context of Euclidean geometry since that is what we have been familiar with. The first postulate of Euclidean geometry states that "a straight line segment can be drawn joining any two points." This is where we begin our investigation. Consider the arc length element

$$ds = \sqrt{(dx)^2 + (dy)^2} \tag{5.3.1}$$

By this element, we can construct a functional $F[x, y, y']$ which is

$$L = \int_{x_0}^{x_1} ds = \int_{x_0}^{x_1} \sqrt{(dx)^2 + (dy)^2} = \int_{x_0}^{x_1} \sqrt{1 + y'^2}\, dx \tag{5.3.2}$$

Where y' is the derivative of y with respect to x. So the functional is $F[x, y, y'] = \int_{x_0}^{x_1} f(x, y, y')dx$, where $f(x, y, y') = \sqrt{1 + y'^2}$. There exist (x, y, y') such that the functional has the extremum, which is the shortest distance. So, the functional satifies

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{\partial \sqrt{1 + y'^2}}{\partial y} - \frac{d}{dx}\frac{\partial \sqrt{1 + y'^2}}{\partial y'} = 0 \tag{5.3.3}$$

Substituting the function $f(x, y, y')$ into (5.3.3), we obtain

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) = 0 \Rightarrow \frac{y'}{\sqrt{1 + y'^2}} = c \tag{5.3.4}$$

where c is a constant. Here, we can obtain an algebraic relationship between y' and c. By rearranging (5.3.4), we express it as a relation between y' and c. That is

$$y'^2 = \frac{c^2}{1 - c^2} \Rightarrow y' = \pm\frac{|c|}{\sqrt{1 - c^2}} := a \tag{5.3.5}$$

By separating the variables, we have

$$dy = a\, dx \Rightarrow y = \int a\, dx = ax + b \tag{5.3.6}$$

Where b is another constant. So, we can finally prove that the straight line is truly the shortest distance between the points.

▷ **Fermat's Principle**

Fermat's principle states that light travels along the path that minimizes the total travel time. In high school, we encountered the familiar equation

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

which is known as the law of refraction. At that time, we proved this equation using geometric arguments and the relationship between incident and refracted angles. However, upon closer inspection, this explanation merely describes the *apparent phenomenon* of how light bends at the interface rather than the *underlying reason* why the bending occurs. In fact, this problem lies within the scope of variational methods and functionals.

We can construct a functional T

$$T[x, y, y', v(x, y)] = \int_{x_1}^{x_2} \frac{ds}{v(x, y)} = \int_{x_1}^{x_2} \frac{\sqrt{1 + y'^2}}{v(x, y)}\, dx \tag{5.3.7}$$

Where v is the speed of light depending on the position n. When the functional has the extremum, it satisfy

$$\frac{\partial\left(\sqrt{1 + y'^2}/v(x, y)\right)}{\partial y} + \frac{d}{dx}\frac{\partial\left(\sqrt{1 + y'^2}/v(x, y)\right)}{\partial y'} = 0 \tag{5.3.8}$$

While $v(x, y) = v \in \mathbb{R}$, it means that the refractive indices are the same on both sides, and the problem reduces to the previous shortest path problem. In other words, when light travels through a medium with a uniform refractive index, it moves in a straight line.

Now, consider a system with an interface at $y = 0$, where the refractive index is n_1 (with speed $v_1 \in \mathbb{R}$) for $y > 0$, and n_2 (with speed $v_2 \in \mathbb{R}$) for $y < 0$. Suppose the point of incidence lies at $(x, 0)$. According to our previous conclusions, the speed remains constant within each homogeneous medium, and the path of the light is a straight line. Therefore, the functional $T[x, y, y', v(x, y)]$ degenerates into a function of a single variable, $T(x)$, which is referred to as a **finite-dimensional problem**. In this case, the extremization of the functional reduces to a standard calculus problem involving an ordinary function, and the distance element ds becomes

$$\Delta S_1 = \sqrt{(x - x_1)^2 + y_1^2}\,, \Delta S_2 = \sqrt{(x - x_2)^2 + y_2^2} \tag{5.3.9}$$

We then have

$$T(x) = \frac{\Delta S_1}{v_1} + \frac{\Delta S_2}{v_2} = \frac{\sqrt{(x - x_1)^2 + y_1^2}}{v_1} + \frac{\sqrt{(x - x_2)^2 + y_2^2}}{v_2} \tag{5.3.10}$$

When the total time has the extemum, we obtain

$$\begin{aligned}\frac{dT(x)}{dx} &= \frac{d}{dx}\frac{\sqrt{(x - x_1)^2 + y_1^2}}{v_1} + \frac{d}{dx}\frac{\sqrt{(x - x_2)^2 + y_2^2}}{v_2} \\ &= \frac{1}{v_1}\frac{x - x_1}{\sqrt{(x - x_1)^2 + y_1^2}} + \frac{1}{v_2}\frac{x - x_2}{\sqrt{(x - x_2)^2 + y_2^2}} = 0\end{aligned}$$

We assume that $x_1 < x < x_2$, and we multiply both sides by c, which is the speed of light constant. Therefore, we obtain

$$\frac{1}{v_1}\frac{x-x_1}{\sqrt{(x-x_1)^2+y_1^2}}+\frac{1}{v_2}\frac{x-x_2}{\sqrt{(x-x_2)^2+y_2^2}}=\frac{c}{v_1}\frac{x-x_1}{\sqrt{(x-x_1)^2+y_1^2}}-\frac{c}{v_2}\frac{x_2-x}{\sqrt{(x-x_2)^2+y_2^2}}=0$$

So, we get

$$\frac{c}{v_1}\frac{x-x_1}{\sqrt{(x-x_1)^2+y_1^2}}=\frac{c}{v_2}\frac{x_2-x}{\sqrt{(x-x_2)^2+y_2^2}} \tag{5.3.11}$$

Where we redefine the constant c/v_1, c/v_2 as the refractive indices

$$\frac{c}{v_1}\coloneqq n_1,\quad \frac{c}{v_2}\coloneqq n_2 \tag{5.3.12}$$

and the term $(x-x_1)/\sqrt{(x-x_1)^2+y_1^2}$ is the sine of the angle θ_1 between the incident ray and the normal to the interface, while $(x_2-x)/\sqrt{(x-x_2)^2+y_2^2}$ is the sine of the angle θ_2 between the refracted ray and the normal to the interface. Rewriting (5.3.11), we can finally obtain the familiar form of Fermat's principle

$$n_1\sin\theta_1 = n_2\sin\theta_2 \tag{5.3.13}$$

▷ Cycloid Problem

A particle of mass m is subject to a uniform gravitational field g. It starts from rest at the point (x_1, y_1) and moves to a lower point (x_2, y_2), where $0 \le y_1 < y_2$. Find the path of the particle that minimizes the time taken for the motion. We define $(x_1, y_1) = (0, 0)$. To

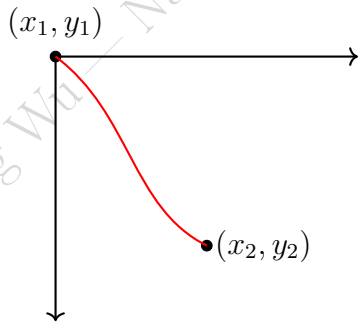

Figure 28: A Diagram of the Position and Trajectory Between Two Points

solve this problem, we need to create a functional. First, we shall find out the velocity(v) which is

$$\frac{1}{2}mv^2 = mgy \Rightarrow v = \sqrt{2gy} \tag{5.3.14}$$

so the infinitesimal displacement is $ds = \sqrt{dx^2+dy^2}$, it takes dt to travel this curve, which is $dt = ds/v$. So the total time is

$$\begin{aligned} T &= \int dt = \int \frac{ds}{v} = \int \frac{\sqrt{dx^2+dy^2}}{\sqrt{2gy}} \\ \Rightarrow T &= \frac{1}{\sqrt{2g}}\int_{y_1}^{y_2}\frac{\sqrt{1+\left(\frac{dx}{dy}\right)^2}}{\sqrt{y}}dy \end{aligned} \tag{5.3.15}$$

So we find the functional T about the total time, in which contains the function $f(x(y), x'(y), y) = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}/\sqrt{y}$. When the time T has the extreme value, it should satisfy the Euler-Lagrange equation.

$$\frac{\partial f}{\partial x(y)} - \frac{d}{dy}\frac{\partial f}{\partial x'(y)} = 0 \tag{5.3.16}$$

where $\frac{\partial f}{\partial x(y)} = 0$, so $\frac{d}{dy}\frac{\partial f}{\partial x'(y)} = 0$, which means the partial derivative $\frac{\partial f}{\partial x'(y)} = C$. WLOG let the constant $C = \frac{1}{\sqrt{2a}} > 0$, then

$$\frac{\partial f}{\partial x'(y)} = \frac{x'(y)}{\sqrt{y(1 + x'(y)^2)}} = \frac{1}{\sqrt{2a}} \Rightarrow x'(y) = \pm\frac{y}{\sqrt{2ay - y^2}} \tag{5.3.17}$$

Separate the variables, we get

$$dx = \pm\frac{y}{\sqrt{2ay - y^2}}dy \Rightarrow x = \int_0^y dy' \frac{y'}{\sqrt{2ay' - y'^2}} \tag{5.3.18}$$

where $x(y = 0) = 0$. Let $y' = a(1 - \cos\theta),\ dy' = a\sin\theta d\theta$, so the equation will be

$$\begin{aligned} x &= \int_0^\theta d\theta' \frac{a\sin\theta' a(1 - \cos\theta')}{\sqrt{2a^2(1 - \cos\theta') - a^2(1 + \cos^2\theta' - 2\cos\theta')}} \\ &= \int_0^\theta d\theta' \frac{a^2\sin\theta'(1 - \cos\theta')}{\sqrt{a^2(1 - \cos^2\theta)}} = a\int_0^\theta d\theta'(1 - \cos\theta') \end{aligned} \tag{5.3.19}$$

After calculation, we can obtain the brachistochrone trajectory

$$\boxed{x = a\theta - a\sin\theta} \tag{5.3.20}$$

Note that we don't care what is a and what is θ. When the trajectory satisfies the equation $x = a\theta - a\sin\theta$ is called **cycloid** in a brachistochrone problem.

▷ Dido Problem

Among all planar curves of fixed length L that connect two points on the x-axis and enclose area above the axis, determine the shape of the curve that maximizes the enclosed area.

This problem is named after Queen Dido of Carthage, who was granted as much land as could be enclosed by a single piece of oxhide. By cutting the oxhide into thin strips and forming a semicircular boundary along a straight shoreline, she maximized the area, a physical realization of a variational principle.

In this problem, there's a constraint condition, which is the length constraint

$$\int_{x_1}^{x_2} \sqrt{1 + (y'(x))^2}\, dx = L \tag{5.3.21}$$

On the other hand, the enclosed area above the x-axis is

$$A = \int_{x_1}^{x_2} y(x)\, dx \tag{5.3.22}$$

However, the function y is subject to a constraint, reducing the problem to a constrained variational problem. In this case, the constraint is given by equation (5.3.21). Therefore, we can combine (5.3.21) with (5.3.22) through the Lagrange multiplier λ and redefine the functional A, which is

$$A[y, y', \lambda] = \int_{x_1}^{x_2} \left(y + \lambda\sqrt{1 + y'^2} \right) dx = \int_{x_1}^{x_2} \tilde{f}(y, y', \lambda) dx \tag{5.3.23}$$

Because $\delta A[y] + \delta L = \delta A[y, y', \lambda] = 0$, the function $\tilde{f}(y, y', \lambda)$ satisfies

$$\frac{\partial \tilde{f}}{\partial y} - \frac{d}{dx}\frac{\partial \tilde{f}}{\partial y'} = \frac{\partial \left(y + \lambda\sqrt{1 + y'^2} \right)}{\partial y} - \frac{d}{dx}\frac{\partial \left(y + \lambda\sqrt{1 + y'^2} \right)}{\partial y'} = 0 \tag{5.3.24}$$

We obtain

$$1 - \frac{d}{dx}\frac{\lambda y'}{\sqrt{1 + y'^2}} = 0 \Rightarrow d\left(\frac{y'}{\sqrt{1 + y'^2}} \right) = \frac{1}{\lambda} dx \tag{5.3.25}$$

Integrating both sides,

$$\frac{y'}{\sqrt{1 + y'^2}} = \frac{x}{\lambda} + C \Rightarrow \left(\frac{y'}{\sqrt{1 + y'^2}} \right)^2 = \left(\frac{x}{\lambda} + C \right)^2 \tag{5.3.26}$$

By arranging (5.3.26), we can thus obtain a differential equation

$$\frac{dy}{dx} = \frac{\left(\frac{x}{\lambda} + C \right)}{\sqrt{1 - \left(\frac{x}{\lambda} + C \right)^2}} \Rightarrow y = \int \frac{\left(\frac{x}{\lambda} + C \right)}{\sqrt{1 - \left(\frac{x}{\lambda} + C \right)^2}} dx \tag{5.3.27}$$

Let $x/\lambda + C = \sin\theta$, $dx = \lambda\cos\theta$. Substituting these into (5.3.27), we can compute

$$y = \int \frac{\sin\theta}{\sqrt{1 - \sin^2\theta}} \lambda\cos\theta d\theta = -\lambda\cos\theta + C' \tag{5.3.28}$$

By $x/\lambda + C = \sin\theta$ and (5.3.28), we obtain the parametric equations for the trajectory that yields the extremum in the Dido problem

$$\begin{cases} x = \lambda\sin\theta - \lambda C \\ y = -\lambda\cos\theta + C' \end{cases} \tag{5.3.29}$$

It's a semicircle. Although this form differs from the standard parametric representation of a circle, it is essentially the result of applying a clockwise rotation matrix followed by a translation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \lambda\cos\theta \\ \lambda\sin\theta \end{pmatrix} + \begin{pmatrix} -\lambda C \\ C' \end{pmatrix} = \begin{pmatrix} \lambda\sin\theta - \lambda C \\ -\lambda\cos\theta + C' \end{pmatrix} \tag{5.3.30}$$

Note that it is not necessary to express θ in terms of inverse trigonometric functions. As long as x and y are related through $\sin\theta$ and $\cos\theta$, the physical nature of the problem ensures that the solution retains the periodic and geometric properties of trigonometric motion.

Chapter 6. Introduction to Linear Algebra

In linear algebra, the main goal is to analyze the structure of matrices, vector spaces in n dimensions, and linear equations. Among these, matrices are the most important concept throughout the entire subject. To better understand matrices, I want to start with the basic formula for the inverse matrix. I will use only algebra and rigorous conditions to make the proof clearer than what is typically found in textbooks on physical mathematics. By doing so, we can explore a different version of the underlying structure. To keep the content concise, **I will skip most of the concepts and interpretations already covered in high school and instead focus on the higher-level mathematical structures**.

6.1 Basic Rules for Matrices

▷ **Matrix Multiplication**

An $m \times n$ **matrix** is a rectangular array consisting of m rows and n columns. The entries in a matrix can be numbers, symbols, or even functions.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \ddots & a_{mn} \end{pmatrix}_{m\times n} \tag{6.1.1}$$

Let $A \in \mathbb{R}^{m\times n}$ be an m-by-n matrix. The ordered tuple $(a_{11}, a_{12}, \cdots, a_{1n})$ is called the first row of the matrix, while $(a_{1j}, a_{2j}, \cdots, a_{mj})$ is called the j-th column.
If we want to refer to the entry in the i-th row and j-th column, we write:

$$A_{ij} = a_{ij} \tag{6.1.2}$$

Moreover, there is a special matrix called the **transpose** of A, denoted A^T, which switches the roles of rows and columns. The transpose of A is defined as:

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1j} & a_{2j} & \cdots & a_{mj} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}_{n\times m} \tag{6.1.3}$$

This definition implies the identity:

$$(A^T)_{ij} = A_{ji} = a_{ji} \tag{6.1.4}$$

So the matrix $A^T \in \mathbb{R}^{n\times m}$ is an n-by-m matrix.
In linear algebra, we consider a vector $\mathbf{v}$ to be a column matrix[6], such as

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \tag{6.1.5}$$

In the context of matrices, the definition of the inner product becomes

$$\mathbf{v}\cdot\mathbf{u} = \sum_i v_i u_i \Rightarrow \mathbf{v}^T\mathbf{u} = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \tag{6.1.6}$$

Suppose that we have vectors $\mathbf{a}_1^T = (a_{11} \quad a_{12} \quad \cdots \quad a_{1m}), \cdots, \mathbf{a}_n^T = (a_{n1} \quad a_{n2} \quad \cdots \quad a_{nm}) \in \mathbb{R}^{1\times m}$, then we define a n-by-m matrix A as

$$A = \begin{pmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_n^T \end{pmatrix}_{n\times m} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \tag{6.1.7}$$

Additionally, we have vectors $\mathbf{b}_1 = (b_{11} \quad b_{21} \quad \cdots \quad b_{m1})^T, \cdots, \mathbf{b}_n = (b_{1n} \quad b_{2n} \quad \cdots \quad b_{mn})^T \in \mathbb{R}^{n\times 1}$ then we define a m-by-n matrix B as

$$B = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{pmatrix}_{m\times n} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & a_{nm} \end{pmatrix} \tag{6.1.8}$$

Now that we have defined these two matrices, we can define their product using the dot product formulation. That is

$$AB = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{pmatrix} \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{pmatrix} \coloneqq \begin{pmatrix} (\mathbf{a}_1\cdot\mathbf{b}_1) & \cdots & (\mathbf{a}_1\cdot\mathbf{b}_n) \\ (\mathbf{a}_2\cdot\mathbf{b}_1) & \cdots & (\mathbf{a}_2\cdot\mathbf{b}_n) \\ \vdots & \ddots & \vdots \\ (\mathbf{a}_n\cdot\mathbf{b}_1) & \cdots & (\mathbf{a}_n\cdot\mathbf{b}_n) \end{pmatrix}_{n\times n} \tag{6.1.9}$$

By the property of the dot product, we obtain

$$\begin{pmatrix} (\mathbf{a}_1\cdot\mathbf{b}_1) & \cdots & (\mathbf{a}_1\cdot\mathbf{b}_n) \\ (\mathbf{a}_2\cdot\mathbf{b}_1) & \cdots & (\mathbf{a}_2\cdot\mathbf{b}_n) \\ \vdots & \ddots & \vdots \\ (\mathbf{a}_n\cdot\mathbf{b}_1) & \cdots & (\mathbf{a}_n\cdot\mathbf{b}_n) \end{pmatrix}_{n\times n} = \begin{pmatrix} \sum_k a_{1k}b_{k1} & \cdots & \sum_k a_{1k}b_{kn} \\ \sum_k a_{2k}b_{k1} & \cdots & \sum_k a_{2k}b_{kn} \\ \vdots & \ddots & \vdots \\ \sum_k a_{nk}b_{k1} & \cdots & \sum_k a_{nk}b_{kn} \end{pmatrix}_{n\times n} \tag{6.1.10}$$

[6]You can also consider it to be a row matrix as long as you have a proper definition.

That is, the matrix multiplication can be rewritten as

$$\boxed{(AB)_{ij} = \sum_k a_{ik} b_{kj}} \tag{6.1.11}$$

We need to ensure that the number of columns in the first matrix matches the number of rows in the second matrix, so that each element can be properly paired during multiplication. So, $A \in \mathbb{R}^{n\times k}$, $B \in \mathbb{R}^{k\times m} \Rightarrow AB \in \mathbb{R}^{n\times m}$.

▷ Linear Transformation

We have explored operations such as vector addition and scalar multiplication. These operations are not limited to pure mathematics; they are, at their core, a language of transformation. In linear algebra, we naturally ask: "Is there a rule that systematically sends each vector to another, while preserving the underlying structure of the vector space?" This is the essence of a **linear transformation**.

In detail, let V, W be vector spaces defined over the same field $\mathbb{F}$, a mapping $T : V \to W$ is called a linear transformation. In other words, we use the mapping rule T to send elements of V into elements of W. Conceptually, this is like a high school student transforming into a university student after taking an entrance exam, still a student (element of a vector space), but now in a different context (space). For all vectors $\mathbf{u}, \mathbf{v} \in V$ and the scalar $c \in \mathbb{F}$, we have

$$\begin{cases} T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \\ T(c\,\mathbf{u}) = cT(\mathbf{u}) \end{cases} \tag{6.1.12}$$

Moreover, every linear transformation between finite-dimensional vector spaces is equivalent to matrix multiplication. That is, for any linear map $T : \mathbb{R}^n \to \mathbb{R}^m$, there exists a unique matrix $A \in \mathbb{R}^{m\times n}$ that completely describes its behavior, such that for all $\mathbf{x} \in \mathbb{R}^n$,

$$T(\mathbf{x}) = A\mathbf{x} \tag{6.1.13}$$

In n dimensions $\mathbb{R}^n$, a vector $\mathbf{x} \in \mathbb{R}^n$ can be written as

$$\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n = \sum_i x_i \mathbf{e}_i \tag{6.1.14}$$

By (6.1.10), the linear transformation T satisfies

$$T(\mathbf{x}) = T\left(\sum_i x_i \mathbf{e}_i\right) = \sum_i x_i T(\mathbf{e}_i) \tag{6.1.15}$$

For each standard basis vector $\mathbf{e}_i$ in the n-dimensional space, the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ maps it to $T(\mathbf{e}_i)$. We define $\mathbf{a}_i \coloneqq T(\mathbf{e}_i) \in \mathbb{R}^m$, which represents the image of the i-th basis vector under the transformation. Then, we can construct a matrix $A = (\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n) \in \mathbb{R}^{m\times n}$ whose columns are precisely the images of the standard basis vectors. Then, for any $\mathbf{x} = (x_1 \quad \cdots \quad x_n)^T$, we have

$$A\mathbf{x} = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \mathbf{a}_1 x_1 + \cdots + \mathbf{a}_n x_n = \sum_i x_i \mathbf{a}_i = T(\mathbf{x}) \tag{6.1.16}$$

In other words, any linear transformation can be written as the action of a matrix A, and this matrix A is unique since the standard basis spans the space $\mathbb{R}^n$, the action of T on $\{\mathbf{e}_i\}$ uniquely determines the transformation.

▷ **Cofactor of a Matrix**

There is a given $m \times n$ matrix A

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \ddots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}_{m \times n}$$

We define a determinant M_{ij} as the determinant of the matrix obtained by removing the i-th row and the j-th column from the original matrix A. This M_{ij} is called the **minor** of A.
Furthermore, we define a new matrix C, constructed from the minors M_{ij} by multiplying each element with $(-1)^{i+j}$, where i and j are the row and column indices. That is, $C_{ij} = (-1)^{i+j} M_{ij}$. This matrix C is called the *cofactor matrix* of A.

However, to avoid introducing unnecessary symbols, we will directly redefine the cofactor in this text as $(-1)^{i+j} C_{ij}$. We adopt this convention partly because the word "cofactor" starts with the letter C :)

This way, we don't need to repeatedly refer back to the minors of the matrix every time, which would be unnecessarily tedious. All subsequent operations follow this convention, which may differ slightly from other textbooks. I kindly ask for your understanding. So, the cofactor

$$\text{Cofactor} \coloneqq (-1)^{i+j} C_{ij} \tag{6.1.17}$$

To calculate the determinant of a $n \times n$ matrix A, we follow the **cofactor expansion method**, also known as **Laplace expansion**.

6.2 Definition of Determinants

▷ **Permutations and Sign of a Permutation**

To simplify the representation of permutations in N–dimensions, we introduce a new notation: the generalized permutation $\sigma(i)$.
Given a number set $(1, 2, \cdots, n)$, there are $n!$ possible permutations, each corresponding to a different σ, for example

$$\sigma_1 = (1, 2, \cdots, n)$$

$$\sigma_2 = (1, n, \cdots, 2)$$

$$\vdots$$

$$\sigma_n = (n, n-1, \cdots, 3)$$

This symbol can be interpreted as a "method of assigning seats" or a specific arrangement of elements. Every different σ means a different arrangement. This notation can also be written as a function $\sigma(i)$, which means that we need to find out an arbitrary element in the $i-$th row. For instance, there is a permutation $\sigma_i = (3, 2, 1)$, the output values of the function are

$$\sigma(1) = 3$$

$$\sigma(2) = 1$$

$$\sigma(3) = 2$$

In this interpretation, the input variable i can be thought of as the "seat number in the classroom," and the output $\sigma(i)$ indicates which "student" is assigned to that seat, that is, which number from the original set is placed in position i.
Like any other function, a permutation $\sigma(i)$ has an inverse function, denoted by $\sigma^{-1}(i)$. The inverse permutation allows us to find the "seat number" given a specific "student."

For example, if the permutation is $\sigma = (3, 2, 1)$, then its inverse permutation is:

$$\sigma^{-1}(1) = 3$$

$$\sigma^{-1}(2) = 2$$

$$\sigma^{-1}(3) = 1$$

So, we can readily verify the following identity:

$$\sigma(\sigma^{-1}(i)) = i \tag{6.2.1}$$

We can define another two permutation notations τ and π, the relationship between σ, τ and π is

$$\pi(i) = \sigma \circ \tau = \sigma(\tau(i)) \Leftrightarrow \sigma(i) = \pi \circ \tau^{-1} = \pi(\tau^{-1}(i)) \tag{6.2.2}$$

In this chain of permutations, τ represents the first "seat arrangement," while σ represents the second arrangement based on the first one. That is, $\sigma \circ \tau$ means we first select elements using the permutation τ, and then apply σ to the result. The final permutation π is thus the total effect of both steps. In other words, we can deduce the second permutation σ from the initial arrangement τ and the resulting permutation. This relationship is quite natural and evident. This property will be widely used in linear algebra.

The final part of the permutation is to define a new function $\text{sgn}(\sigma)$ to substitute the Levi-Civita symbol because it can clearly represent the permutations, which is $+1$ or -1, without writing a lot of indices. The function $\text{sgn}(\sigma)$ is defined to be

$$\text{sgn}(\sigma) = \begin{cases} +1, & \text{When } \sigma \text{ is an even permutation} \\ -1, & \text{When } \sigma \text{ is an odd permutation} \end{cases} \tag{6.2.3}$$

Where $\sigma \in S_n$. Here, S_n denotes the symmetric group of degree n, which contains all possible $n!$ permutations of the set $\{1, 2, \cdots, n\}$.
In addition, there is a beautiful property:

$$\text{sgn}(\sigma_1 \circ \sigma_2) = \text{sgn}(\sigma_1) \cdot \text{sgn}(\sigma_2) \tag{6.2.4}$$

This is because the total parity of the composed permutation $\sigma_1 \circ \sigma_2$ depends on the parities of the two individual permutations. If one is odd and the other is even, their composition is odd; if both are odd or both are even, the composition is even. Therefore, the sign of the composition can be expressed as the product of their individual signs.

By the symmetry of the sign function, the parity of a permutation σ is the same as that of its inverse σ^{-1}. After all, both the arrangement of seats and the assignment of students remain unchanged in terms of parity. This property can be expressed as

$$\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma^{-1}) \tag{6.2.5}$$

▷ Determinant of Order N

We select any row(i-th) of the matrix and use the corresponding cofactors from each column (j-th). That is,

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{i1}(-1)^{i+1}C_{i1} + \cdots + a_{in}(-1)^{i+n}C_{in} \tag{6.2.6}$$

So the determinant of A based on the $j-$th row can be written as

$$\boxed{\det(A) = \sum_j a_{ij}(-1)^{i+j}C_{ij}} \tag{6.2.7}$$

For example, a given 3×3 matrix shown below,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \Rightarrow \det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

However, this method has a limitation: when the matrix has a lot of elements, we can't just use the cofactor $(-1)^{i+j}C_{ij}$ to describe the structure of the minor determinant. Therefore, we must introduce the more general form of the determinant using the sign of a permutation, which is

$$\boxed{\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_i a_{i,\sigma(i)}} \tag{6.2.8}$$

This way, all possible permutations and the ± 1 cofactor are included in the $\operatorname{sgn}(\sigma)$. That's why it was important to introduce these notations earlier. Furthermore, the transpose of A merely switches the order of rows and columns, which has no effect on the value of the determinant. So,

$$\boxed{\det(A) = \det(A^T)} \tag{6.2.9}$$

▷ Determinant Basic Formula

After such a long introduction, we can finally try to deal with one of the most important formulas in linear algebra, the **multiplicative property of determinants**.

There are two different given $n \times n$ matrices $[A]_{ij} = a_{ij}$ and $[B]_{ij} = b_{ij}$; both of them have the non-zero determinant. We define a new matrix λ to be

$$\lambda = AB \Rightarrow \lambda_{ij} = \sum_k a_{ik} b_{kj} \tag{6.2.10}$$

So, we can substitute (6.2.10) into the determinant of λ, which is

$$\det(\lambda) = \det(AB) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_i \lambda_{i,\sigma(i)} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_i \left(\sum_k a_{i,k} b_{k,\sigma(i)} \right) \tag{6.2.11}$$

We redefine the index k to be $\tau(i)$, which means we select the $\tau(i)$-th column in the k-th row, so the product of the summation can be rewritten as

$$\prod_i \left(\sum_k a_{i,k} b_{k,\sigma(i)} \right) = \sum_{\tau \in S_n} \left(\prod_i a_{i,\tau(i)} b_{\tau(i),\sigma(i)} \right) \tag{6.2.12}$$

This is because we need to consider all possible selections of $\tau(i)$. Multiplying everything together after summing over all possibilities is equivalent to listing every possible product first and then summing over all possible arrangements of $\tau(i)$. This way, we account for all possible cases thoroughly. Due to space limitations, I won't list them all here. However, if you have time, I encourage you to try listing them out and verifying the result on your own.

After that, we redefine $\sigma(i)$ is a double permutation

$$\sigma(i) = \pi \ \circ \ \tau = \pi(\tau(i)) \tag{6.2.13}$$

as shown in (6.2.2) and (6.2.4). So (6.2.11) becomes

$$\begin{aligned} \det(\lambda) = \det(AB) &= \sum_{\pi(\tau(i)) \in S_n} \operatorname{sgn}(\sigma) \sum_{\tau \in S_n} \left(\prod_i a_{i,\tau(i)} b_{\tau(i),\pi(\tau(i))} \right) \\ &= \sum_{\pi \in S_n} \operatorname{sgn}(\pi \circ \tau) \sum_{\tau \in S_n} \left(\prod_i a_{i,\tau(i)} b_{\tau(i),\pi(\tau(i))} \right) \\ &= \sum_{\pi,\tau \in S_n} \operatorname{sgn}(\pi)\operatorname{sgn}(\tau) \left(\prod_i a_{i,\tau(i)} b_{\tau(i),\pi(\tau(i))} \right) \end{aligned} \tag{6.2.14}$$

Note that the product $\prod_i b_{\tau(i),\pi(\tau(i))}$ is equivalent to $\prod_i b_{i,\sigma(i)}$ because the permutations τ, σ, π are one-to-one (Just the rearrangement). So, we can rewrite the product of the fourth equality of (6.2.14) into

$$\prod_i a_{i,\tau(i)} b_{\tau(i),\pi(\tau(i))} = \left(\prod_i a_{i,\tau(i)} \right) \left(\prod_i b_{\tau(i),\pi(\tau(i))} \right) \tag{6.2.15}$$

Substituting (6.2.15) into (6.2.14), we have

$$\begin{aligned} \det(AB) &= \sum_{\pi,\tau \in S_n} \operatorname{sgn}(\pi)\operatorname{sgn}(\tau) \left(\prod_i a_{i,\tau(i)} b_{\tau(i),\pi(\tau(i))} \right) \\ &= \sum_{\pi,\tau \in S_n} \operatorname{sgn}(\pi)\operatorname{sgn}(\tau) \left(\prod_i a_{i,\tau(i)} \right) \left(\prod_i b_{\tau(i),\pi(\tau(i))} \right) = \det(A)\det(B) \end{aligned} \tag{6.2.16}$$

This term is exactly the expansion of $\det(A)$, with $\tau \in S_n$ and the product over $a_{i,\tau(i)}$, weighted by $\text{sgn}(\tau)$. The same logic applies to $\det(B)$. We can also verify that the total terms of $\det(AB)$ are that

$$\text{Number of total terms} = \underbrace{n!}_{\tau(i)\text{ permutations}} \times \underbrace{n!}_{\pi(i)\text{ permutations}}$$

Note that we have already applied the permutation τ to the column indices of a, which means the order $\tau(i)$ has been fixed when passed into the row indices of b to match the column permutation of b. Therefore, the degree of freedom for τ in the indices of b has already been constrained.

That's the beautiful proof of the classical formula in linear algebra, which can be discussed by analyzing its algebraic structure. The final conclusion, the determinant of the product of two matrices equals the product of the determinants of each matrix, that is

$$\boxed{\det(AB) = \det(A)\det(B)} \tag{6.2.17}$$

We can use the same method to verify the reverse product $\det(BA) = \det(A)\det(B)$, you can try it by yourself.

6.3 Inverse Matrix

▷ **Meaning of the Identity Matrix**

We have to find a matrix that multiplies another matrix A, the result is still A. We have an obvious equation that the matrix A satisfies

$$\sum_j \delta_{ij} A_{jk} = A_{ik} \quad ; \quad \sum_j A_{kj}\delta_{ji} = A_{ki} \tag{6.3.1}$$

Where $A_{ik},\ A_{ki}$ are the elements of A. Note that the rules of matrix multiplication $A,\ B$ is

$$[AB]_{ik} = \sum_j A_{ij}B_{jk} \tag{6.3.2}$$

So, by (6.3.1), we can thus define a matrix $I_{ij} = \delta_{ij}$ because it follows the rule. And the equation

$$\det(AB) = \det(A)\det(B) \tag{6.3.3}$$

(6.3.3) implies that $\det(IA) = \det(I)\det(A) = \det(A)$, so $\det(I) = 1$. If this feels abstract now, don't worry, this Kronecker delta-based definition will naturally show up again when we derive the inverse matrix formula using cofactors.

▷ **Formula for the Inverse Matrix**

After discussing the meaning and the definition of the identity matrix, we may be back to the original problem. We need to find a $[M^{-1}]_{jk} = \xi_{jk}$ satisfying

$$(MM^{-1})_{ik} = \sum_j m_{ij}\xi_{jk} = \delta_{ik} \tag{6.3.4}$$

Where m_{ij} is the element of M, and ξ_{jk} is the element of M^{-1}. By the definition of the cofactor $(-1)^{i+j}C_{ij}$, the determinant of M can be denoted by

$$\det(M) = \sum_j (-1)^{k+j} m_{kj} C_{kj} \tag{6.3.5}$$

Where k is an arbitrary row of M. Note that this is not the matrix multiplication because of the order of the indices. If we change the index of m, it becomes

$$\sum_j (-1)^{k+j} m_{ij} C_{kj} = \det(M') \tag{6.3.6}$$

We can discuss the relationship between $i,\ k$ to find the value of this summation. When $i = k$, (6.3.4) holds. But, if $i \neq k$, we can redefine an arbitrary matrix M' with the k-th row consisting of the element m_{ij}. Note that the i-th row also consists of the elements m_{ij}. Which is

$$M' = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ m_{i1} & m_{i2} & \cdots & m_{in} \\ \vdots & \vdots & \cdots & \vdots \\ m_{i1} & m_{i2} & \ddots & m_{in} \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix} \tag{6.3.7}$$

By the properties of determinants, if a matrix has two identical rows or columns, then its determinant is zero. So (6.3.5) becomes

$$\det(M') = \sum_j (-1)^{k+j} m_{ij} C_{kj} = 0 \tag{6.3.8}$$

Combining (6.3.4) and (6.3.7), equation (6.3.4) can be rewritten as

$$\sum_j (-1)^{k+j} m_{ij} C_{kj} = \delta_{ik} \det(M) \tag{6.3.9}$$

Dividing both sides by $\det(M)$, (6.3.9) will become

$$\sum_j m_{ij} \left(\frac{1}{\det(M)} (-1)^{k+j} C_{kj} \right) \coloneqq \sum_j m_{ij} \xi_{jk} = \delta_{ik} = I_{ik} \tag{6.3.10}$$

So the matrix $\xi_{jk} = 1/\det(M) \times (-1)^{k+j} C_{kj}$. In this equation, the indices between the two matrices are reversed, indicating that a transposition has been applied. We've known that $A_{ij}^T = A_{ji}$, so

$$\xi_{jk} = (-1)^{k+j} C_{kj} = \left((-1)^{j+k} C_{jk} \right)^T \tag{6.3.11}$$

Note that the transposition doesn't affect the plus and minus signs of the cofactors since $(-1)^{k+j} = (-1)^{j+k}$. Substituting (6.3.11) into (6.3.10),

$$\sum_j m_{ij} \left((-1)^{j+k} C_{jk} \right)^T = (M\xi)_{ik} = I_{ik} \tag{6.3.12}$$

And then we can get the final solution to the proof and verify that

$$\boxed{\xi_{ij} = (M^{-1})_{ij} = \frac{1}{\det(M)}(-1)^{i+j}C_{ij}^T} \tag{6.3.13}$$

The proof for $M^{-1}M$ follows in the same way and will not be repeated here.

Example. *Given the matrix*

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

Find A^{-1}

We need to find the cofactors of each place. That

$$1^{st}\text{row}: \quad \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1; \quad -\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = -2; \quad \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 0$$

$$2^{nd}\text{row}: \quad -\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = 1; \quad \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0; \quad -\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = -1$$

$$3^{rd}\text{row}: \quad \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1; \quad -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1; \quad \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1$$

So the matrix C consisting of these cofactors is

$$C = \begin{pmatrix} 1 & -2 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{pmatrix}$$

The determinant of the original matrix A is

$$\det(A) = 1 \times 1 - 2 \times 0 + 1 \times 0 = 1$$

So the final answer is

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} 1 & -2 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 1 & -1 \\ -2 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

6.4 Linear Dependence and Independence

▷ Linear Relations in a System of Equations

After introducing the fundamental formulas and operations, we now enter one of the core concepts in linear algebra: **linear dependence and independence**.

Consider a set of vectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n \in \mathbb{R}^n$. If there exists a set of scalars $\eta_1, \eta_2, \cdots, \eta_n$, not all zero, such that

$$\sum_i \eta_i \mathbf{x}_i = \mathbf{0} \tag{6.4.1}$$

then the vectors $\mathbf{x}_1, \cdots, \mathbf{x}_n$ are said to be **linearly dependent**. The meaning of linear dependence is that a nontrivial linear combination exists that produces the zero vector. Geometrically, this implies that these vectors can only span a lower-dimensional subspace. In other words, although each $\mathbf{x}_i \in \mathbb{R}^n$, if they are linearly dependent, they do not span the entire $\mathbb{R}^n$, but rather some $\mathbb{R}^k$, where $k < n$.
We now examine the structure of a linear system represented by $A\mathbf{x} = \mathbf{b}$, where $A \in \mathbb{R}^{m\times n}$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^m$, that is

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \Rightarrow \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

where $\mathbf{a}_i = (a_{i1}, \cdots, a_{in})$. We now focus on the determinant of the matrix A alone, we've gotten

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix} = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_i a_{i,\sigma(i)}$$

If the vectors that form the matrix A satisfy the condition of linear dependence defined in (6.4.1), we can rewrite it as:

$$\sum_{i\neq j} \eta_i \mathbf{a}_i = \eta_j \mathbf{a}_j \tag{6.4.2}$$

This implies that the jth vector can be written as a linear combination of the others. By performing row operations (such as those taught in high school) we can reduce the jth row to all zeros. Which is

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix} \tag{6.4.3}$$

According to the properties of the determinant discussed earlier, this means $\det(A) = 0$. In other words, if $\det(A) = 0$, then the column vectors of A are linearly dependent, and the image of the linear transformation degenerates into a lower-dimensional structure.
From the perspective of solving linear systems, a zero determinant means that we cannot uniquely solve for all variables. In $\mathbb{R}^2$, this may correspond to two equations that are either parallel or identical, failing to intersect at a single point. Similarly, in $\mathbb{R}^3$, three equations with linearly dependent normal vectors may represent planes that are coplanar, colinear, or even disjoint, resulting in an unsolvable or underdetermined system.

Consider a set of vectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n \in \mathbb{R}^n$. If there exists a set of scalars $\eta_1, \eta_2, \cdots, \eta_n$, such that

$$\sum_i \eta_i \mathbf{x}_i = \mathbf{0} \Rightarrow \eta_1, \eta_2, \cdots, \eta_n = 0 \tag{6.4.4}$$

then the vectors $\mathbf{x}_1, \cdots, \mathbf{x}_n$ are said to be **linearly independent**. The meaning of linear independence is that there exists no nontrivial linear combination of the vectors that results in the zero vector.

Similarly, consider a system of linear equations $A\mathbf{x} = \mathbf{b}$. If the column vectors of A are linearly independent, then the following condition

$$\sum_{i \neq j} \eta_i \mathbf{a}_i \neq \eta_j \mathbf{a}_j \tag{6.4.5}$$

That is, no vector in the system can be expressed as a linear combination of the others. This ensures that the determinant $\det(A)$ is nonzero. According to the properties of the inverse matrix discussed earlier, any matrix with a nonzero determinant has a unique inverse A^{-1}. Therefore, the solution to the system can be written as

$$A\mathbf{x} = \mathbf{b} \quad \Leftrightarrow \quad \mathbf{x} = A^{-1}\mathbf{b} \tag{6.4.6}$$

which guarantees the existence and uniqueness of the solution, fully determining the solution space.

▷ Wronskian and Abel's Identity

We've discussed the linear relationship between the vectors. In addition, we can also use the previous method to determine the linear independence among functions. Consider an nth-order differential equation as an example

$$y^{(n)} + P_{n-1}(x)y^{(n-1)} + \cdots + P_1(x)y' + P_0(x)y = 0 \tag{6.4.7}$$

The **solution space** of this differential equation is expected to be n-dimensional. Suppose that $y_1(x)$, $y_2(x), \cdots, y_n(x)$ are the solutions to the differential equation. To determine whether they are dependent or not, we express the solution as a linear combination

$$\eta_1 y_1(x) + \eta_2 y_2(x) + \cdots + \eta_n y_n(x) = 0 \tag{6.4.8}$$

Where $\eta_i \in \mathbb{R}$. Assume that y is $(n-1)$-times differentiable. We have

$$\begin{aligned} &\frac{d}{dx}[\eta_1 y_1(x)] + \frac{d}{dx}[\eta_2 y_2(x)] + \cdots + \frac{d}{dx}[\eta_n y_n(x)] \\ &\Rightarrow \eta_1 y_1'(x) + \eta_2 y_2'(x) + \cdots + \eta_n y_n'(x) = 0 \end{aligned} \tag{6.4.9}$$

Similarly, we obtain

$$\begin{cases} \eta_1 y_1'(x) + \eta_2 y_2'(x) + \cdots + \eta_n y_n'(x) = 0 \\ \eta_1 y_1''(x) + \eta_2 y_2''(x) + \cdots + \eta_n y_n''(x) = 0 \\ \quad \vdots \\ \eta_1 y_1^{(n-1)}(x) + \eta_2 y_2^{(n-1)}(x) + \cdots + \eta_n y_n^{(n-1)}(x) = 0 \end{cases}$$

We define $Y = (\mathbf{y}_i(x), \mathbf{y}_i'(x), \cdots, \mathbf{y}_i^{(n-1)}(x))^T$, and η is the column matrix. So, the matrix multiplication

$$\begin{pmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{6.4.10}$$

Here, we define the determinant

$$W(y_1, y_2, \cdots, y_n)(x) \coloneqq \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix} \tag{6.4.11}$$

We call the determinant $W(y_1, y_2, \cdots, y_n)(x)$ as **Wronskian**. If the determinant $W(x) \neq 0$, it means that the inverse matrix of Y exists, so (6.4.10) satisfies

$$Y\eta = \mathbf{0} \Leftrightarrow \eta = Y^{-1}\mathbf{0} = \mathbf{0} \tag{6.4.12}$$

By (6.4.4), we can verify that $y_1(x)$, $y_2(x)$, $\cdots$, $y_n(x)$ are linearly independent. In other words, the solutions of this differential equation span an n-dimensional solution space $\mathbb{R}^n$. Conversely, if the Wronskian $W(x) = 0$, then $y_1(x)$, $y_2(x)$, $\cdots$, $y_n(x)$ are linearly dependent, indicating that the solution space is degenerate and does not satisfy the general solution structure.
Moreover, differentiate the Wronskian $W(x) = 0$ with respect to x,

$$\frac{d}{dx}W(x) = \frac{d}{dx}\det(Y) \tag{6.4.13}$$

We define $\langle A^{j\neq i}, A^i \rangle$ as the determinant of the matrix $A \in \mathbb{R}^{n\times n}$ where only the entries(A^i) in the ith row are differentiated, while all other entries$(A^{j\neq i})$ remain unchanged. So, we have

$$\frac{d}{dx}\det(A) = \sum_i \langle A^{j\neq i}, A^i \rangle \tag{6.4.14}$$

This is because the determinant is computed by summing over all possible permutations of the matrix elements, as given by

$$\frac{d}{dx}\det(A) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\frac{d}{dx}\prod_i a_{i,\sigma(i)}. \tag{6.4.15}$$

According to the properties of differentiation, each product under a given permutation can be decomposed into n separate terms, just like

$$\frac{d}{dx}[a_1(x)a_2(x)a_3(x)] = a_1'(x)a_2(x)a_3(x) + a_1(x)a_2'(x)a_3(x) + a_1(x)a_2(x)a_3'(x).$$

Therefore, we only need to differentiate the entries of a single row while keeping all other entries unchanged. By summing the resulting determinants over all rows, we obtain

$$\frac{d}{dx}W(x) = \sum_i \langle Y^{j\neq i}, Y^i \rangle$$

$$= \begin{vmatrix} y_1' & y_2' & \cdots & y_n' \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1'' & y_2'' & \cdots & y_n'' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} + \cdots$$

$$+ \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$$

Since the first $n-1$ terms contain two identical rows, we can apply the properties of the determinant to conclude that

$$\begin{vmatrix} y_1' & y_2' & \cdots & y_n' \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = \begin{vmatrix} y_1' & y_2' & \cdots & y_n' \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = 0$$

Apply the same logic to the first $n-1$ term; only the last term remains, we can get

$$\frac{d}{dx}W(x) = \sum_i \langle Y^{j\neq i}, Y^i \rangle = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix} \tag{6.4.16}$$

Because $y_1, y_2, \cdots, y_n$ are the solutions to the ODE, so $y_1^{(n)}, y_2^{(n)}, \cdots, y_n^{(n)}$ satisfy

$$y_i^{(n)} = -P_{n-1}(x)y_i^{(n-1)} - \cdots - P_1(x)y_i' - P_0(x)y_i \tag{6.4.17}$$

for $1 \leq i \leq n$. That is to say, $y_1^{(n)}, y_2^{(n)}, \cdots, y_n^{(n)}$ can be written as the linear combination

of the first $n-1$ terms. By performing row operations on the matrix, we can get

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix} = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ -P_{n-1}y_1^{(n-1)} & -P_{n-1}y_2^{(n-1)} & \cdots & -P_{n-1}y_2^{(n-1)} \end{vmatrix}$$

$$= -P_{n-1}(x) \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

$$= -P_{n-1}(x)W(x)$$

Substituting $-P_{n-1}(x)W(x)$ into the original differentiation, we obtain a first-order differential equation for $W(x)$.

$$\begin{aligned} &\frac{d}{dx}W(x) = -P_{n-1}(x)W(x) \\ &\Rightarrow \frac{dW(x)}{W(x)} = -P_{n-1}(x)dx \\ &\Rightarrow W(x) = C \cdot \exp\left(-\int P_{n-1}(x)dx\right) \end{aligned} \tag{6.4.18}$$

The initial condition at the point $x = x_0$, the Wronskian is $W(x_0)$, so we have

$$\Rightarrow W(x_0) = C \cdot \exp\left(-\int_{x_0}^{x_0} P_{n-1}(x)dx\right) = C \tag{6.4.19}$$

Substituting (6.4.19) into (6.4.18),

$$\boxed{W(x) = W(x_0) \cdot \exp\left(-\int_{x_0}^{x} P_{n-1}(x)dx\right)} \tag{6.4.20}$$

This is called **Abel's Identity**. Suppose that $W(x)$ is defined on the interval $I \subseteq \mathbb{R}$, when $W(x) = 0$ on I, then $y_1, y_2, \cdots, y_n$ are linearly dependent, the solution space degenerates; when $W(x_0) \neq 0$ for any $x_0 \in I$, then $y_1, y_2, \cdots, y_n$ are linearly independent, the solutions of this differential equation span an n-dimensional solution space $\mathbb{R}^n$ since

$$\exp\left(-\int_{x_0}^{x_0} P_{n-1}(x)dx\right) > 0,\ W(x_0) \neq 0 \Rightarrow W(x) \neq 0,\ \forall x \in I$$

If $W \neq 0$, then the set of functions is linearly independent. However, it is important to note that if we take an arbitrary set of functions $\{f_1, \cdots, f_n\} \in C^{n-1}(I)$ defined on an interval I, then this condition is not a necessary and sufficient condition. In other words,

we cannot conclude that the functions are linearly dependent if $W = 0$, unless we have the additional assumption that these functions come from the solution space of a linear differential equation.

This means that, even if the functions are locally linearly dependent on every subinterval $I' \subset I$, they may not be globally linearly dependent over the entire interval I. For example,

$$f_1(x) = x^2, \quad f_2(x) = x|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$

are linearly dependent on $x > 0$ and also on $x < 0$, which implies $W(x) = 0$. However, there does not exist a constant $c \in \mathbb{R}$ such that $f_2(x) = cf_1(x)$ holds for all $x \in I$. Thus, they are not globally linearly dependent.

In summary, only when the functions are known to be solutions of a linear differential equation(so that Abel's identity holds) does the condition $W = 0$ become both necessary and sufficient for linear dependence. Otherwise, we can only use $W \neq 0$ to confirm linear independence.

6.5 Orthogonal Matrices

In Chapter 1 and Chapter 2, we discussed the coordinate transformation. In this part, I'll formally introduce the matrices of the transformation. In $\mathbb{R}^n$, we can describe the properties and orientation of a vector originating from the origin using the cosine of the angle it forms with each coordinate axis, as shown in (2.1.13). It is denoted by

$$(\cos\theta_1, \cos\theta_2, \cdots, \cos\theta_n) \coloneqq (\Lambda_1, \Lambda_2, \cdots, \Lambda_n) \tag{6.5.1}$$

Where θ_1, θ_2, $\cdots$, θ_n are the angles between the vector and the axis. When there are two vectors described by the direction cosine $(\Lambda_1^A, \Lambda_2^A, \cdots, \Lambda_n^A)$ and $(\Lambda_1^B, \Lambda_2^B, \cdots, \Lambda_n^B)$, then the angle ϕ between the two satisfies

$$\cos\phi = \sum_i \Lambda_i^A \Lambda_i^B \tag{6.5.2}$$

By the definition in (2.1.7), in a Cartesian coordinate system, the direction cosines of the new axis x_1' with respect to the original coordinate system (x_1, x_2, x_3) are given by $(\lambda_{11}, \lambda_{12}, \lambda_{13})$. Similarly, the direction cosines of x_2' with respect to the original coordinate system (x_1, x_2, x_3) are $(\lambda_{21}, \lambda_{22}, \lambda_{23})$. So by (6.5.2), we can obtain the equation

$$\lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} + \lambda_{13}\lambda_{23} = \cos\left(\frac{\pi}{2}\right) = 0 \tag{6.5.3}$$

since each of the axes is perpendicular to the other in the Cartesian coordinate system. It's obvious that

$$\lambda_{11}^2 + \lambda_{12}^2 + \lambda_{13}^2 = 1 \tag{6.5.4}$$

since the angle between an axis and itself is zero. Therefore, we can obtain an important property

$$\sum_j \lambda_{ij}\lambda_{kj} = \delta_{ik} \tag{6.5.5}$$

According to (2.1.8), we can construct a matrix multiplication that represents the coordinate transformation, namely,

$$x_i' = \sum_j \lambda_{ij} x_j \Rightarrow X' = \boldsymbol{\lambda} X \tag{6.5.6}$$

Where $\boldsymbol{\lambda}_{ij} = \lambda_{ij}$. We expand the matrices, and we can obtain

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

And if we project the coordinate axes from the new system back to the original one, we have

$$x_i = \sum_j \lambda_{ji} x_j' = \sum_j \lambda_{ij}^T x_j' \Rightarrow\Rightarrow X = \boldsymbol{\lambda}^T X' \tag{6.5.7}$$

Similarly,

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{21} & \cdots & \lambda_{n1} \\ \lambda_{12} & \lambda_{22} & \cdots & \lambda_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{1n} & \lambda_{2n} & \cdots & \lambda_{nn} \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix}$$

Therefore, the matrix product of $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^T$ is

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{nn} \end{pmatrix} \begin{pmatrix} \lambda_{11} & \lambda_{21} & \cdots & \lambda_{n1} \\ \lambda_{12} & \lambda_{22} & \cdots & \lambda_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{1n} & \lambda_{2n} & \cdots & \lambda_{nn} \end{pmatrix} = \begin{pmatrix} \sum_j \lambda_{1j}^2 & \sum_j \lambda_{1j}\lambda_{2j} & \cdots & \sum_j \lambda_{1j}\lambda_{nj} \\ \sum_j \lambda_{2j}\lambda_{1j} & \sum_j \lambda_{2j}^2 & \cdots & \sum_j \lambda_{2j}\lambda_{nj} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_j \lambda_{nj}\lambda_{1j} & \sum_j \lambda_{nj}\lambda_{2j} & \cdots & \sum_j \lambda_{nj}^2 \end{pmatrix}$$

By (6.5.5), the result of the matrix product

$$\boldsymbol{\lambda}\boldsymbol{\lambda}^T = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \Rightarrow \left(\boldsymbol{\lambda}\boldsymbol{\lambda}^T\right)_{ij} = \delta_{ij} \tag{6.5.8}$$

By the meaning of the identity matrix and the inverse matrix, we can verify that

$$\boldsymbol{\lambda}\boldsymbol{\lambda}^T = I = \boldsymbol{\lambda}\boldsymbol{\lambda}^{-1} \Leftrightarrow \boxed{\boldsymbol{\lambda}^T = \boldsymbol{\lambda}^{-1}} \tag{6.5.9}$$

From this point, we can formally define such a matrix. If a matrix $\boldsymbol{\lambda}$ satisfies $\boldsymbol{\lambda}^T = \boldsymbol{\lambda}^{-1}$, then we call $\boldsymbol{\lambda}$ as an **orthogonal matrix**. We define a vector $\mathbf{v}$ as a $n \times 1$ column matrix, the dot product of $\mathbf{v}$ with itself is

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{\mathbf{v}^T\mathbf{v}}$$

The result is a 1×1 matrix, which is a scalar and called the *norm* of $\mathbf{v}$. Combine the vector $\mathbf{v}$ and the orthogonal matrix $\boldsymbol{\lambda}$, the norm of the matrix $\boldsymbol{\lambda}\mathbf{v}$ is

$$\|\boldsymbol{\lambda}\mathbf{v}\|^2 = (\boldsymbol{\lambda}\mathbf{v})^T(\boldsymbol{\lambda}\mathbf{v}) = \mathbf{v}^T\boldsymbol{\lambda}^T\boldsymbol{\lambda}\mathbf{v} = \mathbf{v}^T I\mathbf{v} = \|\mathbf{v}\|^2 \tag{6.5.10}$$

Furthermore, we have the property of the determinant that $\det A = \det A^T$, so

$$\det(\boldsymbol{\lambda}\boldsymbol{\lambda}^T) = \det(\boldsymbol{\lambda}^T)\det(\boldsymbol{\lambda}) = [\det(\boldsymbol{\lambda})]^2 = 1 \Rightarrow \boxed{\det \boldsymbol{\lambda} = \pm 1} \tag{6.5.11}$$

Suppose that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are the orthogonal matrices, and we continuously transform the coordinates, which is

$$x'_i = \sum_j \lambda_{ij}x_j \quad x''_k = \sum_i \mu_{ki}x'_i \tag{6.5.12}$$

Substituting $x_i' = \sum_j \lambda_{ij} x_j$ into $x_k'' = \sum_i \mu_{ki} x_i'$, we obtain

$$x_k'' = \sum_i \mu_{ki} x_i' = \sum_i \mu_{ki} \left(\sum_j \lambda_{ij} x_j \right)$$

$$= \sum_{ij} \mu_{ki} \lambda_{ij} x_j = \sum_j \left(\sum_i \mu_{ki} \lambda_{ij} \right) x_j = \sum_j [\mu\lambda]_{kj} x_j$$

The matrix product

$$\begin{aligned} (\boldsymbol{\mu\lambda})^T (\boldsymbol{\mu\lambda}) &= \boldsymbol{\lambda}^T \boldsymbol{\mu}^T \boldsymbol{\mu\lambda} \\ &= \boldsymbol{\lambda}^T I \boldsymbol{\lambda} = I \end{aligned} \tag{6.5.13}$$

So the matrix $\boldsymbol{\mu\lambda}$ is an orthogonal matrix.
In summary, an orthogonal matrix is defined by the condition that its inverse is equal to its transpose, $\boldsymbol{\lambda}^T = \boldsymbol{\lambda}^{-1}$, and its determinant is either $+1$ or -1. Moreover, the product of multiple orthogonal matrices is still an orthogonal matrix.
Most importantly, the action of an orthogonal matrix on coordinates or vectors preserves their length, angles, and other geometric properties. In physics, one of the most significant applications of orthogonal matrices is coordinate transformation within orthogonal systems. For example, the rotation matrix and reflection matrix, both of which we commonly encountered in high school, are classic examples of coordinate transformations in orthogonal systems.
In addition to square matrices, we can also define orthogonality between vectors. If we say that two column vectors $\mathbf{v}$ and $\mathbf{u}$ are orthogonal, then

$$\mathbf{v}^T \mathbf{u} = 0$$

where $\mathbf{v}$, $\mathbf{u} \in \mathbb{R}^{n\times 1}$. This indicates that the two vectors are perpendicular to each other in the n-dimensional real space. However, the vector $\mathbf{v}$ in the complex space, the length of $\mathbf{v}$ is

$$\mathbf{v}^\dagger \mathbf{v} = \|\mathbf{v}\|^2 \tag{6.5.14}$$

Where $\mathbf{v}^\dagger$ is conjugate transpose of $\mathbf{v}$(see Chapter 6.6). That's because the length of a vector $\mathbf{v} = (1-i, 2)$ is defined as $|\mathbf{v}|^2 = (\overline{1-i}, 2) \cdot (1-i, 2) = (1+i, 2) \cdot (1-i, 2) = 6$. So, when two vectors $\mathbf{v}$, $\mathbf{u} \in \mathbb{C}^{n\times 1}$ are perpendicular to each other in the n-dimensional complex space, we have

$$\mathbf{v}^\dagger \mathbf{u} = 0 \tag{6.5.15}$$

6.6 Introduction to Eigenvalues

▷ Definition of Eigenvalues and Eigenvectors

Basically, it's really troublesome to analyze the system by separating it into individual components like $x_1, x_2, x_3, \cdots$. But then we realize: what if we group them into a single vector? If we take all the linear equations that each component satisfies and represent them as a single matrix A, then we start thinking: how can we understand the behavior of this entire vector as a whole?

Suddenly, we Note something powerful that if we multiply the vector by A simply scales it by a constant, then we can study this vector directly, without worrying about the influence of the others. And that's exactly where eigenvalues come in. This allows us to transform a complex, coupled system into a set of simpler, decoupled modes, each evolving independently according to its corresponding eigenvalue.
Consider a vector $\mathbf{v} \neq \mathbf{0} \in \mathbb{R}^n$, there is a linear transformation matrix $A \in \mathbb{R}^{n\times n}$, which satisfies

$$A\mathbf{v} = \lambda\mathbf{v} \tag{6.6.1}$$

where $\lambda \in \mathbb{R}$, then we call the constant λ **eigenvalues** and the vector $\mathbf{v}$ **eigenvector** of A. This means that when a linear transformation is applied to one of its eigenvectors, it only stretches or compresses the vector, without changing its direction. For some nonzero vector $\mathbf{v} \in \mathbb{R}^n$. Rearranging terms, this becomes

$$A\mathbf{v} = \lambda\mathbf{v} \Rightarrow A\mathbf{v} - \lambda I\mathbf{v} = (A - \lambda I)\mathbf{v} = \mathbf{0} \tag{6.6.2}$$

Note that we always see a vector as a column matrix. In linear algebra, if we want the equation $A\mathbf{x} = 0$ to have a nontrivial solution (i.e., $\mathbf{x} \neq \mathbf{0}$), we must ensure that the matrix A is non-invertible. In other words, the necessary and sufficient condition is $\det A = 0$. Here, we've known that $\mathbf{v} \neq \mathbf{0}$, so the determinant of $(A - \lambda I)$ must be 0, namely

$$\boxed{\det(A - \lambda I) = 0} \tag{6.6.3}$$

This is called **characteristic equation**. Let $A_{ij} = a_{ij}$, so we need to compute

$$\begin{vmatrix} a_{11} - \lambda & a_{1n} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n1} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0 \tag{6.6.4}$$

We can express $\det(A - \lambda I)$ as a polynomial in λ, and since $\deg(\det(A - \lambda I)) = n$, it is a degree-$n$ polynomial. According to the Fundamental Theorem of Algebra, this polynomial has n roots, denoted by $\lambda_1, \lambda_2, \cdots, \lambda_n$. Assuming for now that these roots are distinct, each of the n eigenvalues in equation (6.6.4) corresponds to a unique eigenvector. That is, they satisfy

$$\begin{cases} A\mathbf{v}_1 = \lambda_1\mathbf{v}_1 \Rightarrow (A - \lambda_1 I)\mathbf{v}_1 = \mathbf{0} \\ A\mathbf{v}_2 = \lambda_2\mathbf{v}_2 \Rightarrow (A - \lambda_2 I)\mathbf{v}_2 = \mathbf{0} \\ \quad \vdots \\ A\mathbf{v}_n = \lambda_n\mathbf{v}_n \Rightarrow (A - \lambda_n I)\mathbf{v}_n = \mathbf{0} \end{cases}$$

For an $n \times n$ matrix A, if it has n distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, then the corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly independent. Each of them can be written as

$$A\mathbf{v}_i = \begin{pmatrix} a_{11} & a_{1n} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{in} \end{pmatrix} = \begin{pmatrix} \sum_k a_{1k}x_{ik} \\ \sum_j a_{2k}x_{ik} \\ \vdots \\ \sum_j a_{nk}x_{ik} \end{pmatrix} = \begin{pmatrix} \lambda_i x_{i1} \\ \lambda_i x_{i2} \\ \vdots \\ \lambda_i x_{in} \end{pmatrix}$$

Similarly, we define a matrix $\tilde{\mathbf{v}} = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n)$, thus we have

$$A\tilde{\mathbf{v}} = \begin{pmatrix} \sum_k a_{1k}x_{1k} & \sum_k a_{1k}x_{2k} & \cdots & \sum_k a_{1k}x_{nk} \\ \sum_j a_{2k}x_{1k} & \sum_k a_{2k}x_{2k} & \cdots & \sum_k a_{2k}x_{nk} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_j a_{nk}x_{1k} & \sum_k a_{nk}x_{2k} & \cdots & \sum_k a_{nk}x_{nk} \end{pmatrix} = \begin{pmatrix} \lambda_1 x_{11} & \lambda_2 x_{21} & \cdots & \lambda_n x_{n1} \\ \lambda_1 x_{12} & \lambda_2 x_{22} & \cdots & \lambda_n x_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 x_{1n} & \lambda_2 x_{2n} & \cdots & \lambda_n x_{nn} \end{pmatrix}$$

Note that the elements of the matrix on the right-hand side can be written as

$$\sum_i x_{ji}\delta_{ij}\lambda_i \equiv \tilde{\mathbf{v}}\tilde{\Lambda} \tag{6.6.5}$$

Where $\tilde{\lambda}$ is defined as

$$\tilde{\Lambda} \coloneqq \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

Therefore, we have obtained a key equation that characterizes the eigenvectors and eigenvalues.

$$A\tilde{\mathbf{v}} = \tilde{\mathbf{v}}\tilde{\Lambda} \Leftrightarrow A = \tilde{\mathbf{v}}\tilde{\Lambda}\tilde{\mathbf{v}}^{-1} \tag{6.6.6}$$

Here, $\tilde{\Lambda}$ is the diagonal matrix containing the eigenvalues of A. It appears naturally when expressing A in terms of its eigenvectors.

For example, we want to find the eigenvectors and the eigenvalues of the following matrix product

$$A\mathbf{v} = \lambda\mathbf{v} \Rightarrow \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$$

We can further obtain

$$\begin{pmatrix} 5-\lambda & -2 \\ -2 & 2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

We want to get solutions other than $x = y = 0$, so

$$\begin{vmatrix} 5-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 1 \quad \text{or} \quad \lambda = 6$$

We have obtained two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 6$; therefore, we can find two corresponding eigenvectors $\mathbf{v}_1 = (x_1, y_1)$ and $\mathbf{v}_2 = (x_2, y_2)$ that satisfy

$$\begin{cases} (A - \lambda_1 I)\mathbf{v}_1 = \mathbf{0} \Rightarrow \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ (A - \lambda_2 I)\mathbf{v}_2 = \mathbf{0} \Rightarrow \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{cases}$$

We can take the eigenvectors $\mathbf{v}_1 = (1, 2)$ and $\mathbf{v}_2 = (2, -1)$. Namely, the eigenvectors are on the lines

$$\begin{cases} \mathbf{v}_1 = (1, 2) \in L_1 : 2x - y = 0 \\ \mathbf{v}_2 = (2, -1) \in L_2 : x + 2y = 0 \end{cases}$$

▷ **Diagonalization and its Meaning**

There is a vector space $V \subset \mathbb{R}^n$, and we call the set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} \subset V$ is the **vector basis** of the vector space V when

$$\sum_i \eta_i \mathbf{v}_i = 0 \Rightarrow \eta_i = 0, \quad 1 \leq i \leq n \tag{6.6.7}$$

That is, the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ are independent. Furthermore, $\forall \mathbf{x} \in V$, $\exists! (q_1, q_2, \cdots, q_n) \in \mathbb{R}^n$ such that

$$\mathbf{x} = \sum_i q_i \mathbf{v}_i \tag{6.6.8}$$

Which means that any vector can be expressed as a linear combination of the basis vectors of the vector space. Just like $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the vector basis of the $V \subset \mathbb{R}^3$. That is,

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} = \mathbb{R}^n$$

Let $\mathbf{q} = (q_1 \quad q_2 \quad \cdots \quad q_n)^T \in \mathbb{R}^{n\times 1}$ and $C = \tilde{\mathbf{v}} = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n) \in \mathbb{R}^{n\times n}$ since $\mathbf{v}_i = (v_{i1}, \cdots, v_{in})^T$, by equation (6.6.8), we obtain

$$\mathbf{x} = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n) \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} = C\mathbf{q} \Leftrightarrow C^{-1}\mathbf{x} = \mathbf{q} \tag{6.6.9}$$

Then we say that $\mathbf{q}$ is the representation of the original vector $\mathbf{x}$ under the new basis $\mathcal{B}$. Suppose a linear transformation $A \in \mathbb{R}^{n\times n}$, with components $A_{ij} = a_{ij}$, acts on the original vector $\mathbf{x}$. We now define an equivalent linear transformation $D \in \mathbb{R}^{n\times n}$, with components $D_{ij} = d_{ij}$, that acts on $\mathbf{q}$ in the new basis. Then, we have

$$A\mathbf{x} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad D\mathbf{q} = \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} \tag{6.6.10}$$

Because $D\mathbf{q}$ is the linear transformation under the new basis, these linear transformations must be combined with the matrix C through a linear combination to return to the original basis. That is

$$A\mathbf{x} = \sum_{ij} d_{ij} q_j \mathbf{v}_i = CD\mathbf{q} \tag{6.6.11}$$

Substituting $C^{-1}\mathbf{x} = \mathbf{q}$ into (6.6.11), we obtain

$$A\mathbf{x} = CD(C^{-1}\mathbf{x}) = CDC^{-1}\mathbf{x} \Rightarrow \boxed{A = CDC^{-1}} \tag{6.6.12}$$

Similarly, substitute $\mathbf{x} = C\mathbf{q}$ into (6.6.11), and we can also obtain

$$A\mathbf{x} = A(C\mathbf{q}) = AC\mathbf{q} = CD\mathbf{q} \Rightarrow \boxed{C^{-1}AC = D} \tag{6.6.13}$$

In other words, equation (6.6.12) indicates that the linear transformation A in the original basis can be obtained by combining the matrix C with the linear transformation D in the new basis. The resulting transformation acts on vectors expressed in the *original basis*. Conversely, equation (6.6.13) shows that the linear transformation D in the new basis can be obtained by combining C with the linear transformation A in the original basis. Again, the transformation acts on vectors expressed in the *new basis*. These two forms are entirely equivalent; the distinction depends solely on which basis is chosen for the vector space.

Note that all these operations are valid under the assumption that $C = (\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n) \in \mathbb{R}^{n\times n}$ is composed of linearly independent vectors. The underlying transformation remains the same; only its representation changes with the choice of basis.

Let's go back to (6.6.11), when $d_{ij} = \lambda_i\delta_{ij}$, where λ_i is the eigenvalue, namely

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} = \sum_i \lambda_i \mathbf{e}_i\mathbf{e}_i^T \coloneqq \tilde{\Lambda}$$

Where $(\mathbf{e}_i)_{k1} = \delta_{ik}$, $(\sum \mathbf{e}_i\mathbf{e}_i^T)_{jk} = \delta_{jk}$ then (6.6.11) becomes

$$A\mathbf{x} = \sum_{ij} d_{ij}q_j\mathbf{v}_i = \sum_i \lambda_i q_i \mathbf{v}_i = \sum_i \lambda_i \mathbf{x}_i = D\mathbf{x} \Rightarrow A\mathbf{x} = \left(\sum_i \lambda_i \mathbf{e}_i\mathbf{e}_i^T\right)\mathbf{x} \tag{6.6.14}$$

The matrix D has elements not zero only down the main diagonal; it's called a **diagonal matrix**, also called **similar** to A. When we get D consisting of the eigenvalues given A, we claim that we have **diagonalized** A by a **similarity transformation**(Boas,2005). Then the matrix $C = \tilde{\mathbf{v}} = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n)$ is made of eigenvectors, that is, the basis we are working with is known as the **eigenbasis**, it satisfies

$$AC_i = \lambda_i C_i \tag{6.6.15}$$

When we need to obtain D, namely diagonalize A, we need

$$A\tilde{\mathbf{v}}_i = \lambda_i\tilde{\mathbf{v}}_i \Leftrightarrow (A - \lambda_i I)\tilde{\mathbf{v}}_i = 0 \tag{6.6.16}$$

Since $\tilde{\mathbf{v}} = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n)$ are linearly independent, then

$$\det(A - \lambda_i I) = 0 \Rightarrow \lambda_i = \begin{cases} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{cases} \tag{6.6.17}$$

Note that (1)$\lambda_i \in \mathbb{R}$; (2)$\mathbf{x}$, $\mathbf{q}$, $C_i = \tilde{\mathbf{v}}_i \in \mathbb{R}^n$ and (3)$A, C, D, \lambda, \tilde{\mathbf{v}} \in \mathbb{R}^{n\times n}$. So, we "plug" the eigenvalues into the identity matrix, then we can finally get a diagonalized matrix of A

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \Rightarrow AC = CD \Rightarrow \boxed{A\tilde{\mathbf{v}} = \tilde{\mathbf{v}}\tilde{\Lambda} \Rightarrow A = \tilde{\mathbf{v}}\tilde{\Lambda}\tilde{\mathbf{v}}^{-1}}$$

as before. Please do not confuse the following two expressions. The first refers to a single eigenvector, where we seek a real number λ; the second refers to a matrix multiplication involving a set of eigenvectors, where we obtain a diagonalized matrix composed of the eigenvalues $\lambda \in \mathbb{R}$.

$$\begin{cases} A\tilde{\mathbf{v}}_i = \lambda_i \tilde{\mathbf{v}}_i \Rightarrow A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1,\ A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2 \cdots \\ A\tilde{\mathbf{v}} = \tilde{\mathbf{v}}\tilde{\Lambda} \end{cases} \tag{6.6.18}$$

What is written here is a bit different from the previous section. The previous part focused on finding the eigenvalue of a single vector, whereas this part is intended to help you understand the relationship between vectors and matrices. I admit this section is a bit messy, please forgive me :)
In other words, if we find the eigenvalues of an $n \times n$ matrix but cannot obtain n linearly independent eigenvectors, then these vectors cannot span $\mathbb{R}^n$, and thus cannot form an invertible $n \times n$ matrix as a basis for similarity transformation. As a result, the inverse matrix C^{-1}, $\tilde{\mathbf{v}}^{-1}$ does not exist, and the matrix cannot be diagonalized. The failure occurs precisely at the step $\mathbf{x} = C\mathbf{q} \Rightarrow \mathbf{q} = C^{-1}\mathbf{x}$, which is only valid when the matrix C is invertible. This condition fails when the eigenvectors of A are linearly dependent or insufficient in number to span the space.

▷ Diagonalizing Hermitian Matrices(Optional)

Before concluding our discussion of linear algebra, I'd like to introduce a new friend: the **Hermitian matrix**. Although it rarely appears in classical mechanical systems, it plays a central role in the study of quantum mechanics. A Hermitian matrix is defined over the complex field, meaning it is a square matrix $H \in \mathbb{C}^{n\times n}$. It is defined as

$$H = H^\dagger \tag{6.6.19}$$

where $H^\dagger$ is the conjugate transpose of H. The conjugate transpose of a matrix is defined as

$$A^\dagger \coloneqq \left(\overline{A}\right)^T \Rightarrow (a_{ij})^\dagger = \overline{a}_{ji} \tag{6.6.20}$$

So, by the definition of a Hermitian matrix, we get

$$H_{ij} = \overline{H}_{ji} \quad i \neq j \tag{6.6.21}$$

The complex entries occur off the diagonal, while the diagonal entries are necessarily real

$$H_{ii} = \overline{H}_{ii} \quad H_{ii} \in \mathbb{R} \tag{6.6.22}$$

For example, a Hermitian matrix $H \in \mathbb{C}^{2\times 2}$

$$H = \begin{pmatrix} 2 & 1-i \\ 1+i & 3 \end{pmatrix} \rightarrow \overline{H} = \begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix} \rightarrow \overline{H}^T = \begin{pmatrix} 2 & 1-i \\ 1+i & 3 \end{pmatrix}$$

Then we can verify the definition $H = H^\dagger$. After introducing the definition of Hermitian, we are going to check the property of its eigenvalues next. That is, we've had

$$H\mathbf{v} = \lambda\mathbf{v}$$

Now, let us consider the conjugate transpose of the product $H\mathbf{v}$. We have

$$(H\mathbf{v})^\dagger_{ij} = \left((\overline{H\mathbf{v}})_{ij}\right)^T = (\overline{H\mathbf{v}})_{ji} = \sum_k \overline{H}_{jk}\overline{\mathbf{v}}_{ki} = \sum_k \overline{H}^T_{kj}\overline{\mathbf{v}}^T_{ik}$$

$$= \sum_k \overline{\mathbf{v}}^T_{ik}\overline{H}^T_{kj} = \left(\overline{\mathbf{v}}^T\overline{H}^T\right)_{ij} = (\mathbf{v}^\dagger H^\dagger)_{ij}$$

So, we obtain

$$(H\mathbf{v})^\dagger = \mathbf{v}^\dagger H^\dagger \tag{6.6.23}$$

And because λ is a number(we don't know whether it's complex or not), the conjugate transpose of the right-hand side is

$$(\lambda\mathbf{v})^\dagger = \overline{\lambda}\mathbf{v}^\dagger \tag{6.6.24}$$

Combining (6.6.23) with (6.6.24) and comparing them with the original one, we have

$$\begin{cases} \mathbf{v}^\dagger H^\dagger = \overline{\lambda}\mathbf{v}^\dagger \\ H\mathbf{v} = \lambda\mathbf{v} \end{cases} \tag{6.6.25}$$

We multiply both sides of the first equation by the matrix $\mathbf{v}$ on the right, and multiply both sides of the first equation by the matrix $\mathbf{v}^\dagger$ on the left, we get

$$\begin{cases} \mathbf{v}^\dagger H^\dagger\mathbf{v} = \overline{\lambda}\mathbf{v}^\dagger\mathbf{v} \\ \mathbf{v}^\dagger H\mathbf{v} = \mathbf{v}^\dagger\lambda\mathbf{v} = \lambda\mathbf{v}^\dagger\mathbf{v} \end{cases} \tag{6.6.26}$$

By the definition of the Hermitian $H = H^\dagger$, so

$$\lambda\mathbf{v}^\dagger\mathbf{v} = \overline{\lambda}\mathbf{v}^\dagger\mathbf{v} \Rightarrow (\lambda - \overline{\lambda})\mathbf{v}^\dagger\mathbf{v} = 0 \tag{6.6.27}$$

Since the eigenvector $\mathbf{v} \neq \mathbf{0}$, we can prove that $\lambda - \overline{\lambda} = 0 \Rightarrow \lambda = \overline{\lambda}$, that is, $\lambda \in \mathbb{R}$. Therefore, we arrive at an important property: **the eigenvalues of a Hermitian matrix are always real.** With this foundational result, we can further explore the diagonalization of Hermitian matrices.
When there are two distinct eigenvalues λ_1, λ_2 of the Hermitian H, then

$$\begin{cases} H\mathbf{v}_1 = \lambda_1\mathbf{v}_1 \\ H\mathbf{v}_2 = \lambda_2\mathbf{v}_2 \end{cases} \tag{6.6.28}$$

Taking the conjugate transpose of the eigenvalue equation $H\mathbf{v}_1 = \lambda_1\mathbf{v}_1$, we obtain

$$(H\mathbf{v}_1)^\dagger = (\lambda_1\mathbf{v}_1)^\dagger \Rightarrow \mathbf{v}_1^\dagger H = \lambda_1\mathbf{v}_1^\dagger$$

since $H^\dagger = H$ and $\overline{\lambda} = \lambda$. Multiply both sides by $\mathbf{v}_2$ on the right side,

$$\mathbf{v}_1^\dagger H \mathbf{v}_2 = \lambda_1 \mathbf{v}_1^\dagger \mathbf{v}_2 \tag{6.6.29}$$

After this, we can directly multiply both sides by $\mathbf{v}_1^\dagger$ on the left side of the second equation of (6.6.28), we obtain

$$\mathbf{v}_1^\dagger H \mathbf{v}_2 = \mathbf{v}_1^\dagger \lambda_2 \mathbf{v}_2 = \lambda_2 \mathbf{v}_1^\dagger \mathbf{v}_2 \tag{6.6.30}$$

By (6.6.29) and (6.6.30), we have

$$\lambda_1 \mathbf{v}_1^\dagger \mathbf{v}_2 = \lambda_2 \mathbf{v}_1^\dagger \mathbf{v}_2 \Rightarrow (\lambda_1 - \lambda_2)\mathbf{v}_1^\dagger \mathbf{v}_2 = 0 \tag{6.6.31}$$

Because $\lambda_1 \neq \lambda_2$, we can thus deduce that $\mathbf{v}_1^\dagger \mathbf{v}_2 = 0$. By (6.5.15), we can claim that the distinct eigenvectors of the Hermitian matrix are orthogonal.
Having established these key properties, we are now ready to carry out the most important step, the diagonalization. Like (6.6.5), we have

$$H\tilde{\mathbf{v}} = \begin{pmatrix} \sum_k h_{1k}x_{1k} & \sum_k h_{1k}x_{2k} & \cdots & \sum_k h_{1k}x_{nk} \\ \sum_j h_{2k}x_{1k} & \sum_k h_{2k}x_{2k} & \cdots & \sum_k h_{2k}x_{nk} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_j h_{nk}x_{1k} & \sum_k h_{nk}x_{2k} & \cdots & \sum_k h_{nk}x_{nk} \end{pmatrix} = \begin{pmatrix} \lambda_1 x_{11} & \lambda_2 x_{21} & \cdots & \lambda_n x_{n1} \\ \lambda_1 x_{12} & \lambda_2 x_{22} & \cdots & \lambda_n x_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 x_{1n} & \lambda_2 x_{2n} & \cdots & \lambda_n x_{nn} \end{pmatrix}$$

Note that the elements of the matrix on the right-hand side can be written as

$$\sum_i x_{ji}\delta_{ij}\lambda_i \coloneqq \tilde{\mathbf{v}}\tilde{\Lambda}$$

Changing the eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ into **unit vectors**, we redefine the matrix $\tilde{\mathbf{v}}$ as

$$U \coloneqq \frac{\tilde{\mathbf{v}}}{\|\mathbf{v}_i\|} \equiv \begin{pmatrix} \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} & \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} & \cdots & \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|} \end{pmatrix} \tag{6.6.32}$$

Then we rewrite the equation

$$HU = U\tilde{\Lambda} \tag{6.6.33}$$

And we have

$$U^{-1}HU = \tilde{\Lambda} \tag{6.6.34}$$

since $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly dependent. Next, we multiply both sides of (6.6.33) by the matrix $U^\dagger$, then we obtain

$$U^\dagger HU = U^\dagger U\tilde{\Lambda} = \begin{pmatrix} \frac{\mathbf{v}_1^\dagger}{\|\mathbf{v}_1\|} \\ \frac{\mathbf{v}_2^\dagger}{\|\mathbf{v}_2\|} \\ \vdots \\ \frac{\mathbf{v}_n^\dagger}{\|\mathbf{v}_n\|} \end{pmatrix} \begin{pmatrix} \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} & \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} & \cdots & \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|} \end{pmatrix} \tag{6.6.35}$$

(6.6.35) equals

$$\begin{pmatrix} \frac{\mathbf{v}_1^\dagger \mathbf{v}_1}{\|\mathbf{v}_1\|^2} & \frac{\mathbf{v}_1^\dagger \mathbf{v}_2}{\|\mathbf{v}_1\|\|\mathbf{v}_2\|} & \cdots & \frac{\mathbf{v}_1^\dagger \mathbf{v}_n}{\|\mathbf{v}_1\|\|\mathbf{v}_n\|} \\ \frac{\mathbf{v}_2^\dagger \mathbf{v}_1}{\|\mathbf{v}_1\|\|\mathbf{v}_2\|} & \frac{\mathbf{v}_2^\dagger \mathbf{v}_2}{\|\mathbf{v}_2\|^2} & \cdots & \frac{\mathbf{v}_2^\dagger \mathbf{v}_n}{\|\mathbf{v}_n\|\|\mathbf{v}_2\|} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_n^\dagger \mathbf{v}_1}{\|\mathbf{v}_1\|\|\mathbf{v}_n\|} & \frac{\mathbf{v}_n^\dagger \mathbf{v}_2}{\|\mathbf{v}_2\|\|\mathbf{v}_n\|} & \cdots & \frac{\mathbf{v}_n^\dagger \mathbf{v}_n}{\|\mathbf{v}_n\|^2} \end{pmatrix} \tag{6.6.36}$$

By (6.6.31), we know that the eigenvectors of a Hermitian matrix corresponding to distinct eigenvalues are orthogonal. Thus, we have

$$\mathbf{v}_i^\dagger \mathbf{v}_j = \|\mathbf{v}_i\|\|\mathbf{v}_j\|\delta_{ij} \tag{6.6.37}$$

So, (6.6.36) becomes

$$\begin{pmatrix} \frac{\|\mathbf{v}_1\|^2}{\|\mathbf{v}_1\|^2} & 0 & \cdots & 0 \\ 0 & \frac{\|\mathbf{v}_2\|^2}{\|\mathbf{v}_2\|^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\|\mathbf{v}_n\|^2}{\|\mathbf{v}_n\|^2} \end{pmatrix} = U^\dagger U = I \tag{6.6.38}$$

Substituting this into (6.6.35), we can get

$$U^\dagger H U = U^\dagger U \tilde{\Lambda} = I\tilde{\Lambda} = \tilde{\Lambda} \tag{6.6.39}$$

Comparing this result with (6.6.34)

$$\begin{cases} U^\dagger H U = \tilde{\Lambda} \\ U^{-1} H U = \tilde{\Lambda} \end{cases}$$

Thus we obtain

$$U^{-1} = U^\dagger \tag{6.6.40}$$

We call the matrix U a **unitary matrix**. And the transformation $U^\dagger H U = U^{-1} H U = \tilde{\Lambda}$ indicates that the Hermitian matrix H is diagonalized by a **unitary similarity transformation**.

6.7 Applications to the Eigen Problem in Physical Systems

▷ Coupled Spring System

As emphasized throughout this text, the goal is to address the mathematical structures behind problems in mechanics, rather than to delve into physical intuition or diagrams. In the classical coupled spring problem, we consider three springs with identical spring constants k, and two point masses of equal mass m situated between them. Let $x_1(t)$ and $x_2(t)$ denote the displacements of the two masses from equilibrium. The total potential energy of the system is given by

$$V = k(x_1^2 - x_1x_2 + x_2^2) \tag{6.7.1}$$

Then the equations of motion of both point masses are

$$\begin{cases} m\ddot{x}_1 = -\dfrac{\partial V}{\partial x_1} = -2kx_1 + kx_2 \\ m\ddot{x}_2 = -\dfrac{\partial V}{\partial x_2} = -2kx_2 + kx_1 \end{cases} \tag{6.7.2}$$

Let the matrix $\ddot{\mathbf{X}} = (\ddot{x}_1 \quad \ddot{x}_2)^T$ and $\mathbf{X} = (x_1 \quad x_2)^T$, so (6.7.2) can be written as

$$m\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_1 \end{pmatrix} = K\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Leftrightarrow m\ddot{\mathbf{X}} = -kA\mathbf{X} \tag{6.7.3}$$

where the matrix $-kA$ is

$$-kA = \begin{pmatrix} -2k & k \\ k & -2k \end{pmatrix} \Rightarrow A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

By viewing the coupled elements as part of a unified system, we can analyze the second-order linear differential equation in matrix form more easily. This process is known as "decoupling" the system. Therefore, by Chapter 4, we can assume that the solution to the differential equation is

$$\mathbf{X} = \mathbf{v}e^{i\omega t} \tag{6.7.4}$$

Substituting this into (6.7.3), we obtain

$$-m\omega^2\mathbf{v}e^{i\omega t} = -kA\mathbf{v}e^{i\omega t} \Rightarrow A\mathbf{v} = \frac{m\omega^2}{k}\mathbf{v} := \lambda\mathbf{v} \tag{6.7.5}$$

The question becomes a classic eigen-problem. That is,

$$(A - \lambda I)\,\mathbf{v} = 0 \tag{6.7.6}$$

Since $\mathbf{v} \neq 0$, we have

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0 \tag{6.7.7}$$

Then we can get two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$, also, we can obtain their corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. We obtain the frequency

$$\omega_1 = \sqrt{\frac{k\lambda_1}{m}} = \sqrt{\frac{k}{m}} \quad \omega_2 = \sqrt{\frac{k\lambda_2}{m}} = \sqrt{\frac{3k}{m}} \tag{6.7.8}$$

Then we diagonalize the matrix A

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} (\mathbf{v}_1 \quad \mathbf{v}_2) = (\mathbf{v}_1 \quad \mathbf{v}_2) \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \tag{6.7.9}$$

So (6.7.6) can be rewritten as

$$\begin{cases} \begin{pmatrix} 2-1 & -1 \\ -1 & 2-1 \end{pmatrix} \mathbf{v}_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = 0 \\ \begin{pmatrix} 2-3 & -1 \\ -1 & 2-3 \end{pmatrix} \mathbf{v}_2 = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = 0 \end{cases} \tag{6.7.10}$$

By (6.7.10), the eigenvectors of the matrix A are

$$\mathbf{v}_1 = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{6.7.11}$$

From this, we obtain the motion states of the system, which are also referred to as its **modes**. The values ω_1 and ω_2 represent the frequencies associated with the **normal modes** of the system. Hence,

$$\mathbf{X}(t) = C_1 \mathbf{v}_1 \cos\left(\sqrt{\frac{k}{m}}t - \phi_1\right) + C_2 \mathbf{v}_2 \cos\left(\sqrt{\frac{3k}{m}}t - \phi_2\right) \tag{6.7.12}$$

We expand the result of (6.7.12)

$$\begin{aligned} \mathbf{X}(t) = & A_1 \mathbf{v}_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + B_1 \mathbf{v}_1 \sin\left(\sqrt{\frac{k}{m}}t\right) \\ & + A_2 \mathbf{v}_2 \cos\left(\sqrt{\frac{3k}{m}}t\right) + B_2 \mathbf{v}_2 \sin\left(\sqrt{\frac{3k}{m}}t\right) \end{aligned} \tag{6.7.13}$$

Where the constants A_1, A_2, B_1, B_2 are

$$\begin{cases} A_1 = C_1 \cos\phi_1, \ B_1 = C_1 \sin\phi_1 \\ A_2 = C_2 \cos\phi_2, \ B_2 = C_2 \sin\phi_2 \end{cases}$$

To determine the behavior of the system, we need to consider the initial conditions. Let the initial position and velocity of the system be

$$\begin{cases} \mathbf{X}(0) \coloneqq \mathbf{X}_0 = A_1 \mathbf{v}_1 + A_2 \mathbf{v}_2 \\ \dot{\mathbf{X}}(0) \coloneqq \dot{\mathbf{X}}_0 = \sqrt{\frac{3k}{m}} B_1 \mathbf{v}_1 + \sqrt{\frac{3k}{m}} B_2 \mathbf{v}_2 \end{cases} \tag{6.7.14}$$

To determine these constants, we recall the projection formula taught in high school, often referred to as the "length of the orthogonal projection." That is

$$\begin{cases} A_1 = \dfrac{A_1 \left|\mathbf{v}_1^T \mathbf{X}_0\right|}{A_1 \|\mathbf{v}_1\| \|\mathbf{v}_1\|} = \dfrac{\left|\mathbf{v}_1^T \mathbf{X}_0\right|}{\|\mathbf{v}_1\|^2} \\ A_2 = \dfrac{A_2 \left|\mathbf{v}_2^T \mathbf{X}_0\right|}{A_2 \|\mathbf{v}_2\| \|\mathbf{v}_2\|} = \dfrac{\left|\mathbf{v}_2^T \mathbf{X}_0\right|}{\|\mathbf{v}_2\|^2} \end{cases} \tag{6.7.15}$$

Similarly,

$$B_1 = \frac{\left|\mathbf{v}_1^T \dot{\mathbf{X}}_0\right|}{\sqrt{3k/m}\|\mathbf{v}_1\|^2} \quad B_2 = \frac{\left|\mathbf{v}_2^T \dot{\mathbf{X}}_0\right|}{\sqrt{3k/m}\|\mathbf{v}_2\|^2} \tag{6.7.16}$$

From equations (6.7.15) and (6.7.16), we observe that each coefficient is proportional to the projection of the initial condition onto the corresponding eigenvector, that is

$$A_i \propto \mathbf{v}_i^T \mathbf{X}_0, \quad B_i \propto \mathbf{v}_i^T \dot{\mathbf{X}}_0 \tag{6.7.17}$$

Hence, the relative magnitudes of these projections determine which normal mode is more strongly excited. For example, if $|\mathbf{v}_1^T \mathbf{X}_0| \gg |\mathbf{v}_2^T \mathbf{X}_0|$, the system is initially displaced mainly along the direction of $\mathbf{v}_1$, and the motion will closely follow the first mode.

While it is possible to analyze a three-mass coupled system in full generality, the resulting characteristic equation becomes algebraically complicated, and no new conceptual structure emerges. Therefore, we focus our efforts on the two-mass coupled system, which already encapsulates all fundamental features of modal decomposition, eigenvalue-based decoupling, orthogonality of modes, and projection of initial conditions. Higher-dimensional systems follow the same structure, and the essential ideas extend naturally to continuous systems such as the vibrating string.

▷ **Infinite Potential Well**

A particle of mass m is confined in a one-dimensional region $x \in [0, L]$, where the potential outside this region is infinitely large. The potential function $V(x)$ is defined as:

$$V(x) = \begin{cases} 0, & 0 < x < L \\ \infty, & x \leq 0 \text{ or } x \geq L \end{cases} \tag{6.7.18}$$

Since the particle cannot exist outside the potential well, the wavefunction $\psi(x)$ (which represents the probability amplitude) must satisfy the boundary conditions

$$\psi(0) = \psi(L) = 0 \tag{6.7.19}$$

The time-independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x) \tag{6.7.20}$$

By the condition (6.7.18), the potential function $V(x) = 0$, and let $\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$, then (6.7.20) becomes

$$\hat{H}\psi = E\psi \tag{6.7.21}$$

Comparing (6.7.21) and (6.7.5), we observe that

$$\begin{cases} \hat{H}\psi = E\psi \\ A\mathbf{v} = \lambda\mathbf{v} \end{cases} \tag{6.7.22}$$

Here, $\hat{H}$ is a differential operator, meaning it *acts* on the probability wavefunction ψ, while A is a linear transformation matrix that acts on a vector $\mathbf{v}$. Similarly, when the

wavefunction ψ is acted upon by $\hat{H}$, the result differs by a constant factor E; likewise, when the vector $\mathbf{v}$ is acted upon by A, the result differs by a constant λ. In this quantum system, the wavefunction ψ is referred to as an **eigenfunction**, and the energy E is the corresponding **eigenvalue**.

To solve such a problem, let $\lambda^2 = 2mE/\hbar^2$, then (6.7.21) becomes

$$\frac{d^2\psi}{dx} + \lambda^2\psi = 0 \tag{6.7.23}$$

It becomes a second-order differential equation. The solution to this is

$$\psi(x) = C_1 \sin(\lambda x) + C_2 \cos(\lambda x) \tag{6.7.24}$$

By the initial condition,

$$\begin{cases} \psi(0) = C_1 \sin(\lambda x) + C_2 \cos(\lambda x) = 0 \Rightarrow B = 0 \\ \psi(L) = C_1 \sin(L\lambda) = 0 \Rightarrow L\lambda = n\pi \quad n \in \mathbb{N} \end{cases} \tag{6.7.25}$$

So the constant λ is equal to

$$\lambda = \frac{n\pi}{L} \Rightarrow \psi(x) = C_1 \sin\left(\frac{n\pi x}{L}\right) \tag{6.7.26}$$

Substituting this into $\lambda^2 = 2mE/\hbar^2$, we can obtain the energy E, that is, the eigenvalue

$$E = \frac{n^2\hbar^2\pi^2}{2mL^2} \tag{6.7.27}$$

In quantum mechanics, we normalize the wavefunction so that the total probability within the potential well is equal to 1. As for the deeper physical interpretation, I'll leave that to Professor Kuo. :)

$$\begin{aligned} \int_0^L |\psi_n(x)|^2 dx &= C_1^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx \\ &= C_1^2 \int_0^L \left(\frac{1 - \cos\left(\frac{n\pi x}{L}\right)}{2}\right) dx \\ &= C_1^2 \int_0^L \frac{1}{2} dx - \frac{1}{2}\int_0^L \cos\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{LC_1^2}{2} - \frac{1}{2}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L = \frac{LC_1^2}{2} = 1 \end{aligned}$$

So, we obtain

$$C_1 = \sqrt{\frac{2}{L}} \tag{6.7.28}$$

Substituting this into the eigenfunction $\psi(x)$, we can obtain the final solution that

$$\begin{cases} \psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \\ E = \frac{n^2\hbar^2\pi^2}{2mL^2} \end{cases} \quad n \in \mathbb{N} \tag{6.7.29}$$

Much like the eigenvectors in a coupled spring system, these sinusoidal eigenfunctions represent the natural modes of the quantum system.

6.8 Curvilinear Coordinates (Optional)

▷ General Definition of Curvilinear Coordinates

In chapter 1.4, we've discussed the Jacobian matrix and coordinate transformation. In this section, we focus on the transformation from Cartesian coordinates to curvilinear coordinates in three-dimensional system. Our goal is for this transformation to be one-to-one, and we have

$$(x_1, x_2, x_3) \leftrightarrow (u_1, u_2, u_3) \tag{6.8.1}$$

Therefore, the Jacobian between the two coordinate systems represents a bijective structure, which means that they are both linearly independent. We can verify

$$\det(J) = \frac{\partial(x_1, x_2, x_3)}{\partial(u_1, u_2, u_3)} = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \frac{\partial x_1}{\partial u_3} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_2}{\partial u_3} \\ \frac{\partial x_3}{\partial u_1} & \frac{\partial x_3}{\partial u_2} & \frac{\partial x_3}{\partial u_3} \end{vmatrix} = \frac{\partial \mathbf{r}}{\partial u_1} \cdot \left(\frac{\partial \mathbf{r}}{\partial u_2} \times \frac{\partial \mathbf{r}}{\partial u_3} \right) \neq 0 \tag{6.8.2}$$

Where $\mathbf{r} = (x_1, x_2, x_3)$. Similarly,

$$\det(J^{-1}) = \frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix} = \boldsymbol{\nabla} u_1 \cdot (\boldsymbol{\nabla} u_2 \cdot \boldsymbol{\nabla} u_3) \neq 0 \tag{6.8.3}$$

According to the properties of matrix multiplication and inverse matrices,

$$J^{-1}J = \begin{pmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \frac{\partial x_1}{\partial u_3} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_2}{\partial u_3} \\ \frac{\partial x_3}{\partial u_1} & \frac{\partial x_3}{\partial u_2} & \frac{\partial x_3}{\partial u_3} \end{pmatrix} \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{pmatrix} \tag{6.8.4}$$

$$\Rightarrow (J^{-1}J)_{ij} = \sum_k \frac{\partial u_i}{\partial x_k} \frac{\partial x_k}{\partial u_j} = \frac{\partial u_i}{\partial u_j} = \delta_{ij}$$

We can therefore redefine the gradient in the new coordinate system such that it satisfies

$$\boxed{\boldsymbol{\nabla} u_i \cdot \frac{\partial \mathbf{r}}{\partial u_j} = \delta_{ij}} \tag{6.8.5}$$

Here, we must introduce a new symbol and matrix, referred to as **Riemannian metric**. It is defined as

$$g_{ij} \coloneqq \frac{\partial \mathbf{r}}{\partial u_i} \cdot \frac{\partial \mathbf{r}}{\partial u_j} \tag{6.8.6}$$

The matrix form of the Riemannian metric $\mathbf{g}$ can be written as

$$\mathbf{g} = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \tag{6.8.7}$$

where we can denote $\mathbf{g}_{ij} = g_{ij}$. Its inverse matrix $\mathbf{g}^{-1}$ satisfies that

$$\mathbf{g}_{ij}^{-1} \coloneqq g^{ij} \tag{6.8.8}$$

That is,

$$\mathbf{g}\mathbf{g}^{-1} = I \Rightarrow (\mathbf{g}\mathbf{g}^{-1})_{ij} = \sum_k g_{ik} g^{kj} = \delta_{ij} \tag{6.8.9}$$

To better define the directional gradient in the new coordinate system, we make use of the properties stated in equation (6.8.9). Let $\boldsymbol{\nabla} u_i = \sum_k C_k \frac{\partial \mathbf{r}}{\partial u_k}$, by (6.8.5) we have

$$\boldsymbol{\nabla} u_i \cdot \frac{\partial \mathbf{r}}{\partial u_j} = \sum_k C_k \left(\frac{\partial \mathbf{r}}{\partial u_k} \cdot \frac{\partial \mathbf{r}}{\partial u_j} \right) = \delta_{ij}$$

By the definition of (6.8.6) and the property (6.8.9), we can obtain

$$\boldsymbol{\nabla} u_i \cdot \frac{\partial \mathbf{r}}{\partial u_j} = \sum_k C_k \; g_{kj} = \delta_{ij} \Rightarrow C_k = g^{ik} \tag{6.8.10}$$

Then the gradient can be denoted by

$$\boxed{\boldsymbol{\nabla} u_i = \sum_k g^{ik} \frac{\partial \mathbf{r}}{\partial u_k}} \tag{6.8.11}$$

Due to the independence of $\{\partial \mathbf{r}/\partial u_i\}$ and $\{u_i\}$, we can use the set of the tangent vector of u_i to construct the basis vectors $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$, which satisfies

$$\operatorname{span}\{\mathbf{e}_1, \cdots, \mathbf{e}_n\} = \mathbb{R}^n \tag{6.8.12}$$

We define

$$\mathbf{e}_i = \frac{1}{h_i} \frac{\partial \mathbf{r}}{\partial u_i} \quad h_i = \left| \frac{\partial \mathbf{r}}{\partial u_i} \right| \tag{6.8.13}$$

Next, we can use this set of basis vectors in the new coordinate system to derive various elements introduced in Chapter 2, such as the differential arc length and surface area elements. Because the transformation ensures the bijective structure, we can verify that the vector $\mathbf{r}$ can be denoted by

$$\mathbf{r}(x_1, x_2, x_3) = \mathbf{r}(x_1(u_1, u_2, u_3), x_2(u_1, u_2, u_3), x_3(u_1, u_2, u_3)) \tag{6.8.14}$$

By (1.4.14) and (1.4.15), we can apply the chain rule to the vector $\mathbf{r}$

$$\begin{cases} \dfrac{\partial \mathbf{r}}{\partial u_1} = \dfrac{\partial \mathbf{r}}{\partial x_1}\dfrac{\partial x_1}{\partial u_1} + \dfrac{\partial \mathbf{r}}{\partial x_2}\dfrac{\partial x_2}{\partial u_1} + \dfrac{\partial \mathbf{r}}{\partial x_3}\dfrac{\partial x_3}{\partial u_1} \\ \dfrac{\partial \mathbf{r}}{\partial u_2} = \dfrac{\partial \mathbf{r}}{\partial x_1}\dfrac{\partial x_1}{\partial u_2} + \dfrac{\partial \mathbf{r}}{\partial x_2}\dfrac{\partial x_2}{\partial u_2} + \dfrac{\partial \mathbf{r}}{\partial x_3}\dfrac{\partial x_3}{\partial u_2} \\ \dfrac{\partial \mathbf{r}}{\partial u_3} = \dfrac{\partial \mathbf{r}}{\partial x_1}\dfrac{\partial x_1}{\partial u_3} + \dfrac{\partial \mathbf{r}}{\partial x_2}\dfrac{\partial x_2}{\partial u_3} + \dfrac{\partial \mathbf{r}}{\partial x_3}\dfrac{\partial x_3}{\partial u_3} \end{cases} \tag{6.8.15}$$

Then the total derivative of the vector $\mathbf{r}$ can be written as

$$\begin{aligned} d\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial u_1} du_1 + \frac{\partial \mathbf{r}}{\partial u_2} du_2 + \frac{\partial \mathbf{r}}{\partial u_3} du_3 \\ &= h_1 \left(\frac{1}{h_1} \frac{\partial \mathbf{r}}{\partial u_1} \right) du_1 + h_2 \left(\frac{1}{h_2} \frac{\partial \mathbf{r}}{\partial u_2} \right) du_2 + h_3 \left(\frac{1}{h_3} \frac{\partial \mathbf{r}}{\partial u_3} \right) du_3 \\ &= \sum_i h_i \, du_i \, \mathbf{e}_i \end{aligned} \tag{6.8.16}$$

The length element ds is

$$\begin{aligned} ds^2 = d\mathbf{r} \cdot d\mathbf{r} &= \left(\frac{\partial \mathbf{r}}{\partial u_1} du_1 + \frac{\partial \mathbf{r}}{\partial u_2} du_2 + \frac{\partial \mathbf{r}}{\partial u_3} du_3 \right) \cdot \left(\frac{\partial \mathbf{r}}{\partial u_1} du_1 + \frac{\partial \mathbf{r}}{\partial u_2} du_2 + \frac{\partial \mathbf{r}}{\partial u_3} du_3 \right) \\ &= \left(\sum_i \frac{\partial \mathbf{r}}{\partial u_i} du_i \right) \cdot \left(\sum_j \frac{\partial \mathbf{r}}{\partial u_j} du_j \right) = \sum_{ij} \frac{\partial \mathbf{r}}{\partial u_i} \cdot \frac{\partial \mathbf{r}}{\partial u_j} du_i du_j \end{aligned}$$

Substituting the Riemannian metric into the equation above, we can thus obtain

$$ds^2 = \sum_{ij} g_{ij} du_i du_j \tag{6.8.17}$$

Then the length element can be written as

$$\boxed{ds = \sqrt{\sum_{ij} g_{ij} du_i du_j}} \tag{6.8.18}$$

As you did in the Cartesian coordinates, the surface area element is

$$\begin{aligned} dS_i &= \left| \frac{\partial \mathbf{r}}{\partial u_j} \times \frac{\partial \mathbf{r}}{\partial u_k} \right| du_j du_k \\ &= \sqrt{\left(\frac{\partial \mathbf{r}}{\partial u_j} \times \frac{\partial \mathbf{r}}{\partial u_k} \right) \cdot \left(\frac{\partial \mathbf{r}}{\partial u_j} \times \frac{\partial \mathbf{r}}{\partial u_k} \right)} \, du_j du_k \\ &= \sqrt{\left(\frac{\partial \mathbf{r}}{\partial u_j} \cdot \frac{\partial \mathbf{r}}{\partial u_j} \right) \left(\frac{\partial \mathbf{r}}{\partial u_k} \cdot \frac{\partial \mathbf{r}}{\partial u_k} \right) - \left(\frac{\partial \mathbf{r}}{\partial u_j} \cdot \frac{\partial \mathbf{r}}{\partial u_k} \right)^2} \, du_j du_k \end{aligned}$$

We obtain

$$\boxed{dS_i = \sqrt{g_{jj} g_{kk} - g_{jk}^2} \, du_j du_k} \tag{6.8.19}$$

where $\{i, j, k\}$ are permutations of the elements $\{1, 2, 3\}$.

▷ Orthogonal Form of Curvilinear Coordinates

In orthogonal coordinate systems, such as spherical or cylindrical coordinates, the basis (or tangent) vectors are orthogonal to each other. Which is

$$g_{ij} = \frac{\partial \mathbf{r}}{\partial u_j} \cdot \frac{\partial \mathbf{r}}{\partial u_j} = h_i h_j \delta_{ij} \tag{6.8.20}$$

We can also write

$$\mathbf{e}_i \cdot \mathbf{e}_j = \frac{1}{h_i}\frac{\partial \mathbf{r}}{\partial u_j} \cdot \frac{1}{h_j}\frac{\partial \mathbf{r}}{\partial u_j} = \delta_{ij} \tag{6.8.21}$$

By (6.8.20), the Riemannian metric $\mathbf{g}$ satisfies

$$(\mathbf{g})_{ik} = g_{ik} = h_i h_j \delta_{ij} \tag{6.8.22}$$

Then the matrix form of the Riemannian metric becomes

$$\mathbf{g} = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{pmatrix} = \begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix} \tag{6.8.23}$$

By equation (6.8.9), the inverse matrix of the Riemannian metric satisfies

$$(\mathbf{g}\mathbf{g}^{-1})ij = \sum_k g_{ik} g^{kj} = \sum_k h_i h_k \delta_{ik}\, g^{kj} = h_i^2 g^{ij} = \delta_{ij} \tag{6.8.24}$$

Then we obtain

$$g^{ij} = \frac{1}{h_i^2}\delta_{ij} \tag{6.8.25}$$

Substituting this into (6.8.11), the gradient becomes

$$\boldsymbol{\nabla} u_i = \sum_k g^{ik}\frac{\partial \mathbf{r}}{\partial u_k} = g^{ii}\frac{\partial \mathbf{r}}{\partial u_i} = \frac{1}{h_i^2}\frac{\partial \mathbf{r}}{\partial u_i} = \frac{1}{h_1}\mathbf{e}_i \tag{6.8.26}$$

By (6.8.2), the Jacobian can be expressed as the product of the scale factors associated with the three tangent vectors

$$\det(J) = \frac{\partial \mathbf{r}}{\partial u_i} \cdot \left(\frac{\partial \mathbf{r}}{\partial u_j} \times \frac{\partial \mathbf{r}}{\partial u_k}\right) = \pm h_i h_j h_k \tag{6.8.27}$$

The length of the gradient in (6.8.26) is

$$|\boldsymbol{\nabla} u_i| = \frac{1}{h_i^2}\left|\frac{\partial \mathbf{r}}{\partial u_i}\right| = \frac{1}{h_i} = \frac{h_j h_k}{h_i h_j h_k}$$

Substituting (6.8.27) into it, we can finally obtain the gradient in the orthogonal coordinates

$$\boxed{\boldsymbol{\nabla} u_i = \frac{1}{J}\left(\frac{\partial \mathbf{r}}{\partial u_j} \times \frac{\partial \mathbf{r}}{\partial u_k}\right)} \tag{6.8.28}$$

Since

$$\boldsymbol{\nabla} u_i \parallel \frac{\partial \mathbf{r}}{\partial u_j} \times \frac{\partial \mathbf{r}}{\partial u_k}$$

Here, all configurations implicitly follow the right-hand rule, but we will not delve into the orientation details. Readers are advised to pay careful attention to the direction conventions. Furthermore, since the basis vectors are mutually orthogonal, we can easily project the differential arc length onto each basis direction, that is

$$\boxed{ds_i = d\mathbf{r} \cdot \mathbf{e}_i = h_i du_i} \tag{6.8.29}$$

Therefore, the total arc length element is simply the combined contribution from all three coordinate directions

$$\boxed{ds^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2} \tag{6.8.30}$$

Note that this projection-based approach for identifying the contribution along each coordinate direction is only valid in orthogonal coordinate systems. In the general curvilinear coordinates, the basis vectors are not necessarily mutually orthogonal, meaning that each projection may involve contributions from other directions due to the angles between the basis vectors. This is precisely why the Riemannian metric must be introduced to properly account for all directional contributions through the inner products of the basis vectors.

Finally, due to the orthogonality of the coordinate system, where the tangent vectors are mutually perpendicular, the expression for the surface element becomes significantly simpler. We have

$$\boxed{dS_i = \left| \frac{\partial \mathbf{r}}{\partial u_j} \times \frac{\partial \mathbf{r}}{\partial u_k} \right| du_j du_k = h_j h_k \, du_j du_k} \tag{6.8.31}$$

▷ **Gradient in Orthogonal Curvilinear Coordinates**

Just like a direction vector, the function $f(x_1, x_2, x_3)$ can be denoted by

$$f(x_1(u_1, u_2, u_3), x_2(u_1, u_2, u_3), x_3(u_1, u_2, u_3))$$

The gradient of the function f is

$$\boldsymbol{\nabla} f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right)$$

Since we are working within an orthogonal coordinate system, the gradient can be directly projected onto each basis vector $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to express its components along each coordinate direction, that is

$$\begin{aligned}
\boldsymbol{\nabla} f \cdot \mathbf{e}_1 &= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right) \cdot \frac{1}{h_1} \frac{\partial \mathbf{r}}{\partial u_1} \\
&= \frac{1}{h_1} \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right) \cdot \left(\frac{\partial x_1}{\partial u_1}, \frac{\partial x_2}{\partial u_1}, \frac{\partial x_3}{\partial u_1} \right) \\
&= \frac{1}{h_1} \left(\frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial u_1} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial u_1} + \frac{\partial f}{\partial x_3} \frac{\partial x_3}{\partial u_1} \right)
\end{aligned}$$

By (1.4.14), we can rewrite the dot product as

$$\boldsymbol{\nabla} f \cdot \mathbf{e}_1 = \frac{1}{h_1} \frac{\partial f}{\partial u_1} \tag{6.8.32}$$

Similarly,

$$\begin{cases}
\boldsymbol{\nabla} f \cdot \mathbf{e}_2 = \dfrac{1}{h_2} \dfrac{\partial f}{\partial u_2} \\
\boldsymbol{\nabla} f \cdot \mathbf{e}_3 = \dfrac{1}{h_3} \dfrac{\partial f}{\partial u_3}
\end{cases} \tag{6.8.33}$$

In this way, we have identified the components of the gradient along the three basis vectors, so the gradient in orthogonal curvilinear coordinates can be denoted by

$$\boxed{\boldsymbol{\nabla} f = \left(\frac{1}{h_1}\frac{\partial f}{\partial u_1}, \frac{1}{h_2}\frac{\partial f}{\partial u_2}, \frac{1}{h_3}\frac{\partial f}{\partial u_3}\right) = \sum_i \frac{1}{h_i}\frac{\partial f}{\partial u_i}\mathbf{e}_i} \tag{6.8.34}$$

In curvilinear coordinates, the gradient operator is rewritten as

$$\boxed{\boldsymbol{\nabla} = \left(\frac{1}{h_1}\frac{\partial}{\partial u_1}, \frac{1}{h_2}\frac{\partial}{\partial u_2}, \frac{1}{h_3}\frac{\partial}{\partial u_3}\right)} \tag{6.8.35}$$

The following presents the gradient operator in commonly used curvilinear coordinate systems.

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1. **Cylindrical Coordinates**:
   The position vector $\mathbf{r} = (r\cos\phi, r\sin\phi, z)$, then the tangent vectors and their scale factors are
   $$\begin{cases} \dfrac{\partial \mathbf{r}}{\partial r} = (\cos\phi, \sin\phi, 0) \Rightarrow h_r = \left|\dfrac{\partial \mathbf{r}}{\partial r}\right| = 1 \\ \dfrac{\partial \mathbf{r}}{\partial \phi} = (-r\sin\phi, r\cos\phi, 0) \Rightarrow h_\phi = \left|\dfrac{\partial \mathbf{r}}{\partial \phi}\right| = r \\ \dfrac{\partial \mathbf{r}}{\partial z} = (0, 0, 1) \Rightarrow h_z = \left|\dfrac{\partial \mathbf{r}}{\partial z}\right| = 1 \end{cases}$$
   Substituting these into (6.8.34), we can finally obtain
   $$\boxed{\boldsymbol{\nabla} f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi + \frac{\partial f}{\partial z}\mathbf{e}_z} \tag{6.8.36}$$

2. **Spherical Coordinates**:
   The position vector $\mathbf{r} = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$, then the tangent vectors and their scale factors are
   $$\begin{cases} \dfrac{\partial \mathbf{r}}{\partial r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \Rightarrow h_r = \left|\dfrac{\partial \mathbf{r}}{\partial r}\right| = 1 \\ \dfrac{\partial \mathbf{r}}{\partial \theta} = (r\cos\theta\cos\phi, r\cos\theta\sin\phi, -r\sin\theta) \Rightarrow h_\theta = \left|\dfrac{\partial \mathbf{r}}{\partial \theta}\right| = r \\ \dfrac{\partial \mathbf{r}}{\partial \phi} = (-r\sin\theta\sin\phi, r\sin\theta\cos\phi, 0) \Rightarrow h_\phi = \left|\dfrac{\partial \mathbf{r}}{\partial \phi}\right| = r\sin\theta \end{cases}$$
   Substituting these into (6.8.34), we can finally obtain
   $$\boxed{\boldsymbol{\nabla} f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi} \tag{6.8.37}$$

▷ **Divergence in Orthogonal Coordinate Systems**

Consider a vector field $\mathbf{F} = (F_1, F_2, F_3)$ in the orthogonal coordinate systems, which can be denoted by

$$\mathbf{F} = (F_1(u_1, u_2, u_3), F_2(u_1, u_2, u_3), F_3(u_1, u_2, u_3)) \tag{6.8.38}$$

Then the divergence of $\mathbf{F}$ becomes

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \boldsymbol{\nabla} \cdot (F_1\mathbf{e}_1) + \boldsymbol{\nabla} \cdot (F_2\mathbf{e}_2) + \boldsymbol{\nabla} \cdot (F_3\mathbf{e}_3) \tag{6.8.39}$$

By (6.8.26), we've obtained

$$\begin{cases} \boldsymbol{\nabla} u_1 = \dfrac{1}{h_1}\mathbf{e}_1 \Rightarrow \mathbf{e}_1 = h_1\boldsymbol{\nabla} u_1 \\ \boldsymbol{\nabla} u_2 = \dfrac{1}{h_2}\mathbf{e}_2 \Rightarrow \mathbf{e}_2 = h_2\boldsymbol{\nabla} u_2 \\ \boldsymbol{\nabla} u_3 = \dfrac{1}{h_3}\mathbf{e}_3 \Rightarrow \mathbf{e}_3 = h_3\boldsymbol{\nabla} u_3 \end{cases} \tag{6.8.40}$$

Since $\mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_3$, by (6.8.40), we can verify that

$$\mathbf{e}_1 = h_2h_3(\boldsymbol{\nabla} u_2 \times \boldsymbol{\nabla} u_3) \Leftrightarrow \frac{1}{h_2h_3}\mathbf{e}_1 = \boldsymbol{\nabla} u_2 \times \boldsymbol{\nabla} u_3 \tag{6.8.41}$$

Then we can discuss the first term on the right side of (6.8.39) through (6.8.41), we have

$$\begin{aligned} \boldsymbol{\nabla} \cdot (F_1\mathbf{e}_1) &= \boldsymbol{\nabla} \cdot [F_1h_2h_3(\boldsymbol{\nabla} u_2 \times \boldsymbol{\nabla} u_3)] \\ &= \boldsymbol{\nabla}(F_1h_2h_3) \cdot (\boldsymbol{\nabla} u_2 \times \boldsymbol{\nabla} u_3) + F_1h_2h_3\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} u_2 \times \boldsymbol{\nabla} u_3) \\ &= \boldsymbol{\nabla}(F_1h_2h_3) \cdot (\boldsymbol{\nabla} u_2 \times \boldsymbol{\nabla} u_3) \\ &= \boldsymbol{\nabla}(F_1h_2h_3) \cdot \frac{1}{h_2h_3}\mathbf{e}_1 \end{aligned}$$

Substituting the gradient into the equation, we can express that

$$\left(\frac{1}{h_1}\frac{\partial(F_1h_2h_3)}{\partial u_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial(F_1h_2h_3)}{\partial u_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial(F_1h_2h_3)}{\partial u_3}\mathbf{e}_3\right) \cdot \frac{1}{h_2h_3}\mathbf{e}_1 \tag{6.8.42}$$

Since $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$, $\mathbf{e}_1 \cdot \mathbf{e}_3 = 0$, we can obtain the finally result

$$\boldsymbol{\nabla} \cdot (F_1\mathbf{e}_1) = \frac{1}{h_1h_2h_3}\frac{\partial(F_1h_2h_3)}{\partial u_1} \tag{6.8.43}$$

Similarly,

$$\begin{cases} \boldsymbol{\nabla} \cdot (F_2\mathbf{e}_2) = \dfrac{1}{h_1h_2h_3}\dfrac{\partial(F_2h_1h_3)}{\partial u_2} \\ \boldsymbol{\nabla} \cdot (F_3\mathbf{e}_3) = \dfrac{1}{h_1h_2h_3}\dfrac{\partial(F_3h_1h_2)}{\partial u_3} \end{cases} \tag{6.8.44}$$

So the divergence in orthogonal curvilinear coordinates can be denoted by

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial(F_1h_2h_3)}{\partial u_1} + \frac{\partial(F_2h_1h_3)}{\partial u_2} + \frac{\partial(F_3h_1h_2)}{\partial u_3}\right]} \tag{6.8.45}$$

1. **Cylindrical Coordinates**:

$$\boxed{\boldsymbol{\nabla}\cdot\mathbf{F}=\frac{1}{r}\frac{\partial(rF_r)}{\partial r}+\frac{1}{r}\frac{\partial F_\phi}{\partial\phi}+\frac{\partial F_z}{\partial z}} \tag{6.8.46}$$

2. **Spherical Coordinates**:

$$\boxed{\boldsymbol{\nabla}\cdot\mathbf{F}=\frac{1}{r^2}\frac{\partial(r^2F_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(F_\theta\sin\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}} \tag{6.8.47}$$

▷ **Curl in Orthogonal Coordinate Systems**

Consider a vector field $\mathbf{F}(F_1,F_2,F_3)$, as before, we begin by discussing the first term $\boldsymbol{\nabla}\times(F_1\mathbf{e}_1)$. By (6.8.40), we substitute $\mathbf{e}_1=h_1\boldsymbol{\nabla}u_1$ into the curl, we obtain

$$\begin{aligned}\boldsymbol{\nabla}\times(F_1\mathbf{e}_1)&=\boldsymbol{\nabla}\times(F_1h_1\boldsymbol{\nabla}u_1)\\&=\boldsymbol{\nabla}(F_1h_1)\times\boldsymbol{\nabla}u_1+(F_1h_1)\boldsymbol{\nabla}\times\boldsymbol{\nabla}u_1\\&=\boldsymbol{\nabla}(F_1h_1)\times\boldsymbol{\nabla}u_1\end{aligned}$$

Substituting the gradient into the equation, we can express that

$$\left(\frac{1}{h_1}\frac{\partial(F_1h_1)}{\partial u_1}\mathbf{e}_1+\frac{1}{h_2}\frac{\partial(F_1h_1)}{\partial u_2}\mathbf{e}_2+\frac{1}{h_3}\frac{\partial(F_1h_1)}{\partial u_3}\mathbf{e}_3\right)\times\frac{1}{h_1}\mathbf{e}_1 \tag{6.8.48}$$

Therefore, the curl

$$\boldsymbol{\nabla}\times(F_1\mathbf{e}_1)=\frac{1}{h_1h_3}\frac{\partial(F_1h_1)}{\partial u_3}\mathbf{e}_2-\frac{1}{h_1h_2}\frac{\partial(F_1h_1)}{\partial u_2}\mathbf{e}_3 \tag{6.8.49}$$

Similarly,

$$\begin{cases}\boldsymbol{\nabla}\times(F_2\mathbf{e}_2)=\dfrac{1}{h_1h_2}\dfrac{\partial(F_2h_2)}{\partial u_1}\mathbf{e}_3-\dfrac{1}{h_2h_3}\dfrac{\partial(F_2h_2)}{\partial u_3}\mathbf{e}_1\\[2ex]\boldsymbol{\nabla}\times(F_3\mathbf{e}_3)=\dfrac{1}{h_2h_3}\dfrac{\partial(F_3h_3)}{\partial u_2}\mathbf{e}_1-\dfrac{1}{h_1h_3}\dfrac{\partial(F_3h_3)}{\partial u_1}\mathbf{e}_2\end{cases} \tag{6.8.50}$$

Based on the calculations above, we can express the curl using a determinant, written as follows

$$\boxed{\boldsymbol{\nabla}\times\mathbf{F}=\frac{1}{h_1h_2h_3}\begin{vmatrix}h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3\\[1ex]\dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3}\\[1ex]F_1h_1 & F_2h_2 & F_3h_3\end{vmatrix}} \tag{6.8.51}$$

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1. **Cylindrical Coordinates**:
   The curl in the cylindrical coordinates is

$$\boldsymbol{\nabla}\times\mathbf{F}=\frac{1}{r}\begin{vmatrix}\mathbf{e}_r & r\mathbf{e}_\phi & \mathbf{e}_z\\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z}\\ F_r & rF_\phi & F_z\end{vmatrix}$$

   Expanding this determinant, we obtain

$$\boxed{\begin{aligned}\boldsymbol{\nabla}\times\mathbf{F}=&\frac{1}{r}\left(\frac{\partial F_z}{\partial\phi}-r\frac{\partial F_\phi}{\partial z}\right)\mathbf{e}_r+\left(\frac{\partial F_r}{\partial z}-\frac{\partial F_z}{\partial r}\right)\mathbf{e}_\phi\\ &+\frac{1}{r}\left(\frac{\partial(rF_\phi)}{\partial r}-\frac{\partial F_r}{\partial\phi}\right)\mathbf{e}_z\end{aligned}} \tag{6.8.52}$$

2. **Spherical Coordinates**:
   The curl in the cylindrical coordinates is

$$\boldsymbol{\nabla}\times\mathbf{F}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi\\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi}\\ F_r & rF_\theta & r\sin\theta F_\phi\end{vmatrix}$$

   Expanding this determinant, we obtain

$$\boxed{\begin{aligned}\boldsymbol{\nabla}\times\mathbf{F}=&\frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta F_\phi)}{\partial\theta}-\frac{\partial F_\theta}{\partial\phi}\right]\mathbf{e}_r\\ &+\frac{1}{r\sin\theta}\left[\frac{\partial F_r}{\partial\phi}-\sin\theta\frac{\partial(rF_\phi)}{\partial r}\right]\mathbf{e}_\theta\\ &+\frac{1}{r}\left[\frac{\partial(rF_\theta)}{\partial r}-\frac{\partial F_r}{\partial\theta}\right]\mathbf{e}_\phi\end{aligned}} \tag{6.8.53}$$

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