RANDOM WALKS AND BROWNIAN MOTION ON MATRIX LIE GROUPS

AARON MOSER

Contents

1. Introduction

The goal of this project is to review some results on the spectral distribution of random walks and Brownian motion on matrix Lie groups. Lie groups are manifolds with a group structure and often we can represent them as a subgroup of the complex $n \times n$-matrices $\mathrm{Mat}(n, \mathbb{C})$. Thus, as for Brownian motion in Euclidean space, we can think of Brownian motion on matrix Lie groups as a random path on the Lie group. Hence, it makes sense to study the behavior of the eigenvalues as one traverses this random path in the space of matrices. This is particularly interesting since most Lie groups do not purely consist of Hermitian matrices and so the spectrum will generally lie outside of the real axis in $\mathbb{C}$.

The core of this report is to understand the results of the paper [2], which investigates the asymptotic distribution of eigenvalues of random walks and Brownian motion on matrix Lie groups as the dimension becomes large. However, [2] is a sizable article in its own right, and thus we will only present the results that seem most interesting to the author of the present report and not discuss any of the proofs. Moreover, we will liberally skip some details in order to make the presentation not too dense. More precisely, we pursue the following two goals:

(1) In [2] the asymptotic distribution of the eigenvalues for certain matrix Lie groups with non-normal elements are studied. It is show that the eigenvalue distributions of the random walks converge to the Brown measure of a free random walk. The Brown measure is a

certain generalization of the spectral measure for non-normal operators and in this case its support has a bean-like shape, prompting the authors of [2] to term it the "Lima bean law". Again, we computationally verify some of the the results.

(2) Motivated by the Tracy-Widom law for Gaussian ensembles, describing the distribution of the eigenvalue farthest away from 1, we numerically study the largest eigenvalue of matrix random walks. In particular, we find a similar universal behavior in the limit of large dimension, however, the limit is very different.

In the first part of this report we introduce some basics of the theory of matrix Lie groups and their Lie algebras, how to define Brownian motion on these Lie groups, the framework of free probability, and the notion of Brown measure. In the second part we showcase some numerical results on the distribution of the eigenvalue farthest away from 1.

2. Preliminaries

2.1. **Matrix Lie groups and their Lie algebras.** The general definition of a Lie group is a group that is equipped with the structure of a smooth manifold such that multiplication of two group elements and inversion is continuous. The associated Lie algebra of the Lie group is defined as the tangent space on the Lie group at the identity. More abstractly, a Lie algebra is a vector space $\mathfrak{g}$ over some field which is equipped with a bilinear map $[\bullet,\bullet]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, which is called *the Lie bracket* and satisfies the following two properties

(1) *Skew-commutativity:* For all $X\in\mathfrak{g}$ we have $[X,X]=0$.
(2) *Jacobi identity:* For all $X,Y,Z\in\mathfrak{g}$ we have
$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0.$$

Let us write $\mathrm{Mat}(n,\mathbb{C})$ for the set of all $n\times n$-matrices with complex entries. In this report we are only interested in Lie groups which are subgroups of the group $\mathrm{GL}(n,\mathbb{C})$ of invertible $n\times n$-matrices over $\mathbb{C}$. In this case, we will always take the Lie bracket to be the matrix commutator, defined for $X,Y\in\mathrm{Mat}(n,\mathbb{C})$ by
$$[X,Y]=XY-YX.$$
Let us now provide the most important examples of such Lie groups.

Example 2.1. The *group of invertible matrices* $\mathrm{GL}(n,\mathbb{C})$ is a Lie group. Its Lie algebra is given by $\mathfrak{gl}(n,\mathbb{C})=\mathrm{Mat}(n,\mathbb{C})$, the set of all complex $n\times n$-matrices.

Example 2.2. The *symplectic group* $\mathrm{Sp}(2n,\mathbb{C})$ of symplectic $2n\times 2n$-matrices is a Lie group. That is, $X\in\mathrm{Sp}(2n,\mathbb{C})$ if and only of
$$X^TJX=X,\quad\text{where } J=\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}.$$
Its Lie algebra is given by
$$\mathfrak{sp}(2n,\mathbb{C})=\{X\in\mathfrak{gl}(2n,\mathbb{C})\mid JX+X^TJ=0\}.$$
Suppose $X\in\mathfrak{sp}(2n,\mathbb{C})$ and we have the block decomposition
$$X=\begin{pmatrix}A & B\\ C & D\end{pmatrix},$$
where $A,B,C,D\in\mathrm{Mat}(n,\mathbb{C})$. Then
$$0=JX+X^TJ=\begin{pmatrix}C-C^T & A^T+D\\ -A-D^T & -B+B^T\end{pmatrix}$$

showing $A = -D^T$, $B^T = B$, $C^T = C$.

Since we are only interested in Lie groups which are subgroups of some matrix group, we shall not pursue the abstract definition much further but instead specialize to the matrix case. Let then $G \subset \mathrm{GL}(n, \mathbb{C})$ be a Lie group. Then we can compute its Lie algebra $\mathfrak{g}$ is given by the following set

$$\mathfrak{g} = \{P'(0) \mid P\colon (-1,1) \to G \text{ is a smooth path with } P(0) = I\} \subset \mathfrak{gl}(n, \mathbb{C}).$$

This is indeed space of all tangent vectors at the identity matrix I. Using this definition it is possible to verify all of the above examples. Most importantly, it is possible to "project" elements of the Lie algebra $\mathfrak{g}$ back to the Lie group G via the matrix exponential map

$$\exp_G\colon \mathfrak{g} \to G.$$

Most importantly, the exponential map has the property that if $X, Y \in \mathfrak{g}$ with $[X, Y] = 0$, then

$$\exp_G(X + Y) = \exp_G(X) \exp_G(Y).$$

In the case of matrix Lie groups, the exponential map is defined by the convergent power series

$$\exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!}.$$

Indeed, one can check that for each of the above examples, exponentiating any element of the Lie algebra recovers an element of the original Lie group.

2.2. **Brownian motion.** We do not want to spend too much time on the technicalities of Brownian motion in $\mathbb{R}^n$ before moving to Brownian motion on matrix groups, so we will principally be concerned with the construction of Brownian motion from random walks and its properties. The following theorem says that we can recover Brownian motion from random walk approximations.

Theorem 2.3 (Random walk approximation, [1])**.** *Let $(X_j)_{j\in\mathbb{N}}$ be a sequence of i.i.d. centered, L^2-random vectors in $\mathbb{R}^n$ with covariance $\mathbb{E}[X_j X_j^T] = I$. For $t \geq 0$ define*

$$S(t) = \sum_{k=1}^{\lfloor t \rfloor} X_k + (t - \lfloor t \rfloor) X_{\lfloor t \rfloor + 1}.$$

Then for each $T > 0$, the sequence $W_k(t) = S(kt)/\sqrt{k}$ converges weakly on $C([0,T], \mathbb{R}^n)$ to the standard Brownian motion on $\mathbb{R}^n$ as $k \to \infty$.

At this point we should remark that this suggest a "differential" definition of Brownian motion as well. For this let us be a bit imprecise with notation and language. Let $B(t)$ be standard Brownian motion on $\mathbb{R}^n$. If we write X for a standard normal random variable, then we could write the local change $\mathrm{d}B$ in terms of a small change in time Δt and the random variable as

$$B(t + \Delta t) = B(t) + X\sqrt{\Delta t}. \tag{2.1}$$

In particular, we may view $X\mathrm{d}t$ as a normal random variable with infinitesimally small variance. This leads to the characterization of Brownian motion via a *stochastic differential equation*. Since the theory of stochastic differential equations is not so trivial to develop, we will be satisfied with this intuitive notion in terms of difference equations for the rest of this report.

We would like to now transfer a similar approach to Lie groups in order for us to be able to define Brownian motion on them. Taking the equation (2.1) in Euclidean space as a model, Brownian motion on a Lie group should satisfy a stochastic difference equation

$$B(t+\Delta t) = B(t) \cdot \exp\left(\sqrt{\Delta t} X\right), \tag{2.2}$$

where we replaced addition with the group multiplication on the Lie group and where X is a properly chosen element of the tangent space at the identity. The idea is properly construct it is to take small random increments on the Lie algebra and then project this change onto the group via the exponential map. Let us execute this with a bit more detail now.

Choose a basis V of $\mathfrak{g}$. Then we can construct a version of Brownian motion on $\mathfrak{g}$ by defining

$$W(t) \coloneqq \sum_{\xi \in V} W_\xi(t)\xi,$$

where $(W_\xi)_{\xi \in V}$ are i.i.d. standard Brownian motions on $\mathbb{R}$.

Now let us introduce some notation. Fix some $T > 0$ and let $\Pi \coloneqq \{0 = t_0 < t_1 < \cdots < t_k = T\}$ be a partition of $[0, T]$.

(1) t_- is the largest member of Π immediately to the left of t. In particular, $0_- = (t_0)_- = t_0 = 0$ and $(t_j)_j \coloneqq t_{j-1}$ for all $j = 1, \ldots, k$.
(2) $\Delta t \coloneqq t - t_-$ and $|\Pi| \coloneqq \max\{\Delta s \mid s \in \Pi\}$ is the mesh size of Π.
(3) If f is a function from $[0, T]$ to some vector space, then $\Delta_t f \coloneqq f(t) - f(t_-)$.
(4) $\ell_\Pi(s) \coloneqq \sum_{r\in\Pi} r_- \mathbf{1}_{(r_-,r]}(s)$ and $r_\Pi(s) \coloneqq \sum_{r\in\Pi} r \mathbf{1}_{(r_-,r]}(s)$ for all $s \geq 0$.

We approximate the process $W(t)$ that we defined using linear interpolation between the mesh points given in Π by setting

$$W_\Pi(t) \coloneqq W(\ell_\Pi(t)) + (t - \ell_\Pi(t)) \frac{W(r_\Pi(t)) - W(\ell_\Pi(t))}{r_\Pi(t) - \ell_\Pi(t)}$$

for all $t \in [0, T]$. For $t \notin \Pi$ this is a differentiable process with

$$\frac{\mathrm{d}W_\Pi(t)}{\mathrm{d}t} = \frac{W(t_j) - W(t_{j-1})}{t_j - t_{j-1}}$$

for $t \in (t_{j-1}, t_j)$. The W_Π provides us now with an approximation of Brownian motion on the tangent space at the identity in G. To lift this to the Lie group, we solve the stochastic differential equation

$$\frac{\mathrm{d}B_\Pi(t)}{\mathrm{d}t} = B_\Pi(t) \frac{\mathrm{d}W_\Pi(t)}{\mathrm{d}t},$$

which is easily seen to have the solution

$$B_\Pi(t) = \exp(\Delta_{t_1} W) \exp(\Delta_{t_2} W) \cdots \exp(\Delta_{t_{j-1}} W) \exp\left(\frac{t - t_{j-1}}{\Delta t_j} \Delta_{t_j} W\right)$$

for $t \in [t_{j-1}, t_j]$. Note that this is just a version of (2.2).

Theorem 2.4 (Random walk approximation on Lie groups, [3, 4]). *If $(\Pi_k)_{k\in\mathbb{N}}$ is a sequence of partitions of $[0, T]$ such that $|\Pi_k| \to 0$ as $k \to \infty$, then the sequence $(B_{\Pi_k})_{k\in\mathbb{N}}$ of processes converges weakly in $C([0, T], G)$ to the Brownian motion on G.*

For computational purposes, it will be nice to use a uniform partition Π_k, that is where all the $\Delta t_j = T/k$. Then we see that

$$\Delta W_{t_j} = \left(\frac{T}{k}\right)^{1/2} X_j$$

where the $X_1, \ldots, X_k$ are i.i.d. random variables distributed as $W(1)$[1]. The random walk approximation to Brownian motion then has the distribution

$$B_{\Pi_k}(T) \sim \exp\left(\sqrt{\frac{T}{k}} X_1\right) \exp\left(\sqrt{\frac{T}{k}} X_2\right) \cdots \exp\left(\sqrt{\frac{T}{k}} X_k\right).$$

For large k one can ease the computational requirement of computing a matrix exponential by noting that

$$\exp\left(\sqrt{\frac{T}{k}} X_j\right) = I + \sqrt{\frac{T}{k}} X_j + \frac{T}{2k} X_j^2 + o\left(\frac{1}{k}\right),$$

so that we only need to compute X_j^2 to obtain a good approximation. In [2] they even go a step further and even omit the quadratic term.

2.3. **Free probability.** Free probability is often pitched as a "non-commutative version of probability". While this is a very vague statement, let us think about what this could mean.

The "classical" theory of probability is built on the notion of measurability. We fix some measurable space (Ω, Σ), where Ω is a set and Σ is a σ-algebra on Ω. The σ-algebra tells us which subsets of Ω can be detected and we can assign to them a probability by defining a probability measure μ on the measurable space (Ω, Σ). After fixing a measurable space, the focus of classical probability theory then shifts to the study of real- or complex-valued *random variables* on Ω, that is, measurable functions Ω to $\mathbb{R}$ or $\mathbb{C}$. Crucially, $\mathbb{R}$ and $\mathbb{C}$ are equipped with natural measure, the Lebesgue measure.

Suppose we want to build a similar theory for a non-commutative algebra A. We encounter the first issue in the fact there is no obvious measure on A that would allow us to make sense of what it means to be a measurable function $\Omega \to A$. This leaves us a with a choice of how to proceed. Either we find some useful measure on A or we decide that in fact we were not interested in such a general setting from the start. Indeed, often the type of non-commutative algebras one is interested in are operator algebras, which have much more structure. For example, they are usually equipped with some sort of topology, for example the operator norm topology or weak topology. This means that we should restrict the measure on A to be a Borel measure. But we see that this becomes quite tedious at this point. The key insight that simplifies matters is that specifying a Borel measure is equivalent to specifying the value of the integral of each Borel measurable function. So instead of a measure on A, we specify an integral, which we shall call φ, on the set $\mathcal{A}$ of all Borel measurable functions $\Omega \to A$. This is the central idea for the further development of free probability, we just add some additional assumptions on $\mathcal{A}$, which are usually satisfied by operator algebras.

2.3.1. *$*$-probability.* Let us now review some of the relevant free probability theory that we are going to need to understand the Lima bean law.

Definition 2.5 ($*$-probability space)**.** A unital $*$-algebra $\mathcal{A}$ together with a linear functional $\varphi\colon \mathcal{A} \to \mathbb{C}$ is called a *$*$-probability space* if φ is unital, that is, if $\varphi(1_A) = 1$. We call φ *faithful* if $\varphi(a^*a) = 0$ implies $a = 0$ and we call it *tracial* if for all $a, b \in \mathcal{A}$ we have $\varphi(ab) = \varphi(ba)$.

[1] In particular, it will be important to choose the distribution of $W(1)$ properly.

Example 2.6. A useful example of a $*$-probability space to keep in mind is, roughly speaking, the space of random matrices. We let $\mathcal{A}$ be the space of L^∞-random variables with values in $\mathrm{Mat}(n, \mathbb{C})$. Then we can define

$$\varphi(A) = \mathbb{E}\left[\frac{1}{n}\operatorname{tr} A\right]$$

for $A \in \mathcal{A}$. We take the star operation to be taking the adjoint and with this $\mathcal{A}$ together with φ defines a $*$-probability space. Moreover, φ is in fact tracial and faithful.

To state the results later, we will need some more specializied spaces an example of which is also given by the space of random matrices with the above defined state.

Definition 2.7 (W^*-probability space)**.** A W^*-probability spaces is a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is an algebra of bounded operators on a Hilbert space which is closed in the σ-weak operator topology and φ is faithful, tracial and continuous with respect to the σ-weak operator topology.

We now want to introduce a notion of distribution for a $*$-probability space. To expand a little on the philosophical essay in the introduction to this section, let us return to the analogy of measures and integration. In classical probability, when making some mild assumptions, it is generally possible to uniquely identify a probability measure μ with its moments, the value $(\mathbb{E}_\mu[X^k])_{k\in\mathbb{N}}$, where $\mathbb{E}_\mu$ is the expected value with respect to μ. The definition of a $*$-distribution mirrors this.

Definition 2.8 ($*$-distribution)**.** The $*$-*distribution* of a collection $\mathbf{a} = (a_i)_{i\in I} \subset \mathcal{A}$ is the set

$$\{\varphi(a_{i_1}^{\epsilon_1} a_{i_2}^{\epsilon_2} \cdots a_{i_r}^{\epsilon_r}) \mid r \in \mathbb{N}, i_1, \dots, i_r \in I, \epsilon_1, \dots, \epsilon_r \in \{1, *\}\}$$

of all $*$-*moments*. We write $\mathcal{P}_I$ for the algebra generated by all polynomials of the form $a_{i_1}^{\epsilon_1} a_{i_2}^{\epsilon_2} \cdots a_{i_r}^{\epsilon_r}$, where $r \in \mathbb{N}, i_1, \dots, i_r \in I, \epsilon_1, \dots, \epsilon_r \in \{1, *\}$ and define the functional $\varphi_{\mathbf{a}} \colon \mathcal{P}_I \to \mathbb{C}$ by algebraically extending the action of φ to all elements of $\mathcal{P}_I$.

We also need a proper version of convergence.

Definition 2.9 (Convergence in $*$-distribution)**.** Let $(\mathcal{A}, \varphi)$ and $(\mathcal{A}_N, \varphi_N)_{N\in\mathbb{N}}$ be a $*$-probability spaces. We say the collections $\mathbf{a}_N = (a_{N,i})_{i\in I}$ converge in $*$-distribution to the collection $\mathbf{a} = (a_i)_{i\in I}$ and write $\mathbf{a}_N \rightharpoonup \mathbf{a}$ if the functionals $\varphi_{\mathbf{a}_N}$ converge pointwise to $\varphi_{\mathbf{a}}$ on $\mathcal{P}_I$.

One of the most fundamental concepts in classical probability is the notion of independence. This is often stated in terms of the measure, but again there is a useful integral definition for independence, too. Indeed, one can show that if two (classical) random-variables X, Y are independent, then $\mathbb{E}[X^k Y^m] = \mathbb{E}[X^k]\mathbb{E}[Y^m]$ for all $k, m \in \mathbb{N}$. This shall once again serve as a model for the free pendant.

Definition 2.10. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space and $(\mathcal{A}_i)_{i\in I}$ be a family of unital $*$-subalgebras. Then we call $(\mathcal{A}_i)_{i\in I}$ *freely independent* or *free* if

$$\varphi(a_1 \cdots a_N) = 0 \quad \text{whenever } \varphi(a_1) = \varphi(a_2) = \cdots = 0$$

where $a_1 \in \mathcal{A}_{i_1}, \dots, a_N \in \mathcal{A}_{i_N}$ and $i_k \neq i_{k+1}$ for $k = 1, \dots, N-1$.

We call collections $(\mathbf{a}_j)_{j\in\mathbb{N}}$ *free* if the unital $*$-subalgebras that they generate are free.

2.4. **Asymptotic freeness.** Since we are interested in the behavior of random matrices, one might wonder if the $*$-algebra given in Theorem 2.6 has free variables. Unfortunately that is not the case. However, many random matrix ensembles are asymptotically free.

Definition 2.11 (Asymptotic freeness)**.** Let $(\mathcal{A}_N, \varphi_N)_{N\in\mathbb{N}}$ be $*$-probability spaces and let $(\mathbf{a}_{N,i})_{i\in I}$ be collections of random variables with $\mathbf{a}_{N,i} \in \mathcal{A}_N$ for all $N \in \mathbb{N}$ and $i \in I$. We call $\mathbf{a}_{N,i} \in \mathcal{A}_N$ *asymptotically free* if there is a $*$-probability space $(\mathcal{A}, \varphi)$ which contains free collections $(\mathbf{a}_i)_{i\in I}$ such that $\mathbf{a}_{N,i} \rightharpoonup \mathbf{a}_i$ in $*$-distribution as $N \to \infty$.

2.5. **Free Brownian motion.** The limiting object that we are interested in is called *free Brownian motion*. Let $(\mathcal{A}, \varphi)$ be a W^*-probability space that contains many free semicircular random variables. Then $\mathcal{A}$ also contains *free Brownian motion*, namely, a process $((x(t))_{t\geq 0}$ with the properties

(1) $t \mapsto x(t)$ is norm-continuous
(2) For $t > s \geq 0$ the increments $x(t)-x(s)$ are such that their spectrum is distributed according to the semicircle law with variance $t - s$.
(3) If $\mathcal{A}_t$ is the W^*-algebra generated by $\{x(s) \mid s \in [0, t]\}$, then $x(t)-x(s)$ is freely independent from $\mathcal{A}_s$ for $t > s \geq 0$.

This is the free analogue of Brownian motion on $\mathbb{R}$. The corresponding “complexified” object is called *circular Brownian motion*, defined by

$$w(t) = \frac{1}{\sqrt{2}}(x(t) + iy(t))$$

where $x(t)$ and $y(t)$ are freely independent free Brownian motions.

Definition 2.12. The solution to the free stochastic differential equation

$$\mathrm{d}b(t) = b(t)\,\mathrm{d}w(t), \quad b(0) = 1$$

is called *free multiplicative Brownian motion*.

2.6. **The Brown measure.** The Brown measure is a certain generalization to non-normal operators of the spectral measure that one can define for normal operators. It is defined as follows.

Definition 2.13 (Brown measure)**.** Let $(\mathcal{A}, \varphi)$ be a W^*-probability space and let $a \in L^2(\mathcal{A}, \varphi)$. Then the element $|a| = \sqrt{a^*a}$ has a spectral measure, which we shall denote by ν_a. We then define the function $L_a \colon \mathbb{C} \to [-\infty, \infty)$ by

$$L_a(z) := \frac{1}{2\pi}\int_0^\infty \log(x)\,\mathrm{d}\nu_{a-z}(x).$$

The *Brown measure* μ_a *of* a is then defined as

$$\mu_a := 4\partial_z\partial_{\bar{z}} L_a(z),$$

where if $z = x + iy$ we set

$$\partial_z = \frac{1}{2}\left(\partial_x - i\partial_y\right) \quad \text{and} \quad \partial_{\bar{z}} = \frac{1}{2}\left(\partial_x + i\partial_y\right).$$

Note that $4\partial_z\partial_{\bar{z}}$ is just the Laplace operator with respect to the real and imaginary part.

Let us give some heuristics for this definition. We should interpret the function $L_a(z)$ informally as the trace of the logarithm of the absolute value of $a - z$, that is, formally

$$L_a(z) = \frac{1}{2\pi}\operatorname{tr}\left(\log(|a - z|)\right).$$

Note that the taking the logarithm is possible because $|a - z|$ is a Hermitian operator. It is well-known that in the sense of distributions, one has

$$4\partial_z\partial_{\bar{z}} \log(|z - z_0|) = 2\pi\delta_{z_0},$$

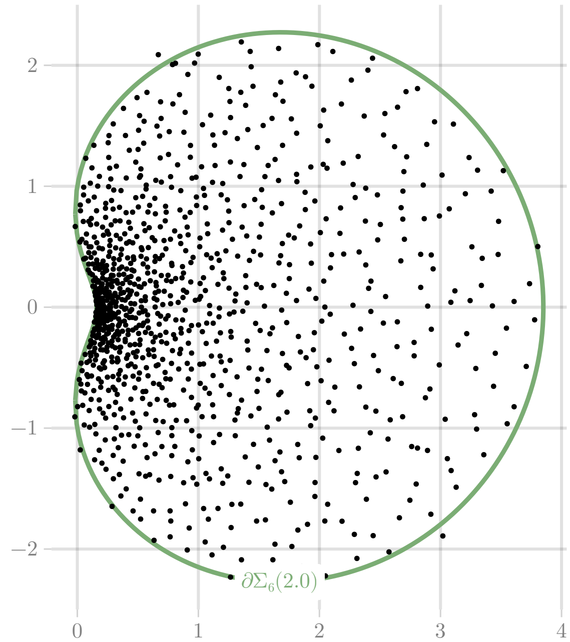


FIGURE 2.1. The support of the Lima bean law for $k = 6$ and $t = 2$ together with the eigenvalues of a single realization of the random walk $B^{\mathfrak{gl}}_{6,1000}(2)$, illustrating Theorem 2.14.

so we expect the measure μ_a to have support on the spectrum of a.

2.7. **The results.** For $n, k \in \mathbb{N}$, $1 \le j \le k$ and $t > 0$ let us write

$$B^{\bullet,(j)}_{k,n}(t) = \left(I + \sqrt{\frac{t}{k}}X_1\right)\cdots\left(I + \sqrt{\frac{t}{k}}X_j\right) \quad \text{and} \quad \mathcal{B}^{\bullet,(j)}_{n,k}(t) \coloneqq \exp\left(\sqrt{\frac{t}{k}}X_1\right)\cdots\exp\left(\sqrt{\frac{t}{k}}X_j\right)$$

where $X_l \sim \mathrm{RM}_\bullet(n, \mathbb{C})$ are independent random matrices and $\bullet \in \{\mathfrak{gl}, \mathfrak{sl}, \mathfrak{sp}\}$.

For ease of notation we shall moreover set

$$B^\bullet_{k,n}(t) = B^{\bullet,(j)}_{k,n}(t) \quad \text{and} \quad \mathcal{B}^\bullet_{k,n}(t) = \mathcal{B}^{\bullet,(j)}_{k,n}(t)$$

The main result is that the spectral distribution of the matrix random walk converges to the Brown measure of the free random walk.

Theorem 2.14 (Convergence of spectrum in distribution)**.** *Under certain assumptions, which include* $\mathrm{RM}_{\mathfrak{gl}}(n, \mathbb{C})$ *and* $\mathrm{RM}_{\mathfrak{sl}}(n, \mathbb{C})$ *step distributions, the empirical spectral distribution of* $B^\bullet_{k,n}(t)$ *converges weakly in probability to the Brown measure of* $b_k(t)$*.*

Remarkably, the support of the Brown measure (and in fact even the density) of $b_k(t)$ have quite explicit expressions.

Theorem 2.15 (Geometric characterization of the Brown measure)**.** *For $k \in \mathbb{N}$, $r > 0$ and $\theta \in (-\pi, \pi]$, define*

$$T_k(r,\theta) \coloneqq \begin{cases} 2(1-\cos\theta) & \text{if } r = 1 \\ \frac{k(r^{2/k}-1)}{r^2-1}(r^2 - 2r\cos\theta + 1) & \text{else.} \end{cases}$$

Then for $\theta \in (-\pi, \pi]$ and $t > 0$

1. *there is a unique $r_k^{\min}(\theta)$ which minimizes the function $r \mapsto T_k(r,\theta)$;*
2. *there is a unique $r_k^+(t,\theta) > r_k^{\min}(\theta)$ so that $T_k(r_k^+(t,\theta),\theta) = t$;*
3. *there is at most one $r_k^-(t,\theta) < r_k^{\min}(\theta)$ so that $T_k(r_k^-(t,\theta),\theta) = t$; if it does not exist define $r^-(t,\theta) = 0$.*

Define furthermore

$$t_k^*(\theta) \coloneqq T_k(r_k^{\min}(\theta),\theta),$$

and the domain

$$\Sigma_k(t) = \begin{cases} \{re^{i\theta} \mid \theta \in (t_k^*)^{-1}([0,t)), r_k^-(t,\theta) < r < r_k^+(t,\theta)\} & \text{if } t \le k \\ \{re^{i\theta} \mid \theta \in (-\pi,\pi], r < r_k^+(t,\theta)\} & \text{if } t > k. \end{cases}$$

Then the closure $\overline{\Sigma_k(t)}$ is the support of the Brown measure of $b_k(t)$. Moreover, there is a limit shape $\Sigma_\infty(t)$, which is the support of the Brown measure of the free Brownian motion $b(t)$.

As shown on Figure 2.2, the $\Sigma_k(t)$ have a shape that resembles a bean for $t \in (0, 4)$. This is intuitively explainable. Indeed, our matrix random walk happens on a Lie group. In particular, this means that at every step the matrix that we obtain is invertible and hence has no eigenvalue at 0. Therefore, the distribution of the eigenvalues is in a way repelled by 0. However, after time $t = 4$ the eigenvalue distribution has spread enough so that the eigenvalues can cross the negative part of the real axis. At this point a hole around 0 forms in the limiting shape.

2.8. **Randomly generating Lie algebra elements.** Before we move to the numerical experiments, we should think about a proper way to randomly generate elements of the Lie algebra. To this end, let us define for $k, l = 1, \dots, n$ the following elementary matrices

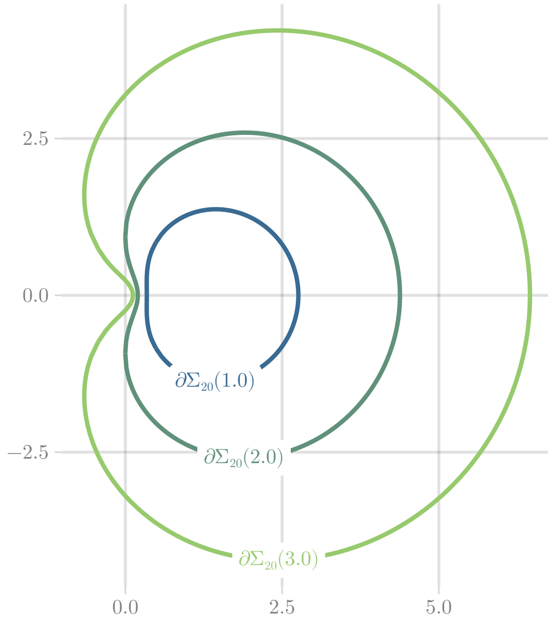
$$E_{kl} = \begin{cases} 1 & \text{at the } (k,l)\text{-entry} \\ 0 & \text{else.} \end{cases}$$

2.8.1. *Random elements of $\mathfrak{gl}(n,\mathbb{C})$.* A well-known method to picking random elements of $\mathrm{Mat}(n,\mathbb{C})$ is by choosing matrices from the *Ginibre ensemble* which consists of random $n \times n$-matrices where the entries are complex Gaussian random variables with mean 0 and variance $1/n$. This means that for each entry, the real and imaginary part are independent real Gaussian random variables with mean 0 and variance $1/(2n)$. We will denote the space of these random matrices by $\mathrm{RM}_{\mathfrak{gl}}(n,\mathbb{C})$. We may write any $A \sim \mathrm{RM}_{\mathfrak{gl}}(n,\mathbb{C})$ as

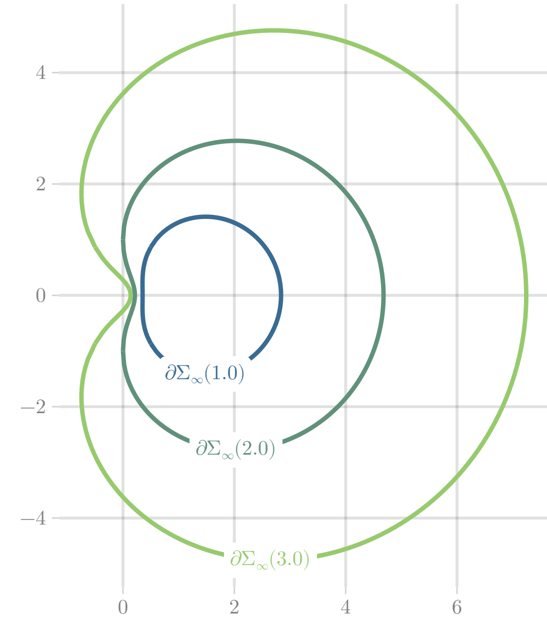
$$A = \sum_{k,l=1}^{n} X_{kl} E_{kl}$$

where the $X_{kl} \sim \mathcal{N}(0, 1/n)$ are independent. Observe that the E_{kl} form an orthonormal basis with respect to the Frobenius inner product given by

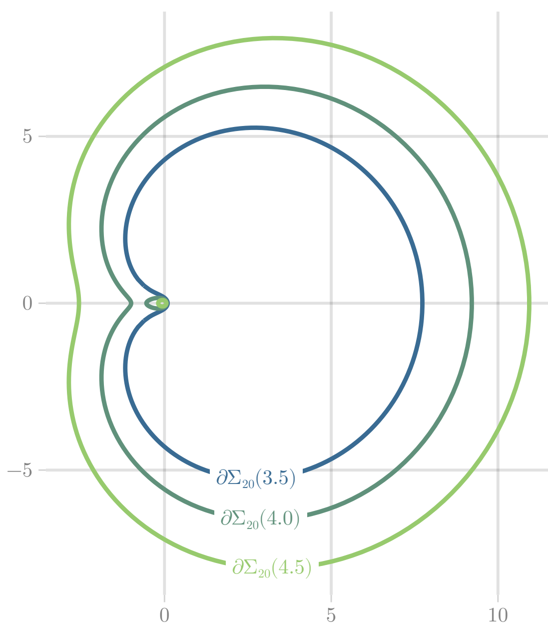
$$\langle A, B \rangle_F = \mathrm{tr}(A^* B) \quad \forall A, B \in \mathrm{Mat}(n,\mathbb{C}).$$



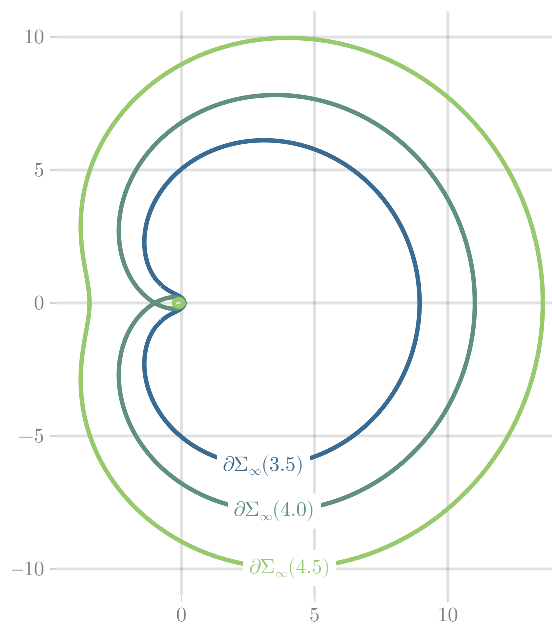
(A) The boundary of the limit shape $\Sigma_{20}(t)$ for $t = 1, 2, 3$.



(B) The boundary of the limit shape $\Sigma_\infty(t)$ for $t = 1, 2, 3$.



(C) The boundary of the limit shape $\Sigma_{20}(t)$ for $t = 3.5, 4, 4.5$.



(D) The boundary of the limit shape $\Sigma_\infty(t)$ for $t = 3.5, 4, 4.5$.

FIGURE 2.2. Boundaries of the support of the Brown measure of the limiting operators for different values of k. Note the topological phase transition that appears between $t = 3.5$ and $t = 4$. For $t < 3.5$ the shape $\Sigma_\infty(t)$ is simply connected but for $t > 4$ it is has a hole around 1.

2.8.2. *Random elements of* $\mathfrak{sl}(n,\mathbb{C})$. Here we have to be a bit more creative. The space $\mathfrak{sl}(n,\mathbb{C})$ consists of all elements of $\mathfrak{gl}(n,\mathbb{C})$, which are traceless. A basis for this space is given by the union

$$\{E_{kl} \mid 1 \le k,l \le n, k \ne l\} \cup \left\{ \frac{1}{\sqrt{2}} \left(E_{kk} - E_{k+1,k+1}\right) \mid k = 1,\dots,n-1 \right\}.$$

Again note that this basis is orthonormal with respect to the Frobenius inner product. A random variable with values in $\mathfrak{sl}(n,\mathbb{C})$ is thus given by

$$\sum_{\substack{k,l=1 \\ k\ne l}}^{n} X_{kl}E_{kl} + \frac{1}{\sqrt{2}} \sum_{k=1}^{n-1} X_{kk} \left(E_{kk} - E_{(k+1)(k+1)}\right)$$

where $(X_{kl})_{k,l=1,\dots,n} \sim \mathrm{RM}_{\mathfrak{gl}}(n,\mathbb{C})$. We denote the space of all random matrices on $\mathfrak{sl}(n,\mathbb{C})$ as constructed above by $\mathrm{RM}_{\mathfrak{sl}}(n,\mathbb{C})$.

2.8.3. *Random elements of* $\mathfrak{sp}(2n,\mathbb{C})$. One could again find an explicit basis which is orthonormal in the Frobenius norm, however, this would be a bit of a mess notationally. Thus, we construct elements of $\mathfrak{sp}(2n,\mathbb{C})$ as follows. Let $X \sim \mathrm{Gin}(2n,\mathbb{C})$ and partition it into $n\times n$-blocks $A_{11}, A_{12}, A_{21}, A_{22}$ as follows

$$X = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Then any realization of the random matrix given by

$$Y = \begin{pmatrix} A_{11} & \frac{1}{\sqrt{2}}(A_{12} + A_{12}^T) \\ \frac{1}{\sqrt{2}}(A_{21} + A_{21}^T) & -A_{11}^T \end{pmatrix}$$

is an element of $\mathfrak{sp}(2n,\mathbb{C})$. One can check that there exists an orthonormal basis with respect to the Frobenius inner product such that Y can be constructed in a similar way as in Section 2.8.2 from this basis. We denote the space of all random matrices in $\mathfrak{sp}(n,\mathbb{C})$ as constructed above by $\mathrm{RM}_{\mathfrak{sp}}(n,\mathbb{C})$.

3. Experiments: Distribution of extremal eigenvalues

Inspired by the Tracy-Widom law, we now want to investigate the behavior of the the extremal eigenvalue for a matrix random walk. Concretely, in this section we will perform a numerical study of the distribution of the eigenvalue that is farthest away from 1 for a random walk on a Lie group. Since our matrix random walks start at the identity, the path that the eigenvalue that is farthest away from 1 could be interpreted as a process similarly to taking the maximum distance point of of a random walk in $\mathbb{R}^n$ at each point in time.

Recall that we defined for $n,k \in \mathbb{N}$, $1 \le j \le k$ and $t > 0$

$$B_{k,n}^{\bullet,(j)}(t) = \left(I + \sqrt{\frac{t}{k}} X_1\right) \cdots \left(I + \sqrt{\frac{t}{k}} X_j\right) \quad \text{and} \quad \mathcal{B}_{n,k}^{\bullet,(j)}(t) := \exp\left(\sqrt{\frac{t}{k}} X_1\right) \cdots \exp\left(\sqrt{\frac{t}{k}} X_j\right),$$

where $X_l \sim \mathrm{RM}_\bullet(n,\mathbb{C})$ are independent random matrices and $\bullet \in \{\mathfrak{gl}, \mathfrak{sl}, \mathfrak{sp}\}$. This is the j-th step in the matrix random walk. Then we define the *extremal eigenvalue processes* follows

$$\sigma_{n,k}^{\bullet,(j)}(t) := \arg\max\Big\{ |\lambda - 1| \mid \lambda \in \mathrm{spec}(B_{n,k}^{\bullet,(j)}(t)) \Big\}$$

and

$$\varsigma_{n,k}^{\bullet,(j)}(t) := \arg\max\Big\{ |\lambda - 1| \mid \lambda \in \mathrm{spec}(\mathcal{B}_{n,k}^{\bullet,(j)}(t)) \Big\}.$$

Our goal here is to understand the distribution of $\varsigma_{n,k}^{\bullet,(j)}(t)$ and $\sigma_{n,k}^{\bullet,(j)}(t)$ as $n \to \infty$ and as $k \to \infty$.

3.1. **Numerical results.** The results of some Monte-Carlo simulations for $\varsigma_{n,k}^{\bullet,(j)}(t)$ for different values of n, k, t and $\bullet \in \{\mathfrak{gl}, \mathfrak{sp}\}$ are depicted on the Figures 3.1 to 3.4. The corresponding results for $\sigma_{n,k}^{\bullet,(j)}(t)$ are shown in Figures 3.5 to 3.8 For reference we also plot the boundaries $\partial\Sigma_k(t)$ for the corresponding times. The Figures 3.9 and 3.10 show the distributions for $\varsigma_{700,k}^{\mathfrak{gl},(j)}(t)$ for different values of k, t and for comparison we also plot the intermediate boundaries $\partial\Sigma_k(t/j)$.

3.2. **Observations and Conjectures.** Figures 3.1 to 3.8 all show very similar features. First, as the n increases, we see that the eigenvalues concentrate around certain regions. In particular, this effect seems to be independent of our choice of Lie algebra. Hence, we would expect some kind of universal behavior. A natural candidate would again be the corresponding spectral quantaties of the free Brownian motion.

Conjecture 3.1 (Universality)**.** *For sufficiently nice step distributions, the distributions of* $\varsigma_{n,k}^{\bullet,(j)}(t)$ *and* $\sigma_{n,k}^{\bullet,(j)}(t)$ *converge to some universal limit as* $n \to \infty$*.*

Next, we see that as k increases, the distances between the steps decrease and that for large k and n, the eigenvalues $\varsigma_{n,k}^{\bullet,(j)}(t)$ and $\sigma_{n,k}^{\bullet,(j)}(t)$ are approximately localized in a cone with cone point at 1.

Conjecture 3.2 (Cone shape)**.** *For sufficiently nice step distributions, the distributions of* $\varsigma_{n,k}^{\infty,(j)}(t)$ *and* $\sigma_{\infty,k}^{\bullet,(j)}(t)$ *converge to distribution that is supported on a cone with cone point at* 1*.*

Now let us move on to Figures 3.9 and 3.10. We note the very good agreement of the eigenvalues $\sigma_{700,3}^{\mathfrak{gl},(3)}(t)$ for $t = 1, 2$ with the outermost boundary $\partial\Sigma_3(t)$. However, the intermediate steps do not agree that well, suggesting that $\partial\Sigma_3(t/j)$ might not be the correct limiting shapes. However, we formulate the following conjecture.

Conjecture 3.3 (Limit of last step for $\sigma_{n,k}^{\bullet,(j)}(t)$)**.** *For sufficiently nice step distributions, the distribution of* $\sigma_{n,k}^{\bullet,(k)}(t)$ *converges to a distribution with support in* $\partial\Sigma_k(t)$*.*

Interestingly, for the $\varsigma_{n,k}^{\mathfrak{gl},(j)}(t)$, we see that the first step has good agreement with the boundary $\partial\Sigma_\infty(t/k)$. However, I suspect that this is becausee for lower t, the boundary $\partial\Sigma_k(t)$ will be very similar to the image of a scaled unit disk in $\mathbb{C}$ under the exponential map, which the asymptotic distribution of the first step $\varsigma_{n,k}^{\mathfrak{gl},(1)}(t)$. Again we see that there is not very good agreement of the distributions of the remaining steps and the boundaries $\partial\Sigma_k(t/j)$, indicating again that these might not be directly related to the actual limiting objects.

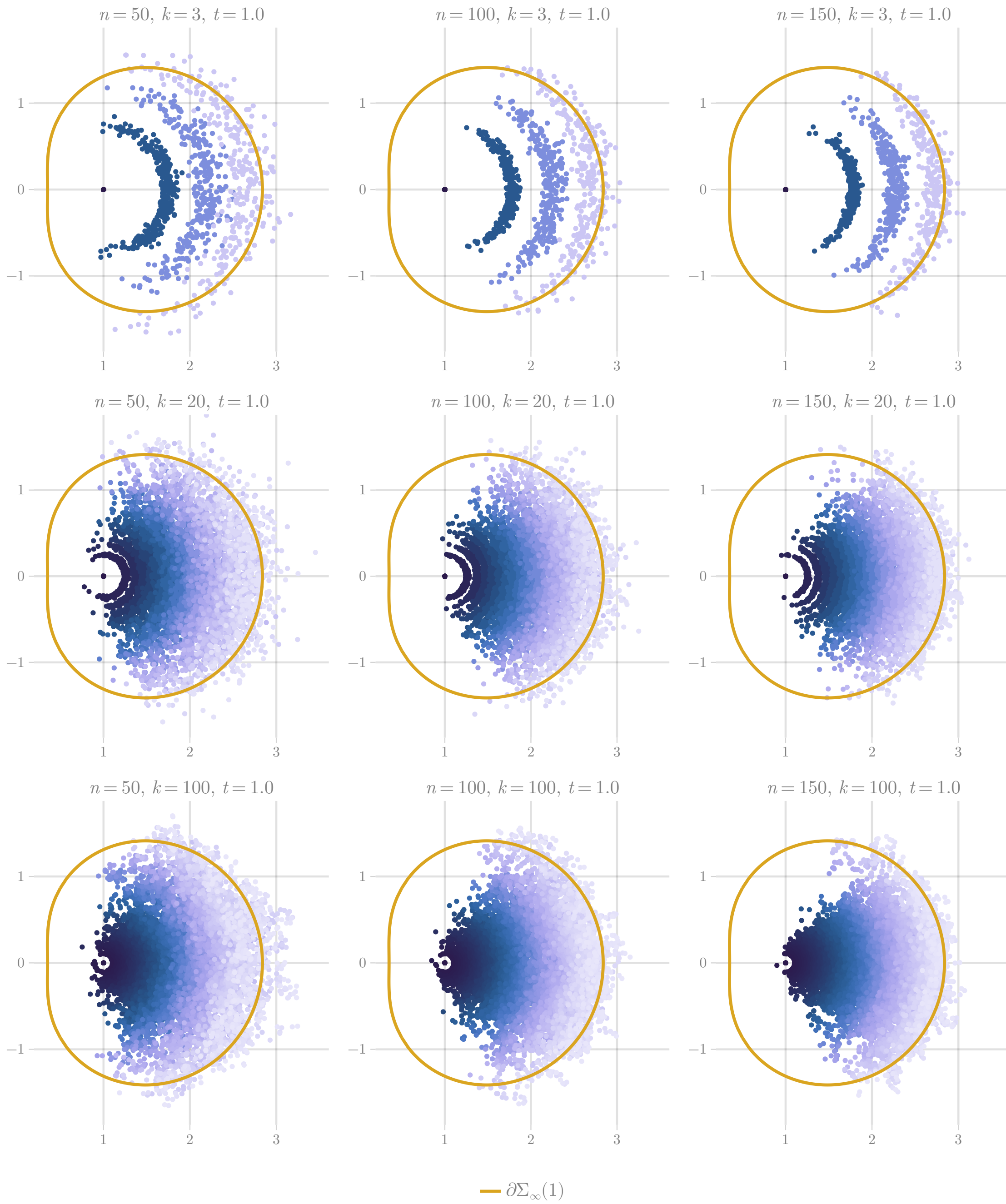


FIGURE 3.1. Distribution of Monte-Carlo samples of the process $\varsigma_{n,k}^{\mathfrak{gl},(j)}(t)$ for different values of n, k and $t = 1$. The color of the markers indicates the value of j, darker color corresponds to lower j. For reference we added the outline of the support of the Brown measure $\partial\Sigma_\infty(1)$

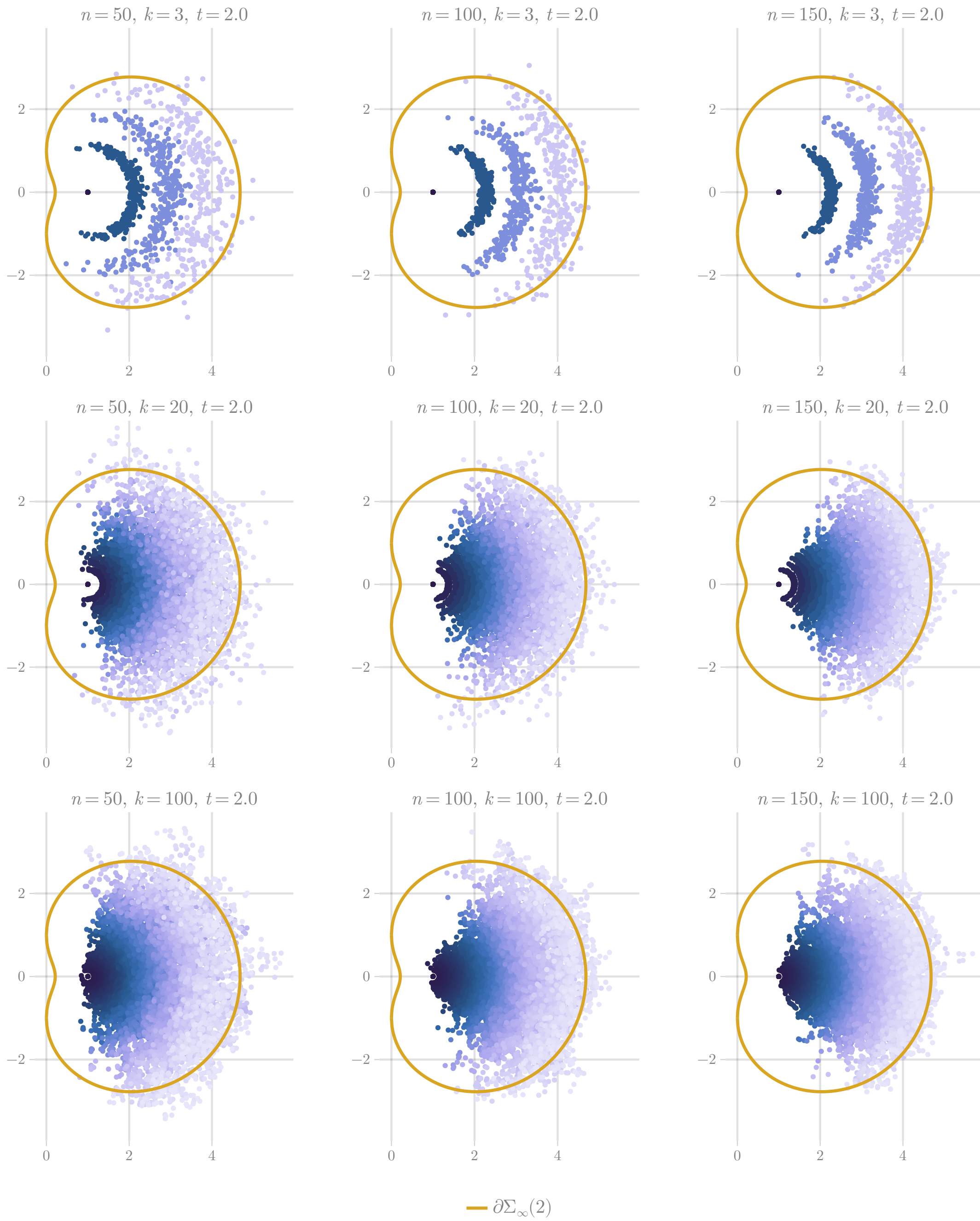


FIGURE 3.2. Distribution of Monte-Carlo samples of the process $\varsigma_{n,k}^{\mathfrak{gl},(j)}(t)$ for different values of n, k and $t = 2$. The color of the markers indicates the value of j, darkest color corresponds to $j = 0$ and the brightest color to $j = k$. For reference we added the outline of the support of the Brown measure $\partial\Sigma_\infty(2)$

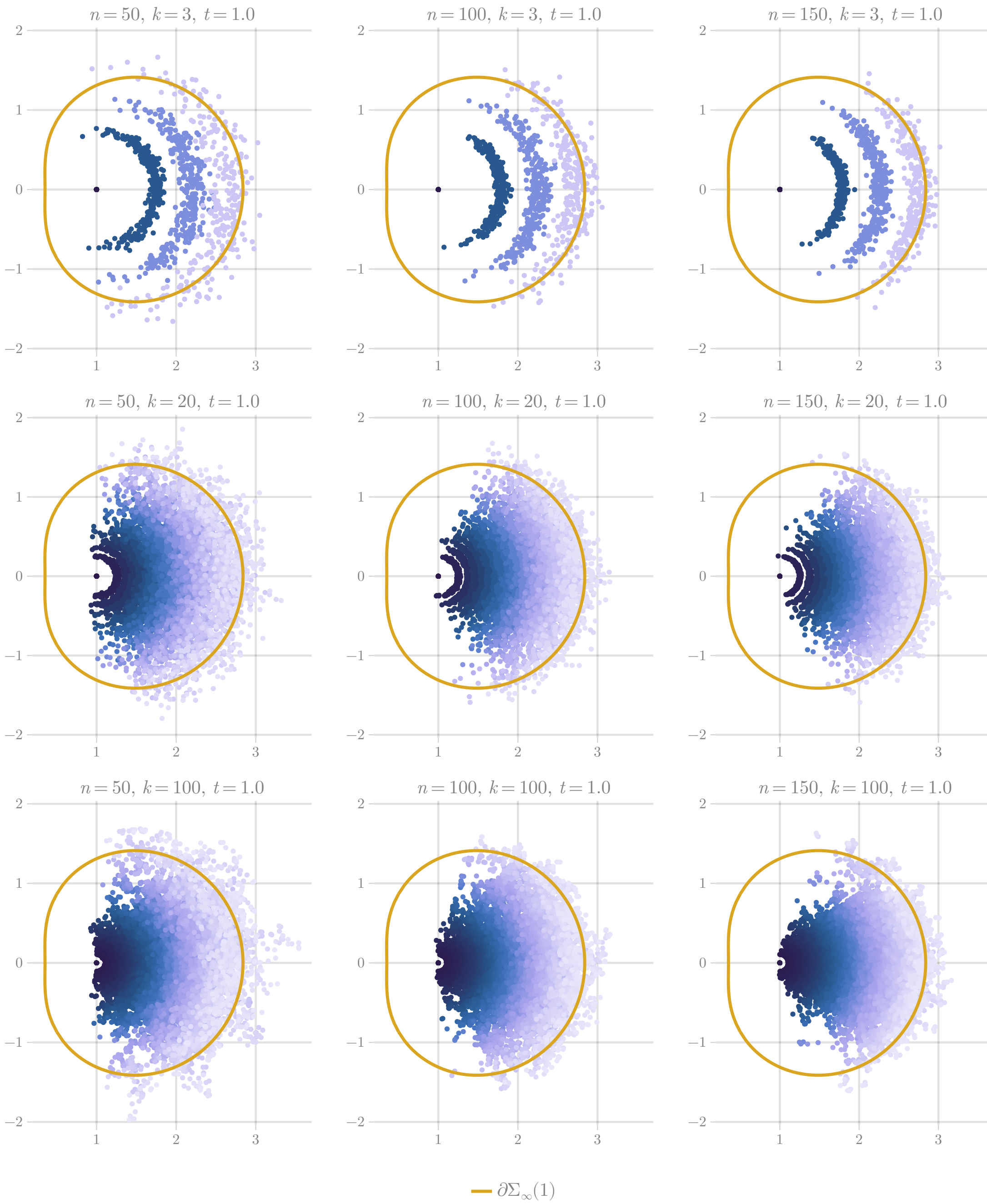


FIGURE 3.3. Distribution of Monte-Carlo samples of the process $\varsigma_{n,k}^{\mathfrak{sp},(j)}(t)$ for different values of n, k and $t = 1$. The color of the markers indicates the value of j, darkest color corresponds to $j = 0$ and the brightest color to $j = k$. For reference we added the outline of the support of the Brown measure $\partial\Sigma_\infty(1)$

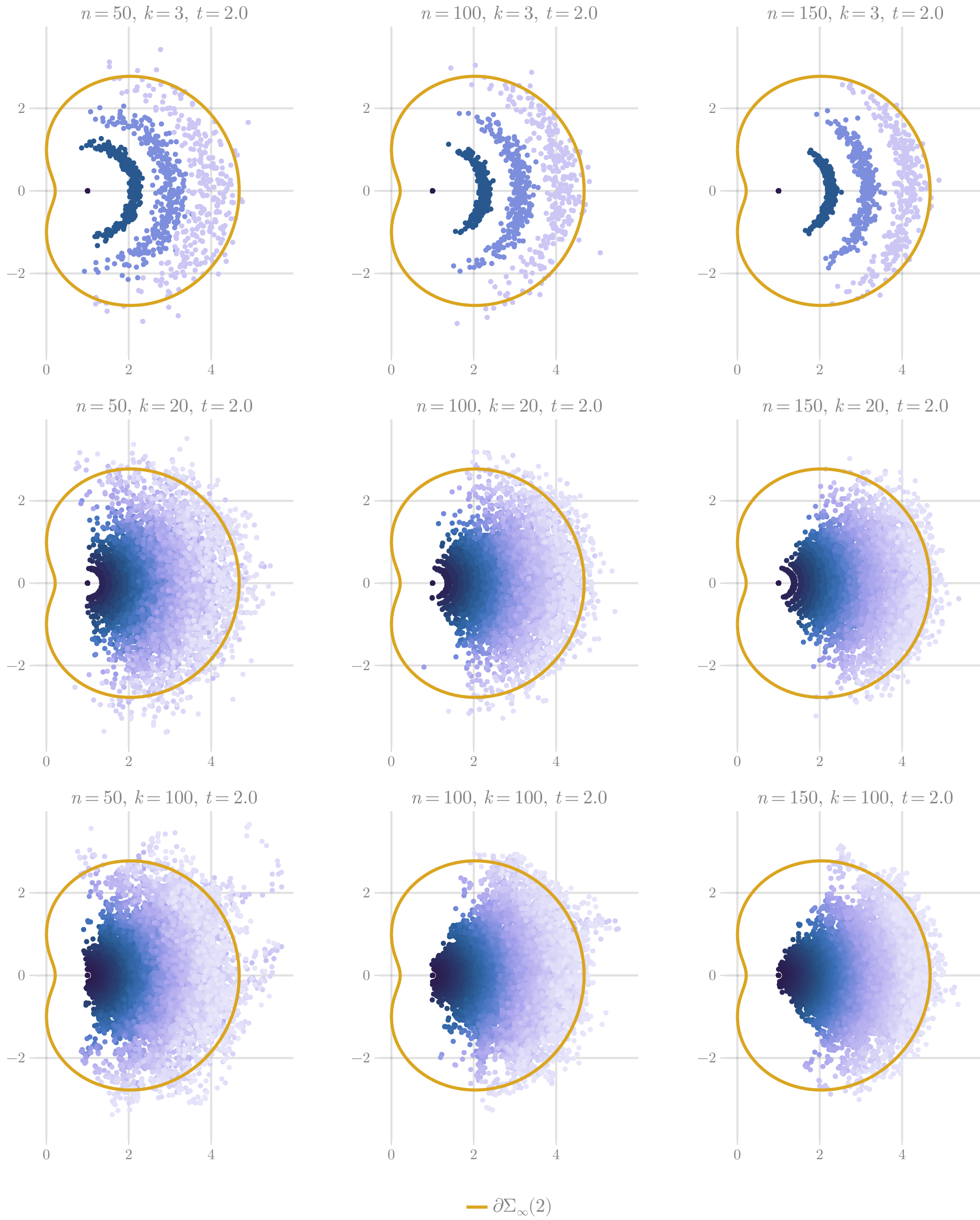


FIGURE 3.4. Distribution of Monte-Carlo samples of the process $\varsigma_{n,k}^{\mathfrak{sp},(j)}(t)$ for different values of n, k and $t = 2$. The color of the markers indicates the value of j, darker color corresponds to lower j. For reference we added the outline of the support of the Brown measure $\partial\Sigma_\infty(2)$

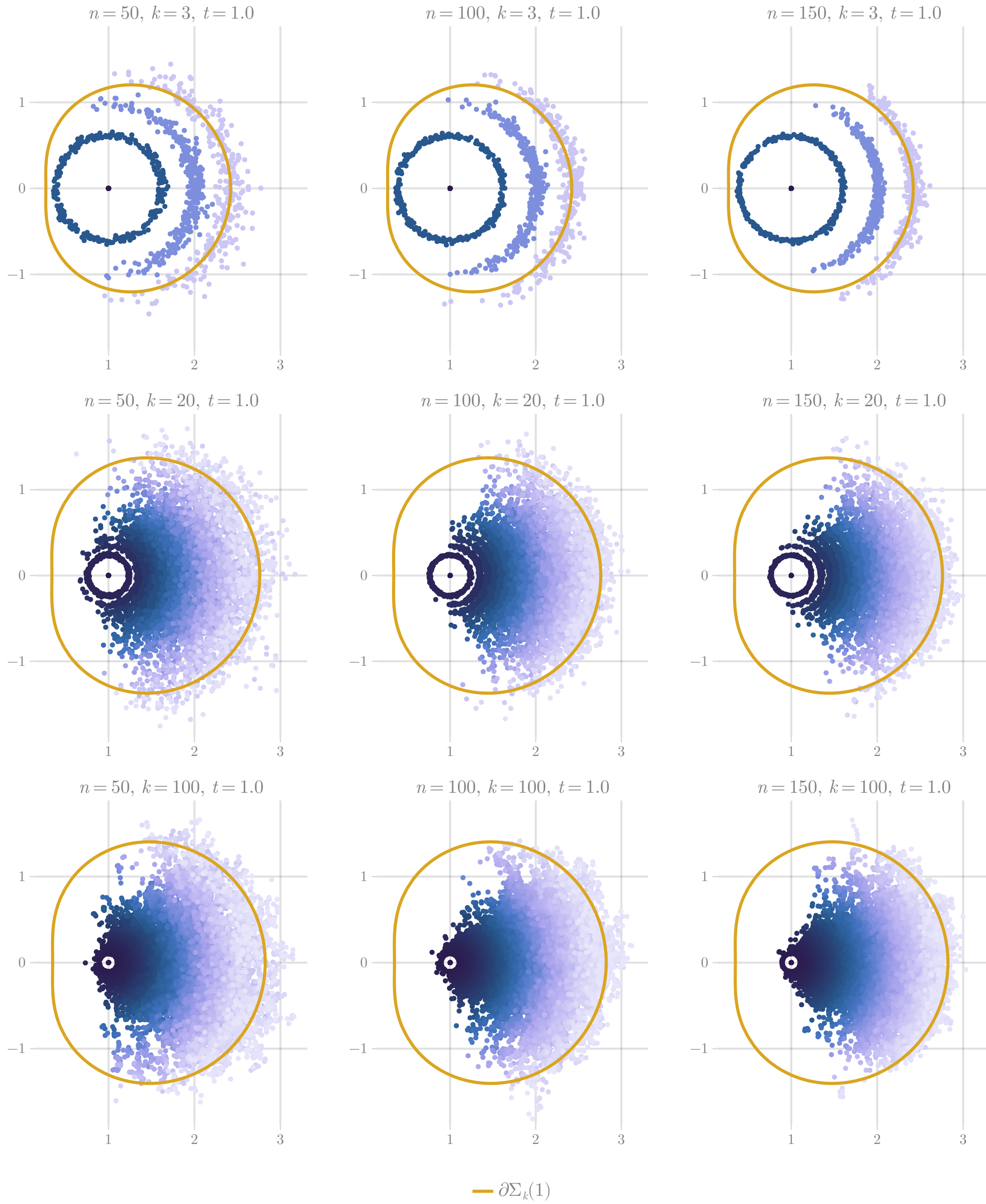


FIGURE 3.5. Distribution of Monte-Carlo samples of the process $\sigma_{n,k}^{\mathfrak{gl},(j)}(t)$ for different values of n, k and $t = 1$. The color of the markers indicates the value of j, darker color corresponds to lower j. For reference we added the outline of the support of the Brown measure $\partial\Sigma_\infty(1)$

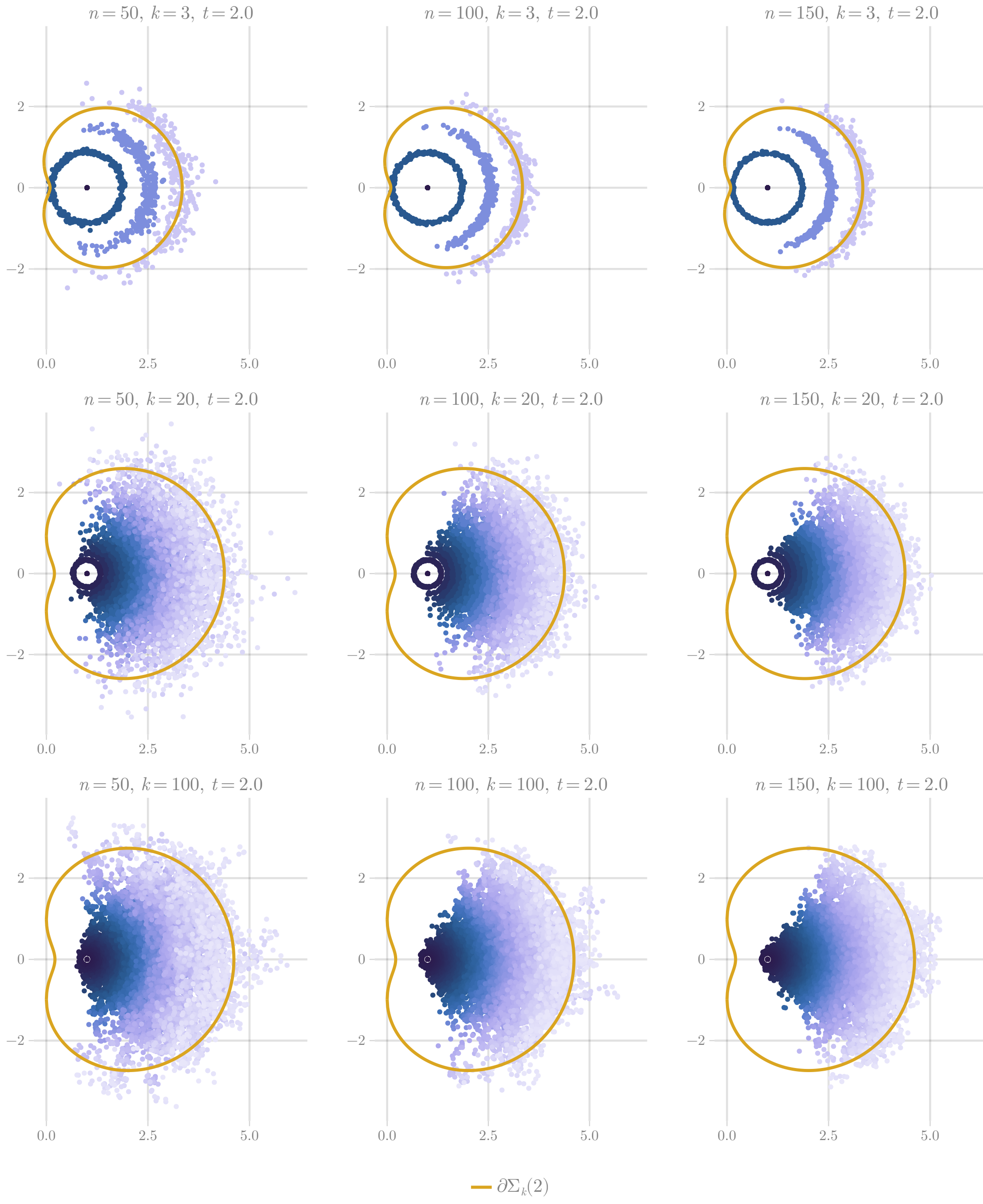


FIGURE 3.6. Distribution of Monte-Carlo samples of the process $\sigma_{n,k}^{\mathfrak{gl},(j)}(t)$ for different values of n, k and $t = 2$. The color of the markers indicates the value of j, darker color corresponds to lower j. For reference we added the outline of the support of the Brown measure $\partial\Sigma_\infty(2)$

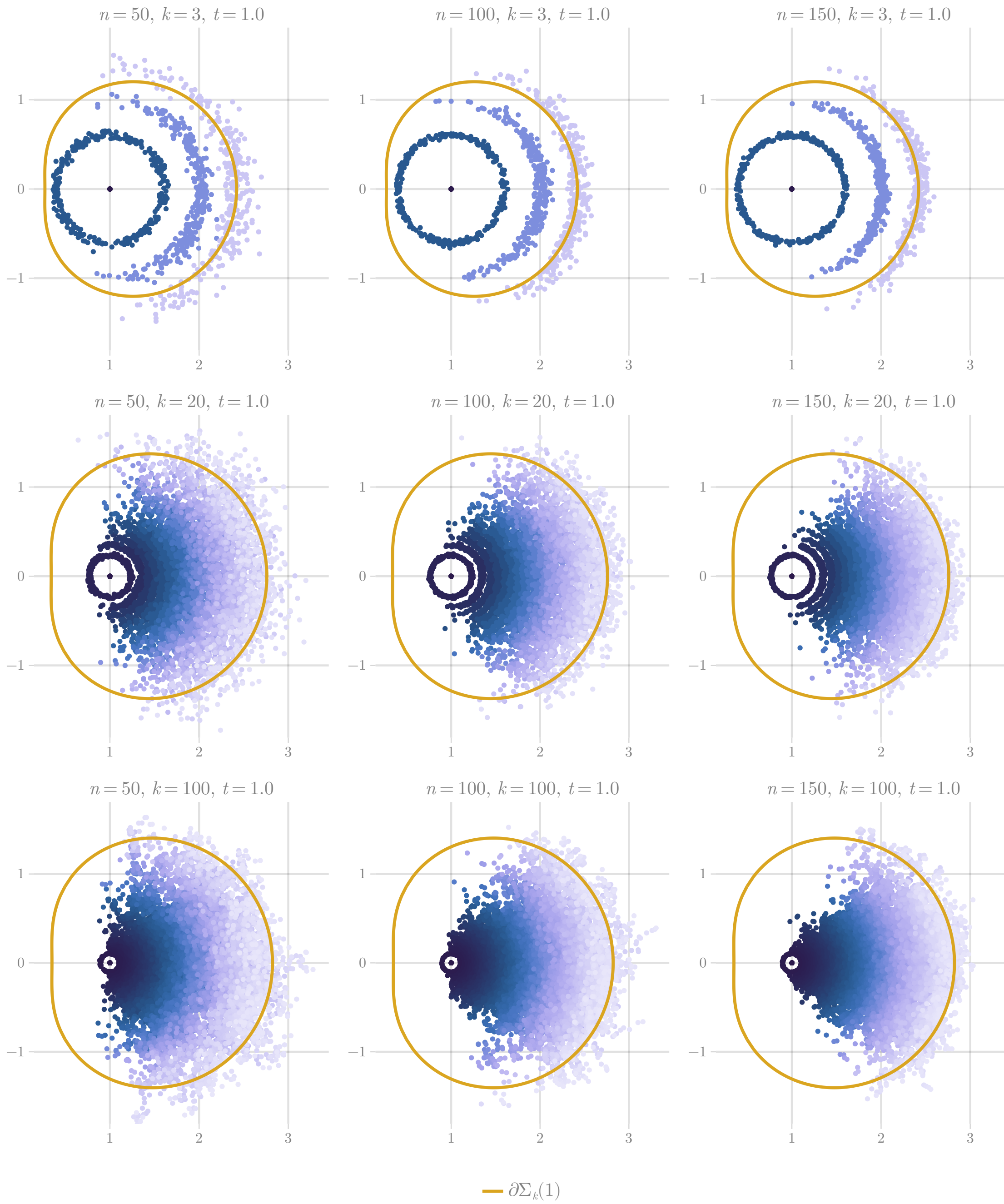


FIGURE 3.7. Distribution of Monte-Carlo samples of the process $\sigma_{n,k}^{\mathfrak{sp},(j)}(t)$ for different values of n, k and $t = 1$. The color of the markers indicates the value of j, darker color corresponds to lower j. For reference we added the outline of the support of the Brown measure $\partial\Sigma_\infty(1)$

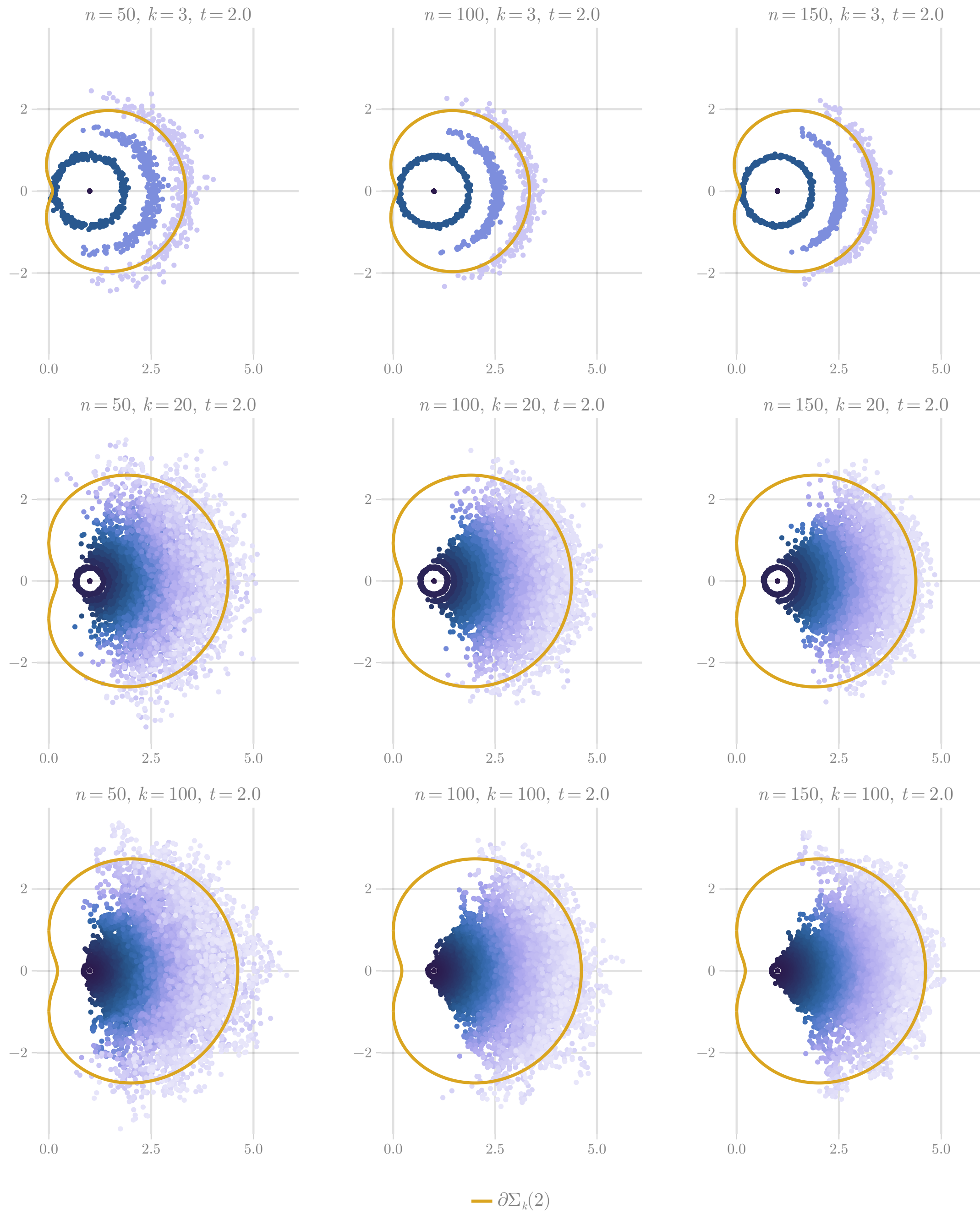
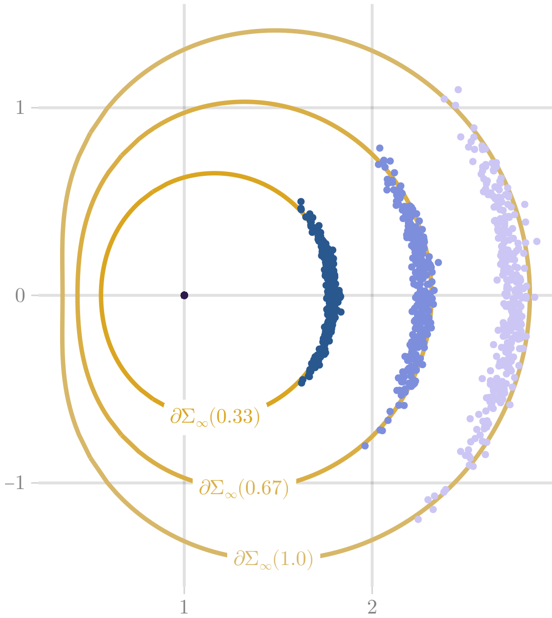
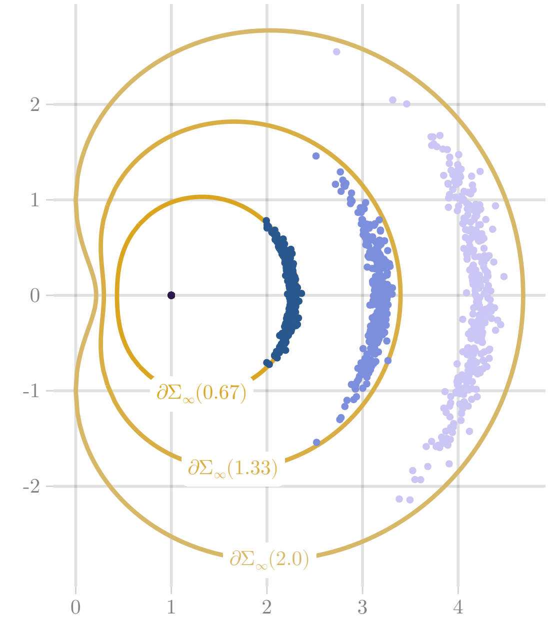


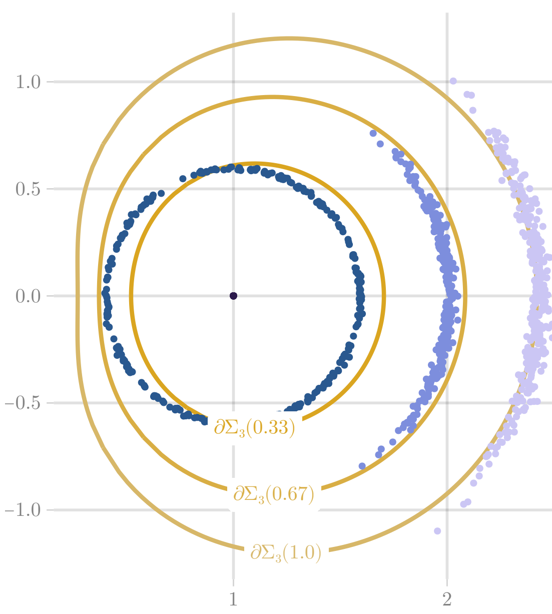
FIGURE 3.8. Distribution of Monte-Carlo samples of the process $\sigma_{n,k}^{\mathfrak{sp},(j)}(t)$ for different values of n, k and $t = 2$. The color of the markers indicates the value of j, darker color corresponds to lower j. For reference we added the outline of the support of the Brown measure $\partial\Sigma_\infty(2)$



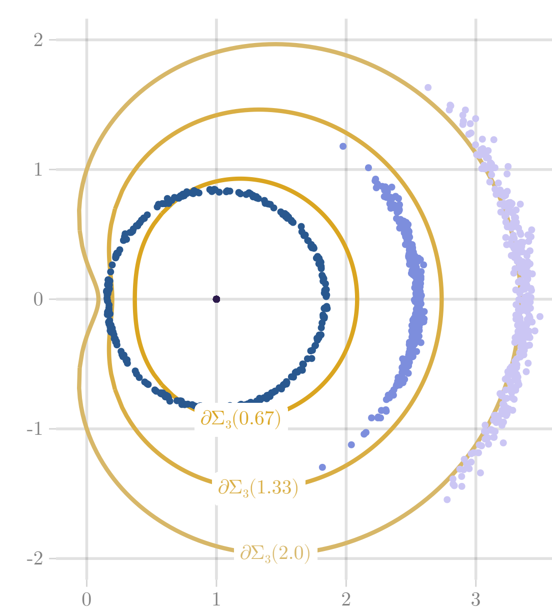
(A) Samples of the distribution of $\varsigma^{\mathfrak{gl},(j)}_{700,3}(1)$, where $j = 0, 1, 2, 3$.



(B) Samples of the distribution of $\varsigma^{\mathfrak{gl},(j)}_{700,3}(2)$, where $j = 0, 1, 2, 3$.

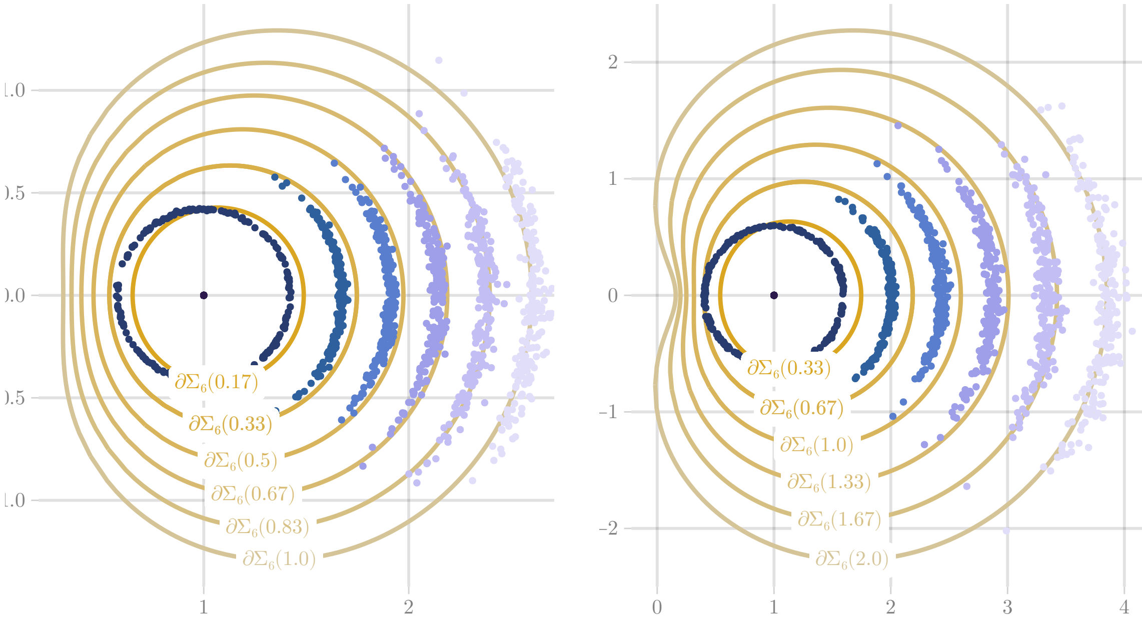


(C) Samples of the distribution of $\sigma^{\mathfrak{gl},(j)}_{700,3}(1)$, where $j = 0, 1, 2, 3$.



(D) Samples of the distribution of $\sigma^{\mathfrak{gl},(j)}_{700,3}(2)$, where $j = 0, 1, 2, 3$.

FIGURE 3.9. Samples of the distribution of $\varsigma^{\mathfrak{gl},(j)}_{700,3}(t)$ and $\sigma^{\mathfrak{gl},(j)}_{700,3}(t)$ for $t = 1, 2$ and $j = 0, 1, 2, 3$, darker shaded points mean lower j. For comparison we also show the boundaries of the limit shape of the support of the limiting Brown measures.



(A) Samples of the distribution of $\sigma^{\mathfrak{gl},(j)}_{700,6}(1)$, where $j = 0, 1, 2, 3$.

(B) Samples of the distribution of $\sigma^{\mathfrak{gl},(j)}_{700,6}(2)$, where $j = 0, 1, 2, 3$.

FIGURE 3.10. Samples of the distribution of $\sigma^{\mathfrak{gl},(j)}_{700,6}(t)$ for $t = 1, 2$ and $j = 0, 1, 2, 3$, darker shaded points mean lower j. For comparison we also show the boundaries of the support of the limiting Brown measures.

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