Introduction to Financial Engineering

Week 40: Portfolio Choices

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Week 40

Making financial decisions

Some financial assets are **risky assets**, meaning that the future cash flows (and/or value) are not known. Only the distribution of returns is known or assumed.

Overall problem

- Which assets to invest in
- How much to invest in each asset

Starting point: Characterizing assets in terms of

- expected return (average)
- variance / standard deviation

Small exercise: Combining risky assets

Market Condition	Return[a] Asset 1	Asset 2	Asset 3	Asset 5	Rainfall	Return[a] Asset 4
Good	15	16	1	16	Plentiful	16
Average	9	10	10	10	Average	10
Poor	3	4	19	4	Poor	4
Mean return	[redacted]					[redacted]
Variance						
Standard deviation						

[a]The alternative returns on each asset are assumed equally likely and, thus, each has a probability of $\frac{1}{3}$.

- Compute expected returns, variance/standard deviation for each asset
- What happens if Asset 2 and Asset 3 are combined?
- What happens if Asset 2 and Asset 5 are combined?
- Is there enough information to say anything about the combination of Asset 2 and Asset 4?

Estimating expected returns and variance

- There is no way of knowing the true distribution of an asset's future returns
- This must be inferred from data (and potentially using sophisticated models)
- The expected return and variance / standard deviation of returns can be estimated from historical data
- Note: Remember the exercise on averaging from week 36 and 37

Other measures of a distribution

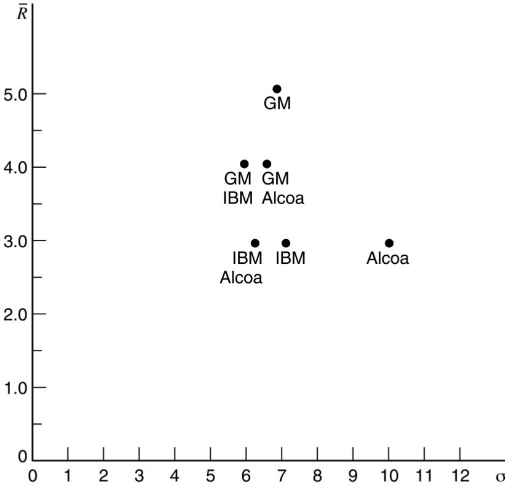
- The expected return and variance of returns is not a full description of a distribution
- Other measures to describe the dispersion (or more generally the distribution) are
  - deviations below the mean
  - the expectation of the squared deviation below the mean is called semivariance (this can be generalized to lower partial moments)
  - quantiles of the distribution (this is equivalent to the Value-at-Risk framework)
- If return distributions are symmetrical, then these other measures give the same ordering of portfolios as variance
- In portfolio literature, much theory is based on average and variance / standard deviation as adequate measures for choosing investments

Combining risky assets – stock market data

Month	IBM	Alcoa	GM	$\frac{1}{2}$ IBM + $\frac{1}{2}$ Alcoa	$\frac{1}{2}$ GM + $\frac{1}{2}$ Alcoa	$\frac{1}{2}$ GM + $\frac{1}{2}$ IBM
1	12.05	14.09	25.20	13.07	19.65	18.63
2	15.27	2.96	2.86	9.12	2.91	9.07
3	−4.12	7.19	5.45	1.54	6.32	0.67
4	1.57	24.39	4.56	12.98	14.48	3.07
5	3.16	0.06	3.72	1.61	1.89	3.44
6	−2.79	6.52	0.29	1.87	3.41	−1.25
7	−8.97	−8.75	5.38	−8.86	−1.69	−1.80
8	−1.18	2.82	−2.97	0.82	−0.08	−2.08
9	1.07	−13.97	1.52	−6.45	−6.23	1.30
10	12.75	−8.06	10.75	2.35	1.35	11.75
11	7.48	−0.70	3.79	3.39	1.55	5.64
12	−.94	8.80	1.32	3.93	5.06	0.19
$\bar{R}$	2.95	2.95	5.16	2.95	4.05	4.05
σ	7.15	10.06	6.83	6.32	6.69	6.02

Correlation Coefficient: IBM and Alcoa = 0.05;
GM and Alcoa = 0.22; IBM and GM = 0.48

Combining risky assets – illustration in (σ, μ) diagram



Investing in different stocks markets

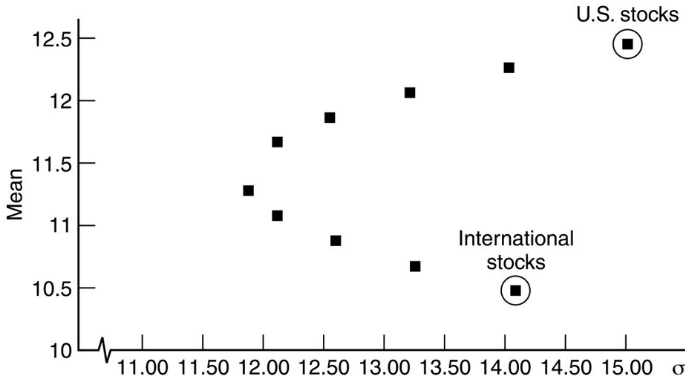
- Assume it is possible to invest in a domestic stock portfolio and in a foreign stock portfolio (each of these is one asset)
- Assume that the domestic stock portfolio has an expected return of 12.5% and the standard deviation is 14.9%
- Assume that the foreign stock portfolio has an expected return of 10.5% and the standard deviation is 14.0%
- The correlation between returns is 0.33
- What happens for different wealth allocations?

Combining stocks – portfolio data

Proportion S&P	Proportion International	Mean Return	Standard Deviation
1	0	12.5	14.90
0.9	0.1	12.3	13.93
0.8	0.2	12.1	13.11
0.7	0.3	11.9	12.46
0.6	0.4	11.7	12.01
0.5	0.5	11.5	11.79
0.45	0.55	11.4	11.76
0.4	0.6	11.3	11.80
0.3	0.7	11.1	12.04
0.2	0.8	10.9	12.50
0.1	0.9	10.7	13.17
0	1	10.5	14.00

Portfolios of US and international stocks

Example (Expected return vs. standard deviation)



Notation and assumptions

- Matrix multiplication makes it easy to formulate the expected return and variance of returns for a portfolio
- Assume that $\boldsymbol{\mu} = [\mu_1 \;\; \mu_2 \;\; \cdots \;\; \mu_n]'$ is a one-column matrix of expected returns on different assets
- Assume that $\boldsymbol{w} = [w_1 \;\; w_2 \;\; \cdots \;\; w_n]'$ is a one-column matrix representing the fractions of the investors wealth invested in each asset
- Denote the covariance matrix between returns by $\boldsymbol{\Sigma}$
- By definition, a covariance matrix is always positive semidefinite, but now it is assumed that it is **positive definite** and thus invertible
- Further, not all coordinates of $\boldsymbol{\mu}$ are equal

Matrix expressions for expected return and variance

- The expected return on such a portfolio is $\mu_P = \boldsymbol{w}^\top \boldsymbol{\mu}$
- The variance of returns for such a portfolio is $\sigma_P^2 = \boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}$, where $\boldsymbol{\Sigma}$ is the covariance matrix of the returns
- In the two-asset case, the variance is $\sigma_P^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 c_{12}$ and the expected return is $\mu_P = w_1 \mu_1 + w_2 \mu_2$.
- If you are struggling with matrix calculations, it might be useful for you to do the derivations for the two-asset case and convince yourself that the derived matrix expressions match.

Stating the optimization problem

- Consider the following problem

$$\min_{\boldsymbol{w}} \tfrac{1}{2}\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}$$

- Under the following constraints:

$$\boldsymbol{w}^\top \boldsymbol{\mu} = \mu_P$$
$$\boldsymbol{w}^\top \boldsymbol{1} = 1$$

- Or in words, given a specific expected rate of return μ_P, what is the allocation of wealth into assets $1, \ldots, n$ that gives this expected return with the smallest variance

Setting up the Lagrange

$$\mathcal{L}(\boldsymbol{w}, \lambda_1, \lambda_2) = \tfrac{1}{2}\boldsymbol{w}^\top \boldsymbol{\Sigma}\boldsymbol{w} - \lambda_1\left(\boldsymbol{w}^\top \boldsymbol{\mu} - \mu_P\right) - \lambda_2\left(\boldsymbol{w}^\top \mathbf{1} - 1\right)$$

FOCs for optimality are

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{w}} = \boldsymbol{\Sigma}\boldsymbol{w} - \lambda_1 \boldsymbol{\mu} - \lambda_2 \mathbf{1} = \mathbf{0} \qquad ([1])$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = \boldsymbol{w}^\top \boldsymbol{\mu} - \mu_P = 0 \qquad ([2])$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_2} = \boldsymbol{w}^\top \mathbf{1} - 1 = 0 \qquad ([3])$$

Solving 1/3

First rearrange [1]:

$$\begin{aligned} \boldsymbol{\Sigma}\boldsymbol{w} &= \begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \Rightarrow \\ \boldsymbol{w} &= \boldsymbol{\Sigma}^{-1} \begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \end{aligned} \tag{[4]}$$

And then [2]-[3]:

$$\begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix}^\top \boldsymbol{w} = \begin{bmatrix} \mu_P \\ 1 \end{bmatrix} \tag{[5]}$$

Solving 2/3

Multiply [4] with $\left[\begin{array}{cc} \boldsymbol{\mu} & \mathbf{1} \end{array}\right]^\top$

$$\left[\begin{array}{cc} \boldsymbol{\mu} & \mathbf{1} \end{array}\right]^\top \boldsymbol{w} = \left[\begin{array}{cc} \boldsymbol{\mu} & \mathbf{1} \end{array}\right]^\top \boldsymbol{\Sigma}^{-1} \left[\begin{array}{cc} \boldsymbol{\mu} & \mathbf{1} \end{array}\right] \left[\begin{array}{c} \lambda_1 \\ \lambda_2 \end{array}\right] \qquad ([6])$$

But [5] has the same left-hand side, meaning that

$$\left[\begin{array}{c} \mu_P \\ 1 \end{array}\right] = \underbrace{\left[\begin{array}{cc} \boldsymbol{\mu} & \mathbf{1} \end{array}\right]^\top \boldsymbol{\Sigma}^{-1} \left[\begin{array}{cc} \boldsymbol{\mu} & \mathbf{1} \end{array}\right]}_{A} \left[\begin{array}{c} \lambda_1 \\ \lambda_2 \end{array}\right] \qquad ([7])$$

If A is positive definite and then invertible, then [7] can be used to isolate λ_1 and λ_2 and this can be inserted in [4] to find $\boldsymbol{w}$.

Solving 3/3

A is invertible, because the coordinates of $\boldsymbol{\mu}$ are not all equal and because Σ and hence Σ^{-1} are invertible (trust this or study it yourself), so

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = A^{-1} \begin{bmatrix} \mu_P \\ 1 \end{bmatrix} \tag{[8]}$$

and finally the optimal portfolio weights for specific expected return μ_P is:

$$\hat{\boldsymbol{w}} = \boldsymbol{\Sigma}^{-1} \begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix} A^{-1} \begin{bmatrix} \mu_P \\ 1 \end{bmatrix} \tag{[9]}$$

Further, it follows that minimal portfolio variance for specific expected return μ_P is:

$$\begin{aligned} \sigma_P^2 &= \hat{\boldsymbol{w}}' \boldsymbol{\Sigma} \hat{\boldsymbol{w}} \\ &= \begin{bmatrix} \mu_P & 1 \end{bmatrix} A^{-1} \begin{bmatrix} \mu_P \\ 1 \end{bmatrix} \end{aligned} \tag{[10]}$$

Some words on A

A is defined

$$A = \begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix}' \boldsymbol{\Sigma}^{-1} \begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix}$$
$$= \begin{bmatrix} \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} & \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{1} \\ \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{1} & \mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1} \end{bmatrix} := \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

so the inverse of A is

$$A^{-1} = \frac{1}{ac - b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix}$$

The minimum variance (or standard deviation) can then be expressed in terms of a, b and c:

$$\sigma_P^2 = \frac{c\mu_P^2 - 2b\mu_P + a}{ac - b^2} \qquad \text{or} \qquad \sigma_P = \sqrt{\frac{c\mu_P^2 - 2b\mu_P + a}{ac - b^2}} \qquad ([11])$$

Efficient frontier

Using the expression for σ_P as a function of μ_P, it's easy to find the portfolio with the smallest variance possible:

$$\frac{d\sigma_P^2}{d\mu_P} = \frac{2c\mu_P - 2b}{ac - b^2} = 0 \Rightarrow$$
$$\mu_{gmv} = b/c \quad \text{with} \quad \sigma_{gmv}^2 = 1/c$$

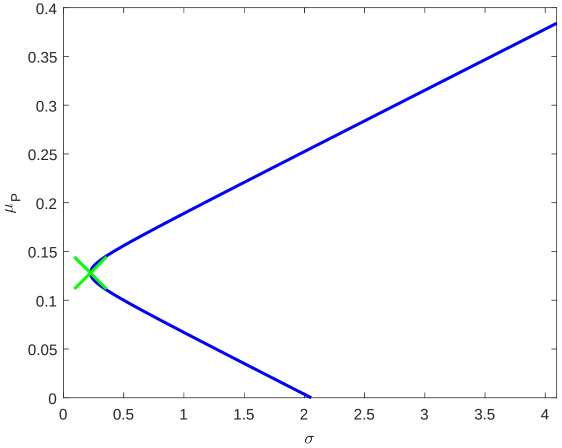
The portfolio weights can be expressed as

$$\hat{\boldsymbol{w}}_{gmw} = \frac{1}{c}\boldsymbol{\Sigma}^{-1}\mathbf{1}$$

In a (standard deviation, mean)-space or in a (variance, mean)-space, the **efficient frontier** or efficient portfolios is the upper half of the curve expressed by [11]. The efficient frontier will have expected specific returns greater than b/c and variances greater than $1/c$.

Risky Assets Only

Example (Minimum variance portfolios)



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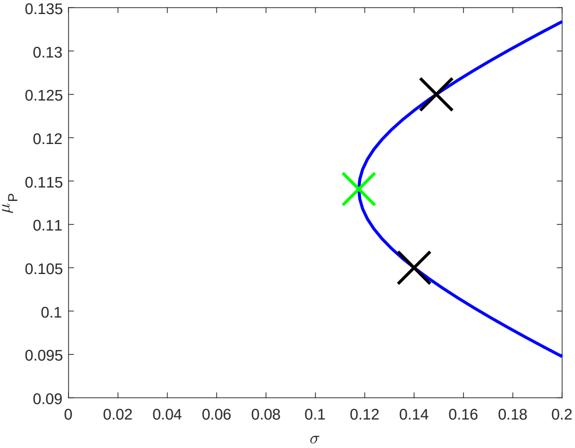
- What happens for different wealth allocations?

Portfolios of stocks and bonds

Example (Expected return vs. standard deviation)



Adding assets and/or constraints

- Inclusion of risk free asset
  - Normally, one can also allocate wealth in a risk free asset
  - The model above can be adjusted, so that solutions are still available in closed form
  - More about this next time
- Constraints on wealth
  - Sometimes, one would like the portfolio to fulfil certain requirements on the allocation of wealth
  - For instance, $w_i \geq 0 \ \ \forall i$
  - Or $w_i \leq 1/4 \ \ \forall i$
  - These are (almost) always not available in closed form, so numerical optimization is needed

What does short selling mean?

- Short selling is when you borrow an asset (for a fee) and sell it
- After a while, you buy back the asset and hand it back to the owner
- This is usually done in anticipation of decreasing prices
- When one or more weights in an efficient portfolio is negative, it's due to short-selling
- For instance, if one has \$100 and invests \$150 in one asset and sell short \$50 of the other asset, the net cost is exactly \$100 and the portfolio weights are 3/2 and -1/2 (and thus still adding up to 1)

Exercises

- Using data for stocks, illustrate different portfolios by their mean/standard deviation and identify certain portfolios
- Program the efficient frontier for a set of means and covariances and analyse how correlation affects the shape in the two-asset case

Next week

Next week, the following material is covered:

- Adding the risk free asset
- The Capital Market Line and the tangent portfolio
- More on constraints
- Different borrowing and lending rates