# Introduction to Financial Engineering

*Week 46: More on CAPM and Factor Models*

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Week 46

# The Capital Asset Pricing Model

- If all investors are mean-variance efficient in a frictionless market and they all have the same expectations, everyone would place themselves on the Capital Market Line
- The Capital Market Line contains all portfolios that are a combination of the risk-free asset and the tangent portfolio
- The tangent portfolio is the combination of risky assets with the highest Sharpe ratio
- If all investors make the optimal choice, then the market portfolio is the tangent portfolio
- In practice, a broad market index is used as a proxy, for instance the S&P500 index
- The Capital Market Line contains all efficient portfolios

# The Capital Market Line

- Every investor places himself on the CML, for instance:
  - Given a certain tolerable risk $\sigma_R$, the proportion $\sigma_R/\sigma_M$ should be invested in the market portfolio and the rest in the risk-free asset.
  - Given a required excess return $\mu_R - \mu_f$, the proportion $(\mu_R - \mu_f)/(\mu_M - \mu_f)$ should be invested in the market portfolio and the rest in the risk-free asset.
  - Or if one additionally makes assumptions about the distribution of returns, a portfolio can be picked such that a loss is not exceeded with a certain probability

Note: Notation from book, chapter 18.

# Expected return of an asset

- All assets in the market portfolios (because it's the tangent portfolio) has the same proportion of excess return to covariance of asset return and market portfolio return
- We can therefore obtain the following expression for the excess expected return of any asset:

$$\mu_j - \mu_f = \underbrace{\frac{cov(r_j, r_M)}{var(r_M)}}_{\beta_j}(\mu_M - \mu_f)$$

- $\beta_j$ measures the covariance of an asset with the market portfolio
- Plotting $\beta$ against expected returns gives the Securities Market Line, a straight line in the ($\beta$, expected return)-space

# $\beta$

- $\beta$ can be obtained by sample covariance divided by sample variance
- Or it can be obtained as the slope, when regressing asset returns on market returns
- If CAPM is supported, then the expected asset return should be a linear function of $\beta$
- High $\beta$s lead to high excess returns, low $\beta$s to low excess returns
- There are several ways of putting labels to $\beta$. The book denotes stocks with $\beta > 1$ as "aggressive" and $\beta < 1$ "not agressive"
- We will talk much more about $\beta$ next time

# Model for returns

- Consider the model for the return of asset $j$ at time $t$:

$$R_{j,t} = \mu_{f,t} + \beta_j (R_{M,t} - \mu_{f,t}) + \epsilon_{j,t},$$

  where $\epsilon_{j,t}$ are $N(0, \sigma^2_{\epsilon,j})$. Further they are uncorrelated across assets.
- How does this match with CAPM?
- The above model implies
  - Expected return of asset is ...
  - Variance of asset is ...
- CAPM says there is no reward for anything except taking market risk

# Diversification

- The risk of asset $j$ has two components: $\beta_j^2 \sigma_M^2$ and $\sigma_{\epsilon,j}^2$
- The first is called market risk or systematic risk
- The second is called unique, non-market or unsystematic risk
- It's not possible to get rid of the market risk by investing in more assets – they are all linked to the market
- But the second component can be reduced by diversification

# Constructing portfolios

Consider the portfolio with weights $w_j$ in asset $j$.

$$R_{P,t} = \sum_{j=1}^{N} w_j R_{j,t}$$

$$\mu_{P,t} = \sum_{j=1}^{N} w_j(\mu_f + \beta_j(\mu_M - \mu_f)) = \mu_f + \beta_P(\mu_M - \mu_f)$$

$$\sigma_P^2 = (w^T\beta)^2\sigma_M^2 + \sum_{j=1}^{N} w_j^2\sigma_{\epsilon,j}^2$$

When $N \to \infty$, the last term diminishes and the standard deviation of the portfolio approaches $\sigma_P = \sqrt{(\sum_{j=1}^{N} w_j\beta_j)^2}\sigma_M = \beta_P\sigma_M$.

Since the residual risk can be eliminated by holding a large portfolio, $\beta_i$ is often used as the measure for the risk of asset $i$.

# Testing CAPM empirically

- Applying CAPM requires a market proxy
- If we are testing the relationship

$$\mu_j - \mu_f = \beta_j(\mu_M - \mu_f)$$

  to see if there is a linear relationship between risk and stock $\beta$s, we can't distinguish between testing if CAPM is correct or if the proxy is in fact the tangency portfolio

- If data supports CAPM, a plot of $\beta$ vs returns should give a straight line with the market risk premium as slope and the risk free rate as intercept
- This is mostly not the case $\rightarrow$ many suggestions for altering the CAPM model and/or criticising the assumptions

# Dimension reduction

- High-dimensional data is often a challenge to handle
- It can be time-consuming and/or computationally difficult
- A standard approach to simplifying the analysis is factor models
  - Principal components analysis (PCA), where "statistical" factors are used to explain variations in data
  - Factor models, where (risk) factors with a certain economic meaning are used to explain variations in data
  - We will look at the latter type – in fact we already did, as CAPM is a factor model

# Model for returns

- A factor model explains the excess return as as

$$R_j, t = \beta_{0,j} + \beta_{1,j} F_{1,t} + \cdots + \beta_{p,j} F_{p,t} + \epsilon_{j,t}$$

  where
  - $R_{j,t}$ is the excess return on asset $j$ at time $t$
  - $F_{p,t}$ is the $p$th factor representing the economy and/or the financial markets at time $t$
  - $\beta_{p,j}$ is sensitivity of asset $j$ to factor $j$ and are often referred to as factor loadings $t$
  - $\epsilon_{j,t}$ is the asset specific risk, which are assumed independent across stocks and with zero mean. Further they are uncorrelated with the factors!
- How can CAPM be specified using the above expression?

# Fama-French Three-Factor model

- The so-called Fama-French Three-Factor model is an example of a fundamental model
- It uses observable characteristics of assets as input
- More specifically it uses:
  - Excess return on the market portfolio
  - SMB: The difference in returns on a portfolio of small stocks and a portfolio of large stocks. Small and big refers to market value of the stock.
  - HML: The difference in returns on a portfolio of stocks with a high book-to-market value and a portfolio with low book-to-market value. Book value means the net worth of the company according to its balance sheet.

# Interpretation of factor loadings

- The Fama-French Three-Factor model is stated:

$$R_j, t - \mu_{f,t} = \beta_{0,j} + \beta_{1,j}(F_{M,t} - \mu_{f,t}) + \beta_{2,j} SML_t + \beta_{3,j} HML_t + \epsilon_{j,t}$$

- The coefficient $\beta_{2,j}$ is high if asset $j$ is a small stock or behaves like a small stock
- The coefficient $\beta_{3,j}$ is positive high if asset $j$ is considered a value stock and negative if asset $j$ is considered a growth stock
  - Stocks with a high book-to-market ratio (or correspondingly low market-to-book value) are called value stocks
  - Stocks with a low book-to-market ratio (or correspondingly high market-to-book value) are called growth stocks because investors are willing to pay a premium expecting future growth

# Obtaining covariances

- A factor model for $n$ assets can be written as

$$R_t = \beta_0 + \beta^T F_t + \epsilon_t$$

where

  - $R_t$ is a $n \times 1$ vector of excess returns at time $t$
  - $\beta_0$ is a $n \times 1$ vector of intercepts
  - $\beta$ is a $p \times n$ matrix of factor loadings
  - $F_t$ is a $p \times 1$ vector of factors at time $t$
  - $\epsilon_t$ is a $n \times 1$ vector of asset specific risks which are i.i.d. with zero mean and diagonal covariance matrix $\Sigma_\epsilon$

# Obtaining covariances

- Given the factor model:

$$R_t = \beta_0 + \beta^T F_t + \epsilon_t$$

- The mean of the excess returns is

$$\mu^e = \beta_0 + \beta^T E(F_t)$$

- The variance-covariance of the excess returns is

$$\Sigma = \beta^T \Sigma_F \beta + \Sigma_\epsilon,$$

where $\Sigma_F$ is the covariance matrix of the factors

- Question: How many parameters needs to be estimated here compared to using the sample covariance matrix?

# Cross-sectional Factor Models

- In CAPM, FF and generally factor modesl, time series of a single asset is used to estimate the loadings
- It does not consider characteristics of the stock itself – like book-to-market value of the specific company
- An alternative is to look at a many assets at a single time point and several characteristics to explain the return of asset $j$
- Characteristics could be industries, book-to-market, dividend yield, size