Introduction to Financial Engineering

Week 38

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Week 38

Finding the yield of a bond

- The question is "which rate should I use for discounting the cash flows to obtain the quoted price of the bond?"
- Or equivalently "which return on investment do I get when buying this bond and holding it to maturity"
- Denote the payment at time t_i from the bond by C_{t_i} for $i = 1, \ldots, N$. Today is time t and maturity is $T = t_N$
- The yield to maturity solves the equation:

$$P_t(T) = \sum_{i=t_1}^{t_N} \frac{C_i}{(1 + y_t(T))^{i-t}}$$

Example

- Today is September 15, 2019 and the bond pays a coupon of 2 mid-November every year and expires in 2020. The current (dirty) bond price is 99.
- $P_t(T) = 99$
- $t_1 = 2/12, t_2 = 14/12$
- $C_{t_1} = 2, C_{t_2} = 102$
- The yield to maturity solves the equation:

$$99 = \frac{2}{(1+y)^{2/12}} + \frac{102}{(1+y)^{14/12}}$$

- Solving this gives a yield of 0.04389

Term structure models

- Generally, each bond gives a different yield
- For similar bonds (e.g., government bonds for the same country), different maturities give different yields
- The term structure of interest rates or the yield curve is the yields as a function of maturity
- As not all maturities are traded, it is convenient to have a parametrized function for the yield curve
- The goal is to find a functional form for $y : T \rightarrow y(T)$
- Once we have the function and the involved parameters, we have interest rates for all maturities

Nelson-Siegel model: Book version

- Nelson-Siegel is one of the most widely used term structure models
- It has four parameters $\theta_0, \theta_1, \theta_2$ and θ_3:

$$y(T) = \theta_0 + \left(\theta_1 + \frac{\theta_2}{\theta_3}\right) \frac{1 - e^{-\theta_3 T}}{\theta_3 T} - \frac{\theta_2}{\theta_3} e^{-\theta_3 T}$$

- The parameters are found by applying routines in Matlab/R/similar programs
- Once we have the parameters, we have interest rates for all maturities

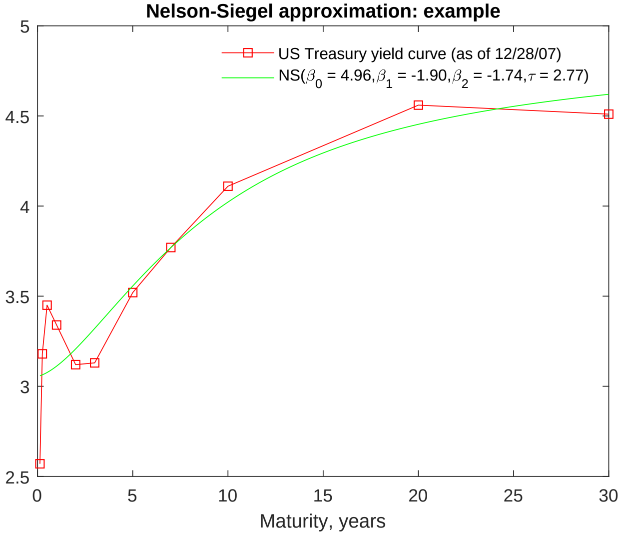
Nelson-Siegel model: Re-parametrising

- The following re-parametrisation is normally used:
- Four parameters $\beta_0, \beta_1, \beta_2$ and τ:

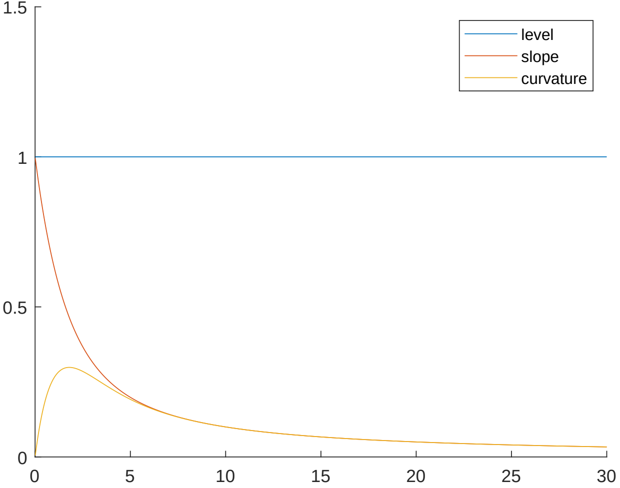
$$y(T) = \beta_0 + \beta_1 \frac{1 - e^{-T/\tau}}{T/\tau} + \beta_2 \left(\frac{1 - e^{-T/\tau}}{T/\tau} - e^{-T/\tau} \right)$$

- You can convince yourself that is it the same model with $\theta_3 = 1/\tau$, $\theta_2 = \beta_2/\tau$, $\theta_0 = \beta_0$ and $\theta_1 = \beta_1$
- The three components that are scaled with βs can be interpreted level, slope and curvature
- τ determines the location of the minimum/maximum value of the curvature component
- In R, λ is $1/\tau$, but that makes no difference, when you use the same program for estimating the model and evaluating the function

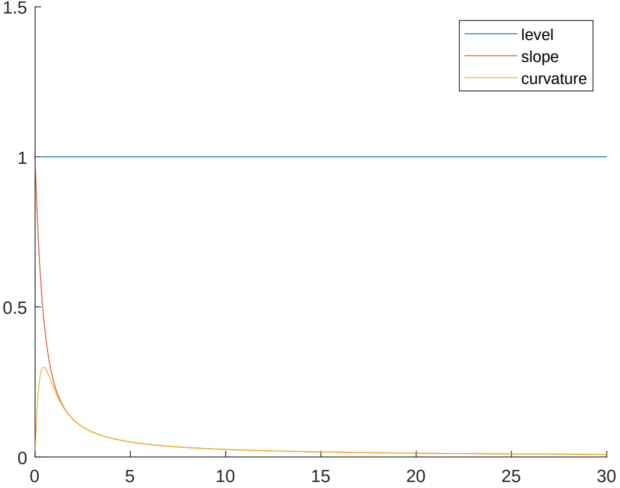
Fitting Nelson-Siegel term structure model



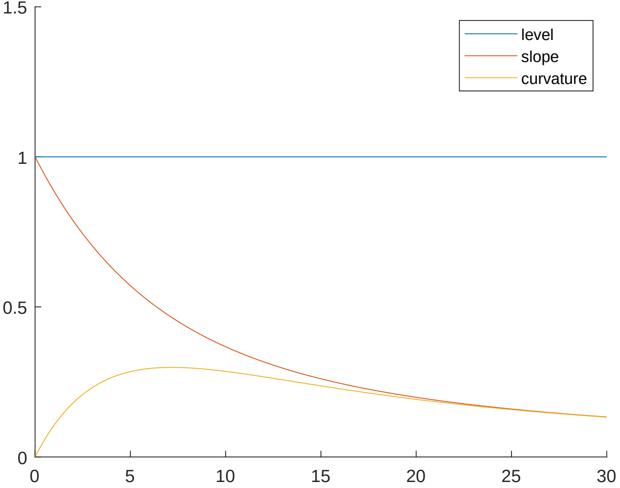
Illustation



Illustation - low τ



Illustation - high τ



Interest rate risk

- The risk that the price/value of a security will change due to movements in the general level of the interest rates
- Obviously, the cash flow of the bond will (most likely) not change
- But the present value of the cash flow can have a different value
- Think of bonds issued by a government, who later wishes to repurchase the bonds instead of paying the coupons. If the interest rates have decreased, then prices of the bond has increased and there is a high cost for buying back the bonds

Price-rate relationship

- The relationship between rates (assume a constant interest rate r and $t = 0$) and the price of a bond is given by

$$P = \sum_{t_i = t_1}^{t_N} \frac{C_{t_i}}{(1+r)^{t_i}}$$

- If the interest rate shift up or down, how does the price change?
- There is an inverse (and non-linear!) relationship between rates and bond prices

Price-rate relationship

- Think of the price of a bond as a function of r
- By a first order Taylor-expansion around initial rate r_0,

$$P(r) - P(r_0) \approx P'(r_0)(r - r_0)$$

- By a second-order Taylor-expansion around initial rate r_0,

$$P(r) - P(r_0) \approx P'(r_0)(r - r_0) + \frac{1}{2}P''(r_0)(r - r_0)^2$$

- Dividing with P on both sides gives the percentage change in price.
- To approximate the price change, we need to know $P'(r)/P$ and $P''(r)/P$ (evaluated at the current level r_0)

Duration

- The duration of a bond can be thought of as the average time of repayment (measured in present value):

$$D^{MAC} = \sum_{t_i=t_1}^{t_N} t_i \frac{C_{t_i}}{(1+r)^{t_i}} / P$$

- What is the duration of a zero coupon bond?
- What is the maximum value of the duration of a bond?

Modified duration

- We now get the clever idea to modify the duration

$$D^{MOD} = D^{MAC}/(1+r)$$

- Compare this with the derivative of the price function:

$$P'(r) = -\frac{1}{1+r}\sum_{t_i=t_1}^{t_N}\frac{t_i C_{t_i}}{(1+r)^{t_i}}$$

- It turns out that the modified duration and the derivative of the price is linked in the following way: $D^{MOD} = -P'(r)/P$.
- So the modified duration is actually (a first order approximation to) the *percentage change* in price with respect to changes in the interest rate

Calculation of duration

- The input to duration calculations are
  - the cash flows of the bond
  - the price
  - the rate/yield
- The yield of the bond could be used as input for duration calculations
- Another option is to use the term structure of rate
- This is particular relevant when looking at portfolio of bonds

Fisher-Weil duration

- Rather than assuming a flat interest rate r, the Fisher-Weil duration uses the zero coupon yield curve (or a similar proxy):

$$D^{FW} = -\frac{1}{P}\sum_{t_i=t_1}^{t_N}\frac{t_i C_{t_i}}{(1+r_{t_i})^{t_i+1}}$$

- Note that if we set r_{t_i} constant, we would get the modified duration
- The FW duration measures a *parallel shift* in the interest rates

Convexity adjustment

- Remember the relationship between price and rate. The first order approximation is not very good for larger changes in interest rates. Convexity is defined as the second order derivative relative to the price:

$$C = \frac{1}{P} \sum_{t_i = t_1}^{t_N} \frac{t_i(t_i + 1) C_{t_i}}{(1 + r_{t_i})^{t_i + 2}}$$

- Convexity adjustments will give a better approximation for changes in the bond price than just using the duration.
- Will duration over- or underestimate the new value of the bond?

Topics for next week

- Today (at 11.45), the project is handed out
- It's due Monday week 40 at 8AM
- No lectures next week, but the TAs are available from 8-12 to help you with the project
- Use Piazza for any non-personal related questions to the project – we'll monitor the site and answer as quickly as we can
- In week 40-41, we'll work with portfolio selection (Note change in program – Danske Bank visit has been moved to week 43)
- Exact readings for week 40-41 will be posted later this week, but in the two weeks we'll cover Chapter 16, a bit of 17 and the Lando-Nielsen-Poulsen note

For todays exercises

- Finish the exercise from last week if you haven't
- Plot the yields as a function of time to maturity – does any of them stand out?
- Fit the Nelson-Siegel term structure to the yields
- For each bond calculate duration and convexity – the definitions can be found in the note/slides