Introduction to Financial Engineering

Week 36

Nina Lange

Management Science, DTU

Week 36

Lectures Wednesdays 8-12

Classes will have the following (rough) structure:

- $\sim 3 \times 30$ minutes of lecturing and sometimes with small classroom discussion/exercises
- $\sim$ two hours of supervised group work doing exercises in R or Matlab

Expected preparation:

- Required readings will be listed $\sim$ one week before class
- Slides and exercises be uploaded $\sim$ one day before class
- Solutions (not always full) for exercises are published with a one-two week delay
- Therefore, make sure to have solved the previous week's exercises before class

Textboook

- Ruppert, D., and Matteson, D.S. (2015). Statistics and Data Analysis for Financial Engineering, Springer.
- We will use parts of the book (chapters 1, 2, 3, 4, 5, 16, 17, 18)
- There are lots data examples and exercises in the book (using R), but the focus is analysis of financial data
- Therefore, the textbook is supplemented with additional readings focusing on a more overall understanding of financial markets

Additional readings

- Elton, E.J., Gruber, M.J., Brown, S.J., and Goetzmann, W.N. (2014). Modern Portfolio Theory and Investment Analysis, Wiley. Chapter 2 and 7 (p. 130-142)
- Grinblatt, M., and Titman, S. (2011). Financial Markets and Corporate Strategy, McGraw-Hill. Chapter 5 (particular 5.8).
- Lando, D., Nielsen, S.E., and Poulsen, R. (2015). Lecture Notes for Finance 1 (and More): Chapter 9, University of Copenhagen http://web.math.ku.dk/ rolf/ifnotes.pdf
- Sections of books, web references and papers uploaded or linked to on Campusnet

Recommended videos

You might find it relevant or enlightening to watch the following videos

- Khan Academy: Short videos explaining different concepts
- The Ascent of Money: A bit on the history of finance and assets – good for watching while on the bus, ironing or being tired after a long week of studying (available on YouTube)

Assessment of course

- Group projects (groups of three students): Handed out in week 38, 43, 47. Due Monday in week 40, 45, 49 at 8AM. The projects needs to be approved in order to take the exam. If a project is not approved, there will be an opportunity to fix the mistakes and hand-in again by a set deadline.
- A written exam on which your course grade is based. The exam takes place on the course allocated exam day.
- Hint: The exam will be a lot easier, if you do the projects well.

Calculus

- Derivatives and partial derivatives
- Finding minima/maxima of functions
- Lagrange multipliers
- Taylor expansions

Probability

- Random variables/probability distributions
- The normal distribution and its' limitations
- Expected values and variance/covariance

Linear algebra

- Solving systems of linear equations
- Add, multiply, transpose and invert matrices, and compute determinants
- And yes, I do expect you to be able to invert a 2-by-2 matrix without the use of a computer

Programming

- In the course and in the project, most calculations will be done in R/Matlab and very few by hand
- You will be required to make small scripts and function to calculate various quantities, import and process financial data, illustrate results in plots etc.
- I don't care if you use R or Matlab. I do most of my work in Matlab. The TA is very good (and probably a better source than me) with R. The textbook and the R book linked to under links on Campusnet is a very good source for doing financial calculations in R.

Course topics

- What are stocks and bonds
- Working with financial data
- Key concepts in bond analysis
- Portfolio choice – risk vs. return (Markowitz-framework)
- Capital Asset Pricing Model
- Analysis/evaluation of portfolios

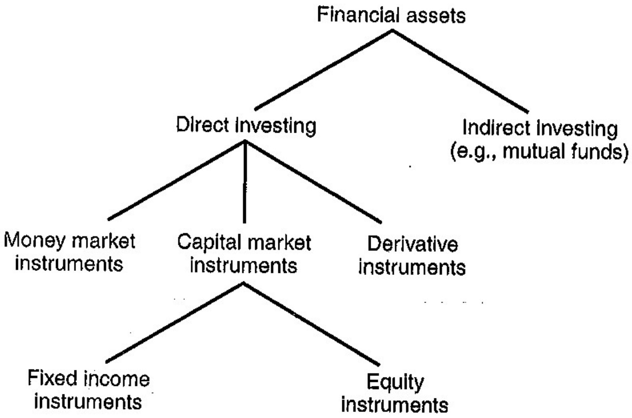
Recordings

- You may not record the lectures and share them with others
- The same goes for photographing examples on the board
- And; it is MUCH better to draw your own notes/drawings step by step while the examples are explained rather than having the final picture

Working with financial data

- Always look at the data to get a feeling for the data. For instance various plots and summary statistics.
- All models are false – but some are useful!
- There is a tradeoff between bias and variance
- Financial markets data are not normally distributed
- Variances (or standard deviation or volatility) are not constant over time

Financial assets



Stocks

What is a stock?

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https://www.khanacademy.org/economics-finance-domain/core-finance/stock-and-bonds/modal/v/what-it-means-to-buy-a-company-s-stock

Bonds

What is a bond?

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https://www.khanacademy.org/economics-finance-domain/core-finance/stock-and-bonds/modal/v/introduction-to-bonds

What affects the prices of stocks?

Spend five minutes researching and/or discussing with the student next to you `https://e.ggtimer.com/5%20minutes`

Other financial assets

Derivatives

- Financial assets whose value is *derived* from something else
- Examples are options, forwards, futures
- The underlying actually doesn't have to be a financial asset, it could be rainfall or the price of a physical good

Funds

- Mutual funds: Pool of money from many investors. Investments in many different assets and professionally managed
- Exchange traded funds: Similar to a mutual fund, but trades on exchanges

The rate of return or the net return

- The rate of return (or just the **net return**) over a time-period is defined as the difference in asset price at the two points in time measured as a fraction of initial value:

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \frac{P_t}{P_{t-1}} - 1$$

- Often in finance, log returns are used instead:

$$r_t = \log\left(\frac{P_t}{P_{t-1}}\right) = p_t - p_{t-1}$$

where $p_t = \log P_t$ is called the log-price. Note: $\log$ means the natural logarithm, sometimes denoted as $\ln$ in some computer programs or calculators

When/why are these two expressions (almost) the same?

Unit and scale

- Returns are scale-free – the unit of prices does not matter
- Returns are *not* unit-free – the their unit is time, i.e., stating a return does not make sense unless the time-frame is specified
- We write the return over the most recent k periods as $R_t(k)$, where

$$\begin{aligned} R_t(k) &= \frac{P_t - P_{t-k}}{P_{t-k}} = \frac{P_t}{P_{t-k}} - 1 \\ &= \frac{P_t}{P_{t-1}} \frac{P_{t-1}}{P_{t-2}} \cdots \frac{P_{t-k+1}}{P_{t-k}} - 1 \\ &= (1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-k+1}) - 1 \end{aligned}$$

Calculating average returns

- Often average returns per unit of time is more relevant that individual returns
- Example:
  - Stock 1 increase by 10% every year for three years
  - Stock 2 increase by 60% the first year, decrease by 50% the next year and increase by 20% the third year
  - Which stock do you prefer? Why?

Geometric average

- The arithmetic average of the two stocks are both 10%
- But the questions is really; at which rate $\bar{R}$ does my stock grow per time period (here a year) on average?

$$(1+\bar{R})^3 = (1+R_1)(1+R_2)(1+R_3)$$

- Or in more general terms: The average rate of return per time period is:

$$\bar{R} = \left(\prod_{i=1}^{N}(1+R_i)\right)^{(1/N)} - 1$$

- What is $\bar{R}$ for the two stocks on the previous slide?

Standard deviations and variance/covariance

- In many applications, we are more interested in the variation of returns than the average rate of return
- For this we use the normal definition of standard deviation and use the build-in functions in the data tool we are using
- The variation of returns gives a (simple) measure for the riskiness of the financial asset
- In today's exercise you will be asked to compute the standard deviation of daily returns
- You will also be asked to compute the covariance matrix and correlation matrix for a group of stocks
- The covariance/correlation matrix says something about how different stocks co-move

Annualising

- In the example before, we used three different annual returns to compute the average *annual* rate of return over those three years
- If we use N daily returns and compute the average, we get a *daily* average rate of return for the period that our daily returns cover
- In practice, we almost always work with daily or weekly returns
- If we have calculated an average daily or weekly return for a period of time (regardless of this time period being half a year or ten years), a relevant question is: "If we get a this return on average per day/week, what would we get in one year?"
- The answer is $(1+\bar{R})\cdots(1+\bar{R})-1$ where we multiply as many times as there are days/weeks in one year:

$$\bar{R}^A = (1+\bar{R}^W)^{52} - 1 = (1+\bar{R}^D)^{252} - 1$$

Annualising log returns

- If we use log returns, then we can just use the simple arithmetic average:

$$\bar{r} = \frac{1}{k} \sum_{i=1}^{N} r_{t-i+1}$$

- Annualising also becomes simpler:

$$\bar{r}^A = 52 \times \bar{r}^W = 252 \times \bar{r}^D$$

Annualising log returns

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Annualising standard deviations

- It can be seen that the standard deviation of annual returns can be extracted from the standard deviation of daily or weekly returns
- At least if we assume that returns are independent, have constant standard deviation and we are using log returns (see argument on blackboard)
- In conclusion, the standard deviation of daily returns are converted into an annual number by

$$\sigma^A = \sqrt{52}\sigma^W = \sqrt{252}\sigma^D$$

- The annualised standard deviation of returns is often also denoted the (annual) volatility

Next week we will talk about

- Cash flow of bonds
- Yield to maturity
- Duration and convexity (mostly covered in next week's slides/lecture)
- Readings: Chapter 3

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Bonds

- Owning a bond is lending to the issuer of the bond
- Bond holders get their money before stock owners
- Government bonds are generally viewed as safe assets, where the bond owner gets a fixed stream of coupons and principal back
- Corporate bonds are generally not completely safe assets, as the company can go bankrupt
- The current market interest rate (and the potential risk) is reflected the bond price

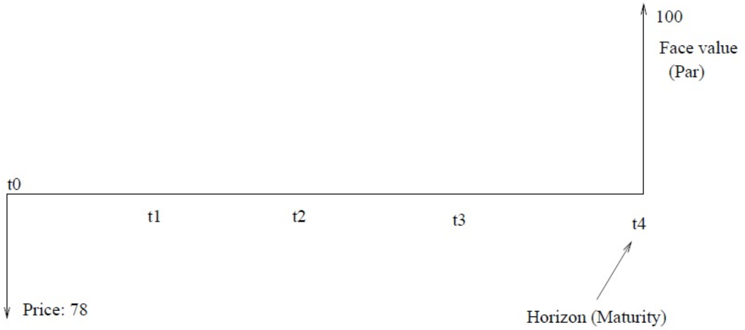
Types of bonds

The basic bond types are

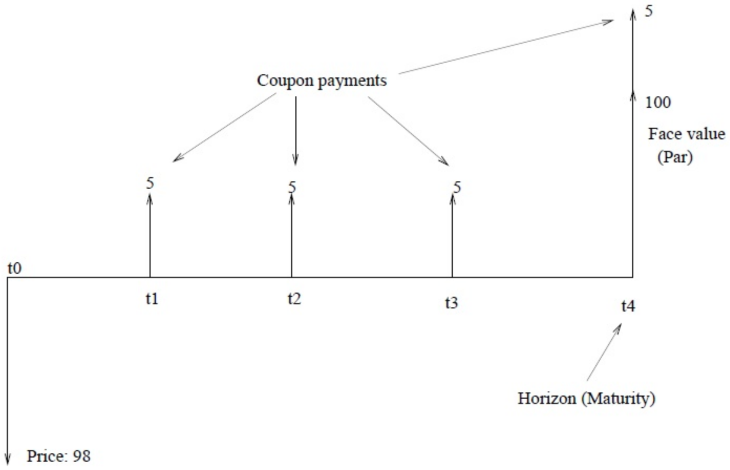
- Zero Coupon Bond
- Bond with coupons
- Annuity
- (Other types, e.g., serial loans)

The bond has a cash flow depending on type, price, coupons, maturity.

The Zero Coupon Bond is the most simple



A bond with coupon has payments before maturity



An annuity pays the same every time

Insert picture from blackboard yourself

For a serial loan repayments are constant

Insert picture from blackboard yourself

Market price of bonds

- A zero coupon bond pays back the par value at maturity
- It usually sells below par
- A coupon bond pays an annual coupon of C plus par value at maturity. In case of coupons being paid semi-annually, $C/2$ is paid twice a year etc.
- Assume that the market interest rate is r with annual compounding
- The market value or present value or market price of a zero coupon bond with maturity T is the par value discounted back at the rate r: $PAR(1+r)^{-T}$
- If the market interest rate change, so does the value of the bond
- Bond prices and interest rates move in the same/opposite direction?

Finding yields from ZCB

- Often, we consider the opposite problem. From prices of bonds, we infer market interest rates (or yields as they are then called)
- From the cash flow, it is possible to extract the *yield to maturity* implied by that bond
- The yield to maturity $y_t(T)$ at time t for a zero coupon bond maturing at time T solves the equation

$$P_t(T) = \frac{PAR}{(1 + y_t(T))^{(T-t)}},$$

  where $P_t(T)$ is the price at time t for a bond maturing at time T.
- Note: The book uses a slightly different notation and assume a semi-annual yield
- We'll stick to annual yields, because the bonds we'll look at pays annual coupons

Finding yields from bonds with coupons

- The principle for extracting yields from coupon-bearing bonds is the same
- The question is "which rate should I use for discounting the cash flows to obtain the quoted price of the bond?"
- Denote the payment at time t_i from the bond by C_{t_i}
- The yield to maturity solves the equation:

$$P_t(T) = \sum_{i=t_1}^{t_N} \frac{C_i}{(1 + y_t(T))^{i-t}}$$

- The yield $y_t(T)$ denotes the yield to maturity at time t for a bond maturing at time T. If it's clear whan t or T is, we might leave it out and just write y

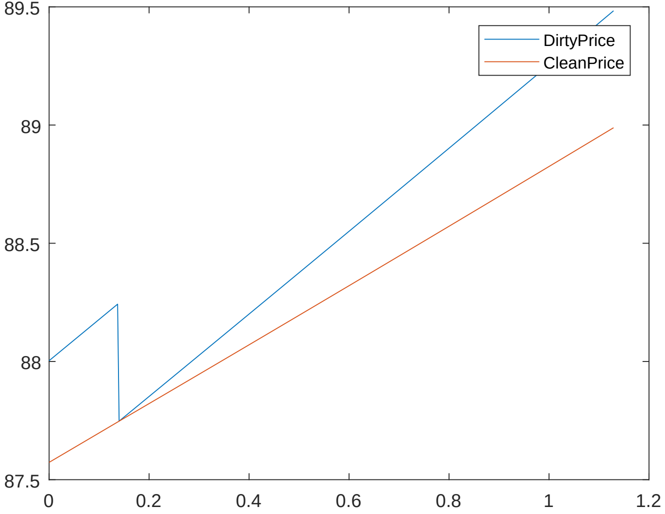
Some additional comments

- For this course, the dependence of T is more relevant than the dynamic dependence t
- The function $T \to y_t(T)$ is called the term structure of yields/interest rates on day t
- As the formulation on the previous slide reveal, yield (and other) calculations can be made on any day regardless of time of payments being an integer. If for instance, the next payment is in a month, then $t_1 = 1/12$ etc.
- There are numerous conventions for how to calculate days when working with bonds. For the purpose of this course, we'll use actual time. If you are curious about this, then NASDAQ Copenhagen has a 24 page long document with guidelines.

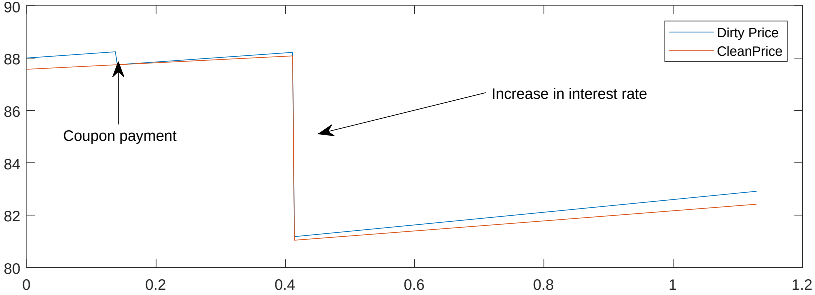
Quoted prices vs present value of cash flows

- There two types of bond prices:
  - Clean price: The price the exchange quotes
  - Dirty price: The price you actually have to pay – it is the present value of all future payments
  - Clean price = Dirty price - accrued interest
  - The accrued interest is the amount of coupon that the previous owner has to be compensated for. The accrued interest is the coupon times time passed since last coupon date.
- When setting up the cash flow and calculating yields, the dirty price needs to be calculated from the clean price and the coupons
- Clean prices are more "clean". When they change, it's because of changes in the market
- The dirty price can change just because of time passing and coupons being paid

Clean vs. dirty prices at constant interest rate



Prices when interest rates increase



When clean prices move/jump, interest rates have changed. When dirty prices change, there is either a coupon payment or something happening with the interest rates.

Topics for next week

- Duration and convexity
- Nelson-Siegel term structure model
- Analysis of interest rate risk
- Reading 1: RM Chapter 11.3. Mainly pay attention to the parametrisation of the yield curve
- Reading 2: Note on duration and convexity in discrete time

For todays exercises

- Finish the exercise from last week if you haven't
- Calculate annual returns/standard deviations
- Extract prices from NASDAQ
- Convert the clean prices into dirty prices
- Set up the cash flow (dates, coupons, principals) as a vector or matrix
- Calculate the yield for each bonds
- Plot the yields as a function of time to maturity – does any of them stand out?

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Week 38

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Week 38

Finding the yield of a bond

- The question is "which rate should I use for discounting the cash flows to obtain the quoted price of the bond?"
- Or equivalently "which return on investment do I get when buying this bond and holding it to maturity"
- Denote the payment at time t_i from the bond by C_{t_i} for $i = 1, \ldots, N$. Today is time t and maturity is $T = t_N$
- The yield to maturity solves the equation:

$$P_t(T) = \sum_{i=t_1}^{t_N} \frac{C_i}{(1 + y_t(T))^{i-t}}$$

Example

- Today is September 15, 2019 and the bond pays a coupon of 2 mid-November every year and expires in 2020. The current (dirty) bond price is 99.
- $P_t(T) = 99$
- $t_1 = 2/12, t_2 = 14/12$
- $C_{t_1} = 2, C_{t_2} = 102$
- The yield to maturity solves the equation:

$$99 = \frac{2}{(1+y)^{2/12}} + \frac{102}{(1+y)^{14/12}}$$

- Solving this gives a yield of 0.04389

Term structure models

- Generally, each bond gives a different yield
- For similar bonds (e.g., government bonds for the same country), different maturities give different yields
- The term structure of interest rates or the yield curve is the yields as a function of maturity
- As not all maturities are traded, it is convenient to have a parametrized function for the yield curve
- The goal is to find a functional form for $y : T \rightarrow y(T)$
- Once we have the function and the involved parameters, we have interest rates for all maturities

Nelson-Siegel model: Book version

- Nelson-Siegel is one of the most widely used term structure models
- It has four parameters $\theta_0, \theta_1, \theta_2$ and θ_3:

$$y(T) = \theta_0 + \left(\theta_1 + \frac{\theta_2}{\theta_3}\right) \frac{1 - e^{-\theta_3 T}}{\theta_3 T} - \frac{\theta_2}{\theta_3} e^{-\theta_3 T}$$

- The parameters are found by applying routines in Matlab/R/similar programs
- Once we have the parameters, we have interest rates for all maturities

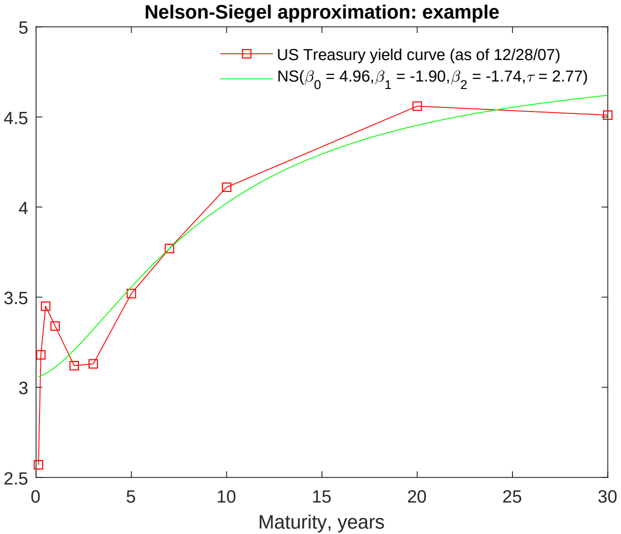
Nelson-Siegel model: Re-parametrising

- The following re-parametrisation is normally used:
- Four parameters $\beta_0, \beta_1, \beta_2$ and τ:

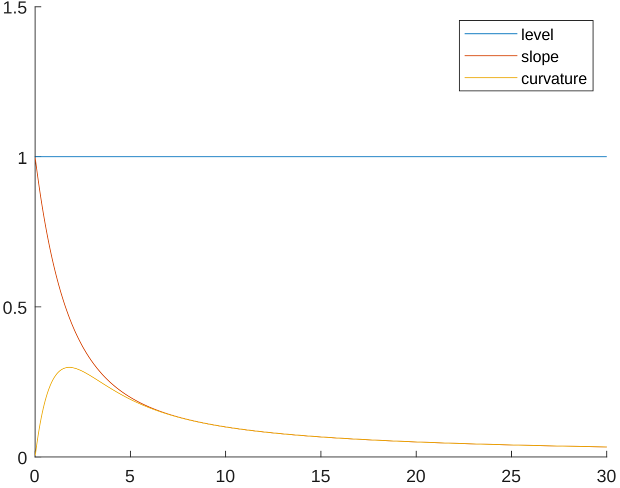
$$y(T) = \beta_0 + \beta_1 \frac{1 - e^{-T/\tau}}{T/\tau} + \beta_2 \left(\frac{1 - e^{-T/\tau}}{T/\tau} - e^{-T/\tau} \right)$$

- You can convince yourself that is it the same model with $\theta_3 = 1/\tau$, $\theta_2 = \beta_2/\tau$, $\theta_0 = \beta_0$ and $\theta_1 = \beta_1$
- The three components that are scaled with βs can be interpreted level, slope and curvature
- τ determines the location of the minimum/maximum value of the curvature component
- In R, λ is $1/\tau$, but that makes no difference, when you use the same program for estimating the model and evaluating the function

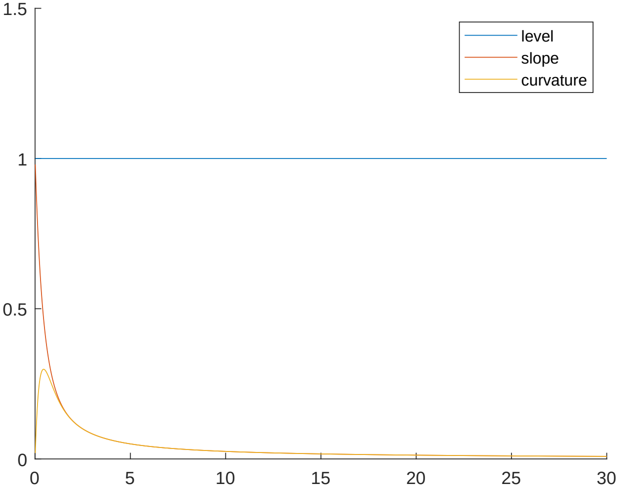
Fitting Nelson-Siegel term structure model



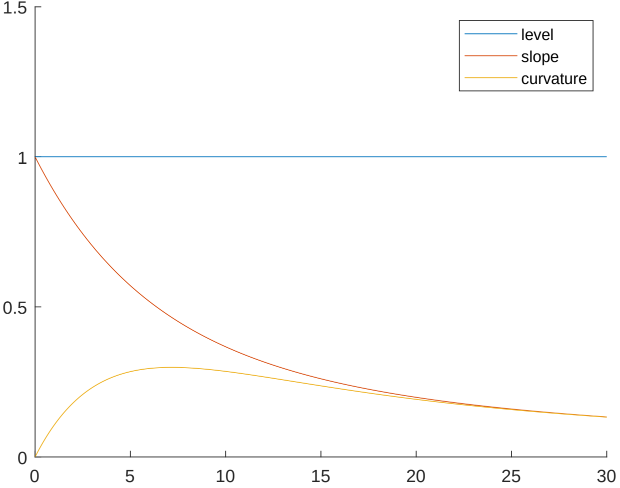
Illustation



Illustation - low τ



Illustation - high τ



Interest rate risk

- The risk that the price/value of a security will change due to movements in the general level of the interest rates
- Obviously, the cash flow of the bond will (most likely) not change
- But the present value of the cash flow can have a different value
- Think of bonds issued by a government, who later wishes to repurchase the bonds instead of paying the coupons. If the interest rates have decreased, then prices of the bond has increased and there is a high cost for buying back the bonds

Price-rate relationship

- The relationship between rates (assume a constant interest rate r and $t = 0$) and the price of a bond is given by

$$P = \sum_{t_i=t_1}^{t_N} \frac{C_{t_i}}{(1+r)^{t_i}}$$

- If the interest rate shift up or down, how does the price change?
- There is an inverse (and non-linear!) relationship between rates and bond prices

Price-rate relationship

- Think of the price of a bond as a function of r
- By a first order Taylor-expansion around initial rate r_0,

$$P(r) - P(r_0) \approx P'(r_0)(r - r_0)$$

- By a second-order Taylor-expansion around initial rate r_0,

$$P(r) - P(r_0) \approx P'(r_0)(r - r_0) + \tfrac{1}{2}P''(r_0)(r - r_0)^2$$

- Dividing with P on both sides gives the percentage change in price.
- To approximate the price change, we need to know $P'(r)/P$ and $P''(r)/P$ (evaluated at the current level r_0)

Duration

- The duration of a bond can be thought of as the average time of repayment (measured in present value):

$$D^{MAC} = \sum_{t_i=t_1}^{t_N} t_i \frac{C_{t_i}}{(1+r)^{t_i}} / P$$

- What is the duration of a zero coupon bond?
- What is the maximum value of the duration of a bond?

Modified duration

- We now get the clever idea to modify the duration

$$D^{MOD} = D^{MAC}/(1+r)$$

- Compare this with the derivative of the price function:

$$P'(r) = -\frac{1}{1+r} \sum_{t_i=t_1}^{t_N} \frac{t_i C_{t_i}}{(1+r)^{t_i}}$$

- It turns out that the modified duration and the derivative of the price is linked in the following way: $D^{MOD} = -P'(r)/P$.
- So the modified duration is actually (a first order approximation to) the *percentage change* in price with respect to changes in the interest rate

Calculation of duration

- The input to duration calculations are
  - the cash flows of the bond
  - the price
  - the rate/yield
- The yield of the bond could be used as input for duration calculations
- Another option is to use the term structure of rate
- This is particular relevant when looking at portfolio of bonds

Fisher-Weil duration

- Rather than assuming a flat interest rate r, the Fisher-Weil duration uses the zero coupon yield curve (or a similar proxy):

$$D^{FW} = -\frac{1}{P} \sum_{t_i=t_1}^{t_N} \frac{t_i C_{t_i}}{(1 + r_{t_i})^{t_i+1}}$$

- Note that if we set r_{t_i} constant, we would get the modified duration
- The FW duration measures a *parallel shift* in the interest rates

Convexity adjustment

- Remember the relationship between price and rate. The first order approximation is not very good for larger changes in interest rates. Convexity is defined as the second order derivative relative to the price:

$$C = \frac{1}{P}\sum_{t_i=t_1}^{t_N} \frac{t_i(t_i+1)C_{t_i}}{(1+r_{t_i})^{t_i+2}}$$

- Convexity adjustments will give a better approximation for changes in the bond price than just using the duration.
- Will duration over- or underestimate the new value of the bond?

Topics for next week

- Today (at 11.45), the project is handed out
- It's due Monday week 40 at 8AM
- No lectures next week, but the TAs are available from 8-12 to help you with the project
- Use Piazza for any non-personal related questions to the project – we'll monitor the site and answer as quickly as we can
- In week 40-41, we'll work with portfolio selection (Note change in program – Danske Bank visit has been moved to week 43)
- Exact readings for week 40-41 will be posted later this week, but in the two weeks we'll cover Chapter 16, a bit of 17 and the Lando-Nielsen-Poulsen note

For todays exercises

- Finish the exercise from last week if you haven't
- Plot the yields as a function of time to maturity – does any of them stand out?
- Fit the Nelson-Siegel term structure to the yields
- For each bond calculate duration and convexity – the definitions can be found in the note/slides

Introduction to Financial Engineering

Week 40: Portfolio Choices

Nina Lange

Management Science, DTU

Week 40

Making financial decisions

Some financial assets are **risky assets**, meaning that the future cash flows (and/or value) are not known. Only the distribution of returns is known or assumed.

Overall problem

- Which assets to invest in
- How much to invest in each asset

Starting point: Characterizing assets in terms of

- expected return (average)
- variance / standard deviation

Small exercise: Combining risky assets

Market Condition	Return[a] Asset 1	Return[a] Asset 2	Return[a] Asset 3	Return[a] Asset 5	Rainfall	Return[a] Asset 4
Good	15	16	1	16	Plentiful	16
Average	9	10	10	10	Average	10
Poor	3	4	19	4	Poor	4
Mean return	[redacted]					[redacted]
Variance						
Standard deviation						

[a]The alternative returns on each asset are assumed equally likely and, thus, each has a probability of $\frac{1}{3}$.

- Compute expected returns, variance/standard deviation for each asset
- What happens if Asset 2 and Asset 3 are combined?
- What happens if Asset 2 and Asset 5 are combined?
- Is there enough information to say anything about the combination of Asset 2 and Asset 4?

Estimating expected returns and variance

- There is no way of knowing the true distribution of an asset's future returns
- This must be inferred from data (and potentially using sophisticated models)
- The expected return and variance / standard deviation of returns can be estimated from historical data
- Note: Remember the exercise on averaging from week 36 and 37

Other measures of a distribution

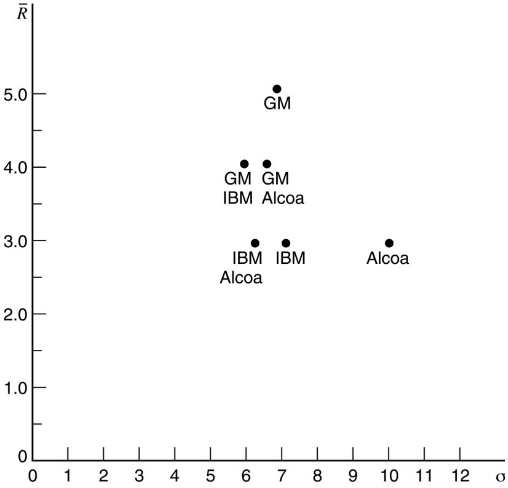
- The expected return and variance of returns is not a full description of a distribution
- Other measures to describe the dispersion (or more generally the distribution) are
  - deviations below the mean
  - the expectation of the squared deviation below the mean is called semivariance (this can be generalized to lower partial moments)
  - quantiles of the distribution (this is equivalent to the Value-at-Risk framework)
- If return distributions are symmetrical, then these other measures give the same ordering of portfolios as variance
- In portfolio literature, much theory is based on average and variance / standard deviation as adequate measures for choosing investments

Combining risky assets – stock market data

Month	IBM	Alcoa	GM	$\frac{1}{2}$ IBM + $\frac{1}{2}$ Alcoa	$\frac{1}{2}$ GM + $\frac{1}{2}$ Alcoa	$\frac{1}{2}$ GM + $\frac{1}{2}$ IBM
1	12.05	14.09	25.20	13.07	19.65	18.63
2	15.27	2.96	2.86	9.12	2.91	9.07
3	−4.12	7.19	5.45	1.54	6.32	0.67
4	1.57	24.39	4.56	12.98	14.48	3.07
5	3.16	0.06	3.72	1.61	1.89	3.44
6	−2.79	6.52	0.29	1.87	3.41	−1.25
7	−8.97	−8.75	5.38	−8.86	−1.69	−1.80
8	−1.18	2.82	−2.97	0.82	−0.08	−2.08
9	1.07	−13.97	1.52	−6.45	−6.23	1.30
10	12.75	−8.06	10.75	2.35	1.35	11.75
11	7.48	−0.70	3.79	3.39	1.55	5.64
12	−.94	8.80	1.32	3.93	5.06	0.19
$\bar{R}$	2.95	2.95	5.16	2.95	4.05	4.05
σ	7.15	10.06	6.83	6.32	6.69	6.02

Correlation Coefficient: IBM and Alcoa = 0.05;
GM and Alcoa = 0.22; IBM and GM = 0.48

Combining risky assets – illustration in (σ, μ) diagram



Investing in different stocks markets

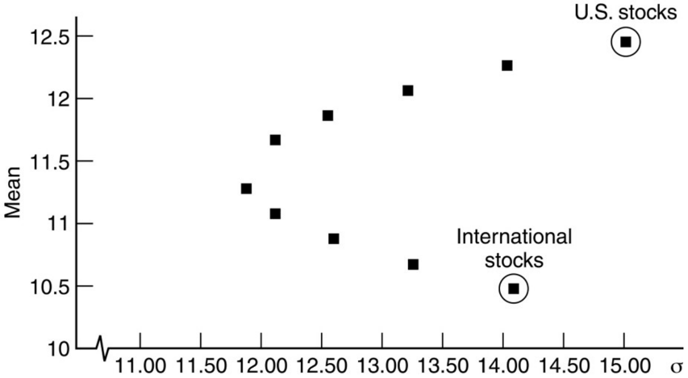
- Assume it is possible to invest in a domestic stock portfolio and in a foreign stock portfolio (each of these is one asset)
- Assume that the domestic stock portfolio has an expected return of 12.5% and the standard deviation is 14.9%
- Assume that the foreign stock portfolio has an expected return of 10.5% and the standard deviation is 14.0%
- The correlation between returns is 0.33
- What happens for different wealth allocations?

Combining stocks – portfolio data

Proportion S&P	Proportion International	Mean Return	Standard Deviation
1	0	12.5	14.90
0.9	0.1	12.3	13.93
0.8	0.2	12.1	13.11
0.7	0.3	11.9	12.46
0.6	0.4	11.7	12.01
0.5	0.5	11.5	11.79
0.45	0.55	11.4	11.76
0.4	0.6	11.3	11.80
0.3	0.7	11.1	12.04
0.2	0.8	10.9	12.50
0.1	0.9	10.7	13.17
0	1	10.5	14.00

Portfolios of US and international stocks

Example (Expected return vs. standard deviation)



Notation and assumptions

- Matrix multiplication makes it easy to formulate the expected return and variance of returns for a portfolio
- Assume that $\boldsymbol{\mu} = [\mu_1 \;\; \mu_2 \;\; \cdots \;\; \mu_n]'$ is a one-column matrix of expected returns on different assets
- Assume that $\boldsymbol{w} = [w_1 \;\; w_2 \;\; \cdots \;\; w_n]'$ is a one-column matrix representing the fractions of the investors wealth invested in each asset
- Denote the covariance matrix between returns by $\boldsymbol{\Sigma}$
- By definition, a covariance matrix is always positive semidefinite, but now it is assumed that it is **positive definite** and thus invertible
- Further, not all coordinates of $\boldsymbol{\mu}$ are equal

Matrix expressions for expected return and variance

- The expected return on such a portfolio is $\mu_P = \boldsymbol{w}^\top \boldsymbol{\mu}$
- The variance of returns for such a portfolio is $\sigma_P^2 = \boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}$, where $\boldsymbol{\Sigma}$ is the covariance matrix of the returns
- In the two-asset case, the variance is $\sigma_P^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2c_{12}$ and the expected return is $\mu_P = w_1\mu_1 + w_2\mu_2$.
- If you are struggling with matrix calculations, it might be useful for you to do the derivations for the two-asset case and convince yourself that the derived matrix expressions match.

Stating the optimization problem

- Consider the following problem

$$\min_{\boldsymbol{w}} \tfrac{1}{2}\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}$$

- Under the following constraints:

$$\boldsymbol{w}^\top \boldsymbol{\mu} = \mu_P$$
$$\boldsymbol{w}^\top \mathbf{1} = 1$$

- Or in words, given a specific expected rate of return μ_P, what is the allocation of wealth into assets $1, \ldots, n$ that gives this expected return with the smallest variance

Setting up the Lagrange

$$\mathcal{L}(\boldsymbol{w}, \lambda_1, \lambda_2) = \tfrac{1}{2}\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w} - \lambda_1 \left(\boldsymbol{w}^\top \boldsymbol{\mu} - \mu_P \right) - \lambda_2 \left(\boldsymbol{w}^\top \mathbf{1} - 1 \right)$$

FOCs for optimality are

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{w}} = \boldsymbol{\Sigma}\boldsymbol{w} - \lambda_1 \boldsymbol{\mu} - \lambda_2 \mathbf{1} = \mathbf{0} \qquad ([1])$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = \boldsymbol{w}^\top \boldsymbol{\mu} - \mu_P = 0 \qquad ([2])$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_2} = \boldsymbol{w}^\top \mathbf{1} - 1 = 0 \qquad ([3])$$

Solving 1/3

First rearrange [1]:

$$\boldsymbol{\Sigma} \boldsymbol{w} = \left[\begin{array}{cc} \boldsymbol{\mu} & \mathbf{1} \end{array} \right] \left[\begin{array}{c} \lambda_1 \\ \lambda_2 \end{array} \right] \Rightarrow$$

$$\boldsymbol{w} = \boldsymbol{\Sigma}^{-1} \left[\begin{array}{cc} \boldsymbol{\mu} & \mathbf{1} \end{array} \right] \left[\begin{array}{c} \lambda_1 \\ \lambda_2 \end{array} \right] \qquad ([4])$$

And then [2]-[3]:

$$\left[\begin{array}{cc} \boldsymbol{\mu} & \mathbf{1} \end{array} \right]^\top \boldsymbol{w} = \left[\begin{array}{c} \mu_P \\ 1 \end{array} \right] \qquad ([5])$$

Solving 2/3

Multiply [4] with $\begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix}^\top$

$$\begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix}^\top \boldsymbol{w} = \begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix}^\top \boldsymbol{\Sigma}^{-1} \begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \qquad ([6])$$

But [5] has the same left-hand side, meaning that

$$\begin{bmatrix} \mu_P \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix}^\top \boldsymbol{\Sigma}^{-1} \begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix}}_{A} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \qquad ([7])$$

If A is positive definite and then invertible, then [7] can be used to isolate λ_1 and λ_2 and this can be inserted in [4] to find $\boldsymbol{w}$.

Solving 3/3

A is invertible, because the coordinates of $\boldsymbol{\mu}$ are not all equal and because Σ and hence Σ^{-1} are invertible (trust this or study it yourself), so

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = A^{-1} \begin{bmatrix} \mu_P \\ 1 \end{bmatrix} \qquad ([8])$$

and finally the optimal portfolio weights for specific expected return μ_P is:

$$\hat{\boldsymbol{w}} = \boldsymbol{\Sigma}^{-1} \begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix} A^{-1} \begin{bmatrix} \mu_P \\ 1 \end{bmatrix} \qquad ([9])$$

Further, it follows that minimal portfolio variance for specific expected return μ_P is:

$$\begin{aligned} \sigma_P^2 &= \hat{\boldsymbol{w}}' \boldsymbol{\Sigma} \hat{\boldsymbol{w}} \\ &= \begin{bmatrix} \mu_P & 1 \end{bmatrix} A^{-1} \begin{bmatrix} \mu_P \\ 1 \end{bmatrix} \qquad ([10]) \end{aligned}$$

Some words on A

A is defined

$$A = \begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix}' \boldsymbol{\Sigma}^{-1} \begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} & \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{1} \\ \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{1} & \mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1} \end{bmatrix} := \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

so the inverse of A is

$$A^{-1} = \frac{1}{ac - b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix}$$

The minimum variance (or standard deviation) can then be expressed in terms of a, b and c:

$$\sigma_P^2 = \frac{c\mu_P^2 - 2b\mu_P + a}{ac - b^2} \quad \text{or} \quad \sigma_P = \sqrt{\frac{c\mu_P^2 - 2b\mu_P + a}{ac - b^2}} \qquad ([11])$$

Efficient frontier

Using the expression for σ_P as a function of μ_P, it's easy to find the portfolio with the smallest variance possible:

$$\frac{d\sigma_P^2}{d\mu_P} = \frac{2c\mu_P - 2b}{ac - b^2} = 0 \Rightarrow$$

$$\mu_{gmv} = b/c \quad \text{with} \quad \sigma_{gmv}^2 = 1/c$$

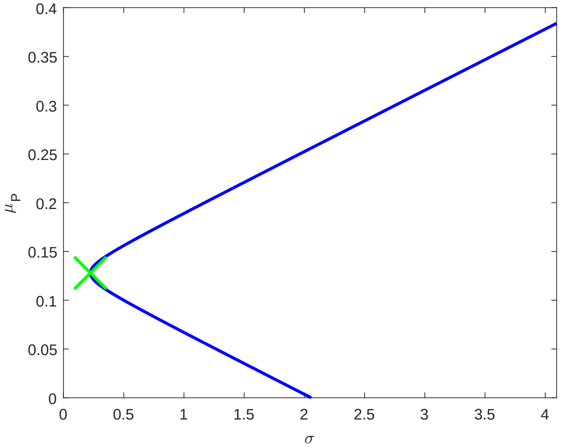
The portfolio weights can be expressed as

$$\hat{\boldsymbol{w}}_{gmw} = \frac{1}{c}\boldsymbol{\Sigma}^{-1}\mathbf{1}$$

In a (standard deviation, mean)-space or in a (variance, mean)-space, the **efficient frontier** or efficient portfolios is the upper half of the curve expressed by [11]. The efficient frontier will have expected specific returns greater than b/c and variances greater than $1/c$.

Risky Assets Only

Example (Minimum variance portfolios)

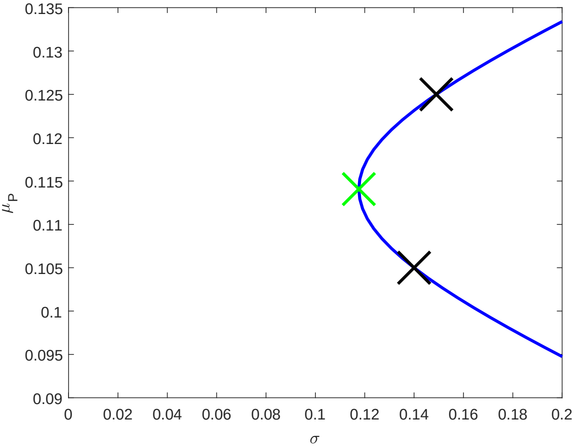


Investing in different stocks markets

- Assume it is possible to invest in a domestic stock portfolio and in a foreign stock portfolio (each of these is one asset)
- Assume that the domestic stock portfolio has an expected return of 12.5% and the standard deviation is 14.9%
- Assume that the foreign stock portfolio has an expected return of 10.5% and the standard deviation is 14%
- The correlation between returns is 0.33
- What happens for different wealth allocations?

Portfolios of stocks and bonds

Example (Expected return vs. standard deviation)



Adding assets and/or constraints

- Inclusion of risk free asset
  - Normally, one can also allocate wealth in a risk free asset
  - The model above can be adjusted, so that solutions are still available in closed form
  - More about this next time
- Constraints on wealth
  - Sometimes, one would like the portfolio to fulfil certain requirements on the allocation of wealth
  - For instance, $w_i \geq 0 \ \ \forall i$
  - Or $w_i \leq 1/4 \ \ \forall i$
  - These are (almost) always not available in closed form, so numerical optimization is needed

What does short selling mean?

- Short selling is when you borrow an asset (for a fee) and sell it
- After a while, you buy back the asset and hand it back to the owner
- This is usually done in anticipation of decreasing prices
- When one or more weights in an efficient portfolio is negative, it's due to short-selling
- For instance, if one has \$100 and invests \$150 in one asset and sell short \$50 of the other asset, the net cost is exactly \$100 and the portfolio weights are 3/2 and -1/2 (and thus still adding up to 1)

Exercises

- Using data for stocks, illustrate different portfolios by their mean/standard deviation and identify certain portfolios
- Program the efficient frontier for a set of means and covariances and analyse how correlation affects the shape in the two-asset case

Next week

Next week, the following material is covered:

- Adding the risk free asset
- The Capital Market Line and the tangent portfolio
- More on constraints
- Different borrowing and lending rates

Introduction to Financial Engineering

Week 41: Portfolio Choices

Nina Lange

Management Science, DTU

Week 41

Making financial decisions

Some financial assets are **risky assets**, meaning that the future cash flows (and/or value) are not known. Only the distribution of returns is known or assumed.

Overall problem

- Which assets to invest in
- How much to invest in each asset

Starting point: Characterizing assets in terms of

- expected return (average)
- variance / standard deviation

Notation and assumptions

- Matrix multiplication makes it easy to formulate the expected return and variance of returns for a portfolio
- Assume that $\boldsymbol{\mu} = [\mu_1 \ \ \mu_2 \ \ \cdots \ \ \mu_n]'$ is a one-column matrix of expected returns on different assets
- Assume that $\boldsymbol{w} = [w_1 \ \ w_2 \ \ \cdots \ \ w_n]'$ is a one-column matrix representing the fractions of the investors wealth invested in each asset
- Denote the covariance matrix between returns by $\boldsymbol{\Sigma}$
- By definition, a covariance matrix is always positive semidefinite, but now it is assumed that it is **positive definite** and thus invertible
- Further, not all coordinates of $\boldsymbol{\mu}$ are equal

Matrix expressions for expected return and variance

- The expected return on such a portfolio is $\mu_P = \boldsymbol{w}^\top \boldsymbol{\mu}$
- The variance of returns for such a portfolio is $\sigma_P^2 = \boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}$, where $\boldsymbol{\Sigma}$ is the covariance matrix of the returns
- In the two-asset case, the variance is $\sigma_P^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2c_{12}$ and the expected return is $\mu_P = w_1\mu_1 + w_2\mu_2$.
- If you are struggling with matrix calculations, it might be useful for you to do the derivations for the two-asset case and convince yourself that the derived matrix expressions match.

Stating the optimization problem

- Consider the following problem

$$\min_{\boldsymbol{w}} \tfrac{1}{2}\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}$$

- Under the following constraints:

$$\boldsymbol{w}^\top \boldsymbol{\mu} = \mu_P$$
$$\boldsymbol{w}^\top \mathbf{1} = 1$$

- Or in words, given a specific expected rate of return μ_P, what is the allocation of wealth into assets $1, \ldots, n$ that gives this expected return with the smallest variance

Efficient frontier

The efficient frontier can be described as

$$\sigma_P^2 = \frac{c\mu_P^2 - 2b\mu_P + a}{ac - b^2} \qquad \text{or} \qquad \sigma_P = \sqrt{\frac{c\mu_P^2 - 2b\mu_P + a}{ac - b^2}} \qquad ([1])$$

$$\mu_{gmv} = b/c \quad \text{with} \quad \sigma_{gmv}^2 = 1/c$$

where a, b and c are obtained from the definition of the A-matrix. The portfolio weights for a given required return can be expressed as

$$\hat{\boldsymbol{w}} = \boldsymbol{\Sigma}^{-1} \begin{bmatrix} \boldsymbol{\mu} & \mathbf{1} \end{bmatrix} A^{-1} \begin{bmatrix} \mu_P \\ 1 \end{bmatrix}, \qquad \hat{\boldsymbol{w}}_{gmw} = \frac{1}{c}\boldsymbol{\Sigma}^{-1}\mathbf{1} \qquad ([2])$$

In a (standard deviation, mean)-space, the **efficient frontier** or efficient portfolios is the upper half of the curve expressed by [1]. The efficient frontier will have expected returns greater than b/c and standard deviations greater than $1/\sqrt{c}$.

Inclusion of risk free asset

- Assume that a risk free asset exists with return μ_0
- Express returns as excess returns
$\boldsymbol{\mu}^e = [\mu_1 - \mu_0, \mu_2 - \mu_0, \ldots, \mu_n - \mu_0]^\top$
- Assume that $\boldsymbol{w} = [w_1, w_2, \ldots, w_n]^\top$ are the fractions of the investors wealth invested in each risky asset
- Assume that $w_0 = 1 - \boldsymbol{w}^\top \mathbf{1}$ is invested in the risk free asset.
- If $w_0 > 0$, the investor has put money in the bank. If $w_0 < 0$, the investor has borrowed money.
- For a given expected excess return μ_P^e, the objective is to find the portfolio with the lowest variance (or standard deviation)

Stating the optimization problem

- Consider the following problem

$$\min_{\boldsymbol{w}} \tfrac{1}{2}\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}$$

- Under the following constraints:

$$\boldsymbol{w}^\top \boldsymbol{\mu}^e = \mu_P^e$$

- Or in words, given a specific expected excess rate of return μ_P^e, what is the allocation of wealth into assets $1, \ldots, n$ that gives this expected excess return with the smallest variance

Setting up the Lagrange

$$\mathcal{L}(\boldsymbol{w}, \lambda) = \frac{1}{2}\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w} - \lambda_1 \left(\boldsymbol{w}^\top \boldsymbol{\mu}^e - \mu_P^e \right)$$

FOCs for optimality are

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{w}} = \boldsymbol{\Sigma} \boldsymbol{w} - \lambda \boldsymbol{\mu}^e = \boldsymbol{0} \qquad ([3])$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \boldsymbol{w}^\top \boldsymbol{\mu}^e - \mu_P^e = 0 \qquad ([4])$$

Solving 1/3

First rearrange [3]:

$$\begin{aligned} \boldsymbol{\Sigma} \boldsymbol{w} &= \lambda \boldsymbol{\mu}^e \Rightarrow \\ \boldsymbol{w} &= \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e \lambda \end{aligned} \qquad ([5])$$

And then [4]:

$$(\boldsymbol{\mu}^e)^\top \boldsymbol{w} = \mu_P^e \qquad ([6])$$

Solving 2/3

Multiply [5] with $(\boldsymbol{\mu}^e)^\top$

$$(\boldsymbol{\mu}^e)^\top \boldsymbol{w} = (\boldsymbol{\mu}^e)^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e \lambda \tag{[7]}$$

But [6] has the same left-hand side, meaning that

$$\mu_P^e = (\boldsymbol{\mu}^e)^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e \lambda \tag{[8]}$$

Because Σ is positive definite and then invertible, then $(\boldsymbol{\mu}^e)^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e > 0$ and λ can be inserted in [5] to find $\boldsymbol{w}$.

Solving 3/3

So

$$\lambda = \frac{\mu_P^e}{(\boldsymbol{\mu}^e)^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e} \quad ([9])$$

and the optimal portfolio weights for specific expected excess return μ_P^e is:

$$\boldsymbol{w} = \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e \frac{\mu_P^e}{(\boldsymbol{\mu}^e)^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e} \quad ([10])$$

Further, it follows that minimal portfolio variance for specific expected excess return μ_P^e is:

$$\sigma_P^2 = \frac{(\mu_P^e)^2}{(\boldsymbol{\mu}^e)^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e} \quad ([11])$$

Capital Market Line

The link between σ_P and of μ_P^e is:

$$\sigma_P = \frac{\mu_P^e}{\sqrt{(\boldsymbol{\mu}^e)^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e}}$$

or equivalently (this is called the Capital Market Line)

$$\mu_P = \sigma_P \sqrt{(\boldsymbol{\mu}^e)^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e} + \mu_0$$

- It is reasonable to assume than $\mu_0 < \mu_{gmv}$ (Why?)
- The CML touches the efficient frontier without the risk free asset in exactly one point (Why?)

Tangent Portfolio

The portfolio where everything is invested in risky assets is called the **tangent portfolio**. The excess return of the tangent portfolio is

$$\mu^e_{tan} = \frac{(\boldsymbol{\mu}^e)^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e} \tag{[12]}$$

with

$$\sigma_{tan} = \frac{\sqrt{(\boldsymbol{\mu}^e)^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e} \tag{[13]}$$

$$\boldsymbol{w}_{tan} = \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e \frac{1}{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e} \tag{[14]}$$

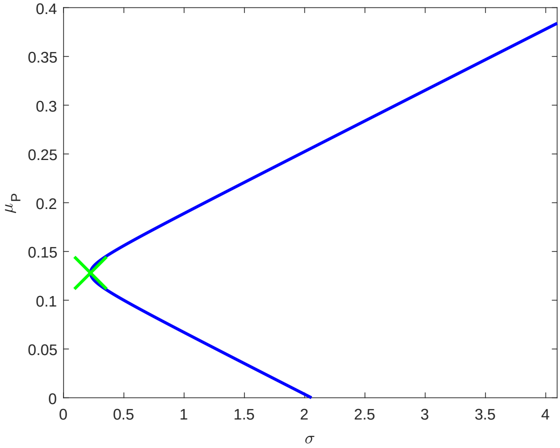
One lending and one borrowing rate

If there is one borrowing rate and one lending rate, then the efficient portfolios will be

- The CML using the lending rate for $\sigma_P < \sigma_{tan}^l$
- The CML using the borrowing rate for $\sigma_P > \sigma_{tan}^b$
- The efficient frontier for the risky assets only $\sigma_{tan}^l < \sigma_P < \sigma_{tan}^b$

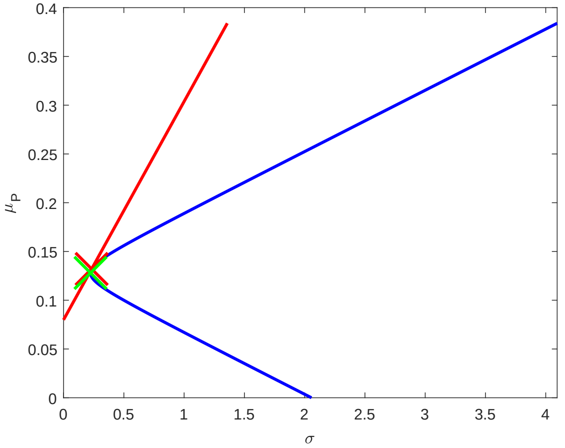
Risky Assets Only

Example (Minimum variance portfolios)



Including Risk Free Asset

Example (Minimum variance portfolios)

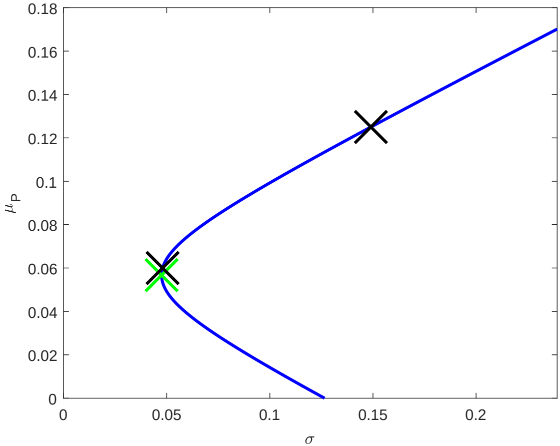


Investing in two stocks

- Assume it is possible to invest in two different stocks
- Assume that stock 1 has an expected return of 12.5% and the standard deviation is 14.9%
- Assume that stock 2 has an expected return of 6% and the standard deviation is 4.8%
- The correlation between returns is 0.45

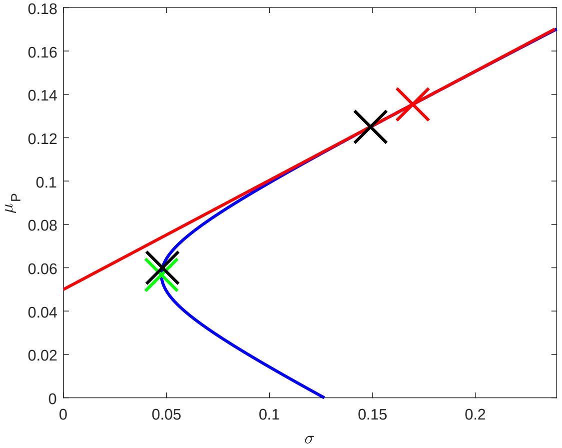
The efficient frontier

Example (Expected return vs. standard deviation)



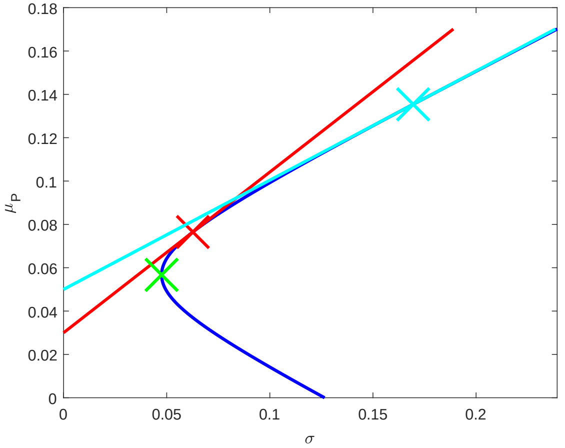
The efficient frontier and the CML

Example (Expected return vs. standard deviation)



Portfolios of stocks and bonds and two different rates

Example (Expected return vs. standard deviation)



Two risky assets case

- For two risky assets, both assets will be on the parabola of minimum-variance portfolios
- They will not necessarily be on the efficient frontier – the upper half curve. The position will depend on the assets (correlation, expected return, standard deviation)
- If short-selling is not allowed, the possible minimum-variance portfolios is on the parabola of minimum-variance portfolios between the two assets
- The efficient and feasible portfolios consist of the upper half curve for μ-values between $\max\{\min(\mu_1, \mu_2), \mu_{gmw}\}$ and $\max(\mu_1, \mu_2)$

Multiple risky assets

- For multiple risky assets, the assets will generally not be on the parabola of minimum-variance portfolios
- If short-selling is not allowed, the possible minimum-variance portfolios must be found by solving the optimization problem with the additional constraint that $w_i \geq 0$ for $i = 1, \ldots, n$
- This leads to portfolios with at least the same variance as without the additional constraint
- The highest feasible return is $\max(\mu_1, \mu_2, \ldots, \mu_n)$

Constraints on the risk free asset

- A risk free asset with possible different interest rates can also be incorporated in the optimization problem
- Risk less lending and no borrowing could for instance be incorporated by removing constraint $\mathbf{1}'\boldsymbol{w} = 1$ and instead constraining $w_0 \geq 0$ together with the previous condition of no short-selling of risky assets $w_i \geq 0$ for $i = 1, \ldots, n$
- Borrowing with no lending can also be incorporated by varying the constraints

Portfolios of particular interest

- The minimum variance portfolio: The portfolio with the lowest variance
- The maximum return portfolio: The portfolio with the highest expected return
- The tangent portfolio: This portfolio actually has the highest **Sharpe Ratio** $SR_P = \frac{\mu_P - \mu_0}{\sigma_P}$ among the risky asset only portfolios
- Choosing according to risk aversion: Finding the portfolio that (for instance) maximizes $\mu_P - \lambda\sigma_P^2$ for a given λ. If λ is high, the investor dislikes risk ("risk averse"). If λ is low, he favors return over risk.

Structure of Chapter 16

- The book covers more or less the same content as the Lando-Poulsen note, but the latter has more general, and closed form solutions
- The book is a good source for putting the maths in a simpler form
- It finds the tangent portfolio for two risky assets and for N assets by programming – we did it for N in closed form
- The Capital Market Line is presented as a combination of investing in the tangent portfolio and the risk free asset – we derived it from N risky assets and on risk free asset, but it is the same result!
- Section 16.2.1 highlight issues in estimating expected return and risk. We have used sample mean and variance for this, but it's possible to do more sophisticated models (like CAPM, Factor models, GARCH models etc.)
- The code example in Chapter 16.6 is good inspiration for adding constraints

Next week

Next week, the following material is covered:

- Chapter 4 up til and including 4.4
- Chapter 5 up til and including 5.4
- Danske Bank will visit from around 9.30/10 and present some cases, of which some are related to the above material and some of more general data analysis character

Introduction to Financial Engineering

Week 43: Basic Data Analysis

Nina Lange

Management Science, DTU

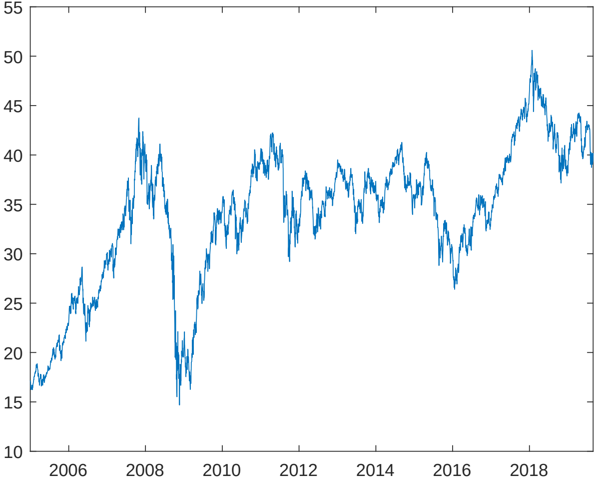
Week 43

Using financial data

- We wish to use financial data to make financial decisions based on given criteria
- The distribution of data provide information about potential risks and possible rewards
- The objective is to say something about the distribution of a financial variable in a day, a week, a month (or every future time point)
- A starting point is to look at historical data and assume that the future is well described by the past
- In later courses, projects or in real life, things needs to be a bit more sophisticated
- But remember; the theory on efficient frontier, CML etc. does not hinge on how estimates were obtained. It just says that if we assume certain mean and covariances, we know how to compose optimal portfolios.

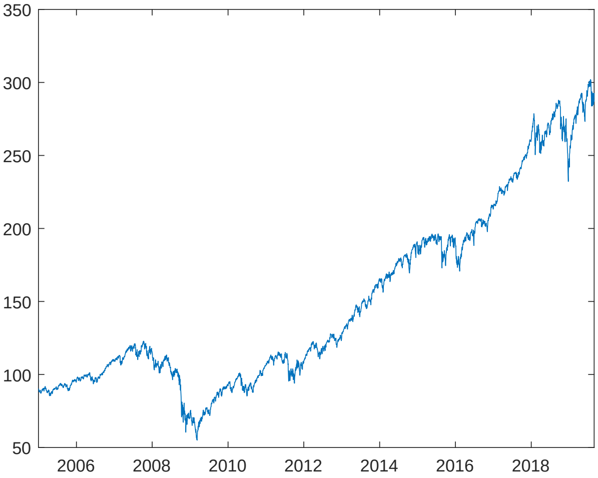
Plot of data

Example (iShares MSCI Emerging Markets ETF (EEM))



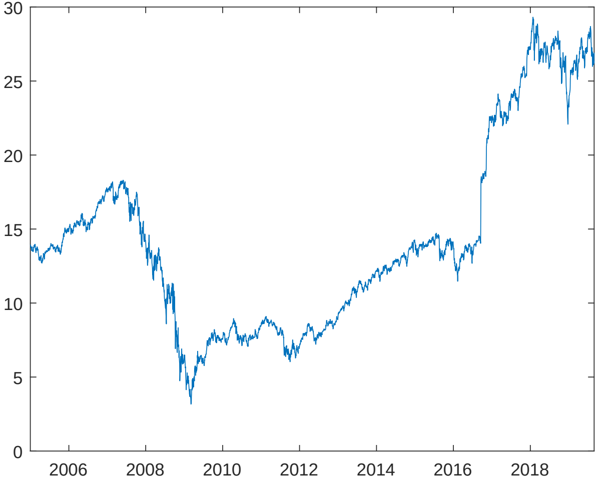
Plot of data

Example (SPDR S&P 500 trust (SPY))



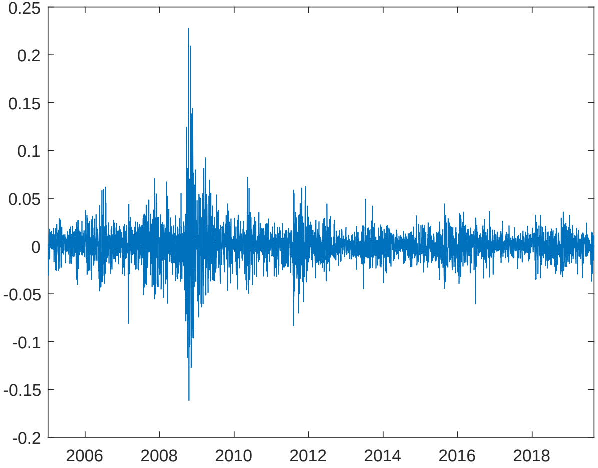
Plot of data

Example (The Financial Select Sector SPDR Fund (XLF))



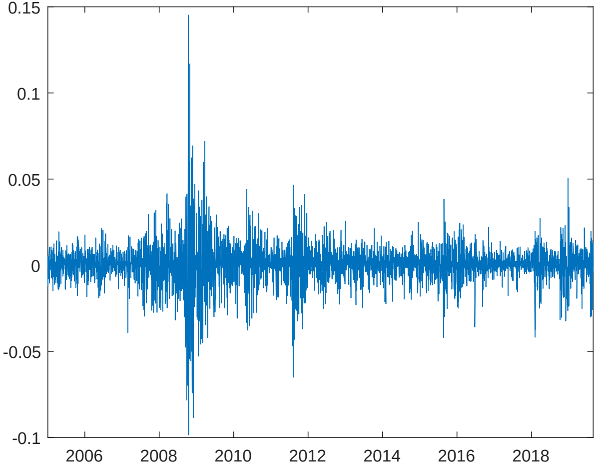
Plot of returns

Example (iShares MSCI Emerging Markets ETF (EEM))



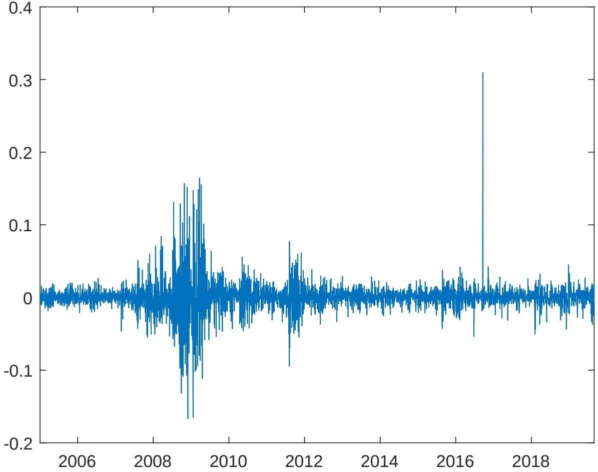
Plot of returns

Example (SPDR S&P 500 trust (SPY))



Plot of returns

Example (The Financial Select Sector SPDR Fund (XLF))



Histogram

Example (The Financial Select Sector SPDR Fund (XLF))

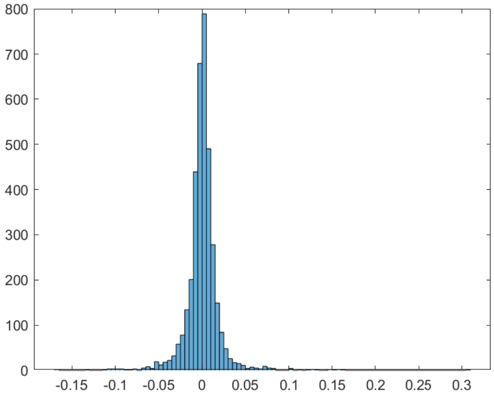
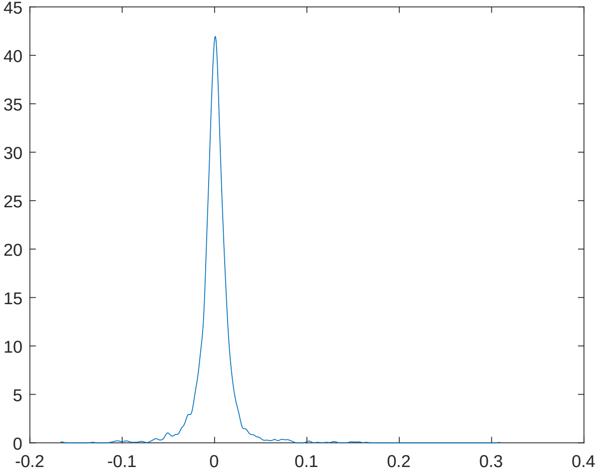


Figure:

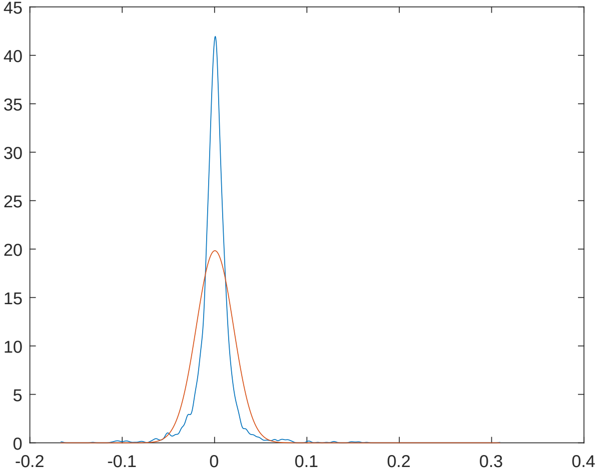
Empirical density

Example (The Financial Select Sector SPDR Fund (XLF))



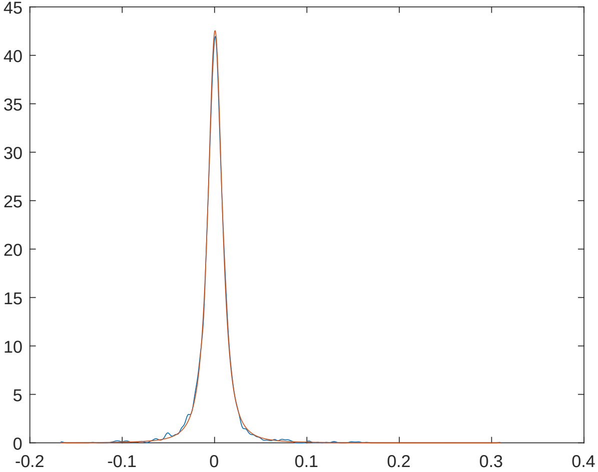
Empirical density vs Normal distribution

Example (The Financial Select Sector SPDR Fund (XLF))



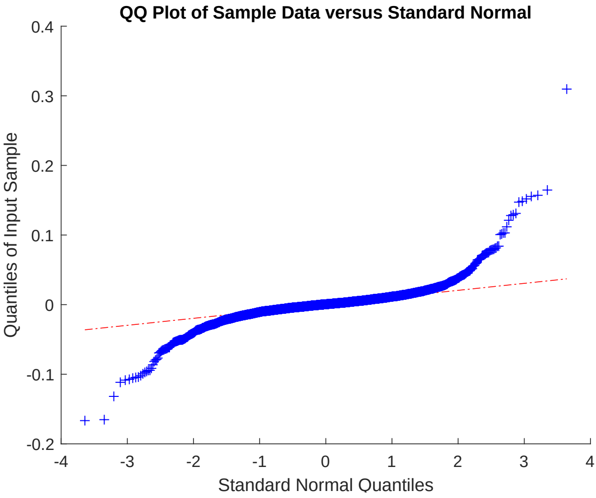
Empirical density vs t-distribution

Example (The Financial Select Sector SPDR Fund (XLF))



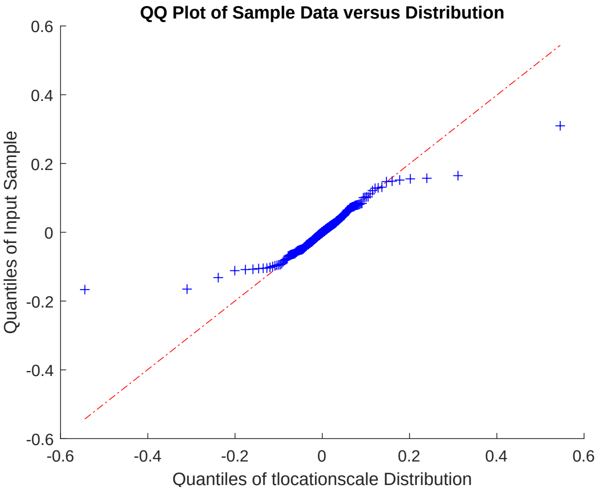
Normal probability plot – testing normality

Example (The Financial Select Sector SPDR Fund (XLF))



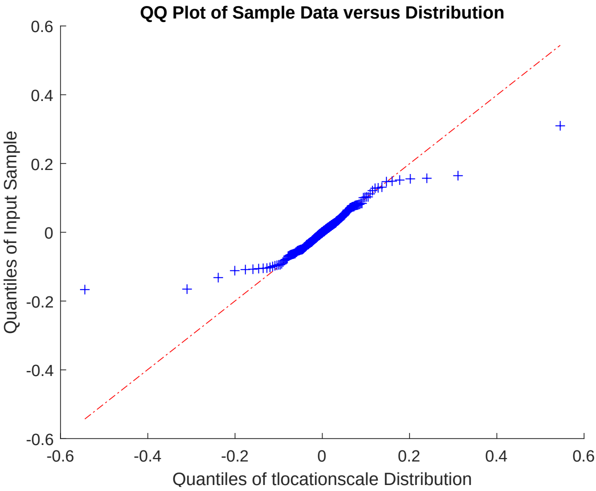
Normal probability plot – testing t-distribution

Example (The Financial Select Sector SPDR Fund (XLF))



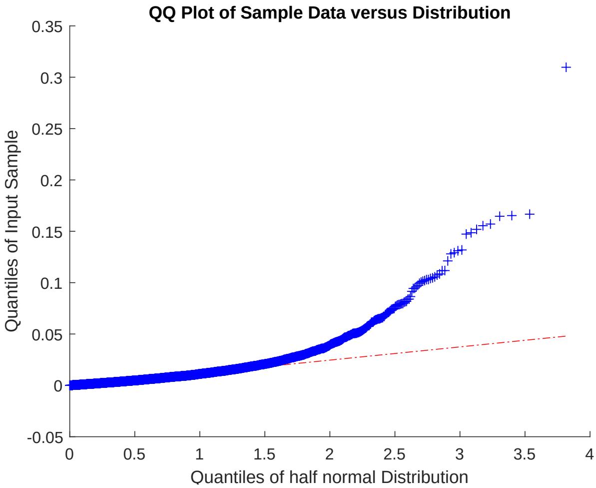
Normal probability plot – testing t-distribution

Example (The Financial Select Sector SPDR Fund (XLF))



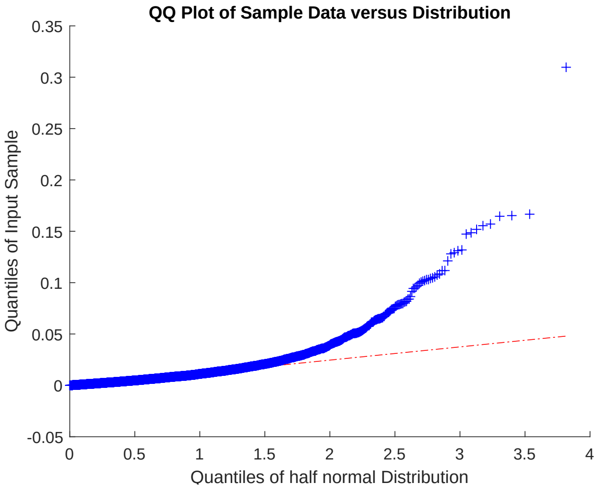
Half-normal plot – outliers

Example (The Financial Select Sector SPDR Fund (XLF))



Half-normal plot – outliers

Example (The Financial Select Sector SPDR Fund (XLF))



Overview of data

Besides graphical inspection and presentation, it's often relevant to display a summary of data. For instance:

- The historical average rate of return (annualised)
- The historical variance or standard deviation of returns (again, annualised)
- The correlation of returns
- Higher moments, such as (excess) kurtosis and skewness
- Statistical tests for normality, for instance Jarque-Bera or Shapiro-Wilk
- Auto-correlation (and autocorrelatioin of squared returns)

Jarque-Bera test

- The Jarque-Bera test is (one of many) test(s) for normality
- Normally distributed data has skewness S of 0 and excess kurtosis of 0 (kurtosis K equal to 3)
- The JB-test tests if data can be assumed to have zero skewness and excess kurtosis
- $JB = n\left(S^2/6 + (K-3)^2/24\right)$
- The test-statistic converges to a $\chi^2(2)$-distribution.
- The test (and many other) are available in R and Matlab

More advanced modelling

- The analysis done so far has dealt with *unconditional* modelling of the relevant variable
- Sometimes this is also referred to as a marginal distribution, i.e., the distribution of a future value given no knowledge about the past
- It would be relevant to include information about past returns in order to capture auto-correlation or volatility ($\sim$ standard deviation) clustering
- If you want to explore this further, Financial Products, Financial Risk Management, Time Series Analysis and others are relevant courses

Introduction to Financial Engineering

Week 45: Capital Asset Pricing Model

Nina Lange

Management Science, DTU

Week 45

1 Choosing portfolios
- Efficient frontier
- Two-fund separation
- Illustration
- Tangent portfolio
- Market portfolio

2 CAPM

Mean-Variance Analysis

- Risk is quantified in terms of variance or standard deviation
- Not the only way, but widely used and fairly simple
- There is a trade-off between return and risk: Investors prefer to have higher returns for the same risk or lower risk for the same return
- In the absence of a risk-free asset, this leads to combinations of risk and return that takes a hyperbolic form (in a (standard deviation, mean)-space)

Two-fund seperation

- The (upper half) of the efficient frontier is where the investor wants to be
- It is where he gets the most return for the risk he's willing to take
- Or equivalently, it is where he gets the lowest risk for the return he wants
- All portfolios on the efficient frontier can be obtained as a weighted average of two other portfolios on the efficient frontier
- This can be proved using the expression of portfolio weights or verified numerically (see today's exercise 1)

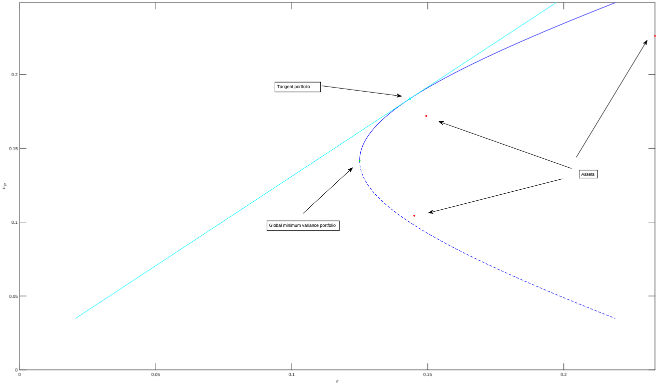
Adding a risk free asset

- Adding a risk-free asset expands the feasible set of portfolios
- The choice of optimal portfolios become a straight line in the (standard deviation, mean)-space
- The two-fund separation theorem now tells us that we only need to identify one other efficient portfolio to get all optimal portfolios
- For instance; the portfolio with all wealth invested in risky assets, i.e., the tangent portfolio
- All efficient portfolios can now be expressed as

$$\mu_P = \mu_0 + \frac{\mu_{tan} - \mu_0}{\sigma_{tan}} \sigma_P$$

- This can be obtained by investing $\frac{\mu_P - \mu_{tan}}{\mu_0 - \mu_{tan}}$ in the risk free asset and the rest in the tangent portfolio

Capital Market Line and Tangent Portfolio



Properties of the tangent portfolio

- The tangent portfolio is relevant to characterise optimal portfolios
- The ratio of excess return for an asset to its' covariance with the tangent portfolio is constant:

$$\frac{\mu_i - \mu_0}{cov(r_i, r_{tan})} = \text{constant}$$

- Intuition: This is reward to marginal variance ratio. If they are not constant, then it's better to put more money in the asset with a high ratio and less money in the asset with a low ratio. This will increase the expected return, while lowering the variance. See numerical example 5.4
- Using the expression for the tangent portfolio derived earlier, it can also be proved that the above ratio is constant and equal to $\mathbf{1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^e$

How expected return relates to the tangent portfolio

- The constant ratio condition specifically holds for the tangent portfolio itself:

$$\frac{\mu_i - \mu_0}{cov(r_i, r_{tan})} = \frac{\mu_{tan} - \mu_0}{cov(r_{tan}, r_{tan})}$$

- We can isolate μ_i to obtain:

$$\mu_i - \mu_0 = \frac{cov(r_i, r_{tan})}{var(r_{tan})}(\mu_{tan} - \mu_0)$$

- We denote $\frac{cov(r_i, r_{tan})}{var(r_{tan})}$ as the β of the asset. It is usually obtained as the slope of a regression of asset returns against the tangent portfolio
- β measures the covariance of an asset with the tangent portfolio (scaled with a constant)

From the tangent portfolio to the market portfolio

- Theoretically (and in practice for a few assets) it is possible to find the tangent portfolio
- But in practice it is complex and inaccurate (and backward-looking)
- With a few sound theoretical assumptions, we can derive the tangency portfolio and thereby obtain a link between risk and expected return
- This approach is called the Capital Asset Pricing Model, CAPM
- Besides the assumptions from the mean-variance theory on frictionless markets and that investors only care about mean and variance, we assume that all investors have the same beliefs and information
- CAPM tells us that the tangent portfolio must be the market portfolio

What is the market portfolio

- The market portfolio is the portfolio, where the weights on each asset is the market value of that asset divided by the entire market value of risky assets
- Example 5.7:

Stock	Stock price	No of shares	Market value	w
HP	33	2 billion	66 billion	$66/268 = 0.25$
IBM	95	1.758 billion	167 billion	$167/268 = 0.62$
CPQ	20.25	1.7 billion	35 billion	$35/268 = 0.13$

- In practice, this would be an index like S&P 500 or similar

Why is the tangency portfolio the market portfolio

- Assume there are two investors in the market from the previous slide, Jack and Jill
- Mean-variance analysis states that both Jack and Jill will invest a proportion of their wealth in the tangent portfolio and the rest in the risk-free asset.
- Since they agree on the tangency portfolio, there must be a positive holding of each stock in the tangency portfolio
- Since they agree on the tangency portfolio, they must hold a portfolio where assets have the same relative weight
- This can only be obtained if the tangency portfolio is the same as the market portfolio
- This can be generalised to many assets and many investors

CAPM

- CAPM tells us that the tangency portfolio is the market portfolio
- The market portfolio can be proxied by a index like the S&P 500
- The relation between expected return of a portfolio/asset and risk is

$$\mu_P - \mu_0 = \beta_P(\mu_M - \mu_0)$$

  where β_P is obtained using the market portfolio
- The β of a stock is the covariance with the market portfolio scaled with the variance of the market portfolio
- The β of a portfolio is the portfolio-weighted average of the βs of individual assets (covariance is linear)

Some words on β

- CAPM tells us that we don't need to find the tangency portfolio
- The market portfolio can be used instead
- The market portfolio is practically impossible to obtain, but we use proxies like S&P 500
- After choosing the proxy, we need to estimate βs, risk-free return and market risk premium (or market expected return) to implement the CAPM
- βs are only estimated - they are not true values of β
- There are ways to get "better" βs

Testing CAPM empirically

- Applying CAPM requires a market proxy
- If we are testing the relationship

$$\mu - \mu_0 = \beta(\mu_M - \mu_0)$$

  to see if there is a linear relationship between returns and the stocks' βs, we can't distinguish between testing if CAPM is correct or if the proxy is in fact the tangency portfolio
- If data supports CAPM, a plot of β vs returns should give a straight line with the market risk premium as slope and the risk free rate as intercept
- If we further regress on other characteristics such as size, momentum etc., there should be no explanatory power if CAPM is true
- This is mostly not the case $\rightarrow$ many suggestions for altering the CAPM model and/or criticising the assumptions

Introduction to Financial Engineering

Week 46: More on CAPM and Factor Models

Nina Lange

Management Science, DTU

Week 46

The Capital Asset Pricing Model

- If all investors are mean-variance efficient in a frictionless market and they all have the same expectations, everyone would place themselves on the Capital Market Line
- The Capital Market Line contains all portfolios that are a combination of the risk-free asset and the tangent portfolio
- The tangent portfolio is the combination of risky assets with the highest Sharpe ratio
- If all investors make the optimal choice, then the market portfolio is the tangent portfolio
- In practice, a broad market index is used as a proxy, for instance the S&P500 index
- The Capital Market Line contains all efficient portfolios

The Capital Market Line

- Every investor places himself on the CML, for instance:
  - Given a certain tolerable risk σ_R, the proportion σ_R/σ_M should be invested in the market portfolio and the rest in the risk-free asset.
  - Given a required excess return $\mu_R - \mu_f$, the proportion $(\mu_R - \mu_f)/(\mu_M - \mu_f)$ should be invested in the market portfolio and the rest in the risk-free asset.
  - Or if one additionally makes assumptions about the distribution of returns, a portfolio can be picked such that a loss is not exceeded with a certain probability

Note: Notation from book, chapter 18.

Expected return of an asset

- All assets in the market portfolios (because it's the tangent portfolio) has the same proportion of excess return to covariance of asset return and market portfolio return

- We can therefore obtain the following expression for the excess expected return of any asset:

$$\mu_j - \mu_f = \underbrace{\frac{cov(r_j, r_M)}{var(r_M)}}_{\beta_j}(\mu_M - \mu_f)$$

- β_j measures the covariance of an asset with the market portfolio

- Plotting β against expected returns gives the Securities Market Line, a straight line in the (β, expected return)-space

β

- β can be obtained by sample covariance divided by sample variance
- Or it can be obtained as the slope, when regressing asset returns on market returns
- If CAPM is supported, then the expected asset return should be a linear function of β
- High βs lead to high excess returns, low βs to low excess returns
- There are several ways of putting labels to β. The book denotes stocks with $\beta > 1$ as "aggressive" and $\beta < 1$ "not agressive"
- We will talk much more about β next time

Model for returns

- Consider the model for the return of asset j at time t:

$$R_{j,t} = \mu_{f,t} + \beta_j(R_{M,t} - \mu_{f,t}) + \epsilon_{j,t},$$

  where $\epsilon_{j,t}$ are $N(0, \sigma^2_{\epsilon,j})$. Further they are uncorrelated across assets.
- How does this match with CAPM?
- The above model implies
  - Expected return of asset is ...
  - Variance of asset is ...
- CAPM says there is no reward for anything except taking market risk

Diversification

- The risk of asset j has two components: $\beta_j^2\sigma_M^2$ and $\sigma_{\epsilon,j}^2$
- The first is called market risk or systematic risk
- The second is called unique, non-market or unsystematic risk
- It's not possible to get rid of the market risk by investing in more assets – they are all linked to the market
- But the second component can be reduced by diversification

Constructing portfolios

Consider the portfolio with weights w_j in asset j.

$$R_{P,t} = \sum_{j=1}^{N} w_j R_{j,t}$$

$$\mu_{P,t} = \sum_{j=1}^{N} w_j(\mu_f + \beta_j(\mu_M - \mu_f)) = \mu_f + \beta_P(\mu_M - \mu_f)$$

$$\sigma_P^2 = (w^T \beta)^2 \sigma_M^2 + \sum_{j=1}^{N} w_j^2 \sigma_{\epsilon,j}^2$$

When $N \to \infty$, the last term diminishes and the standard deviation of the portfolio approaches $\sigma_P = \sqrt{(\sum_{j=1}^{N} w_j \beta_j)^2} \sigma_M = \beta_P \sigma_M$.

Since the residual risk can be eliminated by holding a large portfolio, β_i is often used as the measure for the risk of asset i.

Testing CAPM empirically

- Applying CAPM requires a market proxy
- If we are testing the relationship

$$\mu_j - \mu_f = \beta_j(\mu_M - \mu_f)$$

  to see if there is a linear relationship between risk and stock βs, we can't distinguish between testing if CAPM is correct or if the proxy is in fact the tangency portfolio

- If data supports CAPM, a plot of β vs returns should give a straight line with the market risk premium as slope and the risk free rate as intercept
- This is mostly not the case $\rightarrow$ many suggestions for altering the CAPM model and/or criticising the assumptions

Dimension reduction

- High-dimensional data is often a challenge to handle
- It can be time-consuming and/or computationally difficult
- A standard approach to simplifying the analysis is factor models
  - Principal components analysis (PCA), where "statistical" factors are used to explain variations in data
  - Factor models, where (risk) factors with a certain economic meaning are used to explain variations in data
  - We will look at the latter type – in fact we already did, as CAPM is a factor model

Model for returns

- A factor model explains the excess return as as

$$R_j, t = \beta_{0,j} + \beta_{1,j} F_{1,t} + \cdots + \beta_{p,j} F_{p,t} + \epsilon_{j,t}$$

  where
  - $R_{j,t}$ is the excess return on asset j at time t
  - $F_{p,t}$ is the pth factor representing the economy and/or the financial markets at time t
  - $\beta_{p,j}$ is sensitivity of asset j to factor j and are often referred to as factor loadings t
  - $\epsilon_{j,t}$ is the asset specific risk, which are assumed independent across stocks and with zero mean. Further they are uncorrelated with the factors!
- How can CAPM be specified using the above expression?

Fama-French Three-Factor model

- The so-called Fama-French Three-Factor model is an example of a fundamental model
- It uses observable characteristics of assets as input
- More specifically it uses:
  - Excess return on the market portfolio
  - SMB: The difference in returns on a portfolio of small stocks and a portfolio of large stocks. Small and big refers to market value of the stock.
  - HML: The difference in returns on a portfolio of stocks with a high book-to-market value and a portfolio with low book-to-market value. Book value means the net worth of the company according to its balance sheet.

Interpretation of factor loadings

- The Fama-French Three-Factor model is stated:

$$R_j, t - \mu_{f,t} = \beta_{0,j} + \beta_{1,j}(F_{M,t} - \mu_{f,t}) + \beta_{2,j} SML_t + \beta_{3,j} HML_t + \epsilon_{j,t}$$

- The coefficient $\beta_{2,j}$ is high if asset j is a small stock or behaves like a small stock
- The coefficient $\beta_{3,j}$ is positive high if asset j is considered a value stock and negative if asset j is considered a growth stock
  - Stocks with a high book-to-market ratio (or correspondingly low market-to-book value) are called value stocks
  - Stocks with a low book-to-market ratio (or correspondingly high market-to-book value) are called growth stocks because investors are willing to pay a premium expecting future growth

Obtaining covariances

- A factor model for n assets can be written as

$$R_t = \beta_0 + \beta^T F_t + \epsilon_t$$

  where
  - R_t is a $n \times 1$ vector of excess returns at time t
  - β_0 is a $n \times 1$ vector of intercepts
  - β is a $p \times n$ matrix of factor loadings
  - F_t is a $p \times 1$ vector of factors at time t
  - ϵ_t is a $n \times 1$ vector of asset specific risks which are i.i.d. with zero mean and diagonal covariance matrix Σ_ϵ

Obtaining covariances

- Given the factor model:

$$R_t = \beta_0 + \beta^T F_t + \epsilon_t$$

- The mean of the excess returns is

$$\mu^e = \beta_0 + \beta^T E(F_t)$$

- The variance-covariance of the excess returns is

$$\Sigma = \beta^T \Sigma_F \beta + \Sigma_\epsilon,$$

where Σ_F is the covariance matrix of the factors

- Question: How many parameters needs to be estimated here compared to using the sample covariance matrix?

Cross-sectional Factor Models

- In CAPM, FF and generally factor modesl, time series of a single asset is used to estimate the loadings
- It does not consider characteristics of the stock itself – like book-to-market value of the specific company
- An alternative is to look at a many assets at a single time point and several characteristics to explain the return of asset j
- Characteristics could be industries, book-to-market, dividend yield, size