Duration and convexity of bonds

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Abstract

This note is meant to replace/supplement Section 3.8 in Ruppert & Matteson. Everything is formulated in discrete time for the purpose of this course. The continuous-time notation as seen in the textbook might be relevant for you in other courses.

1 Quantifying interest rate risk

Assume that the term structure of interest rates is flat at r and the payment stream of a bond is denoted $(c_1, c_2, \ldots, c_T)^1$. In the case of a coupon bearing bond with annual coupons C, $c_T = C + PAR$ and $c_t = C$ for $t < T$. The present value of a bond is

$$P(r) = \sum_{t=1}^{T} \frac{c_t}{(1+r)^t} \tag{1}$$

1.1 Macaulay Duration

For a payment stream $(c_1, c_2, \ldots, c_T)$, the Macaulay duration D^{MAC} is given by

$$\begin{aligned} D^{MAC}(r) &= \frac{1}{P(r)} \sum_{t=1}^{T} t \frac{c_t}{(1+r)^t} & (2) \\ &= \sum_{t=1}^{T} t \frac{\frac{c_t}{(1+r)^t}}{P(r)} & (3) \end{aligned}$$

$\frac{\frac{c_t}{(1+r)^t}}{P(r)}$ is the fraction that is repaid at time t (measured in present value terms). This expression explains the term duration – it explains the weighted

[1] To ease the notation, the subscript refers to a payment in t years, where t is an integer number of years. There is nothing stopping us from making calculations for a bond, where payments occur in non-integer number of years. In fact, this would almost always be the case.

average time before a bondholder gets the bond's cash flows. Example: A zero-coupon bond has Macaulay duration T. The bond holder has to wait until maturity to get the cash flow from the bond.

1.2 Modified Duration

The modified duration is the (negative) of the percentage change in price with respect to the change in interest rates, i.e., $D^{MOD} = -P'(r)/P(r)$. As there is a negative relationship between price and interest rate, the negative sign in the definition results in the modified duration being positive by definition. The derivative of the price with respect to the change in interest rates is derived as follows:

$$P'(r) = \sum_{t=1}^{T} -t \frac{c_t}{(1+r)^{t+1}} \tag{4}$$

$$= -\frac{1}{(1+r)} \sum_{t=1}^{T} t \frac{c_t}{(1+r)^t} \tag{5}$$

From the above equation, it can be seen why term modified duration arise. The modified duration is simply the Macauley duration divided with one plus the interest rate: $D^{MOD} = D^{MAC}/(1+r)$. Or calculated from cash flows and timing:

$$D^{MOD}(r) = \frac{1}{P(r)} \sum_{t=1}^{T} t \frac{c_t}{(1+r)^{t+1}} \tag{6}$$

1.3 Taylor expanding the price function around current interest rate

Assuming that the yield changes from r_0 to r. The relative effect on the price can be approximated by a first order Taylor approximation:

$$\frac{P(r) - P(r_0)}{P(r)} \approx \frac{P'(r_0)}{P(r)}(r - r_0) \tag{7}$$

$$= -D^{MOD}(r - r_0) \tag{8}$$

The first order approximation can be improved by adding a second term. The effect on the price can be approximated by:

$$\frac{P(r) - P(r_0)}{P(r)} \approx \frac{P'(r_0)}{P(r)}(r - r_0) + \frac{1}{2}\frac{P''(r_0)}{P(r)}(r - r_0)^2 \tag{9}$$

$$= -D^{MOD}(r - r_0) + \frac{1}{2}CON(r - r_0)^2 \tag{10}$$

where CON is defined as the second order derivative of the price with respect to the interest rate relative to the price of the bond $CON = P''(r)/P(r)$. $P''(r)$ is calculated as

$$P''(r) = \frac{d}{dr}\left(-\sum_{t=1}^{T} t\frac{c_t}{(1+r)^{t+1}}\right) \quad (11)$$

$$= \left(\sum_{t=1}^{T} \frac{t(t+1)c_t}{(1+r)^{t+2}}\right) \quad (12)$$

1.4 Relaxing the assumption of flat interest rate

The above calculations all assumed a flat term structure of interest rates. If the yield of the bond is used to calculate duration and convexity, then the interest rate curve is flat by construction. In reality a non-flat term structure of interest rates are seen and the formulas above should therefore be adjusted. For instance, the Fisher-Weil duration measures the effect of a parallel shift in the zero coupon yield curve (obtained by for instance the Nelson-Siegel model):

$$D^{FW}(r) = \frac{1}{P(r)}\sum_{t=1}^{T} t\frac{c_t}{(1+r_t)^{t+1}} \quad (13)$$

Also the convexity can be calculated using a general term structure of interest rates by replacing r with r_t.