CHAPTER 5

Mean-Variance Analysis and the Capital Asset Pricing Model

Learning Objectives

After reading this chapter, you should be able to:

1. Understand the importance of the mean-standard deviation diagram and know how to locate within the diagram the efficient frontier of risky assets, the capital market line, the minimum variance portfolio, and the tangency portfolio.
2. Compute and use both the tangency portfolio and the efficient frontier of risky assets.
3. Understand the linkage between mean-variance efficiency and risk-expected return equations.
4. Describe how to compute the beta of a portfolio given the betas of individual assets in the portfolio and their respective portfolio weights.
5. Comprehend what the market portfolio is, what assumptions are needed for the market portfolio to be the tangency portfolio—that is, for the Capital Asset Pricing Model (CAPM) to hold—and the empirical evidence on the CAPM.

Quantal International, an investment advisory firm, provides estimates of the covariances between Dow Jones stock indexes in the United States, Japan, the United Kingdom, and Canada. The model also estimates the mean returns of the four stock indexes. In November 2000, the model suggested that the covariances between the stock indexes for the four countries were as follows:

	Covariance with		
	Japan	United Kingdom	Canada
United States	*.0007*	*.0103*	*.0259*
Japan		*.0041*	*.0082*
United Kingdom			*.0180*

The annualized means and standard deviations for the stock index returns associated with the four countries were estimated as follows:

	U.S.	Japan	United Kingdom	Canada
Mean	*13.2%*	*8.8%*	*10.9%*	*14.7%*
Standard deviation	*17.3*	*18.0*	*16.4*	*27.2*

Based on this model and these data, many Quantal clients manage international portfolios.

When most investors think about quantifying the risk of their portfolios, they think about the variance or standard deviation of their portfolio's return. While variance is not the only way to quantify risk, it is the most widely used measure of it. This chapter analyzes the portfolio selection problem of an investor who uses variance as the sole measure of a portfolio's risk. In other words, the investor wishes to select a portfolio that has the maximum expected return given the variance of its future returns. To do this, he or she must understand the trade-off between mean and variance.

Chapter 4 introduced the analysis of this trade-off, known as "mean-variance analysis." To analyze problems like those in the opening vignette, however, one needs to develop mean-variance analysis in more depth. For example, based on the data in the vignette, one can show that a portfolio 67 percent invested in U.S. stocks, 13 percent in U.K. stocks, and 20 percent in Canadian stocks could earn the same mean return as a portfolio 100 percent invested in the U.S. stocks. At the same time, this multinational stock portfolio would have a standard deviation of about 16.3 percent per year compared with the U.S. stock portfolio's standard deviation of 17.3 percent per year. Mastery of certain portions of this chapter is a requirement for understanding how to come up with superior portfolio weights in situations like this.

As one of the cornerstones of financial theory, mean-variance analysis is significant enough to have been mentioned in the award of two Nobel Prizes in economics: to James Tobin in 1981 and Harry Markowitz in 1990. While an important tool in its own right, mean-variance analysis also indirectly generated a third Nobel Prize for William Sharpe in 1990 for his development of the Capital Asset Pricing Model (CAPM), a model of the relation of risk to expected return.[1] This chapter will examine this model, which follows directly from mean-variance analysis.

The chapter is organized into three major parts. After a brief introduction to applications of mean-variance analysis and the CAPM in use today, the first part focuses on the trade-off between mean and variance and uses the tools developed in this chapter to design optimal portfolios. The second part looks at the risk-expected return relation derived from mean-variance analysis, focusing on the CAPM as a special case. The last part examines how to implement the CAPM and analyzes the empirical evidence about the CAPM.

[1] Sharpe (1964) shares credit for the CAPM with Lintner (1965).

5.1 Applications of Mean-Variance Analysis and the CAPM in Use Today

Mean-variance analysis and the Capital Asset Pricing Model (CAPM) have practical applications for both professional investors and individuals working in corporate finance.

Investment Applications of Mean-Variance Analysis and the CAPM

As tools for illustrating how to achieve higher average returns with lower risk, mean-variance analysis and the CAPM are routinely applied by brokers, pension fund managers, and consultants when formulating investment strategies and giving financial advice. For example, mean-variance analysis is widely used in making decisions about the allocation of assets across industries, countries, and asset classes, such as bonds, stocks, cash, and real estate.

Corporate Applications of Mean-Variance Analysis and the CAPM

A firm grasp of mean-variance analysis and the CAPM also is becoming increasingly important for the corporate manager. In a world where managers of firms with declining stock prices are likely to lose their jobs in a takeover or restructuring, the need to understand the determinants of share value, and what actions to take to increase this value in response to the pressures of stockholders and directors, has never been greater.

For example, corporations can use mean-variance analysis to hedge their risks optimally and diversify their portfolios of real investment projects. However, one of the lessons of the CAPM is that while diversifying investments can reduce the variance of a firm's stock price, it does not reduce the firm's **cost of capital**, which is a weighted average of the expected rates of return required by the financial markets for a firm's debt and equity financing. As a result, a corporate diversification strategy can create value for a corporation only if the diversification increases the expected returns of the real asset investments of the corporation.[2]

Corporations also use the CAPM and mean-variance analysis to evaluate their capital expenditure decisions. Financial managers use the insights of mean-variance analysis and the CAPM not only to derive important conclusions about how to value real assets, but also to understand how debt financing affects the risk and the required return of a share of stock.[3]

5.2 The Essentials of Mean-Variance Analysis

To illustrate how to use the mean-standard deviation diagram for investment decisions, it is necessary to first understand where all possible investments lie in the diagram. The first subsection discusses what the diagram implies about the feasible mean-standard deviation outcomes that can be achieved with portfolios. The second subsection analyzes the assumptions of mean-variance analysis and discusses which feasible mean-standard deviation outcomes are desirable.

[2]Chapters 21 and 22 provide further discussion of this application.
[3]Chapters 11 and 13 provide further discussion of this application.

The Feasible Set

The **feasible set** of the mean-standard deviation diagram—the blue shaded area in Exhibit 5.1 and its boundary—is the set of mean and standard deviation outcomes, plotted with mean return on the vertical axis and standard deviation on the horizontal axis,[4] that are achieved from all feasible portfolios.

To simplify exposition, Exhibit 5.1 assumes that the feasible set is formed from portfolios of only four stocks, the four points inside the hyperbolic-shaped boundary of the blue shaded area. This hyperbolic shape occurs whenever a risk-free security or portfolio is not available. This and the next few sections analyze the problem of optimal investment when a risk-free investment both is and is not available.

Chapter 4 noted that the mean and variance of the return of a portfolio are completely determined by three characteristics of each security in the portfolio:

- The mean return of each security, also known as the expected return.
- The variance of the return of each security.
- The covariances between the return of each security and the returns of other securities in the portfolio.

Hence, knowing means, variances, and covariances for a group of investments is all that is needed to obtain a figure like that found in Exhibit 5.1.

As seen in Exhibit 5.1, investors achieve higher means and lower variances by "moving to the northwest," or up and to the left, while staying within the feasible set. One of the goals of this chapter is to learn how to identify the weights of the portfolios on the upper-left or "northwest" boundary of this blue shaded area. These portfolios are known as **mean-variance efficient portfolios**.[5]

The Assumptions of Mean-Variance Analysis

The identification of the weights of the portfolios on the northwest boundary is useful only if investors prefer to be on the northwest boundary and if there are no frictions or impediments to forming such portfolios. Thus, it should not be surprising that the two assumptions of mean-variance analysis are as follows:

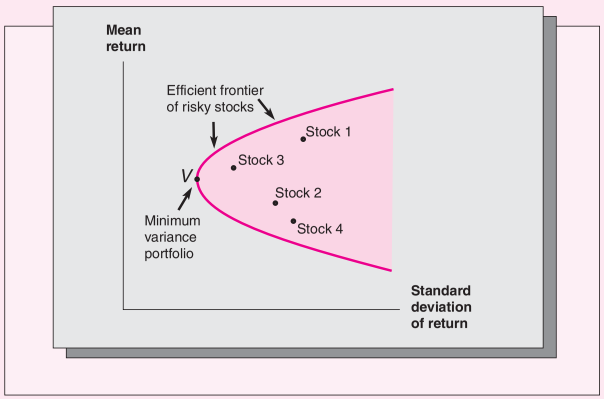
- In making investment decisions today, investors care only about the means and variances of the returns of their portfolios over a particular period (for example, the next week, month, or year). Their preference is for higher means and lower variances.
- Financial markets are *frictionless* (to be defined shortly).

These assumptions allow us to use the mean-standard deviation diagram to draw conclusions about which portfolios are better than others. As a tool in the study of optimal

[4]At one time, mean-variance analysis was conducted by studying a diagram where the axes were the mean and variance of the portfolio return. However, certain important portfolio combinations lie on a straight line in the mean-standard deviation diagram but on a curved line in the mean-variance diagram. Although the focus today is on the simpler mean-standard deviation analysis, the name "mean-variance analysis" has stuck with us.

[5]Because of the popularity of mean-variance analysis, a number of commercially available software packages include an option to find mean-variance efficient portfolios. An example is the domestic and international risk management and optimization system developed by Berkeley, CA-based Quantal International Inc. Details and examples of how such systems are used can be found at *http://www.quantal.com*.

EXHIBIT 5.1 The Feasible Set



investment, the diagram can help to rule out **dominated portfolios**, which are plotted as points in the diagram that lie below and to the right (that is, to the southeast) of some other feasible portfolio. These portfolios are dominated in the sense that other feasible portfolios have higher mean returns and lower return variances and thus are better.

The Assumption That Investors Care Only about the Means and Variances of Returns. The first assumption of mean-variance analysis—that investors care only about the mean and variance[6] of their portfolio return and prefer higher means and lower variances—is based on the notion that investors prefer portfolios that generate the greatest amount of wealth with the lowest risk. Mean-variance analysis assumes that the return risk, or uncertainty, that concerns investors can be summarized entirely by the return variance. Investors prefer a higher mean return because it implies that, on average, they will be wealthier. A lower variance is preferred because it implies that there will be less dispersion in the possible wealth outcomes. Investors are generally thought to be *risk averse;* that is, they dislike dispersion in their possible wealth outcomes.

Statisticians have shown that the variance fully summarizes the dispersion of any normally distributed return. The motivation behind the use of variance as the proper measure of dispersion for analyzing investment risk is the close relation between the observed distribution of many portfolio returns and the normal distribution.

[6] The standard deviation, the square root of the variance, can be substituted for "variance" in this discussion and vice versa.

The Assumption That Financial Markets Are Frictionless. The second assumption of mean-variance analysis—frictionless markets—is actually a collection of assumptions designed to simplify the computation of the feasible set. In **frictionless markets**, all investments are tradable at any price and in any quantity, both positive or negative (that is, there are no short sales restrictions). In addition, there are no transaction costs, regulations, or tax consequences of asset purchases or sales.

How Restrictive Are the Assumptions of Mean-Variance Analysis? Because both of these assumptions are strong, the simplicity gained from making them comes at some cost. For example, Fama (1976) noted that the returns of individual stocks are not distributed normally (although the returns of portfolios tend to be more normally distributed). Moreover, investors can generate returns that are distinctly nonnormal, for example, by buying index options or by using option-based portfolio insurance strategies.[7] Given two portfolio strategies with the same mean and variance, an investor who cares primarily about large losses might prefer the investment with the smallest maximum loss. The variance does not capture precisely the risk that these investors wish to avoid.

An additional objection to mean-variance analysis, which applies even if returns are distributed normally, is that investors do not view their portfolio's return in isolation, as the theory suggests. Most investors are concerned about how the pattern of their portfolio returns relates to the overall economy as well as to other factors affecting their well-being. Some investors, for example, might prefer a portfolio that tends to have a high return in the middle of a recession, when the added wealth may be needed, to an otherwise equivalent portfolio that tends to do well at the peak of a business cycle. The former investment would act as insurance against being laid off from work. Similarly, retirees living off the interest on their savings accounts might prefer an investment that does well when short-term interest rates decline.

Because it is a collection of assumptions, the frictionless markets assumption may or may not be critical, depending on which assumption one focuses on. Relaxing portions of this collection of assumptions often leads to basically the same results, but at the cost of much greater complexity. In other cases, relaxing some of these assumptions leads to different results. In most instances, however, the basic intuitive lessons from this chapter's relatively simple treatment remain the same: Portfolios dominate individual assets; covariances are more important than variances for risk-return equations; and optimal portfolios, in a mean-variance sense, can generally be found if one knows the inputs.

5.3 The Efficient Frontier and Two-Fund Separation

The top half of the boundary in Exhibit 5.1 is sometimes referred to as the efficient frontier of risky stocks. The **efficient frontier** represents the means and standard deviations of the mean-variance efficient portfolios. The efficient frontier is the most efficient trade-off between mean and variance. By contrast, an inefficient portfolio, such as a 100 percent investment in U.S. technology stocks, wastes risk by not maximizing the mean return for the risk it contains. A more efficient portfolio weighting scheme can earn a higher mean return and have the same variance (or, alternatively, the same mean and a lower variance).

[7] Options and portfolio insurance strategies are discussed in Chapter 8.

The Quest for the Holy Grail: Optimal Portfolios

Based on the assumptions of the last section, the efficient frontier is the "holy grail"—that is, the efficient frontier is where an investor wants to be. Of course, the efficient frontier contains many portfolios; which of these portfolios investors select depends on their personal trade-off between mean and variance. For example, the leftmost point of Exhibit 5.1's frontier is point *V*, which characterizes the mean and standard deviation of the minimum variance portfolio. This portfolio will attract only those investors who dislike variance so much that they are willing to forgo substantial mean return to minimize variance. Other investors, who are willing to experience higher variance in exchange for higher mean returns, will select portfolios on the efficient frontier that are above point *V* in Exhibit 5.1.

Chapter 4 noted that point *V*, the minimum variance portfolio, is a unique portfolio weighting that can be identified by solving a set of equations. In most instances, each mean standard-deviation point on the boundary is achieved with a *unique* portfolio of stocks. On the interior of the feasible set, however, many combinations of stocks can achieve a given mean-standard deviation outcome.

Because investors who treat variance as the sole measure of risk want to select mean-variance efficient portfolios, it is useful to learn how to construct them. The task of identifying these special portfolios is greatly simplified by learning about an important property known as "two-fund separation."

Two-Fund Separation

Two-fund separation means that it is possible to divide the returns of all mean-variance efficient portfolios into weighted averages of the returns of two portfolios. As one moves along the efficient frontier, the weights may change, but the two separating portfolios remain the same.

This insight follows from a slightly more general result:

Result 5.1 All portfolios on the mean-variance efficient frontier can be formed as a weighted average of any two portfolios (or funds) on the efficient frontier.

Result 5.1 can be generalized even further. Two funds generate not only the northwest boundary of efficient portfolios, but all of the portfolios on the boundary of the feasible set: northwest plus southwest (or lower left boundary). This implies that once any two funds on the boundary are identified, it is possible to create *all* other mean-variance efficient portfolios from these two funds!

Exhibit 5.2 highlights four boundary portfolios, denoted A, B, C, and D, and the minimum variance portfolio, *V*. All of the portfolios on the western (or left-hand) boundary of the feasible set are weighted averages of portfolios A and B, as well as averages of C and D, B and D, or B and *V*, and so on. Moreover, *any* weighted average of two boundary portfolios is itself on the boundary.

Example 5.1 provides an illustration of two-fund separation.

Example 5.1: Two-Fund Separation and Portfolio Weights on the Boundary

Consider a mean-standard deviation diagram constructed from five stocks. One of its boundary portfolios has the weights

$$x_1 = .2,\ x_2 = .3,\ x_3 = .1,\ x_4 = .1, \quad \text{and} \quad x_5 = .3$$

The other portfolio has equal weights of .2 on each of the five stocks. Determine the weights of the five stocks for all other boundary portfolios.

Answer: The remaining boundary portfolios are described by the weighted averages of the two portfolios (or funds). Portfolio weights (.2, .3, .1, .1, .3) describe the first fund. Weights (.2, .2, .2, .2, .2) describe the second fund. Thus, letting *w* denote the weight on the first fund, it is possible to define all boundary portfolios by portfolio weights x_1, x_2, x_3, x_4, and x_5 that satisfy the equations

$$\begin{aligned} x_1 &= .2w + .2(1 - w) \\ x_2 &= .3w + .2(1 - w) \\ x_3 &= .1w + .2(1 - w) \\ x_4 &= .1w + .2(1 - w) \\ x_5 &= .3w + .2(1 - w) \end{aligned}$$

For example, if the new boundary portfolio is equally weighted between the two funds ($w = .5$), its portfolio weights on stocks 1–5 are, respectively, .2, .25, .15, .15, and .25. For $w = -1.5$, the boundary portfolio weights on stocks 1–5 are, respectively, .2, .05, .35, .35, and .05.

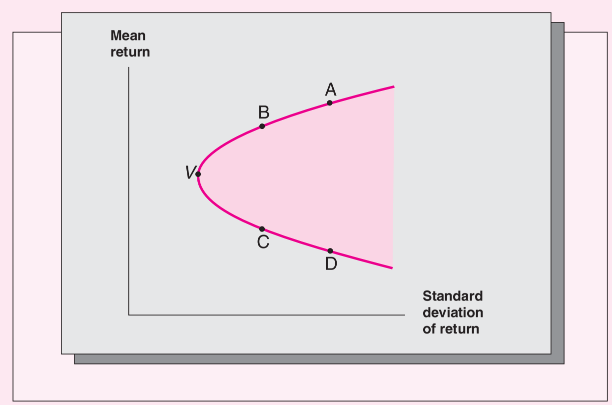
Example 5.2 finds a specific portfolio on the boundary generated in the last example.

Example 5.2: Identifying a Specific Boundary Portfolio

If the portfolio weight on stock 3 in Example 5.1 is −.1, what is the portfolio weight on stocks 1, 2, 4, and 5?

Answer: Solve for the *w* that makes $.1w + .2(1 - w) = -.1$. The answer is $w = 3$. Substituting this value into the other four portfolio weight equations in Example 5.1 yields a boundary portfolio with respective weights of .2, .5, −.1, −.1, and .5.

EXHIBIT 5.2 Two-Fund Separation and the Boundary of the Feasible Set



One insight gained from Examples 5.1 and 5.2 is that, whenever a stock has the same weight in two portfolios on the boundary, as stock 1 does in the last two examples, it must have the same weight in all portfolios on the boundary. More typically, as with other stocks in these examples, observe that:

- Some stocks have a portfolio weight that continually increases as w increases.
- Other stocks have a portfolio weight that continually decreases as w increases.

Although these insights help to characterize the boundary, they do not precisely identify the portfolios on the boundary. This chapter will address this topic after describing how a risk-free asset affects the analysis.

5.4 The Tangency Portfolio and Optimal Investment

So far, this chapter has studied how to invest optimally by looking at the portfolios formed only from risky stocks. Generally, whenever an additional asset is added to the set of investments that can be held in a portfolio, the feasible set of the mean-standard deviation diagram expands. The risk-free asset is no exception, but it is notable for the manner in which it changes the shape of the feasible set and the efficient frontier.

Chapter 4 indicated that portfolios of a risk-free investment and a risky investment lie on a straight line in the mean-standard deviation diagram. Because of this, the addition of a risk-free asset to the analysis of risky stocks not only greatly expands the feasible set, but also it changes the shape of the efficient frontier from a hyperbola to a straight line. This greatly reduces our search for the optimal portfolio.

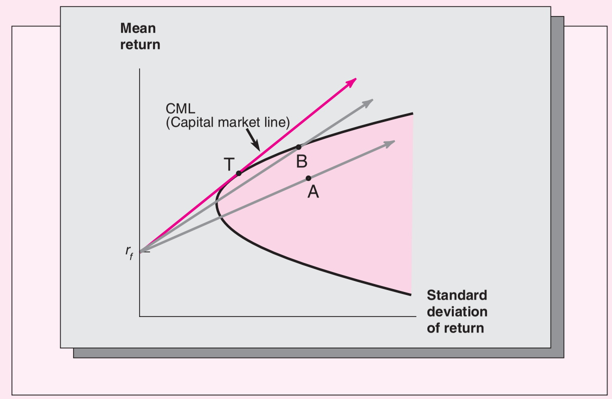
Indeed, as this chapter will show, when there is risk-free investment, only one "key" portfolio needs to be found because of the principle of two-fund separation (that is, the efficient frontier can be generated by only two portfolios). This was illustrated in the last section where, because the analysis precluded the existence of a risk-free investment, the two portfolios that generated the efficient frontier were necessarily risky portfolios; that is, they had positive variance. However, a risk-free investment, if one exists, will be the minimum variance investment, and thus must be on the efficient frontier. This greatly simplifies the problem of the optimal investment mix because we now need to be concerned with only one efficient risky portfolio. Because of two-fund separation, this efficient risky portfolio and the risk-free asset generate the entire boundary of the feasible set.

The analysis that follows illustrates how to derive the weights of this efficient risky portfolio. For reasons that will become clear shortly, it is appropriate to call this the **tangency portfolio**. This portfolio represents the unique optimal portfolio that contains no investment in the risk-free asset.

Optimal Investment When a Risk-Free Asset Exists

The blue shaded region and its black boundary in Exhibit 5.3 represents the feasible portfolios composed only of risky stocks. Consider three risky stock portfolios represented by points A, B, and T, and combine each of them separately with a risk-free investment. As we saw in the last chapter, such combinations generate a straight line. Point T identifies the tangency portfolio. The line connecting the risk-free return with point T is designated as the **capital market line**, or **CML**. As we discuss below, the capital market line represents the portfolios that optimally combine all investments.

Exhibit 5.3 Combining Risky Portfolios with a Risk-Free Portfolio



The Tangency Portfolio and the Capital Market Line. It should be clear that portfolio T is the best of the three stock portfolios since line CML, which connects T with the risk-free investment (at point r_f), lies above the other two lines. More generally, the line going through portfolio T is tangent to the efficient frontier of *risky* investments; as a result, no feasible portfolio lies northwest of this line. Investors want to invest in portfolios that have the best trade-off between mean and variance and such portfolios all lie on this line. Result 5.2 summarizes this important point.

Result 5.2 Under the assumptions of mean-variance analysis, and assuming the existence of a risk-free asset, all investors will select portfolios on the capital market line.

Result 5.2 states that the capital market line is the key to optimal investment for every investor interested in maximizing expected return for a given amount of variance risk. Investors who are extremely risk averse will select a portfolio close to the risk-free asset, achieving a low expected return but also a low variance for their future wealth. Investors who are only slightly risk averse will select a portfolio high up on the CML, possibly above T if they choose to sell short the risk-free asset. They would achieve a higher expected return than more risk-averse investors, but the larger variance also makes the *possibility* of realizing large losses quite high.

The Equation of the Capital Market Line. Line CML is represented by the equation:

$$\bar{R}_p = r_f + \frac{\bar{R}_T - r_f}{\sigma_T}\sigma_p \qquad (5.1)$$

where $\bar{R}_T$ and σ_T are, respectively, the mean and standard deviation of the tangency portfolio's return and r_f is the return of the risk-free asset. As the steepest sloped

EXHIBIT 5.4 Means, Standard Deviations, Risk Premiums, and Mean-Standard Deviation Slopes

Portfolio	*Mean Return*[a]	*Risk Premium*[a]	*Standard Deviation*[a]	*Slope*
S&P 500	13.3%	9.5%	20.1%	.47
Small cap stocks	17.6	13.8	33.6	.41
Long-term corporate bonds	5.9	2.1	8.7	.24
Long-term government bonds	5.5	1.7	9.3	.18

[a]Source: © Computed using data from *Stocks, Bonds, Bills & Inflation 2000 Yearbook*™, Ibbotson Associates, Chicago, IL. Used with permission. All rights reserved. Data are from 1926 to 1999. Risk premiums are based on the average T-bill return of 3.8 percent. Means are averages of annual returns. Standard deviations are sample standard deviations of annual returns.

line available from combining a risk-free investment with any risky investment, the CML plots the set of mean-variance efficient portfolios that can be achieved by combining the risky stocks with a risk-free investment. All of the portfolios above r_f on line CML—as weighted averages of the weights of portfolio T and the risk-free asset—have the same relative proportions invested in any two risky stocks as portfolio T. For example, if the ratio of portfolio T's weight on Hewlett Packard (HP) to its weight on IBM is 1 to 6, all portfolios on line CML have relative weights on these two stocks in ratios of 1 to 6. The actual portfolio weights on HP and IBM, however, will scale down proportionately as one moves down line CML toward the risk-free asset.

Empirical Estimates of the Slope of the Capital Market Line. The slope of the capital market line, $(\bar{R}_T - r_f)/\sigma_T$, is a ratio that measures the trade-off between risk and return. Larger ratios imply that the financial markets offer greater expected return improvements for given increases in risk.

The historical returns of some popular investment portfolios help to characterize the numerator and denominator of the ratio. The numerator of the ratio, $\bar{R}_T - r_f$, is often referred to as a "risk premium."[8] Using an average T-bill return of 3.8 percent as the risk-free rate, Exhibit 5.4 notes that the risk premium of the S&P 500 has been about 9.5 percent per year while the standard deviation has been about 20.1 percent per year, generating a ratio of .47. For a portfolio of small-capitalization stocks, that is, small companies, the ratio is .41; for corporate bonds, the ratio is .24. Some portfolios of these investments have even steeper slopes. Hence, the capital market line, which is based on the optimal combination of all investments, should have a ratio greater than .47. In other words, for investments on the capital market line, an increase in standard deviation from 10 percent to 20 percent per year will generate more than a 4.7 percent per year increase in expected return.

An interesting feature of Exhibit 5.4 is that the investment portfolios with the larger mean returns generally have the larger standard deviations. This is not true, however, for all investment portfolios. For example, the portfolio of government bonds has a lower mean return and a higher standard deviation than the portfolio of

[8]The **risk premium** of a stock or portfolio is its expected return less the risk-free return.

corporate bonds. In general, one learns from the mean-standard deviation diagram that there is no necessary relation between mean and standard deviation. Portfolios with the same standard deviation in the diagram can have very different means. As a result, it pays to identify efficient portfolios, which maximize mean return for a given standard deviation.

If standard deviations have no relation to mean returns, what then determines mean returns? Here, our study of the tangency portfolio serves double duty because not only is this portfolio useful from the perspective of identifying proper investment weights, but also it is useful for addressing what determines mean returns, as will be emphasized later in this chapter.

Identification of the Tangency Portfolio

Because the tangency portfolio is generally a unique combination of individual stocks and is the key to identifying the other portfolios on the capital market line, determining the weights of the tangency portfolio is an important and useful exercise.

The Algebraic Formula for Finding the Tangency Portfolio. For all investments—efficient ones, such as the portfolios on the capital market line, and the dominated investments—the following result applies.

Result 5.3 The ratio of the risk premium of every stock and portfolio to its covariance with the tangency portfolio is constant; that is, denoting the return of the tangency portfolio as $\tilde{R}_T$

$$\frac{\bar{r}_i - r_f}{cov(\tilde{r}_i, \tilde{R}_T)}$$

is identical for all stocks.

Result 5.3 suggests an algebraic procedure for finding the tangency portfolio which is similar to the technique used to find the minimum variance portfolio. Recall that to find the minimum variance portfolio, it is necessary to find the portfolio that has equal covariances with every stock. To identify the tangency portfolio, find the portfolio that has a covariance with each stock that is a constant proportion of the stock's risk premium. This proportion, while unknown in advance, is the same across stocks and is whatever proportion makes the portfolio weights sum to 1. This suggests that to derive the portfolio weights of the tangency portfolio:

1. Find "weights" (they do not need to sum to 1) that make the covariance between the return of each stock and the return of the portfolio constructed from these weights equal to the stock's risk premium.
2. Then rescale the weights to sum to 1 to obtain the tangency portfolio.

A Numerical Example Illustrating How to Apply the Algebraic Formula. Example 5.3 illustrates how to use Result 5.3 to compute the tangency portfolio's weights.

Example 5.3: Identifying the Tangency Portfolio

Amalgamate Bottlers, from Example 4.19, wants to find the tangency portfolio of capital investments from franchising in three less developed countries (LDCs). Recall that covariances between franchising operations in India (investment 1), Russia (investment 2), and China (investment 3) were:

	Covariance with		
	India	*Russia*	*China*
India	.002	.001	0
Russia	.001	.002	.001
China	0	.001	.002

Find the tangency portfolio for the three investments when they have expected returns of 15 percent, 17 percent, and 17 percent, respectively, and the risk-free return is 6 percent per year.

Answer: Step 1: Solve for the portfolio "weights" that make the portfolio's covariance with each stock equal to their risk premiums. These weights are not true weights because they do not necessarily sum to 1. The first step is the simultaneous solution of the three equations:

$$.002x_1 + .001x_2 + 0x_3 = .15 - .06$$
$$.001x_1 + .002x_2 + .001x_3 = .17 - .06$$
$$0x_1 + .001x_2 + .002x_3 = .17 - .06$$

The left-hand side of the first equation is the covariance of a portfolio with weights x_1, x_2, and x_3 with stock 1. Thus, the first equation states that this covariance must equal .09. The other two equations make analogous statements about covariances with stocks 2 and 3.

Using the substitution method, the first and third equations can be rewritten to read:

$$x_1 = 45 - \frac{x_2}{2}$$

$$x_3 = 55 - \frac{x_2}{2}$$

Upon substitution into the second equation, they yield:

$$.001x_2 = .01, \text{ or } x_2 = 10$$

Substituting this value for x_2 into the remaining two equations implies:

$$x_1 = 40 \quad x_3 = 50$$

Step 2: Rescale the portfolio weights so that they add to 1. After rescaling, the solution for the weights of the tangency portfolio is:

$$x_1 = .4 \quad x_2 = .1 \quad x_3 = .5$$

The Intuition for the Algebraic Formula. Why is the ratio of the risk premium to the covariance so relevant? Consider the case where the ratio of a stock's risk premium to its covariance with the candidate tangency portfolio differs from stock to stock. In this case, it is possible to alter the weights of the portfolio slightly to increase its mean return while lowering its variance. This can be done by slightly increasing the weight on a stock that has a high ratio of risk premium to marginal variance, while slightly lowering the weight on a stock with a low ratio and altering the weight on the risk-free asset so that the weights add up to 1. This action implies that the candidate tangency portfolio was not on the capital market line to begin with.

A Numerical Illustration of How to Generate a Mean-Variance Improvement. Example 5.4 demonstrates how to achieve this mean-variance improvement by taking a portfolio for which the ratio condition in Result 5.3 is violated and constructing a new portfolio that is mean-variance superior to it.

Example 5.4: Developing a Superior Portfolio When Risk Premiums Are Not Proportional to Covariances

The return of ACME Corporation stock has a covariance with Henry's portfolio of .001 per year and a mean return of 20 percent per year, while ACYOU Corporation stock has a return covariance of .002 with the same portfolio and a mean return of 40 percent per year. The risk-free rate is 10 percent per year. Prove that Henry has not chosen the tangency portfolio.

Answer: To prove this, construct a **self-financing** (that is, zero-cost) investment of ACME, ACYOU, and the risk-free asset that has a negative marginal variance and a positive marginal mean. Adding this self-financing investment to Henry's portfolio generates a new portfolio with a higher expected return and lower variance. Letting the variable m represent a small number per dollar invested in Henry's portfolio, this self-financing investment is long \$.99m in the ACYOU Corporation, short \$1.99m in the ACME corporation, and long \$$m$ in the risk-free asset. When added to Henry's portfolio, this self-financing investment increases the expected return by

$$.99m(40\%) - 1.99m(20\%) + m(10\%) = 9.8m\% \text{ per year}$$

However, if m is sufficiently small, the addition of this portfolio to the existing portfolio reduces return variance because the covariance of the self-financing portfolio of the three assets with Henry's portfolio is

$$.99m(.002) - 1.99m(.001) = -.00001m$$

A negative covariance means a negative marginal variance when the added portfolio is sufficiently small.

The risk premium-to-covariance ratio from Result 5.3 should be the same whether the ratio is measured for individual stocks, projects that involve investment in real assets, or portfolios. For example, using the tangency portfolio in place of stock i, the ratio of the tangency portfolio's risk premium to its covariance with itself (i.e., its variance) should equal the ratio in Result 5.3, or[9]

$$\frac{\bar{r}_i - r_f}{cov(\tilde{r}_i, \tilde{R}_T)} = \frac{\bar{R}_T - r_f}{var(\tilde{R}_T)} \qquad (5.2)$$

5.5 Finding the Efficient Frontier of Risky Assets

One can reasonably argue that no risk-free asset exists. While many default-free securities such as U.S. Treasury bills are available to investors, even a one-month U.S. T-bill fluctuates in value unpredictably from day to day. Thus, when the investment horizon is shorter than a month, this asset is definitely not "risk-free." In addition, foreign investors would not consider the U.S. Treasury bill a risk-free asset. An Italian investor, for example, views the certain dollar payoff at the maturity of the T-bill as risky because it must be translated into Italian lire at an uncertain exchange rate. Even to a U.S. investor with a one-month horizon, the purchasing power of an asset, not just its nominal value, is critical. Thus, the inflation-adjusted returns of Treasury bills are risky, even when calculated to maturity. Also, in many settings there may be no risk-free asset. For example, a variety of investment and corporate finance problems preclude investment in a risk-free asset. For these reasons, it is useful to learn how to compute all of the portfolios on the (hyperbolic-shaped) boundary of the feasible set of risky investments, detailed in Exhibit 5.1. This section uses the insights from Section 5.4 to find this boundary.

[9]Section 5.7 develops more intuition for equation (5.2) and (the equivalent) Result 5.3.

Because of two-fund separation, the identification of any two portfolios on the boundary is enough to construct the entire set of risky portfolios that minimize variance for a given mean return. Use the minimum variance portfolio of the risky assets as one of the two portfolios since computing its weights is so easy. For the other portfolio, note that (with one exception)[10] it is possible to draw a tangent line from every point on the vertical axis of the mean-standard deviation diagram to the hyperbolic boundary. We will refer to the point of tangency as the "hypothetical tangency portfolio." Hence:

1. Select any return that is less than the expected return of the minimum-variance portfolio.
2. Compute the hypothetical tangency portfolio by pretending that the return in step 1 is the risk-free return, even if a risk-free asset does not exist.
3. Take weighted averages of the minimum variance portfolio and the hypothetical tangency portfolio found in step 2 to generate the entire set of mean-variance efficient portfolios. The weight on the minimum variance portfolio must be less than 1 to be on the top half of the hyperbolic boundary.

Example 5.5 illustrates this three-step technique.

Example 5.5: Finding the Efficient Frontier When No Risk-Free Asset Exists

Find the portfolios on the efficient frontier constructed from investment in the three franchising projects in Example 5.3 (and Example 4.19).

Answer: Solve for the portfolio "weights" that make the portfolio's covariance with each stock equal to the stock's risk premium. (In this example, "risk premium" refers to the expected return less some hypothetical return that you select.) Then, rescale the weights so that they sum to one. If the hypothetical return is 6 percent, the weights (unscaled) are given by the simultaneous solution of the three equations

$$\begin{aligned} .002x_1 + .001x_2 + \quad 0x_3 &= .15 - .06 \\ .001x_1 + .002x_2 + .001x_3 &= .17 - .06 \\ 0x_1 + .001x_2 + .002x_3 &= .17 - .06 \end{aligned}$$

The solution to these equations, when rescaled, generate portfolio weights of

$$x_1 = .4 \quad x_2 = .1 \quad x_3 = .5$$

(Not surprisingly, with a hypothetical risk-free return in this example that is identical to the risk-free return in Example 5.3, the weights in the two examples, .4, .1, and .5, match. Alternatively, instead of subtracting .06 from the expected returns on the right-hand side of the first three equations, you could have subtracted other numbers (for example, .04 or zero). If .04 had been used in lieu of .06, the right-hand side of the first three equations would be .11, .13, and .13, respectively, instead of .09, .11, and .11. If zero had been used, the right-hand side would be .15, .17, and .17, respectively.)

For the other portfolio, use the minimum variance portfolio (which was computed in Example 4.19 to be)

$$x_1 = .5 \quad x_2 = 0 \quad x_3 = .5$$

Thus, the portfolios on the boundary of the feasible set are described by

$$\begin{aligned} x_1 &= .4w + .5(1 - w) \\ x_2 &= .1w \\ x_3 &= .5w + .5(1 - w) = .5 \end{aligned}$$

Those with $w > 0$ are on the top half of the boundary and are mean-variance efficient.

[10]The exception is at the expected return of the minimum variance portfolio, point *V* in Exhibit 5.2.

Since the financial markets contain numerous risky investments available to form a portfolio, finding the efficient frontier of risky investments in realistic settings is best left to a computer. Examples in this chapter, like the one above, which are simplified so that these calculations can be performed by hand, illustrate basic principles that you can apply to solve more realistic problems.[11]

5.6 How Useful Is Mean-Variance Analysis for Finding Efficient Portfolios?

One difficulty in employing mean-variance analysis to find mean-variance efficient portfolios is that means and covariances are generally unobservable. The real world requires that they be estimated. Since there is an incredibly large number of stocks and other investments to choose from, the full implementation of mean-variance analysis as a tool for portfolio management seems limited. First, the calculation of the necessary inputs seems to be almost a heroic undertaking, given that almost 10,000 stocks are traded in the U.S. market alone. Second, the estimated means and covariances will differ from the true means and covariances for virtually all of these securities.

The difficulties in applying mean-variance analysis to determine the efficient portfolios of *individual* stocks can be overcome with additional assumptions. These assumptions, when added to those of mean-variance analysis, enable the analyst to deduce the efficient portfolios rather than to compute them from historical covariances and historical means. (We will explore one theory based on additional assumptions, the CAPM, in this chapter.[12])

Moreover, these considerations are far less important for the applicability of mean-variance analysis to smaller types of problems, such as asset allocation across asset classes (that is, what fraction of the investor's wealth should be in bonds, stocks, cash, and so on), countries (that is, what fraction of wealth should be in Japan, Europe, the United States, etc.), or industries. These simpler problems are both more manageable from a computational standpoint and generally have estimated covariances and means closer to their true values because the fundamental "assets" in this case are broad-based portfolios rather than individual stocks.[13] These considerations also do not limit the use of mean-variance analysis for hedging in corporate finance.[14]

[11]Many software packages, including spreadsheets, can be used to obtain a numerical solution to this type of problem. The solution usually requires inverting the covariance matrix, an important step that the computer uses to solve systems of linear equations. Then sum "weighted" columns of the inverted covariance matrix, which is the same as taking weighted sums of the elements in each row of the matrix, where the "weight" on column i is the risk premium of investment i. Next, rescale the "weighted" sum of the columns, itself a column, so that its entries sum to 1. Entry j of the rescaled column is the weight on investment j in the tangency portfolio. In Microsoft Excel, the function MINVERSE inverts a matrix and the function MMULT multiplies the inverted matrix and the column of risk premia, which is the same as summing weighted columns.

[12]A second approach known as factor modeling, discussed in Chapter 6, is a statistical method for reducing the problem of estimating covariances to one of manageable size in order to derive insights about optimal portfolios. If the statistical assumptions correspond to reality, the covariance estimates obtained may be precise. Moreover, with this second approach, the mean returns necessary to find the optimal portfolios reduce to a problem of estimating the means of a few broad-based portfolios of large numbers of stocks. As suggested in our discussion of asset allocation, mean-variance analysis is more feasible in this case.

[13]Estimates of portfolio means and standard deviations are more accurate than those for individual stocks because random estimation errors across stocks tend to cancel one another in a portfolio.

[14]The practice of hedging is discussed in Chapter 22.

5.7 The Relation Between Risk and Expected Return

A secondary benefit of identifying a mean-variance efficient portfolio is that it generates an equation that relates the risk of an asset to its expected return. Knowing the relation between risk and expected return has a variety of applications; among these are the evaluation of the performance of professional fund managers, the determination of required rates of return in order to set fair rates for regulated utilities, and the valuation of corporate investment projects.

As an example of the last application, suppose that Dell Computer wants to expand by developing factories in the Far East. It has estimated the expected future cash flows from such an investment. To determine whether this expansion improves the firm's value, the expected future cash flows from the expansion need to be translated into a value in today's dollars. Dell can compute this by discounting the expected future cash flows at the rate of return required by the financial markets on investments of similar risk.[15]

A popular assumption in the estimation of this required rate of return is that the expansion project has the same required return as Dell's common stock. However, it would be foolish for Dell to estimate this required rate of return by taking the average of Dell's historical stock returns. Dell's stock appreciated in value by more than 200-fold in the ten years following its initial public offering in the late 1980s. It would be highly unusual for this incredible track record to be repeated. In other words, because of its remarkable performance in its first ten years as a publicly traded company, the average historical return of Dell's stock substantially exceeds the expected rate of return required by investors looking ahead.

Because stock returns have such high variances, Dell's problem is common to many corporations. If historical data provide unreliable estimates of the true expected rates of return of the stocks of individual corporations, how do these companies obtain such estimates? Fortunately, the difficulty in estimating expected returns is not shared by measures of return risk. Reasonably accurate estimates of return risk can be obtained if one knows the tangency portfolio. In this case, a theory that relates the variables that one can estimate well (the risk measures) to the variables that are problematic to estimate (the expected returns of individual companies) could be useful to companies like Dell. Later parts of this chapter estimate Dell's risk and expected return with such a theory.

Relevant Risk and the Tangency Portfolio

When a risk-free asset exists, the relation between the relevant risk of an investment and its expected return can be derived directly from equation (5.2). Specifically, equation (5.3) is obtained by moving the covariance of the investment with the tangency portfolio to the right-hand side of equation (5.2):

$$\bar{r} - r_f = \frac{cov(\tilde{r}, \tilde{R}_T)}{var(\tilde{R}_T)}(\bar{R}_T - r_f) \tag{5.3}$$

(For simplicity in notation, we have dropped the i subscript.)

Equation (5.3) describes the relation between the expected return of an investment and a measure of its risk. In this case, the relevant measure of risk is the covariance between the returns of the tangency portfolio and the investment.[16]

Example 5.6 illustrates how to apply equation (5.3).

[15] Discounting means dividing by a power of the sum: one plus a rate of return. It is formally defined in Chapter 9. Chapter 11 explains how to derive this rate of return in great detail.

[16] Section 5.8 discusses how this equation is altered by the absence of a risk-free asset in the economy.

Example 5.6: Implementing the Risk-Return Equation

The risk-free return is 8 percent. The return of General Motors stock has a covariance with the return of the tangency portfolio that is 50 percent larger than the corresponding covariance for Disney stock. The expected return of Disney stock is 12 percent per year. What is the expected return of General Motors stock?

Answer: The risk premium of Disney stock is 4 percent per year (the expected return minus the risk-free return: 12 percent less 8 percent). General Motors' risk premium must be 50 percent larger or 6 percent per year. Adding the risk-free return to this number yields 14 percent, the expected return of General Motors.

Betas

The first factor in the product on the right-hand of equation (5.3) is commonly referred to as **beta** and typically denoted by the Greek letter β; that is

$$\beta = \frac{cov(\tilde{r}, \tilde{R}_T)}{var(\tilde{R}_T)}$$

This notation is used because the right-hand of equation (5.3) also happens to be the formula for the slope coefficient in a regression, which commonly uses β to denote the slope. With this notation, equation (5.3) becomes:

$$\bar{r} - r_f = \beta(\bar{R}_T - r_f) \tag{5.4}$$

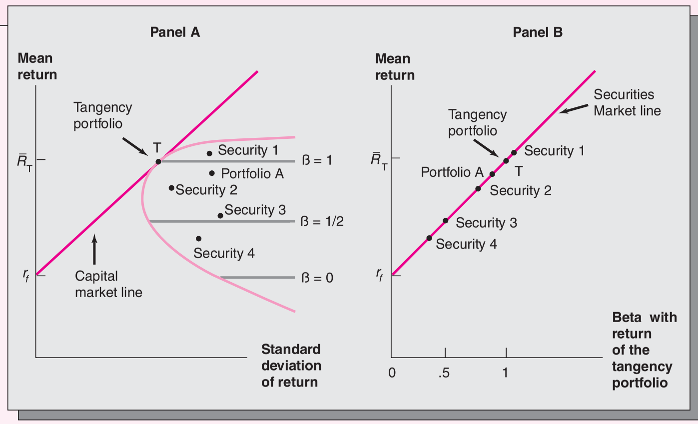
The Securities Market Line versus the Mean-Standard Deviation Diagram. Panel A of Exhibit 5.5 plots the familiar mean-standard deviation diagram. For the same financial market, panel B to the right of panel A plots what is commonly known as the securities market line. The **securities market line** is a line relating two important attributes for all the investments in the securities market. In equation (5.4), it is the graphical representation of mean versus beta. The four securities singled out in both panels are the same securities in both diagrams. The tangency portfolio is also the same portfolio in both panels.

Note in panel B that the tangency portfolio has a beta of 1 because the numerator and denominator of the ratio used to compute its beta are identical. The risk-free asset necessarily has a beta of zero; being constant, its return cannot covary with anything. Each portfolio on the capital market line (see panel A), a weighted average of the tangency portfolio and the risk-free asset, has a location on the securities market line found by taking the same weighted average of the points corresponding to the tangency portfolio and the risk-free asset.[17] What is special about the securities market line, however, is that all investments in panel A lie on the line: both the efficient portfolios on the capital market line and the dominated investments to the right of the capital market line.

Exhibit 5.5 purposely places the two graphs side by side to illustrate the critical distinction between the securities market line and the mean-standard deviation diagram. The difference between the graphs in panels A and B is reflected on the horizontal axis. Panel A shows the standard deviation on this axis while panel B shows the beta with the return of the tangency portfolio, which is proportional to the marginal variance. Thus, while investments with the same mean can have different standard deviations, as seen

[17] For this weighting of two portfolios, think of the risk-free asset as a portfolio with a weight of 1 on the risk-free asset and 0 on all the other assets.

EXHIBIT 5.5 The Relation between the Mean-Standard Deviation Diagram and a Mean-Beta Diagram



in panel A, they must have the same beta, as seen in panel B. For example, in panel A, all of the points on the grey line to the right of point T, labeled "$\beta = 1$," are portfolios with the same beta as the tangency portfolio. In panel B, all of these portfolios—even though they are distinct in terms of their portfolio weights and standard deviations—plot at exactly the same point as the tangency portfolio. For the same reason, all points on the grey horizontal line to the right of the risk-free asset in the mean-standard deviation diagram, designated "$\beta = 0$" are portfolios with a beta of 0 even though they have positive and differing standard deviations. In the mean-beta diagram, which graphs the securities market line in panel B, these portfolios plot at the same point as r_f.[18]

Portfolio Betas. An important property of beta is found in Result 5.4.[19]

Result 5.4 The beta of a portfolio is a portfolio-weighted average of the betas of its individual securities; that is

[18]Because mean return and beta plot on a straight line (see panel B), all investments with the same mean have the same beta and all investments with the same beta have the same mean.

[19]Because β is merely the covariance of security i with the tangency portfolio divided by a constant, Result 5.4 is a direct extension of Result 4.5 in Chapter 4: the covariance of the return of a portfolio with the return of a stock is the portfolio-weighted average of the covariances of the investments in the portfolio with the stock return.

$$\beta_p = \sum_{i=1}^{N} x_i \beta_i$$

where

$$\beta_i = \frac{cov(\tilde{r}_i, \tilde{R}_T)}{var(\tilde{R}_T)}$$

Thus, a portfolio that is 75 percent invested in a stock with a beta of 1.2 and 25 percent invested in a stock with a beta of 0.8 has a beta of 1, since

$$1.1 = .75(1.2) + .25(.8).$$

Contrasting Betas and Covariances. Note that betas and covariances are essentially the same measure of marginal variance. Beta is simply the covariance divided by the same constant for every stock. For historical reasons as well as the ease of estimation with regression, beta has become the more popular scaling of marginal variance. In principle, however, both are equally good as measures of marginal risk.

Marginal Variance versus Total Variance

Previously, this text defined a portfolio's risk as the variance of its return. However, to determine the expected rate of return on an investment, the relevant risk is beta (or covariance) computed with respect to the tangency portfolio.

Beta versus Variance as a Measure of Risk. Why is it that the beta and not the variance is the relevant measure of risk? An analogy from economics may shed some light on this question. A central tenet of economics is that the market price of a good is equal to the marginal cost of producing one more unit of the good. Thus, the total cost of production or the average cost of production does not matter for pricing—only the *marginal* cost matters. In finance, the marginal variance (that is, the covariance of an investment with the return of the optimal portfolio of an investor) determines the incremental risk from adding a small amount of the investment to the portfolio. Therefore, it is not surprising that required rates of return on risky investments are determined by their marginal variances.

Tracking Portfolios in Portfolio Management and as a Theme for Valuation

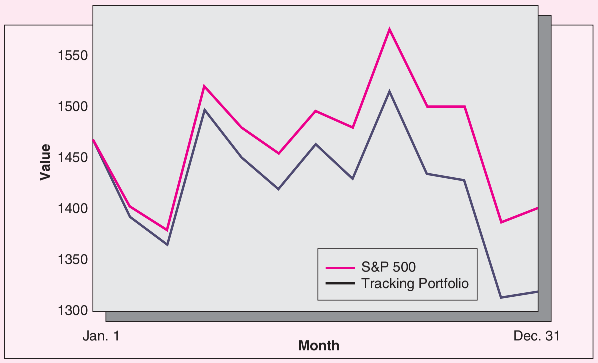
Investment professionals often use **tracking portfolios**, portfolios designed to match certain attributes of target portfolios, for the purpose of managing an index fund at a low transaction cost. This text generalizes the concept of a tracking portfolio to develop valuation models. Understanding the concept of a tracking portfolio is critical for understanding the role of portfolios and portfolio theory in corporate finance and strategy.

Uses of Tracking Portfolios in Investment Management. Investment professionals often need to **track**, that is, replicate the return characteristics of large portfolios using a relatively small number of stocks. For example, a number of portfolio managers are asked to create portfolios that "track" the S&P 500 Index in hopes of beating it. In what follows, it will sometimes be important to distinguish between perfect and imperfect tracking.

Portfolio A tracks portfolio B perfectly if the difference in the returns of the portfolios is a constant.

Exhibit 5.6 illustrates the value in 2000 of the S&P 500 and a tracking portfolio that has a higher return.

EXHIBIT 5.6 Value of the S&P 500 and Its Tracking Portfolio in 2000



In a typical tracking strategy, the investment professional selects about 50 stocks that he or she believes are underpriced and then weights each of the underpriced stocks in a portfolio in a way that minimizes the variance of the difference between the return of the managed portfolio and the return of the S&P 500. While such a portfolio may not track the S&P 500 perfectly, it can come close to doing so. As we will see, tracking strategies can lead to important insights about the risk-expected return equation.

Uses of Tracking Portfolios as a Theme in This Text. One of the key themes in this text is that it is possible to think about tracking almost any investment. One can track an individual stock, a real asset like a factory, or an index of stocks like the S&P 500. In contrast to the almost perfect tracking of broad-based portfolios like the S&P 500 or of derivatives (which have complicated tracking strategies), the tracking of an individual stock or real asset generally involves substantial tracking error.[20] In these cases, the best tracking portfolio is one that comes as close as possible to matching the returns of the tracked investment. This kind of tracking portfolio minimizes the variance of the tracking error.

In this chapter, where stock k is tracked with a portfolio of the risk-free asset (weight $1 - b$) and the tangency portfolio (weight b), the best tracking portfolio is the one whose tracking weight, b, is the same as the stock's beta, that is:

$$b = \beta_k$$

A tracking portfolio generated with this "best" tracking weight has the same marginal variance as stock k.[21] The main insight derived from this best tracking portfolio is summarized in Result 5.5.

[20]The tracking of broad-based portfolios is discussed in Chapter 6, that of derivatives in Chapters 7 and 8.

[21]This conclusion, which implies that tracking portfolios with these weights are optimal hedges, is proved in Chapter 22.

Result 5.5 If a stock and its tracking portfolio have the same *marginal variance* with respect to the tangency portfolio, then the stock and its tracking portfolio must have the same *expected return*.

This result was partly illustrated in Exhibit 5.5, where we learned that all investments with the same beta have the same expected return. However, it is critical that this beta be computed with respect to a mean-variance efficient portfolio like the tangency portfolio. Portfolios that lie inside the efficient frontier of risky assets, like Portfolio A or Security 2 in Exhibit 5.5, will not do.

5.8 The Capital Asset Pricing Model

Implementing the risk-expected return relation requires observation of the tangency portfolio. However, it is impossible to derive the tangency portfolio simply from observed historical returns on large numbers of assets. First, such an exercise would be extremely complex and inaccurate, requiring thousands of covariance estimates. Moreover, using historical average returns to determine means, and historical return data to estimate the covariances and variances, would only create a candidate tangency portfolio that would be useless for generating forward-looking mean returns. The required rates of return derived from a risk-return relation based on betas with respect to such a tangency portfolio would simply be the average of the historical rates of return used to find the tangency portfolio. In the case of Dell, this procedure would generate Dell's large historical average return as its expected return which, as suggested earlier, is a bad estimate. Clearly, other procedures for identifying the true tangency portfolio are needed.

To put some economic substance into the risk-expected return relation described by equation (5.4), it is necessary to develop a theory that identifies the tangency portfolio from sound theoretical assumptions. This section develops one such theory, generally referred to as the **Capital Asset Pricing Model (CAPM)**, which, as noted earlier, is a model of the relation of risk to expected returns.

The major insight of the CAPM is that the variance of a stock by itself is *not* an important determinant of the stock's expected return. What is important is the market beta of the stock, which measures the covariance of the stock's return with the return on a market index, scaled by the variance of that index.

Assumptions of the CAPM

As noted earlier, the two assumptions of mean-variance analysis are that:

1. Investors care only about the mean and variance of their portfolio's returns.
2. Markets are frictionless.

To develop the CAPM, one additional assumption is needed:

3. Investors have **homogeneous beliefs**, which means that all investors reach the same conclusions about the means and standard deviations of all feasible portfolios.

The assumption of homogeneous beliefs implies that investors will not be trying to outsmart one another and "beat the market" by actively managing their portfolios. On the other hand, the assumption does not imply that investors can merely throw darts to pick their portfolios. A scientific examination of means, variances, and covariances may still be of use, but every person will arrive at the same conclusions about the mean and standard deviation of each feasible portfolio's return after his or her own scientific examination.

The Conclusion of the CAPM

From these three assumptions, theorists were able to develop the Capital Asset Pricing Model, which concludes that the tangency portfolio must be the market portfolio. The next section details what this portfolio is and how practitioners implement it in the Capital Asset Pricing Model.

The Market Portfolio

The **market portfolio** is a portfolio where the weight on each asset is the market value (also called the **market capitalization**) of that asset divided by the market value of all risky assets.

Example 5.7: Computing the Weights of the Market Portfolio

Consider a hypothetical economy with only three investments: the stocks of Hewlett Packard (HP), IBM, and Compaq (CPQ). As of mid-December, 2000, the approximate prices per share of these three stocks are HP 5 $33, IBM 5 $95, and CPQ 5 $20.25. The approximate number of shares outstanding for the three firms are 2 billion (HP), 1.758 billion (IBM), and 1.7 billion (CPQ). What are the portfolio weights of the market portfolio?

Answer: The market capitalization of the stocks is

$$\begin{aligned} \text{HP} &= \$33 \times 2 \text{ billion} &&= \$66 \text{ billion} \\ \text{IBM} &= \$95 \times 1.758 \text{ billion} &&= \$167 \text{ billion} \\ \text{CPQ} &= \$20.25 \times 1.7 \text{ billion} &&= \$35 \text{ billion} \\ &\text{Total market capitalization} &&= \$268 \text{ billion} \end{aligned}$$

The market portfolio's weights (with decimal approximations) on the three stocks are therefore

$$\text{HP} = \frac{\$66 \text{ billion}}{\$268 \text{ billion}} = 0.25$$

$$\text{IBM} = \frac{\$167 \text{ billion}}{\$268 \text{ billion}} = 0.62$$

$$\text{CPQ} = \frac{\$35 \text{ billion}}{\$268 \text{ billion}} = 0.13$$

In Example 5.7, the return of the market portfolio, $.25\tilde{r}_{HP} + .62\tilde{r}_{IBM} + .13\tilde{r}_{CPQ}$, is the relevant return with which one computes the betas of the three stocks if the CAPM is true. The betas determine the expected returns of the three stocks. Of course, the world contains many investment assets, not just three stocks, implying that the actual market portfolio has a weight on every asset in the world.

With all the world's assets to consider, the task of calculating the market portfolio is obviously impractical. As claims to the real assets of corporations, all stocks and corporate bonds listed on all world exchanges and those traded over the counter would have to be included along with all real estate.

Since many of these investments are not traded frequently enough to obtain prices for them, one must use a proxy for the market portfolio. A frequently used proxy is the S&P 500, a **value-weighted portfolio**, meaning that the portfolio weight on each of its 500 typically larger market capitalization stocks—traded on the New York Stock Exchange (NYSE), the American Stock Exchange (AMEX), and the Nasdaq over-the-counter market—is proportional to the market value of that stock. Another commonly used proxy is the value-weighted portfolio of all stocks listed on the NYSE, Nasdaq, and AMEX. Still, these proxies ignore vast markets (for example, U.S. residential and commercial real estate, the Tokyo Stock Exchange, and the Tokyo real estate market), making them poor substitutes for the true market portfolio.

Why the Market Portfolio Is the Tangency Portfolio

Example 5.7 considered a hypothetical world that contained only three risky investments: the stocks of Hewlett Packard, IBM, and Compaq. Suppose that there also is a risk-free asset available for investment and that only two investors exist in this world: Jack and Jill.

What portfolio will Jack select? Mean-variance analysis implies that Jack will hold the tangency portfolio along with either a long or a short position in the risk-free investment. The proportion of his portfolio in the risk-free investment will depend on Jack's aversion to risk. Jill also will invest in some combination of the tangency portfolio and the risk-free investment.

In Example 5.7, the market values of all HP, IBM, and CPQ stock are $66 billion, $167 billion and $35 billion, respectively. Since Jack and Jill are the only two investors in the world, their joint holdings of HP, IBM, and CPQ also must total $66 billion, $167 billion, and $35 billion. It should be apparent that the tangency portfolio must contain some shares of HP. Otherwise, neither Jack nor Jill will hold HP shares, implying that the supply of HP stock ($66 billion) would not equal its demand ($0).

If supply does equal demand and Jack and Jill both hold the tangency portfolio, Jack must hold the same fraction of all the outstanding shares of the three stocks. That is, *if* Jack holds one-half of the 2 billion shares of HP stock, he must also hold one-half of the 1.758 billion shares of IBM stock, and one-half of the 1.7 billion shares of CPQ. In this case, Jill would own the other half of all three stocks. The respective proportions of their total stock investment spent on each of the three stocks will thus be the market portfolio's proportions; that is, approximately 0.25, 0.62, and 0.13 (see Example 5.7).

To understand why this must be true, consider what would happen if Jack held one-half of the shares of HP, but only one-third of IBM's shares. In this case, the ratio of Jack's portfolio weight on HP to his weight on IBM would *exceed* the 0.25/0.62 ratio of their weights in the market portfolio. This implies that Jill would have to hold one-half of the shares of HP and two-thirds of the shares of IBM for supply to equal demand. But then the ratio of Jill's weight on HP to her weight on IBM, being *less than* the 0.25/0.62 ratio of the market portfolio, would differ from Jack's, implying that Jack and Jill could not both be on the capital market line.

In short, because both Jack and Jill hold the tangency portfolio, they hold the risky investments in the exact same proportions. Because their holdings of risky stocks add up to the economy's supply of risky stocks, the market portfolio must also consist of risky investments allocated with these same proportions. It follows that the market portfolio is the tangency portfolio. The same conclusion, summarized in Result 5.6, is reached whether there are two investors in the world or billions.

Result 5.6 Under the assumptions of the CAPM, and if a risk-free asset exists, the market portfolio is the tangency portfolio and, by equation (5.4), the expected returns of financial assets are determined by

$$\bar{r} - r_f = \beta(\bar{R}_M - r_f) \tag{5.5}$$

where $\bar{R}_M$ is the mean return of the market portfolio, and β is the beta computed against the return of the market portfolio.

Equation (5.5) is a special case of equation (5.4) with the market portfolio used as the tangency portfolio. By identifying the tangency portfolio, the CAPM provides a risk-return relation that is not only implementable, but is implemented in practice.

Result 5.6 is not greatly affected by the absence of a risk-free asset or by different borrowing and lending rates. In these cases, the market portfolio is still mean-variance efficient with respect to the feasible set of portfolios constructed solely from risky assets. Equation (5.5) remains the same except that r_f is replaced by the expected return of a risky portfolio with a beta of zero.

Implications for Optimal Investment

In addition to the implementable relation between risk and expected return, described by equation (5.5), the CAPM also implies a rule for optimal investment:

Result 5.7 Under the assumptions of the CAPM, if a risk-free asset exists, every investor should optimally hold a combination of the market portfolio and a risk-free asset.

According to the CAPM, the major difference between the portfolios of Jack and Jill derives entirely from their differing weights on the risk-free asset. This is demonstrated in Example 5.8.

Example 5.8: Portfolio Weights That Include the Risk-Free Asset

Consider one-month U.S. Treasury bills as the risk-free asset. Ten million T-bills are issued for $9,900 each. Jack holds 4 million T-bills, and Jill holds 6 million. If Jack has $200 billion in wealth, what are the portfolio weights of Jack and Jill given the data in the previous example, which indicated that the aggregate wealth invested in risky assets is $268 billion?

Answer: The total wealth in the world is the value of the risky assets, $268 billion, plus the value of the T-bills, $99 billion, which sum to a total of $367 billion. Thus, if Jack has $200 billion, Jill has $167 billion. Jack spends $39.6 billion on T-bills, which makes his portfolio weight on T-bills $39.6 billion/$200 billion = .20. Jill spends $59.4 billion on T-bills, making her T-bill portfolio weight approximately .36. Thus, Jack owns $160.4 billion/$268 billion of the shares of the three risky assets and Jill owns $107.6 billion/$268 billion. After some calculation, the four portfolio weights (respectively, the risk-free asset, HP, IBM, and CPQ) for Jack are approximately (.20, .20, .50, .10) and the weights for Jill are (.36, .16, .40, .08).

Note, from Example 5.8, that the last three weights in Jack's and Jill's portfolios—that is, weights of .20, .50, and .10 (Jack) and .16, .40, and .08 (Jill) on HP, IBM, and CPQ, respectively—are the market portfolio's weights if they are rescaled to sum to 1. Obviously, this result follows from both Jack's and Jill's portfolios being combinations of the tangency (market) portfolio and the risk-free asset.

To understand the importance of Result 5.7, think again about the inputs needed to find the tangency portfolio. With thousands of securities to choose from, an investor would need to calculate not only thousands of mean returns, but also millions of covariances. Such a daunting task would surely require a professional portfolio manager. However, the CAPM suggests that none of this is necessary; investors can do just as well by investing in the market portfolio.

The 1970s, 1980s, and 1990s witnessed tremendous growth in the use of passively managed index portfolios as vehicles for investment in the pension fund, mutual fund, and life insurance industries. These portfolios attempted to mimic the return behavior of value-weighted portfolios like the S&P 500. One of the major reasons behind this trend was the popularization of the CAPM, a theory which suggested that the mean-standard deviation trade-off from investing in the market portfolio cannot be improved upon.

5.9 Estimating Betas, Risk-Free Returns, Risk Premiums, and the Market Portfolio

To implement the risk-expected return relation of the Capital Asset Pricing Model, it is necessary to estimate its parameters. These include the risk-free return, beta, and the market risk premium. Obviously, we also have to know the composition of the market portfolio to compute the latter two parameters.

Risk-Free or Zero-Beta Returns

Most academic studies of the CAPM have used short-term Treasury bill returns as proxies for the risk-free return. However, as Black, Jensen, and Scholes (1972), among others, have noted, this rate seems to be lower than the typical average return of a zero-beta risky stock. An alternative is to use the zero-beta expected return estimate that comes from fitting the intercept in the risk-expected return equation to all stocks. Interestingly, the risk-free rate employed in derivative securities pricing models, which is the London interbank offered rate (LIBOR),[22] appears to be much closer to this fitted number.

Beta Estimation and Beta Shrinkage

Beta, as mentioned previously, is the notation for the covariance divided by the variance because this ratio is the appropriate slope coefficient in a regression. In practice, one never obtains the true beta, but it is possible to obtain an estimate. Estimation with historical data is easy after recognizing that the ratio of covariance to variance is a slope coefficient, which can be obtained from a linear regression. The left-hand variable in the regression is the return of the stock on which beta is being estimated; the right-hand side is a proxy for the market return (for example, the return of the S&P 500). Many software packages and calculators have built-in regression routines that will use these data to estimate beta as the regression slope coefficient.

Example 5.9 provides real-world data and illustrates both a beta calculation and the estimation of expected return using beta.

Example 5.9: Estimating Beta and the Expected Return for Dell Computer

Historical quarterly returns (in %) for Dell Computer and the S&P 500 are given below:[a]

	Dell				S&P 500			
	Q1	*Q2*	*Q3*	*Q4*	*Q1*	*Q2*	*Q3*	*Q4*
1990	38.57	65.62	−30.72	111.53	−3.81	5.32	−14.52	7.90
1991	54.07	−14.05	36.25	−23.24	13.63	−1.08	4.50	7.54
1992	41.96	−25.26	57.94	67.67	−3.21	1.10	2.37	4.29
1993	−26.83	−46.61	−11.33	36.07	3.66	−0.25	1.86	1.64
1994	11.60	4.46	41.96	9.50	−4.43	−0.34	4.15	−0.74
1995	6.71	37.43	41.36	−18.53	9.02	8.80	7.28	5.39
1996	−3.24	51.86	52.83	36.65	4.80	3.89	2.49	7.77
1997	27.30	73.66	64.98	−13.29	2.21	16.91	7.02	2.44
1998	61.31	36.99	41.68	11.31	13.53	2.91	−10.30	20.87
1999	11.70	−9.48	13.01	21.97	4.65	6.71	−6.56	14.54

[22]See Chapter 2.

a. What is the annualized expected return required by investors in Dell Computer stock as estimated by averaging the 40 quarterly returns from 1991 through the end of 1999 and multiplying by 4?

b. What is the annualized expected return required by investors in Dell Computer stock as estimated from the CAPM, using the S&P 500 as the market portfolio, 4.9 percent for the risk-free (or zero-beta) return, and the ten-year average return of the S&P 500 less 4.9 percent as the market portfolio's risk premium?

Answer: *a.* Averaging the 40 quarterly returns of Dell and multiplying by 4 generates an annualized expected return of 94.54 percent.

b. The beta estimated by regressing the 40 returns of Dell on the 40 returns of the S&P 500 is 1.56. The annualized average return of the S&P over the 40 quarters is 15.4 percent. Hence, using equation (5.5), the expected return of Dell is

$$21.3\% = 4.9\% + 1.56\ (15.4\% - 4.9\%)$$

[a]Source: Dell and S&P 500 returns are computed using price data from quote.yahoo.com.

A variety of statistical methods can improve the beta estimate. These methods usually involve taking some weighted average of 1 and the beta estimated with a software package.

Improving the Beta Estimated from Regression

Example 5.9 estimated the beta of Dell Computer with a simple regression of 40 quarterly Dell stock returns on the corresponding returns of a proxy for the market portfolio. The better beta estimates, alluded to above, account for estimation error. One source of estimation error arises simply because Dell's stock returns are volatile; therefore, estimates based on those returns are imprecise.[23] A second source of estimation error arises because price changes for some stocks (usually the smaller capitalization stocks) seem to lag the changes of other stocks either because of nontrading or stale limit orders, that is, limit orders that were executed as a result of the investor failing to update the order as new information about the stock became available.

To understand the importance of estimation error, consider a case where last year's returns are used to estimate the betas of four very similar firms, denoted as firms A, B, C, and D. The estimated betas are $\beta_A = 1.4$, $\beta_B = .8$, $\beta_C = .6$, and $\beta_D = 1.2$. However, because these are estimated betas, they contain estimation error. As a result, the true betas are probably not as divergent as the estimated betas. Given these estimates, it is likely that stock A has the highest beta and stock C the lowest. Our best guess, however, is that the beta of stock A is overestimated and the beta of stock C is underestimated.[24]

[23]Just as a coin tossed 10 times can easily have a "heads" outcome 60 percent of the time (or 6 times), even if the true probability of a "heads" outcome is 50 percent, the average historical returns of stocks are rarely equal to their true mean returns.

[24]To understand why this is true, think about your friends who scored 770 on their GMATs or SATs. While it is true that most people who score 770 are smart, scoring that high might also require some luck; thus, those with the best scores may not be quite as smart as their 770 score would indicate. Similarly, the stock with the highest estimated beta in a given group may not really be as risky as its beta would indicate. The stock with the highest estimated beta in a group is likely to have a high estimation error in addition to having a high actual beta.

The Bloomberg Adjustment. Bloomberg, an investment data service, adjusts estimated betas with the following formula

$$\text{Adjusted beta} = .66 \times \text{Unadjusted beta} + .34$$

which would reduce Dell's beta of 1.56 to 1.37.

In general, the **Bloomberg adjustment** formula lowers betas that exceed 1 and increases betas that are under 1.

The Rosenberg Adjustment. A number of data services provide beta adjustments of this type to portfolio managers. One was started by a former University of California, Berkeley, finance professor, Barr Rosenberg, who was one of the first to develop ways to improve beta estimates. Rosenberg, Reid, and Lanstein (1985) showed that using historical betas as predictors of future betas was much less effective than using alternative beta prediction techniques. Rosenberg first used a shrinkage factor similar to what Bloomberg is now using. Rosenberg later refined his prediction technique to incorporate fundamental variables—an industry variable and a number of company descriptors. Rosenberg sold his company, known as BARRA, which later expanded this approach into a successful risk management product.

Adjusting for the Lagging Reaction of the Prices of Small Company Stocks to Market Portfolio Returns. It also may be necessary to make additional adjustments to the betas of small firms because the returns of the stocks of small companies tend to react to market returns with a lag. This delayed reaction creates a downward bias in the beta estimates of these smaller capitalization stocks, since only part of the effect of market movements on the returns of these stocks are captured by their contemporaneous covariances. The bias can be significant when one estimates the betas from daily stock returns. For this reason, analysts should avoid daily returns and instead estimate betas with weekly or monthly returns where the effect of delayed reaction tends to be less severe. However, a paper by Handa, Kothari, and Wasley (1989) suggests that the monthly betas of small capitalization stocks also may be underestimated compared with yearly betas.[25]

The following simple procedure for adjusting the betas of smaller cap stocks can compensate for the lagged adjustment: Add the lagged market return as an additional right-hand-side variable in the beta estimation regression. Then, sum the two slope coefficients in the regression—the slope coefficient on the market return that is contemporaneous with the stock return and the slope coefficient on the lagged market return—to obtain the adjusted beta. One could further refine this adjusted beta with the Bloomberg or Rosenberg techniques.

A Result to Summarize the Beta Adjustments. Result 5.8 summarizes this subsection.

Result 5.8 Betas estimated from standard regression packages may not provide the best estimates of a stock's true beta. Better beta estimates can be obtained by taking into account the lead-lag effect in stock returns and the fact that relatively high beta estimates tend to be overestimates and relatively low beta estimates tend to be underestimates.

[25]In the absence of these considerations, the smaller the return horizon, the more precise is the beta estimate. Our preferred compromise horizon for large firms is weekly data. An alternative is to employ daily data but make a statistical correction to the beta estimation procedure. See Scholes and Williams (1977) and Dimson (1979) for details on these statistical corrections.

Estimating the Market Risk Premium

Assuming one knows the composition of the market portfolio, averaging its return over a long historical time series to compute an expected return on the market portfolio has the advantage of generating a better statistical estimate if the market portfolio's expected return also is stable over time. However, some empirical evidence suggests that the mean returns of portfolios like the S&P 500 change over time, providing an argument for the use of a shorter historical time series, although the ten years used in Example 5.9 may be too short. In addition, changes in the expected return of the market portfolio appear to be predictable from variables such as the level of interest rates, the aggregate dividend yield, and the realized market return over the previous three to five years. To the extent that a model predicting the market's expected return is accurate and holds over long periods of time, one should estimate the parameters of such a model with as much historical data as possible, and then use current levels of the predictor variables to generate a forecast of the market's expected return.

To compute the market portfolio's risk premium, subtract a risk-free return from the expected return estimate. It also is possible to estimate the risk premium directly by averaging the market portfolio's historical **excess returns**, which are its returns in excess of the risk-free return. However, this is sensible only if the risk premium is stable over time. Empirical evidence suggests that the mean of the market return itself is more stable than the mean of the excess return. Hence, we do not recommend averaging historical excess returns to estimate the risk premium.

Identifying the Market Portfolio

Of course, the entire analysis here presumes that the analyst can identify the weights of the market portfolio. Previously, our discussion focused on several common proxies for the market portfolio, which were selected because, like the market portfolio, they were value-weighted portfolios. However, they contain only a small set of the world's assets. Hence, always keep in mind that these are merely proxies and that the usefulness of the CAPM depends on whether these proxies work or not. The next section discusses the evidence about how well these proxies do as candidates for the tangency portfolio.

5.10 Empirical Tests of the Capital Asset Pricing Model

In Part III of this text the CAPM is used as a tool for obtaining the required rates of return needed to evaluate corporate investment projects. The relevance of CAPM applications is determined by the ability of the theory to accurately predict these required rates of return. Given the importance of this topic, financial economists have conducted hundreds of studies that examine the extent to which the expected returns predicted by the CAPM fit the data. This section describes the results of these studies.

In empirical tests of the CAPM, the returns of low-beta stocks are much too high relative to its predictions, and the returns of high-beta stocks are much too low. More importantly, a number of stock characteristics explain historical average returns much better than the CAPM beta does. These characteristics include, among others, the firm's market capitalization; that is, the market value of the firm's outstanding shares, the ratio of the firm's market value to book value or **market-to-book ratio**, and **momentum**, defined as the stock's return over the previous six months.[26] Interestingly, investment

[26]Momentum investment strategies that rank stocks based on their returns in the past 3 months, 9 months, or 12 months seem to work about equally well. The choice of 6 months for our discussion is arbitrary.

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funds exist to exploit all three characteristics. For example, Dimensional Fund Advisors is perhaps the most famous fund to exploit what has come to be known as the small firm effect. The current interpretation of these empirical findings, discussed in greater depth below, is that the CAPM does not properly describe the relation between risk and expected return.

Can the CAPM Really Be Tested?

Applications and tests of the CAPM require the use of market proxies like the S&P 500 because, as noted earlier, the exact composition of the market portfolio is unobservable. In an influential article, Roll (1977) pointed out that the unobservability of the market portfolio made the CAPM inherently untestable. As such, previous tests that used proxies for the market provided almost no evidence that could lead one to either accept or reject the CAPM. Roll's logic was as follows:

1. A portfolio always exists with the property that the expected returns of all securities are related linearly to their betas, calculated with respect to that portfolio. (Equation 5.4 shows this is a mean-variance efficient or tangency portfolio.)
2. Even if the theory is wrong, the portfolio used as a market proxy may turn out to be mean-variance efficient, in which case, the tests will incorrectly support the theory.
3. Alternatively, the proxy may be not be mean-variance efficient even though the theory is correct, in which case, the theory is incorrectly rejected.

Applications of the CAPM in corporate finance and portfolio management share this problem of observability with the CAPM tests. Is it possible to apply a model like the CAPM if it cannot be tested because its most crucial components cannot be observed?

Although academics have debated whether the CAPM is testable without arriving at a consensus, the model is applied by practitioners, using various portfolios as proxies for the market. In these industry applications, appropriate expected returns are obtained from any market proxy that is mean-variance efficient, whether the CAPM actually holds or, equivalently, whether the "true" market portfolio is mean-variance efficient. Therefore, the appropriateness of the various applications of the CAPM rests not on whether the CAPM actually holds, but on whether the S&P 500, or whatever market proxy one uses to apply the CAPM, is mean-variance efficient. Purported tests of the CAPM that use these proxies for the market are in fact tests of the mean-variance efficiency of the proxies and are of interest for exactly this reason.

We summarize this discussion as follows:

Result 5.9 Testing the CAPM may be problematic because the market portfolio is not directly observable. Applications of the theories use various proxies for the market. Although the results of empirical tests of the CAPM that use these proxies cannot be considered conclusive, they provide valuable insights about the appropriateness of the theory as implemented with the specific proxies used in the test.

Example 5.10 illustrates why it is important to test the CAPM using proxy portfolios that are applied in practice.

Example 5.10: Using a Proxy to Estimate the Cost of Capital

Upstart Industries would like to obtain an estimate of the expected rate of return on its stock. Analysts estimate that the stock's beta with respect to the S&P 500 is about 1, and the expected rate of return on the S&P 500 is estimated to be 13.5 percent. Is the expected return on Upstart Industries' stock 13.5 percent?

Answer: If the S&P 500 is a mean-variance efficient portfolio, 13.5 percent is a good estimate of Upstart's expected return, regardless of whether the CAPM is correct. If the S&P 500 is not mean-variance efficient, the estimate will not be valid.

Most tests of the CAPM used the value-weighted portfolio of all NYSE and AMEX stocks as a proxy for the market portfolio. (Nasdaq data was unavailable at the time many of these tests were performed.) This value-weighted portfolio is highly correlated with the S&P 500 stock index described earlier. If empirical tests strongly reject the mean-variance efficiency of this value-weighted portfolio, then you must be skeptical of applications that use similar portfolios (for example, the S&P 500) to calculate expected returns. However, if these same tests provide strong support for the model, then you can be comfortable with applications that use the S&P 500.

Our view of this debate is that if the true picture of the world is that the portfolio used as a market proxy is mean-variance efficient, then even if the theory is wrong—Roll's situation (2)—we can throw out the CAPM and use the proxy even if there is no initially apparent theoretical justification for its use. The danger is that a hunt for a mean-variance efficient proxy, which is based largely on empirical fit and not on sound theoretical footing, is unlikely to have the same fit in the future. However, some balance between theory and empirical fit may be the best we can do.[27]

Is the Value-Weighted Market Index Mean-Variance Efficient?

To understand the nature of the various tests of the CAPM, it is useful to first contrast the model with the empirical tests that use historical data. The CAPM provides predictions about how the expected rates of return of securities relate to their betas. Unfortunately, the analyst does not observe either the expected returns or the betas. The tests assume that in large samples the average historical return of each stock approximates its expected return and the estimated betas approximate the true betas. Of course, the CAPM will not hold exactly with these estimated betas and estimated expected returns. Research on the CAPM performs statistical tests to determine whether the observed deviations from the model occurred because of estimation error (for example, the averages of the realized returns may have been very different from the expected returns) or because the model was wrong.

Cross-Sectional Tests of the CAPM

In the early 1970s, extensive tests were conducted to determine if the CAPM was consistent with the observed distribution of the returns of NYSE-listed stocks. One of the earliest procedures used to test the CAPM involved a two-step approach. First, betas were estimated with a set of time-series regressions, one for each security. (In a **time series regression**, each data observation corresponds to a date in time; for example, the returns on IBM stock and the S&P 500 in January 2001 might be the respective left-hand-side and right-hand-side values for a single observation.) Each of these regressions, one for each security j, can be represented by the equation

$$r_{jt} = \alpha_j + \beta_j R_{Mt} + \epsilon_{jt} \tag{5.6}$$

[27]This will be discussed further in Chapter 6 where inquiries into efficiency are a bit less dismal than those presented here.

where

α_j = the regression's intercept
β_j = the regression's slope coefficient
r_{jt} = the month t return of stock j
R_{Mt} = the month t return of the value-weighted portfolio of NYSE and AMEX stocks
ϵ_{jt} = the month t regression residual for stock j

Exhibit 5.7 graphs the data and line of best fit for a beta regression involving the returns of Dell Computer stock, using the quarterly data from the end of 1991 through the end of 1999 (see Example 5.9). The slope of the line of best fit is the beta.[28]

The second step obtains estimates of the intercept and slope coefficient of a single **cross-sectional regression**, in which each data observation corresponds to a stock. (For example, IBM's average return and beta might be the respective left- and right-hand-side values for a single observation.)[29] This equation can be represented algebraically as

$$\bar{r}_j = \gamma_0 + \gamma_1 \hat{\beta}_j + \gamma_2 \text{CHAR}_j + \delta_j \tag{5.7}$$

where

$\bar{r}_j$ = average monthly historical return of stock j, each j representing a NYSE-listed stock
$\hat{\beta}_j$ = estimated slope coefficient from the time series regression described in equation (5.9)
CHAR_j = a characteristic of stock j unrelated to the CAPM, like firm size
γs = intercept and slope coefficients of the regression
δ_j = stock j regression residual

If the CAPM is true, the second step regression, equation (5.7), should have the following features:

- The intercept, γ_0, should be the risk-free return.
- The slope, γ_1, should be the market portfolio's risk premium.
- γ_2 should be zero since variables other than beta (for example, return variance or firm size), represented as CHAR_j, should not explain the mean returns once beta is accounted for.

Exhibit 5.8, which illustrates hypothetical data and fit for the cross-sectional regression, is indicative of data that is consistent with the CAPM. In particular, the intercept is the risk-free return and the slope is the risk premium of the market portfolio. In contrast, the four panels in Exhibit 5.9 portray data that are inconsistent with the theory. In panel A, the intercept is wrong; in panel B, the slope is wrong; in panel C, securities appear to lie on a curve rather than on a line; in panel D, the deviations of the mean returns from the securities market line are plotted against firm size. The evidence of a relationship between returns (after accounting for beta) and firm size would imply rejection of the CAPM.

[28]To list the large volume of data in Example 5.9, we used quarterly data. In practice, monthly or even weekly return data would be better.

[29]Fama and MacBeth (1973) used an innovative procedure to overcome a bias in statistical inference arising from correlated residuals in the cross-sectional regression. This involves running one cross-sectional regression for each observation in time, and averaging the slope coefficients. Also, these researchers ran their tests on beta-grouped portfolios rather than on individual stocks.

EXHIBIT 5.7 Dell Computer: Quarterly Returns (Quarter 1, 1990–Quarter 4, 1999)

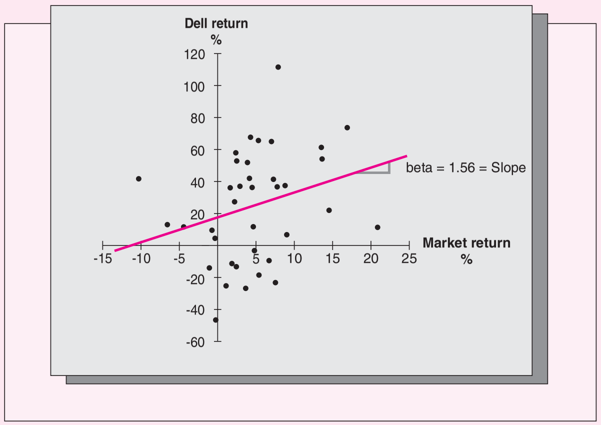


EXHIBIT 5.8 Second Step Regression Data Consistent with the CAPM

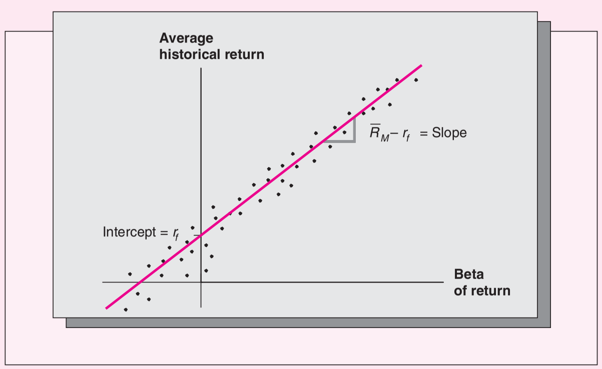
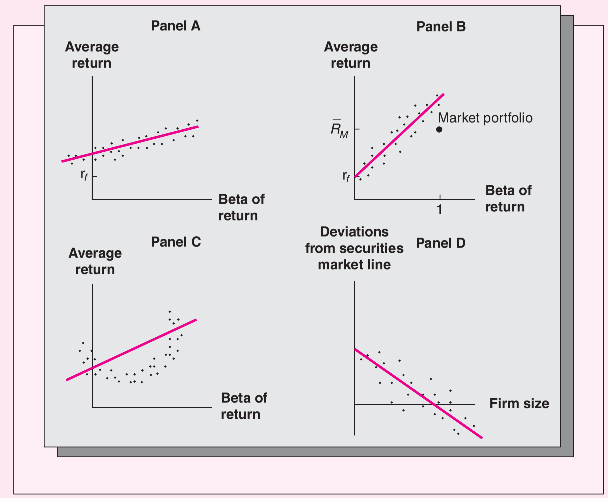


EXHIBIT 5.9 Second Step Regression Data Inconsistent with the CAPM



Time-Series Tests of the CAPM

A second set of CAPM tests, introduced by Black, Jensen and Scholes (1972), examine the restrictions on the intercepts of time-series market model regressions. Consider the regression:

$$r_{jt} - r_f = \alpha_j + \beta_j(R_{Mt} - r_f) + z_{jt} \tag{5.8}$$

It is substantially identical to the regression in equation (5.6)[30] except that excess returns are used in lieu of returns. The CAPM implies that the intercept, α_j, in equation (5.8) is zero for every stock or portfolio. Researchers have tested the CAPM with this approach, using the returns of portfolios formed from characteristics such as the stock's prior beta, firm size, and the ratio of the market value of a stock to its book value to estimate the coefficients in equation (5.8). For example, one could test the CAPM by regressing the excess returns of a portfolio consisting of the 100 stocks on the NYSE with the smallest capitalization on the excess returns of a market proxy. The CAPM predicts that the intercepts from such regressions should be zero. However, if the CAPM underestimates the returns of small-capitalization stocks, the intercepts in regressions that involve small company stocks will be positive.

[30]Betas estimated from both regressions are almost always nearly identical.

Results of the Cross-Sectional and Time-Series Tests: Size, Market-to-Book, and Momentum

Both the time-series and cross-sectional tests find evidence that is not supportive of the CAPM. The following are the most noteworthy violations of the CAPM:

- The relation between estimated beta and average historical return is much weaker than the CAPM suggests. Some researchers have found no relation between beta and average return.[31]
- The market capitalization or size of a firm is a predictor of its average historical return (see Exhibit 5.10). This relation cannot be accounted for by the fact that smaller capitalization stocks tend to have higher betas.[32]
- Stocks with low market-to-book ratios tend to have higher returns than stocks with high market-to-book ratios (see Exhibit 5.11). Again, differences in beta do not explain this difference.
- Stocks that have performed well over the past six months tend to have high expected returns over the following six months (see Exhibit 5.12).[33]

There may even be a negative relation between beta and stock returns after controlling for firm size (more precisely, capitalization). The top row of Exhibit 5.13 presents the returns of three portfolios, the value-weighted portfolio of all stocks (left column), a value-weighted portfolio of stocks ranked by beta in the lowest decile (middle column), and a portfolio of stocks ranked by beta in the highest decile (right column). The figures presented in this row reveal that portfolios of low-beta stocks had returns that were slightly higher than the returns of high-beta stocks. When the portfolios include only small capitalization stocks (middle row), the average returns are essentially the same; that is, low beta implies only slightly higher returns. However, for the large capitalization stocks (bottom row), there is a large penalty associated with beta. Large capitalization stocks with high betas realized very poor returns over the 1963–1990 time period.

Exhibit 5.14 provides the monthly returns of nine portfolios formed on the basis of capitalization and market-to-book ratios (ME/BE). Reading across the first row, one sees that for the sample of all firms, low market-to-book stocks realize much higher returns than high market-to-book stocks. However, the lower two rows show that the effect of market-to-book ratios on stock returns is substantially stronger for the small cap firms. What Exhibit 5.14 shows is that an investor who bought and held small capitalization stocks with low market-to-book ratios would have realized a yearly return of close to

[31]Kothari, Shanken, and Sloan (1995) pointed out that the relation between beta and expected return is much stronger when using annual returns instead of monthly returns to estimate betas. Moreover, they argued that the data used to measure market-to-book ratios presents an inaccurate picture of true expected returns because the data vendor is more likely to provide data on successful firms—what is called "backfill bias." However, as Fama and French (1996) countered, size is still an important determinant of expected returns in Kothari, Shanken, and Sloan's data, which violates the CAPM.

[32]The fact that small capitalization stocks have historically outperformed large capitalization stocks seems to be an international phenomenon. For example, Ziemba (1991) found that in Japan, the smallest stocks outperformed the largest stocks by 1.2 percent per month, 1965–1987. In the United Kingdom, Levis (1985) found that small capitalization stocks outperformed large capitalization stocks by 0.4 percent, 1958–1982. However, one of the biggest size premiums was in Australia, where Brown, Kleidon, and Marsh (1983) found a premium of 5.73 percent per month from 1958 to 1981.

[33]Hong, Lim, and Stein (2000) have shown that some of this effect is due to slow information diffusion in that it is more pronounced among stocks with small analyst followings. Lee and Swaminathan (2000) have shown that the effect is more pronounced among firms with high trading volume. Moskowitz and Grinblatt (1999) have shown that some of this is due to the fact that stocks in past-winning industries tend to outperform stocks from past-losing industries.

EXHIBIT 5.10 Average Annualized Returns, Beta, and Firm Size for Value-Weighted Portfolios of NYSE and AMEX Stocks[a]

Size Portfolio	*Annualized Mean Return (percent)*	*Beta*	*Market Capitalization ($millions)*
Smallest	19.8%	1.17	9.7
2	17.8	1.19	23.2
3	16.1	1.15	41.4
4	15.4	1.17	68.0
5	16.0	1.11	109.8
6	14.5	1.05	178.9
7	14.4	1.04	291.4
8	14.8	1.03	502.3
9	13.0	1.01	902.1
Largest	11.9	0.95	3983.0

[a]Source: Reprinted from Gabriel Hawawini and Donald Keim, "On the Predictability of Common Stock Returns, World Wide Evidence," 1995, Chapter 17 in *Handbooks in Operations Research and Management Science*, Vol. 9, *Finance*, edited by R. Jarrow, V. Maksimovic, and W. Ziemba, with kind permission of Elsevier Science-NL, Sara Burgerhartstreet 25, 1055 KV Amsterdam, The Netherlands. Data are formed on the basis of market capitalization (Apr. 1951–Dec. 1989).

EXHIBIT 5.11 Average Annualized Returns, Market-to-Book Ratios (ME/BE), Beta, and Firm Size for Value-Weighted Portfolios of NYSE and AMEX Stocks[b]

ME/BE Portfolio	*Annualized Mean Return (percent)*	*Beta*	*Market Capitalization ($millions)*
Negative	19.9%	1.29	118.6
Lowest > 0	17.9	1.04	260.5
2	17.5	0.95	401.2
3	13.0	0.90	619.5
4	13.4	0.83	667.7
5	11.5	0.90	641.1
6	9.8	0.91	834.6
7	10.3	0.98	752.3
8	11.2	1.02	813.0
9	10.2	1.11	1000.8
Highest	10.9	1.05	1429.8

[b]Source: Reprinted from Gabriel Hawawini and Donald Keim, "On the Predictability of Common Stock Returns, World Wide Evidence," 1995, Chapter 17 in *Handbooks in Operations Research and Management Science*, Vol. 9, *Finance*, edited by R. Jarrow, V. Maksimovic, and W. Ziemba, with kind permission of Elsevier Science-NL, Sara Burgerhartstreet 25, 1055 KV Amsterdam, The Netherlands. Data formed on the basis of the ratio of market price/book price (Apr. 1962–Dec. 1989).

EXHIBIT 5.12 Average Annualized for Returns for Portfolios Grouped by Six-Month Momentum and Held for Six Months[a]

Momentum-Ranked Portfolio (by decile)	*Average Annualized Returns (percent)*
Portfolio 1 (minimum momentum)	9.48%
Portfolio 2	13.44
Portfolio 3	15.00
Portfolio 4	14.88
Portfolio 5	15.36
Portfolio 6	16.08
Portfolio 7	16.32
Portfolio 8	17.16
Portfolio 9	18.36
Portfolio 10 (maximum momentum)	20.88
Portfolio 10 minus Portfolio 1	11.40

[a]Source: Reprinted with permission from *The Journal of Finance*, Narasimhan Jegadeesh and Sheridan Titman, "Returns to Buying Winners and Selling Losers: Implications for Stock Market Efficiency," Vol. 48, pp. 65–91. Data are from Jan. 1965–Dec. 1989.

EXHIBIT 5.13 Average Annualized Returns Based on Size and Beta Grouping[b]

	All Firms (%)	*Low Betas (%)*	*High Betas (%)*
All firms	15.0%	16.1%	13.7%
Small cap firms	18.2	20.5	17.0
Large cap firms	10.7	12.1	6.7

[b]Source: Reprinted with permission from *The Journal of Finance* 47, "The Cross-section of Expected Stock Returns," by Eugene Fama and Kenneth R. French, pp. 427–465. Data are from July 1963–Dec. 1990.

EXHIBIT 5.14 Average Annualized Returns (%) Sorted by Size and Market Equity/Book Equity[c]

	All Firms (%)	*High ME/BE (%)*	*Low ME/BE (%)*
All firms	14.8%	7.7%	19.6%
Small cap firms	17.6	8.4	23.0
Large cap firms	10.7	11.2	14.2

[c]Source: Reprinted with permission from *The Journal of Finance* 47, "The Cross-section of Expected Stock Returns," by Eugene Fama and Kenneth R. French, pp. 427–465. Data are from July 1963–Dec. 1990.

23 percent. Compounded over time, a $1,000 investment made at the beginning of 1963 would grow at a 23 percent annual rate, to about $267,000 by the end of 1989.[34]

Result 5.10 summarizes the results of this subsection:

Result 5.10 Research using historical data indicates that cross-sectional differences in stock returns are related to three characteristics: market capitalization, market-to-book ratios, and momentum. Controlling for these factors, these studies find no relation between the CAPM beta and returns over the historical time periods studied.

Interpreting the CAPM's Empirical Shortcomings

There are two explanations for the poor ability of the CAPM to explain average stock returns. The first explanation has to do with the possibility that the various proxies for the market portfolio do not fully capture all of the relevant risk factors in the economy. According to this explanation, firm characteristics, such as size and market-to-book ratios, are highly correlated with the sensitivities of stocks to risk factors not captured by proxies for the market portfolio. For example, Jagannathan and Wang (1996) suggested that **human capital**—that is, the present value of a person's future wages—is an important component of the market portfolio that is not included in the various market portfolio proxies. Since investors would like to insure against the possibility of losing their jobs, they are willing to accept a lower rate of return on those stocks that do relatively well in the event of layoffs.[35] In addition, if investors believe that large firms are likely to benefit more or be harmed less than small firms from economic factors that lead to increased layoffs or wage reductions, then they may prefer investing in the stocks of the large firms even if the expected returns of large firms are lower.

A second explanation of the CAPM's poor performance is that it is simply a false theory because investors have behavioral biases against classes of stocks that have nothing to do with the mean and marginal risk of the returns on stocks. The smaller capitalization stocks and stocks with lower market-to-book ratios may require a higher expected rate of return if investors shy away from them for behavioral reasons. For example, firms with low market-to-book ratios are generally considered to have poor prospects. Indeed, that is why their market values are so low relative to their book values. By contrast, firms with high market-to-book ratios are considered to be those with the brightest futures.

One behavioral explanation for the higher returns of the distressed, or near-bankrupt, firms is that some portfolio managers find it more costly to lose money on distressed firms. There used to be a saying on Wall Street that "portfolio managers can't get fired buying IBM stock." The basic argument was that if portfolio managers had bought IBM stock in the late 1970s and then lost money it wasn't really their fault. All the other professional money managers were buying IBM stock and the conventional wisdom was that the firm was doing great. Contrast this with Chrysler in the late 1970s. The newspapers were full of stories about Chrysler's imminent demise, and its stock was selling for less than $5 a share. Since it should have been obvious to anyone reading a newspaper that Chrysler was on the verge of bankruptcy, money managers who invested in Chrysler would probably have been fired if Chrysler did go bankrupt. What this means is that a money manager might find it riskier, from a personal perspective,

[34]This assumes that 23 percent is the growth rate each year. If there is variation in the growth rate, but the average is 23 percent, one ends up with much less than $267,000 at the end of 1989. See the appendix to Chapter 11 for further discussion of the impact of growth rate variation on long-term growth.

[35]See Mayers (1972).

to buy a distressed stock even if the actual return distribution of the stock is similar to that of a financially healthy firm. If this were true, the distressed stock would require a higher expected rate of return.

A more recent situation arose with Microsoft and Apple Computer. In 1995, Microsoft came out with a new operating system which was a critical success and the newspapers were full of articles suggesting that the firm had almost unlimited growth opportunities. Apple, on the other hand, was having problems and a number of newspapers predicted its demise.

What should portfolio managers do in this situation? They may not be willing to invest in Apple, even if—understanding the risks associated with Apple—they believe that the market may have overreacted to its misfortunes. If portfolio managers buy Apple stock and it does poorly, they have to explain why they bought the stock despite all the predictions of the company's demise. However, if they buy Microsoft stock and it does poorly, they might be blamed less for the poor performance, given all the great publicity that Microsoft was having at the time the stock was purchased. Because of these more personal risks, portfolio managers may require a much higher expected return to invest in Apple than in Microsoft, even if both stocks have the same beta.

A study by Lakonishok, Shleifer, and Vishny (1994), using data from April 1968 to April 1990, examined the long-term success of what they called value and glamour stocks, and reported evidence in favor of the behavioral story over the missing risk factor story. Although they do not select stocks explicitly in terms of market-to-book ratios, their value stocks are generally low market-to-book stocks and their glamour stocks are generally high market-to-book stocks.

The authors provided two important pieces of evidence suggesting that the higher returns of the value stocks are due to some kind of behavioral bias. First, value stocks tend to consistently dominate glamour stocks.[36] If hidden risk factors are driving the expected return difference, then a glamour investment strategy should occasionally beat a value strategy. It is difficult to conclude that a strategy that almost never seems to lose requires a big risk premium. Second, the story of the missing risk factor would be more credible if the value stocks performed relatively poorly during recessions, but they do not.[37]

Are These CAPM Anomalies Disappearing?

There is evidence that suggests that at least some of the anomalies described in this section seem to be disappearing with each passing year as the participants in the financial markets become more sophisticated. For example, small firms have not outperformed large firms in the 20 years since the small-firm effect was publicized in the early 1980s and the market-to-book effect seems to have disappeared shortly after it was publicized in the late 1980s and early 1990s. However, in contrast, past winning stocks have outperformed past losing stocks throughout most of the 1990s, even though the momentum effect was well publicized in the early 1990s.

[36]Lakonishok, Shleifer, and Vishny formed value and glamour portfolios in each year between 1968 and 1989 and tracked the performance of the portfolios for the next 1, 3, or 5 years. For every formation year, the value portfolio was worth more than the glamour portfolio after five years. In other words, in this 21-year period, a patient investor would always do better with a value strategy than with a glamour strategy.

[37]Four recessions—December 1969 to November 1970, November 1973 to March 1975, January 1980 to July 1980, and July 1981 to November 1982—took place during the study's sample period. The authors found that value stocks do well during recessions and beat the glamour stocks in three out of four of those recessions.

5.11 Summary and Conclusions

This chapter analyzed various features of the mean-standard deviation diagram, developed a condition for deriving optimal portfolios, and showed that this condition has important implications for valuing financial assets. The chapter also showed that the beta used in this valuation relation, the relation between risk and expected return, is an empty concept unless it is possible to identify the tangency portfolio. The Capital Asset Pricing Model is a theory that identifies the tangency portfolio as the market portfolio, making it a powerful tool for valuing financial assets. As later chapters will illustrate, it also is useful for valuing real assets.

Valuation plays a major role in corporate finance. Part III will discuss that when firms evaluate capital investment projects, they need to come up with some estimate of the project's required rate of return. Moreover, valuation often plays a key role in determining how corporations finance their new investments. Many firms are reluctant to issue equity when they believe their equity is undervalued. To value their own equity, they need an estimate of the expected rate of return on their stock; the Capital Asset Pricing Model is currently one of the most popular methods used to determine this.

The empirical tests of the CAPM contradict its predictions. Indeed, the most recent evidence fails to find a positive relation between the CAPM beta and average rates of return on stocks from 1960 to 1990, finding instead that stock characteristics, like firm size and market-to-book ratios, provide very good predictions of a stock's return. Does this mean that the CAPM is a useless theory?

One response to the empirical rejection of the CAPM is that there are additional aspects of risk not captured by the market proxies used in the CAPM tests. If this is the case, then the CAPM should be used, but augmented with additional factors that capture those aspects of risk. Chapter 6, which discusses multifactor models, considers this possibility. The other possibility is that the model is fundamentally unsound and that investors do not act nearly as rationally as the theory suggests. If this is the case, however, investors should find beta estimates even more valuable because they can be used to construct portfolios that have relatively low risk without giving up anything in the way of expected return.

This chapter provides a unique treatment of the CAPM in its concentration on the more general proposition that the CAPM "works" as a useful financial tool if its market proxy is a mean-variance efficient portfolio and does not work if the proxy is a dominated portfolio. The empirical evidence to date has suggested that the proxies used for the CAPM appear to be dominated. This does not preclude the possibility that some modification of the proxy may ultimately prove to be useful. As you will see in Chapter 6, research along these lines currently appears to be making great strides.

We also should stress that testing the CAPM requires examining data in the fairly distant past, to a period prior to when Wall Street professionals, portfolio managers, and corporations used the CAPM. However, corporations that are interested in valuing either investment projects or their own stocks have no interest in the historical relevance of the CAPM; they are interested only in whether the model provides good current estimates of required rates of return. As economists, we like to think that investors act as if they are trained in the nuances of modern portfolio theory even though beta was a relatively unknown Greek letter to the Wall Street of the 1960s. As educators, however, we like to believe that we do make a difference and that our teaching and research has made a difference. Given those beliefs, we at least want to entertain the possibility that current required rates of return reflect the type of risk suggested by the CAPM, even if rates of return did not reflect this type of risk in the past. Some evidence of this is that the size effect has greatly diminished, if it has not disappeared entirely, since the early 1980s, when practitioners began to focus on the academic research in this subject. The same is true of the market-to-book effect.

Key Concepts

Result 5.1: All portfolios on the mean-variance efficient frontier can be formed as a weighted average of any two portfolios (or funds) on the efficient frontier.

Result 5.2: Under the assumptions of mean-variance analysis, and assuming the existence of a risk-free asset, all investors will select portfolios on the capital market line.

Result 5.3: The ratio of the risk premium of every stock and portfolio to its covariance with the tangency portfolio is constant; that is, denoting the return of the tangency portfolio as $\tilde{R}_T$

$$\frac{\bar{r}_i - r_f}{cov(\tilde{r}_i, \tilde{R}_T)}$$

is identical for all stocks.

Result 5.4: The beta of a portfolio is a portfolio-weighted average of the betas of its individual securities; that is

$$\beta_p = \sum_{i=1}^{N} x_i \beta_i$$

where

$$\beta_i = \frac{cov(\tilde{r}_i, \tilde{R}_T)}{var(\tilde{R}_T)}$$

Result 5.5: If a stock and its tracking portfolio have the same *marginal variance* with respect to the tangency portfolio, then the stock and its tracking portfolio must have the same *expected return*.

Result 5.6: Under the assumptions of the CAPM, and if a risk-free asset exists, the market portfolio is the tangency portfolio and, by equation (5.4), the expected returns of financial assets are determined by

$$\bar{r} - r_f = \beta(\bar{R}_M - r_f) \qquad (5.5)$$

where $\bar{R}_M$ is the mean return of the market portfolio, and β is the beta computed against the return of the market portfolio.

Result 5.7: Under the assumptions of the CAPM, if a risk-free asset exists, every investor should optimally hold a combination of the market portfolio and a risk-free asset.

Result 5.8: Betas estimated from standard regression packages may not provide the best estimates of a stock's true beta. Better beta estimates can be obtained by taking into account the lead-lag effect in stock returns and the fact that relatively high beta estimates tend to be overestimates and relatively low beta estimates tend to be underestimates.

Result 5.9: Testing the CAPM may be problematic because the market portfolio is not directly observable. Applications of the theories use various proxies for the market. Although the results of empirical tests of the CAPM that use these proxies cannot be considered conclusive, they provide valuable insights about the appropriateness of the theory as implemented with the specific proxies used in the test.

Result 5.10: Research using historical data indicates that cross-sectional differences in stock returns are related to three characteristics: market capitalization, market-to-book ratios, and momentum. Controlling for these factors, these studies find no relation between the CAPM beta and returns over the historical time periods studied.

Key Terms

beta 147
Bloomberg adjustment 157
Capital Asset Pricing Model (CAPM) 151
capital market line (CML) 138
cost of capital 132
cross-sectional regression 161
dominated portfolios 134
efficient frontier 135
excess returns 158
feasible set 133
frictionless markets 135
homogeneous beliefs 151
human capital 167
market capitalization 152
market portfolio 152
market-to-book ratio 158
mean-variance efficient portfolios 133
momentum 158
risk premium 140
securities market line 147
self-financing 143
tangency portfolio 138
time-series regression 160
track 149
tracking portfolio 149
two-fund separation 136
value-weighted portfolio 152

Exercises

5.1. Here are some general questions and instructions to test your understanding of the mean standard deviation diagram.
a. Draw a mean-standard deviation diagram to illustrate combinations of a risky asset and the risk-free asset.
b. Extend this concept to a diagram of the risk-free asset and all possible risky portfolios.
c. Why does one line, the capital market line, dominate all other possible portfolio combinations?
d. Label the capital market line and tangency portfolio.
e. What condition must hold at the tangency portfolio?

Exercises 5.2–5.9 make use of the following information about the mean returns and covariances for three stocks. The numbers used are hypothetical.

Stock	Covariance with *AOL*	*Microsoft*	*Intel*	*Mean Return*
AOL	.002	.001	0	15%
Microsoft	.001	.002	.001	12
Intel	0	.001	.002	10

5.2. Compute the tangency portfolio weights assuming a risk-free asset yields 5 percent.
5.3. How does your answer to exercise 5.2 change if the risk-free rate is 3 percent? 7 percent?
5.4. Draw a mean-standard deviation diagram and plot AOL, Microsoft, and Intel on this diagram as well as the three tangency portfolios found in exercises 5.2 and 5.3.
5.5. Show that an equally weighted portfolio of AOL, Microsoft, and Intel can be improved upon with marginal variance-marginal mean analysis.
5.6. Repeat exercises 5.2 and 5.3, but use a spreadsheet to solve for the tangency portfolio weights of AOL, Microsoft, and Intel in the three cases. The solution of the system of equations requires you to invert the matrix of covariances above, then post multiply the inverted covariance matrix by the column of risk premiums. The solution should be a column of cells, which needs to be rescaled so that the weights sum to 1. *Hint:* See footnote 11.
5.7. *a*. Compute the betas of AOL, Microsoft, and Intel with respect to the tangency portfolio found in exercise 5.2.
b. Then compute the beta of an equally weighted portfolio of the three stocks.
5.8. Using the fact that the hyperbolic boundary of the feasible set of the three stocks is generated by any two portfolios:
a. Find the boundary portfolio that is uncorrelated with the tangency portfolio in exercise 5.2.
b. What is the covariance with the tangency portfolio of all inefficient portfolios that have the same mean return as the portfolio found in part *a*?
5.9. What is the covariance of the return of the tangency portfolio from exercise 5.2 with the return of all portfolios that have the same expected return as AOL?
5.10. Using a spreadsheet, compute the minimum variance and tangency portfolios for the universe of three stocks described below. Assume the risk-free return is 5 percent. Hypothetical data necessary for this calculation are provided in the table below. See exercise 5.6 for detailed instructions.

Stock	*Standard Deviation*	*Mean Return*	*Correlation with Bell South*	*Correlation with Caterpillar*
Apple	.20	.15	.8	−.1
Bell South	.30	.10	1.0	.2
Caterpillar	.25	.12	.2	1.0

5.11. The Alumina Corporation has the following simplified balance sheet (based on market values)

Assets	Liabilities and Equity
$10 billion	Debt $6 billion Common Stock $4 billion

a. The debt of Alumina, being risk-free, earns the risk-free return of 6 percent per year. The equity of Alumina has a mean return of 12 percent per year, a standard deviation of 30 percent per year, and a beta of .9. Compute the mean return, beta, and standard deviation of the assets of Alumina. *Hint:* View the assets as a portfolio of the debt and equity.
b. If the CAPM holds, what is the mean return of the market portfolio?

c. How does your answer to part *a* change if the debt is risky, has returns with a mean of 7 percent, has a standard deviation of 10 percent, a beta of .2, and has a correlation of .3 with the return of the common stock of Alumina?

5.12. The following are the returns for Exxon (which later merged with Mobil) and the corresponding returns of the S&P 500 market index for each month in 1994.

Month	*Exxon Return (%)*	*S&P 500 Return (%)*
January	5.35%	3.35%
February	−1.36	−2.70
March	−3.08	−4.35
April	0.00	1.30
May	−1.64	1.63
June	−7.16	−2.47
July	4.85	3.31
August	1.21	4.07
September	−3.36	−2.41
October	9.35	2.29
November	−2.78	−3.67
December	0.62	1.46

Using a spreadsheet, compute Exxon's beta. Then apply the Bloomberg adjustment to derive the adjusted beta.

5.13. What value must ACYOU Corporation's expected return be in Example 5.4 to prevent us from forming a combination of Henry's portfolio, ACME, ACYOU, and the risk-free asset that is mean-variance superior to Henry's portfolio?

5.14. Assume that the tangency portfolio for stocks allocates 80 percent to the S&P 500 index and 20 percent to the Nasdaq composite index. This tangency portfolio has an expected return of 13 percent per year and a standard deviation of 8.8 percent per year. The beta for the S&P 500 index, computed with respect to this tangency portfolio, is .54. Compute the expected return of the S&P 500 index, assuming that this 80%/20% mix really is the tangency portfolio when the risk-free rate is 5 percent.

5.15. Exercise 5.14 assumed that the tangency portfolio allocated 80 percent to the S&P 500 index and 20 percent to the Nasdaq composite index. The beta for the S&P 500 index with this tangency portfolio is .54. Compute the beta of a portfolio that is 50 percent invested in the tangency portfolio and 50 percent invested in the S&P 500 index.

5.16. Using data only from 1991–1995, redo Example 5.9. Which differs more from the answer given in Example 5.9: the expected return estimated by averaging the quarterly returns or the expected return obtained by estimating beta and employing the risk-expected return equation? Why?

5.17. Estimate the Bloomberg-adjusted betas for the following companies.

	Unadjusted Beta
Delta Air Lines	0.84
Procter & Gamble	1.40
Coca-Cola	0.88
Gillette	0.90
Citigroup	1.32
Caterpillar	1.00
ExxonMobil	0.64

5.18. Compute the tangency and minimum variance portfolios assuming that there are only two stocks: Nike and McDonald's. The expected returns of Nike and McDonald's are .15 and .14, respectively. The variances of their returns are .04 and .08, respectively. The covariance between the two is .02. Assume the risk-free rate is 6%.

5.19. There exists a portfolio P, whose expected return is 11%. Stock I has a covariance with P of .004, and Stock II has a covariance with P of .005. If the expected returns on Stocks I and II are 9% and 12%, respectively, and the risk-free rate is 5%, then is it possible for portfolio P to be the tangency portfolio?

5.20. The expected return of the S&P 500, which you can assume is the tangency portfolio, is 16% and has a standard deviation of 25% per year. The expected return of Microsoft is unknown, but it has a standard deviation of 20% per year and a covariance with the S&P 500 of 0.10. If the risk-free rate is 6 percent per year,

a. Compute Microsoft's beta.

b. What is Microsoft's expected return given the beta computed in part *a*?

c. If Intel has half the expected return of Microsoft, then what is Intel's beta?

d. What is the beta of the following portfolio?

.25 in Microsoft
.10 in Intel
.75 in the S&P 500
− .20 in GM (where $\beta_{GM} = .80$)
.10 in the risk-free asset

e. What is the expected return of the portfolio in part *d*?

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