Introduction to Financial Engineering

Week 41: Portfolio Choices

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Week 41

Making financial decisions

Some financial assets are **risky assets**, meaning that the future cash flows (and/or value) are not known. Only the distribution of returns is known or assumed.

Overall problem

- Which assets to invest in
- How much to invest in each asset

Starting point: Characterizing assets in terms of

- expected return (average)
- variance / standard deviation

Notation and assumptions

- Matrix multiplication makes it easy to formulate the expected return and variance of returns for a portfolio
- Assume that $\boldsymbol{\mu} = [\mu_1 \ \ \mu_2 \ \ \cdots \ \ \mu_n]'$ is a one-column matrix of expected returns on different assets
- Assume that $\boldsymbol{w} = [w_1 \ \ w_2 \ \ \cdots \ \ w_n]'$ is a one-column matrix representing the fractions of the investors wealth invested in each asset
- Denote the covariance matrix between returns by $\boldsymbol{\Sigma}$
- By definition, a covariance matrix is always positive semidefinite, but now it is assumed that it is **positive definite** and thus invertible
- Further, not all coordinates of $\boldsymbol{\mu}$ are equal

Matrix expressions for expected return and variance

- The expected return on such a portfolio is $\mu_P = \boldsymbol{w}^\top \boldsymbol{\mu}$
- The variance of returns for such a portfolio is $\sigma_P^2 = \boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}$, where $\boldsymbol{\Sigma}$ is the covariance matrix of the returns
- In the two-asset case, the variance is $\sigma_P^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2c_{12}$ and the expected return is $\mu_P = w_1\mu_1 + w_2\mu_2$.
- If you are struggling with matrix calculations, it might be useful for you to do the derivations for the two-asset case and convince yourself that the derived matrix expressions match.

Stating the optimization problem

- Consider the following problem

$$\min_{\boldsymbol{w}} \frac{1}{2}\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}$$

- Under the following constraints:

$$\boldsymbol{w}^\top \boldsymbol{\mu} = \mu_P$$
$$\boldsymbol{w}^\top \mathbf{1} = 1$$

- Or in words, given a specific expected rate of return μ_P, what is the allocation of wealth into assets $1, \ldots, n$ that gives this expected return with the smallest variance

Efficient frontier

The efficient frontier can be described as

$$\sigma_P^2 = \frac{c\mu_P^2 - 2b\mu_P + a}{ac - b^2} \qquad \text{or} \qquad \sigma_P = \sqrt{\frac{c\mu_P^2 - 2b\mu_P + a}{ac - b^2}} \qquad ([1])$$

$$\mu_{gmv} = b/c \quad \text{with} \quad \sigma_{gmv}^2 = 1/c$$

where a, b and c are obtained from the definition of the A-matrix. The portfolio weights for a given required return can be expressed as

$$\hat{\boldsymbol{w}} = \boldsymbol{\Sigma}^{-1} \left[\begin{array}{cc} \boldsymbol{\mu} & \mathbf{1} \end{array} \right] A^{-1} \left[\begin{array}{c} \mu_P \\ 1 \end{array} \right], \qquad \hat{\boldsymbol{w}}_{gmw} = \frac{1}{c} \boldsymbol{\Sigma}^{-1} \mathbf{1} \qquad ([2])$$

In a (standard deviation, mean)-space, the **efficient frontier** or efficient portfolios is the upper half of the curve expressed by [1]. The efficient frontier will have expected returns greater than b/c and standard deviations greater than $1/\sqrt{c}$.

Inclusion of risk free asset

- Assume that a risk free asset exists with return μ_0
- Express returns as excess returns
  $\boldsymbol{\mu}^e = [\mu_1 - \mu_0, \mu_2 - \mu_0, \ldots, \mu_n - \mu_0]^\top$
- Assume that $\boldsymbol{w} = [w_1, w_2, \ldots, w_n]^\top$ are the fractions of the investors wealth invested in each risky asset
- Assume that $w_0 = 1 - \boldsymbol{w}^\top \mathbf{1}$ is invested in the risk free asset.
- If $w_0 > 0$, the investor has put money in the bank. If $w_0 < 0$, the investor has borrowed money.
- For a given expected excess return μ_P^e, the objective is to find the portfolio with the lowest variance (or standard deviation)

Stating the optimization problem

- Consider the following problem

$$\min_{\boldsymbol{w}} \tfrac{1}{2}\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}$$

- Under the following constraints:

$$\boldsymbol{w}^\top \boldsymbol{\mu}^e = \mu_P^e$$

- Or in words, given a specific expected excess rate of return μ_P^e, what is the allocation of wealth into assets $1, \dots, n$ that gives this expected excess return with the smallest variance

Setting up the Lagrange

$$\mathcal{L}(\boldsymbol{w}, \lambda) = \frac{1}{2}\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w} - \lambda_1 \left(\boldsymbol{w}^\top \boldsymbol{\mu}^e - \mu_P^e \right)$$

FOCs for optimality are

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{w}} = \boldsymbol{\Sigma} \boldsymbol{w} - \lambda \boldsymbol{\mu}^e = \boldsymbol{0} \qquad ([3])$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \boldsymbol{w}^\top \boldsymbol{\mu}^e - \mu_P^e = 0 \qquad ([4])$$

Solving 1/3

First rearrange [3]:

$$\begin{aligned} \boldsymbol{\Sigma} \boldsymbol{w} &= \lambda \boldsymbol{\mu}^e \Rightarrow \\ \boldsymbol{w} &= \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e \lambda \end{aligned} \qquad ([5])$$

And then [4]:

$$(\boldsymbol{\mu}^e)^\top \boldsymbol{w} = \mu_P^e \qquad ([6])$$

Solving 2/3

Multiply [5] with $(\boldsymbol{\mu}^e)^\top$

$$(\boldsymbol{\mu}^e)^\top \boldsymbol{w} = (\boldsymbol{\mu}^e)^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e \lambda \tag{[7]}$$

But [6] has the same left-hand side, meaning that

$$\mu_P^e = (\boldsymbol{\mu}^e)^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e \lambda \tag{[8]}$$

Because Σ is positive definite and then invertible, then $(\boldsymbol{\mu}^e)^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e > 0$ and λ can be inserted in [5] to find $\boldsymbol{w}$.

Solving 3/3

So

$$\lambda = \frac{\mu_P^e}{(\boldsymbol{\mu}^e)^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e} \qquad ([9])$$

and the optimal portfolio weights for specific expected excess return μ_P^e is:

$$\boldsymbol{w} = \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e \frac{\mu_P^e}{(\boldsymbol{\mu}^e)^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e} \qquad ([10])$$

Further, it follows that minimal portfolio variance for specific expected excess return μ_P^e is:

$$\sigma_P^2 = \frac{(\mu_P^e)^2}{(\boldsymbol{\mu}^e)^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e} \qquad ([11])$$

Capital Market Line

The link between σ_P and of μ_P^e is:

$$\sigma_P = \frac{\mu_P^e}{\sqrt{(\boldsymbol{\mu}^e)^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e}}$$

or equivalently (this is called the Capital Market Line)

$$\mu_P = \sigma_P \sqrt{(\boldsymbol{\mu}^e)^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e} + \mu_0$$

- It is reasonable to assume than $\mu_0 < \mu_{gmv}$ (Why?)
- The CML touches the efficient frontier without the risk free asset in exactly one point (Why?)

Tangent Portfolio

The portfolio where everything is invested in risky assets is called the **tangent portfolio**. The excess return of the tangent portfolio is

$$\mu^e_{tan} = \frac{(\boldsymbol{\mu}^e)^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e}{\mathbf{1}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e} \qquad ([12])$$

with

$$\sigma_{tan} = \frac{\sqrt{(\boldsymbol{\mu}^e)^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e}}{\mathbf{1}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e} \qquad ([13])$$

$$\boldsymbol{w}_{tan} = \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e \frac{1}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}^e} \qquad ([14])$$

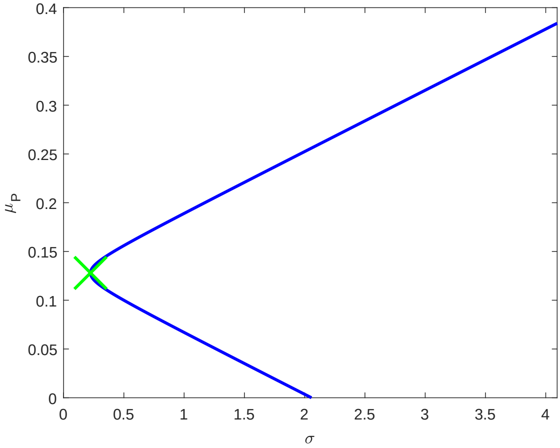
One lending and one borrowing rate

If there is one borrowing rate and one lending rate, then the efficient portfolios will be

- The CML using the lending rate for $\sigma_P < \sigma_{tan}^l$
- The CML using the borrowing rate for $\sigma_P > \sigma_{tan}^b$
- The efficient frontier for the risky assets only $\sigma_{tan}^l < \sigma_P < \sigma_{tan}^b$

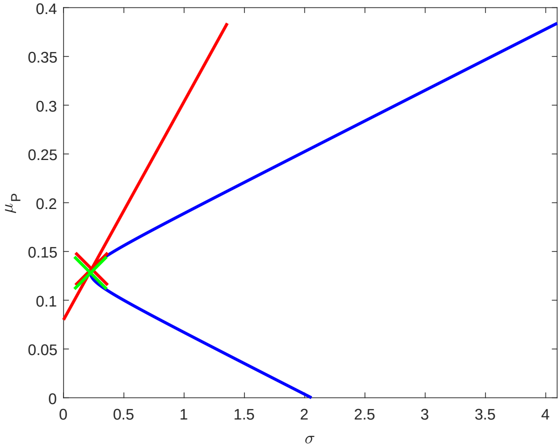
Risky Assets Only

Example (Minimum variance portfolios)



Including Risk Free Asset

Example (Minimum variance portfolios)

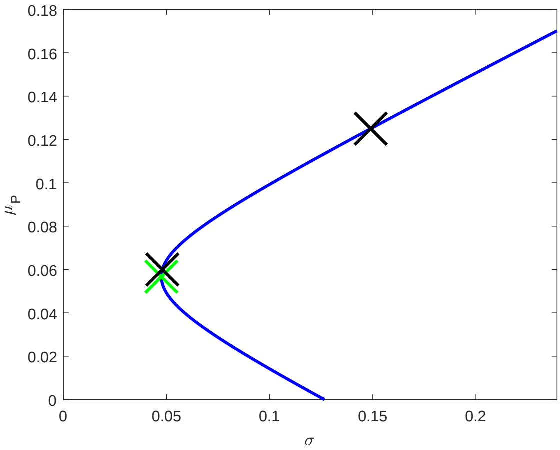


Investing in two stocks

- Assume it is possible to invest in two different stocks
- Assume that stock 1 has an expected return of 12.5% and the standard deviation is 14.9%
- Assume that stock 2 has an expected return of 6% and the standard deviation is 4.8%
- The correlation between returns is 0.45

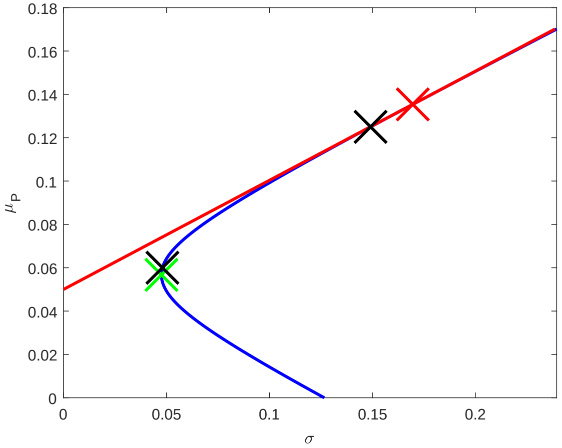
The efficient frontier

Example (Expected return vs. standard deviation)



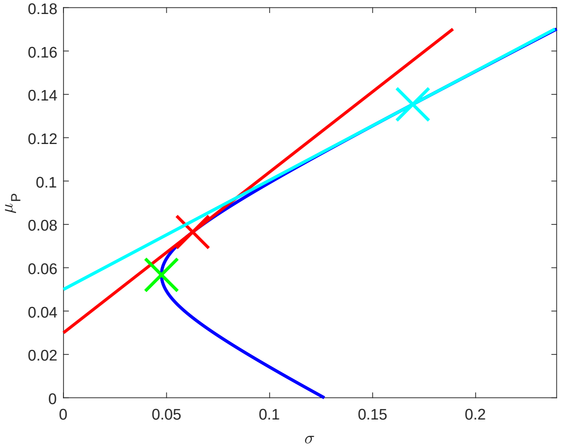
The efficient frontier and the CML

Example (Expected return vs. standard deviation)



Portfolios of stocks and bonds and two different rates

Example (Expected return vs. standard deviation)



Two risky assets case

- For two risky assets, both assets will be on the parabola of minimum-variance portfolios
- They will not necessarily be on the efficient frontier – the upper half curve. The position will depend on the assets (correlation, expected return, standard deviation)
- If short-selling is not allowed, the possible minimum-variance portfolios is on the parabola of minimum-variance portfolios between the two assets
- The efficient and feasible portfolios consist of the upper half curve for μ-values between $\max\{\min(\mu_1, \mu_2), \mu_{gmw}\}$ and $\max(\mu_1, \mu_2)$

Multiple risky assets

- For multiple risky assets, the assets will generally not be on the parabola of minimum-variance portfolios
- If short-selling is not allowed, the possible minimum-variance portfolios must be found by solving the optimization problem with the additional constraint that $w_i \geq 0$ for $i = 1, \ldots, n$
- This leads to portfolios with at least the same variance as without the additional constraint
- The highest feasible return is $\max(\mu_1, \mu_2, \ldots, \mu_n)$

Constraints on the risk free asset

- A risk free asset with possible different interest rates can also be incorporated in the optimization problem
- Risk less lending and no borrowing could for instance be incorporated by removing constraint $\mathbf{1}'\boldsymbol{w} = 1$ and instead constraining $w_0 \geq 0$ together with the previous condition of no short-selling of risky assets $w_i \geq 0$ for $i = 1, \ldots, n$
- Borrowing with no lending can also be incorporated by varying the constraints

Portfolios of particular interest

- The minimum variance portfolio: The portfolio with the lowest variance
- The maximum return portfolio: The portfolio with the highest expected return
- The tangent portfolio: This portfolio actually has the highest **Sharpe Ratio** $SR_P = \frac{\mu_P - \mu_0}{\sigma_P}$ among the risky asset only portfolios
- Choosing according to risk aversion: Finding the portfolio that (for instance) maximizes $\mu_P - \lambda\sigma_P^2$ for a given λ. If λ is high, the investor dislikes risk ("risk averse"). If λ is low, he favors return over risk.

Structure of Chapter 16

- The book covers more or less the same content as the Lando-Poulsen note, but the latter has more general, and closed form solutions
- The book is a good source for putting the maths in a simpler form
- It finds the tangent portfolio for two risky assets and for N assets by programming – we did it for N in closed form
- The Capital Market Line is presented as a combination of investing in the tangent portfolio and the risk free asset – we derived it from N risky assets and on risk free asset, but it is the same result!
- Section 16.2.1 highlight issues in estimating expected return and risk. We have used sample mean and variance for this, but it's possible to do more sophisticated models (like CAPM, Factor models, GARCH models etc.)
- The code example in Chapter 16.6 is good inspiration for adding constraints

Next week

Next week, the following material is covered:

- Chapter 4 up til and including 4.4
- Chapter 5 up til and including 5.4
- Danske Bank will visit from around 9.30/10 and present some cases, of which some are related to the above material and some of more general data analysis character