

Problem 5.4

Which of the following, if any, is a valid way to prove $P(n) \rightarrow P(n+1)$.

(i) Let's see what happens if $P(n+1)$ is T. \rightarrow Valid derivations \rightarrow Look! $P(n)$ is True.

(ii) Let's see what happens if $P(n+1)$ is F. \rightarrow Valid derivations \rightarrow Look! $P(n)$ is False.

Only the second example is a valid way to show that $P(n) \rightarrow P(n+1)$

The first is flawed because one cannot assume $P(n+1)$ to be true when it is what you're trying to prove. As shown in class with the " $7=4$ " example, one can do perfectly valid derivations and still get a result that doesn't make logical sense. Additionally, the first statement only attempts to show that $P(n+1) \rightarrow P(n)$ which is not the same as $P(n) \rightarrow P(n+1)$.

The second statement is a successful use of proof by contradiction. By showing that if $P(n+1)$ is false then $P(n)$ can never be true, you are also proving that if $P(n)$ is true then $P(n+1)$ can never be false.

Problem 5.3(c)

For which n is $P(n)$ true? Explain by showing the chain of implications.

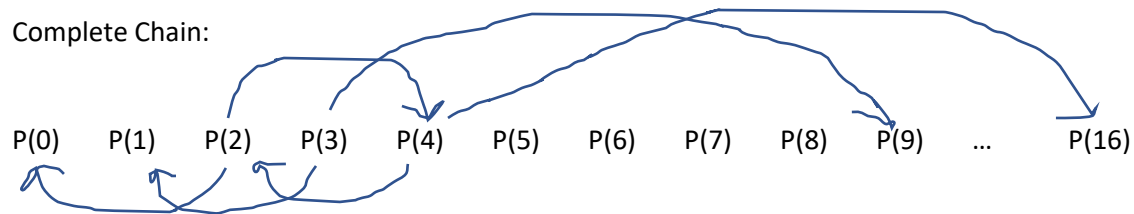
(c) $P(2)$ is true and $p(n) \rightarrow (P(n^2) \text{ AND } P(n-2))$ is True for n greater or equal to 2.

$P(2) \rightarrow P(4) \text{ AND } P(0)$

$P(3) \rightarrow P(9) \text{ AND } P(1)$

$P(4) \rightarrow P(16) \text{ AND } P(2)$

Complete Chain:



As demonstrated above, as n gets bigger, $P(n-2)$ statements will get "filled in." $P(2)$ being true also proves $P(0)$ is true. $P(3)$ being true also proves $P(1)$ is true. Etc etc. Thus n is true for all numbers greater or equal to zero.

Problem 5.11(b)

The n th Harmonic number is $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$. Prove:

(b) $1 + \frac{1}{2} \log_e n \leq H_n \leq 1 + \log_e n$.

Problem 5.43a

(a) A robot moves one diagonal move at a time. Prove that no sequence of moves takes the robot to the shaded square.

Initial predicate: $P(x, y)$ = A robot starting at position (x, y) cannot move to the point $(x+1, y)$ when restricted to moving on diagonals.

Proof by induction:

We'll start with the base case $P(1, 1)$.

The robot wants to move to the point $(2, 1)$, so we can consider all possible moves to see if that point is reachable.

If the robot moves to the upper right diagonal, then it's coordinates will be $(2, 2)$.

If the robot moves to the upper left diagonal, then it's coordinates will be $(0, 2)$.

If the robot moves to the lower right diagonal, then it's coordinates will be $(2, 0)$.

If the robot moves to the lower left diagonal, then it's coordinates will be $(0, 0)$.

None of the possible moves result in moving to the shaded area. Reaching the shaded area will also be impossible because any move to the right $(x+1)$ to get on the same x value of the shaded area must also be accompanied by a move in the y direction, putting the robot either above or below the shaded area.

Now to prove that $P(x, y) \rightarrow P(x \pm 1, y \pm 1)$. I'll proof by contraposition.

Assume $P(x \pm 1, y \pm 1)$ is false.

Now, say that $n = x + y$.

$P(x \pm 1, y \pm 1) = P(n)$ OR $P(n+2)$ OR $P(n-2)$ and $P(x, y) = P(n)$.

When $P(x \pm 1, y \pm 1) = P(n)$, since $P(x, y) = P(n)$, $P(x \pm 1, y \pm 1) = P(x, y)$.

When $P(x \pm 1, y \pm 1) = P(n+2)$, the robot can just move to the lower left diagonal and the sum of its new coordinates will be n .

When $P(x \pm 1, y \pm 1) = P(n-2)$, the robot can just move to the upper right diagonal and the sum of its new coordinates will equal n .

In all instances, $P(x \pm 1, y \pm 1)$ is either equivalent to or can be derived to be equal to $P(x, y)$. Since we assumed $P(x \pm 1, y \pm 1)$ to be false and showed that $P(x, y)$ is equivalent to $P(x \pm 1, y \pm 1)$, $P(x, y)$ will also be false.

This is a proof by contraposition because if $P(x \pm 1, y \pm 1)$ is false then $P(x, y)$ can never be true, then if $P(x, y)$ is true $P(x \pm 1, y \pm 1)$ can never be false.

Thus, by induction, $P(x, y) \rightarrow P(x \pm 1, y \pm 1)$ which means $P(x, y)$ is true for all coordinates.

Problem 5.43b

(b) One of the moves changed. Now prove that any square (m, n) can be reached by a finite sequence of moves.

Exercise 6.2

Use induction to prove the claim $P(n)$: $n^3 < 2^n$, for n greater or equal to 10.

Exercise 6.4

Show that if the missing (blackened out) square is at position (n, n) in the 2^n by 2^n , you can still L-tile the patio. The patios for $n = 1, 2, 3, 4$ are illustrated.