DMC Exercise 7.12(b)

Give recursive definitions for the set S in each of the following cases.

(b) S = {all strings which are palindromes}, for example 010, 110011, 000 are palindromes but 011 is not a palindrome.

Start with the base case of ε ∈ S

Next, some tinkering to look for patterns

Length 1	Length 2	Length 3	Length 4	Length 5
0	00	010	1111	00000
1	11	000	0000	11111
		101	1001	01110
			0110	10001
				11011
				00100
				01010
				10101

It seemed as if every palindrome could be created by adding stuff to the middle of an existing palindrome.

I thought 1 • x • 1 AND 0 • x • 0 could be a constructor, but that would exclude palindromes with odd lengths.

Then I realized that I could adjust the base case to account for this.

Final Answer:
1. ε, 0, 1 ∈ S [base]
2. x ∈ S → 0 • x • 0 ∈ S AND 1 • x • 1 ∈ S [constructor]
3. Nothing else is in S. [minimality]

DMC Problem 8.6(a)

The set P of parenthesis strings has a recursive definition as shown. (By default, nothing else is in P – minimality.)

1. ε ∈ P
2. x ∈ P → [x] ∈ P
x, y ∈ P → xy ∈ P

(a) Which of [[[]]]][, [][[]][[]], and [[][][] are in P. Give derivations of those that are.

[][[]][[]] is the only string that is in P. You can derive it like so:

X = ε	→ []	X = []	→ [[]]	X = [[]]	→ [[]][[]]	X = []	→ [][[]][[]]
	Via x ∈ P → [x] ∈ P		Via x ∈ P → [x] ∈ P	Y = [[]]	Via x, y ∈ P → xy ∈ P	Y = [[]][[]]	x, y ∈ P → xy ∈ P

DMC Problem 6.17

For x ∈ R, suppose $\mathbf{x + \frac{1}{x}}$ ∈ Z. Prove $\mathbf{x^n + \frac{1}{x^n}}$ ∈ Z for n ≥ 1.

P(n): For x ∈ R, $x^n + \frac{1}{x^n} \in Z$. I will prove P(n) for all n ≥ 1.

Strengthen the induction like so: Q(n): P(1) ∧ P(2) ∧ P(3) … ∧ P(n) are all true.

Base Cases:
Q(0) = $x^0 + \frac{1}{x^0} = 2$ for all x. Since 2 ∈ Z, Q(0) is true.
Q(1): P(0) is true as proven above. We also know P(1) is true because it is given in the question prompt ("suppose $\mathrm{x} + \frac{1}{\mathrm{x}}$ ∈ Z").

Induction Step:
I tinkered around with the binomial theorem: $(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots$ and decided to start do the proof in the form Q(n-1) → Q(n) as this made more sense to me. Since we have the base case Q(0), this can also form a complete induction chain for all values of n ≥ 1.

Prove Q(n-1) → Q(n) by direct proof:

Assume $x^{n-1} + \frac{1}{x^{n-1}}$ is true. Also, $x^1 + \frac{1}{x^1}$ is true (it's the base case Q(1)). Also, an integer times an integer always yields an integer.

$$x^{n-1} + \frac{1}{x^{n-1}} \times \left(x^1 + \frac{1}{x^1}\right) = x^{n-1}x + \frac{x}{x^{n-1}} + \frac{x^{n-1}}{x} + \frac{1}{x^{n-1}x}$$
$$= x^n + \frac{1}{x^{n-2}} + x^{n-2} + \frac{1}{x^n}$$

$\frac{1}{x^{n-2}} + x^{n-2}$ is true because of the strong induction in the original assumption.

$$= x^n + \frac{1}{x^n} + x^{n-2} + \frac{1}{x^n}$$

Below is the final proof. It puts Q(n) in terms of Q(n-1), Q(1), and Q(n-2) all of which are true statements because of the strong induction in our initial assumption. This means Q(n) has to be an integer because Q(n-1), Q(1), and Q(n-2) are also integers and integers operated on by integers always yield integers.

$$Q(n) = x^n + \frac{1}{x^n} = \left(x^{n-1} + \frac{1}{x^{n-1}}\right)\left(x^1 + \frac{1}{x^1}\right) - \left(x^{n-2} + \frac{1}{x^{n-2}}\right)$$

DMC Problem 7.23(a)

Give pseudocode for a recursive function that computes 3^{2^n} on input n.

(a) Prove that your function correctly computes 3^{2^n} for every n ≥ 0.

First I made a table of values

n	0	1	2	3
3^{2^n}	3	9	81	6561

Every value after 0 is the product of the previous value times itself (starting at 3). Thus, we can write a program like so:

```
Out=foo(n) {
        If (n==0) out=3
        Else out = foo(n-1)^2
}
```

Now the proof:

My claim is P(n): 3^{2^n}= foo(n). I'll prove P(n) for n ≥ 0 by induction.

Base case: P(0) = foo(0).

Foo(0) clearly equals 3. P(0) = 3^{2^0} = 3^1 = 3.

Induction step: show that P(n) → P(n+1)

I'll assume P(n): foo(n) = 3^{2^n} is True.
I'll prove f(n+1) = $3^{2^{n+1}}$is true via a direct proof.

Foo(n+1) = foo(n-1+1) * f(n-1+1) via the definition above
= f(n) * f(n)
= 3^{2^n}* 3^{2^n} via induction hypothesis
= $3^{2^{n+1}}$

Thus Foo(n+1) = $3^{2^{n+1}}$
Which means P(n) → P(n+1)
Which means P(n) is true for all n ≥ 0.

DMC Problem 7.23(b)

(b) Obtain a recurrence for the runtime T_n. Guess and prove a formula for T_n.

T_n for foo(0):
Testing to see if n=0 is one step
Outputting 3 is another step.
Two steps total.

T_n for foo(1)
Testing to see if n=0 is one step
Computing foo(0) is another 2 steps
Computing the square of foo(0) is another step
Outputting 9 is another step.
Five steps total

T_n for foo(2):
Testing to see if n=0 is one step
Computing foo(1) is 5 steps
Computing the square of foo(1) is another step
Outputting 81 is another step
Eight steps total

I did the same for further values of n to fill out the table below.

N	0	1	2	3
T_n	2	5	8	11

It's clear that the runtime starts at 2 and increases by 3 with each successive n.

I came up with the recurrence:
$T_0 = 2$; $T_n = T_{n-1} + 3$

This can be written as the formula: $f(n) = 3n + 2$.

Now I will prove P(n): $T_n = T_{n-1} + 3$ is equal to $f(n) = 3n + 2$ for $n \geq 0$ by induction.

Base Case: $T_0 = f(0)$
T_0 is equal to 2 which is easily obtained from the recurrence above.
$F(0) = 3(0) + 2 = 2$.

Induction Step: Prove $P(n) \rightarrow P(n+1)$
I'll assume P(n): $T_n = f(n)$ is true

$T_{n+1} = f(n+1)$
$T_{n-1+1} + 3 = 3(n+1) + 2$
$T_n + 3 = 3n + 3 + 2$
Sub in for T_n via the induction hypothesis
$3n + 2 + 3 = 3n + 3 + 2$

Proved that $T_{n+1} = f(n+1)$

Which means $P(n) \rightarrow P(n+1)$
Which means $P(n)$ is true for all $n \geq 0$.

DMC Problem 5.36

(a) Prove by induction on the number of people: Every tournament has a ranking.
(b) Prove: Every tournament has a ranking with the first player having the most wins.
(c) Give a tournament with 5 people that has no inconsistency.
(d) Prove by induction: In every tournament with no inconsistency, one player beats all other players and one player loses to all other players.
(e) Prove by induction: Every tournament with an inconsistency has a 3-person inconsistency.

DMC Problem 6.12

Show that every tournament has a ranking by strong induction using the following approach. Remove an arbitrary player v and consider the sub-tournament of all players who lost to v, and the sub-tournament of all players who beat v.