FK and IK of four axes

CY

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1 FK

$$
{}^{0}T_1 = \begin{bmatrix} & & 0 \\ Ry\left(\frac{\pi}{2}\right) Rz(\theta_1) & & 0 \\ & & 0 \\ 0 & & 1 \end{bmatrix} \tag{1}
$$

$$
{}^{1}T_2 = \begin{bmatrix} & & 0 \\ Rx\left(\frac{\pi}{2}\right) Rz\left(\theta_2\right) & & 0 \\ & & 0 \\ 0 & & 1 \end{bmatrix} \tag{2}
$$

$$
{}^{2}T_3 = \begin{bmatrix} & & 0 \\ Rx\left(-\frac{\pi}{2}\right) Rz(\theta_3) & & 0 \\ & & 0 \\ 0 & & 1 \end{bmatrix} \tag{3}
$$

$$
{}^{3}T_4 = \begin{bmatrix} & & l_x \\ Rx\left(\frac{\pi}{2}\right) Rz(\theta_4) & & 0 \\ & & d_1 \\ 0 & & 1 \end{bmatrix} \tag{4}
$$

$$
{}^{4}T_f = \begin{bmatrix} & -l_x \\ E & d_2 \\ & 0 \\ 0 & 1 \end{bmatrix} \tag{5}
$$

$$
{}^{0}T_f = {}^{0}T_1 \cdot {}^{1}T_2 \cdot {}^{2}T_3 \cdot {}^{3}T_4 \cdot {}^{4}T_f \tag{6}
$$

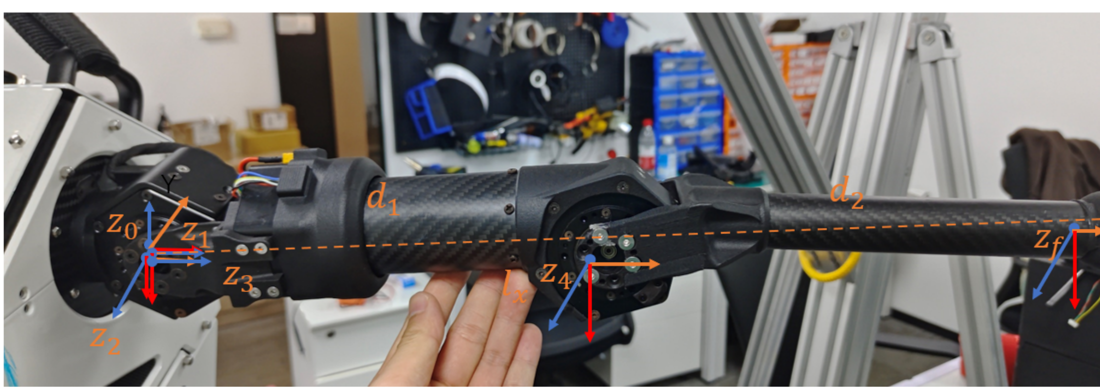

Figure 1: THEMIS Arm Frame Attachment. All joints are in zero position.

2 IK

$$
{}^0T_f = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{aligned}
r_{11} &= c_2 s_4 + c_3 c_4 s_2 && (7)\\
r_{21} &= c_4 \left(c_1 s_3 + c_2 c_3 s_1\right) - s_1 s_2 s_4 && (8)\\
r_{31} &= c_1 s_2 s_4 + c_4 \left(-c_1 c_2 c_3 + s_1 s_3\right) && (9)\\
r_{12} &= c_2 c_4 - c_3 s_2 s_4 && (10)\\
r_{22} &= -c_4 s_1 s_2 - s_4 \left(c_1 s_3 + c_2 c_3 s_1\right) && (11)\\
r_{22} &= c_1 c_4 s_2 - s_4 \left(-c_1 c_2 c_3 + s_1 s_3\right) && (12)\\
r_{13} &= s_2 s_3 && (13)\\
r_{23} &= s_1 c_2 s_3 - c_1 c_3 && (14)\\
r_{33} &= -c_1 c_2 s_3 - s_1 c_3 && (15)\\
p_x &= c_2(d_2 c_4 - lx s_4 + d_1) + s_2 c_3(lx - lx c_4 - d_2 s_4) && (16)\\
p_y &= c_1 s_3(-lx c_4 - d_2 s_4 + lx) + s_1 c_2 c_3(-lx c_4 - d_2 s_4 + lx) + s_1 s_2(-d_2 c_4 + lx s_4 - d_1) && (17)\\
p_z &= c_1 c_2 c_3(lx c_4 + d_2 s_4 - lx) + c_1 s_2(d_2 c_4 - lx s_4 + d_1) + s_1 s_3(-lx c_4 - d_2 s_4 + lx) && (18)
\end{aligned}
$$

$$
\begin{aligned}
A &= d_2 c_4 - lx s_4 + d_1 && (19)\\
B &= lx - lx c_4 - d_2 s_4 && (20)
\end{aligned}
$$

The equations used in the solution are as follows:

$$
\begin{aligned}
r_{13} &= s_2 s_3 && (21)\\
r_{23} &= s_1 c_2 s_3 - c_1 c_3 && (22)\\
r_{33} &= -c_1 c_2 s_3 - s_1 c_3 && (23)\\
p_x &= c_2 A + s_2 c_3 B && (24)\\
p_y &= (c_1 s_3 + s_1 c_2 c_3)B - s_1 s_2 A && (25)\\
p_z &= (s_1 s_3 - c_1 c_2 c_3)B + c_1 s_2 A && (26)
\end{aligned}
$$

By observing, we can obtain

$$
p_x^2 + p_y^2 + p_z^2 = d_1^2 - 2l_x^2 c_4 + d_2^2 + 2l_x^2 - 2d_1 l_x s_4 - 2d_2 l_x s_4 + 2d_1 d_2 c_4 \qquad (27)
$$

2.1 Solving θ_4

By formula (27), we can obtain:

$$
a\cos\theta_4 + b\sin\theta_4 = c
$$

where

$$
\begin{aligned}
a &= -2l_x^2 + 2d_1 d_2 && (28)\\
b &= -2d_1 l_x - 2d_2 l_x && (29)\\
c &= p_x^2 + p_y^2 + p_z^2 - d_1^2 - d_2^2 - 2l_x^2 && (30)
\end{aligned}
$$

- **Solution**
  - **Two Solutions**

$$
\theta = A\tan 2\left(\pm\sqrt{a^2 + b^2 - c^2}, c\right) + A\tan 2(b, a)
$$

- **Conditions for Solutions**
  * For a solution to exist: $a^2 + b^2 - c^2 > 0$
  * No solution (outside of the workspace): $a^2 + b^2 - c^2 < 0$
  * One solution (singularity): $a^2 + b^2 - c^2 = 0$

2.2 Solving θ_2

According to formula (21) (24),we can obtain

$$P_x = \gamma A + \alpha B \tag{31}$$
$$r_{13} = \beta \tag{32}$$
$$\alpha^2 + \beta^2 + \gamma^2 = 1 \tag{33}$$

where

$$\alpha = s_2 c_3 \tag{34}$$
$$\beta = s_2 s_3 \tag{35}$$
$$\gamma = c_2 \tag{36}$$

Simplify to obtain

$$k_1 \gamma^2 + k_2 \gamma + k_3 = 0 \tag{37}$$

where

$$k_1 = \frac{A^2}{B^2} + 1 \tag{38}$$
$$k_2 = -\frac{2P_x A}{B^2} \tag{39}$$
$$k_3 = \frac{P_x^2}{B^2} + r_{13}^2 - 1 \tag{40}$$

Now finally we can solve

$$\gamma = \frac{-k_2 \pm \sqrt{k_2^2 - 4k_1 k_3}}{2k_1}. \tag{41}$$

We have to determine the sign after comparing both solutions in the end. And then we can simply derive

$$\theta_2 = \operatorname{atan2}\left(\pm\sqrt{1-\gamma^2}, \gamma\right). \tag{42}$$

If $B = 0$, now $\theta_4 = 0$,we need to select another base to avoid singularity, e.g.,

$$p_x = c_2 A \tag{43}$$

$$b = p_x / A \tag{44}$$
$$\theta = A\tan 2\left(\pm\sqrt{1-b^2}, b\right) \tag{45}$$
$$\theta_2 = \theta \tag{46}$$
$$\theta_2 = -\theta \tag{47}$$

2.3 Solving θ_3

According to formula(21),we can obtain

$$\sin\theta = a \quad a \in [-1, 1] \tag{48}$$

$$\cos\theta = b \quad a \in [-1, 1] \tag{49}$$

where

$$a = \frac{r_{13}}{s_2}, \quad b = \frac{p_x - c_2 A}{s_2 B} \tag{50}$$

$$\theta_3 = \text{atan2}\,(a, b) \tag{51}$$

If $s_2 = 0$, we need to select another base to avoid singularity, e.g.,

$$\theta_3 = initial \tag{52}$$

2.4 Solving θ_1

Formulae (25) and (26) are sorted into a target form.

$$a\cos\theta_1 + c\sin\theta_1 = d \tag{53}$$

$$e\cos\theta_1 + f\sin\theta_1 = g \tag{54}$$

where

$$\begin{aligned} a &= s_3 B \\ c &= c_2 c_3 B - s_2 A \\ d &= p_y \\ e &= s_2 A - c_2 c_3 B \\ f &= s_3 B \\ g &= p_z \end{aligned}$$

For a solution to exist, the determinant must not be zero

$$af - ce \neq 0 \tag{55}$$

Solution

$$\begin{cases} \sin\theta = \frac{ag-de}{af-ce} \\ \cos\theta = \frac{df-cg}{af-ce} \end{cases} \Rightarrow \theta_1 = \begin{cases} \text{atan2}(ag - de, df - cg), & af - ce > 0 \\ \text{atan2}(de - ag, cg - df), & af - ce < 0 \end{cases} \tag{56}$$