Kinematics

1.1 Leg Kinematics

1.1.1 Forward Kinematics

Position & Orientation

Figure 1.1 shows the frame attachment. Note that both legs share the same frame attachment except the value of l_y^{leg}. The individual transformation matrices are

$$
{}^{b}\boldsymbol{T}_1 = \begin{bmatrix} & l_x \\ \boldsymbol{R}_z\left(\theta_1\right) & l_y^{leg} \\ & l_z \\ \boldsymbol{0} & 1 \end{bmatrix}, \tag{1}
$$

$$
{}^{1}\boldsymbol{T}_2 = \begin{bmatrix} & 0 \\ \boldsymbol{R}_x\left(\frac{\pi}{2}\right)\boldsymbol{R}_z\left(\theta_2 - \frac{\pi}{2}\right) & 0 \\ & 0 \\ \boldsymbol{0} & 1 \end{bmatrix}, \tag{2}
$$

$$
{}^{2}\boldsymbol{T}_3 = \begin{bmatrix} & 0 \\ \boldsymbol{R}_x\left(-\frac{\pi}{2}\right)\boldsymbol{R}_z\left(\theta_3\right) & 0 \\ & 0 \\ \boldsymbol{0} & 1 \end{bmatrix}, \tag{3}
$$

$$
{}^{3}\boldsymbol{T}_{4} = \begin{bmatrix} & & d_3 \\ \boldsymbol{R}_x\left(\frac{\pi}{2}\right)\boldsymbol{R}_z\left(\theta_4\right) & & 0 \\ & & 0 \\ & \mathbf{0} & 1 \end{bmatrix}, \tag{4}
$$

$$
{}^{4}\boldsymbol{T}_{5} = \begin{bmatrix} & d_4 \\ \boldsymbol{R}_z\left(\theta_5\right) & 0 \\ & 0 \\ \mathbf{0} & 1 \end{bmatrix}, \tag{5}
$$

$$
{}^{5}\boldsymbol{T}_{f} = \begin{bmatrix} & f_x \\ \boldsymbol{R}_y\left(-\frac{\pi}{2}\right)\boldsymbol{R}_z\left(\frac{\pi}{2}\right) & f_y \\ & 0 \\ \mathbf{0} & 1 \end{bmatrix}. \tag{6}
$$

The overall transformation matrix can be thus determined by

$$
{}^{b}\boldsymbol{T}_{f} = {}^{b}\boldsymbol{T}_{1}\,{}^{1}\boldsymbol{T}_{2}\,{}^{2}\boldsymbol{T}_{3}\,{}^{3}\boldsymbol{T}_{4}\,{}^{4}\boldsymbol{T}_{5}\,{}^{5}\boldsymbol{T}_{f} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{7}
$$

where

$$
r_{11} = +c_1 c_2 c_{45} + \left(s_1 s_3 - c_1 s_2 c_3\right) s_{45}, \tag{8}
$$

$$
r_{21} = +s_1 c_2 c_{45} - \left(c_1 s_3 + s_1 s_2 c_3\right) s_{45}, \tag{9}
$$

$$
r_{31} = +s_2 c_{45} + c_2 c_3 s_{45} \tag{10}
$$

$$
r_{12} = -s_1 c_3 - c_1 s_2 s_3, \tag{11}
$$

$$r_{22} = +c_1 c_3 - s_1 s_2 s_3, \tag{12}$$

$$r_{32} = +c_2 s_3, \tag{13}$$

$$r_{13} = -c_1 c_2 s_{45} + (s_1 s_3 - c_1 s_2 c_3)\, c_{45}, \tag{14}$$

$$r_{23} = -s_1 c_2 s_{45} - (c_1 s_3 + s_1 s_2 c_3)\, c_{45}, \tag{15}$$

$$r_{33} = -s_2 s_{45} + c_2 c_3 c_{45} \tag{16}$$

$$p_x = l_x - f_x r_{13} + f_y r_{11} + d_4 c_1 c_2 s_4 - (d_3 + d_4 c_4)(s_1 s_3 - c_1 s_2 c_3) \tag{17}$$

$$p_y = l_y - f_x r_{23} + f_y r_{21} + d_4 s_1 c_2 s_4 + (d_3 + d_4 c_4)(c_1 s_3 + s_1 s_2 c_3) \tag{18}$$

$$p_z = l_z - f_x r_{33} + f_y r_{31} + d_4 s_2 s_4 - (d_3 + d_4 c_4)\, c_2 c_3. \tag{19}$$

Angular Velocity

The angular velocity Jacobian ${}^{b}_{\omega}\boldsymbol{J}_f\,(\boldsymbol{\theta})$ can be determined by the following construction:

$${}^{b}_{\omega}\boldsymbol{J}_f = \begin{bmatrix} {}^{b}\boldsymbol{z}_1 & {}^{b}\boldsymbol{z}_2 & {}^{b}\boldsymbol{z}_3 & {}^{b}\boldsymbol{z}_4 & {}^{b}\boldsymbol{z}_5 \end{bmatrix}. \tag{20}$$

Note that ${}^{b}_{\omega}\boldsymbol{J}_f\dot{\boldsymbol{\theta}}$ will actually give us the angular velocity of frame $\{5\}$ described in the base frame $\{b\}$ and frame $\{f\}$ shares the same angular velocity since they are relatively stationary. Differentiating each term yields

$$\frac{\mathrm{d}}{\mathrm{d}t}{}^{b}\boldsymbol{z}_i = \frac{\mathrm{d}}{\mathrm{d}t}\left({}^{b}\boldsymbol{R}_i\,{}^{i}\boldsymbol{z}_i\right) = {}^{b}\boldsymbol{R}_i\,\widehat{{}^{i}\boldsymbol{\omega}_i}\,{}^{i}\boldsymbol{z}_i = {}^{b}\boldsymbol{\omega}_i \times {}^{b}\boldsymbol{z}_i, \tag{21}$$

where the angular velocity is the accumulation of precedent angular motions:

$${}^{b}\boldsymbol{\omega}_i = \sum_{j=1}^{i} \dot{\theta}_j\,{}^{b}\boldsymbol{z}_j. \tag{22}$$

Linear Velocity

The linear velocity Jacobian ${}^{b}_{v}\boldsymbol{J}_f\left(\boldsymbol{\theta}\right)$ can be determined by the following construction:

$$
{}^{b}_{v}\boldsymbol{J}_f = \left[\begin{array}{cccc} {}^{b}\boldsymbol{z}_1 \times \left({}^{b}\boldsymbol{p}_f - {}^{b}\boldsymbol{p}_1\right) & {}^{b}\boldsymbol{z}_2 \times \left({}^{b}\boldsymbol{p}_f - {}^{b}\boldsymbol{p}_2\right) & \cdots & {}^{b}\boldsymbol{z}_5 \times \left({}^{b}\boldsymbol{p}_f - {}^{b}\boldsymbol{p}_5\right) \end{array}\right]. \tag{23}
$$

Differentiating each term yields

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{b}\boldsymbol{z}_i \times \left({}^{b}\boldsymbol{p}_f - {}^{b}\boldsymbol{p}_i\right)\right) = \frac{\mathrm{d}}{\mathrm{d}t}{}^{b}\boldsymbol{z}_i \times \left({}^{b}\boldsymbol{p}_f - {}^{b}\boldsymbol{p}_i\right) + {}^{b}\boldsymbol{z}_i \times \left(\frac{\mathrm{d}}{\mathrm{d}t}{}^{b}\boldsymbol{p}_f - \frac{\mathrm{d}}{\mathrm{d}t}{}^{b}\boldsymbol{p}_i\right), \tag{24}
$$

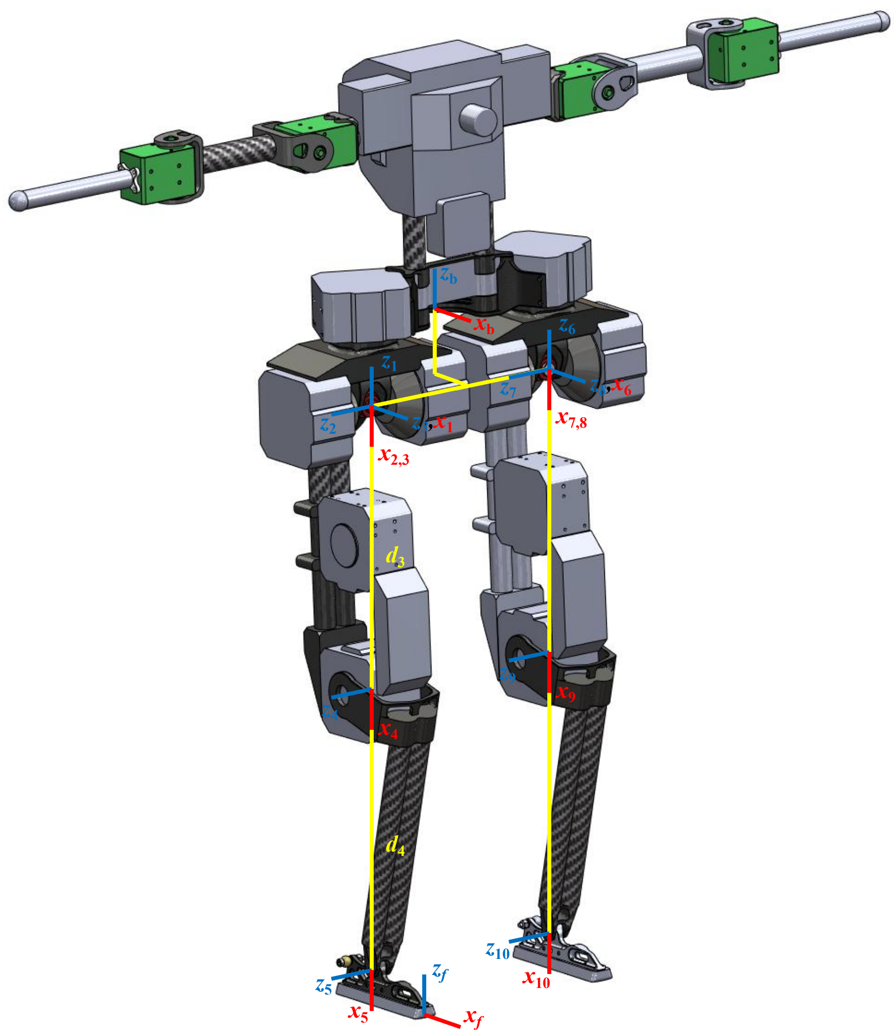

Figure 1.1: BRUCE Leg Frame Attachment. All joints are in zero position.

where

$$\frac{\mathrm{d}}{\mathrm{d}t}{}^{b}\boldsymbol{p}_i = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{i}{}^{b}\boldsymbol{R}_{j-1}{}^{j-1}\boldsymbol{p}_j = \sum_{j=1}^{i}{}^{b}\boldsymbol{\omega}_{j-1}\times\left({}^{b}\boldsymbol{R}_{j-1}{}^{j-1}\boldsymbol{p}_j\right) = \sum_{j=1}^{i}{}^{b}\boldsymbol{\omega}_{j-1}\times\left({}^{b}\boldsymbol{p}_j - {}^{b}\boldsymbol{p}_{j-1}\right). \quad (25)$$

1.1.2 Inverse Kinematics

Position & Orientation

Before we even start, note that each leg of BRUCE is deficient since it only has five DoFs, which is less than six. As a result, we cannot achieve any arbitrary special configurations of both position and orientation. For our application, we are most interested in the position configuration with three DoFs as well as the direction of the x_f–axis (foot pointing direction) with two DoFs. Therefore, the IK problem is that given r_{11}, r_{21}, r_{31}, p_x, p_y, p_z, solve θ_1, θ_2, θ_3, θ_4, θ_5. In addition, for simplicity, it is assumed $f_x = 0$ because otherwise I cannot find an analytical solution ☹. It is good for most scenarios though since we usually focus on the ankle joint instead of some point on the foot.

Now by OBSERVATION ☺, we can obtain

$$\underbrace{\left(p_x - l_x - f_y r_{11}\right)}_{r_x}^2 + \underbrace{\left(p_y - l_y^{leg} - f_y r_{21}\right)}_{r_y}^2 + \underbrace{\left(p_z - l_z - f_y r_{31}\right)}_{r_z}^2 = d_3^2 + d_4^2 + 2d_3d_4c_4 \quad (26)$$

$$\Rightarrow c_4 = \frac{r_x^2 + r_y^2 + r_z^2 - d_3^2 - d_4^2}{2d_3d_4}, \quad (27)$$

which solves

$$\theta_4 = \mathrm{atan2}\left(-\sqrt{1 - c_4^2}, c_4\right) \quad (28)$$

since $\theta_4 \leq 0$.

By another OBSERVATION ☺, we can obtain

$$r_x c_1 c_2 + r_y s_1 c_2 + r_z s_2 = d_4 s_4, \tag{29}$$

$$\underbrace{(r_z r_{21} - r_y r_{31})}_{t_x} c_1 c_2 + \underbrace{(r_x r_{31} - r_z r_{11})}_{t_y} s_1 c_2 + \underbrace{(r_y r_{11} - r_x r_{21})}_{t_z} s_2 = 0. \tag{30}$$

Setting $\alpha = c_1 c_2$, $\beta = s_1 c_2$, $\gamma = s_2$ yields

$$r_x \alpha + r_y \beta + r_z \gamma = d_4 s_4, \tag{31}$$

$$t_x \alpha + t_y \beta + t_z \gamma = 0, \tag{32}$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1. \tag{33}$$

Solving the first two equations yields

$$\alpha = k_1 \gamma + k_2, \tag{34}$$

$$\beta = k_3 \gamma + k_4, \tag{35}$$

where

$$k_1 = \frac{r_y t_z - r_z t_y}{r_x t_y - r_y t_x}, \quad k_2 = \frac{d_4 s_4 t_y}{r_x t_y - r_y t_x}, \tag{36}$$

$$k_3 = \frac{r_z t_x - r_x t_z}{r_x t_y - r_y t_x}, \quad k_4 = \frac{-d_4 s_4 t_x}{r_x t_y - r_y t_x}. \tag{37}$$

Substituting them into the third equation yields

$$k_5 \gamma^2 + k_6 \gamma + k_7 = 0, \tag{38}$$

where

$$k_5 = k_1^2 + k_3^2 + 1, \tag{39}$$

$$k_6 = 2\left(k_1 k_2 + k_3 k_4\right), \tag{40}$$

$$k_7 = k_2^2 + k_4^2 - 1. \tag{41}$$

Now finally we can solve

$$\gamma = \frac{-k_6 \pm \sqrt{k_6^2 - 4k_5 k_7}}{2k_5}. \tag{42}$$

We have to determine the sign after comparing both solutions in the end. And then we can simply derive

$$\theta_2 = \operatorname{atan2}\left(\gamma, \sqrt{1-\gamma^2}\right) \tag{43}$$

since $\theta_2 \sim 0$. If $r_x t_y - r_y t_x = 0$, we need to select another base to avoid singularity, e.g.,

$$\beta = k_1 \alpha + k_2, \tag{44}$$

$$\gamma = k_3 \alpha + k_4, \tag{45}$$

if $r_y t_z - r_z t_y \neq 0$ or

$$\alpha = k_1 \beta + k_2, \tag{46}$$

$$\gamma = k_3 \beta + k_4, \tag{47}$$

if $r_x t_z - r_z t_x \neq 0$. These three will not be zero simultaneously. If $c_2 = 0$, singularity will lead to infinite numbers of solution for $\theta_1 = 0$. One trick is just adopting the previous solution if we are continuously solving IK. Another one is consider $c_2 = \pm 0.001$ when solving θ_1. Luckily, it will never happen during nominal operation. Otherwise,

$$\theta_1 = \operatorname{atan2}\left(\beta / c_2, \alpha / c_2\right). \tag{48}$$

By last OBSERVATION ☺, we can obtain

$$s_3\left(d_3+d_4c_4\right)=r_yc_1-r_xs_1, \tag{49}$$

$$c_3\left(d_3+d_4c_4\right)=\left(r_xc_1+r_ys_1\right)s_2-r_zc_2, \tag{50}$$

$$s_3s_{45}=r_{11}s_1-r_{21}c_1, \tag{51}$$

$$c_3s_{45}=r_{31}c_2-\left(r_{11}c_1+r_{21}s_1\right)s_2, \tag{52}$$

$$c_{45}=\left(r_{11}c_1+r_{21}s_1\right)c_2+r_{31}s_2. \tag{53}$$

Solving the first two equations yields

$$\theta_3=\operatorname{atan2}\left(s_3,c_3\right). \tag{54}$$

Solving the last three equations yields

$$\theta_5=\operatorname{atan2}\left(s_{45},c_{45}\right)-\theta_4. \tag{55}$$

Angular Velocity

We can utilize the angular velocity Jacobian ${}^{b}_{\omega}\boldsymbol{J}_f\left(\boldsymbol{\theta}\right)$ for the inverse, e.g., ${}^{b}_{\omega}\boldsymbol{J}_f\left(\boldsymbol{\theta}\right)\dot{\boldsymbol{\theta}}={}^{b}\boldsymbol{\omega}_f$.

Linear Velocity

We can utilize the linear velocity Jacobian ${}^{b}_{v}\boldsymbol{J}_f\left(\boldsymbol{\theta}\right)$ for the inverse, e.g., ${}^{b}_{v}\boldsymbol{J}_f\left(\boldsymbol{\theta}\right)\dot{\boldsymbol{\theta}}={}^{b}\dot{\boldsymbol{p}}_f$.

1.1.3 Transformation Between Joint Space and Actuator Space

Position

For the right leg, we have

$$\begin{cases} \theta_1 = -\varphi_1 \\ \theta_2 = +\frac{1}{2}(\varphi_2 - \varphi_3) \\ \theta_3 = -\frac{1}{2}(\varphi_2 + \varphi_3) \\ \theta_4 = -\varphi_4 \\ \theta_5 = +\varphi_4 + \varphi_5 \end{cases} \Rightarrow \begin{cases} \varphi_1 = -\theta_1 \\ \varphi_2 = +\theta_2 - \theta_3 \\ \varphi_3 = -\theta_2 - \theta_3 \\ \varphi_4 = -\theta_4 \\ \varphi_5 = +\theta_4 + \theta_5 \end{cases}, \tag{56}$$

where φ_i denotes the rotational position of the ith BEAR actuator after homing calibration. Similarly, for the left leg, we have

$$\begin{cases} \theta_6 = -\varphi_6 \\ \theta_7 = +\frac{1}{2}(\varphi_7 - \varphi_8) \\ \theta_8 = -\frac{1}{2}(\varphi_7 + \varphi_8) \\ \theta_9 = +\varphi_9 \\ \theta_{10} = -\varphi_9 - \varphi_{10} \end{cases} \Rightarrow \begin{cases} \varphi_6 = -\theta_6 \\ \varphi_7 = +\theta_7 - \theta_8 \\ \varphi_8 = -\theta_7 - \theta_8 \\ \varphi_9 = +\theta_9 \\ \varphi_{10} = -\theta_9 - \theta_{10} \end{cases}. \tag{57}$$

Velocity

Now taking the time derivatives of them gives the velocity mapping:

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & +0.5 & -0.5 & 0 & 0 \\ 0 & -0.5 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 & +1 \end{bmatrix} \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \\ \dot{\varphi}_3 \\ \dot{\varphi}_4 \\ \dot{\varphi}_5 \end{bmatrix}, \tag{58}$$

$$\begin{bmatrix} \dot{\theta}_6 \\ \dot{\theta}_7 \\ \dot{\theta}_8 \\ \dot{\theta}_9 \\ \dot{\theta}_{10} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & +0.5 & -0.5 & 0 & 0 \\ 0 & -0.5 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} \dot{\varphi}_6 \\ \dot{\varphi}_7 \\ \dot{\varphi}_8 \\ \dot{\varphi}_9 \\ \dot{\varphi}_{10} \end{bmatrix}. \tag{59}$$

We can easily verify that these two Jacobians are invertible.

Torque

Finally, the transpose of these two Jacobians yields the static torque mapping:

$$\begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \\ \kappa_4 \\ \kappa_5 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & +0.5 & -0.5 & 0 & 0 \\ 0 & -0.5 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & 0 & +1 \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5 \end{bmatrix}, \tag{60}$$

$$\begin{bmatrix} \kappa_6 \\ \kappa_7 \\ \kappa_8 \\ \kappa_9 \\ \kappa_{10} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & +0.5 & -0.5 & 0 & 0 \\ 0 & -0.5 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \tau_6 \\ \tau_7 \\ \tau_8 \\ \tau_9 \\ \tau_{10} \end{bmatrix} \tag{61}$$

where κ_i denotes the BEAR actuator output torque and τ_i denotes the joint torque.