FK and IK of four axes

CY

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1 FK

$$
{}^{b}T_1 = \begin{bmatrix} Ry\left(\frac{\pi}{4}\right) Rz\left(\theta_1\right) & \begin{matrix} l_x \\ l_y^{\text{leg}} \\ l_z \end{matrix} \\ 0 & 1 \end{bmatrix} \tag{1}
$$

$$
{}^{1}T_2 = \begin{bmatrix} Ry\left(\frac{\pi}{-2}\right) Rz\left(\theta_2 - \frac{\pi}{2}\right) & \begin{matrix} d_1 \\ 0 \\ 0 \end{matrix} \\ 0 & 1 \end{bmatrix} \tag{2}
$$

$$
{}^{2}T_3 = \begin{bmatrix} Rx\left(-\frac{\pi}{4}\right) Ry\left(\frac{\pi}{2}\right) Rz(\theta_3) & \begin{matrix} d_2^{\text{leg}} \\ 0 \\ 0 \end{matrix} \\ 0 & 1 \end{bmatrix} \tag{3}
$$

$$
{}^{3}T_4 = \begin{bmatrix} Rz(\theta_4) & \begin{matrix} d_3 \\ 0 \\ 0 \end{matrix} \\ 0 & 1 \end{bmatrix} \tag{4}
$$

$$
{}^{4}T_5 = \begin{bmatrix} E & \begin{matrix} d_4 \\ 0 \\ 0 \end{matrix} \\ 0 & 1 \end{bmatrix} \tag{5}
$$

$$
{}^{b}T_5 =^{b} T_1 \cdot^{1} T_2 \cdot^{2} T_3 \cdot^{3} T_4 \cdot^{4} T_5
$$

2 IK

$$
bT_0^{-1} bT_5 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

with

$$
bT_0 = \begin{bmatrix} Ry\left(\frac{\pi}{4}\right) & \begin{matrix} l_x \\ l_y^{\text{leg}} \\ l_z \end{matrix} \\ 0 & 1 \end{bmatrix} \tag{6}
$$

$$
bT_0^{-1} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} l_x + \frac{\sqrt{2}}{2} l_z \\ 0 & 1 & 0 & -l_y^{\text{leg}} \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} l_x - \frac{\sqrt{2}}{2} l_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

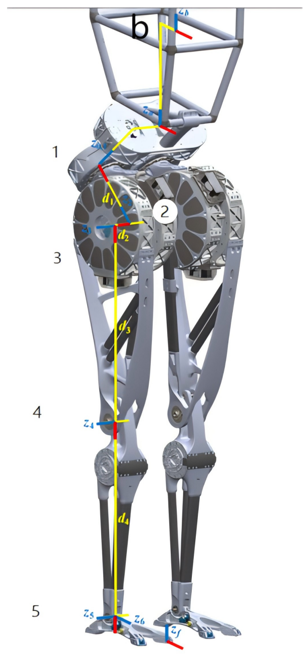

Figure 1: THEMIS Leg Frame Attachment. All joints are in zero position..

However, the result is still too complicated to handle. One trick by observation is that we can intentionally add an offset of $\pi/4$ to θ_3 so that the frames are aligned, i.e., $\theta_3 = \theta_3' - \pi/4$, which can simplify the computation significantly. Substituting θ_3 with θ_3' yields

$$
\begin{aligned}
r_{11} &= c_1 s_{34} + s_1 s_2 c_{34} && (7)\\
r_{21} &= s_1 s_{34} - c_1 s_2 c_{34} && (8)\\
r_{31} &= -c_2 c_{34} && (9)\\
r_{12} &= c_1 c_{34} - s_1 s_2 s_{34} && (10)\\
r_{22} &= s_1 c_{34} + c_1 s_2 s_{34} && (11)\\
r_{32} &= c_2 s_{34} && (12)\\
r_{13} &= c_2 s_1 && (13)\\
r_{23} &= -c_1 c_2 && (14)\\
r_{33} &= s_2 && (15)\\
px &= c_1(d_1 + d_3 s_3 + d_4 s_{34}) - s_1(-c_2 d_2^{\text{leg}} - s_2(d_3 c_3 + d_4 c_{34})) && (16)\\
py &= s_1(d_1 + d_3 s_3 + d_4 s_{34}) + c_1(-c_2 d_2^{\text{leg}} - s_2(d_3 c_3 + d_4 c_{34})) && (17)\\
pz &= d_2^{\text{leg}} s_2 - c_2(d_3 c_3 + d_4 c_{34}) && (18)
\end{aligned}
$$

2.1 Soving θ_2

- **Equation**

$$\sin\theta = r33, \quad r33 \in [-1, 1]$$
$$\cos\theta = \pm\sqrt{1 - r33^2}$$

- **Solution**

$$\theta = \text{atan2}(r33, \pm\sqrt{1 - r33^2})$$

  - *Two Solutions*

$$\theta_2 = \theta$$

or

$$= 180^\circ - \theta$$

– *Singularity at the Boundary*

$$\text{When } \theta_2 = \pm 90^\circ, \quad |a| = 1$$

2.2 Soving θ_1

By (7) and (8), we can obtain

$$\theta_1 = \text{atan2}(r13/c2, -r23/c2)$$

where θ_2 does not reach 90 degrees, $c2 \neq 0$. .

2.3 Soving θ_4 and θ_3

$$k_8 = d_3 s_3 + d_4 s_{34} = A,$$

$$k_9 = d_3 c_3 + d_4 c_{34} = B$$

where where

$$B = \frac{d_2^{\text{leg}} \cdot s_2 - p_z}{c_2}$$
$$A = \frac{p_x + s_1 \cdot \left(-c_2 \cdot d_2^{\text{leg}} - s_2 \cdot B\right)}{c_1} - d_1$$

Sum of squares of them yields

$$k_8^2 + k_9^2 = d_3^2 + d_4^2 + 2 d_3 d_4 c_4 \Rightarrow c_4 = \frac{k_8^2 + k_9^2 - d_3^2 - d_4^2}{2 d_3 d_4},$$

which solves

$$\theta_4 = \text{atan2}\left(-\sqrt{1 - c_4^2}, c_4\right),$$

$$\theta_3' = \text{atan2}\left((d_3 + d_4 c_4) k_8 - d_4 s_4 k_9, (d_3 + d_4 c_4) k_9 + d_4 s_4 k_8\right) \Rightarrow \theta_3 = \theta_3' - \frac{\pi}{4},$$

since $\theta_4 \leq 0$.

$$p_1 = l_x + \frac{\sqrt{2} \cdot p_x}{2} + \frac{\sqrt{2} \cdot p_z}{2}$$
$$p_2 = l_y + p_y$$
$$p_3 = l_z - \frac{\sqrt{2} \cdot p_x}{2} + \frac{\sqrt{2} \cdot p_z}{2}$$

$$p_x = \frac{p_1 - p_3 - l_x + l_z}{\sqrt{2}}$$
$$p_y = p_2 - l_y$$
$$p_z = \frac{p_1 + p_3 - l_x - l_z}{\sqrt{2}}$$

$$r_{13} = \frac{\sqrt{2} \cdot r_{13}}{2} - \frac{\sqrt{2} \cdot r_{33}}{2}$$
$$p_x = -\frac{\sqrt{2} \cdot l_x}{2} + \frac{\sqrt{2} \cdot l_z}{2} + \frac{\sqrt{2} \cdot p_x}{2} - \frac{\sqrt{2} \cdot p_z}{2}$$
$$r_{23} = r_{23}$$
$$p_y = -l_y + p_y$$
$$r_{33} = \frac{\sqrt{2} \cdot r_{13}}{2} + \frac{\sqrt{2} \cdot r_{33}}{2}$$
$$p_z = -\frac{\sqrt{2} \cdot l_x}{2} - \frac{\sqrt{2} \cdot l_z}{2} + \frac{\sqrt{2} \cdot p_x}{2} + \frac{\sqrt{2} \cdot p_z}{2}$$