MATH 1052 Notes

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Contents

1 Infinite Series

Theorem 14.6. The **Comparison Test**
Let $\sum a_n$ be a series of non-negative terms.

1. If $\sum a_n$ converges and $\forall n\ |b_n| \leq a_n$, then b_n converges.
2. If $\sum a_n = \infty$ and $\forall n\ b_n \geq a_n$, then $\sum b_n = \infty$.

Proof. Case 1:
Let $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} b_k$ be the sequence of partial sums. Since $\sum a_n$ converges, s_n converges, so s_n is a Cauchy sequence.
Thus, let $\varepsilon > 0$ be given. Then, $\exists N$ s.t. $n, m > N \implies |s_n - s_m| < \varepsilon$. We take $n > m$. Then,

$$\begin{aligned}
|t_n - t_m| &= \left|\sum_{k=m+1}^{n} b_k\right| \\
&\leq \sum_{k=m+1}^{n} |b_k| && \text{(By Triangle Inequality)} \\
&\leq \sum_{k=m+1}^{n} a_k \\
&= |s_n - s_m| < \varepsilon
\end{aligned}$$

Hence (t_n) is Cauchy, and so converges, therefore $\sum b_n$ converges.
Case 2:
With s_n and t_n as above, we have

$$\begin{aligned}
t_n &= \sum_{k=1}^{n} b_k \\
&\geq \sum_{k=1}^{n} a_k \\
&= s_n
\end{aligned}$$

Since $\sum a_n = \infty$, $\lim s_n = \infty$, then by Exercise 9.9a), $\lim t_n = \infty$. Thus, $\sum b_n = \infty$. $\square$

Proposition 1.1. If $\sum a_n$ converges, then $\sum k a_n = k \sum a_n$.
If $\sum a_n$ diverges, and $k \neq 0$, then $\sum k a_n$ diverges.

Examples. Consider

$$\sum_{k=1}^{\infty} \frac{2n+1}{3n^2+n}$$

By rough intuition, this is roughly equal to $\frac{1}{n}$, which diverges. We have

$$\begin{aligned}
\frac{2n+1}{3n^2+n} &> \frac{2n}{4n^2} \\
&= \frac{2}{n} \\
&> \frac{1}{n}
\end{aligned}$$

Since $\sum \frac{1}{n}$ diverges, by the comparison test this series diverges.

Consider

$$\sum_{k=1}^{\infty} \frac{2n+1}{3n^3+n}$$

We have

$$\begin{aligned} \frac{2n+1}{3n^3} &< \frac{2n+n}{3n^3} \\ &= \frac{1}{n^2} \end{aligned}$$

Since $\sum \frac{1}{n^2}$ converges, by the comparison test this series converges.

Definition 1.2. A series $\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges. If $\sum a_n$ converges, but $\sum |a_n|$ diverges, then $\sum a_n$ **converges conditionally**.

Theorem 14.7. Absolutely convergent series are convergent.

Proof. Suppose $\sum |a_n|$ is convergent. Since $|a_n| \leq |a_n|$, and $|a_n|$ converges, then $|a_n|$ converges by the comparison test. $\square$

Examples. Consider

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

Since $\sum \left|\frac{(-1)^n}{n^2}\right| = \sum \frac{1}{n^2}$ is a convergent series, this series converges absolutely.
Consider

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

Since the absolute value of this expression is equal to $\frac{1}{n}$, then this series does not converge absolutely. However, since this is an alternating sequence this series does converge conditionally.

Proposition 1.3. If $\sum a_n$ converges, then $\sum ka_n = k\sum a_n$.
If $\sum a_n$ and $\sum b_n$ converge absolutely, then

$$\sum(a_n + b_n) = \sum a_n + \sum b_n$$

Example. Consider

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{4^{n+1}}{5^n} &= 4\sum_{n=2}^{\infty} \frac{4^n}{5^n} \\ &= 4\sum_{n=2}^{\infty} (\frac{4}{5})^n \\ &= 4(\frac{4}{5})^2 \sum_{n=2}^{\infty} (\frac{4}{5})^{n-2} \\ \text{Let } m = n-2 & \\ &= 4(\frac{4}{5})^2 \sum_{m=0}^{\infty} (\frac{4}{5})^m \\ &= 4(\frac{4}{5})^2 (\frac{1}{1-\frac{4}{5}}) \end{aligned}$$

An alternative way to solve this is

$$
\begin{aligned}
4\sum_{n=2}^{\infty}(\frac{4}{5}) &= 4\left(\sum_{n=0}^{\infty}(\frac{4}{5})^n - \sum_{n=0}^{1}(\frac{4}{5})^n\right) \\
&= 4\left(\sum_{n=0}^{\infty}(\frac{4}{5})^n - 1 - \frac{4}{5}\right) \\
&= 4\left(\frac{1}{1-\frac{4}{5}} - 1 - \frac{4}{5}\right)
\end{aligned}
$$

Theorem 14.8. The **Ratio Test**
Let $\sum a_n$ be a series of non-zero terms for which

$$L = \left|\frac{a_{n+1}}{a_n}\right|$$

exists.
Then,

$$\sum a_n \begin{cases} \text{converges absolutely, if } L < 1 \\ \text{diverges, if } L > 1 \\ \text{can either converge or diverge, if } L = 1 \end{cases}$$

Proof. Since $\left|\frac{a_{n+1}}{a_n}\right| > 0$, we have $L \geq 0$ (by Textbook exercise 9.9c)
Case 1:
Since $0 \leq L < 1$, we can choose k such that $0 \leq l < k < 1$ by Theorem 4.7. Thus, $k - L > 0$. Therefore, $\exists N$ s.t.

$$n > N \implies \left||\frac{a_{n+1}}{a_n}| - L\right| < k - L$$

Thus, $-(k-L) < \left|\frac{a_{n+1}}{a_n}\right| < k - L$ and so $\left|\frac{a_{n+1}}{a_n}\right| < k$. Therefore, $|a_{n+1}| < k|a_n|$. Hence, $|a_{N+2}| < k|a_{N+1}|$ (Recall this holds for $n > N$ from the epsilon proof).
As a result, we also have $|a_{N+3}| < k|a_{N+2}| < k^2|a_{N+1}|$ and so on. Thus in general, we have $|a_{N+j}| < k^{j-1}|a_{N+1}|$ for some $j \in \mathbb{N}$ with $j \geq 2$. Therefore, we have

$$
\begin{aligned}
\sum_{j=2}^{\infty} k^{j-1}|a_{N+1}| &= |a_{N+1}|\sum_{j=2}^{\infty} k^{j-1} \\
&= |a_{N+1}|\frac{1}{k}\sum_{j=2}^{\infty} k^j
\end{aligned}
$$

This is a convergent series since $0 < k < 1$. Thus by the comparison test, $\sum a_n$ converges absolutely.
Case 2:
Since $L > 1$, we can choose $1 < k < L$. Then $L - k > 0$. Therefore, $\exists N$ s.t.

$$n > N \implies \left||\frac{a_{n+1}}{a_n}| - L\right| < L - k$$

Thus, $-(L-k) < \left|\frac{a_{n+1}}{a_n}\right| < L - k$ and so $k < \left|\frac{a_{n+1}}{a_n}\right|$. Therefore, $|a_{n+1}| > k|a_n|$. Thus in general, we have $|a_{N+j}| > k^{j-1}|a_{N+1}|$ for some $j \in \mathbb{N}$ with $j \geq 2$. Since $\lim k^{j-1}|a_{N+1}| = |a_{N+1}|\frac{1}{k}\lim_{j\to\infty} k^j = \infty$, by Exercise 9.9a, $\lim |a_n| = \infty$, thus $\lim a_n \neq 0$. Thus this sequence diverges. $\square$

Example. Consider

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

We have

$$\lim_{n\to\infty}\left|\frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}}\right| = \lim_{n\to\infty}\left|(\frac{n+1}{n})^2(\frac{2^n}{2^{n+1}})\right|$$
$$= \frac{1}{2}\lim\left(\frac{n+1}{n}\right)^2$$
$$= \frac{1}{2}$$

Since $0 < \frac{1}{2} < 1$, this series is convergent.

Theorem 14.9. The **Root Test**
Let $\sum a_n$ be a series for which $L = \lim |a_n|^{\frac{1}{n}}$ exists. Then,

$$\sum a_n \begin{cases} \text{converges absolutely, if } L < 1 \\ \text{diverges, if } L > 1 \\ \text{can either converge or diverge, if } L = 1 \end{cases}$$

Proof. Case 1:
Since $0 \leq L < 1$, we can choose k such that $0 \leq l < k < 1$ by Theorem 4.7. Thus, $k - L > 0$. Therefore, $\exists N$ s.t.

$$n > N \implies \left||a_n|^{\frac{1}{n}} - L\right| < k - L$$

Thus, $-(k-L) < |a_n|^{\frac{1}{n}} < k - L$ and so $|a_n|^{\frac{1}{n}} < k$. This is equal to $|a_n| < k^n$. Since $0 < k < 1$, $\sum k^n$ is convergent. Thus by the comparison test, $|a_n|$ is also convergent.
Case 2:
Since $L > 1$, we can choose $1 < k < L$. Then $L - k > 0$. Therefore, $\exists N$ s.t.

$$n > N \implies \left||a_n|^{\frac{1}{n}}\right| < L - k$$

Thus, $-(L-k) < |a_n|^{\frac{1}{n}} < L - k$ and so $k < |a_n|^{\frac{1}{n}}$. Therefore, $|a_n| > k^n$. Since $k > 1$, $\sum k^n$ is divergent. Thus by the comparison test, $|a_n|$ is divergent. $\square$

Example. Consider

$$\sum_{n=0}^{\infty}\frac{(-1)^n n^{2n}}{2^n(n^2+1)^n}$$

We have

$$\sum_{n=0}^{\infty}\frac{(-1)^n n^{2n}}{2^n(n^2+1)^n} = \sum\left(\frac{-1\cdot n^2}{2\cdot(n^2+1)}\right)^n$$
$$\text{Since } \lim\left|\left(\frac{-1\cdot n^2}{2\cdot(n^2+1)}\right)^n\right|^{\frac{1}{n}} = \lim\frac{n^2}{2n^2+2}$$
$$= \frac{1}{2}$$

Since $\frac{1}{2} < 1$, this converges.

Theorem 15.3. The **Alternating Series Test**
If (a_n) is a decreasing sequence of non-negative terms and has a limit of 0, then the alternating series $\sum(-1)^{n+1}a_n$ converges.

Proof. Let s_n be the series of partial sums, or

$$s_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$$

Then, s_n converges $\Leftrightarrow$ (a_n) converges.
Consider the subsequence s_{2n}, which is all the even terms. Then

$$\begin{aligned}
&S_{2(n+1)} - S_{2n} && \text{(Every term except for } n+1 \text{ and } n+2 \text{ cancels out)} \\
&= -a_{2n+2} + a_{2n+1} && ((-1)^{n+1} \text{ is negative for } n=2\text{, but positive for } n=1) \\
&- a_{2n+2} + a_{2n+1} \geq 0 && ((a_n) \text{ is decreasing})
\end{aligned}$$

Thus, because (a_n) is decreasing, (s_{2n}) is increasing.
Consider the subsequence s_{2n+1}, which is all the odd terms. Then

$$\begin{aligned}
S_{2(n+1)+1} - S_{2n+1} &= a_{2n+3} - a_{2n+2} \\
-a_{2n+2} + a_{2n+1} &\leq 0 && ((a_n) \text{ is decreasing})
\end{aligned}$$

Thus, because (a_n) is decreasing, (s_{2n+1}) is decreasing.
We now prove that

$$s_{2m} \leq s_{2n+1} \; \forall m, n \in \mathbb{N}. \tag{1}$$

Since $s_{2m+1} - s_{2n} = a_{2n+1} \geq 0$, we have $s_{2n} \leq s_{2n+1}$.
If $m \leq n$, then $s_{2m} \leq s_{2n}$ (Since even subsequences are increasing), so $s_{2m} \leq s_{2n+1}$ by transitivity.
If $m > n$, then $s_{2n+1} \geq s_{2m+1}$ (Since odd subsequences are decreasing), so $s_{2n+1} \geq s_{2m}$ Thus, (1) holds.
Therefore, (s_{2n}) is increasing, but it's bounded above by any odd partial sum (1), and bounded below by its first term. Thus, s_{2n} is convergent to some s.
Therefore, (s_{2n+1}) is decreasing, but it's bounded above by its first term, and bounded below by any even partial sum (1). Thus, s_{2n+1} is convergent to some t.
We have

$$\begin{aligned}
t - s &= \lim s_{2n+1} - \lim s_{2n} \\
&= \lim(s_{2n+1} - s_{2n}) \\
&= \lim(a_{2n+1}) \\
&= 0 && (\lim(a_n) = 0 \text{ by def.})
\end{aligned}$$

Thus $s = t$, and so both s_{2n} and s_{2n+1} is convergent to s. Thus, $\lim s_n = s$.

□

Example. Consider

$$\sum \left(\frac{(-1)^{n+1}}{n} \right)$$

Note that the limit of this sequence is 0, and the terms are decreasing. Then this series converges.

Remark. Note that in the above series, the absolute value of the sequence, which is $\frac{1}{n}$ diverges, so this series only converges conditionally. As an aside, the above series converges to $\ln 2$.

2 Functions

Definition 2.1. A **function** $f : A \to B$ is a relation from A to B where each element of A maps to exactly one element in B, called $f(x)$.
A is called the **domain** of f, written dom (f).
B is called the **co-domain** of f.
If no domain for f is given, it is understood that it is the **natural domain**, the largest subset of the reals on which f is defined.

Example. The function f defined by $f(x) = \frac{1}{x^2}$ has natural domain $\{x \in \mathbb{R} \mid x \neq 0\}$.
The function g defined by $g(x) = \sqrt{9 - x^2}$ has natural domain

$$\begin{aligned} 9 - x^2 &\geq 0 \\ 9 &\geq x^2 \\ 3 &\geq |x| \end{aligned}$$

$\{x \in \mathbb{R} \mid -3 \leq x \leq 3\}$ or $[-3, 3]$.

Definition 2.2. Let f be a real-valued function with dom $(f) \subseteq \mathbb{R}$ and let $a \in \mathbb{R}$ be the limit of some sequence in the domain. Let $L \in \mathbb{R}$. Then, $\lim_{x\to a} = L$ iff $\forall \varepsilon > 0$, there is $\delta > 0$ such that $x \in$ dom (f) and $0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$.
i.e. We can bring $f(x)$ arbitrarily close to L (within ε) by taking x close enough to a (within δ).

Examples. Prove

$$\lim_{x\to 4}(5x - 3) = 17$$

Given $\varepsilon > 0$, find $\delta > 0$ such that $x \in \mathbb{R}$ and

$$0 < |x - 4| < \delta \implies |5x - 3 - 17| < \varepsilon$$

Note that

$$\begin{aligned} |5x - 3 - 17| &= |5x - 20| \\ &= 5|x - 4| \\ &< 5\delta \\ &\leq \varepsilon \end{aligned}$$

Thus, we can choose delta as $\frac{\varepsilon}{5}$.
Therefore, let $\varepsilon > 0$ be given and $\delta = \frac{\varepsilon}{5}$. Then $x \in \mathbb{R}$ and $0 < |x - 4| < \delta \implies$

$$\begin{aligned} |5x - 3 - 17| &= 5|x - 4| \\ &< 5\delta \\ &= \varepsilon \end{aligned}$$

as required. $\square$
Prove

$$\lim_{x\to 2}(3x^2 - 1) = 11$$

Note that $\operatorname{dom}(f) = \mathbb{R}$.
Given $\varepsilon > 0$, find $\delta > 0$ such that $0 < |x-2| < \delta \implies \left|3x^2 - 1 - 11\right| < \varepsilon$.
Rough Work: We have

$$\begin{aligned}\left|3x^2 - 1 - 11\right| &= 3\left|x^2 - 4\right| \\ &= 3|x-2||x+2|\end{aligned}$$

We get to choose δ, so choose it such that $\delta < 1$.
Then,

$$\begin{aligned}&|x-2| < 1 \\ &\implies -1 < x-2 < 1 \\ &\implies 3 < x+2 < 5 \\ &\implies |x+2| < 5\end{aligned}$$

Another way to show is is to use the triangle inequality.

$$\begin{aligned}|x+2| &= |x-2+4| \\ &\leq |x-2| + 4 \\ &< 1 + 4 \\ &= 5\end{aligned}$$

Thus,

$$\begin{aligned}3|x-2||x+2| &< 15|x-2| && \text{(Provided } \delta < 1\text{)} \\ &< 15\delta && \text{(We want this expression to be } \leq \varepsilon\text{)}\end{aligned}$$

Take $\delta \leq \frac{\varepsilon}{15}$, or $\delta = \min(\frac{\varepsilon}{15}, 1)$.
Proof: Let $\varepsilon > 0$ be given. Choose $\delta = \min(\frac{\varepsilon}{15}, 1)$. Then, $x \in \mathbb{R}$ and $0 < |x-2| < \delta$ implies both

$$\begin{aligned}|x-2| < 1 &\implies -1 < x-2 < 1 \\ &\implies 3 < x+2 < 5 \\ &\implies |x+2| < 5\end{aligned}$$

and

$$|x-2| < \delta \implies |x-2| < \frac{\varepsilon}{15}$$

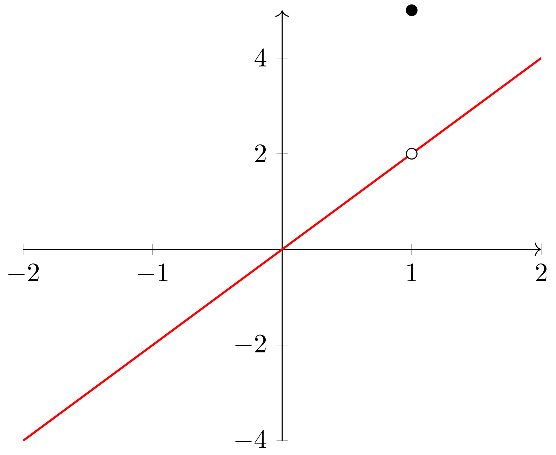
We then have

$$\begin{aligned}|3x - 1 - 11| &= 3|x-2||x+2| \\ &< 3(\frac{\varepsilon}{15})(5) \\ &= \varepsilon\end{aligned}$$

$\square$

Example 3:
Let

$$f(x) = \begin{cases} 2x, & \text{if } x \neq 1 \\ 5, & \text{if } x = 1 \end{cases}$$



Note that $|x-1| > 0 \implies x \neq 1$.
Thus,

$$\begin{aligned} |f(x) - 2| &= |2x - 2| \\ &= 2|x-1| \\ &< 2\delta \\ &(\leq \varepsilon) \end{aligned}$$

Choose $\delta = \frac{\varepsilon}{2}$.
Proof: Let $\varepsilon > 0$ be given, choose $\delta = \frac{\varepsilon}{2}$. Then, $x \in \mathbb{R}$ and $0 < |x-1| < \delta \implies$

$$\begin{aligned} |f(x) - 2| &= |2x - 2| && (x \neq 1) \\ &= 2|x-1| \\ &< 2\delta \\ &= 2 \cdot \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

$\square$

Example 4:
Let

$$f(x) = \frac{x^2 - 1}{x - 1}$$

Prove $\lim_{x \to 1} f(x) = 2$. The domain of f is $\{x \in \mathbb{R} \mid x \neq 1\}$.
Note that $\frac{x^2-1}{x-1} = x + 1$ if $x \neq 1$.
Given $\varepsilon > 0$, find $\delta > 0$ such that $x \in \text{dom}\,(f)$ and $0 < |x-1| < \delta \implies |f(x) - 2| < \varepsilon$.

Since $x \neq 1$,

$$\begin{aligned} |f(x) - 2| &= |\frac{x^2 - 1}{x - 1} - 2| \\ &= |x + 1 - 2| \\ &= |x - 1| \\ &< \delta \\ &(\leq \varepsilon) \end{aligned}$$

Choose $\delta = \varepsilon$.
Proof: Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon$. Then, $0 < |x - 1| < \delta \implies x \neq 1$. We have

$$\begin{aligned} |f(x) - 2| &= |\frac{x^2 - 1}{x - 1} - 2| \\ &= |x + 1 - 2| \\ &= |x - 1| \\ &< \delta \\ &= \varepsilon \end{aligned}$$

□

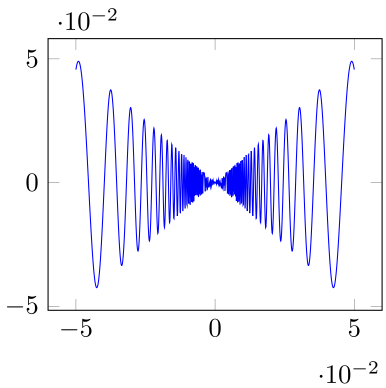
Definition 2.3. Let f be a real-valued function with dom $(f) \subseteq \mathbb{R}$. Let $a \in$ dom (f). Then f is **continuous** at a iff

$$\lim_{x \to a} f(x) = f(a)$$

If f is continuous on each $a \in S$, for some $S \subseteq$ dom (f), then f is continuous on S.

Examples. Let $f(x) = 5x - 3$. We've proved that $\lim_{x \to 4} f(x) = 17$. Since $f(4) = 17$, f is continuous at $x = 4$.
Let $g(x) = 3x^2 - 1$. We've proved that $\lim_{x \to 2} g(x) = 11$. Since $g(2) = 11$, g is continuous at $x = 2$.
Let $h(x) = \begin{cases} 2x, & \text{if } x \neq 1 \\ 5, & \text{if } x = 1 \end{cases}$. We've proved that $\lim_{x \to 1} h(x) = 2$. Since $h(1) = 5$, h is discontinuous at $x = 1$.
Let $k(x) = \frac{x^2 - 1}{x - 1}$. Since $1 \notin$ dom (k), k is discontinuous at $x = 1$.
Example 2:
Let

$$f(x) = f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$



We prove that f is continuous at $x = 0$. We must show $\lim_{x \to 0} f(x) = f(0) = 0$.
Given $\varepsilon > 0$, find δ such that

$$0 < |x - 0| < \delta \implies |f(x) - 0| < \varepsilon$$

Note that $x > 0 \implies x \neq 0$. Thus,

$$\begin{aligned}|f(x) - 0| &= |x \sin \frac{1}{x}| \\ &= |x|| \sin \frac{1}{x}| \\ &\leq |x| && (|\sin a| \leq 1) \\ &< \delta \\ &(\leq \varepsilon)\end{aligned}$$

Choose $\delta = \varepsilon$.
Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon$. Then $0 < |x| < \delta \implies x \neq 0$, so

$$|f(x) - 0| = |x \sin \frac{1}{x}| \leq |x| < \delta = \varepsilon$$

Theorem 17.2. Let f be a real valued function with dom $(f) \subseteq R$. Let $a \in$ dom (f). Then, f is continuous at a if and only if for every sequence (x_n) in dom (f) converging to a, we have

$$\lim_{n\to\infty} f(x_n) = f(a)$$

Example. Prove that $f(x) = 3x^3 - 2x^2 + x + 1$ is continuous on $\mathbb{R}$.
The old way to prove this would be to show that for an arbitrary element of $x_0 \in \mathbb{R}$, $\lim_{x\to x_0} f(x_0) = x_0$. However, we can use Theorem 17.2. Suppose

$$\lim_{n\to\infty} x_n = x_0$$

Then,

$$\begin{aligned}\lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} (3x_n^3 - 2x_n^2 + x_n + 1) \\ &= 3(\lim_{n\to\infty} x_n)^3 - 2(\lim_{n\to\infty} x_n)^2 + \lim_{n\to\infty} x_n + \lim_{n\to\infty} 1 \\ &= 3{x_0}^3 - 2{x_0}^2 + x_0 + 1 \\ &= f(x_0)\end{aligned}$$

Proof. We show the forwards direction.
Suppose f is continuous at a, and let (x_n) be a sequence in the domain of f with $\lim_{n\to\infty} x_n = a$. Then, we show

$$\lim f(x_n) = f(a)$$

Let $\varepsilon > 0$ be given. Since f is continuous at a, $\exists \delta > 0$ such that $x \in$ dom (f) and

$$|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon. \tag{1}$$

Since $\lim x_n = a$, there is N such that $n > N \implies |x_n - a| < \delta$. (Take $\varepsilon = \delta$)
Take $n > N$, then $|x_n - a| < \delta$ and so by (2) with $x = x_n \implies$

$$|f(x_n) - f(a)| < \varepsilon$$

as required.
We show the reverse direction. Suppose for any sequence (x_n) in dom (f) convergent to a,

$$\lim_{n\to\infty} f(x_n) = f(a) \tag{2}$$

but f is not continuous. Then, $\exists \varepsilon > 0$ such that $\forall \delta > 0$, the implication

$$x \in \text{dom }(f) \text{ and } 0 < |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon$$

is false. In particular, $\forall n \in \mathbb{N}$, $\exists x_n \in \text{dom }(f)$ such that $0 < |x_n - a| < \frac{1}{n}$, but $|f(x) - f(a)| \geq \varepsilon$. Since $0 < |x_n - a| < \frac{1}{n}$ and $\lim \frac{1}{n} = 0$, by squeeze theorem $\lim(x_n - a) = 0$, so $\lim x_n = a$. However, since $|f(x_n) - f(a) \geq \varepsilon > 0$, $\lim f(x_n) \neq f(a)$, which contradicts (3). □

Definition 2.4. Let f, g be real valued functions and let $k \in \mathbb{R}$. Then,

$$(kf)(x) = kf(x)\ \forall x \in \text{dom }(f)$$

Then,

$$(f + g)(x) = f(x) + g(x)\ \forall x \in \text{dom }(f) \cap \text{dom }(g)$$

Then,

$$(fg)(x) = f(x)g(x)\ \forall x \in \text{dom }(f) \cap \text{dom }(g)$$

Then,

$$(\frac{f}{g})(x) = \frac{f(x)}{g(x)}\ \forall x \in \text{dom }(f) \cap \text{dom }(g) \text{ and for where } g(x) \neq 0$$

Theorem 17.3 & 17.4. Let f and g be real valued functions continuous at x_0, and let $k \in \mathbb{R}$. Then the following are continuous at x_0.

- kf
- $|f|$
- $f + g$
- fg
- $\frac{f}{g}$ if $g(x_0) \neq 0$

Proof. Let (x_n) be a sequence in dom (f) with $\lim x_n = x_0$. Since f is continuous at x_0, by theorem

$$\lim f(x_n) = f(x_0)$$

Then,

$$\lim_{n \to \infty} kf(x_n) = k \lim f(x_n) = kf(x_0)$$

Thus kf is continuous at x_0. □

Definition 2.5. Let f, g be real valued functions. Define the function $g \circ f$ ("g composed with f") by

$$(g \circ f)(x) = g(f(x))$$

The domain is $x \in \text{dom }(f) \cap f(x) \in \text{dom }(g)$.

Example. Let $f(x) = x + 1$, $g(x) = x^2$

$$(g \circ f)(x) = g(f(x)) = g(x + 1) = (x + 1)^2$$

$$(f \circ g)(x) = f(g(x)) = f(x^2) = x^2 + 1$$

Theorem 17.5. If f is continuous at x_0, and g is continuous at $f(x_0)$, the $g \circ f$ is continuous at x_0.

Proof. Let x_n be a sequence in the domain of $g \circ f$ convergent to x_0. Since f us continuous at x_0, it follows that $(f(x_n))$ is convergent to $f(x_0)$. Since g is continuous at $f(x_0)$, it follows that $(g(f(x_n))$ is convergent $g(f(x_0))$. That is, $(g \circ f(x_n))$ converges to $(g \circ f(x_0))$. It follows that $g \circ f$ is convergent at x_0. □

Remark. We will take for granted that $f(x) = e^x$ is continuous on $\mathbb{R}$.

Example. Let $f(x) = e^x$ and $g(x) = \frac{x-1}{x^2+1}$. By theorem 17.5, $(g \circ f)(x) = \frac{e^x-1}{e^2x+1}$ is continuous on $\mathbb{R}$.

Definition 2.6. A real valued function f is **bounded** if the set

$$\{f(x) \mid x \in \text{dom}\ (f)\}$$

is bounded, that is $\exists M$ such that

$$|f(x)| \leq M\ \forall x \in \text{dom}\ (f)$$

Theorem 18.1. Let f be a continuous real valued function on a closed interval $[a, b]$. Then f is bounded and f assumes a minimum and maximum value in $[a, b]$.

Proof. Suppose f is continuous on $[a, b]$, but not bounded, i.e. $\exists x_n \in [a, b]$ such that $|f(x)| > n$.
Since x_n is bounded ($a \leq x_n \leq b$), by Bolzano-Weierstrass Theorem, there is a convergent subsequence (x_{n_k}) with limit x_0. By Exercise 8.9, $x_0 \in [a, b]$. Since f is continuous on $[a, b]$, $\lim_{k\to\infty} f(x_{n_k}) = f(x_0)$. But since $|f(x_n)| > n$, we have $\lim_{k\to\infty} |f(x_{n_k})| = \infty$, which is a contradiction. Thus f is bounded on $[a, b]$.
Let $M = \sup\{f(x) \mid x \in [a, b]\}$. Since f is bounded, M exists. $\forall n \in \mathbb{N}$, we can find $y_n \in [a, b]$ such that $M - \frac{1}{n} < f(y_n) \leq M$, since $M - \frac{1}{n}$ is not an upper bound. By the squeeze theorem, $\lim_{n\to\infty} f(y_n) = M$.
Since (y_n) is bounded ($a \leq y_n \leq b$), by the Bolzano-Weierstrass Theorem, there is a convergent subsequence (y_{n_k}) with limit $y_0 \in [a, b]$. Since f is continuous, $\lim_{k\to\infty} f(y_{n_k}) = f(y_0)$. But, $f(y_{n_k})$ is a subsequence of the convergent sequence $(f(y_n))$, and so $f(y_n) = \lim f(y_{n_k}) = \lim f(y_n) = M$.
Thus $f(y_0) = M$ for $y_0 \in [a, b]$ and so f assumes the maximum. □

Theorem 18.2. The Intermediate Value Theorem
Let f be a continuous real valued function on an interval I. Let $a, b \in I$ with $a < b$ and suppose y lies between $f(a)$ and $f(b)$. Then, there exists at least one $x \in (a, b)$ such that $f(x) = y$.

Proof. Assume $f(a) < y < f(b)$ WLOG (almost). Let $S = \{x \in [a, b] \mid f(x) < y\}$ Note $a \in S$, since $f(a) < y$, so S is non-empty. Since S is non-empty, it is bounded, so $x_0 = \sup S \in [a, b]$.
Then $\forall n \in \mathbb{N}, \exists s_n \in S$ such that $x_0 - \frac{1}{n} \leq s_n \leq x_0$. By squeeze theorem, $x_n \to x_0$.
Since $s_n \in S$, $f(s_n) < y$, and hence $f(x_0) = \lim f(s_n) \leq y$.
Now let $t_n = \min(b, x_0 + \frac{1}{n})$ so that $t_n \in [a, b]$.
Since $x_0 \leq t_n \leq x_0 + \frac{1}{n}$, by squeeze theorem $t_n \to x_0$. Each t_n is in $[a, b]$, but not in S, so $t_n > y$. Thus $f(x_0) = \lim f(t_n) \geq y$. Since $y \geq f(x_0) \geq y$, $f(x_0) = y$. □

Example. Show that $f(x) = x^3 + x^2 + 1 = 0$ has at least one solution. Note that f is continuous on $\mathbb{R}$. When $x = 2, f(-2) = -3$, and when $x = -1, f(-1) = 1$. As $f(-2) < 0 < f(-1)$, by the IVT $\exists x \in (-2, -1)$ such that $f(x) = 0$. Thus f has at least one solution.

Proposition 2.7. Let f be continuous with $f : [0, 1] \to [0, 1]$. Then f must have at least one fixed point $(x_0 = f(x_0))$.

Proof. Consider $g(x) = f(x) - x$. By Theorem, g is continuous on $[0, 1]$. Note that $g(0) = f(0) - 0 \geq 0$ and $g(1) = f(1) - 1 \leq 0$.
If $g(0) = 0$, then $f(0) = 0$. If $g(1) = 0$, then $f(1) = 1$. Thus we have a fixed point. Otherwise, $g(0) > 0$ and $g(1) < 0$.
By the IVT, $\exists x_0 \in (0, 1)$ with $g(x_0) = 0$. Then $0 = f(x_0) - x_0$, so $f(x_0) = x_0$. □

2.1 Uniform Continuity

Example. Prove that $f(x) = \frac{1}{x}$ is continuous on $(0, \infty)$ with the precise definition of a limit.
Let $x_0 \in (0, \infty)$. Given $\varepsilon > 0$, we find $\delta > 0$ such that $x > 0$ and $|x - x_0| < \delta \implies |\frac{1}{x} - \frac{1}{x_0}| < \varepsilon$. We have:

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \left|\frac{x_0 - x}{xx_0}\right| = \frac{|x - x_0|}{xx_0} < \frac{\delta}{xx_0}$$

We can do this since $x \neq 0$.
Say we ensure that $\delta \leq L$ for some L, a fixed constant. Can we take L independent of x_0?
Then,

$$\begin{aligned} |x - x_0| < \delta \leq L &\implies -L < x - x_0 < L \\ &\implies x_0 - L < x < x_0 + L \end{aligned}$$

but for any L we choose, there will be x_0 close to L, making $x_0 - L < 0$.
It seems L will have to depend on x_0. Taking $L = \frac{x_0}{2}$ (so $\delta \leq \frac{x_0}{2}$) implies $x > x_0 - \frac{x}{2} = \frac{x_0}{2} > 0$ and so $\frac{1}{x} < \frac{2}{x_0}$. Then $|\frac{1}{x} - \frac{1}{x_0}| < \frac{\delta}{xx_0} < \frac{2\delta}{x_0^2}$. We choose $\delta \leq \frac{x_0^2 \varepsilon}{2}$. Take $\delta = \min(\frac{x_0}{2}, \frac{x_0^2\varepsilon}{2})$. It seems our choice of δ depends on x_0. This is okay, but it will be useful to know where δ can be chosen to depend only on ε and S.

Definition 2.8. Let f be a real valued function defined on a set S. Then f is **uniformly continuous** on S iff $\forall \varepsilon > 0, \exists \delta > 0$ such that $x, y \in S$ and $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$.
Compare with normal cont. on S:

$$\forall y \in S, \forall \varepsilon > 0, \exists \delta > 0 s.t. \forall x \in S \text{ and } |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

Remark. If a function is uniformly cts., it is also cts.

Example. Let $b > 0$ be a fixed real number. Then $f(x) = \frac{1}{x}$ is uniformly cts. on $[b, \infty)$.
Rough Work: Let $\varepsilon > 0$ be given. Have to find δ such that $|x - y| < \delta$ and $x, y \geq b \implies |\frac{1}{x} - \frac{1}{y}| < \varepsilon$. We have:

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x - y|}{xy} < \frac{\delta}{xy}.$$

But $x, y \geq b \implies \frac{1}{x} \leq \frac{1}{b}$ and $\frac{1}{y} \leq \frac{1}{b}$. So $\frac{1}{x} - \frac{1}{y} < \frac{\delta}{b^2}$.
Proof: Let $\varepsilon > 0$ and take $\delta = b^2\varepsilon$. Then $x, y \geq b$ and

$$|x - y| < \delta \implies \left|\frac{1}{x} - \frac{1}{y}\right| < \frac{\delta}{b^2} = \varepsilon$$

Theorem 19.2. If f is cts. on $[a, b]$, then f is uniformly cts. on $[a, b]$.

Proof. Let f be cts. on $[a, b]$.
Suppose f is not uniformly cts. on $[a, b]$. Then $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$, $\exists x, y \in [a, b]$ where $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$.
Then $\forall n \in \mathbb{N}$, there are $x_0, y_0 \in [a, b]$ for which

$$|x_n - y_n| < \frac{1}{n} \text{ and } |f(x_n) - f(y_n)| \geq \varepsilon. \tag{1}$$

Since $x_n \in [a, b]$, (x_n) is bounded, and by Bolzano-Weierstrass, this implies it has a convergent subsequence (x_{n_k}).
Also, if we write $\lim_{k\to\infty} x_{n_k} = x_0$, then by exercise 8.9, $x_0 \in [a, b]$.
Since $|x_n - y_n| < \frac{1}{n}$, we have $x_n - \frac{1}{n} < y_n < x_n + \frac{1}{n}$, and so $\lim_{k\to\infty} y_{n_k} = x_0$ by squeeze theorem.
Since f is cts. at x_0, we have $f(x_0) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} f(y_{n_k})$, and so $\lim_{k\to\infty}(f(x_{n_k}) - f(y_{n_k}) = 0$.
However, (1) implies $\lim_{k\to\infty}(f(x_{n_k}) - f(y_{n_k}) \geq \varepsilon > 0$, a contradiction. $\square$

Theorem 19.4. Let f be uniformly cts. on a set S. Then if (s_n) is a Cauchy sequence, then $f(s_n)$ is a Cauchy sequence.
Given f defined on S, if we can find a Cauchy sequence (s_n) in S such that $f(s_n)$ is not Cauchy, then f is not uniformly cts. on S.

Proof. Let (s_n) be a Cauchy sequence in S, and let $\varepsilon > 0$ be given. Since f is uniformly cts. on S, $\exists \delta > 0$ s.t. $x, y \in S$, and

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon \tag{2}$$

Since s_n is Cauchy, $\exists N$ s.t. $m, n > N \implies |s_m - s_n| < \varepsilon$. Then $m, n > N \implies |f(s_n) - f(s_m)| < \varepsilon$ by (2). So $f(s_n)$ is Cauchy. $\square$

Example. Consider $f(x) = \frac{1}{x}$ on $(0, \infty)$. Let $s_n = \frac{1}{n}$. This converges, thus is Cauchy. We have $f(s_n) = \frac{1}{s_n} = \frac{1}{\frac{1}{n}} = n$, which is not Cauchy. Thus f is not uniformly cts. on $(0, \infty)$.

2.2 Left and Right Hand Limits

Remark. Recall:
Let a be the limit of some sequence in dom (f) and let $L \in \mathbb{R}$. Then

$$\lim_{x \to a} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } x \in \text{dom } (f) \text{ and } 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$$

What if we only consider x on one side of a?

Definition 2.9. Let $L \in \mathbb{R}$ and a be the limit of some sequence in dom (f) consisting of terms $\geq a$. Then:

$$\lim_{x \to a^+} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } x \in \text{dom } (f) \text{ and } a < x < a + \delta \implies |f(x) - L| < \varepsilon$$

Now let a be the limit of some sequence in dom (f) consisting of terms $\leq a$. Then:

$$\lim_{x \to a^-} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } x \in \text{dom } (f) \text{ and } a - \delta < x < a \implies |f(x) - L| < \varepsilon$$

Example. Let

$$f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ -x, & x > 1 \end{cases}$$

Then one can show:

$$\begin{aligned} &\lim_{x \to 1^-} f(x) = 1 \\ &\lim_{x \to 1^+} f(x) = -1 \\ &\lim_{x \to 1} f(x) \text{ does not exist.} \end{aligned}$$

Left and Right Limits Agree. Let I be an open interval containing a, and let f be defined on I except possibly at a. Then

$$\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$$

Proof. If $\lim_{x \to a} f(x) = L$, then it is clear that $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$ since the premise $a < x < a + \delta$ and $a - \delta < x < a$ are implied by $0 < |x - a| < \delta$.
Now suppose $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$. Consider $\varepsilon > 0$. Then $\exists \delta_1, \delta_2$ s.t. $a < x < a + \delta_1 \implies |f(x) - L| < \varepsilon$ and $a - \delta_2 < x < a \implies |f(x) - L| < \varepsilon$. Take $\delta = \min(\delta_1, \delta_2)$. Then $0 < |x - a| < \delta \implies x \neq a$ and $a - \delta < x < a + \delta$. This implies $|f(x) - L| < \varepsilon$. $\square$

Definition 2.10. We define $\lim_{x \to s} f(x) = \infty$, and $\lim_{x \to s} f(x) = -\infty$ where s is one of a, a^+, a^-.

$$\lim_{x \to s} f(x) = \infty \Leftrightarrow \forall M > 0, \exists \delta > 0 \text{ s.t. } \mathbf{A} \implies f(x) > M$$

If $s = a$, $\mathbf{A}$ is $0 < |x - a| < \delta$. If $s = a^-$, $\mathbf{A}$ is $a - \delta < x < a$. If $s = a^+$, $\mathbf{A}$ is $a < x < a + \delta$.

$$\lim_{x \to s} f(x) = -\infty \Leftrightarrow \forall M < 0, \exists \delta > 0 \text{ s.t. } \mathbf{A} \implies f(x) < M$$

where $\mathbf{A}$ is the same as above.

Example. Show $\lim_{x\to 0^+} \frac{1}{x} = +\infty$.
Rough Work: Given $M > 0$ and $0 < x < \delta$, we want $\frac{1}{x} > M$. We know $\frac{1}{x} > \frac{1}{M}$, so choose $\delta = \frac{1}{M}$.
Proof: Given $M > 0$, choose $\delta = \frac{1}{M}$. Then $0 < x < \delta \implies 0 < x < \frac{1}{M}$, and so $\frac{1}{x} > \frac{1}{M}$.

Example. Show $\lim_{x\to 0^-} \frac{1}{x} = -\infty$.
Rough Work: Given $M < 0$ and $-\delta < x < 0$, we want $\frac{1}{x} < M$.

$$\begin{aligned} -\delta < x < 0 &\implies \delta > -x > 0 \\ &\implies \frac{1}{\delta} < -\frac{1}{x} < 0 \\ &\implies -\frac{1}{\delta} > \frac{1}{x} > 0 \end{aligned}$$

We choose $\delta = -\frac{1}{M}$. Note that since $M < 0$, $-\frac{1}{M} > 0$.
Proof: Given $M < 0$, choose $\delta = -\frac{1}{M}$. Then $-\delta < x < 0 \implies \frac{1}{M} < x < 0$, and so $1 > Mx$. Thus $\frac{1}{x} < M$.

Example. Show $\lim_{x\to 0} \frac{1}{x^2} = \infty$.
Rough: Given $M > 0$ and $0 < |x| < \delta \implies \frac{1}{x^2} > M$. We have

$$\begin{aligned} 0 < |x| < \delta &\implies 0 < x^2 < \delta^2 \\ &\implies \frac{1}{x^2} > \frac{1}{\delta^2}. \end{aligned}$$

Since we want $M = \frac{1}{\delta^2}$, so choose $\delta = \frac{1}{\sqrt{M}}$.
Proof: Given $M > 0$, choose $\delta = \frac{1}{\sqrt{M}}$. Then $0 < |x| < \frac{1}{\sqrt{M}}$, and so $0 < x^2 < \frac{1}{M}$. Thus, $\frac{1}{x^2} > M$.

2.3 Limits at Infinity

Definition 2.11.

$$\lim_{x\to\infty} f(x) = L \Leftrightarrow \forall \mathbf{A}, \exists \alpha \in \mathbb{R} \text{ s.t. } x > \alpha \implies \mathbf{B}$$

If $L \in \mathbb{R}$, $\mathbf{A}$ is $\varepsilon > 0$ and B is $|f(x) - L| < \varepsilon$.
If $L = \infty$, $\mathbf{A}$ is $M > 0$ and B is $f(x) > M$.
If $L = -\infty$, $\mathbf{A}$ is $M > 0$ and B is $f(x) < M$.
The definition of $x \to -\infty$ is similar, we change $x > \alpha$ to $x < \alpha$.

Example. Show $\lim_{x\to\infty} \frac{1}{x} = 0$. Given $\varepsilon > 0$, we must find $\alpha \in \mathbb{R}$ such that $x > \alpha \implies |\frac{1}{x} - 0| < \varepsilon$. We can ensure that $\alpha > 0$, so that $|\frac{1}{x} - 0| = \frac{1}{x}$.
Since $0 < \frac{1}{x} < \varepsilon \Leftrightarrow x > \frac{1}{\varepsilon}$, choose $\alpha = \frac{1}{\varepsilon}$. Then $x > \alpha \implies x > \frac{1}{\varepsilon} \implies \frac{1}{x} < \varepsilon$.

Example. Show $\lim_{x\to\infty} x^2 = \infty$.
Given $M > 0$, we must find $\alpha \in \mathbb{R}$ such that $x > \alpha \implies x^2 > M$. Choose $\alpha = \sqrt{M}$.
Then $x > \alpha \implies x > \sqrt{M} \implies x^2 > M$.

2.4 Limit Laws

Theorem 2.12. Let f be defined on a set S and let a be the limit of some sequence in S (including the possibility that $a = \infty$ or $-\infty$). Let $L \in \mathbb{R}$. Then

$$\lim_{x\to a} f(x) = L$$

if and only if for every sequence $(x_n) \in S$ with limit a for which $x_n \neq a$ for all n, we have

$$\lim_{n\to\infty} f(x_n) = L$$

Proof. We prove the forward direction.
Suppose $\lim_{x\to a} f(x) = L$. Let (x_n) be a sequence in S with $x_n \neq a$ for all n and with $lim_{n\to\infty} x_n = a$. We must prove that $\lim_{n\to\infty} f(x_n) = L$
Case 1:
$a \in \mathbb{R}$.
Let $\varepsilon > 0$ be given, since $\lim_{x\to a} f(x) = L$, there is a $\delta > 0$ such that $x \in S$ and $0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon$. Since $\lim_{n\to\infty} x_n = a$, there is an N such that $n > N$ implies $|x_n - a| < \delta$. Since $x_n \neq a$, we have $0 < |x_n - a|$. Thus $n > N \implies 0 < |x_n - a| < \delta$, and so $|f(x_n) - L| < \varepsilon$ as required.
Case 2:
$a = \infty$.
Let $\varepsilon > 0$ be given. Since $\lim_{x\to\infty} f(x) = L$, there is $\alpha > 0$ such that $x \in S$ and $x > \alpha$ implies $|f(x) - L| < \varepsilon$.
Since $\lim_{n\to\infty} x_n = \infty$, there is an N such that $n > N$ implies $x_n > \alpha$. Thus $n > N$ implies $|f(x_n) - L| < \varepsilon$ as required.
Case 3:
$a = -\infty$.
Left as exercise to the reader.
Now we show the reverse direction.
Suppose that for any sequence $(x_n) \in S$ with $x_n \neq a$ for all n and with $\lim_{n\to\infty} x_n = a$, we have $\lim_{n\to\infty} f(x_n) = L$.
Suppose the limit is not L for contradiction.
Case 1:
$a \in \mathbb{R}$.
Then $\exists \varepsilon > 0, \forall \delta > 0$, $x \in S$ and $0 < |x - a| < \delta \implies |f(x) - L| \geq \varepsilon$. Take $\delta = \frac{1}{n}$.
Since $0 < |x_n - a| < \frac{1}{n}$, we have $a - \frac{1}{n} < x_n < a + a + \frac{1}{n}$. By the squeeze theorem, we have $\lim_{n\to\infty} x_n = a$. However, since $|f(x_n) - L| \geq \varepsilon > 0$, we have $\lim_{n\to\infty} f(x_n) \neq L$, a contradiction.
Case 2:
$a = \infty$.
Then $\exists \varepsilon > 0, \forall \alpha \in \mathbb{R}$, $x_n \in S$ and $x_n > \alpha \implies |f(x_n) - L| \geq \alpha$. Take $\alpha = n \in \mathbb{N}$. Since $x_n > n$ by Exercise 9.9a, we have $\lim_{n\to\infty} x_n = \infty$. However, since $|f(x_n) - L| \geq \varepsilon > 0$, we have $\lim_{n\to\infty} f(x_n) \neq L$, a contradiction.
Case 3:
$a = -\infty$.
Left as exercise to the reader. □

Theorem 20.4. Let f and g be functions defined on a set S for which

$$\lim_{x\to a} f(x) = L_1$$

and

$$\lim_{x\to a} g(x) = L_2$$

exist with $L_1, L_2 \in \mathbb{R}$ (we may have $a \in \mathbb{R}$ or $a = \pm\infty$) Then

1. $\lim_{x\to a}(f + g)(x) = L_1 + L_2$.
2. $\lim_{x\to a}(fg)(x) = L_1 L_2$.
3. $\lim_{x\to a} \left(\frac{f}{g}\right)(x) = \frac{L_1}{L_2}$ provided $L_2 \neq 0$.

Proof. Let (x_n) be a sequence in S with limit a and by the previous theorem,

$$\lim_{n\to\infty} f(x_n) = L_1$$

and

$$\lim_{x\to\infty} g(x_n) = L_2$$

By Theorem 9.3, we have

$$\lim_{n\to\infty}(f + g)(x) = \lim_{n\to\infty} f(x_n) + \lim_{n\to\infty} g(x_n) = L_1 + L_2$$

Since this holds for any sequence (x_n) in S with limit a, by the previous theorem we have

$$\lim_{x\to a}(f+g)(x) = L_1 + L_2$$

Proof is similar for mul. and div. $\square$

Theorem 20.5. Let f be a function defined on S for which

$$\lim_{x\to a} f(x) = L$$

exists with $L \in \mathbb{R}$. Let g be a function defined on $\{f(x) \mid x \in S\} \cup \{L\}$ that is cts. at L. Then

$$\lim_{x\to a}(g \circ f)(x) = g(L)$$

Proof. Let (x_n) be a sequence in S with limit a. (where $a \in \mathbb{R}$ or $a = \pm\infty$) Since

$$\lim_{x\to a} f(x) = L$$

we have

$$\lim_{n\to\infty} f(x_n) = L$$

Then

$$\lim_{n\to\infty}(g\circ f)(x_n) = \lim n \to \infty g(f(x_n)) = g(L)$$

since g is cts. at L. $\square$

Example. Let's take $g_1(x) = e^x$ and $g_2(x) = \sin x$ which are both cts.
Then,

$$(g_1 \circ f)(x) = g_1(f(x)) = e^{f(x)}$$

and

$$(g_2 \circ f)(x) = g_2(f(x)) = \sin(f(x))$$

If $\lim_{x\to a} f(x)$ exists, then by Theorem 20.5 we have

$$\lim_{x\to a} e^{f(x)} = e^{\lim_{x\to a} f(x)}$$

and

$$\lim_{x\to a} \sin f(x) = \sin\left(\lim_{x\to a} f(x)\right)$$

For example,

$$\lim_{x\to\infty} \sin\left(\frac{1}{x}\right) = \sin\left(\lim_{x\to\infty}\frac{1}{x}\right) = \sin 0 = 0$$

3 Derivatives

Definition 3.1. Let f be a real valued function defined on an open interval I containing a point a. Then f is **differentiable** at a if

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

exists and is finite. We then write $f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$.
Note that the dom $f'(a)$ will be the points in dom f for which the limit exists and is finite.

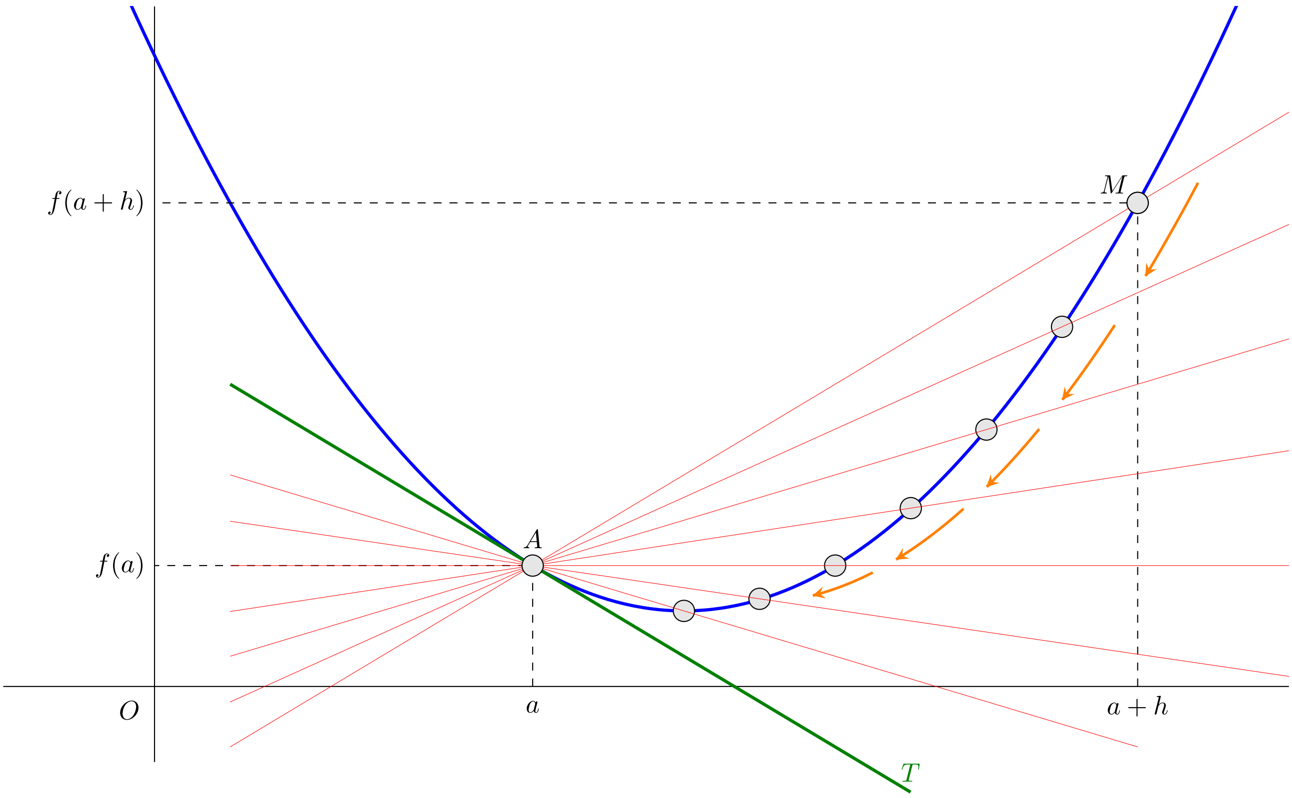
Remark. The intuitive idea is that you take x near a. The slope of the secant line between x and a is

$$\frac{f(x)-f(a)}{x-a}$$

which is the average rate of change of f between x and a.

As we move x near a, the secant line will approach the tangent line at a. Thus

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\ &= \text{slope of tangent line at } a \\ &= \text{instantaneous rate of change} \end{aligned}$$



Proposition 3.2. Note that $f'(a)$ is also equal to

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

by letting $x = a+h$ so $h = x-a$.

Example. Let $f(x) = x^2$. Then

$$\begin{aligned} f'(2) &= \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} \\ &= \lim_{x\to 2} \frac{x^2-4}{x-2} \\ &= \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} \\ &= \lim_{x\to 2} x+2 \\ &= 4 && \text{(Since } f \text{ is cts.)} \end{aligned}$$

In fact,

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\
&= \lim_{x\to a} \frac{x^2-a^2}{x-a} \\
&= \lim_{x\to a} \frac{(x-a)(x+a)}{x-a} \\
&= \lim_{x\to a} x+a \\
&= 2a && \text{(Since } f \text{ is cts.)}
\end{aligned}$$

So if $f(x) = x^2$, then $f'(x) = 2x$.

Example. Let $f(x) = x^2$. Then for $a > 0$,

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a} \\
&= \lim_{x\to a} \frac{1}{\sqrt{x}+\sqrt{a}} && \text{(Rationalize the Numerator)} \\
&= \frac{1}{2\sqrt{a}}
\end{aligned}$$

Suppose $a = 0$, then

$$\begin{aligned}
f'(a) &= \lim_{x\to 0^+} \frac{\sqrt{x}-0}{x-0} \\
&= \lim_{x\to a} \frac{1}{\sqrt{x}} \\
&= \infty
\end{aligned}$$

Thus $f'(x)$ is not differentiable at $x - 0$.

3.1 Limit Laws

Derivative of a constant. Let $f(x) = c$ be a constant function. Then $f'(x) = 0$.

Proof. Let $a \in \mathbb{R}$. Then

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\
&= \lim_{x\to a} \frac{c-c}{x-a} \\
&= 0
\end{aligned}$$

□

Power Rule v1. Let $n \in \mathbb{N}$. Consider $f(x) = x^n$ on $\mathbb{R}$. Then $f'(x) = nx^{n-1}$.

Proof. Let $a \in \mathbb{R}$. Then,

$$\begin{aligned}
f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
&= \lim_{x \to a} \frac{x^n - a^n}{x - a} \\
&= \lim_{x \to a} \frac{(x-a)(x^{n-1} + ax^{n-2} + \cdots + a^{n-1})}{x - a} \\
&= \lim_{x \to a} (x^{n-1} + ax^{n-2} + \cdots + a^{n-1}) \\
&= a^{n-1} + a^{n-1} + \dots && \text{(polynomials are cts.)} \\
&= na^{n-1}
\end{aligned}$$

Note n sweeps from $n - 1$ to 0, so n terms in total. □

Theorem 28.3. Let f and g be differentiable at a. Let c be a constant. Then $cf, f + g, fg, f/g$ (with $g(a) \neq 0$) are diff. at a.
They have the following derivatives:

1. $(cf)'(a) = cf'(a)$
2. $(f + g)'(a) = f'(a) + g'(a)$
3. $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$
4. $(\frac{f}{g})'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$

Proof.

$$\begin{aligned}
(cf)'(a) &= \lim_{x \to a} \frac{(cf)(x) - (cf)(a)}{x - a} \\
&= \lim_{x \to a} \frac{cf(x) - cf(a)}{x - a} \\
&= c \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
&= cf'(a)
\end{aligned}$$

$$\begin{aligned}
(f + g)'(a) &= \lim_{x \to a} \frac{(f + g)(x) - (f + g)(a)}{x - a} \\
&= \lim_{x \to a} \frac{f(x) + g(x) - f(a) - g(a)}{x - a} \\
&= \lim_{x \to a} \frac{f(x)f(a)}{x - a} + \lim_{x \to a} \frac{g(x) - g(a)}{x - a} \\
&= f'(a) + g'(a)
\end{aligned}$$

$$\begin{aligned}
(fg)'(a) &= \lim_{x \to a} \frac{(fg)(x) - (fg)(a)}{x - a} \\
&= \lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x - a} \\
&= \lim_{x \to a} \frac{f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)}{x - a} \\
&= \lim_{x \to a} \frac{g(x)(f(x) - f(a)) + f(a)(g(x) - g(a))}{x - a} \\
&= (\lim_{x \to a} g(x)) \left(\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right) + (\lim_{x \to a} \frac{f(a)(g(x) - g(a))}{x - a}) \\
&= f'(a)g(a) + f(a)g'(a)
\end{aligned}$$

$\square$

Example. Prove that $f(x) = |x|$ is not differentiable at $x = 0$. Consider the left and right hand limits.

$$\begin{aligned}\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0^-} \frac{|x|}{x} \\ &= \lim_{x\to 0^-} \frac{-x}{x} \\ &= -1\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0^+} \frac{|x|}{x} \\ &= \lim_{x\to 0^+} \frac{x}{x} \\ &= 1\end{aligned}$$

Thus the derivative does not exist at $x = 0$.

Theorem 28.2. If f is differentiable at a, then f is cts. at a.

Proof. Suppose f is differentiable at a, so that $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists and is finite. We have

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}\left((x-a)\left(\frac{f(x)-f(a)}{x-a}\right)+f(a)\right) && \text{(This is equal to } f(x)\text{)} \\ &= \lim_{x\to a}(x-a)\lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)+\lim_{x\to a} f(a) \\ &= 0\cdot f'(x)+f(a) \\ &= f(a)\end{aligned}$$

$\square$

Proof. Now we can prove the Quotient Rule:

$$\left(\frac{f}{g}\right)(a) = \frac{f'(a)g(a)-f(a)g'(a)}{(g(a))^2}$$

for $g(a) \neq 0$.
Since $g(a) \neq 0$ and $g(a)$ is a cts. function, $\exists I$ s.t. $a \in I$ for which $g(x) \neq 0 \forall x \in I$.
For $x \in I$, we have

$$\begin{aligned}\left(\frac{f}{g}\right)(x)-\left(\frac{f}{g}\right)(a) &= \frac{f(x)}{g(x)}-\frac{f(a)}{g(a)} \\ &= \frac{f(x)g(a)-f(a)g(x)}{g(x)g(a)} \\ &= \frac{f(x)g(a)-g(a)f(a)+g(a)f(a)-f(a)g(x)}{g(x)g(a)} \\ &= \frac{g(a)(f(x)-f(a))-f(a)(g(x)-g(a))}{g(x)g(a)}\end{aligned}$$

Now we take the divide by $x - a$ and take the limit.

$$\begin{aligned}\lim_{x\to a} \frac{g(a)\frac{f(x)-f(a)}{x-a}-f(a)\frac{g(x)-g(a)}{x-a}}{g(x)g(a)} &= \frac{g(a)f'(x)-f(a)g'(x)}{g(x)g(a)} && \text{(distribute the limits)} \\ &= \frac{f'(a)g(a)-f(a)g'(a)}{(g(a))^2} && (g \text{ is cts.})\end{aligned}$$

$\square$

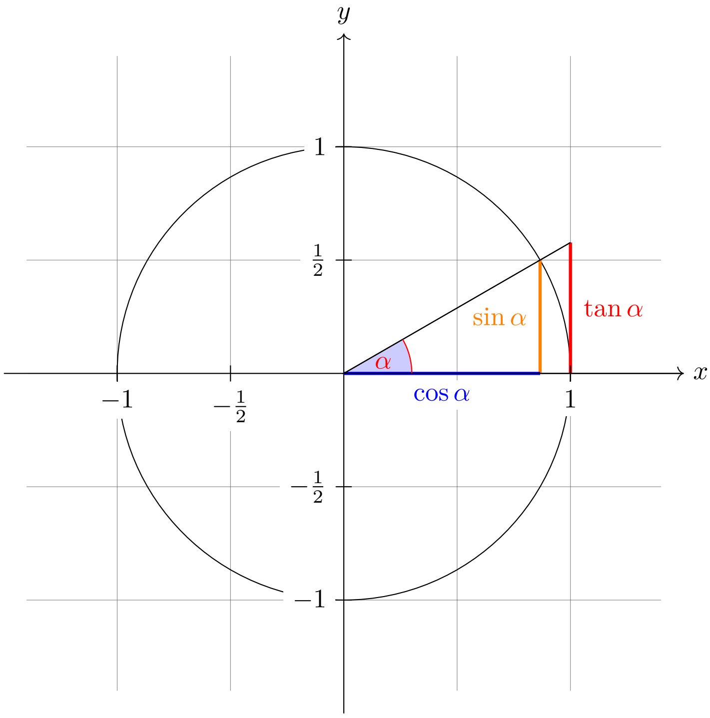
Power Rule v2. Let $m \in \mathbb{N}$ and consider $f(x) = x^{-m}$ for $x \neq 0$.
Then $f(x) = \frac{1}{x^m} = \frac{g(x)}{h(x)}$ where $g(x) = 1$ and $h(x) = x^m$. Then,

$$\begin{aligned} f'(x) &= \frac{0(x^m) - 1(mx^{m-1}}{x^{2m}} \\ &= \frac{-mx^{m-1}}{x^2 m} \\ &== mx^{-m-1} \end{aligned}$$

Let $n = -m$, then $f'(x) = nx^{n-1}$

Lemma 3.3. For $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $x \neq 0$, we have

$$\cos x \leq \frac{\sin x}{x} \leq 1$$



Proof. Consider the image for $0 < \alpha < \frac{\pi}{2}$. Let A be the origin, B be the point (1, 0), and C the point where the hypotenuse touches the circle. (IDK how to graph)
Then the area of triangle ABC is: $\frac{1}{2}(1)(\sin\alpha) = \frac{\sin\alpha}{2}$.
Since the circumference of the circle is 2π, and the arc length of the slice is α, the slice accounts for $\frac{\alpha}{2\pi}$ of the circumference of the circle.
Since the area of the circle is π, the area of the curve slice is $\left(\frac{\alpha}{2\pi}\right)(\pi) = \frac{\alpha}{2}$.
Also, the area of the large triangle is $\frac{1}{2}(1)(\tan\alpha) = \frac{1}{2}\tan\alpha$.
Since

$$\begin{aligned} \text{area of ABC} &\leq \text{area of pie slice} \\ &\leq \text{area of ABD} \end{aligned}$$

we have

$$\begin{aligned} &\frac{1}{2}\sin x \leq \frac{x}{2} \leq \frac{1}{2}\tan x \\ &\implies 0 < \sin x \leq x \leq \tan x \\ &\implies \frac{\cos x}{\sin x} \leq \frac{1}{x} \leq \frac{1}{\sin x} \\ &\implies \cos x \leq \frac{\sin x}{x} \leq 1 \end{aligned}$$

For the negative part, since $\cos(-x) = \cos x$ and $\frac{\sin(-x)}{-x} = \frac{\sin x}{x}$, we can replace the α with its negation.

$\square$

Theorem 3.4.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

and

$$\lim_{x\to 0} \frac{\cos x - 1}{x}$$

Proof. For $-\frac{\pi}{2} < x < \frac{\pi}{2}$, we have $\cos x \leq \frac{\sin x}{x} \leq 1$. Since $\lim_{x\to 0} \cos x = 1$, by squeeze theorem $\lim_{x\to 0} \frac{\sin x}{x} x = 1$. Note that

$$\begin{aligned}
\frac{\cos x - 1}{x} &= \frac{(\cos x - 1)(\cos x + 1)}{x(\cos x + 1)} \\
&= \frac{\cos^2 x - 1}{x(\cos x + 1)} \\
&= \frac{-\sin^2 x}{x(\cos x + 1)} && \text{(Pythagorean Identity)} \\
&= \left(\frac{\sin x}{x}\right)\left(\frac{-\sin x}{x}\right)
\end{aligned}$$

Taking limits, we have $\lim \left(\frac{\sin x}{x}\right)\left(\frac{-\sin x}{x}\right) = 1\left(\frac{-0}{2}\right) = 0$.

$\square$

Theorem 3.5. $\sin x$ and $\cos x$ are differentiable on $\mathbb{R}$ with

$$(\sin x)' = \cos x$$

and

$$(\cos x)' = -\sin x$$

Proof. If $f(x) = \sin x$, then

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{\sin(a+h) - \sin a}{h} \\
&= \lim_{h\to 0} \frac{\sin a \cos h + \cos a \sin h - \sin a}{h} \\
&= \lim_{h\to 0} (\sin a)\left(\frac{\cos h - 1}{h}\right) + \cos a \left(\frac{\sin h}{h}\right) \\
&= \sin a \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos a \frac{\sin h}{h} \\
&= 0 + \cos a(1) \\
&= \cos a
\end{aligned}$$

If $f(x) = \cos x$, then

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{\cos(a+h) - \cos a}{h} \\
&= \lim_{h\to 0} \frac{\cos a \cos h - \sin a \sin h - \cos a}{h} \\
&= \lim_{h\to 0} \cos a \left(\frac{\cos h - 1}{h}\right) - \sin a \left(\frac{\sin h}{h}\right) \\
&= \cos a(0) - \sin a(1) \\
&= -\sin a
\end{aligned}$$

$\square$

Corollary 3.6. Now we can differentiate the other trig functions. For example, let $f(x) = \tan x$. Then,

$$\begin{aligned} h'(x) &= \frac{(\sin x)' \cos x - \sin x(\cos x)'}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \sec^2 x \end{aligned}$$

Theorem 28.4 (Chain Rule). If f is differentiable at a, and g is differentiable at $f(a)$, then

$$((g \circ f)(a))' = g'(f(a))f'(a)$$

Proof. Notice that

$$\frac{(g \circ f)(x) - (g \circ f)(a)}{x - a} = \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot \frac{f(x) - f(a)}{x - a}$$

Can we use limits? We have

$$\left(\lim_{x \to a} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)}\right)\left(\lim_{x \to a} \frac{f(x) - f(a)}{x - a}\right)$$

Since f is cts. at a, we have $\lim_{x \to a} f(x) = f(a)$.
Thus it looks like the limit is going to $g'(f(a))f'(a)$.
However, this proof doesn't work since we can be dividing by 0 with $f(x) = f(a)$.
(Full proof on brightspace) $\square$

Example. Let $f(x) = \sin\left(x^2 + x\right)$. Then, $f'(x) = \cos\left(x^2 + x\right) \cdot (2x + 1)$
Let $f(x) = \left(\sin\left(x^2\right)\right)^2$. Then $f'(x) = 2\sin\left(x^2\right) \cdot \cos\left(x^2\right) \cdot 2x$
Let $f(x) = x^2 \sin\left(\frac{1}{x}\right)$. Then $f'(x) = 2x(\sin\left(\frac{1}{x}\right)) + (x^2)(\cos\left(\frac{1}{x}\right)) \cdot (-x^{-2})$

Theorem 29.1. Let f be defined on an open interval I containing x_0. Suppose f assumes its max/min and f is differentiable on I. Then, $f'(x_0) = 0$.

Proof. Suppose f is defined on $(a, b) = I$ with $x_0 \in I$. WLOG, assume f hits its max at x_0.
If $f'(x_0) > 0$, then we have $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} > 0$. Let $\varepsilon = f'(x_0)$. Then $\exists \delta > 0$ s.t. $a < x_0 - \delta < x_0 < x_0 + \delta < b$ and

$$0 < |x - x_0| < \delta \implies \left|\frac{f(x) - f(x_0)}{x - x_0} - f'(x)\right| < f'(x_0) = \varepsilon$$

Thus

$$-f(x_0) < \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) < f'(x_0)$$

and so

$$\frac{f(x) - f(x_0)}{x - x_0} > 0$$

Taking $x_0 < x < x_0 + \delta$, we have

$$\begin{aligned} f(x) - f(x_0) &> 0 \\ f(x) &> f(x_0) \end{aligned}$$

This is a contradiction, since f takes its max at x_0.
Suppose $f'(x) < 0$. Let $\varepsilon = -f(x_0)$, then $\exists \delta > 0$ s.t.

$$0 < |x - x_0| < \delta \implies \left|\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)\right| < f'(x_0)$$

Thus we have

$$\frac{f(x) - f(x_0)}{x - x_0} < 0$$

Taking $x_0 - \delta < x < x_0$ so that $x - x_0 < 0$, we have

$$\begin{aligned} f(x) - f(x_0) &> 0 \\ f(x) &> f(x_0) \end{aligned}$$

This is a contradiction, since f takes its max at x_0.
Thus we must have $f'(x) = 0$. □

Corollary 3.7. Thus if we have a function on a closed interval, by Theorem 18.1, it must have a max/min and then f can only hit those points at the endpoints, any non-differentiable points, or where the derivative is 0.

Example. Find the max and min of $f(x) = x^5 + x^4 - 5x^3 - x^2 + 8x - 4$ on $[-2, 2]$.
Since polynomials are cts. everywhere, we only need to look at the endpoints and the derivative.
We have $f'(x) = 5x^4 + 4x^3 - 15x^2 - 2x + 8$. By factor theorem and synthetic division, we have

$$f'(x) = (x-1)^2(x+2)(5x+4)$$

There are roots at $x = 1, -2, -\frac{4}{5}$.
Thus

$$\begin{aligned} f(-2) &= 0 \\ f(-\frac{4}{5}) &= \frac{-2^2 \cdot 3^8}{5^3} \\ f(1) &= 0 \\ f(2) &= 16 \end{aligned}$$

Thus the max and min are at $x = 2, -\frac{4}{5}$

Example. Find the max and min of $f(x) = x^{\frac{2}{3}}$ on $[-1, 1]$.
We have $f'(x) = \frac{2}{3}x^{\frac{-1}{3}}$ when $x \neq 0$. For $x = 0$, it can be shown the limit does not exist.
Thus we must check $x = 0$. Now we set $f'(x) = 0$. We have

$$\frac{2}{3x^{\frac{1}{3}}} = 0$$

This is never true. Thus there are no additional points to check.
We have

$$\begin{aligned} f(1) &= 1 \\ f(0) &= 0 \\ f(-1) &= 1 \end{aligned}$$

3.2 Mean Value Theorem

Theorem 29.2 (Rolle's Theorem). Let f be cts. on $[a, b]$, differentiable on (a, b). Suppose $f(a) = f(b)$. Then $\exists x \in (a, b)$ s.t. $f'(x) = 0$.

Proof. By 18.1, $\exists x_0, y_0 \in [a, b]$ s.t. $f(x_0) \leq f(x) \leq f(y_0) \forall x \in [a, b]$.
If x_0, y_0 are the endpoints, then f is a constant function on the interval since $f(x_0) = f(y_0)$ and so $f'(x) = 0 \forall x$.
Otherwise, f assumes a max/min at $x \in (a, b)$ and so by Theorem 29.1, $f'(x) = 0$. □

Theorem 29.3 (Mean Value Theorem). Let f be cts. on $[a, b]$ and differentiable on (a, b), then $\exists x \in (a, b)$ such that $f'(x) = \frac{f(b)-f(a)}{b-a}$. This is the average rate of change between a and b, and the slope of the secant line, but $f'(x)$ is the instantaneous rate of change at x, and the slope of the tangent line.

Proof. Let $L(x)$ be the function whose graph is a straight line connecting $(a, f(a))$ and $(b, f(b))$ (secant line). Thus $L(a) = f(a)$ and $L(b) = f(b)$. Furthermore, $L'(x) = \frac{f(b)-f(a)}{b-a}$ for all x.
Let $g(x) = f(x) - L(x)$ for $x \in [a, b]$. It follows that g is cts. on $[a, b]$, diff. on (a, b), and $g(a) = 0 = g(b)$. By Rolle's Theorem, there is $x \in (a, b)$ with $g'(x) = 0$. Since $g'(x) = f'(x) - L'(x)$, we have $f'(x) = L'(x)$, or

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

□

Corollary 3.8. (29.4)
Let f be differentiable on (a, b), with $f'(x) = 0 \forall x \in (a, b)$. Then f must be constant on (a, b).

Proof. Suppose $f'(x) = 0$ on (a, b) but f is not constant. Then there are $x_1, x_2 \in (a, b)$ and with $x_1 < x_2$, where $f(x_1) \neq f(x_2)$. By the MVT, there is $x \in (x_1, x_2)$, with $f'(x) = \frac{f(x_2)-f(x_1)}{x_2-x_1} \neq 0$, a contradiction. □

Corollary 3.9. (29.5) Let f, g be diff. on (a, b) that have the same derivative, i.e. $f' = g'$ on (a, b). Then there is a constant c such that $f(x) = g(x) + c$ for all $x \in (a, b)$.

Proof. Let $h = f - g$, then $h' = f' - g' = 0$. By previous corollary, h is constant on (a, b). □

Example. Find all functions whose derivative is x^2.
We can find $\frac{1}{3}x^3$ by inspection. Thus $\frac{1}{3}x^3 + c$ will have derivative x^2 for any constant c. By previous corollary, no other such functions can unique.

Definition 3.10. Let f be defined on I.
f is **strictly increasing** on I if $x_1, x_2 \in I$ with $x_1 < x_2$ imply $f(x_1) < f(x_2)$.
f is **increasing** on I if $x_1, x_2 \in I$ with $x_1 < x_2$ imply $f(x_1) \leq f(x_2)$.
f is **strictly decreasing** on I if $x_1, x_2 \in I$ with $x_1 < x_2$ imply $f(x_1) > f(x_2)$.
f is **decreasing** on I if $x_1, x_2 \in I$ with $x_1 < x_2$ imply $f(x_1) \geq f(x_2)$.

Corollary 3.11. (29.7)
Let f be diff. on (a, b). Then

1. f is strictly increasing on (a, b) if $f'(x) > 0$ on (a, b).
2. f is increasing on (a, b) if $f'(x) \geq 0$ on (a, b).
3. f is strictly decreasing on (a, b) if $f'(x) < 0$ on (a, b).
4. f is decreasing on (a, b) if $f'(x) \leq 0$ on (a, b).

Proof. Proof of 3:
Consider $x_1, x_2 \in (a, b)$ with $x_1 < x_2$. By MVT, there is $x \in (x_1, x_2)$ with $\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(x) < 0$. Thus $f(x_2) - f(x_1) < 0$ and so $f(x_1) > f(x_2)$.
Proof of others is almost identical. □

3.3 Inverse Functions

Definition 3.12. A function $f : A \to B$ is **injective**, or one-to-one, if for all $x, y \in A$, then $f(x) = f(y) \implies x = y$.

Exercise. If f is strictly increasing or decreasing, then f is injective.

Remark. If $f : A \to \operatorname{range} f$ is injective, then f is invertible. There exists a function $f^{-1} : \operatorname{range} f \to A$ such that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ for all x.

Theorem 29.9. If f is injective and cts. on I, and differentiable at $x_0 \in I$, then f^{-1} is cts. on I and diff. at $f(x_0)$ except $f'(x_0) \neq 0$.

Remark. What is $(f^{-1})'(y_0)$?
Since $(f^{-1} \circ f)(x) = x$ for all $x \in A$, so $(f^{-1} \circ f)'(x) = 1$. Thus $1 = (f^{-1}(f(x)))' = (f^{-1})'(f(x)) \circ f'(x)$.
Thus $1 = (f^{-1})'(f(x_0))f'(x_0)$, so

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

Example. Let $n \in \mathbb{N}$. Consider $g(x) = x^{\frac{1}{n}}$.

$$\text{dom } g = \begin{cases} [0, \infty) & \text{if } n \text{ is even} \\ \mathbb{R} & \text{if } n \text{ is odd} \end{cases}$$

It can be shown that g is cts. on its domain and g is injective.
Thus $g^{-1} = f$ exists, for $f(x) = x^n$. We know $f'(x) = nx^{n-1}$. Also $f^{-1} = g$.
Let $y_0 \in \text{dom } g$. Then $g'(y_0) = (f^{-1})'(y_0) = \frac{1}{n(y_0^{\frac{1}{n}})^{n-1}} = \frac{1}{ny_0^{\frac{n-1}{n}}} = \frac{1}{n} y_0^{\frac{1}{n}-1}$.

Power Rule v3. Now suppose $F(x) = x^{\frac{m}{n}}$ for $m, n \in \mathbb{Z}, n > 0$.
Then $F = h \circ g$ for $h(x) = x^m$ and $g(x) = x^{\frac{1}{n}}$. We know $h'(x) = mx^{m-1}$ and $g'(x) = \frac{1}{n}x^{\frac{1}{n}-1}$. Thus $F'(x) = m(x^{\frac{1}{n}})^{m-1}(\frac{1}{n}x^{\frac{1}{n}-1} = \frac{m}{n}x^{\frac{m}{n}-1}$
Let $r \in Q$. Then the derivative of x^r is rx^{r-1} where defined.

3.4 Trig Functions

Example. $f(x) = \sin x$ is injective on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with range $[-1, 1]$.
We write $f^{-1}(x) = \arcsin x$ or $f^{-1}(x) = \sin^{-1} x$.
We know $f'(x) = \cos x$. Let $y_0 \in [-1, 1]$. If $x_0 = \arcsin y_0$, then $y_0 = \sin x_0$.

$$(\arcsin)'(y_0) = \frac{1}{\cos x_0} = \frac{1}{\sqrt{1 - y_0^2}}$$

since $\cos x_0 = \sqrt{1 - \sin^2 x_0}$.
Thus the derivative of $\arcsin x$ is $\frac{1}{\sqrt{1-x^2}}$.