MATH 226 Formulae

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1 Vectors

1.1 Projection

Definition 1.1. The vector projection of \mathbf{u} onto \mathbf{v} is

$$\mathbf{u}_{\mathbf{v}} = \frac{\mathbf{u} \cdot \mathbf{v}}{\left\|\mathbf{v}\right\|^2} \mathbf{v}$$

and the scalar projection is

$$s = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|}$$

1.2 Lines and Planes

Proposition 1.2. The equation of a plane containing a point (x_0, y_0, z_0) and perpendicular to a vector (A, B, C) can be written as

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Proposition 1.3. The equation of a line containing a point (x_0, y_0, z_0) and with direction vector (a, b, c) can be written as

$$(x - x_0, y - y_0, z - z_0) = t(a, b, c)$$

Corollary 1.4. The above is equivalent to the system of equations

$$\begin{cases} x = x_0 + at \\ y = y_0 + at \\ z = z_0 + at \end{cases}$$

or

$$(x, y, z) = (a_0, b_0, c_0) + t(a, b, c)$$

2 Differentiation

2.1 Limits

Definition 2.1. If $L \in \mathbb{R}$ and (a,b) is a point in \mathbb{R}^2 , then f has **limit** L at (a,b) written as $\lim_{(x,y)\to(a,b)} f(x,y) = L$ if

$$\forall \varepsilon > 0, \exists \delta > 0, \sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |f(x,y) - L| < \varepsilon$$

Theorem 2.2. If the limits along different paths approaching (a, b) are different, then f does not have a limit there.

2.2 Partial Derivatives

Definition 2.3. The partial derivative f wrt x_j is the function f_j defined by

$$f_j(\mathbf{x}) = \lim_{h \to 0} \frac{1}{h} \left(f(x_1, \dots, x_j + h, \dots, x_n) - f(\mathbf{x}) \right)$$

Example. For 2 variables, the partials are

$$f_1(x,y) = \lim_{h \to 0} \frac{1}{h} (f(x+h,y) - f(x,y))$$

$$f_2(x,y) = \lim_{h \to 0} \frac{1}{h} (f(x,y+h) - f(x,y))$$

2.3 Differentiability

Definition 2.4. A function f(x,y) is **differentiable** if

$$\lim_{(x,y)\to(a,b)} \frac{|f(x,y) - L(x,y)|}{\sqrt{(x-a)^2 + (y-b)^2}} = 0$$

where

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Theorem 2.5. If f(x,y) is C^1 on a neighbourhood of (a,b), it is differentiable at (a,b).

Theorem 2.6. If f is differentiable at (a,b), then both f_x, f_y exist at (a,b) and f is continuous at (a,b).

Corollary 2.7. If f_x, f_y exist at (a, b) or f is continuous at (a, b), then f(x, b) is continuous at x = a and f(a, y) is continuous at y = b.

2.4 Tangent Planes

Theorem 2.8. The normal vector to the tangent plane is

$$-f_x(a,b)\mathbf{i} - f_y(a,b)j + k$$

Theorem 2.9. The equation of the tangent plane is

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Proposition 2.10. The linearization of a function at a point is the same equation as for the tangent plane.

2.5 Directional Derivatives

Definition 2.11. The directional derivative of a function in the direction of a unit vector **u** is defined as

$$D_{\mathbf{u}}f(a,b) = \lim_{h \to 0} \frac{1}{h} (f(a + hu_1, b + hu_2) - f(a,b))$$

Theorem 2.12.

$$D_{\mathbf{u}}f(a,b) = \nabla f(a,b) \cdot \mathbf{u} = |\nabla f(a,b)| \cos \theta$$

Corollary 2.13. The direction of steepest ascent is $\mathbf{u} = \frac{\nabla f}{|\nabla f|}$, and similarly, the direction of steepest descent is $\mathbf{u} = -\frac{\nabla f}{|\nabla f|}$

2.6 Implicit Functions

Theorem 2.14. If F(x,y) = 0, then if y = y(x), we can use chain rule to yield

$$0 = \frac{d}{dx}F(x, y(x)) = F_1(x, y(x)) + F_2(x, y(x))\frac{dy}{dx}$$

so that $\frac{dy}{dx} = -\frac{F_1}{F_2}$. Similarly for 3 variables, $\frac{\partial z}{\partial x} = -\frac{F_1}{F_3}$ and $\frac{\partial z}{\partial y} = -\frac{F_2}{F_3}$.

Theorem 2.15. If we have the system of equations u = f(x, y) and v = g(x, y), and x = x(u, v), y = y(u, v), we can take the derivative wrt u yielding

$$1 = f_1(x, y) \cdot \frac{\partial x}{\partial u} + f_2(x, y) \cdot \frac{\partial y}{\partial u}$$

and

$$0 = g_1(x, y) \cdot \frac{\partial x}{\partial y} + g_2(x, y) \cdot \frac{\partial y}{\partial y}$$

We can then solve for $\frac{\partial x}{\partial u}$ and $\frac{\partial y}{\partial u}$.

2.7 Minimum and Maximum

Conditions for a local min/max. We must have at least one of

- 1. $\nabla f(a,b) = 0$
- 2. $\nabla f(a,b)$ DNE
- 3. (a, b) is a boundary point of the domain D

Definition 2.16 (Taylor Polynominals). We have

$$f(\mathbf{x}) \approx f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \frac{1}{2} (\mathbf{x} - \mathbf{a}) \cdot \mathcal{H}_{f(\mathbf{a})} (\mathbf{x} - \mathbf{a})^t$$

where

$$\mathcal{H} = \begin{pmatrix} f_{11} & f_{12} & \dots & f_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ f_{n1} & f_{n2} & \dots & f_{nn} \end{pmatrix}$$

Example. For a polynominal in 2 variables,

$$p_2(x,y) = f(a,b) + f_1(a,b)(x-a) + f_2(a,b)(y-b) + \frac{1}{2}f_{11}(a,b)^2 + f_{12}(a,b)(x-a)(y-b) + \frac{1}{2}f_{22}(a,b)(y-b)^2$$

Classifying Critical Points with the Second Derivative Test. We consider the behaviour of the principal minors of \mathcal{H} evaluated at \mathbf{a} . We have

$$D_1 = f_{11}, D_2 = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}, D_3 = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

- 1. If all $D_i > 0$, then \mathcal{H} is positive definite and so f has a local minimum at a
- 2. If $D_i < 0$ when i is even and $D_i > 0$ when i odd, then \mathscr{H} is negative definite and so f has a **local maximum** at a
- 3. If $D_n = \det(\mathcal{H}) \neq 0$ (the largest one) but the sign is not all positive or alternating with negative first, then \mathcal{H} is indefinite and so f has a **saddle point** at **a**
- 4. If $D_n = 0$, then the test is inconclusive

Definition 2.17. A domain $X \subset \mathbb{R}^n$ is **compact** if it is bounded and closed.

Procedure for finding Global Extrema on Compact Domains. If X is compact and f cts., then it must attain a global min/max on X.

If a point is a global minimum, it has to be a local one as well or on the boundary, thus we can follow a similar process as before

- 1. Find all critical/singular points inside X of the function f
- 2. Determine the min/max values of f on the boundary of X and note the points where these min/max values are

attained

3. Compute $f(\mathbf{a})$ at all the points found, and pick the biggest and smallest

Easier to just compute and not use 2nd derivative test.

2.8 Lagrange Multipliers

Remark. We want to determine the extreme values of f subject to a constraint $g(\mathbf{x}) = c$. We want ∇f parallel to ∇g .

Theorem 2.18. Want to find all points where

- $\nabla f(x,y) = \lambda \nabla g(x,y)$ for some $\lambda \in \mathbb{R}$
- $\nabla g(x,y) = 0$
- $\nabla f(x,y)$ or $\nabla g(x,y)$ DNE

3 Integration

Determining Convergence of Improper Integrals. Let $D \subset \mathbb{R}^2$ be a region.

- If $f \ge g \ge 0$ on D, then $\iint_D g(x,y) dA = \infty \implies \iint_D f(x,y) = \infty$.
- If $0 \le f \le g$ on D, then $\iint_D g(x,y) < \infty \implies \iint_D f(x,y) < \infty$.

Proposition 3.1. A double integral over a region D can be interpreted as

$$\iint_D f(x,y)dA = \text{volume under the graph of } z = f(x,y) \text{ above } D$$

Theorem 3.2. The average value of f(x, y) on F is

$$\frac{\iint_D f(x,y)dA}{\iint_D 1dA}$$

since $\iint_D 1 dA$ is the area of D.

Definition 3.3. The **centroid** of D is the point $(\overline{x}, \overline{y})$ where

$$\overline{x} = \frac{\iint_D x dA}{\iint_D 1 dA}$$

and similarly for \overline{y} .

3.1 Cylindrical Coordinates

Definition 3.4. Cylindrical coordinates are obtained by the change of variables $(x,y) \to (r,\theta)$, where

$$x = r\cos\theta, y = r\sin\theta$$

and

$$dV = dx dy dz = r dr d\theta dz$$

3.2 Spherical Coordinates

Definition 3.5. Consider a point P. Then P can be represented uniquely by

- \bullet R is the radius of the sphere
- $0 \le \theta \le 2\pi$ is the projection of the point onto the xy-axis and taking the angle from the x axis to P'.
- $0 \le \phi \le \pi$ is the angle made by P and the z-axis.

3.3 Change of Variables

Theorem 3.6. The conversion for area is

$$dA = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| \begin{matrix} x_u & x_v \\ y_u & v_v \end{matrix} \right|$$

It is also equivalent to

$$\left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1}$$

Remark. Remember to flip and take the absolute value!!!