MATH 342

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Contents

1 Linear Codes

Definition 1.1. A **linear code** C is a subspace of $V(n, q)$ for some positive n. Thus, C is linear iff

1. $u, v \in C \implies u + v \in C$
2. $u \in C, a \in GF(q) \implies au \in C$

If C is a k-dimensional subspace of $V(n, q)$ then C is called an $[n, k]$ or $[n, k, d]$ code.

Remark. A q-ary $[n, k, d]$ code is also a q-ary (n, q^k, d) code but converse is not true.
0 must be in C.

Definition 1.2. The **weight** of a vector x is defined to be the number of non-zero entries of x.

Lemma 1.3. If $x, y \in V(n, q)$ then $d(x, y) = w(x - y)$.

Proof. The vector $x - y$ has non-zero entries in those places where x, y differ. □

Theorem 1.4. Let C be linear. Then

$$d(C) = \min_{x \in C} w(x)$$

Proof. There are codewords x, y such that $d(x, y) = c = d(C)$. (As otherwise the distance of the codeword could never be c).
Then, $d(C) = w(x - y) \geq \min_{x \in C} w(x)$ since $x - y$ is a codeword of C. However, for some $x \in C$ $\min_{x \in C} w(x) = w(x) = d(x, 0) \geq d(C)$. Thus have both inequalities. □

Definition 1.5. A $k \times n$ matrix whose rows form a basis of a linear $[n, k]$ code is called a **generator matrix** of the code.

Theorem 1.6. Let G be a generator matrix of an $[n, k]$ code. Using EROs, G can be transformed into **standard form**

$$[I_k \mid A]$$

where I_k is the $k \times k$ identity and $A is k \times (n - k)$

1.1 Encoding with a Linear Code

Definition 1.7. Let C be $[n, k]$-code over $GF(q)$ with generator G. C contains q^k codewords, so that is the max possible number of distinct messages.
A message is a k-tuple of $V(k, q)$. Encode a message vector $u = u_1 u_2 \dots u_k$ by multiplying as uG.
Note that this is a map $V(k, q) \to C \subset V(n, q)$

Corollary 1.8. If G is in standard form, then the encoding $x = uG = (x_1, x_2, x_3, \dots, x_k, x_{k+1}, \dots, x_n)$ has $x_i = u_i$ for $1 \leq i \leq k$ (called **message digits**) and $x_{k+i} = \sum_{j=1}^{k} a_{ji} u_j$ for $1 \leq i \leq n - k$ (called **check digits**).

Example. Let C be binary $[7, 4]$ code. Let

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

be its generator matrix.

A message vector u_1, u_2, u_3, u_4 is encoded as

$$(u_1, u_2, u_3, u_4, u_1 + u_2 + u_3, u_2 + u_3 + u_4, u_1 + u_2 + u_4)$$

For instance 1000 is encoded as 1000101.

1.2 Decoding with a Linear Code

Definition 1.9. Suppose the codeword x is sent and the codeword y is recieved. Define the **error vector** e as $e = y - x$.

2 Cyclic Codes

Definition 2.1. A code C is cyclic if

1. C is linear.
2. C is closed under shift S, i.e., $w \in C \implies S(w) \in C$.

Example. The code $C = \{000, 101, 011, 110\}$ is cyclic.

Remark. Note that a shift is a right shift, but left shifts can be simulated by shifting right $n-1$ times where n is the length of the codeword.

Remark. Can view a cyclic code as a polynominal, where the digits of the codeword are coefficients for a polynominal of degree $n-1$, $a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1}$.

Definition 2.2. In a cyclic code, declare $x^n \equiv 1 \Leftrightarrow x^n - 1 = 0$.
Then a cyclic code C is a subspace of $Z_p[x]/(x^n - 1)$ such that C is closed under multiplication with x.

2.1 The Ring of Polynomials

Definition 2.3. Let F be a field. Then $F[x]$ is the set of polynominals with coefficients in F. Let $\deg f = d$, and f is **monic** if the term with highest degree has coefficient one.

Definition 2.4. Define division as

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$

where $\deg r < \deg g$.

Definition 2.5. g divides f if $\frac{f(x)}{g(x)}$ has $r(x) = 0$.

Definition 2.6. Let $f(x)$ be a fixed polynominal in $F[x]$. Two polynominals g, h are said to be **congruent** modulo f symbolized by $g(x) = h(x) \pmod{f(x)}$ if $g(x) - h(x)$ is divisible by $f(x)$.

Remark. Any polynominal $a(x)$ is congruent modulo $f(x)$ to a unique polynominal $r(x)$, which is the principal remainder when $a(x)$ is divided by $f(x)$.

Definition 2.7. The GCD of f, g is the monic polynominal of highest degree that divides them.

Definition 2.8. $\alpha \in F$ is a **root** of f if $f(\alpha) = 0$

Theorem 2.9. α is a root of f if and only if $x - \alpha$ divides f.

Corollary 2.10. If $\deg f = n$, then f can have at most n roots.

Definition 2.11. $f(x) \in F[x]$ is **irreducible** if $f(x) \neq g(x)h(x)$ where $\deg g, \deg h < \deg f$. Usually take irreducibles monic.

Theorem 2.12. Every f can be factored as the product of irreducibles.

Corollary 2.13. Irreducible degree 3 or less polynominals must have no roots.

Remark. To find all monic irreducibles, helpful to list all polynominals (There are p^n of them, where p is the modulus, and n the degree), then count the reducible ones and subtract. Use stars and bars formula.

Definition 2.14. $Z_p[x]/f(x) = \{\text{all principal remainders divided by } f\} = \{\text{all } r(x) \text{ such that } \deg r < \deg f\}$ which is the set $\{a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}\}$.

Example. The ring $Z_2[x]/(x^3 + x + 1)$ has the elements $\{0, 1, x, x+1, x^2, x^2+1, x^2+x, x^2+x+1\}$.

2.1.1 Inverses

Definition 2.15. g^{-1} is the element such that $gg^{-1} = 1$ in $Z_p[x]/f(x)$. Note that the inverse may not always exist.

Theorem 2.16. g^{-1} exists if and only $\gcd(f, g) = 1$ in a ring mod f.

Theorem 2.17. $Z_p[x]/f(x)$ is a field if and only if $f(x)$ is irreducible in $Z_p[x]$

Example. $Z_2[x]/(x^2 + x + 1)$ is a field with four elements, also called $\mathbb{F}_4$.
$Z_2[x]/(x^3 + x + 1)$ is a field with eight elements, also called $\mathbb{F}_8$.

Proposition 2.18. Every field with finite number of elements has form $\mathbb{F}_q$ where $q = p^n$ for some prime p.

Remark. If two fields have the same modulus p and the same degree modular polynominal, then they are isomorphic.

Proposition 2.19. Every $\mathbb{F}_{p^n}$ has a primitive element g such that the powers $g, g^2, \ldots, g^{p^n-1} = 1$ are distinct.

2.2 Ideals

Definition 2.20. Let $R_n = F[x]/(x^n - 1)$ where $F = F_q$ be implicit.

Definition 2.21. A simpler definition for a cyclic code $C \subset R_n$ is

1. $0 \in C$
2. $g(x), h(x) \in C \implies g(x) + h(x) \in C$
3. $g(x) \in C, r(x) \in \mathbb{R}_n \implies r(x)g(x) \in C$

C is called an **ideal** in the ring R_n.

Proof. Suppose C is cyclic in R_n. C is thus linear and so the additive closure condition holds.
Let $a(x) \in C$ and $r(x) = r_0 + r_1 x + \cdots + r_{n-1}x^{n-1} \in R_n$. Since multiplication by x corresponds to a cyclic shift, we have $xa(x) \in C$ and $x^2 a(x) \in C$ and so on. Hence $r(x)a(x) = r_0 a(x) + r_1 x a(x) + \cdots + r_{n-1}x^{n-1}a(x)$ is also in C since each term is in C. This the multiplicative closure condition holds.
Taking $r(x) = 0$ satifies the zero condition. □

Remark. Note that taking $r(x) = c$ in the above proof implies C is linear and $r(x) = x$ implies C is cyclic.

Definition 2.22. Let R be a ring. I is **principal** if $I = \langle g \rangle = R \cdot g$.

Example. $R = \mathbb{Z}, I = \langle 12, 18 \rangle$, $I = \{a \cdot 12 + b \cdot 18 \mid a, b \in \mathbb{Z}\}$. Claim $6 \in I$, since 6 is the gcd. Claim $I = \langle 6 \rangle$ because $12a + 18b = \mathbb{Z} \cdot 6$

Proposition 2.23. Every ideal in $\mathbb{Z}$ is principal. $\langle g_1, g_2, \ldots, g_n \rangle = \langle d \rangle$ where $d = \gcd(g_i)$.

Example. Given an ideal $I \subset \mathbb{Z}$, know that $I = \langle d \rangle$. Find d by taking the smallest positive number in I.

Theorem 2.24. Every cyclic code (non-zero) is of the form $\langle g(x) \rangle \subset R_n$ where g divides $x^n - 1$. The generator g is unique if it is monic.

Corollary 2.25. Cyclic codes are 1-1 with monic g that divide $x^n - 1$

2.3 Generator Polynomials

Definition 2.26.

$$\langle f(x) \rangle = \{r(x)f(x) \mid r(x) \in R_n\}$$

Theorem 2.27. For any $f(x) \in R_n$, the set $\langle f(x) \rangle$ is a cyclic code, and it is called the code generated by $f(x)$.

Proof. Check the conditions of 2.21

1. Let $a(x)f(x), b(x)f(x) \in \langle f(x) \rangle$. then $a(x)f(x) + b(x)f(x) = (a(x) + b(x))f(x) \in \langle f(x) \rangle$
2. Let $a(x)f(x) \in \langle f(x) \rangle$, $r(x) \in R_n$, then

$$r(x)(a(x)f(x)) = (r(x)a(x)) \in \langle f(x) \rangle$$

$\square$

Example. Consider the code $C = \langle 1 + x^2 \rangle$ in R_3 with $F = GF(2) = \mathbb{Z}_2$. Multiplying by each of the 8 elements of R_3, (i.e. $0, 1, x, x+1, x^2, x^2+1, x^2+x, x^2+x+1$) and reducing modulo $x^3 - 1$ produces only 4 distinct codewords, namely $0, 1+x, 1+x^2, x+x^2$. Thus C is the code $\{000, 110, 101, 011\}$.

Theorem 2.28. Let C be a non-zero cyclic code in R_n. Then,

1. There exists a unique monic polynominal $g(x)$ of smallest degree in C.
2. $C = \langle g(x) \rangle$
3. $g(x)$ is a factor of $x^n - 1$

Definition 2.29. The polynominal given by the above is called the **generator polynominal** of C.

Example. We find all the binary cyclic codes of length $n = 3$. Factor $x^3 - 1$ as $(x-1)(x^2+x+1)$, both irreducible. By 2.28. Thus, a list of binary cyclic codes is:

1. Generator Polynomial: 1, Code in R_3: All of R_3, Corresponding Code in $V(3,2)$: all of $V(3,2)$
2. Generator Polynomial: $x+1$, Code in R_3: $\{0, 1+x, x+x^2, 1+x^2\}$, Corresponding Code in $V(3,2)$: $\{000, 110, 011, 101\}$.
3. Generator Polynomial: x^2+x+1, Code in R_3: $\{0, 1+x+x^2\}$, Corresponding Code in $V(3,2)$: $\{000, 111\}$.
4. Generator Polynomial: $x^3 - 1 (= 0)$, Code in R_3: $\{0\}$, Corresponding Code in $V(3,2)$: $\{000\}$.

Lemma 2.30. Let $g(x) = g_0 + g_1 x + \cdots + d_r x^r$ be generator polynominal of a cyclic code. Then $g_0 \neq 0$.

Example. Let $p = 2$, $n = 7$ (binary codes of length 7). Find all cyclic codes. Must find all $g(x) \mid x^n - 1$ since they will be generators (also multiplied togeher). Have $x^n - 1 = (x-1)(x^3+x+1)(x^3+x^2+1)$. Let $a(x), b(x), c(x)$ correspond to these three polynominals. Then,

$\deg g$	$g(x)$	$\dim C$
0	1	7
1	$a(x)$	6
2	–	5
3	$b(x), c(x)$	4
4	$a(x)b(x), a(x)c(x)$	3
5	–	2
6	$b(x)c(x)$	1
7	$a(x)b(x)c(x)$	0

Note that $\dim C = n - \deg g(x)$.
Confused here and next theorem.

2.4 Generator and Parity Check Matrices

Theorem 2.31. Let $C = \langle g(x) \rangle$. Then every codeword in C can be written as $w(x) = a(x)g(x)$ where $\deg a(x) < n - \deg g(x)$. Moreover, such a is unique.

Example. Continued from previous example. Pick $g(x) = (x+1)(x^3+x+1)$. Note that $\deg g = 4$. Then any codeword $w(x) = (a_0 + a_1 x + a_2 x^2)g(x) = a_0 g + a_1 xg + a_2 x^2 g$. Thus $g, xg, x^2 g$ are a basis for C.

Theorem 2.32. Let $C = \langle g \rangle$. Let $\deg g = d$. Then $g, xg, x^2 g \dots x^{n-d-1} g$ form a basis for C. There are $n-d$ elements.

Theorem 2.33. Let C be cyclic with generator

$$g(x) = g_0 + g_1 x + \cdots + g_r x^r$$

of degree r. Then $\dim C = n - r$ with generator matrix

$$G = \begin{bmatrix} g_0 & g_1 & g_2 & \cdots & \cdots & g_r & 0 & 0 & 0 \\ 0 & g_0 & g_1 & g_2 & \cdots & \cdots & g_r & 0 & 0 \\ 0 & 0 & \vdots & \vdots & \vdots & \cdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & g_0 & g_1 & g_2 & \cdots & \cdots & g_r \end{bmatrix}$$

This matrix is $\dim C \times n$.

Example. 12.13

Example. Let $n = 12$ and work in $GF(2)$. How many cyclic codes of dimension $k = 8$?
This is the same as saying how many $g(x)$ of degree 4 since $\dim C = n - \deg g$.
Factor $x^{12} - 1$ into $(x+1)^4(x^2+x+1)^4$.
Thus there are 3 such codes comprised of the polynominals $(x+1)^4, (x^2+x+1)^2, (x^2+x+1)(x+1)^2$.

Definition 2.34. Let C be cyclic $[n,k]$-code with generator $g(x)$. By theorem, $g(x)$ is a factor of $x^n - 1$ and so

$$x^n - 1 = g(x)h(x)$$

for some polynominal h. Note that h is monic since g is. $g(x)$ has degree $n-k$ from 2.33 so h has degree k. The polynominal h is called the **check polynominal** of C.

Theorem 2.35. Suppose C is cyclic in R_n with generator g and check polynominal h. Then an element $c(x)$ is a codeword of C iff $c(x)h(x) = 0$.

Example. Let $n = 7$ working in $GF(2)$, with $g(x) = (x-1)(x^3+x+1)$. Then $h(x) = x^3 + x^2 + 1$. Want to know if $w(x) = x^6 + x^3 + x^2 + x \in C$. Multiplying with $h(x)$ yields 0 so it is a codeword.

Theorem 2.36. Suppose C is cyclic $[n, k]$-code with check polynominal $h(x) = h_0 + h_1 x + \cdots + d_k x^k$. Then,

1. a parity-check matrix for C is

$$H = \begin{bmatrix} h_k & k_{k-1} & \cdots & h_0 & 0 & 0 & 0 & 0 \\ 0 & h_k & k_{k-1} & \cdots & h_0 & 0 & 0 & 0 \\ 0 & 0 & h_k & k_{k-1} & \cdots & h_0 & 0 & 0 \\ 0 & 0 & 0 & h_k & k_{k-1} & \cdots & h_0 & 0 \\ 0 & 0 & 0 & 0 & h_k & k_{k-1} & \cdots & h_0 \end{bmatrix}$$

2. $C^\perp$ is a cyclic code generated by the polynominal $\overline{h}(x) = h_k + h_{k-1}x + \cdots + d_0 x^k$

The size of H is $n \times \dim C^\perp = k$ since $\deg C^\perp = \deg \overline{h} = \deg h = (n - \deg g) = (n - (n-k)) = k$ assuming $\deg g = n - k$.

Definition 2.37. The polynominal $\overline{h}(x) = x^k h(x-1) = h_k + h_{k-1}x + \cdots + h_0 x^k$ is called the **reciprocal polynominal** of h, its coefficients are those of h in reverse order.
We may regard $\overline{h}$ as the generator of $C^\perp$ though we should multiply by h_0^-1 to make it monic.

Remark. The polynominal $h(x-1) = x^{n-k}\overline{h}(x)$ is a member of $C^\perp$