# MATH 2052 Notes

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# Contents

# 1 Integration

## 1.1 Darboux Sums

**Remark.** See OneNote for graphs and figures for this section.

**Definition 1.1.** Let $f$ be a bounded function on $[a, b]$. Letting $S \subseteq [a, b]$, we let

$$M(f, S) = \sup\{f(x) \mid x \in S\}$$
$$m(f, S) = \inf\{f(x) \mid x \in S\}$$

**Definition 1.2.** A **partition** of the closed interval $[a, b]$ is a finite sequence $(t_n)$ where $t_0 = a$ and $t_n = b$ with $t_0 < t_1 < \ldots < t_n$.

**Definition 1.3.** The **upper Darboux sum** $U(f, P)$ of a function $f$ with respect to a partition $P$ is

$$U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k])(t_k - t_{k-1})$$

and the **lower Darboux sum** $L(f, P)$ is

$$L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k])(t_k - t_{k-1})$$

See Fig. 1 thru 4 in Note.

**Remark.** We see that the LDS is underestimating the area under the curve while the UDS is overestimating it.

**Example.** Let $f(x) = x - 1$ on $[0, 2]$ with partitions $t_0 = 0, t_1 = \frac{1}{2}, t_2 = \frac{3}{2}, t_3 = 2$.
We have

$$\begin{aligned} U(f, P) &= \sum_{k=1}^{3} M(f, [t_{k-1}, t_k])(t_k - t_{k-1}) \\ &= M\left(f, \left[0, \frac{1}{2}\right]\right)\left(\frac{1}{2} - 0\right) + M\left(f, \left[\frac{1}{2}, \frac{3}{2}\right]\right)\left(\frac{3}{2} - \frac{1}{2}\right) + M\left(f, \left[\frac{3}{2}, 2\right]\right)\left(2 - \frac{3}{2}\right) \\ &= \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)(1) + (1)\left(\frac{1}{2}\right) \\ &= \frac{3}{4} \end{aligned}$$

Note that changing the partitions will change the result, for example:
Let $t_0 = 0, t_1 = \frac{3}{2}, t_2 = \frac{7}{4}, t_3 = 2$.
Then,

$$\begin{aligned} U(f, P) &= \sum_{k=1}^{3} M(f, [t_{k-1}, t_k])(t_k - t_{k-1}) \\ &= M\left(f, \left[0, \frac{3}{2}\right]\right)\left(\frac{3}{2} - 0\right) + M\left(f, \left[\frac{3}{2}, \frac{7}{4}\right]\right)\left(\frac{7}{4} - \frac{3}{2}\right) + M\left(f, \left[\frac{7}{4}, 2\right]\right)\left(2 - \frac{7}{4}\right) \\ &= \left(\frac{1}{2}\right)\left(\frac{3}{2}\right) + \left(\frac{3}{4}\right)\left(\frac{1}{4}\right) + (1)\left(\frac{1}{4}\right) \\ &= \frac{19}{16} \end{aligned}$$

**Corollary 1.4.** We note that

$$\begin{aligned}
U(f,P) &= \sum_{k=1}^{n} M(f,[t_{k-1},t_k])(t_k - t_{k-1}) && \text{(Note that the second term is always greater than 0)}\\
&\le \sum_{k=1}^{n} M(f,[a,b])(t_k - t_{k-1}) && (\text{if } A \subseteq B, \text{ then } \sup A \le \sup B)\\
&= M(f,[a,b]) \sum_{k=1}^{n} (t_k - t_{k-1})\\
&= M(f,[a,b])\left((t_1 - t_0) + (t_2 - t_1) + \cdots + (t_n - t_{n-1})\right)\\
&= M(f,[a,b])(-t_0 + t_n)\\
&= M(f,[a,b])(b-a)
\end{aligned}$$

Similarly, $L(f,P) \ge m(f,[a,b])(b-a)$.

**Proposition 1.5.** Thus,

$$m(f,[a,b])(b-a) \le L(f,P) \le U(f,P) \le M(f,[a,b])(b-a) \tag{1}$$

**Definition 1.6.** The **upper Darboux integral** $U(f)$ of $f$ over $[a,b]$ is

$$U(f) = \inf\{U(f,P) \mid P \text{ is a partition of } [a,b]\}$$

and the **lower Darboux integral** $L(f)$ of $f$ over $[a,b]$ is

$$L(f) = \sup\{L(f,P) \mid P \text{ is a partition of } [a,b]\}$$

We are taking the "most accurate" Darboux sums to get the area under the curve.

**Lemma 1.7.** By (1), $U(f,P), L(f,P)$ are bounded and so $U(f), L(f)$ exists.

**Remark.** We will eventually prove $L(f) \le U(f)$, but this is not obvious.

**Definition 1.8.** If $L(f) = U(f)$, then we say $f$ is integrable on $[a,b]$, and write $\int_a^b f(x)dx$ or $\int_a^b f$.

**Example.** Let $f(x) = c$ on $[a,b]$. Then for all sub-intervals $[t_{k-1},t_k]$ of a partition $P$,

$$M(f,[t_{k-1},t_k]) = c = m(f,[t_{k-1},t_k])$$

and so for any partition $P$,

$$\begin{aligned}
U(f,P) &= \sum_{k=1}^{n} c(t_k - t_{k-1})\\
&= c\sum_{k=1}^{n} (t_k - t_{k-1})\\
&= c(b-a)\\
&= L(f,P) && \text{(By similar argument)}
\end{aligned}$$

therefore, $U(f) = \inf\{c(b-a) \mid P \text{ is a partition of } [a,b]\} = c(b-a) = L(f)$. Thus, $\int_a^b c\,dx = c(b-a)$.

**Remark.** Recall:

$$\sum_{k=1}^{n} = \frac{n(n+1)}{2}$$

**Example.** Let $f(x) = x$ on $[0, b]$ with $b > 0$.
For any sub-interval $[t_{k-1}, t_k]$ of any partition $P$, $M(f, [t_{k-1}, t_k]) = t_k$ and $m(f, [t_{k-1}, t_k]) = t_{k-1}$. Thus,

$$U(f, P) = \sum_{k=1}^{n} t_k(t_k - t_{k-1})$$
$$L(f, P) = \sum_{k=1}^{n} t_{k-1}(t_k - t_{k-1})$$

How do we find the inf of an infinite number of possible partitions?
Consider the family of partitions $P_n$ with $t_k = \frac{kb}{n}$. There will be $n$ sub-intervals of the same width. For those $P_n$,

$$\begin{aligned} U(f, P_n) &= \sum_{k=1}^{n} \frac{kb}{n}\left(\frac{kb}{n} - \frac{(k-1)b}{n}\right) \\ &= \frac{b^2}{n^2}\sum_{k=1}^{n} k(k-(k-1)) \\ &= \frac{b^2}{n^2}\sum_{k=1}^{n} k(1) \\ &= \frac{b^2}{n^2}\left(\frac{n(n+1)}{2}\right) \\ &= \frac{b^2}{2}\left(\frac{n^2+n}{n^2}\right) \end{aligned}$$

Since $\frac{n^2+n}{n^2} > \frac{n^2}{n^2} = 1$ and $\lim_{n\to\infty} \frac{n^2+n}{n^2} = 1$,

$$\inf\{\frac{b^2}{2}\left(\frac{n^2+n}{n^2}\right) \mid n \in \mathbb{N}\} = \frac{b^2}{2}$$

Thus $U(f) \leq \frac{b^2}{2}$ as $U$ is the infimum over all partitions.
Similarly for these $P_n$,

$$\begin{aligned} L(f, P_n) &= \sum_{k=1}^{n} \frac{(k-1)b}{n}\left(\frac{kb}{n} - \frac{(k-1)b}{n}\right) \\ &= \frac{b^2}{n^2}\sum_{k=1}^{n} k-1(k-(k-1)) \\ &= \frac{b^2}{n^2}\sum_{k=1}^{n} k-1 \\ &= \frac{b^2}{n^2}\left(\frac{n(n+1)}{2} - n\right) \\ &= \frac{b^2}{2}\left(\frac{n^2-n}{n^2}\right) \end{aligned}$$

Since $\frac{n^2-n}{n^2} < \frac{n^2}{n^2} = 1$ and $\lim_{n\to\infty} \frac{n^2-n}{1}$,

$$\sup\{\frac{b^2}{2}\left(\frac{n^2-n}{n^2}\right) \mid n \in \mathbb{N}\} = \frac{b^2}{2}$$

Thus $L(f) \geq \frac{b^2}{2}$ as $L$ is the supremum over all partitions.
Thus we have $\frac{b^2}{2} \leq L(f) \leq U(f) \leq \frac{b^2}{2}$ and so $L(f) = U(f) = \frac{b^2}{2}$.
Thus $\int_b^a x\,dx = \frac{b^2}{2}$.

**Example.** Consider

$$f(x) = \begin{cases} 1, & x \in Q \\ 0, & x \in \overline{Q} \end{cases}$$

Note that between any 2 distinct real numbers, there exists both a rational and irrational number between them.
Thus, $M(f, [t_{k-1}, t_k]) = 1$ and $m(f, [t_{k-1}, t_k]) = 0$.
Thus for any $P$,

$$\begin{aligned} U(f, P) &= \sum_{k=1}^{n} (t_k - t_{k-1}) \\ &= b - a \\ L(f, P) &= \sum_{k=1}^{n} (0)(t_k - t_{k-1}) \\ &= 0 \end{aligned}$$

Hence $U(f) = b - a$ and $L(f) = 0$, and so $U(f) \neq L(f)$, and thus $\int_b^a f(x)\,dx$ does not exist.

**Lemma 32.2.** Let $f$ be a bounded function on $[a, b]$. If $P, Q$ are partitions of $[a, b]$ with $P \subseteq Q$ (i.e., Q is a "finer" partition), then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$$

**Proof.** $L(f, Q) \leq U(f, Q)$ is clear by definition. We show $L(f, P) \leq L(f, Q)$ as the proof of the other case is similar. Assume that $Q$ has 1 more point than $P$, for we could then apply this lemma repeatedly to get the general result. If $P$ consists of $a = t_0, t_1, \ldots, t_n = b$, let $Q$ consist of $a = t_0, t_1, \ldots, t_{k-1}, u, t_k, \ldots, t_n = b$ for some $k$ with $1 \leq k \leq n$. Then most terms in $L(f, P)$ and $L(f, Q)$ are the same. In particular, we have

$$L(f, Q) - L(f, P) = m(f, [t_{k-1}, u])(u - t_{k-1}) + m(f, [u, t_n])(t_n - u) - m(f, [t_{k-1}, t_k])(t_k - t_{k-1}) \tag{2}$$

Since $[t_{k-1}, u] \subseteq [t_{k-1}, t_k]$, we have $\inf\{f(x) \mid x \in [t_{k-1}, u]\} \geq \inf\{f(x) \mid x \in [t_{k-1}, t_k]\}$ and so $m(f, [t_{k-1}, u]) \geq m(f, [t_{k-1}, t_k])$, similarly $m(f, [u, t_k]) \geq m(f, [t_{k-1}, t_k])$. Thus, by (2), we have

$$\begin{aligned} L(f, Q) - L(f, P) &\geq m(f, [t_{k-1}, t_k])(u - t_{k-1} + t_k - u - (t_k - t_{k-1})) \\ &= 0 \end{aligned}$$

Thus $L(f, Q) \geq L(f, P)$ as required. □

**Lemma 32.3.** Let $f$ be bounded on $[a, b]$, and let $P, Q$ be partitions of $[a, b]$. Then $L(f, P) \leq U(f, Q)$.

**Proof.** Note that $P \cup Q$ is a partition of $[a, b]$. Since $P \subseteq P \cup Q$ and $Q \subseteq P \cup Q$, by Lemma 32.2, we have

$$\begin{aligned} L(f, P) &\leq L(f, P \cup Q) \\ &\leq U(f, P \cup Q) \\ &\leq U(f, Q) \end{aligned}$$

□

**Theorem 32.4.** Let $f$ be bounded on $[a, b]$. Then $L(f) \leq U(f)$.

**Proof.** Fix a partition $P$ of $[a, b]$. By Lemma 32.3, $L(f, P)$ is a lower bound of the set $\{U(f, Q) \mid Q \text{ is a partition of } [a, b]\}$. Thus for any $P$, $L(f, P) \leq U(f)$ since $U(f)$ is the infimum. Thus implies $U(f)$ is an upper bound of $\{L(f, P) \mid P \text{ is a partition of } [a, b]\}$. Since $L(f) = \sup\{L(f, P)\}$, $L(f) \leq U(f)$. □

## 1.2 Integration Formulas

**Theorem 32.5.** A bounded function $f$ on $[a, b]$ is integrable iff

$$\forall \varepsilon > 0, \exists P \text{ of } [a, b] \text{ s.t. } U(f, P) - L(f, P) < \varepsilon$$

**Proof.** We prove the forwards direction:
Suppose $f$ is integrable. Let $\varepsilon > 0$ be given. Then there exists partitions $P_1, P_2$ with $L(f, P_1) > L(f) - \frac{\varepsilon}{2}$ and $U(f, P_2) < U(f) + \frac{\varepsilon}{2}$. Recall that $L(f)$ and $U(f)$ are the sup/inf of all the $L/U(f, P)$.
Let $P = P_1 \cup P_2$. Then use Lemma 32.2 to get

$$\begin{aligned}
U(f, P) - L(f, P) &\leq U(f, P_2) - L(f, P_1) && \text{(Since } P_1, P_2 \subseteq P) \\
&< U(f) + \frac{\varepsilon}{2} - \left(L(f) - \frac{\varepsilon}{2}\right) \\
&= U(f) - L(f) + \varepsilon \\
&= \varepsilon && (U(f) = L(f) \text{ in integrable functions})
\end{aligned}$$

We now show the reverse:
Suppose $\forall \varepsilon > 0, \exists P$ of $[a, b]$ s.t. $U(f, P) - L(f, P) < \varepsilon$ holds for some partition $P$. Then $U(f) \leq U(f, P)$ and $L(f) \geq L(f, P)$. Then, we have

$$\begin{aligned}
U(f, P) &= U(f, P) - L(f, P) + L(f, P) \\
&< \varepsilon + L(f, P) \\
&\leq \varepsilon + L(f)
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $U(f) \leq L(f)$, for if $U(f) > L(f)$, we can set $\varepsilon = U(f) - L(f)$ and we will get $U(f) < U(f)$. But since $L(f) \leq U(f)$, we must have $U(f) = L(f)$. Thus $f$ is integrable. □

**Definition 1.9.** A function $f$ is monotonic on an interval $I$ if it is either increasing or decreasing on $I$. In other words,

$$x < y \implies f(x) \leq f(y)$$

(or $\geq$)

**Definition 1.10.** The **mesh** of a partition $P$ is the maximum length of the sub-intervals comprising $P$.

**Theorem 33.1.** Every monotonic function $f$ on $[a, b]$ is integrable.

**Proof.** We prove the increasing case.
We may assume that $b > a \implies f(b) > f(a)$, for otherwise $f$ is constant. Since $f(a) < f(x) < f(b) \forall x \in [a, b]$, $f$ is bounded on $[a, b]$.
Let $\varepsilon > 0$ be given. Take a partition of $[a, b]$ with mesh

$$\{t_k - t_{k-1} \mid 1 \leq k \leq n\} < \frac{\varepsilon}{f(b) - f(a)}$$

Then

$$
\begin{aligned}
U(f,P) - L(f,P) &= \sum_{k=1}^{n} (M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k])(t_k - t_{k-1}) \\
&= \sum_{k=1}^{n} (f(t_k) - f(t_{k-1}))(t_k - t_{k-1}) \\
&< \sum_{k=1}^{n} (f(t_k) - f(t_{k-1})) \left(\frac{\varepsilon}{f(b) - f(a)}\right) \\
&= \left(\frac{\varepsilon}{f(b) - f(a)}\right) \sum_{k=1}^{n} (f(t_k) - f(t_{k-1})) \\
&= \left(\frac{\varepsilon}{f(b) - f(a)}\right) (-f(a) + f(b)) && \text{(Telescoping Series)} \\
&= \varepsilon
\end{aligned}
$$

By Theorem 32.5, $f$ is integrable. □

**Example.** The following functions are integrable:

- $\sqrt{x}$
- $\frac{1}{x}$
- $\left(\frac{1}{x}\right)^n$
- $\ln x$
- $e^x$
- $\frac{1}{\ln x}$
- $\frac{1}{e^x}$
- $\lfloor x \rfloor$
- $\lceil x \rceil$
- $\tan x$

**Theorem 33.2.** Every continuous function $f$ on $[a,b]$ is integrable.

**Proof.** Let $\varepsilon > 0$ be given. Since $f$ is cts. on $[a,b]$, then by Theorem 19.2, $f$ is uniformly cts. on $[a,b]$. Thus there is a

$$\delta > 0 \text{ such that } x, y \in [a,b] \text{ and } |x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{b - a}$$

Let $P$ be a partition with mesh less than $\delta$.
By Theorem 18.1, $f$ assumes its max and min in each closed sub-interval. Thus the above implies

$$M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k]) < \frac{\varepsilon}{b - a}$$

and $M(f,[t_{k-1},t_k]) = f(x)$ for some $x \in [t_{k-1},t_k]$ and the same for $m$ with $y$.
Thus

$$
\begin{aligned}
U(f,P) - L(f,P) &= \sum_{k=1}^{n} (M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k]))(t_k - t_{k-1}) \\
&< \sum_{k=1}^{n} \frac{\varepsilon}{b - a}(t_k - t_{k-1}) \\
&= \frac{\varepsilon}{b - a} b - a \\
&= \varepsilon
\end{aligned}
$$

Thus $f$ is integrable on $[a,b]$. □

**Example.** The following functions are integrable:

- $\sin x$
- $\cos x$
- $\frac{p(x)}{q(x)}$
- $e^{-x^2}$
- $\frac{\sin x \ln x}{x^2+1}$

**Theorem 33.3.** Let $f, g$ be integrable on $[a, b]$ and let $c$ be constant. Then,

1. $cf$ is integrable with $\int_a^b cf = c\int_a^b f$
2. $f + g$ is integrable with $\int_a^b (f + g) = \int_a^b f + \int_a^b g$
3. One can show $fg$ is integrable, but there is not a nice formula for this

**Lemma 1.11.**
1. If $c > 0$, then $\inf\{cs \mid s \in S\} = c \inf S$ and $\sup\{cs \mid s \in S\} = c \sup S$
2. $\inf\{-s \mid S \in S\} = -\sup S$ and $\sup\{-s \mid s \in S\} = -\inf S$
3. $\inf\{f(x) + g(x) \mid x \in S\} \geq \inf\{f(x) \mid x \in S\} + \inf\{g(x) \mid x \in S\}$ and $\sup\{f(x) + g(x) \mid x \in S\} \leq \sup\{f(x) \mid x \in S\} + \sup\{g(x) \mid x \in S\}$

**Proof.** Proof of (1):
If $c = 0$, the result is clear. First suppose $c > 0$, then for all sub-intervals we have $M(cf, [t_{k-1}, t_k]) = cM(f, [t_{k-1}, t_k])$ and $m(cf, [t_{k-1}, t_k]) = cm(f, [t_{k-1}, t_k])$. Thus for all partitions $P$, we have $U(cf, P) = cU(f, P)$ and $L(cf, P) = cL(f, P)$. Lemma (i) implies $U(cf) = cU(f)$ and $L(cf) = cL(f)$. We then have

$$\begin{aligned} L(cf) &= cL(f) \\ &= cU(f) && (f \text{ is integrable}) \\ &= U(cf) \end{aligned}$$

thus $cf$ is integrable with integral $\int_a^b cf = U(cf) = cU(f) = c\int_a^b f$.
Now take $c = -1$. Then Lemma (ii) implies $M(-f, [t_{k-1}, t_k]) = -m(f, [t_{k-1}, t_k])$ and $m(-f, [t_{k-1}, t_k]) = -M(f, [t_{k-1}, t_k])$. Then $U(-f, P) = L(f, P)$ and $L(-f, P) = U(f, P)$. Thus

$$\begin{aligned} U(-f) &= \inf\{U(f, P) \mid P \text{ is a partition of } [a, b]\} \\ &= \inf\{-L(f, P) \mid P \text{ is a partition of } [a, b]\} \\ &= \sup\{L(f, P) \mid P \text{ is a partition of } [a, b]\} \\ &= -L(f) \end{aligned}$$

One can similarly show $L(-f) = -U(f)$. Hence,

$$\begin{aligned} U(-f) &= -L(f) \\ &= -U(f) \\ &= L(-f) \end{aligned}$$

Hence, $-f$ is integrable with integral $\int_a^b -f = U(-f) = -U(f) = -\int_a^b f$.

Finally suppose $c < 0$. Then,

$$\begin{aligned}\int_b^a cf &= -\int_a^b (-c)f \\ &= -(-c)\int_a^b f \\ &= c\int_a^b f\end{aligned}$$

□

**Proof.** Proof of (2):
We use Theorem 32.5. Let $\varepsilon > 0$ be given. Then there exist partitions $P_1, P_2$ of $[a,b]$ with $U(f,P_1) = L(f,P_2) < \frac{\varepsilon}{2}$ and $U(g,P_2) - L(g,P_2) < \frac{\varepsilon}{2}$. Let $P = P_1 \cup P_2$, and using Lemma 32.2 yields $U(f,P) - L(f,P) < \frac{\varepsilon}{2}$ and $U(g,P) - L(g,P) < \frac{\varepsilon}{2}$. By Lemma (iii), we have $m(f+g,[t_{k-1},t_k]) \geq m(f,[t_{k-1},t_k]) + m(g,[t_{k-1},t_k])$ and so $L(f+g,P) \geq L(f,P) + L(g,P)$. Similarly, $U(f+g,P) \leq U(f,P) + U(g,P)$. Thus,

$$\begin{aligned}U(f+g,P) - L(f+g,P) &\leq U(f,P) + U(g,P) - L(f,P) - L(g,P) \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

Thus by Theorem 32.5, $f+g$ is integrable. We have

$$\begin{aligned}\int_b^a (f+g) = U(f+g) &\leq U(f+g,P) \\ &\leq U(f,P) + U(g,P) \\ &< L(f,P) + L(g,P) + \varepsilon \\ &\leq L(f) + L(g) + \varepsilon \\ &= \int_a^b f + \int_a^b g + \varepsilon\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $\int_b^a (f+g) \leq \int_a^b f + \int_a^b g$. Also,

$$\begin{aligned}\int_a^b (f+g) = L(f+g) &\geq L(f+g,P) \\ &\geq L(f,P) + L(g,P) \\ &> U(f,P) + U(g,P) - \varepsilon \\ &\geq U(f) + U(g) - \varepsilon \\ &= \int_a^b f + \int_a^b g - \varepsilon\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $\int_a^b (f+g) \geq \int_a^b f + \int_a^b g$. Thus $\int_a^b (f+g) = \int_a^b f + \int_a^b g$. □

**Theorem 33.4.**
- If $f, g$ are integrable on $[a,b]$ and $f(x) \leq g(x)\ \forall x \in [a,b]$ then $\int_a^b f \leq \int_a^b g$
- If $g$ is a cts. non-negative function on $[a,b]$ with $\int_a^b g = 0$, then $g(x) = 0\ \forall x \in [a,b]$

**Proof.** Theorem 33.3 implies that $h = g - f$ is integrable on $[a,b]$. Since $h(x) \geq 0\ \forall x \in [a,b]$, we have $L(h,P) \geq 0\ \forall$ partitions $P$ of $[a,b]$.

Thus $\int_a^b h = L(h) \geq 0$. Thus since $g = f + h$, we have

$$\int_a^b g = \int_a^b (f+h) = \int_a^b f + \int_a^b h \geq \int_a^b f$$

□

**Theorem 33.5.** If $f$ is integrable on $[a, b]$, then $|f|$ is integrable with

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

**Proof.** Since $-|f| \leq f \leq |f|$, Theorem 33.3, 33.4 implies $-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|$ hence $\left|\int_a^b f\right| \leq \int_a^b |f|$. □

**Theorem 33.6.** Let $f$ defined on $[a, b]$ and let $a < c < b$. If $f$ is integrable on $[a, c]$ and $[c, b]$, then $f$ is integrable on $[a, b]$ with

$$\int_a^b f = \int_a^c f + \int_c^b f$$

**Proof.** Since $f$ is bounded on $[a, c]$ and $[c, b]$, so $f$ is bounded on $[a, b]$. Let $\varepsilon > 0$ be given. Theorem 32.5 implies there exist partitions $P_1$ of $[a, c]$ and $P_2$ of $[c, b]$ with

$$U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{2} \text{ and } U(f, P_2) - L(f, P_2) < \frac{\varepsilon}{2}$$

Then $P = P_1 \cup P_2$ is a partition of $[a, b]$.
Thus,

$$U(f, P) - L(f, P) = U(f, P_1) + U(f, P_2) - L(f, P_1) - L(f, P_1) < \varepsilon$$

Thus $f$ is integrable on $[a, b]$.
Also, we have

$$\begin{aligned} \int_b^a &\leq U(f, P) \\ &= U(f, P_1) + U(f, P_2) \\ &< L(f, P_1) + L(f, P_2) + \varepsilon \\ &\leq \int_a^c f + \int_c^b f + \varepsilon \end{aligned}$$

for any $\varepsilon > 0$, and hence

$$\int_a^b f \leq \int_a^c f + \int_c^b f$$

Also, we have

$$\begin{aligned} \int_b^a &\geq L(f, P) \\ &= L(f, P_1) + L(f, P_2) \\ &> U(f, P_1) + U(f, P_2) - \varepsilon \\ &\geq \int_a^c f + \int_c^b f - \varepsilon \end{aligned}$$

for any $\varepsilon > 0$, and hence

$$\int_a^b f \geq \int_a^c f + \int_c^b f$$

Thus

$$\int_a^b f = \int_a^c f + \int_c^b f$$

$\square$

**Definition 1.12.** A function $f$ is **piecewise monotonic** on $[a, b]$ if there is a partition $P$ such that $f$ is monotonic on each open subinterval $(t_{k-1}, t_k)$.

**Definition 1.13.** A function $f$ is **piecewise continuous** if $f$ is uniformly continuous on each open subinterval $(t_{k-1}, t_k)$.

**Theorem 33.8.** If $f$ is piecewise cts. or is a bounded piecewise monotonic function on $[a, b]$, then $f$ is integrable on $[a, b]$.

**Proof.** We first show in either case, if we consider $f$ on the open subinterval $(t_{k-1}, t_k)$, we can extend it to an integrable function $f_k$ defined on the closed interval $[t_{k-1}, t_k]$. If $f$ is uniformly cts. on $(t_{k-1}, t_k)$, then $f$ can be extended to a cts. function on $[t_{k-1}, t_k]$. Since $f_k$ is cts. on $[t_{k-1}, t_k]$, by 33.2 it is integrable here.
If $f$ is bounded and monotonic on $(t_{k-1}, t_k)$, say $f$ is increasing, then we can set $f_k(t_{k-1}) = \inf\{f(x) \mid x \in (t_{k-1}, t_k)\}$ and $f_k(t_k) = \sup\{f(x) \mid x \in (t_{k-1}, t_k)\}$ to yield an increasing function $f_k$ on $[t_{k-1}, t_k]$. A similar extension can be made for a decreasing function. Since the resulting function $f_k$ is monotonic on $[t_{k-1}, t_k]$, by 33.1 it is integrable there.
In either case, we have $f = f_k$ on $[t_{k-1}, t_k]$ except possibly at the endpoints, and $f_k$ is integrable on the closed subinterval. By Exercise 32.7, $f$ is integrable on $[t_{k-1}, t_k]$ since they differ at only finitely many points. 33.6 then implies that $f$ is integrable over $[a, b]$. $\square$

## 1.3 Fundamental Theorem of Calculus I

**FTC I.** If $g$ is cts on $[a, b]$, diff-able on $(a, b)$ and if $g'$ is integrable on $[a, b]$, then

$$\int_b^a g' = g(b) - g(a)$$

**Proof.** Let $\varepsilon > 0$ be given. Since $g'$ is integrable, by 32.5 there is a partition $P$ of $[a, b]$ with

$$U(g', P) - L(g', P) < \varepsilon$$

Since $g$ is cts and diff-able, we can apply MVT to $g$ on each sub-interval $t_{k-1}, t_k$ to get $x_k \in (t_{k-1}, t_k)$ with

$$g'(x_k) = \frac{g(t_k) - g(t_{k-1})}{t_k - t_{k-1}}$$

Thus,

$$g'(x_k)(t_k - t_{k-1}) = g(t_k) - g(t_{k-1})$$

Summing this over $k$ yields

$$\sum_{k=1}^{n} g'(x_k)(t_k - t_{k-1}) = \sum_{k=1}^{n} (g(t_k) - g(t_{k-1})) = g(b) - g(a)$$

Since $x_k \in [t_{k-1}, t_k]$, we have

$$m(g', [t_{k-1}, t_k]) \leq g'(x_k) \leq M(g', [t_{k-1}, t_k])$$

Multiplying by $t_k - t_{k-1} > 0$ yields

$$m(g', [t_{k-1}, t_k])(t_k - t_{k-1}) \leq g'(x_k)(t_k - t_{k-1}) \leq M(g', [t_{k-1}, t_k])(t_k - t_{k-1})$$

Summing this over $k$ and using the above equality and definitions yields

$$L(g', P) \leq g(b) - g(a) \leq U(g', P)$$

On the other hand, we have

$$L(g', P) \leq \int_a^b g' \leq U(g', P)$$

Subtracting the first from the second yields

$$L(f', P) - U(g', P) \leq \int_a^b g' - (g(b) - g(a)) \leq U(g', P) - L(g', P)$$

However, recall that we have $U(g', P) - L(g', P) < \varepsilon$. Using this, we have

$$-\varepsilon < \int_a^b g' - (g(b) - g(a)) < \varepsilon$$

Thus

$$0 \leq \left| \int_a^b g' - (g(b) - g(a)) \right| < \varepsilon$$

and so

$$\left| \int_a^b g' - (g(b) - g(a)) \right| = 0$$

and so

$$\int_a^b g' = g(b) - g(a)$$

□

**Remark.** If we are trying to integrate an integrable function $f$, FTC(I) shows us that we should look for a function $F$ such that $F' = f$. Then we'd have

$$\int_a^b f = F(b) - F(a)$$

**Definition 1.14.** Let $f$ be an integrable function. $F$ is the function where $F' = f$, and is called the **antiderivative** of $f$. It is common to write

$$F(x)\Big|_a^b$$

for $F(b) - F(a)$.

**Example.** Consider $\int_0^b x^3 \, dx$. Can we find $F$ such that $F'(x) = x^3$? Let $F(x) = \frac{1}{4}x^4$. We thus have

$$\int_0^b x^3 \, dx = \frac{1}{4}x^4\Big|_0^b = \frac{1}{4}b^4 - \frac{1}{4}0^4 = \frac{1}{4}b^4$$

**Proposition 1.15.** In general, if $n \neq -1$ and if in the case that $n < 0$, we have either $a, b > 0$ or $a, b < 0$, then

$$\int_a^b x^n \, dx = \frac{1}{n+1}x^{n+1}\Big|_a^b = \frac{1}{n+1}b^{n+1} - \frac{1}{n+1}a^{n+1}$$

**Example.** Consider $\int_0^1 \sqrt{x} \, dx$ which is the area under $f(x) = \sqrt{x}$ between $x = 0$ and $x = 1$. We have

$$\int_0^1 \sqrt{x} \, dx = \int_0^1 x^{\frac{1}{2}} \, dx = \frac{2}{3}x^{\frac{3}{2}}\Big|_0^1 = \frac{2}{3}$$

**Remark.** We know how to integrate $x^n$ for $n \neq -1$. What if $n = -1$? For $a, b > 0$ we have

$$\int_a^b x^{-1} \, dx = \int_a^b \frac{1}{x} \, dx = \ln x\Big|_a^b = \ln b - \ln a$$

**Remark.** Consider $\int_a^b f$. Suppose it is unknown if $f$ is integrable, but there is a function $F$ with $F' = f$. Does this imply $f$ is integrable?
No! Consider

$$F(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

We have

$$f = F' = \begin{cases} 2x \sin\left(\frac{1}{x^2}\right) - \frac{2}{x}\cos\left(\frac{1}{x^2}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

So if we were to start with $f$, then $F$ is a diff-able function with $F' = f$. However, $f$ is unbounded near 0 and hence not integrable.

**Definition 1.16.** We call the integral $\int_a^b f(x)\,dx$ where $a, b \in \mathbb{R}$ the **definite integral**.

**Remark.** If 2 functions have the same derivative, they must differ by at most a constant. Thus if $F$ is an antiderivative of $f$, so is $F(x) + c$ for any constant c, and these give all the antiderivatives for $f$.

**Definition 1.17.** $F(x) + c$ is called the **indefinite integral** of $f(x)$, and is written

$$\int f(x)\,dx$$

**Example.**

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

$$\int x^2\,dx = \frac{1}{3}x^3 + c$$

$$\int 2x^3 - 7x + 3\,dx = 2\int x^3\,dx - 7\int x\,dx + \int 3\,dx = \frac{1}{2}x^4 - \frac{7}{2}x^2 + 3x + c$$

$$\int e^x\,dx = e^x + c$$

$$\int e^{2x}\,dx = \frac{1}{2}e^{2x} + c$$

$$\int e^{-x^2}\,dx$$

has no obvious antiderivative.

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

**Example.** What is the area under one bump of the sine curve?

$$\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = -\cos\pi - (-\cos 0) = 2$$

What is the area between 0 and $2\pi$?

$$\int_0^{2\pi} \sin x\,dx = -\cos x\Big|_0^{2\pi} = -\cos 2\pi - (-\cos 0) = 0$$

## 1.4 Integration by Parts

**Integration by Parts.** Suppose $u, v$ are cts. on $[a, b]$, diff-able on $(a, b)$, and suppose $u', v'$ are integrable on $[a, b]$.

Then,

$$\int_a^b u(x)v'(x)\ dx = u(x)v(x)\Big|_a^b - \int_a^b u'(x)v(x)\ dx$$

This is often written as

$$\int udv = uv - \int vdu$$

**Proof.** Let $g(x) = u(x)v(x)$, so that $g'(x) = u'(x)v(x) + u(x)v'(x)$. Since $u, v$ are cts. then they are integrable. By assumption $u', v'$ are integrable, so it follows that $u'v$ and $uv'$ are integrable.
BY FTC I, we have

$$\int_a^b g'(x)\ dx = g(x)\Big|_a^b = u(x)v(x)\Big|_a^b$$

By 33.3, we have

$$\begin{aligned}\int_a^b g'(x)\ dx &= \int_a^b u'(x)v(x) + u(x)v'(x)\ dx \\ &= \int_a^b u'(x)v(x)\ dx + \int_a^b u(x)v'(x)\ dx\end{aligned}$$

Equating yields

$$\int_a^b u'(x)v(x)\ dx + \int_a^b u(x)v'(x)\ dx = u(x)v(x)\Big|_a^b$$

and rearranging this yields the theorem. □

**Remark.** To use integration by parts, we view the function that we are integrating as a product, ideally one where one factor simplifies after it is differentiated, and the other factor doesn't get too much more complicated when integrating it.

**Example.** Evaluate $\int_0^\pi x \sin x\ dx$.
Let's let $u(x) = x$, $v'(x) = \sin x$. Then $u'(x) = 1$ and $v(x) = -\cos x$. Then

$$\begin{aligned}\int_0^\pi x \sin x\ dx &= -x\cos x\Big|_0^\pi - \int_0^\pi -\cos x\ dx \\ &= -\pi(-1) - 0 + \int_0^\pi \cos x\ dx \\ &= \pi + \sin x\Big|_0^\pi \\ &= \pi + 0 - 0 \\ &= \pi\end{aligned}$$

and

$$\begin{aligned}\int x\sin x\ dx &= -x\cos x - \int -\cos x\ dx \\ &= -x\cos x + \int \cos x\ dx \\ &= -x\cos x + \sin x + c\end{aligned}$$

**Example.** Find $\int \ln x\ dx$. The trick here is to view the integrand as $(1)(\ln x)$. So $u(x) = \ln x$, $v'(x) = 1$. Then $u'(x) = \frac{1}{x}$ and $v(x) = x$. Integration by parts yields

$$\begin{aligned}\int \ln x\ dx &= x\ln x - \int (x)\left(\frac{1}{x}\right)\ dx \\ &= x\ln x - x + c\end{aligned}$$

Now let's find

$$\int_1^e \ln x \, dx = (x \ln x - e)\Big|_1^e = e \ln e - e - (\ln 1 - 1) = 1$$

**Example.** Find

$$\int xe^x \, dx$$

Let $u(x) = x$ and $v'(x) = e^x$. Then $u'(x) = 1$ and $v(x) = e^x$.
Integration by parts yields

$$\begin{aligned}\int xe^x \, dx &= xe^x - \int e^x \, dx \\ &= xe^x - e^x + c\end{aligned}$$

**Example.** Find $\int \arctan x \, dx$.
Recall that $(\arctan x)' = \frac{1}{x^2+1}$. Let $u(x) = \arctan x$ and $v'(x) = 1$. Then $u'(x) = \frac{1}{x^2+1}$ and $v(x) = x$. We have

$$\begin{aligned}\int \arctan x \, dx &= x \arctan x - \int \frac{x}{x^2+1} \, dx \\ &= x \arctan x - \frac{1}{2} \ln\big(x^2+1\big) + c\end{aligned}$$

**Example.** Find $\int (\ln x)^2 \, dx$.
Let $u(x) = (\ln x)^2$ and $v'(x) = 1$. Then $u'(x) = 2 \ln x \frac{1}{x}$ and $v(x) = x$. Then

$$\begin{aligned}\int (\ln x)^2 \, dx &= x(\ln x)^2 - 2 \int \ln x \, dx \\ &= x(\ln x)^2 - 2(x \ln x - x) + c \\ &= x((\ln x)^2 - 2 \ln x + 2) + c\end{aligned}$$

**Example.** Find $\int x^2 \cos(2x) \, dx$. Let $u(x) = x^2, v'(x) = \cos(2x)$. Thus $u'(x) = 2x$ and $v(x) = \frac{1}{2} \sin(2x)$. We have

$$\int x^2 \cos(2x) \, dx = \frac{x^2}{2} \sin(2x) - \int x \sin(2x) \, dx$$

For the integral on the right side, we can use integration by parts again with $u(x) = x$ and $v'(x) = \sin(2x)$. Then we have $u'(x) = 1$ and $v(x) = \frac{-1}{2} \cos(2x)$. Thus

$$\begin{aligned}\int x^2 \cos(2x) \, dx &= \frac{x^2}{2} \sin(2x) - \left(-\frac{x}{2} \cos(2x) - \int \left(\frac{-1}{2} \cos(2x)\right)\right) dx \\ &= \frac{x^2}{2} \sin(2x) + \frac{x}{2} \cos(2x) - \frac{1}{2} \int \cos(2x) \, dx \\ &= \frac{x^2}{2} \sin(2x) + \frac{x}{2} \cos(2x) - \frac{1}{4} \sin(2x) + c\end{aligned}$$

## 1.5 Fundamental Theorem of Calculus II

**Definition 1.18.** If $b > a$, define

$$\int_a^b f = -\int_b^a f$$

**Corollary 1.19.** We have

$$\begin{aligned}\int_a^a f(x)\,dx &= \int_a^b f(x)\,dx + \int_b^a f(x)\,dx \\ &= \int_a^b f(x)\,dx - \int_a^b f(x)\,dx \\ &= 0\end{aligned}$$

**FTC II.** Let $f$ be integrable on $[a,b]$. For a point $x \in [a,b]$, define $F(x) = \int_a^x f(t)\,dt$. Then $f$ is continuous on $[a,b]$. Furthermore, if $f$ is continuous at $x_0 \in [a,b]$, then $F$ is diff-able at $x_0$ and $F'(x_0) = f(x_0)$.
In Leibniz notation,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

**Proof.** Since $f$ is integrable on $I := [a,b]$, it is bounded there, thus $\exists B > 0$ with $|f(x)| \leq B\ \forall x \in I$.
Let $\varepsilon > 0$ be given, and suppose $x, y \in I$ with $|x - y| < \frac{\varepsilon}{B}$. WLOG, suppose $x < y$. Then

$$\begin{aligned}|F(y) - F(x)| &= \left|\int_a^y f(t)\,dt - \int_a^x f(t)\,dt\right| \\ &= \left|\int_a^y f(t)\,dt + \int_x^a f(t)\,dt\right| \\ &= \left|\int_x^y f(t)\,dt\right| && \text{(Theorem 33.6)} \\ &\leq \int_x^y |f(t)|\,dt && \text{(Theorem 33.5)} \\ &\leq \int_x^y B\,dt \\ &= B(y-x) \\ &< B\frac{\varepsilon}{B} \\ &= \varepsilon\end{aligned}$$

Thus $f$ is uniformly continuous on $I$.
For $x = x_0$, we have

$$\begin{aligned}F(x) - F(x_0) &= \int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt \\ &= \int_a^x f(t)\,dt + \int_{x_0}^a f(t)\,dt \\ &= \int_{x_0}^x f(t)\,dt\end{aligned}$$

and

$$\frac{F(x) - F(x_0)}{x - x_0} = \frac{1}{x - x_0}\int_{x_0}^x f(t)\,dt$$

as

$$\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = F'(x_0)$$

We also have

$$\begin{aligned}
\frac{1}{x-x_0}\int_{x_0}^{x} f(t)\,dt &= \frac{f(x_0)}{x-x_0}\int_{x_0}^{x} 1\,dt \\
&= \frac{f(x_0)}{x-x_0}(x-x_0) \\
&= f(x_0)
\end{aligned}$$

Thus $\frac{F(x)-F(x_0)}{x-x_0} - F(x_0) = \frac{1}{x-x_0}\int_{x_0}^{x}(f(t)-f(x_0))\,dt$.
Let $\varepsilon > 0$ be given. Since $f$ is continuous at $x_0$, then $\exists\delta > 0$ s.t. $t \in (a,b)$ and $|t-x_0| < \delta \implies |f(t)-f(x_0)| < \varepsilon$.
In the case $x > x_0$, we have

$$\begin{aligned}
\left|\frac{F(x)-F(x_0)}{x-x_0} - F(x_0)\right| &= \frac{1}{|x-x_0|}\left|\int_{x_0}^{x}(f(t)-f(x_0))\,dt\right| \\
&\le \frac{1}{x-x_0}\int_{x_0}^{x}|f(t)-f(x_0)|\,dt && \text{(Theorem 33.5)} \\
&\le \frac{1}{x-x_0}\int_{x_0}^{x}\varepsilon\,dt && \text{(Theorem 33.4 (i))} \\
&= \varepsilon(x-x_0)\left(\frac{1}{x-x_0}\right) \\
&= \varepsilon
\end{aligned}$$

In the other case

$$\begin{aligned}
\left|\frac{F(x)-F(x_0)}{x-x_0} - F(x_0)\right| &= \frac{1}{|x-x_0|}\left|\int_{x_0}^{x}(f(t)-f(x_0))\,dt\right| \\
&\le \frac{1}{x_0-x}\int_{x_0}^{x}|f(t)-f(x_0)|\,dt && \text{(Theorem 33.5)} \\
&\le \frac{1}{x_0-x}\int_{x_0}^{x}\varepsilon\,dt && \text{(Theorem 33.4 (i))} \\
&= \frac{-1}{x_0-x}\int_{x}^{x_0}\varepsilon\,dt \\
&= \varepsilon(x_0-x)\left(\frac{1}{x_0-x}\right) \\
&= \varepsilon
\end{aligned}$$

Thus by a definition of a limit, we've shown that

$$\lim_{x\to x_0}\frac{F(x)-F(x_0)}{x-x_0} = f(x_0)$$

which implies $F'(x_0) = f(x_0)$. □

**Example.** Let

$$g(x) = \int_{1}^{x} t^2\,dt$$

By FTC II, $g'(x) = x^2$.

**Example.** Let

$$p(x) = \int_{1}^{x} e^{-t^2}\,dt$$

By FTC II, $p'(x) = e^{-x^2}$.

**Example.** Let $F(x) = \int_{2x}^{x^2} e^{-t^2}\,dt$. Find $F'$.
Let $p(x) = \int_{0}^{x} e^{-t^2}\,dt$, then by FTC II $p'(x) = e^{-x^2}$.

Thus

$$\begin{aligned}
F(x) &= \int_{2x}^{0} e^{-t^2}\ dt + \int_0^{x^2} e^{-t^2}\ dt \\
&= -\int_0^{2x} e^{-t^2}\ dt + \int_0^{x^2} e^{-t^2}\ dt \\
&= -p(2x) + p(x^2)
\end{aligned}$$

Thus $F'(x) = -2p'(x) + 2xp'(x^2)$. Since $2x, x^2, p(x)$ are diff-able, we have

$$F'(x) = -2e^{-4x^2}2xe^{-x^4}$$

**Example.** Let $Li(x) = \int_2^x \frac{1}{\ln t}\ dt$. Then $Li'(x) = \frac{1}{\ln x}$.

**Change of Variables.** Let $u$ be a diff-able function on an open interval $J$ with $u'$ continuous. Let $I$ be an open interval with $u(x) \in I$ for $x \in J$. If $f$ is continuous on $I$, then $f \circ u$ is continuous on $J$ with

$$\int_a^b (f \circ u)(x)u'(x)\ dx = \int_{u(a)}^{u(b)} f(u)\ du$$

for $a, b \in J$.

**Proof.** $f \circ u$ is continuous by Theorem 17.5. Let $c \in I$, define

$$F(u) = \int_c^u f(t)\ dt$$

By FTC II, $F'(u) = f(u)$ for all $u \in I$. Let $g = f \circ u$, then $g'(x) = F'(u(x))u'(x) = f(u(x))u'(x)$.
Hence,

$$\begin{aligned}
\int_a^b (f \circ u)(x)u'(x)\ dx &= \int_a^b g'(x)\ dx \\
&= g(b) - g(a) \\
&= F(u(b)) - F(u(a)) \\
&= \int_c^{u(b)} f(t)\ dt - \int_c^{u(a)} f(t)\ dt \\
&= \int_c^{u(b)} f(t)\ dt + \int_{u(a)}^{c} f(t)\ dt \\
&= \int_{u(a)}^{u(b)} f(t)\ dt
\end{aligned}$$

□

**Remark.** Look for integrals in the form

$$\int f(u(x))u'(x)\ dx$$

**Example.** Find

$$\int_0^2 xe^{-x^2}\ dx$$

We have $u(x) = -x^2$, so $u'(x) = -2x$. Then $f(u) = -\frac{1}{2}e^u$, this yields $f(u(x))u'(x) = -\frac{1}{2}e^{-x^2} \cdot -2x = xe^{-x^2}$. Then

$$\begin{aligned}
\int_{u(0)=0}^{u(2)=4} -\frac{1}{2}e^u\ du &= -\frac{1}{2}e^u\Big|_0^4 \\
&= -\frac{1}{2}(e^{-4} - 1)
\end{aligned}$$

We often write this as $u = -x^2, du = -2xdx$ so $-\frac{1}{2}du = xdx$

**Example.** Find

$$\int_0^2 \frac{x^2}{\sqrt{x^3+1}}\ dx$$

Let $u = x^3+1, du = 3x^2\ dx$ and $\frac{1}{3}du = x^2\ dx$. Then we have

$$\begin{aligned}\int_0^2 \frac{x^2}{\sqrt{x^3+1}}\ dx &= \frac{1}{3}\int_1^9 \frac{1}{\sqrt{u}}\ du \\ &= \frac{2}{3}u^{\frac{1}{2}}\Big|_1^9 \\ &= \frac{2}{3}(3-1) \\ &= \frac{4}{3}\end{aligned}$$

**Example.** Find

$$\int xe^{-x^2}\ dx$$

Let $u = -x^2 \implies du = -2x\ dx \implies = \frac{1}{2}du = x\ dx$. Then we have

$$\begin{aligned}\int xe^{-x^2}\ dx &= \int -\frac{1}{2}e^u\ du \\ &= -\frac{1}{2}e^u + c \\ &= -\frac{1}{2}e^{-x^2} + c\end{aligned}$$

**Example.** Find

$$\int x\sqrt{4-x}\ dx$$

Let $u = 4-x \implies du = -1\ dx$ and $x = 4-u$. Then we have

$$\begin{aligned}\int x\sqrt{4-x}\ dx &= -\int 4-u\ du \\ &= -\int 4u^{0.5} - u^{1.5}\ du \\ &= -(4\cdot\frac{2}{3}u^{1.5} - \frac{2}{5}u^{\frac{5}{2}}) + c \\ &= -\frac{8}{3}u^{\frac{3}{2}} + \frac{2}{5}u^{\frac{5}{2}} + c \\ &= -\frac{8}{3}(4-x)^{\frac{3}{2}} + \frac{2}{5}(4-x)^{\frac{5}{2}} + c\end{aligned}$$

**Example.** Find

$$\int \frac{x+1}{x^2+1}\ dx$$

Let $u = x^2 + 1 \implies du = 2x\,dx \implies \frac{1}{2}du = x\,dx$. Then we have

$$\begin{aligned}
\int \frac{x+1}{x^2+1}\,dx &= \int \left(\frac{x}{x^2+1} + \frac{1}{x^2+1}\right)\,dx \\
&= \frac{1}{2}\int \frac{1}{u}\,du + \arctan x \\
&= \frac{1}{2}\ln|u| + \arctan x \\
&= \frac{1}{2}\ln|x^2+1| + \arctan x + c
\end{aligned}$$

**Example.** Find

$$\int \frac{1}{1+e^x}\,dx$$

Let $u = e^{-x+1} \implies du = -e^{-x}\,dx \implies -du = e^{-x}\,dx$. Then we have

$$\begin{aligned}
\int \frac{1}{1+e^x}\,dx &= \int \frac{e^{-x}}{e^{-x}+1}\,dx \\
&= \int \frac{-1}{u}\,du \\
&= -\ln|u| + c \\
&= -\ln|e^{-x}+1| + c
\end{aligned}$$

**Example.** Find

$$\int \sec x\,dx$$

Note that $\sec' x = \sec x \tan x$ and $\tan' x = \sec^2 x$ so $\sec' x + \tan' x = \sec x(\tan x + \sec x)$. Let $u = \sec x + \tan x \implies du = \sec x(\tan x + \sec x)\,dx$. Then we have

$$\begin{aligned}
\int \sec x\,dx &= \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} \\
&= \int \frac{1}{u}\,du \\
&= \ln|u| + c \\
&= \ln|\sec x + \tan x| + c
\end{aligned}$$

**Example.** Find

$$\int \cos(\ln x)\,dx$$

Let $t = \ln x \implies x = e^t \implies dt = \frac{1}{x}\,dx \implies dx = xdt = e^t dt$.
Then we have

$$\begin{aligned}
\int \cos(\ln x)\,dx &= \int e^t \cos t\,dt \\
&= \frac{1}{2}e^t(\cos t + \sin t) + c \\
&= \frac{1}{2}x(\cos\ln x + \sin\ln x) + c
\end{aligned}$$

**Example.** Find

$$\int x^5 e^{x^2}\,dx$$

Let $t = x^2 \implies dt = 2x\,dx \implies xdx = \frac{1}{2}dt$.

Then we have

$$\begin{aligned}\int x^5 e^{x^2}\, dx &= \frac{1}{2}\int t^2 e^t\, dt \\ &= \frac{1}{2}\left(t^2e^t - 2\int te^t\, dt\right) && \text{(Integration by parts)} \\ &= \frac{1}{2}t^2e^t - \left(te^t - \int e^t\, dt\right) \\ &= \frac{1}{2}e^{x^2}(x^4 - 2x^2 + 2) + c\end{aligned}$$

## 1.6 Trig Substitution

**Trig substitution.** For factors like $\sqrt{a^2 - x^2}, \sqrt{a^2 + x^2}, \sqrt{x^2 - a^2}$ for some constant $a$.
Recall that $\sin^2\theta + \cos^2\theta = 1$, which yields $1 - \sin^2\theta = cos^2\theta$, and $\tan^2\theta + 1 = \sec^2\theta$. We use these identities to remove square roots.

**Example.** Find

$$\int \frac{1}{\sqrt{1-x^2}}\, dx$$

Let $x = \sin\theta \implies dx = \cos\theta\, d\theta$, and $1 - x^2 = 1 - \sin^2\theta = \cos^2\theta$ and so $\sqrt{1-x^2} = \sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta$ since $\cos\theta$ is positive for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, and for these $\theta$, $-1 < x < 1$, which is our domain.
Then we have

$$\begin{aligned}\int \frac{1}{\sqrt{1-x^2}}\, dx &= \int \frac{\cos\theta}{\cos\theta}\, d\theta \\ &= \int 1\, d\theta \\ &= \theta + c \\ &= \arcsin x + c\end{aligned}$$

**Example.** Find

$$\int_2^{2\sqrt{2}} \sqrt{x^2-4}\, dx$$

Let $x = 2\sec\theta \implies x^2 - 4 = 4(\sec^2\theta - 1) = 4\tan^2\theta$ and $dx = 2\sec\theta\tan\theta\, d\theta$.
Then we have

$$\begin{aligned}\sqrt{x^2-4} &= \sqrt{4\tan^2\theta} \\ &= 2|\tan\theta|\end{aligned}$$

and we can drop the absolute value if theta is in the 1st or 3rd quadrant.
Also,

$$\frac{\pi}{2} = \sec\theta = \frac{1}{\cos\theta} \implies \cos\theta = \frac{2}{x}$$

When $x = 2, \cos\theta = 1 \implies \theta = 0$.
When $x = 2\sqrt{2}, \cos\theta = \frac{1}{\sqrt{2}} \implies \theta = \frac{\pi}{4}$.

The integral becomes:

$$
\begin{aligned}
\int_2^{2\sqrt{2}} \sqrt{x^2-4}\,dx &= \int_0^{\frac{\pi}{4}} (2\tan\theta)(2\sec\theta\tan\theta)\,d\theta \\
&= 4\int_0^{\frac{\pi}{4}} \sec\theta\tan^2\theta\,d\theta \\
&= 4\int_0^{\frac{\pi}{4}} \sec\theta(\sec^2\theta-1)\,d\theta \\
&= 4\int_0^{\frac{\pi}{4}} \sec^3\theta-\sec\theta)\,d\theta \\
&= 4(\frac{1}{2}\sec\theta\tan\theta+\frac{1}{2}\ln|\sec\theta+\tan\theta| = -\ln|\sec\theta+\tan\theta|)\Big|_0^{\frac{\pi}{4}} \\
&= 2(\sec\theta\tan\theta = \ln|\sec\theta+\tan\theta|)\Big|_0^{\frac{\pi}{4}} \\
&= 2(\sqrt{2}-\ln\Big(\sqrt{2}+1\Big))
\end{aligned}
$$

Say we want $\int \sqrt{x^2-4}\,dx$. By above, this would be

$$2(\sec\theta\tan\theta-\ln|\sec\theta+\tan\theta|)+c$$

Our substitution was $\sec\theta = \frac{x}{2}$. We also have $\tan^2\theta = \sec^2 -1 = \frac{x^2-4}{4} \implies \tan\theta = \frac{\sqrt{x^2-4}}{2}$.
Then,

$$
\begin{aligned}
\int \sqrt{x^2-4}\,dx &= 2\left(\frac{x}{2}\cdot\frac{\sqrt{x^2=4}}{2} - \ln\left|\frac{x}{2}+\frac{\sqrt{x^2-4}}{2}\right|\right)+c \\
&= \frac{x}{2}\sqrt{x^2-4} - 2\ln|x+\sqrt{x^2-4}| + 2\ln 2 + c \\
&= \frac{x}{2}\sqrt{x^2-4} - 2\ln|x+\sqrt{x^2-4}| + c && \text{(constant can be absorbed into } c\text{)}
\end{aligned}
$$

**Example.** Let's find the area of a circle, $x^2+y^2=r^2 \implies y^2=r^2-x^2 \implies y=\pm\sqrt{r^2-x^2}$.
Integrate $y=\sqrt{r^2-x^2}$ from $0\to r$ and multiply this by 4 to get the area of the whole circle.
Let $x=r\sin\theta \implies dx = r\cos\theta\,d\theta$ and $r^2-x^2=r^2-r^2\sin^2\theta = r^2(1-\sin^2\theta)=r^2\cos\theta$. Thus $\sqrt{r^2-x^2}=r\cos\theta$.
Then we have

$$
\begin{aligned}
A &= 4\int_0^r \sqrt{r^2-x^2}\,dx \\
&= 4\int_0^{\frac{\pi}{2}} r\cos\theta r\cos\theta\,d\theta && (x=0 \implies \theta=0,\, x=r \implies \theta=\tfrac{\pi}{2}) \\
&= 4r^2\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta \\
&= 4r^2\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\,d\theta \\
&= 2r^2\int_0^{\frac{\pi}{2}} 1+\cos 2\theta\,d\theta \\
&= 2r^2(\theta+\frac{1}{2}\sin 2\theta)\Big|_0^{\frac{\pi}{2}} \\
&= 2r^2(\frac{\pi}{2}+0-0) \\
&= \pi r^2
\end{aligned}
$$

**Example.** Find

$$\int \frac{x^3}{(4x^2+9)^{\frac{3}{2}}}\,dx$$

We use $\tan^2\theta + 1 = \sec^2\theta$.
Let $x = \frac{3}{2}\tan\theta$, $\implies dx = \frac{3}{2}\sec^2\theta\, d\theta$, so $4x^2+9 = 9\tan^2\theta + 9 = 9(\tan^2\theta/+1) = 9\sec^2\theta$.
We have $(4x^2+9)^{\frac{3}{2}} = (9\sec^2\theta)^{\frac{3}{2}} = 27\sec^3\theta$. So $x^3 = \frac{27}{8}\tan^3\theta$.
Then we have

$$\begin{aligned}
\int \frac{x^3}{(4x^2+9)^{\frac{3}{2}}}\,dx &= \frac{27}{8}\cdot\frac{3}{2}\cdot\frac{1}{27}\int \frac{\tan^3\theta\sec^2\theta}{\sec^3\theta}\,d\theta \\
&= \frac{3}{16}\int \frac{\tan^3\theta}{\sec\theta}\,d\theta \\
&= \frac{3}{16}\int \frac{\sin^3\theta}{\cos^2\theta}\,d\theta \\
&= \frac{3}{16}\int \frac{\sin^2\theta(1-\cos^2\theta)}{\cos^2\theta}\,d\theta && (\text{Let } u = \cos\theta \implies du = -\sin\theta\, d\theta) \\
&= -\frac{3}{16}\int \frac{1-u^2}{u^2}\,du \\
&= -\frac{3}{16}\int \frac{1}{u^2} - 1\,du \\
&= -\frac{3}{16}\left(-\frac{1}{u} - u\right) + c \\
&= \frac{3}{16}\left(\frac{1}{u} + u\right) + c \\
&= \frac{3}{16}\left(\frac{u^2+1}{u}\right) + c \\
&= \frac{3}{16}\left(\frac{\cos^2\theta+1}{\cos\theta}\right) + c && (\tan\theta = \tfrac{2x}{3}) \\
&= \frac{3}{16}\left(\frac{\left(\frac{3}{\sqrt{4x^2+9}}\right)^2+1}{\frac{3}{\sqrt{4x^2+9}}}\right) + c
\end{aligned}$$

## 1.7 Partial Fractions

**Remark.** Let's find $A, B$ with

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$

We multiply by the denominator which yields

$$\begin{aligned}
1 &= A(x+1) + B(x-1) \\
&= Ax + A + Bx - B \\
&= (A+B)x + (A-B)
\end{aligned}$$

so $A+B=0$ and $A-B=1$. We can solve this using linear algebra techniques, or we can substitute values for $x$. Let $x=1$, then $1 = A(x+1)+B(x-1) \implies 1 = A(2)+B(0) \implies A = \frac{1}{2}$. Similarly, setting $x=-1$ yields $B = -\frac{1}{2}$. Note that the above method assumes an unique solution exists.

**Partial Fraction Decomposition.** Consider a ratio of polynomials $f(x)/g(x)$. If $\deg f \geq \deg g$, we may use long division to write the ratio as $h(x) + \frac{f_1(x)}{g(x)}$, with $h$ a polynominal and $\deg f_1 < \deg g$.
Thus we may assume $\deg f < \deg g$. Suppose $g$ factors over $\mathbb{R}$ as follows,

$$g(x) = p_1(x)^{n_1}\dots p_k(x)^{n_k}$$

where each $p_i$ is co-prime and irreducible and $n_i \in \mathbb{Z}^+$. By the Fundamental Theorem of Algebra, each $p_i$ will either be degree 1 or 2. We may then write

$$\frac{f(x)}{g(x)} = \sum_{i=1}^{k} T_i(x)$$

where $T_i$ is as follows:

1. If $\deg p_i = 1$, then $T_i(x) = \frac{A_1}{p_i(x)} + \cdots + \frac{A_{n_i}}{P_i(x)^{n_i}}$ for $A_j \in \mathbb{R}$.
2. If $\deg p_i = 2$, then $T_i(x) = \frac{A_1 x + B_1}{p_i(x)} + \cdots + \frac{A_{n_i} x + B_{n_i}}{P_i(x)^{n_i}}$ for $A_j, B_j \in \mathbb{R}$.

**Example.** Find the PFD of

$$\frac{2x-1}{x^2+x-2} = \frac{2x-1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

We multiply by the denominator and get

$$2x+1 = A(x+2) + B(x-1)$$

Let $x = 1$, this yields $3 = 3A \implies A = 1$.
Let $x = -2$, this yields $-3 = -3B \implies B = 1$.
Thus

$$\frac{2x-1}{x^2+x-2} = \frac{1}{x-1} + \frac{1}{x+2}$$

and

$$\int \frac{2x-1}{x^2+x-2}\,dx = \int \frac{1}{x-1} + \frac{1}{x+2}\,dx = \ln|x-1| + \ln|x+2| + c$$

**Example.** Find the PFD of

$$\frac{4}{x^3+x^2-x-1} = \frac{4}{(x-1)(x+1)^2} == \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

Note we get the above by factoring by grouping.
We multiply by the denominator and get

$$4 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

Let $x = -1$, this yields $4 = 4A \implies A = 1$.
Let $x = 1$, this yields $4 = -2C \implies C = -2$.
Let $x = 0$, this yields $4 = 1 - B + 2 \implies B = -1$.
Thus

$$\frac{4}{x^3+x^2-x-1} = \frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{(x+1)^2}$$

and

$$\int \frac{4}{x^3+x^2-x-1}\,dx = \int \frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{(x+1)^2}\,dx = \ln|x-1| + \ln|x+1| + \frac{2}{x+1}$$

**Example.** Find the PFD of

$$\frac{6x-3}{x^3-1} = \frac{6x-3}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

We multiply by the denominator and get

$$6x-3 = A(x^2+x+1) + (Bx+C)(x-1)$$

Let $x = 1$, this yields $3 = 3A \implies A = 1$.
Let $x = 0$, this yields $-3 = (1)(1) - C \implies C = 4$.
Let $x = 2$, this yields $9 = 7 + (2B+4)(1) \implies B = -1$.
Thus

$$\frac{6x-3}{x^3-1} = \frac{1}{x-1} + \frac{4-x}{x^2+x+1}$$

How do we integrate $-\int \frac{x-4}{x^2+x+1}\,dx$?

The ideal substitution would be $u = x^2 + x + 1 \implies du = 2x + 1\ dx$. Thus we will try to get this to appear.

$$\begin{aligned}-\int \frac{x-4}{x^2+x+1}\,dx &= -\frac{1}{2}\int \frac{2x+1-9}{x^2+x+1}\,dx \\ &= -\frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx + \frac{9}{2}\int \frac{1}{x^2+x+1}\,dx\end{aligned}$$

Let's integrate the first term for now:

$$\begin{aligned}-\frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx &= -\frac{1}{2}\int \frac{1}{u}\,du \\ &= -\frac{1}{2}\ln|u| \\ &= -\frac{1}{2}\ln\bigl(x^2+x+1\bigr)\end{aligned}$$

Now for the second term, we complete the square: Note that

$$x^2+x+1 = \left(x^2+x+\frac{1}{4}\right)+\frac{3}{4} = \left(x+\frac{1}{2}\right)^2+\frac{3}{4}$$

$$\begin{aligned}\frac{9}{2}\int \frac{1}{x^2+x+1}\,dx &= \frac{9}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx \\ &= \frac{9}{2}\int \frac{1}{\frac{3}{4}\left(\frac{4}{3}\left(x+\frac{1}{2}\right)^2+1\right)}\,dx \\ &= 6\int \frac{1}{\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1}\,dx\end{aligned}$$

Let $u = \frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right) \implies du = \frac{2}{\sqrt{3}}dx \implies dx = \frac{\sqrt{3}}{2}$. Thus,

$$\begin{aligned}6\int \frac{1}{\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1}\,dx &= 6\cdot\frac{\sqrt{3}}{2}\int \frac{1}{u^2+1}\,du \\ &= 3\sqrt{3}\arctan u + c \\ &= 3\sqrt{3}\arctan\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)+c\end{aligned}$$

Thus,

$$-\int \frac{x-4}{x^2+x+1}\,dx = -\frac{1}{2}\ln|x^2+x+1| + 3\sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+c$$

and so

$$\int \frac{6x=3}{x^3-1}\,dx = \ln|x-1| - \frac{1}{2}\ln|x^2+x+1| + 3\sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+c$$

# 2 Differential Equations

**Definition 2.1.** A **differential equation** is an equation that relates a function to its derivative.

**Example.** Solve $y' = ky$ for some fixed $k \in \mathbb{R}$.

Note that $y(x) = 0$ is a solution, and for non-zero $y$ we can divide it out. This yields

$$
\begin{aligned}
\frac{1}{y}y' &= k \\
\implies \int \frac{1}{y}y' \, dx &= \int k \, dx \\
\implies \int \frac{1}{u} \, du &= kx + c_1 && \text{(with } u = y \implies du = y' \, dx\text{)} \\
\implies \ln|u| + c_2 &= kx + c_1 \\
\implies \ln|y| + c_2 &= kx + c_1 \\
\implies \ln|y| &= kx + c_3 && \text{(where } c_3 = c_1 - c_2\text{)} \\
\implies |y| &= e^{kx+c_3} \\
&= e^{kx} \cdot e^{c_3} \\
\implies y &= \pm e^{kx} \cdot e^{c_3} \\
&= ce^{kx} && \text{(where } c = \pm e^{c_3}\text{, and since 0 is a solution, we have } c \in \mathbb{R}\text{)}
\end{aligned}
$$

**Remark.** This works because we could rearrange the differential equation to the form

$$F(y)y' = G(x)$$

Such an equation is called **separable**. Integration wrt $x$ yields

$$
\begin{aligned}
\int F(y)y' \, dx &= \int G(x) \, dx \\
\implies \int F(y) \, dy &= \int G(x) \, dx
\end{aligned}
$$

**Example.** Solve $y' = \frac{-x}{y}$. We have

$$
\begin{aligned}
yy' &= -x \\
\implies \int yy' \, dx &= \int -x \, dx \\
\implies \int y \, dy &= \int -x \, dx \\
\implies \frac{1}{2}y^2 &= -\frac{1}{2}x^2 + c \\
\implies x^2 + y^2 &= 2c
\end{aligned}
$$

This is a circle with radius $\sqrt{2c}$. So, the slope of the tangent lines on a circle are $-\frac{x}{y}$

# 3 Series and Sequences of Functions

**Integral Test.** Let $f$ be a continuous, non-negative, and decreasing function on $[k, \infty)$ and let $f(n) = a_n$, then

$$\int_k^\infty f(x) \, dx \text{ is convergent } \Leftrightarrow \sum_{n=k}^\infty a_n \text{ converges}$$

## 3.1 Power Series

**Definition 3.1.** Given a sequence $(a_n)_{n=0}^{\infty}$, the series $\sum_{n=0}^{\infty} = a_n x^n$ is a **power series** centered at 0 with coefficients $a_n$ and $x$ is a variable. This series may converge for some values of $x$ and diverge for others. Since

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

the series converges to $a_0$ when $x = 0$.

**Example.**

$$\sum_{n=0}^{\infty} n^n x^n$$

so

$$a_n = \begin{cases} 0 & n = 0 \\ n^n & n \geq 1 \end{cases}$$

Let's fix $x$ and use the root test. Have $\lim_{n\to\infty} |n^n x^n|^{\frac{1}{n}} = \lim_{n\to\infty} n|x|$. We need this $< 1$ to converge, which only happens when $x = 0$. Thus the series converges only at $x = 0$.

**Example.**

$$\sum_{n=0}^{\infty} x^n$$

so $a_n = 1$ is a power series. This is the geometric series! Proved in 1052 this converges for $|x| < 1$ and diverges for $|x| > 1$. Know that it converges to $\frac{1}{1-x}$ when it converges.

**Example.**

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

so $a_n = \frac{1}{n!}$. Let's fix $x$ and use the ratio test.

$$\lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)!} x^{n+1}}{\frac{1}{n!} x^n} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} |x| = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

Thus this power series converges absolutely for all $x$.

**Theorem 3.2.** For the power series $\sum_{n=0}^{\infty} a_n x^n$, let

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

Then the power series converges absolutely for $|x| < R$ and diverges for $|x| > R$. $R$ is called the **radius of convergence**.

**Proof.** We use the ratio test on $\sum_{n=0}^{\infty} a_n x^n$. Have

$$\lim_{n\to\infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |x|$$

Case 1: $0 < R < \infty$

Then by Theorem 9.6 implies $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{\left| \frac{a_{n+1}}{a_n} \right|} = \frac{1}{R}$ and so the limit above is $\frac{|x|}{R}$.

By the ratio test, the series converges absolutely if $\frac{|x|}{R} < 1 \Leftrightarrow |x| < R$, and diverges if $|x| > R$.

Case 2: $R = \infty$

Theorem 9.10 implies $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{\left| \frac{a_n}{a_{n+1}} \right|} = 0$ and so the limit above is $0 < 1 \forall x$, so the series converges absolutely for all $x$.

Case 3: $R = 0$. Theorem 9.10 implies $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{\left|\frac{a_n}{a_{n+1}}\right|} = \infty$. Thus the limit above is $\infty$ for all $x$ and so the series converges only for $x = 0$. $\square$

**Example.**

$$\sum_{n=1}^{\infty} \frac{1}{n} x^n$$

so $a_n = \begin{cases} 0 & n = 0 \\ \frac{1}{n} & n = 1 \end{cases}$

We have $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{n}}{\frac{1}{n+1}}\right| = \lim_{n\to\infty} \frac{n+1}{n} = 1$.

This power series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$.

What about for $x = \pm 1$? When $x = 1$, the series is $\sum_{n=1}^{\infty} \frac{1}{n}$ which is the harmonic series and diverges.

When $x = -1$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges conditionally. Thus the series converges for $x \in [-1, 1)$.

**Remark.** In general, the endpoints must be checked separately.

**Definition 3.3.** The interval on which a power series converges is called the **interval of convergence**.

**Example.**

$$\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$$

We have $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{n^2}}{\frac{1}{(n+1)^2}}\right| = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = 1$. Thus it converges for $|x| < 1$. At $x = 1$, the series is $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and at $x = -1$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ which both converge.

The interval of convergence is $[-1, 1]$.

**Example.**

$$\sum_{n=0}^{\infty} 3^n x^{2n}$$

so $a_{2n} = 3^n$ and $a_{n+1} = 0$ (the series is $3^0 + 3x^2 + 3^2x^4 + \dots$)

We have $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ which doesn't exist as every second term is 0. Let's fix $x$ and use the ratio test, $\lim_{n\to\infty} \left|\frac{3^{n+1}x^{2(n+1)}}{3^n 2^{2n}}\right| = \lim_{n\to\infty} 3x^2 = 3x^2$. So, we must have $3x^2 < 1 \implies |x| < \frac{1}{\sqrt{3}}$ for it to converge. At $x = \frac{1}{\sqrt{3}}$, series is $\lim_{n\to\infty} 3^n \left(\frac{1}{\sqrt{3}}\right)^{2n} = \sum_{n=0}^{\infty} 1$ which diverges, at $x = -\frac{1}{\sqrt{3}}$, series is $\sum_{n=0}^{\infty} 3^n \left(-\frac{1}{\sqrt{3}}\right)^{2n} = \sum_{n=0}^{\infty} 1$ which diverges.

Thus the radius of convergence is $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

An alternative method to solve is let $y = x^2$, so series is $\sum_{n=0}^{\infty} 3^n y^n$. Then $R = \lim_{n\to\infty} \frac{3^n}{3^{n+1}} = \frac{1}{3}$. Then series converges for $|y| < \frac{1}{\sqrt{3}} \implies \left|x^2\right| < \frac{1}{3} \implies |x| < \frac{1}{\sqrt{3}}$ as before.

### 3.1.1 Series not Centered at 0

**Definition 3.4.** The power series $\sum_{n=0}^{\infty} a_n(x - x_0)$ centered at $x_0$ will still have a radius of convergence of $R$ and will converge for $|x - x_0| < R$. That is, for $x_0 - R < x < x_0 + R$. It for diverge for $|x - x_0| > R$. Convergent of endpoints needs to be checked separately.

**Example.**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n$$

so $a_n = \begin{cases} 0 & n = 1 \\ \frac{(-1)^{n+1}}{n} & n \geq 1 \end{cases}$ and $x_0 = 1$.

Have

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}}{n}}{\frac{(-1)^{n+2}}{n+2}}\right| &= \lim_{n\to\infty}\left|\frac{n+1}{n}\right| \\
&= 1 \\
&= R
\end{aligned}$$

and so this converges for $|x-1|<1$ or $0<x<2$.
At $x=0$, series is

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(-1)^n &= \sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n} \\
&= -\sum_{n=1}^{\infty}\frac{-1}{n}
\end{aligned}$$

which diverges.
At $x=2$, series is

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(1)^n = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$$

which converges conditionally.
Thus the interval of convergence is $(0,2]$.

**Remark.** A power series is a function of $x$ with domain its interval of convergence. Want to know is it continuous, differentiable, integrable.
The partial sums of a power series are polynomials, which are continuous. If the series converges, its sum is the limit of the sequence of partial sums.
Is the limit of a convergent sequence of continuous functions necessarily continuous?
The answer is no.

**Example.** Let $f_n(x)=x^n$ for $x\in[0,1]$. These are the functions $x, x^2, x^3, \ldots$. Each of these functions is continuous on $[0,1]$. For $x<1$, have $\lim_{n\to\infty} x^n = 0$, but at $x=1$, have $\lim_{n\to\infty} 1^n = 1$. Thus the sequence of continuous functions $f_n(x)$ converges to the discontinuous function $f(x)=\begin{cases}0 & x<1\\ 1 & x=1\end{cases}$
We are essentially fixing a point $x$ and looking at the sequence $f_n(x)$ separately for each value of $x$. This is called point-wise convergence.

**Definition 3.5.** The sequence of functions $f_n$ on $S\subseteq\mathbb{R}$ converges **pointwise** to a function $f$ on $S$ if for all $x\in S$, $\lim_{n\to\infty} f_n(x)=f(x)$. Formally,

$$\forall x\in S, \forall\varepsilon>0, \exists N, n>N \implies |f_n(x)-f(x)|<\varepsilon$$

Here, $N$ may depend on $\varepsilon$ and/or $x$. That is, can choose $N$ differently for different $x$.

**Example.** In the previous example, given a fixed $\varepsilon=0.1$, increasingly large $N$ are needed for $x$ near 1.

At $x = \frac{1}{2}$, we have

$$\begin{aligned} \left(\frac{1}{2}\right)^n &< 0.1 \\ \implies 2^n &> 10 \\ \implies n &> 4 \end{aligned}$$

so take $N = 4$ for this $x$ and $\varepsilon$.
Now with the same $\varepsilon$ take $x = \frac{9}{10}$. Have

$$\begin{aligned} \left(\frac{9}{10}\right)^n &< 0.1 \\ \implies \left(\frac{10}{9}\right)^n &> 10 \\ \implies n \ln \frac{10}{9} &> \ln 10 \\ \implies n &> \frac{\ln 10}{\ln \frac{10}{9}} \approx 21.8 \end{aligned}$$

so take $N = 22$ for this $x$ and $\varepsilon$.
Now with the same $\varepsilon$ take $x = \frac{99}{100}$. Have

$$\begin{aligned} \left(\frac{99}{100}\right)^n &< 0.1 \\ \implies n &> \frac{\ln 10}{\ln \frac{100}{99}} \approx 229.1 \end{aligned}$$

so take $N = 230$ for this $x$ and $\varepsilon$.
At $x = 1$, sequence is $1, 1, 1, 1, 1, \ldots$. It seems that $N$ will not change wrt $x$. We can strengthen the definition so that one $N$ must work for all $x$.

## 3.2 Uniform Convergence

**Definition 3.6.** The sequence of functions $f_n$ on $S \subseteq \mathbb{R}$ **converges uniformly** to a function $f$ on $S$ if

$$\forall \varepsilon > 0, \exists N, \forall x \in S, n > N \implies |f_n(x) - f(x)| < \varepsilon$$

**Remark.** If $f_n$ converges to $f$ uniformly, it also does pointwise as well.

**Example.** Let's return to our example.
We have $f(x) = \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$
Does this converge uniformly?
Suppose so for contradiction. Take $\varepsilon = \frac{1}{2}$. The definition implies we have

$$\exists N, \forall x \in [0, 1], n > N \implies |x^n - f(x)| < \frac{1}{2}$$

Take $x \in [0, 1)$ so we can simplify $f(x)$ as it is always 0. Take $n = N + 1$. We then have $x^{N+1} < \frac{1}{2}$.
Let $x_1 = \frac{1}{2^{\frac{1}{N+1}}}$, and thus have $0 < x_1 < 1$, but $x_1^{N+1} = \frac{1}{2}$, so we have $\frac{1}{2} < \frac{1}{2}$ from the $\varepsilon - N$ definition, which is a contradiction. Thus these functions do not converge uniformly.

**Corollary 3.7.** Any uniformly convergent sequence is pointwise convergent. (trivial)

**Example.** Let $f(x) = \frac{1}{n}\sin(nx)$ on $\mathbb{R}$.
Have $f_1(x) = \sin x$, $f_2(x) = \frac{1}{2}\sin 2x$, and so on. The period and the amplitude are decreasing.
Looks like $f(x)$ approaches the constant zero function. We can show this is convergent uniformly on $\mathbb{R}$. Have

$$\begin{aligned} |f_n(x) - 0| &= \frac{1}{n}|\sin nx| \\ &< \frac{1}{n}(1) \\ &< \varepsilon \end{aligned}$$

so $n > \frac{1}{\varepsilon} = N$.
Let $\varepsilon > 0$ be given. Let $N = \frac{1}{\varepsilon}$. Then for $x \in \mathbb{R}$ and $n > N$ implies

$$|f_n(x) - 0| < \varepsilon$$

as required.

**Theorem 24.3.** The uniform limit of continuous functions is continuous. In other words, let $f_n \to f$ be uniformly convergent on $S \subseteq \mathbb{R}$ where $f$ is a function on $S$. If each $f_n$ is continuous at $x_0 \in S$, then $f$ is continuous at $x_0$.

**Proof.** We want $|f(x) - f(x_0)|$ small when $x$ is near $x_0$. Since $f_n \to f$ uniformly, can make $|f_n(x) - f(x)|$ and $|f_n(x_0) - f(x_0)|$ small for large enough $n$. Since each $f_n$ is continuous, $|f_n(x) - f_n(x_0)|$ is small provided $x$ close to $x_0$.
Have

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \end{aligned}$$

Let $\varepsilon > 0$ be given. Since $f_n$ converges uniformly to $f$, $\exists N$ such that $\forall x \in S$, including $x_0$, $n > N \implies |f_n(x) - f(x)| < \frac{\varepsilon}{3}$. In particular,

$$|f_{N+1}(x) - f(x)| < \frac{\varepsilon}{3} \tag{1}$$

Since $f_{N+1}$ is continuous at $x_0$, $\exists \delta > 0$ such that $x \in S$ and

$$|x - x_0| < \delta \implies |f_{N+1}(x) - f_{N+1}(x_0)| < \frac{\varepsilon}{3} \tag{2}$$

Thus $x \in S$ and $|x - x_0| < \delta \implies$

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f_{N+1}(x)| + |f_{N+1}(x) - f_{N+1}(x_0)| + |f_{N+1}(x_0) - f(x_0)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} && \text{(From 1 and 2 above)} \\ &= \varepsilon \end{aligned}$$

□

**Corollary 3.8.** If $f$ is discontinuous, and $f_n$ is continuous, then $f_n$ does not uniformly converge.

**Example.** Let $f_n = (1 - |x|)^n$ on $(-1, 1)$.
If $x = 0$, then $(1 - |x|) = 1$ and so $f_n = 1^n = 1$ as $n \to \infty$.
If $x \neq 0$, then $(1 - |x|) < 1$ and so $f_n = 0$ as $n \to \infty$.
Thus, $f_n \to f$ where $f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$
Since $f$ is not continuous, it does not uniformly converge.

**Example.** Consider $f_n(x) = x^n$ on $[0, 1)$.
The pointwise limit is $f(x) = 0$ which is continuous.
However, this does not imply that $f_n$ converges uniformly.
This is the same example as before.

**Example.** Let $0 < b < 1$ be fixed and consider $f_n = x^n$ on $[0, b]$. Then $f_n \to 0$ uniformly.
Let $\varepsilon > 0$ be given. Find $N$ first.
Have

$$\begin{aligned} |x^n - 0| &= x^n \\ &< b^n && \text{(We want } b^n < \varepsilon) \\ &< \varepsilon \end{aligned}$$

Then

$$\begin{aligned} b^n &< \varepsilon \\ n \ln b &< \ln \varepsilon \\ n &> \frac{\ln \varepsilon}{\ln b} && (\ln b < 0 \text{ since } b < 1 < e) \end{aligned}$$

Let $\varepsilon > 0$ be given. Take $N = \frac{\ln \varepsilon}{\ln b}$. Then $\forall x \in [0, b]$, $n > N \implies$

$$\begin{aligned} |x^n - 0| &= x^n \\ &\le b^n \\ &< b^N && (b < 1\text{, so lower powers are higher}) \\ &= b^{\frac{\ln \varepsilon}{\ln b}} \\ &= e^{\ln \varepsilon} \\ &= \varepsilon \end{aligned}$$

Thus, let $f_n(x) = x^n$, so $f(x) = \begin{cases} 0 & x \neq 1 \\ 1 & x = 1 \end{cases}$
On $[0, 1]$, $f_n \to f$ pointwise only.
On $[0, 1)$, $f_n \to f$ pointwise only.
On $[0, b]$, where $b < 1$, $f_n \to f$ pointwise uniformly.

**Theorem 25.2.** Let $(f_n)$ be a sequence of continuous functions on $[a, b]$ that converges uniformly to $f$ on $[a, b]$. Then $\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx$

**Proof.** Theorem 24.3 implies that $f$ is continuous. Thus $f_n \to f$ is continuous and integrable on $[a, b]$ for all $n$.
Let $\varepsilon > 0$ be given. Since $f_n \to f$ uniformly on $[a, b]$, then $\exists N, \forall x \in [a, b], n > N \implies |f_n(x) - f(x)| < \frac{\varepsilon}{b-a}$.

Thus, have

$$\begin{aligned}
\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| &= \left|\int_a^b f_n(x) - f(x)\,dx\right| \\
&\le \int_a^b |f_n(x) - f(x)|\,dx \\
&< \int_a^b \frac{\varepsilon}{b-a}\,dx \\
&= \frac{\varepsilon}{b-a}\int_a^b 1\,dx \\
&= \frac{\varepsilon}{b-a} b - a \\
&= \varepsilon
\end{aligned}$$

□

**Example.** Consider $f_n(x) = x^n$ on $[0, b]$ for $b < 1$. By 25.2,

$$\begin{aligned}
\lim_{n\to\infty}\int_0^b f_n(x)\,dx &= \int_0^b f(x)\,dx \\
&= \int_0^b 0\,dx \\
&= 0
\end{aligned}$$

Let's check.

$$\begin{aligned}
\lim_{n\to\infty}\int_0^b f_n(x)\,dx &= \lim_{n\to\infty}\frac{1}{n+1}x^{n+1}\Big|_0^b \\
&= \lim_{n\to\infty}\frac{b^{n+1}}{n+1} \\
&= 0 && (0 < b < 1)
\end{aligned}$$

## 3.3 Cauchy Convergence

**Definition 3.9.** Let $(f_n)$ be a sequence of functions $S \subseteq \mathbb{R}$. The sequence is **uniformly Cauchy** on $S$ if

$$\forall \varepsilon > 0, \exists N, \forall x \in S, m, n > N \implies |f_n(x) - f_m(x)| < \varepsilon$$

**Theorem 3.10.** Suppose $f_n \to f$ uniformly on $S$. Then $f_n$ is uniformly Cauchy on $S$.

**Proof.** Let $\varepsilon > 0$ be given. Then $\exists N, x \in S, n > N \implies |f_n(x) - f(x)| < \frac{\varepsilon}{2}$.
Then $m, n > N$ implies

$$\begin{aligned}
|f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| \\
&\le |f_n(x) - f(x)| + |f(x) - f_m(x)| \\
&\le |f_n(x) - f(x)| + |f_m(x) - f(x)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

□

**Theorem 25.4.** Let $(f_n)$ be a uniformly convergent sequence of functions on $S \subseteq \mathbb{R}$, then $\exists f$ on $S$ such that $f_n \to f$ uniformly.

**Proof.** We must first find $f$. Since $(f_n)$ is uniformly Cauchy, given $\varepsilon > 0$, have $\exists N, \forall x \in S, m, n > N \implies |f_n(x) - f_m(x)| < \varepsilon$.
Fix $x_0 \in S$. Then the above for $x = x_0$, the sequence $(f_n(x_0))$ is a Cauchy sequence of numbers that converge. Thus $\exists \lim_{n\to\infty} f_n(x_0)$. We define $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in S$. Thus, $f_n \to f$ converges pointwise. We now show that $f_n \to f$ uniformly on $S$. Let $\varepsilon > 0$ be given. Since $(f_n)$ is uniformly Cauchy, $\exists N, \forall x \in S, m, n > N \implies |f_n(x) - f_m(x)| < \frac{\varepsilon}{2}$. Let $n > N$ and $x \in S$, the above implies

$$f_n(x) - \frac{\varepsilon}{2} < f_m(x) < f_n(x) + \frac{\varepsilon}{2}$$

for all $m > N$ and

$$f_n(x) - \frac{\varepsilon}{2} < f(x) < f_n(x) + \frac{\varepsilon}{2}$$

since $\lim_{m\to\infty} f_m(x) = f$. Thus, have $|f(x) - f_n(x)| \le \frac{\varepsilon}{2} < \varepsilon$ for all $x \in S$ and $n > N$. Thus, it converges uniformly. □

## 3.4 Series of Functions

**Definition 3.11.** We say that the series of functions $\sum_{k=0}^{\infty} g_k(x)$ converges to a function $g$ if and only if $\lim_{n\to\infty} \sum_{k=0}^{n} g_k(x) = g$.
If the sequence of partial sums converges uniformly on $S$, then we say the series converges uniformly on $S$. If the sequence of partial sums diverges to $\pm\infty$, then we say the series diverges to $\pm\infty$. Otherwise the series has no meaning.

**Remark.** A power series $\sum_{k=0}^{\infty} a_k x^k$ is a series of functions with $g_k(x) = a_k x^k$.
$\sum_{k=0}^{\infty} \frac{x^k}{1+x^k}$ is a series of functions that is not a power series as written.

**Theorem 25.5.** Let $\sum_{k=0}^{\infty} g_k(x)$ be a series on $S \subseteq \mathbb{R}$. If then $g_k(x)$ is continuous on $S$ and the series converges uniformly on $S$, then $\sum_{k=0}^{\infty} g_k(x)$ is continuous on $S$.

**Proof.** The partial sum $\sum_{k=0}^{n} g_k(x)$ is a finite sum of continuous functions and are continuous. Thus the sequence of partial sums is a uniformly convergent sequence of continuous functions. Then 24.3 implies that the limit $\sum_{k=0}^{\infty} g_k(x)$ is continuous. □

**Corollary 3.12.** Let's write the Cauchy criterion for the sequence of partial sums of the series $\sum_{k=0}^{\infty} g_k(x)$ is uniformly convergent on $S$ if and only if $\forall \varepsilon > 0, \exists N, \forall x \in S, n \ge m > N$ (WLOG) $\implies |\sum_{k=m}^{n} g_k(x)| < \varepsilon$.

**Weierstrass M-test for uniform convergence.** Let $(M_k)$ be a sequence of non-negative numbers with $\sum M_k < \infty$. If $|g_k(x)| \le M_k \forall x \in S$, then $\sum g_k(x)$ converges uniformly on $S$.

**Proof.** Let $\varepsilon > 0$ be given. Since $\sum M_k$ converges, the sequence of partial sums $\sum_{k=0}^{n} M_k$ is Cauchy. Then, $\exists N$ such that $n \ge m > N \implies \sum_{k=m}^{n} M_k < \varepsilon$. Thus, if $n \ge m > N$ and $x \in S$, have

$$\left|\sum_{k=m}^{n} g_k(x)\right| \le \sum_{k=m}^{n} |g_k(x)| \le \sum_{k=m}^{n} M_k < \varepsilon$$

Thus $\sum g_k$ is Cauchy and thus uniformly convergent on $S$. □

**Lemma 3.13.** If $\sum g_k(x)$ converges uniformly on $S$, then $\lim_{n\to\infty} \sup_{x\in S}\{|g_n(x)| \mid x \in S\} = 0$

**Proof.** Let $\varepsilon > 0$ be given. Since $\sum g_k(x)$ is Cauchy, $\exists N$ such that $\forall x \in S, n \ge m \implies |\sum_{k=m}^{n} g_k(x)| < \frac{\varepsilon}{2}$. Let $m = n$, have that $\forall x \in S$, $n > N \implies |g_n(x)| < \frac{\varepsilon}{2}$. Then $n > N \implies \sup_{x\in S}\{|g_n(x)| \mid x \in S\} \le \frac{\varepsilon}{2}$. Since $\forall \varepsilon > 0, \exists N$ such that $n > N \implies 0 \le \sup_{x\in S}\{|g_n(x)| \mid x \in s\} \le \frac{\varepsilon}{2} < \varepsilon$, and so $\lim_{n\to\infty}(\sup_{x\in S}\{|g_n(x)| \mid x \in S\}$. □

**Example.** Consider $\sum_{n=0}^{\infty} 3^{-n} x^n$. Have $R = \lim_{n\to\infty} \left|\frac{3^{-n}}{3^{-(n+1)}}\right| = 3$. At $x = \pm 3$, the series diverges by $n$-th term test. Thus, the interval of pointwise convergence is $(-3, 3)$.
Let $0 < b < 3$. For $x \in [-b, b]$, we have $|3^{-n} x^n| \le 3^{-n} b^n = \left(\frac{b}{3}\right)^n$. Note that $\sum \left(\frac{b}{3}\right)$ converges since it is a geometric

series with $\left|\frac{b}{3}\right| < \frac{3}{3} < 1$. Thus, the Weierstrass M-test implies $\sum_{n=0}^{\infty} 3^{-n}x^n$ converges uniformly on $[-b, b]$. By Theorem 25.5, since $3^{-n}x^n$ is continuous on all $x$ for each $n$, the sum is continuous on $[-b, b]$. Since this holds for all $b < 3$, given any $-3 < x_0 < 3$ we can find $b$ with $x_0 < b < 3$ and so $x_0 \in [-b, b]$. Thus, the sum is continuous on $(-3, 3)$.
However, we note that $\sup_{x\in(-3,3)}\{|3^{-n}x^n| \mid x \in (-3, 3)\} = 1 \forall n$. Thus $\lim_{n\to\infty}(\sup_{x\in(-3,3)}\{|3^{-n}x^n| \mid x \in (-3, 3)\} = 1 \neq 0$. By previous lemma, the power series does not converge uniformly on $(-3, 3)$. In summary, $\sum_{n=0}^{\infty} 3^{-n}x^n$ converge pointwise $(-3, 3)$ to a continuous function. It converges uniformly on $[-b, b]$ for any $0 < b < 3$, but does not converge uniformly on $(-3, 3)$. In fact, since we know that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$, we can replace $x$ with $\frac{x}{3}$ to get $\sum_{n=0}^{\infty} 3^{-n}x^n = \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n = \frac{1}{1-\frac{x}{3}} = \frac{3}{3-x}$ for $\left|\frac{x}{3}\right| < 1$.

**Theorem 26.1 + Corollary 26.2.** Let $\sum_{n=0}^{n} a_n x^n$ be a power series with $R > 0$ (possibly $\infty$). If $0 < R_1 < R$, then the power series converges uniformly on $[-R_1, R_1]$ and converges to a continuous function on $(-R, R)$.

**Proof.** The series $\sum a_n x^n$ and $\sum |a_n| x^n$ have the same $R$ of convergence (23.1). Since $|R_1| < R$, $R_1$ is in the interval of convergence and so $\sum |a_n| R_1^n < \infty$.
Since $\forall x \in [-R_1, R_1]$, we have $|a_n x^n| \leq |a_n| R_1^n$, the W-M test implies $\sum a_n x^n$ converges uniformly on $[-R_1, R_1]$ By 25.5, the limit function is continuous on this interval. If $x_0 \in (-R, R)$, then $x_0 \in [-R_1, R_1]$ for some $R_1 < R$. Thus, the above implies the limit function is continuous at $x_0$. □

**Lemma 26.3.** If $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R$, then $\sum_{n=1}^{\infty} n a_n x^{n-1}$ and $\sum_{n=\infty}^{\infty} \frac{a_n}{n+1} x^{n+1}$ also have radius of convergence $R$.
Note that the interval of convergence may change.

**Proof.** Note that $\sum n a_n x^{n-1}$ and $\sum n a_n x^n$ have the same $R$ as do $\sum \frac{a_n}{n+1} x^{n+1}$ and $\sum \frac{a_n}{n+1} x^n$.
We have

$$\lim_{n\to\infty} \left|\frac{n a_n}{(n+1)(a_{n+1}}\right| = \left(\lim_{n\to\infty} \frac{n}{n+1}\right)\left(\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|\right)$$
$$` = 1 \cdot R$$

and

$$\lim_{n\to\infty} \left|\frac{\frac{a_n}{n+1}}{\frac{a_{n+1}}{n+2}}\right| = \left(\lim_{n\to\infty} \frac{n+2}{n+1}\right)\left(\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|\right) = 1(R)$$

Note if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}}$ does not exist, the ratio test can be used to complete the proof. □

**Theorem 26.4.** Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Then

$$\int_0^x f(t)\, dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

**Proof.** Fix $|x| < R$. We prove the case where $x > 0$. By 26.1, $\sum_{n=0}^{\infty} a_n t^n$ converges uniformly to $f(t)$ on $[0, x]$ and so the sequence of partial sums $\sum_{k=0}^{n} a_k t^k$ with $t \in [0, x]$ converges uniformly to $f(t)$. Thus

$$\begin{aligned}
\int_0^x f(t)\, dt &= \lim_{n\to\infty} \int_0^x \sum_{k=0}^{n} a_k t^k \\
&= \lim_{n\to\infty} \sum_{n=0}^{n} a_k \int_0^x t_k\, dt \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} a_k \left( \frac{x^{k+1} - 0^{k+1}}{k+1} \right) \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{a_k}{k+1} x^{k+1} \\
&= \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}
\end{aligned} \tag{33.3}$$

□

**Example.** $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < R$. Integrate term by term from $0 \to x$.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} &= \int_0^x \frac{1}{1-t}\, dt \\
&= -\ln|1-t| \Big|_0^x \\
&= -\ln|1-x| - (-\ln|1|) \\
&= -\ln(1-x) && \text{(Since } -1 < x < 1\text{)}
\end{aligned}$$

Thus, $\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x)$ for $|x| < 1$. If we let $x = \frac{1}{2}$, we get $\sum_{n=1}^{\infty} \frac{1}{n2^n} = \ln 2$.

**Remark.** We can integrate term-by-term, but can we also differentiate?
E.g.,

$$\frac{d}{dx} \frac{1}{n} \sin(nx) = \cos(nx)$$

The first function converges but the second one does not.

**Theorem 26.5.** Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Then $f$ is differentiable on $(-R, R)$ with $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$

**Proof.** Consider $g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ which will converge for $|x| < R$ (Lemma 26.5). By 26.4, can integrate $G$ term by term. $\int_0^x g(t)\, dt = \sum_{n=1}^{\infty} a_n x^n = f(x) - a_0$ for $|x| < R$. For $R_1$ with $0 < R_1 < R$, we have

$$\begin{aligned}
\int_{-R_1}^{x} g(t)\, dt &= \int_{-R_1}^{0} g(t)\, dt + \int_0^x g(t)\, dt \\
&= \int_{-R_1}^{0} g(t)\, dt + f(x) - a_0
\end{aligned}$$

These are both constants, so $f(x) = \int_{-R_1}^{x} g(t)\, dt + k$. By 26.1, $g(x) = \sum_{n=1}^{n} n a_n x^{n-1}$ is continuous. $f$ is differentiable on $(-R_1, R_1)$ with $f'(x) = g(x)$. Since $R_1 < R$ is arbitrary, $f'(x) = \sum_{n=1}^{n} n a_n x^{n-1}$ for $|x| < R$. □

**Theorem 26.6 - Abel's Theorem.** Let $f$ be a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with radius of convergence $R < \infty$. If the series converges at $x = \pm R$, then $f$ is continuous there.

**Example.** We saw that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$. Differentiating yields

$$\sum_{n=1}^{\infty} n x^{n-1} = -\frac{1}{(1-x)^2}$$

for the same $R$. Differentiating again,

$$\sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{2}{(1-x)^3} = \sum_{n=0}(n+1)(n+2)x^n$$

We can multiply the first equation by $x$,

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$$

for $|x| < 1$.
Let $x = \frac{1}{2}$. Then $\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2$.
Now we integrate the series term by term

$$\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} = \int_0^x \frac{1}{t}\, dt = -\ln|1-x|$$

Hence, $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$. $R$ is still 1, but now the series converges at $x = -1$. By Abel's Theorem, $-\sum_{n=1}^{\infty} \frac{x^n}{n}$ is continuous at $x = -1$, as is $\ln(1-x)$. Thus they must be equal at $x = -1$.
That is,

$$\ln(1-(-1)) = \ln 2 = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

and so

$$\sum_{n=1} \infty \frac{(-1)^{n+1}}{n} = \ln 2$$

**Example.** Consider $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$. This converges $\forall x$. Thus,

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} \frac{n}{n!} x^{n-1} \\ &= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^{n-1} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} x^n \\ &= f(x) \end{aligned}$$

We have $f'(x) = f(x)$! This can only happen if $f(x) = ce^x$. Letting $x = 0$ yields $\frac{1}{0!} = ce^0 \implies c = 1$. Thus,

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$$

For example, when $x = 1$, $\sum_{n=0}^{\infty} \frac{1}{n!} = e$.

**Example.** Consider the Fibonacci numbers $F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Let's form the power series with coefficents $F_n$, called a generating series for $F_n$.

$$g(x) = \sum_{n=1}^{\infty} F_n x^n$$

We have

$$
\begin{aligned}
g(x) &= F_1x + F_2x^2 + \sum_{n=3}^{\infty}(F_{n-1}+F_{n-2})x^n \\
&= x + x^2 + x\sum_{n=3}^{\infty} F_{n-1}x^{n-1} + x^2 \sum n = 3^{\infty} F_{n-2}x^{n-2} \\
&= x + x\sum_{n=2}^{\infty} F_n x^n + x^2 \sum n = 3^{\infty} F_n x^n \\
&= x + x\sum_{n=1}^{\infty} F_n x^n + x^2 \sum n = 3^{\infty} F_n x^n \\
&= x + xg(x) = x^2 g(x)
\end{aligned}
$$

So

$$g(x) = \frac{x}{1-x-x^2} = \frac{-x}{x^2-x-1}$$

Thus,

$$
\begin{aligned}
\frac{-x}{x^2-x-1} &= \frac{-x}{(x+r_1)(x+r_2)} \\
&= \frac{A}{x+r_1} + \frac{B}{x+r_2} \\
\implies A &= -\frac{r_1}{\sqrt{5}}, B = \frac{r_2}{\sqrt{5}}
\end{aligned}
$$

where $r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$.
Thus,

$$
\begin{aligned}
g(x) &= \frac{1}{\sqrt{5}}\left(\frac{r_2}{x+r_2} - \frac{r_1}{x+r_1}\right) \\
&= \frac{1}{\sqrt{5}}\left(\frac{1}{1+\frac{x}{r_2}} - \frac{1}{1+\frac{x}{r_1}}\right) \\
&= \frac{1}{\sqrt{5}}\left(\frac{1}{1-r_1x} - \frac{1}{1-r_2x}\right) && \text{(Using } r_1r_2 = -1\text{)} \\
&= \frac{1}{\sqrt{5}}\left(\sum_{n=0}^{n}(r_1x)^n - \sum_{n=0}^{n}(r_2x)^n\right) \\
&= \frac{1}{\sqrt{5}}\left(\sum_{n=0}^{n}(r_1^n - r_2^n)x^n\right)
\end{aligned}
$$

Since $g(x) = \sum_{n=1}^{\infty} F_n x^n$, we have

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

## 3.5 Taylor Series

**Remark.** Suppose have power series centered at $c$. $\sum_{n=0}^{\infty} a_n(x-c)^n$ for $|x-c| < R$.

We have

$$\begin{aligned}
f(x) &= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots &&= \sum_{n=0}^{\infty} a_n(x-c)^n \\
f'(x) &= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \dots &&= \sum_{n=1}^{\infty} na_n(x-c)^{n-1} \\
f''(x) &= 2a_2 + 6a_3(x-c) + \dots &&= \sum_{n=2}^{\infty} n(n-1)a_n(x-c)^{n-2} \\
f'''(x) &= 6a_3 + \dots &&= \sum_{n=3}^{\infty} n(n-1)(n-2)a_n(x-c)^{n-3}
\end{aligned}$$

Let $x = c$. Then,

$$\begin{aligned}
f(c) &= a_0 \\
f'(c) &= a_1 \\
f''(c) &= 2a_2 \\
f'''(c) &= 6a_3
\end{aligned}$$

Thus starting with the power series, there is a formula for its coefficients $a_n = \frac{f^{(n)}(c)}{n!}$.
What if we start with $f(x)$, not necessarily a power series and form the power series $\sum a_n(x-c)^n$ with $a_n = \frac{f^{(n}(c)}{n!}$.
Will it converge to $f$?

**Definition 3.14.** Let $f$ be defined on $(a, b)$ with $c \in (a, b)$ and suppose all order of derivatives of $f$ exist at $c$. Then the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

is the **Taylor Series** for $f$ centered at $c$.
A Taylor Series centered at 0 is called a **Maclaurin Series**.
For $n \geq 1$, the remainder is $R_n(x) = f(x) - \sum_{k=0}^{n} -1 \frac{f^{(k)}(c)}{k!}(x-c)^n$.
Thus, $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \Leftrightarrow \lim_{n\to\infty} R_n = 0$.

**Example.** Find the Taylor Series of $\sin x$ centered at 0.

$$\begin{aligned}
f(x) &= \sin x &&\implies f(0) &&= 0 \\
f'(x) &= \cos x &&\implies f'(0) &&= 1 \\
f''(x) &= -\sin x &&\implies f''(0) &&= 0 \\
f'''(x) &= -\cos x &&\implies f'''(0) &&= -1
\end{aligned}$$

and this pattern repeats.
Thus,

$$f^{(k)} = \begin{cases} 0 & \text{k=2n} \\ 1 & \text{k = 4n+1} \\ -1 & \text{k = 4n+3} \end{cases}$$

Thus, the Taylor series centered at 0 is

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k = \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(c)}{(2n+1)!}$$

where $n = 2k+1$.
We then have

$$f^{(2n+1)}(0) = \begin{cases} 1 & \text{n is even} \\ -1 & \text{n is odd} \end{cases} = (-1)^n$$

Thus the series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Let's find the radius of convergence. We have

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}}{(2n+3)!} x^{2n+3}}{\frac{(-1)^n}{(2n+1)!} x^{2n+1}} \right| &= x^2 \lim_{n\to\infty} \left| \frac{(2n+1)!}{(2n+3)!} \right| \\ &= x^2 \lim_{n\to\infty} \left| \frac{1}{(2n+2)(2n+3)} \right| \\ &= 0 \end{aligned}$$

This converges for all $x$.

**Remark.** If we know that a function $f$ is equal to a power series, then we saw that $a_n = \frac{f^{(n)}}{n!}$ and so that power series is the Taylor Series centered at $c$.

**Example.** We saw that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ on $\mathbb{R}$ so the Taylor series for $e^x$ centered at 0 is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and converges.
Find the Taylor series for $e^{-x^2}$ centered at 0 and for $e^x$ centered at 2.
By above, we have $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$. Also, $e^{x-2} = \sum_{n=0}^{\infty} \frac{1}{n!}(x-2)^n = e^x e^{-2}$. So $e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$.

**Taylor's Theorem (31.3).** If $f$ is defined in $(a,b)$ with $a < c < b$ not necessarily finite. Suppose the $n$th derivative $f^{(n)}(x)$ exists on the interval. Then $\forall x \in (a,b) \neq c$, there is some $y$ in between $c$ and $x$ such that $R_n(x) = \frac{f^{(n)}(y)}{n!}(x-c)^n$.

**Lemma 3.15.** Let $b > 0$ be constant. Then

$$\lim_{n\to\infty} \frac{b^n}{n!} = 0$$

**Corollary 3.16.** Let $f$ be defined on $(a,b)$ with $a < c < b$. If all derivatives $f^{(n)}(x)$ exist on $(a,b)$ and are bounded by a single constant $B$, then

$$\lim_{n\to\infty} R_n(x) = 0$$

for all $x \in (a,b)$ where $R_n$ is the remainder for the Taylor Series centered at $c$.

**Example.** Recall that we found the Taylor series for $\sin x$ which converges everywhere. Since all derivatives of $\sin x$ are bounded by $1 \in \mathbb{R}$, by corollary $R_n \to 0$ as $n \to \infty$. Thus, the Taylor series converges to $\sin x$ on $\mathbb{R}$.

**Example.** Find the Taylor series for $\cos x$ centered at 0.
We can use the same technique for sin, but we can just differentiate instead.

$$\cos x = \sum_{n\to\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

**Example.** Show that the Taylor series for $e^x$ converges to $e^x$ (centered at 0).
We have $f(x) = e^x \implies f^{(n)}(x) = e^x \implies f^{(n)}(0) = e^0 = 1$. Thus, the Taylor series is $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$. By usual formula for $R$, we have $R = \infty$. On $(-b,b)$, $\left|f^{(n)}(x)\right| < e^b$ so by corollary, the Taylor series converges to $e^x$ on $(-b,b)$. Since $b$ is arbitrary in $\mathbb{R}$, it converges pointwise to $e^x$ for all $x$.

**Example.** This example is of a Taylor series for a function $f$ centered at 0 that does not converge to $f$ on any interval $(-b, b)$.

Let $f(x) = \begin{cases} e^{\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$

We will show that $f^{(n)}(0) = 0 \forall n \in \mathbb{Z}^+$. This implies that the Taylor series for $f$ centered at 0 is $\sum \frac{f^{(n)}(0)}{n!} x^n = 0$. However, in any interval $(-b, b)$, $\exists x$ for $f(x) \neq 0$. It is clear that $f$ has derivatives of all orders for $x \neq 0$, namely

$$f'(x) = e^{-\frac{1}{x}} \frac{1}{x^2}$$

and

$$f''(x) = e^{-\frac{1}{x}} \left( \frac{1}{x^4} - \frac{2}{x^3} \right)$$

We claim that for each $n$, there is a polynominal $P_n$ of degree $2n$ such that $f^{(n)}(x) = e^{-\frac{1}{x}} P_n(\frac{1}{x})$. For example, $P_1(t)m = t^2$ and $P_2(t) = t^4 - 2t^3$. Suppose for $x > 0$, the $n$-th derivative of $f$ is $f^{(n)}(x) = e^{-\frac{1}{x}} P_n(\frac{1}{x})$ where $P_n(t) = a_0 + a_1 t + \cdots + a_{2n} t^{2n}$ and $a_{2n} \neq 0$. Then, $f^{(n)}(x) = e^{-\frac{1}{x}} \sum_{k=0}^{2} n a_k \cdot \frac{1}{x^k}$. Differentiating, $f^{(n+1)}(x) = e^{-\frac{1}{x}} \cdot -\frac{1}{x^2} \sum_{k=0}^{2} n \frac{a_k}{x^k} + e^{-\frac{1}{x}} \sum_{k=0}^{2} n \frac{-k a_k}{x^{k+1}}$ so $p_{n+1} = t^{-2} \sum_{k=0}^{2n} a_k t_k - \sum_{k=0}^{2n} k a_k t^{k+1}$.
We now show that $f^{(n)}(0)$. Suppose $f^{(n)}(0)$. Want to prove $f^{(n+1)}(0) = 0$. Have

$$\begin{aligned} f^{(n+1)}(0) &= \lim_{x \to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} \\ &= \lim_{x \to 0} \frac{f^{(n)}(x)}{x} \\ &= 0 \end{aligned}$$

We also show that $\lim_{x \to 0} e^{-\frac{1}{x}} q(\frac{1}{x}) = 0$ for all polynomials $q$. Since $f^{(n+1)}(x) = \lim_{x \to 0} e^{-\frac{1}{x}} p(\frac{1}{x})$. This is a polynomial evaluated at $\frac{1}{x}$. This implies $\lim_{x \to 0} \frac{f^{(n)}(0)}{x}(0) = 0$. Since $q(\frac{1}{x})$ is finite sum of form $\frac{b_k}{x^k}$. We show $lim_{x \to 0} \frac{e^{-\frac{1}{x}}}{x^k} = 0$. Let $g = \frac{1}{x}$ As $x \to 0, g \to \infty$. Thus $\lim_{x \to 0^+} = \frac{e^{-\frac{1}{x}}}{x^k} = \lim_{g \to \infty} g^k e^{-g} = \lim_{g \to \infty} \frac{g^k}{e^g}$. Apply L'Hopital's rule $k$ times to get 0.