# MATH 226 Formulae

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# 1 Vectors

## 1.1 Projection

**Definition 1.1.** The vector projection of $\mathbf{u}$ onto $\mathbf{v}$ is

$$\mathbf{u}_{\mathbf{v}} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}$$

and the scalar projection is

$$s = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|}$$

## 1.2 Lines and Planes

**Proposition 1.2.** The equation of a plane containing a point $(x_0, y_0, z_0)$ and perpendicular to a vector $(A, B, C)$ can be written as

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

**Proposition 1.3.** The equation of a line containing a point $(x_0, y_0, z_0)$ and with direction vector $(a, b, c)$ can be written as

$$(x - x_0, y - y_0, z - z_0) = t(a, b, c)$$

**Corollary 1.4.** The above is equivalent to the system of equations

$$\begin{cases} x &= x_0 + at \\ y &= y_0 + at \\ z &= z_0 + at \end{cases}$$

or

$$(x, y, z) = (a_0, b_0, c_0) + t(a, b, c)$$

# 2 Differentiation

## 2.1 Limits

**Definition 2.1.** If $L \in \mathbb{R}$ and $(a, b)$ is a point in $\mathbb{R}^2$, then $f$ has **limit** L at $(a, b)$ written as $\lim_{(x,y)\to(a,b)} f(x, y) = L$ if

$$\forall\varepsilon > 0, \exists\delta > 0, \sqrt{(x - a)^2 + (y - b)^2} < \delta \implies |f(x, y) - L| < \varepsilon$$

**Theorem 2.2.** If the limits along different paths approaching $(a, b)$ are different, then $f$ does not have a limit there.

## 2.2 Partial Derivatives

**Definition 2.3.** The partial derivative $f$ wrt $x_j$ is the function $f_j$ defined by

$$f_j(\mathbf{x}) = \lim_{h\to 0} \frac{1}{h} \left(f(x_1, \ldots, x_j + h, \ldots, x_n) - f(\mathbf{x})\right)$$

**Example.** For 2 variables, the partials are

$$f_1(x, y) = \lim_{h\to 0} \frac{1}{h}(f(x + h, y) - f(x, y))$$

and

$$f_2(x,y) = \lim_{h\to 0} \frac{1}{h}(f(x,y+h) - f(x,y))$$

## 2.3 Differentiability

**Definition 2.4.** A function $f(x,y)$ is **differentiable** if

$$\lim_{(x,y)\to(a,b)} \frac{|f(x,y) - L(x,y)|}{\sqrt{(x-a)^2 + (y-b)^2}} = 0$$

where

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

**Theorem 2.5.** If $f(x,y)$ is $C^1$ on a neighbourhood of $(a,b)$, it is differentiable at $(a,b)$.

**Theorem 2.6.** If $f$ is differentiable at $(a,b)$, then both $f_x, f_y$ exist at $(a,b)$ and $f$ is continuous at $(a,b)$.

**Corollary 2.7.** If $f_x, f_y$ exist at $(a,b)$ or $f$ is continuous at $(a,b)$, then $f(x,b)$ is continuous at $x = a$ and $f(a,y)$ is continuous at $y = b$.

## 2.4 Tangent Planes

**Theorem 2.8.** The normal vector to the tangent plane is

$$-f_x(a,b)\mathbf{i} - f_y(a,b)j + k$$

**Theorem 2.9.** The equation of the tangent plane is

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

**Proposition 2.10.** The linearization of a function at a point is the same equation as for the tangent plane.

## 2.5 Directional Derivatives

**Definition 2.11.** The **directional derivative** of a function in the direction of a unit vector $\mathbf{u}$ is defined as

$$D_{\mathbf{u}}f(a,b) = \lim_{h\to 0} \frac{1}{h}(f(a+hu_1, b+hu_2) - f(a,b))$$

**Theorem 2.12.**

$$D_{\mathbf{u}}f(a,b) = \nabla f(a,b) \cdot \mathbf{u} = |\nabla f(a,b)| \cos\theta$$

**Corollary 2.13.** The direction of steepest ascent is $\mathbf{u} = \frac{\nabla f}{|\nabla f|}$, and similarly, the direction of steepest descent is $\mathbf{u} = -\frac{\nabla f}{|\nabla f|}$

## 2.6 Implicit Functions

**Theorem 2.14.** If $F(x,y) = 0$, then if $y = y(x)$, we can use chain rule to yield

$$0 = \frac{d}{dx}F(x,y(x)) = F_1(x,y(x)) + F_2(x,y(x))\frac{dy}{dx}$$

so that $\frac{dy}{dx} = -\frac{F_1}{F_2}$. Similarly for 3 variables, $\frac{\partial z}{\partial x} = -\frac{F_1}{F_3}$ and $\frac{\partial z}{\partial y} = -\frac{F_2}{F_3}$.

**Theorem 2.15.** If we have the system of equations $u = f(x, y)$ and $v = g(x, y)$, and $x = x(u, v), y = y(u, v)$, we can take the derivative wrt $u$ yielding

$$1 = f_1(x, y) \cdot \frac{\partial x}{\partial u} + f_2(x, y) \cdot \frac{\partial y}{\partial u}$$

and

$$0 = g_1(x, y) \cdot \frac{\partial x}{\partial u} + g_2(x, y) \cdot \frac{\partial y}{\partial u}$$

We can then solve for $\frac{\partial x}{\partial u}$ and $\frac{\partial y}{\partial u}$.

## 2.7 Minimum and Maximum

**Conditions for a local min/max.** We must have at least one of

1. $\nabla f(a, b) = 0$
2. $\nabla f(a, b)$ DNE
3. $(a, b)$ is a boundary point of the domain $D$

**Definition 2.16** (Taylor Polynominals)**.** We have

$$f(\mathbf{x}) \approx f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \frac{1}{2}(\mathbf{x} - \mathbf{a}) \cdot \mathscr{H}_{f(\mathbf{a})}(\mathbf{x} - \mathbf{a})^t$$

where

$$\mathscr{H} = \begin{pmatrix} f_{11} & f_{12} & \dots & f_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ f_{n1} & f_{n2} & \dots & f_{nn} \end{pmatrix}$$

**Example.** For a polynominal in 2 variables,

$$p_2(x, y) = f(a, b) + f_1(a, b)(x - a) + f_2(a, b)(y - b) + \frac{1}{2} f_{11}(a, b)^2 + f_{12}(a, b)(x - a)(y - b) + \frac{1}{2} f_{22}(a, b)(y - b)^2$$

**Classifying Critical Points with the Second Derivative Test.** We consider the behaviour of the **principal minors** of $\mathscr{H}$ evaluated at $\mathbf{a}$. We have

$$D_1 = f_{11}, D_2 = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}, D_3 = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

1. If all $D_i > 0$, then $\mathscr{H}$ is positive definite and so $f$ has a **local minimum** at $\mathbf{a}$
2. If $D_i < 0$ when $i$ is even and $D_i > 0$ when $i$ odd, then $\mathscr{H}$ is negative definite and so $f$ has a **local maximum** at $\mathbf{a}$
3. If $D_n = \det(\mathscr{H}) \neq 0$ (the largest one) but the sign is not all positive or alternating with negative first, then $\mathscr{H}$ is indefinite and so $f$ has a **saddle point** at $\mathbf{a}$
4. If $D_n = 0$, then the test is inconclusive

**Definition 2.17.** A domain $X \subset \mathbb{R}^n$ is **compact** if it is bounded and closed.

**Procedure for finding Global Extrema on Compact Domains.** If $X$ is compact and $f$ cts., then it must attain a global min/max on $X$.
If a point is a global minimum, it has to be a local one as well or on the boundary, thus we can follow a similar process as before.

1. Find all critical/singular points inside $X$ of the function $f$
2. Determine the min/max values of $f$ on the boundary of $X$ and note the points where these min/max values are

attained

3. Compute $f(\mathbf{a})$ at all the points found, and pick the biggest and smallest

Easier to just compute and not use 2nd derivative test.

## 2.8 Lagrange Multipliers

**Remark.** We want to determine the extreme values of $f$ subject to a constraint $g(\mathbf{x}) = c$. We want $\nabla f$ parallel to $\nabla g$.

**Theorem 2.18.** Want to find all points where

- $\nabla f(x, y) = \lambda \nabla g(x, y)$ for some $\lambda \in \mathbb{R}$
- $\nabla g(x, y) = 0$
- $\nabla f(x, y)$ or $\nabla g(x, y)$ DNE

# 3 Integration

**Determining Convergence of Improper Integrals.** Let $D \subset \mathbb{R}^2$ be a region.

- If $f \geq g \geq 0$ on D, then $\iint_D g(x, y) dA = \infty \implies \iint_D f(x, y) = \infty$.
- If $0 \leq f \leq g$ on D, then $\iint_D g(x, y) < \infty \implies \iint_D f(x, y) < \infty$.

**Proposition 3.1.** A double integral over a region $D$ can be interpreted as

$$\iint_D f(x, y) dA = \text{volume under the graph of } z = f(x, y) \text{ above } D$$

**Theorem 3.2.** The average value of $f(x, y)$ on $F$ is

$$\frac{\iint_D f(x, y) dA}{\iint_D 1 dA}$$

since $\iint_D 1 dA$ is the area of $D$.

**Definition 3.3.** The **centroid** of $D$ is the point $(\overline{x}, \overline{y})$ where

$$\overline{x} = \frac{\iint_D x dA}{\iint_D 1 dA}$$

and similarly for $\overline{y}$.

## 3.1 Cylindrical Coordinates

**Definition 3.4.** Cylindrical coordinates are obtained by the change of variables $(x, y) \to (r, \theta)$, where

$$x = r \cos \theta, y = r \sin \theta$$

and

$$dV = dx dy dz = r dr d\theta dz$$

## 3.2 Spherical Coordinates

**Definition 3.5.** Consider a point $P$. Then $P$ can be represented uniquely by

- $R$ is the radius of the sphere
- $0 \le \theta \le 2\pi$ is the projection of the point onto the $xy$-axis and taking the angle from the $x$ axis to $P'$.
- $0 \le \phi \le \pi$ is the angle made by $P$ and the $z$-axis.

## 3.3 Change of Variables

**Theorem 3.6.** The conversion for area is

$$dA = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} x_u & x_v \\ y_u & v_v \end{vmatrix}$$

It is also equivalent to

$$\left( \frac{\partial(u,v)}{\partial(x,y)} \right)^{-1}$$

**Remark.** Remember to flip and take the absolute value!!!