Chapter 1

Preliminaries

1.1 Linear Rotation in $\mathbb{R}^n$

Before introducing rotations, we recall the standard notions of inner product and norm in $\mathbb{R}^n$. For two vectors $v_1 = (a_1, a_2, a_3, \ldots, a_n)$ and $v_2 = (b_1, b_2, b_3, \ldots, b_n)$ in $\mathbb{R}^n$, the inner product (dot product) is defined by

$$v_1 \cdot v_2 = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_n b_n$$

The norm (or length) of a vector $v \in \mathbb{R}^n$ is then defined using the inner product:

$$||v|| = \sqrt{v \cdot v}$$

Definition 1.1.1. Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Since $\mathbb{R}^n$ is finite-dimensional, L always has a $n \times n$ matrix representation with respect to the standard basis. From here on, we will mostly treat L as a matrix rather than an operator. L is a linear rotation if:

1. It preserve lengths:
$$||Lv|| = ||v|| \quad \text{for all } v \in \mathbb{R}^n$$
2. Its determinant of matrix representation is one
$$\det(L) = 1$$

This definition gives a precise way to describe what we mean by a rotation in $\mathbb{R}^n$ in the linear case. The first condition makes sure that lengths stay the same, while the second condition rules out reflections. Together, these two points capture the basic idea of a rotation as a transformation that moves vectors around without changing their size. Also, notice that a linear rotation always leaves the origin fixed.Later, we will broaden this idea to include rotations that are not necessarily centered at the origin and we will explore how to handle such case.

1.2 The special orthogonal group $SO(n)$

Definition 1.2.1. Let F be a field. Then the general linear group $GL_n(F)$ is the group of invertible $n \times n$ matrices with entries in F under matrix multiplication. Equivalently,

$$GL_n(F) := \{A \in M_n(F) \mid \det(A) \neq 0\}$$

Every invertible linear transformation $T : V \to V$ on an n-dimensional vector space over a field F can be represented, with respect to a basis, by an invertible $n \times n$ matrix. The collection of all such invertible matrices form the general linear group, $GL_n(F)$. Thus, $GL_n(F)$ provides the matrix representation of all invertible linear transformation of F^n.

Definition 1.2.2. A square matrix $A \in \mathbb{R}^{n\times n}$ is orthogonal if $A^T A = AA^T = I$. The special orthogonal group $SO(n)$ is defined as the subgroup of the general linear group $GL_n(\mathbb{R})$ consisting of $n \times n$ real matrices that are orthogonal and have determinant equal to 1. Equivalently,

$$SO(n) := \{A \in GL_n(\mathbb{R}) \mid A^T A = AA^T = I_n,\ \det(A) = 1\}$$

Lemma 1.2.3. $SO(n)$ is a subgroup of $GL_n(\mathbb{R})$

Proof. First, $I_n \in SO(n)$ because $I_n^T I_n = I_n I_n^T = I_n$ and $\det(I) = 1$, so the set is nonempty.

Let $A, B \in SO(n)$. Then

$$\det(AB^{-1}) = \det(A)\det(B^{-1}) = 1 \cdot 1 = 1.$$

Moreover,

$$(AB^{-1})^T (AB^{-1}) = (B^T)^{-1}(A^T A)B^{-1} = (B^T B)^{-1} = I_n,$$

and similarly,

$$(AB^{-1})(AB^{-1})^T = A(B^T B)^{-1}A^T = AA^T = I_n.$$

Thus $AB^{-1} \in SO(n)$, and so $SO(n)$ is a subgroup of $GL_n(\mathbb{R})$. □

Proposition 1.2.4. The set of all linear rotation in $\mathbb{R}^n$ is exactly the $SO(n)$ matrices.

Proof. Suppose L is a linear rotation as in 1.1. By condition (1), for every $v \in \mathbb{R}^n$ we have

$$||Lv||^2 = ||v||^2$$

Writing this in terms of the inner product,

$$(Lv) \cdot (Lv) = v \cdot v$$

Using the standard matrix form of the dot product, this becomes

$$(Lv)^T(Lv) = v^T v$$

Expanding the left-hand side,

$$v^T(L^T L)v = v^T v$$

Bringing terms together,

$$v^T(L^T L)v - v^T v = 0$$

or equivalently,

$$v^T(L^T L - I)v = 0 \quad \text{for all } v$$

Since the left hand side equation vanishes for all v, the only possibility is

$$L^T L - I = 0$$

so

$$L^T L = I$$

Together with the condition $\det(L) = 1$, This shows That L is in $SO(n)$. Now for the converse, suppose $L \in SO(n)$ and $v \in \mathbb{R}^n$

$$||Lv|| = \sqrt{Lv \cdot Lv} = \sqrt{v^T L^T L v} = \sqrt{v^T v} = \sqrt{v \cdot v} = ||v||$$

so L preserve lengths. Since $\det(L) = 1$ by assumption, L is a linear rotation. $\square$

1.3 The special unitary group $SU(n)$

Definition 1.3.1. A square matrix $U \in \mathbb{C}^{n\times n}$ is unitary if its inverse is equal to its conjugate transpose. The special unitary group $SU(n)$ is defined as the subgroup of the general linear group $GL_n(\mathbb{C})$ consisting of $n \times n$ complex matrices that are unitary and have determinant equal to 1. Equivalently,

$$SU(n) := \{U \in GL_n(\mathbb{C}) \mid U^* U = I_n,\ \det(U) = 1\}$$

Where U^* denotes the conjugate transpose of U.

Lemma 1.3.2. $SU(n)$ is a subgroup of $GL_n(\mathbb{C})$

Proof. First, note that the identity matrix $I \in SU(2)$ since

$$I^*I = II^* = I, \quad \det(I) = 1$$

Thus, $SU(2)$ is non empty.

Next, let $U, V \in SU(2)$. Then

$$\begin{aligned}
(UV^{-1})^*(UV) &= \overline{(UV^{-1})^T}(UV^{-1}) \\
&= \overline{((V^{-1})^T U^T)}(UV^{-1}) \\
&= \overline{((V^{-1})^T)}\,\overline{(U^T)}UV^{-1} \\
&= (\overline{V^T})^{-1}U^*UV^{-1} \\
&= (VV^*)^{-1} \\
&= I
\end{aligned}$$

Similarly we have $(UV)(UV)^* = I$. Hence UV is unitary. Moreover,

$$\det(UV) = \det(U)\det(V) = 1 \cdot 1 = 1$$

Thus, $UV^{-1} \in SU(2)$ and we concluded that $SU(2)$ is a subgroup of $GL_n(\mathbb{C})$ □

Proposition 1.3.3. Every $U \in SU(2)$ can be written as

$$U = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \quad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1.$$

Proof. Let

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$$

Because U is unitary, we have $U^{-1} = U^*$. We have

$$U^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

and for the conjugate transpose we have

$$U^* = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}$$

Entry wise this yields

$$\overline{a} = d, \quad \overline{c} = -b, \quad \overline{b} = -c, \quad \overline{d} = a$$

Subtituting these into U gives

$$U = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$$

for the determinant condition, we need to have $|a|^2 + |b|^2 = 1$. □

1.4 General rotation in $\mathbb{R}^n$

Definition 1.4.1. Linear rotations, as defined in 1.1, always keep the origin fixed. In geometry, however, rotations are not always centered at the origin they can occur anywhere in $\mathbb{R}^n$. To describe this more general situation, we extend the definition. A general rotation in $\mathbb{R}^n$ is a map of the form

$$R(v) = a + Lv, \quad v \in \mathbb{R}^n$$

where L is a linear rotation and a is some point in $\mathbb{R}^n$

Recall that for a linear operator T between vector spaces, $T : V \to W$, The kernel of T is defined as

$$\ker(T) = \{v \in V \mid T(v) = 0_W\}$$

Where 0_W is the zero vector of vector space W

Lemma 1.4.2. Let $T : V \to W$ be a linear operator between vector spaces, Then $\ker(T)$ is a subspace of V.

Proof. $0_V \in \ker(V)$ so $\ker(T)$ is non empty because

$$\begin{aligned} T(0_V) &= T(0_V) + T(0_V) - T(0_V) \\ &= T(0_V + 0_V) - T(0_V) \\ &= T(0_V) - T(0_V) \\ &= 0_W \end{aligned}$$

Next, let $u, v \in \ker(T)$. Then

$$T(u + v) = T(u) + T(v) = 0_W + 0_W = 0_W,$$

so $u + v \in \ker(T)$. Similarly, for any scalar c,

$$T(cu) = cT(u) = 0_W$$

so $cu \in \ker(T)$. Thus, $\ker(T)$ is a subspace of V $\square$

Lemma 1.4.3. let $A \in \mathbb{R}^{n\times n}$ and $\lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{C}$ be the eigenvalues of A. Then

$$\det(A) = \lambda_1\lambda_2 \cdots \lambda_n$$

Proof. A scalar $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if there exist a nonzero vector v such that

$$Av = \lambda v$$

this condition is equivalent to

$$(A - \lambda I)v = 0$$

which hold for if and only if $A - \lambda I$ is singular. Therefore, λ is an eigenvalues of A if and only if

$$\det(A - \lambda I) = 0$$

The polynomial

$$p_A(\lambda) = \det(A - \lambda I)$$

is called the charateristic polynomial of A. Since $p_A(\lambda)$ is a degree-n polynomial with real coefficient, its root are precisely the eigenvalues of A. By Fundamental Theorem of Algebra, $p_A(\lambda)$ can be completely factored over $\mathbb{C}$ into linear factors.

$$p_A(\lambda) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$$

where the factor $(-1)^n$ arises because the leading term of $\det(A - \lambda I)$ is obtained by taking $-\lambda$ from each diagonal entry, giving $(-\lambda)^n = (-1)^n\lambda^n$

Finally, evaluating at $\lambda = 0$ yields

$$\det(A) = p_A(0) = (-1)^n(-\lambda_1)(-\lambda_2)\cdots(-\lambda_n) = \lambda_1\lambda_2\cdots\lambda_n$$

as required.

□

Theorem 1.4.1. let $L \in SO(3)$. Then the eigenvalues of L are

$$\{1, e^{i\theta}, e^{-i\theta}\}$$

for some $\theta \in [0, 2\pi)$

Proof. First, we show that 1 is always an eigenvalue. Since L is orthogonal and $\det(L) = 1$ and L is 3×3 matrix,

$$\begin{aligned}
\det(L - I) &= \det(L - LL^T) \\
&= \det(L)\det(I - L^T) \\
&= \det\left((I - L^T)^T\right) \\
&= \det(I - L) \\
&= (-1)^3\det(L - I) \\
&= -\det(L - I)
\end{aligned}$$

which forces $\det(L - I) = 0$. Hence 1 is an eigenvalue of L.

Now let the other two eigenvalues be λ_2, λ_3. By lemma 1.4.3 the determinant equals the product of the eigenvalues,

$$\det(L) = \lambda_2\lambda_3 = 1$$

Next, we show that all eigenvalues of L lie on the unit circle. Suppose λ is an eigenvalue so there exist nonzero vector v such that $Lv = \lambda v$. Taking norms,

$$||Lv|| = ||\lambda v|| = |\lambda| ||v||$$

But since L is orthogonal, $||Lv|| = ||v||$. Therefore $|\lambda| = 1$, This show all eigenvalues of $L \in SO(3)$ lies on unit circle in $\mathbb{C}$.

So, we may write

$$\lambda_2 = e^{i\theta_2}, \quad \lambda_3 = e^{i\theta_3}$$

for some real $\theta_2, \theta_3 \in [0, 2\pi)$. The product condition becomes

$$e^{i(\theta_2+\theta_3)} = 1$$

which implies $\theta_3 = -\theta_2$. Hence, the eigenvalues of L are

$$\{1, e^{i\theta}, e^{-i\theta}\}$$

as required.

□

We now consider the operator $L - I$ where $L \in SO(3)$. Its kernel is

$$\ker(L - I) = \{v \in \mathbb{R}^3 \mid (L - I)v = 0\} = \{v \in \mathbb{R}^3 \mid Lv = v\}$$

Thus, $\ker(L - I)$ is the set of all vector in $\mathbb{R}^3$ that is fixed by L

Proposition 1.4.4. let $L \in SO(3)$ with $L \neq I$. Then ker(L - I) is 1 dimensional subspace of $\mathbb{R}^3$

Proof. Since $L - I$ is a linear operator on $\mathbb{R}^3$ by Lemma 1.4.2 its kernel is a subspace of $\mathbb{R}^3$. By Theorem 1.4.1, 1 is an eigenvalue of L. Hence, there exist a nonzero vector $u \in \mathbb{R}^3$ such that $Lu = u$, which is equivalent to $(L - I)u = 0$. Therefore $u \in \ker(L - I)$, and so

$$\dim(\ker(L - I)) \geq 1$$

If $\dim(\ker(L - I)) = 3$, then $\ker(L - I) = \mathbb{R}^3$, which implies $(L - I)v = 0$ for all $v \in \mathbb{R}^3$. Hence $L = I$, contradicting the assumption $L \neq I$. Thus

$$\dim(\ker(L - I)) \neq 3$$

suppose for contradiction that $\dim(\ker(L - I)) = 2$. Then we may choose an orthogonal basis $\{u, v\}$ for $\ker(L - I)$ via Gram-Schmidt process. Extending this to an orthogonal basis $\{u, v, w\}$ of $\mathbb{R}^3$ via Gram-Schmidt process, we have

$$Lu = u, \quad Lv = v$$

Since L is orthogonal,

$$\begin{aligned} Lw \cdot u &= Lw \cdot Lu \\ &= w^T L^T L u \\ &= w^T u \\ &= w \cdot u \\ &= 0 \end{aligned}$$

and similarly $Lw \cdot v = 0$. Therefore Lw is orthogonal to both u and v, which implies, $Lw \in \text{span}\{w\}$. Hence, $Lw = \lambda w$ for some $\lambda \in \mathbb{R}$

Thus, $\{u, v, w\}$ is a set of eigenvectors of L with eigenvalues $1, 1, \lambda$ respectively. By Lemma 1.4.3,

$$\det(L) = 1 \cdot 1 \cdot \lambda = \lambda$$

But since $L \in SO(3)$, we have $\det(L) = 1$, which forces $\lambda = 1$. Thus $\{u, v, w\}$ is a basis of $\mathbb{R}^3$ consisting of eigenvectors of L, each with eigenvalue 1. In particular,

$$Lu = u, \quad Lv = v, \quad Lw = w$$

Since a linear operator is uniquely determined by its action on basis, it follows that L acts as the identity on all of $\mathbb{R}^3$. Hence $L = I$, contradicting the assumption $L \neq I$. Therefore, we conclude

$$\dim(\ker(L - I)) = 1$$

as required $\square$

This automatically implies that $a + \ker(L - I)$ is an affine line in $\mathbb{R}^3$ where a is a point in $\mathbb{R}^3$

Definition 1.4.5. Axis, angle and orientation of rotation for non identity rotation L.

1. The one-dimensional subspace

$$\ker(L - I) = \{v \in \mathbb{R}^3 \mid Lv = v\}$$

   is called the axis of rotation of L. Every nonzero vector in $\ker(L - I)$ is fixed by L, and hence represents the direction of the axis.

2. By Theorem 1.4.1, the remaining two eigenvalues of L are complex conjugates $\{e^{i\theta}, e^{-i\theta}\}$ for some $\theta \in (0, 2\pi)$. This θ is called the angle of rotation of L.

3. The orientation of the rotation is determined as follows:
   If $\{u, v, w\}$ is a right handed orthonormal basis of $\mathbb{R}^3$ with u along the axis of rotation, Then L acts as a rotation by angle θ in the oriented plane $\text{span}\{v, w\}$

A fundamental property of rotations in $\mathbb{R}^3$ is that every non-identity rotation has an axis of rotation. This is a line of fixed points that remains invariant under the transformation. In this report, we will primarily consider rotations whose axis passes through the origin (linear rotation). If the axis of rotation does not pass through the origin, then the rotation is no longer a purely linear map but an affine transformation. In such cases, we may reduce the problem to the linear case:

1. Translate the space so that a point on the axis is moved to the origin.
2. Perform the rotation about the translated axis (which now passes through the origin).
3. Translate back to the original position.

In formula :

$$R(v) = p + L(v - p),$$

where L is purely linear rotational map and p is some point on the original axis.

1.5 Topological Preliminaries

Definition 1.5.1 (Topological Space)**.** Let X be a set. A collection of subsets $\tau_X \subseteq \mathcal{P}(X)$ (the power set of X) is called a topology on X if it satisfies the following:

1. The empty set and the whole set are in τ_X:
$$\varnothing \in \tau_X \quad \text{and} \quad X \in \tau_X.$$
2. τ_X is closed under unions: whenever $\{U_\alpha\}_{\alpha \in A}$ is any family of sets in τ_X, the union
$$\bigcup_{\alpha \in A} U_\alpha$$
is also in τ_X.
3. τ_X is closed under finite intersections: whenever $U_1, U_2 \in \tau_X$, the intersection
$$U_1 \cap U_2$$
is also in τ_X.

The pair (X, τ_X) is then called a topological space, and the sets in τ_X are called open sets.

Definition 1.5.2 (Subspace Topology)**.** Let (X, τ_X) be a topological space and let $Y \subseteq X$ be any subset. The subspace topology on Y is defined by

$$\tau_Y = \{\, U \cap Y \mid U \in \tau_X \,\}.$$

The pair (Y, τ_Y) is called a subspace of X.

Proof. We check the topology axioms.

1. Since $\varnothing \in \tau_X$, we have

$$\varnothing \cap Y = \varnothing \in \tau_Y.$$

Also, since $X \in \tau_X$,

$$X \cap Y = Y \in \tau_Y.$$

2. Let $\{V_\alpha\}_{\alpha\in A}$ be any family of sets in τ_Y. By definition of τ_Y, for each α there is some $U_\alpha \in \tau_X$ such that

$$V_\alpha = U_\alpha \cap Y.$$

Then

$$\bigcup_{\alpha\in A} V_\alpha = \bigcup_{\alpha\in A} (U_\alpha \cap Y) = \left(\bigcup_{\alpha\in A} U_\alpha\right) \cap Y.$$

Since $\bigcup_{\alpha\in A} U_\alpha \in \tau_X$, its intersection with Y is in τ_Y.

3. If $V_1, V_2 \in \tau_Y$, then there exist $U_1, U_2 \in \tau_X$ such that

$$V_1 = U_1 \cap Y, \qquad V_2 = U_2 \cap Y.$$

Thus

$$V_1 \cap V_2 = (U_1 \cap Y) \cap (U_2 \cap Y) = (U_1 \cap U_2) \cap Y.$$

Since $U_1 \cap U_2 \in \tau_X$, its intersection with Y lies in τ_Y.

Hence τ_Y satisfies all topology axioms and is a topology on Y. □

Definition 1.5.3 (Continuous Map)**.** Let (X, τ_X) and (Y, τ_Y) be topological spaces. A function $f : X \to Y$ is said to be continuous if for every open set $V \in \tau_Y$, the preimage

$$f^{-1}(V) = \{\, x \in X \mid f(x) \in V \,\}$$

is open in X.

Definition 1.5.4 (Homeomorphism)**.** Let (X, τ_X) and (Y, τ_Y) be topological spaces. A function $f : X \to Y$ is called a homeomorphism if

1. f is a bijection,
2. f is continuous, and
3. the inverse map $f^{-1} : Y \to X$ is continuous.

If such a map exists, we say that X and Y are homeomorphic.

Definition 1.5.5 (Connected Space)**.** A topological space (X, τ_X) is said to be connected if there do not exist nonempty open sets U and V in τ_X such that

$$X = U \cup V \quad \text{and} \quad U \cap V = \varnothing.$$

A connected space cannot be written as the union of two disjoint nonempty open sets.

Definition 1.5.6 (Standard Topology on $\mathbb{R}^n$ and $\mathbb{C}^n$)**.** For $n \geq 1$, the standard topology on $\mathbb{R}^n$ (and similarly on $\mathbb{C}^n$) is the topology generated by all open balls

$$B(x, r) = \{ y \mid \|y - x\| < r \},$$

where $x \in \mathbb{R}^n$, $r > 0$, and $\| \cdot \|$ denotes the Euclidean norm on $\mathbb{R}^n$. A subset $U \subseteq \mathbb{R}^n$ is open in this topology if for every point $x \in U$ there exists $r > 0$ such that $B(x, r) \subseteq U$. In this sense, $\mathbb{C}^n$ is viewed as $\mathbb{R}^{2n}$ with the same Euclidean norm.

Definition 1.5.7 (Topology Induced by a Bijection)**.** Let (X, τ_X) be a topological space and let Y be a set. Suppose $f : X \to Y$ is a bijection. We define a collection of subsets of Y by

$$\tau_Y := \{ U \subseteq Y \mid f^{-1}(U) \in \tau_X \}.$$

Then τ_Y is a topology on Y, called the topology induced by the bijection f. With this topology, (Y, τ_Y) becomes a topological space, and the map

$$f : (X, \tau_X) \to (Y, \tau_Y)$$

is a homeomorphism.

In this report, whenever there is a clear bijection between a space and a set, we equip the set with the topology induced by that bijection. For example, the space $M_n(\mathbb{R})$ of real $n \times n$ matrices is in bijection with $\mathbb{R}^{n^2}$ by listing the matrix entries in any fixed order. We therefore give $M_n(\mathbb{R})$ the topology induced by this bijection, so that $M_n(\mathbb{R})$ becomes a topological space homeomorphic to $\mathbb{R}^{n^2}$.

Chapter 2

SU(2) as a double cover of SO(3)

In this report, we aim to show that SU(2) is a double cover of SO(3). Specifically we will construct an explcit group homomorphism from SU(2) to SO(3) and show that it is surjective with kernel $\{\pm I\}$, therefore establishing that the map is 2-to-1.

2.1 Pauli matrices

The pauli matrices $\sigma_1, \sigma_2, \sigma_3$ are defined by the 2×2 matrices :

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Pauli matrices has the following properties :

$$\begin{aligned} \sigma_i^2 &= I, \quad \text{for } i = 1, 2, 3 \\ \sigma_i \sigma_j + \sigma_j \sigma_i &= 0, \quad \text{for } i \neq j \end{aligned}$$

Claim. The Paulie matrices together with the identity I form a basis of $M_2(\mathbb{C})$ over $\mathbb{C}$.

Proof. A generic matrix $M \in M_2(\mathbb{C})$ has the form of

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{C}$$

We will show that there exist complex numbers $\alpha, \beta, \gamma, \delta$ such that

$$M = \alpha I + \beta \sigma_1 + \gamma \sigma_2 + \delta \sigma_3$$

Compute the right-hand side:

$$\alpha I + \beta\sigma_1 + \gamma\sigma_2 + \delta\sigma_3 = \begin{pmatrix} \alpha+\delta & \beta-i\gamma \\ \beta+i\gamma & \alpha-\delta \end{pmatrix}$$

Equating entries with M gives the linear system

$$\begin{cases} \alpha+\delta = a \\ \alpha-\delta = d \\ \beta-i\gamma = b \\ \beta+i\gamma = c \end{cases}$$

From the first two equations,

$$\alpha = \frac{a+d}{2}, \qquad \delta = \frac{a-d}{2}$$

From the last two,

$$\beta = \frac{b+c}{2}, \qquad \gamma = \frac{c-b}{2i}$$

These formulae give a solution for $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Hence the four matrices span $M_2(\mathbb{C})$. If we now consider the zero matrix,

$$M = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Then the same decomposition

$$\alpha I + \beta\sigma_1 + \gamma\sigma_2 + \delta\sigma_3 = M$$

forces $\alpha = \beta = \gamma = \delta = 0$.

This shows that the set $\{I, \sigma_1, \sigma_2, \sigma_3\}$ is linearly independent. Therefore the Pauli matrices together with the identity form a basis of $M_2(\mathbb{C})$ over $\mathbb{C}$

$\square$

We recall that the definition of the traces of a matrix and some of its properties. For a square matrix $A = (a_{ij}) \in M_n(\mathbb{C})$, the trace of A is defined by

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii}$$

That is the trace is the sum of the diagonal entries of A

Properties of the trace.
For any $A, B \in M_n(\mathbb{C})$ and any scalar $c \in \mathbb{C}$, the trace satisfies :

1. Linearity:

$$\mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B), \quad \text{and} \quad \mathrm{Tr}(cA) = c\,\mathrm{Tr}(A)$$

2. Cyclic property:

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$$

Proof. let $A = (a_{ij})$ and $B = (b_{ij})$. Then

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ji}a_{ij} = \sum_{j}^{n} (BA)_{jj} = \mathrm{Tr}(BA)$$

$\square$

3. Transpose invariance:

$$\mathrm{Tr}(A^T) = \mathrm{Tr}(A)$$

4. Conjugate transpose:

$$\mathrm{Tr}(A^*) = \overline{\mathrm{Tr}(A)}$$

5. Trace of identity:

$$\mathrm{Tr}(I_n) = n$$

The proofs of the remaining properties are straightforward and follow directly from the definitions, so we omit them here.

We now consider the real subspace V of $M_2(\mathbb{C})$ defined by

$$V = \mathrm{span}_{\mathbb{R}}\{\sigma_1, \sigma_2, \sigma_3\}$$

That is,

$$V = \{x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 \mid x_1, x_2, x_3 \in \mathbb{R}\}$$

Every element $A \in V$ can be expressed uniquely as

$$A = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}$$

Proposition 2.1.1. The space V can equivalently be described as the subset of $M_2(\mathbb{C})$ consisting of all Hermitian and traceless matrices, That is,

$$V = \{A \in M_2(\mathbb{C}) \,|\, A^* = A,\, \mathrm{Tr}(A) = 0\}$$

Where a matrix is called Hermitian if it is equal to its conjugate transpose, that is, $A^* = A$.

We can consider A as a map

$$\begin{aligned} A:\ & \mathbb{R}^3 \to V, \\ & \vec{x} \mapsto x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 \end{aligned}$$

Proposition 2.1.2. The map $A : \mathbb{R}^3 \to V$ is a real vector space isomorphism.

Proof.

1. Linearity:
   For any $\vec{x} = (x_1, x_2, x_3), \vec{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$ and $c \in \mathbb{R}$,

$$\begin{aligned} A(\vec{x} + \vec{y}) &= A(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\ &= (x_1 + y_1)\sigma_1 + (x_2 + y_2)\sigma_2 + (x_3 + y_3)\sigma_3 \\ &= x_1\sigma_1 + y_1\sigma_2 + x_2\sigma_3 + y_2\sigma_1 + x_3\sigma_2 + y_3\sigma_3 \\ &= x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 + y_1\sigma_1 + y_2\sigma_2 + y_3\sigma_3 \\ &= A(\vec{x}) + A(\vec{y}) \end{aligned}$$

$$\begin{aligned} A(c\vec{x}) &= A(cx_1, cx_2, cx_3) \\ &= cx_1\sigma_1 + cx_2\sigma_2 + cx_3\sigma_3 \\ &= c(x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3) \end{aligned}$$

2. Injectivity:
   Suppose $A(\vec{x}) = A(\vec{y})$. Then

$$x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 = y_1\sigma_1 + y_2\sigma_2 + y_3\sigma_3$$

   Moving all terms to the left hand side gives

$$(x_1 - y_1)\sigma_1 + (x_2 - y_2)\sigma_2 + (x_3 - y_3)\sigma_3 = 0$$

   Since $\sigma_1, \sigma_2, \sigma_3$ are linearly independent, each coefficient must be zero. Therefore,

$$x_1 = y_1, \quad x_2 = y_2, \quad x_3 = y_3.$$

   Thus A is injective.

3. Surjectivity:
   By definition, every element of V is a real linear combination of $\sigma_1, \sigma_2, \sigma_3$, hence for any $B \in V$, there exist $\vec{x} \in \mathbb{R}^3$ with $B = A(\vec{x})$. Thus A is surjective.

Since A is linear, injective and surjective, it is a bijective linear map. Hence an isomorphism of real vector spaces. $\square$

We now compute the trace inner product of the image under A.
For any $\vec{x} = (x_1, x_2, x_3), \vec{y} = (y_1, y_2, y_3) \in R^3$, we have

$$A(\vec{x})A(\vec{y}) = (x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3)(y_1\sigma_1 + y_2\sigma_2 + y_3\sigma_3) = \sum_{1\le i,j\le 3} x_i y_j \sigma_i \sigma_j$$

Taking the trace yield

$$\mathrm{Tr}(A(\vec{x})A(\vec{y})) = \sum_{1\le i,j\le 3} x_i y_j \mathrm{Tr}(\sigma_i\sigma_j)$$

Using the relation

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA) \quad \text{and} \quad \sigma_i\sigma_j + \sigma_j\sigma_i = 0 \quad \text{for } i \neq j$$

Since $\sigma_i^2 = I$ it follows that,

$$\mathrm{Tr}(A(\vec{x})A(\vec{y})) = 2\sum_{i=1}^{3} x_i y_i = 2\,\vec{x}\cdot\vec{y}$$

Therefore we can define an inner product in V that is equivalent to the one in $\mathbb{R}^3$ by

$$A(\vec{x})\cdot A(\vec{y}) = \frac{1}{2}\mathrm{Tr}(A(\vec{x})A(\vec{y}))$$

Now let us define a transformation T_U on V where $U \in \mathrm{SU}(2)$ by

$$T_U(A(\vec{x})) = UA(\vec{x})U^{-1} = UA(\vec{x})U^*.$$

The map T_U indeed yields an element of V, as shown below. We regard V as the subspace of $M_2(\mathbb{C})$ consisting of all Hermitian, traceless matrices. These two properties completely charaterize V,as established earlier. Now observe that

$$\begin{cases} (T_U(A(\vec{x})))^* = (UA(\vec{x})U^*)^* = UA(\vec{x})U^* = T_U(A(\vec{x})), \\ \mathrm{Tr}(T_U(A(\vec{x})) = \mathrm{Tr}(UA(\vec{x})U^{-1}) = \mathrm{Tr}(U^{-1}UA(\vec{x})) = \mathrm{Tr}(A(\vec{x})) = 0. \end{cases}$$

so $T_U(A(\vec{x})) \in V$. We can consider the inner product of $T_U(A(\vec{x}))$ and $T_U(A(\vec{y}))$,

$$\begin{aligned} T_U(A(\vec{x}))\cdot T_U(A(\vec{y})) &= \frac{1}{2}\mathrm{Tr}(UA(\vec{x})U^{-1}UA(\vec{y})U^{-1}) \\ &= \frac{1}{2}\mathrm{Tr}(UA(\vec{x})A(\vec{y})U^{-1}) \\ &= \frac{1}{2}\mathrm{Tr}(U^{-1}UA(\vec{x})A(\vec{y})) \\ &= \frac{1}{2}\mathrm{Tr}(A(\vec{x})A(\vec{y})) \\ &= A(\vec{x})\cdot A(\vec{y}) \end{aligned}$$

This shows that T_U preserve the inner product on V. Now, consider the corresponding transformation on $\mathbb{R}^3$ instead of V. Define

$$\phi_U = A^{-1} \circ T_U \circ A,$$

which represents T_U as a transformation action on $\mathbb{R}^3$. From the previous equality, we have

$$\begin{aligned}
\phi_U(\vec{x}) \cdot \phi_U(\vec{y}) &= \frac{1}{2}\mathrm{Tr}(A(\phi_U(\vec{x}))\, A(\phi_U(\vec{y}))) \\
&= \frac{1}{2}\mathrm{Tr}(A(A^{-1}(T_U(A(\vec{x}))))\, A(A^{-1}(T_U(A(\vec{y}))))) \\
&= \frac{1}{2}\mathrm{Tr}(T_U(A(\vec{x}))\, T_U(A(\vec{y}))) \\
&= \frac{1}{2}\mathrm{Tr}(A(\vec{x})\, A(\vec{y})) \\
&= \vec{x} \cdot \vec{y}
\end{aligned}$$

for all $\vec{x}, \vec{y} \in \mathbb{R}^3$. Hence ϕ_U preserves the inner product on $\mathbb{R}^3$. Any linear map on $\mathbb{R}^3$ that preserves the inner product is orthogonal, and therefore

$$\phi_U \in O(3).$$

Accordingly, we introduce the map

$$\phi : SU(2) \to O(3), \qquad U \mapsto \phi_U,$$

where

$$\phi_U = A^{-1} \circ T_U \circ A.$$

Proposition 2.1.3. The map $\phi : SU(2) \to SO(3)$ is continuous. We establish this by first writing an explicit expression for $\phi(U)$, and then we verify that each coordinate function is continous.

Proof. We regard $SU(2)$ as a subspace of $M_2(\mathbb{C})$, which is identified with $\mathbb{R}^8$ with the usual topology. Likewise, $O(3)$ is a subspace of $M_3(\mathbb{R}) \cong \mathbb{R}^9$. A function into $M_3(\mathbb{R})$ is continuous if and only if each of its nine matrix-entry coordinate functions is continuous.

Let e_1, e_2, e_3 be the standard basis of $\mathbb{R}^3$. The map $A : \mathbb{R}^3 \to V \subset M_2(\mathbb{C})$ satisfies

$$x \cdot y = \frac{1}{2}\mathrm{Tr}(A(x)A(y)).$$

For $i, j \in \{1, 2, 3\}$, the (i, j)-entry of the matrix $\phi(U)$ is

$$(\phi(U))_{ij} = e_i \cdot \phi_U(e_j) = \frac{1}{2}\mathrm{Tr}(A(e_i)\, A(\phi_U(e_j)))\,.$$

Since

$$A(\phi_U(e_j)) = U\, A(e_j)\, U^*,$$

we obtain

$$(\phi(U))_{ij} = \frac{1}{2}\mathrm{Tr}(A(e_i)\, U\, A(e_j)\, U^*) = \frac{1}{2}\mathrm{Tr}(\sigma_i\, U\, \sigma_j\, U^*)\,.$$

The expression on the right side is built from the following operations: matrix multiplication, the adjoint map $U \mapsto U^*$, and the trace. All of these operations are continuous in the matrix entries of U. Therefore each coordinate function $(\phi(U))_{ij}$ is continuous on $SU(2)$.

Since all nine coordinate functions of ϕ are continuous, the map

$$\phi : SU(2) \longrightarrow M_3(\mathbb{R})$$

is continuous. Because $O(3)$ has the subspace topology from $M_3(\mathbb{R})$, the restricted map

$$\phi : SU(2) \to O(3)$$

is continuous as well. □

We now return to the fact that $\varphi_U \in O(3)$. This does not imply that every element of $O(3)$ is obtained from some $U \in SU(2)$, so it is more accurate to regard the codomain of φ as its image $\mathrm{Ran}(\varphi)$.

Earlier in this report, we have already shown that every

$$U \in SU(2)$$

can be written as

$$U = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \qquad \alpha, \beta \in \mathbb{C}, \qquad |\alpha|^2 + |\beta|^2 = 1.$$

Writing

$$\alpha = a_1 + ia_2, \qquad \beta = b_1 + ib_2,$$

with $a_1, a_2, b_1, b_2 \in \mathbb{R}$, the condition

$$|\alpha|^2 + |\beta|^2 = 1$$

becomes

$$a_1^2 + a_2^2 + b_1^2 + b_2^2 = 1.$$

This is precisely the equation of the 3-sphere $S^3 \subset \mathbb{R}^4$. Therefore, under this identification, $SU(2)$ is homeomorphic to S^3. Since S^3 is a connected topological space, it follows that $SU(2)$ is connected as well.

Proposition 2.1.4. Let $f : X \to Y$ be a continuous function between topological spaces. If X is connected, then the image $\mathrm{Ran}(f)$ is connected.

Proof. Assume for contradiction that $\mathrm{Ran}(f)$ is not connected. Then there exist nonempty open sets $A, B \subseteq \mathrm{Ran}(f)$ such that

$$A \cup B = \mathrm{Ran}(f), \qquad A \cap B = \varnothing.$$

Consider the preimages $f^{-1}(A)$ and $f^{-1}(B)$. We make the following observations.

1. $f^{-1}(A)$ and $f^{-1}(B)$ are nonempty. Since A and B are nonempty subsets of $\mathrm{Ran}(f)$, there exist $a, b \in X$ such that $f(a) \in A$ and $f(b) \in B$. Thus $a \in f^{-1}(A)$ and $b \in f^{-1}(B)$.

2. $f^{-1}(A)$ and $f^{-1}(B)$ are open in X. Because f is continuous and A and B are open in $\mathrm{Ran}(f)$ (with the subspace topology), their preimages are open in X.

3. $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint. If $x \in f^{-1}(A) \cap f^{-1}(B)$, then

$$f(x) \in A \quad \text{and} \quad f(x) \in B,$$

which contradicts $A \cap B = \varnothing$. Hence

$$f^{-1}(A) \cap f^{-1}(B) = \varnothing.$$

4. Their union is all of X. Let $x \in X$. Since $f(x) \in \mathrm{Ran}(f) = A \cup B$, we have $f(x) \in A$ or $f(x) \in B$. Thus

$$x \in f^{-1}(A) \cup f^{-1}(B),$$

so

$$f^{-1}(A) \cup f^{-1}(B) = X.$$

Together, these four properties show that $f^{-1}(A)$ and $f^{-1}(B)$ form a separation of X, contradicting the assumption that X is connected. Therefore $\mathrm{Ran}(f)$ must be connected. $\square$

Proposition 2.1.5. The subsets $\mathrm{SO}(3)$ and $-\mathrm{SO}(3) = \{A \in O(3) : \det(A) = -1\}$ form a separation of $O(3)$. In particular, both $\mathrm{SO}(3)$ and $-\mathrm{SO}(3)$ are nonempty, disjoint, open (in the subspace topology of $O(3)$), and

$$\mathrm{SO}(3) \cup \big(-\mathrm{SO}(3)\big) = O(3).$$

Hence $O(3)$ is disconnected and has exactly these two connected components.

Proof. We verify the required properties in turn.

Nonemptiness. The identity matrix $I \in \mathrm{SO}(3)$, so $\mathrm{SO}(3) \neq \varnothing$. Also $-I \in O(3)$ and $\det(-I) = -1$, so $-\mathrm{SO}(3) \neq \varnothing$.

Disjointness. If $A \in \mathrm{SO}(3) \cap (-\mathrm{SO}(3))$, then $\det(A) = 1$ and $\det(A) = -1$ simultaneously, which is impossible. Hence

$$\mathrm{SO}(3) \cap \big(-\mathrm{SO}(3)\big) = \varnothing.$$

Union. For any $A \in O(3)$ one has $\det(A) = \pm 1$. Thus every $A \in O(3)$ lies in $\mathrm{SO}(3)$ or in $-\mathrm{SO}(3)$, so

$$\mathrm{SO}(3) \cup \big(-\mathrm{SO}(3)\big) = O(3).$$

Openness. We use the fact that the determinant map

$$\det : M_3(\mathbb{R}) \longrightarrow \mathbb{R}$$

is continuous. The values of det on $O(3)$ are exactly $\{1, -1\}$. Choose disjoint open intervals $U_1, U_{-1} \subset \mathbb{R}$ with $1 \in U_1$ and $-1 \in U_{-1}$ (for example $U_1 = (\frac{1}{2}, \frac{3}{2})$ and $U_{-1} = (-\frac{3}{2}, -\frac{1}{2})$).

By continuity of det, the preimages $\det^{-1}(U_1)$ and $\det^{-1}(U_{-1})$ are open in $M_3(\mathbb{R})$. Since $\det(O(3)) = \{1, -1\}$ and the intervals U_1, U_{-1} are chosen to separate 1 and -1, we have

$$\mathrm{SO}(3) = \det^{-1}(\{1\}) \cap O(3) = \big(\det^{-1}(U_1) \cap O(3)\big),$$

and similarly

$$-\mathrm{SO}(3) = \det^{-1}(\{-1\}) \cap O(3) = \big(\det^{-1}(U_{-1}) \cap O(3)\big).$$

Hence both $\mathrm{SO}(3)$ and $-\mathrm{SO}(3)$ are open in the subspace topology of $O(3)$.

Since $\mathrm{SO}(3)$ and $-\mathrm{SO}(3)$ are nonempty, disjoint, open in $O(3)$ and their union equals $O(3)$, they form a separation of $O(3)$. Therefore $O(3)$ is disconnected and its two connected components are precisely $\mathrm{SO}(3)$ and $-\mathrm{SO}(3)$. $\square$

Proposition 2.1.6. Let X be a topological space with a separation A and B. If $Y \subseteq X$ is connected, then Y is contained entirely in A or entirely in B.

Proof. Suppose for contradiction that Y intersects both A and B. Then there exist points $y_A \in Y \cap A$ and $y_B \in Y \cap B$.

Define

$$A_Y = Y \cap A, \qquad B_Y = Y \cap B.$$

Both sets are nonempty. Since A and B are open in X, the intersections A_Y and B_Y are open in the subspace Y. They are disjoint because A and B are disjoint, and their union is

$$A_Y \cup B_Y = (Y \cap A) \cup (Y \cap B) = Y \cap (A \cup B) = Y.$$

Thus A_Y and B_Y form a separation of Y, contradicting the fact that Y is connected. Therefore Y must lie entirely in A or entirely in B. $\square$

Proposition 2.1.7. Let $\phi : SU(2) \to O(3)$ be the continuous map constructed above. Then $\mathrm{Ran}(\phi) \subseteq \mathrm{SO}(3)$.

Proof. Since ϕ is continuous and $SU(2)$ is connected (homeomorphic to S^3), the image $\mathrm{Ran}(\phi) = \phi(SU(2))$ is a connected subset of $O(3)$.

Recall that $O(3) = \mathrm{SO}(3) \cup \big(-\mathrm{SO}(3)\big)$ and that $\mathrm{SO}(3)$ and $-\mathrm{SO}(3) = \{A \in O(3) : \det(A) = -1\}$ form a separation of $O(3)$ (they are nonempty, disjoint, open in the subspace topology, and their union is $O(3)$). By the preceding proposition on connected subspaces, any connected subset of $O(3)$ must lie entirely in one of the two separated pieces. Hence

$$\mathrm{Ran}(\phi) \subseteq \mathrm{SO}(3) \quad \text{or} \quad \mathrm{Ran}(\phi) \subseteq -\mathrm{SO}(3).$$

Finally, note that $\phi(I) = I$, because $T_I(A(x)) = IA(x)I^{-1} = A(x)$ and thus $\phi_I = \mathrm{id}_{\mathbb{R}^3}$. In particular $I \in \mathrm{Ran}(\phi)$ and $I \in \mathrm{SO}(3)$, so $\mathrm{Ran}(\phi)$ cannot be contained in $-\mathrm{SO}(3)$. Therefore

$$\mathrm{Ran}(\phi) \subseteq \mathrm{SO}(3),$$

as required. $\square$

2.2 Open neighborhoods generate connected topological groups, and application to $\phi : SU(2) \to SO(3)$

Definition 2.2.1 (Topological group)**.** A *topological group* is a group G equipped with a topology such that the multiplication map $G \times G \to G,\ (x, y) \mapsto xy$ and the inversion map $G \to G,\ x \mapsto x^{-1}$ are continuous.

Lemma 2.2.2. Let (X, τ) be a topological space. If X is connected then the only subsets of X which are both open and closed (clopen) are $\varnothing$ and X itself.

Proof. Let $C \subset X$ be clopen. If $C \neq \varnothing$ and $C \neq X$ then C and $X \setminus C$ are nonempty disjoint open sets whose union is X, contradicting connectedness. Hence any clopen subset must be either $\varnothing$ or X. $\square$

Definition 2.2.3 (Symmetric neighborhood)**.** Let G be a topological group with identity e. An open neighborhood $V \subset G$ of e is called *symmetric* if $V^{-1} = \{v^{-1} : v \in V\} = V$.

Lemma 2.2.4. If U is any open neighborhood of the identity in a topological group G, then $V := U \cap U^{-1}$ is an open symmetric neighborhood of the identity contained in U.

Proof. The inversion map is continuous, so U^{-1} is open. The intersection $V = U \cap U^{-1}$ is therefore open, contains e, and satisfies $V^{-1} = (U \cap U^{-1})^{-1} = U^{-1} \cap U = V$. $\square$

Proposition 2.2.5. Let G be a topological group and let $V \subset G$ be an open symmetric neighborhood of the identity e. For each integer $k \geq 1$ define

$$V^k := \underbrace{V \cdot V \cdots V}_{k \text{ times}}, \qquad H := \bigcup_{k \geq 1} V^k.$$

Then H is an open and closed subgroup of G.

Proof. (Subgroup). If $x \in V^m$ and $y \in V^n$ then $xy \in V^{m+n} \subset H$, so H is closed under products. Since V is symmetric, $(V^m)^{-1} = V^m$, hence $x^{-1} \in V^m \subset H$. The identity $e \in V \subset H$. Associativity is inherited from G. Thus H is a subgroup.

(Openness). We prove by induction that each V^k is open. The base $V^1 = V$ is open by assumption. Assume V^k is open. Then

$$V^{k+1} = V \cdot V^k = \bigcup_{a \in V} aV^k.$$

For each fixed $a \in V$ the left-translation $L(a) : G \to G$, $L(a)(x) = ax$, is a homeomorphism (since multiplication is continuous and $L(a)^{-1} = L(a^{-1})$ is continuous). Hence $aV^k = L(a)(V^k)$ is open. As a union of open sets, V^{k+1} is open. By induction all V^k are open, and so $H = \bigcup_{k \geq 1} V^k$ is open.

(Closedness). Each left coset gH is open because H is open and left translations are homeomorphisms. Therefore the complement $G \setminus H = \bigcup_{g \notin H} gH$ is a union of open sets, hence open. Thus H is closed as well.

Combining the above, H is an open-and-closed subgroup of G, as required. □

Corollary 2.2.6. If G is a connected topological group and U is any open neighborhood of the identity e, then

$$G = \bigcup_{k \geq 1} U^k.$$

Equivalently, every open neighborhood of e topologically generates G.

Proof. By the previous lemma we may replace U by the symmetric open neighborhood $V = U \cap U^{-1} \subset U$. Form $H = \bigcup_{k \geq 1} V^k$. By Proposition 2.2.5, H is a nonempty open-and-closed subgroup of G. Since G is connected and $H \neq \varnothing$, the only possibility is $H = G$. Because $V \subset U$ we have $\bigcup_{k \geq 1} U^k \supset \bigcup_{k \geq 1} V^k = G$, hence $\bigcup_{k \geq 1} U^k = G$. □

Application to $\phi : SU(2) \to SO(3)$.

Let $\phi : SU(2) \to SO(3)$ be the continuous group homomorphism constructed earlier. We state without proof that there exist open neighborhoods $A \subset SU(2)$ of the identity and $B \subset SO(3)$ of the identity such that the restricted map $\phi|_A : A \to B$ is a homeomorphism. Then:

- $B = \phi(A) \subset \mathrm{Ran}(\phi)$.
- Since ϕ is a homomorphism, for every $k \geq 1$ we have $\phi(A^k) = \phi(A)^k = B^k$, so $B^k \subset \mathrm{Ran}(\phi)$.
- Therefore $\bigcup_{k \geq 1} B^k \subset \mathrm{Ran}(\phi)$.

- By Corollary 2.2.6 (applied to $G = SO(3)$, which is connected), $\bigcup_{k\geq 1} B^k = SO(3)$.

Hence $SO(3) \subset \mathrm{Ran}(\phi)$. Together with the previously established reverse inclusion $\mathrm{Ran}(\phi) \subset SO(3)$, we conclude

$$\mathrm{Ran}(\phi) = SO(3).$$

$\square$