Derivatives
FIN-404

The VIX and related derivatives
Project Report

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1 Documentation

Volatility has traditionally been viewed as a measure of market risk, but it has evolved into a tradable asset class on its own. As investors' interest in managing exposure to market uncertainty has grown, a variety of financial instruments linked to volatility and variance have been developed, especially related to broad equity indices such as the S&P 500.

At the core of volatility trading lies the distinction between *realized volatility* and *implied volatility*. Realized volatility measures the actual fluctuations of the underlying asset's returns over a past period, while implied volatility is derived from the prices of options and reflects the market's expectation of future volatility. Because implied volatility tends to trade above subsequent realized volatility, investors willing to sell variance earn a persistent variance risk premium, which motivates many hedging and carry strategies [1].

A standard annualized estimate of realized variance over $[t, T]$ is

$$\mathrm{RV}_{t,T} = \frac{252}{N} \sum_{i=1}^{N} \left(\ln S_{t_i} - \ln S_{t_{i-1}}\right)^2,$$

where the log returns are sampled N times (typically at daily frequency) and multiplied by 252 trading days. In a variance swap, this realized variance is compared to a fixed strike variance K_{var} at maturity [2].

The market for variance derivatives began to take shape in the early 1990s. In 1993, UBS executed the first over-the-counter (OTC) variance swap on the FTSE 100 index [3, 4]. This contract allowed investors to gain direct exposure to the realised variance of an index over a specified period. Its payoff equals the difference between the realised variance (1) and a pre-agreed strike variance level. The innovation enabled market participants to hedge volatility risk more precisely and independently from directional market movements.

Moreover, in 1993, the Chicago Board Options Exchange (CBOE) introduced the VIX index as a measure of market implied volatility. Originally based on near-the-money options on the S&P 100 index, the VIX was redesigned in 2003 to use a model free approach relying on a wide range of S&P 500 option prices [5, 6].

Since 2003, the VIX index is defined by:

$$\left(\frac{\mathrm{VIX}_t}{100}\right)^2 \eta = 2 \sum_i \left[\left(\frac{\Delta K_i}{K_i^2}\right) e^{r(T-t)} \mathcal{O}_t(T, K_i) - \left(\frac{F_t(T)}{K_i} - 1\right)^2 \right],$$

where $\eta = \frac{30}{252}$ is expressed in years, $F_t(T)$ is the forward price of the S&P500 (SPX) for delivery at date $T = t + \eta$, K_0 is the largest strike below $F_t(T)$ among existing thirty-day SPX options, $K_{\pm i}$ is the ith strike above (below) the reference strike K_0, and $\mathcal{O}_t(T, K_i)$ is the price of an OTM SPX option with maturity T and strike K_i.

The episode widely known as *"Volmageddon"* occurred on 5 February 2018 when the VIX more than doubled in a single day, its largest percentage jump on record. The surge forced massive, end of day rebalancing by inverse volatility ETNs such as Credit Suisse's XIV, which lost about 97% of its value overnight [7] and was subsequently liquidated. A feedback loop ensued: the jump in VIX futures compelled holders of short volatility products to slash exposure, which pushed volatility even higher. The S&P 500 therefore endured its biggest one day drop since 2011, exposing hidden risks in popular short vol strategies. Afterwards, exchanges and issuers reduced leverage limits, redesigned volatility products, and regulators began monitoring the build up of short vol positions to avoid a repeat of the event.

Since the VIX is an index and not a tradable asset itself, the CBOE launched futures on the VIX in 2004 and options on those futures in 2006 [5]. These exchange traded contracts let investors express views on future volatility, hedge against market stress, or run carry trades based on the variance risk premium captured in the gap between implied and realised volatility.

As shown in Figure 1, the VIX spikes sharply when the S&P 500 falls, underscoring their strong negative correlation. Events such as the 2008 Financial Crisis, Brexit, the COVID crash, and periods of rapid rate hikes all triggered large VIX jumps. This inverse relationship explains why VIX linked contracts are popular hedges during equity sell offs.

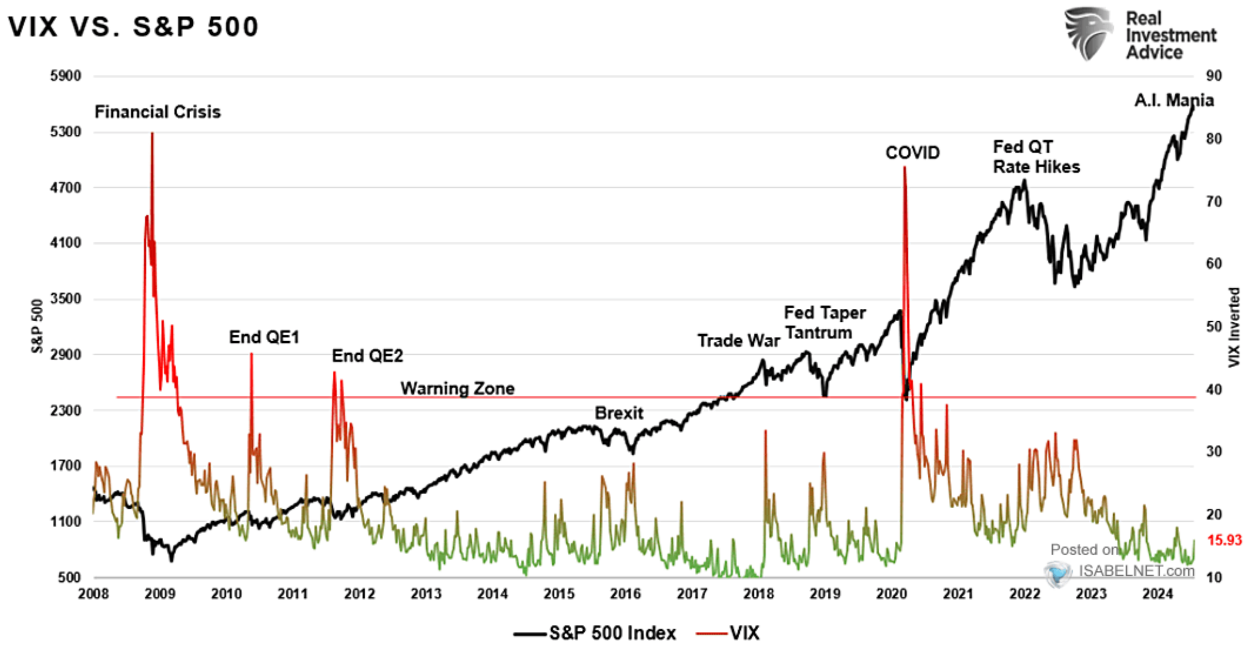


Figure 1: VIX vs. S&P 500 (2008-2024). Source: Real Investment Advice via ISABELNET.

Beyond the VIX and its derivatives, other variance related instruments have emerged. Variance swaps remain widely used in OTC markets to gain pure exposure to realized variance without the nonlinear payoffs of options. More specialized products, such as corridor variance swaps, gamma swaps, and timer options, offer additional flexibility to tailor volatility exposure to specific needs [8]. It is important to clearly distinguish between these instruments. The VIX index is a forward looking measure of implied volatility, calculated from a broad range of option prices and reflecting the market's expectation of future volatility over the next 30 days [9] as shown in . However, the VIX itself is not directly tradable; investors gain exposure to it via futures and options contracts.

In contrast, a variance swap is an OTC derivative whose payoff depends on the realized variance of the underlying asset during the life of the contract, minus a fixed strike variance level agreed at inception [8]. This allows investors to obtain direct and pure exposure to actual realized volatility, independent from the underlying asset price movements.

Variance futures, on the other hand, are standardized, exchange traded contracts that provide exposure to the expected variance over a future period. While variance swaps settle at maturity based on realized variance, variance futures are marked to market daily and offer greater liquidity and transparency but less customization [1].

Overall, derivatives related to the variance and volatility of the S&P 500 have become essential tools for market participants. They allow hedging of volatility risk, speculation on future market uncertainty, and extraction of risk premia, contributing to more efficient and sophisticated financial markets [1].

2 The Carr-Madan formula

2.1 Analytical Expression via Integration by Parts

Let $H : \mathbb{R}_+ \to \mathbb{R}$ be a function that is piecewise twice continuously differentiable on the positive real line.
By the fundamental theorem of calculus:

$$\int_{x_0}^{x} H'(k)dk = H(x) - H(x_0)$$
$$\implies H(x) = H(x_0) + \int_{x_0}^{x} H'(k)dk$$

We integrate by parts the term $\int_{x_0}^{x} H'(k)\,dk$, setting $u = H'(k)$ and $v = k - x$ (so that $dv = dk$, $du = H''(k)\,dk$).

$$\begin{aligned}
\int_{x_0}^{x} \underbrace{H'(k)}_{u}\underbrace{dk}_{dv} &= [\underbrace{H'(k)}_{u}\underbrace{(k-x)}_{v}]_{k=x_0}^{k=x} - \int_{x_0}^{x} \underbrace{(k-x)}_{v}\underbrace{H''(k)dk}_{du} \\
&= H'(x)(x-x) - H'(x_0)(x_0-x) + \int_{x_0}^{x} H''(k)(x-k)dk \\
&= H'(x_0)(x-x_0) + \int_{x_0}^{x} H''(k)(x-k)dk
\end{aligned}$$

Therefore we have:

$$\boxed{H(x) = H(x_0) + (x-x_0)H'(x_0) + \int_{x_0}^{x} H''(k)(x-k)dk}$$

2.2 Alternative Representation Using the Positive Part Function

We know that $(x-y) = (x-y)^+ - (y-x)^+$
We have:

$$H(x) = H(x_0) + H'(x_0)(x-x_0) + \int_{x_0}^{x} H''(k)(x-k)dk$$

We transform the term $\int_{x_0}^{x} H''(k)(x-k)dk$:

$$\begin{aligned}
\int_{x_0}^{x} H''(k)(x-k)dk &= \int_{x_0}^{x} H''(k)((x-k)^+ - (k-x)^+)dk \\
&= \int_{x_0}^{x} H''(k)(x-k)^+dk - \int_{x_0}^{x} H''(k)(k-x)^+dk \\
&= \int_{x_0}^{x} H''(k)(x-k)^+dk + \int_{x}^{x_0} H''(k)(k-x)^+dk \\
&= \int_{x_0}^{\infty} H''(k)(x-k)^+dk - \int_{x}^{\infty} H''(k)\underbrace{(x-k)^+}_{=0\forall k\geq x}dk \\
&\quad + \int_{0}^{x_0} H''(k)(k-x)^+dk - \int_{0}^{x} H''(k)\underbrace{(k-x)^+}_{=0\forall k<x}dk \\
&= \int_{x_0}^{\infty} H''(k)(x-k)^+dk + \int_{0}^{x_0} H''(k)(k-x)^+dk
\end{aligned}$$

Therefore we have:

$$\boxed{H(x) = H(x_0) + H'(x_0)(x-x_0) + \int_{x_0}^{\infty} H''(k)(x-k)^+dk + \int_{0}^{x_0} H''(k)(k-x)^+dk}$$

2.3 Carr-Madan Replication Formula and Financial Interpretation

We assume a constant risk-free rate r and a continuous dividend yield δ on the underlying asset. We have:

$$H(x) = H(x_0) + H'(x_0)(x - x_0) + \int_0^{x_0} H''(k)(k - x)^+ dk + \int_{x_0}^{\infty} H''(k)(x - k)^+ dk$$

We interpret at maturity for $x = S_T$:

$$H(S_T) = H(x_0) + H'(x_0)(S_T - x_0) + \int_0^{x_0} H''(k)(k - S_T)^+ dk + \int_{x_0}^{\infty} H''(k)(S_T - k)^+ dk \tag{1}$$

Under the risk-neutral measure, the price at time 0 of a payoff X_T is:

$$V_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}[X_T]$$

We apply it to $H(S_T)$:

$$\begin{aligned}
V_0 &= e^{-rT}\mathbb{E}^{\mathbb{Q}}[H(S_T)] \\
&= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[H(x_0) + H'(x_0)(S_T - x_0) + \int_0^{x_0} H''(k)(k - S_T)^+ dk + \int_{x_0}^{\infty} H''(k)(S_T - k)^+ dk\right] \\
&= e^{-rT}\left(H(x_0) + H'(x_0)(\mathbb{E}^{\mathbb{Q}}[S_T] - x_0) + \int_0^{x_0} H''(k)\mathbb{E}^{\mathbb{Q}}[(k - S_T)^+]dk + \int_{x_0}^{\infty} H''(k)\mathbb{E}^{\mathbb{Q}}[(S_T - k)^+]dk\right)
\end{aligned}$$

We know:

$\mathbb{E}^{\mathbb{Q}}[S_T] = F_0(T) = S_0 e^{(r-\delta)T}$

$\mathbb{E}^{\mathbb{Q}}[(k - S_T)^+] := p(T;k)$ is the price at time 0 of a European put with maturity T and strike k

$\mathbb{E}^{\mathbb{Q}}[(S_T - k)^+] := c(T;k)$ is the price at time 0 of a European call with maturity T and strike k

Putting this into the V_0 expression:

$$\begin{aligned}
V_0 &= H(x_0)e^{-rT} + H'(x_0)\big(S_0 e^{-\delta T} - x_0 e^{-rT}\big) + \int_0^{x_0} H''(k)p(T;k)\,dk + \int_{x_0}^{\infty} H''(k)c(T;k)\,dk \qquad (2) \\
&= \big(H(x_0) - x_0 H'(x_0)\big)e^{-rT} + H'(x_0)S_0 e^{-\delta T} + \int_0^{x_0} H''(k)p(T;k)\,dk + \int_{x_0}^{\infty} H''(k)c(T;k)\,dk
\end{aligned}$$

Thus, the static replication strategy is:

- $a_0 = \big(H(x_0) - x_0 H'(x_0)\big)e^{-rT}$ invested at the risk-free rate
- $n_0 = H'(x_0)e^{-\delta T}$ units of the underlying asset
- $w(k) = H''(k)\,dk$ units of puts with maturity T and strike k, for all $k \leq x_0$
- $w(k) = H''(k)\,dk$ units of calls with maturity T and strike k, for all $k > x_0$

If we replace $x_0 = F_0(T) = S_0 e^{(r-\delta)T}$ in (2), we get:

$$V_0 = H(F_0(T))e^{-rT} + H'(F_0(T))\big(S_0 e^{-\delta T} - F_0(T)e^{-rT}\big) + \int_0^{F_0(T)} H''(k)p(T;k)\,dk + \int_{F_0(T)}^{\infty} H''(k)c(T;k)\,dk$$

But note that:

$$F_0(T)e^{-rT} = S_0 e^{-\delta T} \implies S_0 e^{-\delta T} - F_0(T)e^{-rT} = 0$$

So the second term vanishes, and:

$$V_0 = H(F_0(T))e^{-rT} + \int_0^{F_0(T)} H''(k)p(T;k)\,dk + \int_{F_0(T)}^{\infty} H''(k)c(T;k)\,dk$$

The static replication strategy becomes:

- $a_0 = H(F_0(T))e^{-rT}$ invested at the risk-free rate
- $n_0 = 0$ units of the underlying asset
- $w(k) = H''(k)\, dk$ units of puts with maturity T and strike k, for all $k \leq x_0$
- $w(k) = H''(k)\, dk$ units of calls with maturity T and strike k, for all $k > x_0$

We do not have a position in the underlying asset anymore.
Therefore, the strategy becomes delta-neutral.

2.4 Static Replication of Power Payoffs

For a power derivative with payoff $H(x) = x^p$, we have:

$$H(x) = x^p, \qquad H'(x) = px^{p-1}, \qquad H''(x) = p(p-1)x^{p-2}$$

Thus, the static replication srategy is:

- $a_0 = (1-p)x_0^p e^{-rT}$ invested at the risk-free rate
- $n_0 = px_0^{p-1}$ units of the underlying asset
- $w(k) = p(p-1)k^{p-2}\, dk$ units of put with maturity T and strike k $\forall k \leq x_0$
- $w(k) = p(p-1)k^{p-2}\, dk$ units of call with maturity T and strike k $\forall k > x_0$

2.5 Practical Limitations of the Carr-Madan Approach

The Carr-Madan formula requires integrating over all strikes $k \in (0, \infty)$ using infinitesimal weights $w(k)$. But the market offers only a discrete set of strikes. Moreover, the formula assumes that the option payoff function is at least twice differentiable with respect to the strike.

3 The VIX index

3.1 Realized Variance and Log-Return Identity

We aim to show that $\overline{V}_{t,T} := \frac{1}{T-t}\int_t^T V_s\, ds$ satisfies $x\overline{V}_{t,T} = \left(\int_t^T \frac{dS_s}{S_s} - \log \frac{S_T}{S_t}\right)$ for $x > 0$ to determine.
Under the risk-neutral measure $\mathbb{Q}$, the S&P 500 index price S_s evolves as

$$\frac{dS_s}{S_s} = (r-\delta)\, ds + \sqrt{V_s}\, dB_s^{\mathbb{Q}}.$$

By Itô's formula,

$$d\log S_s = \frac{dSs}{Ss} - \frac{1}{2}\frac{(dSs)^2}{Ss^2}$$

and so:

$$d\log S_s = (r-\delta)\, ds - \frac{1}{2}V_s\, ds + \sqrt{V_s}\, dB_s^{\mathbb{Q}},$$

$$\frac{dS_s}{S_s} = d\log S_s + \frac{1}{2}V_s\, ds.$$

Integrating from t to T gives

$$\int_t^T \frac{dS_s}{S_s} = \log\frac{S_T}{S_t} + \frac{1}{2}\int_t^T V_s\, ds = \log\frac{S_T}{S_t} + \frac{1}{2}\overline{V}_{t,T}.$$

Rearranging yields

$$\boxed{\overline{V}_{t,T} = 2\left(\int_t^T \frac{dS_s}{S_s} - \log\frac{S_T}{S_t}\right)} \qquad (3)$$

Hence, the constant in the identity is $x = \frac{1}{2}$.

3.2 Expectation of Log-Returns Under the Risk-Neutral Measure

We aim show that $\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \frac{dS_u}{S_u}\right] = \alpha(T-t)$ with α to be determined.
Under the risk-neutral measure $\mathbb{Q}$, the S&P 500 index follows

$$\frac{dS_u}{S_u} = (r-\delta)\,du + \sqrt{V_u}\,dB_u^{\mathbb{Q}}.$$

Taking expectations conditional on time t, we have:

$$\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \frac{dS_u}{S_u}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T (r-\delta)\,du + \int_t^T \sqrt{V_u}\,dB_u^{\mathbb{Q}}\right].$$

Linearity of expectation and the fact that the stochastic integral is a martingale with zero mean under $\mathbb{Q}$ yield:

$$\boxed{\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \frac{dS_u}{S_u}\right] = (r-\delta)(T-t)} \tag{4}$$

Hence, the constant in the identity is:

$$\alpha = r-\delta$$

3.3 Expected Log-Return via Option Prices

From the general Carr-Madan replication formula recalled in Part 2.3, equation (1), we now specialize to the case $H(x) = \log(x)$, whose derivatives are: $H'(x) = \frac{1}{x}$, $\quad H''(x) = -\frac{1}{x^2}$.
Plugging this into the formula with $x_0 = K_0$ and subtracting $\log(S_t)$ from both sides, we obtain:

$$\log\left(\frac{S_T}{S_t}\right) = \log\left(\frac{K_0}{S_t}\right) + \frac{S_T-K_0}{K_0} - \int_0^{K_0} \frac{(k-S_T)^+}{k^2}\,dk - \int_{K_0}^{\infty} \frac{(S_T-k)^+}{k^2}\,dk.$$

We multiply both sides by $e^{-r(T-t)}$:

$$\log\left(\frac{S_T}{S_t}\right) e^{-r(T-t)} = \left[\log\left(\frac{K_0}{S_t}\right) + \frac{S_T-K_0}{K_0} - \int_0^{K_0} \frac{(k-S_T)^+}{k^2}\,dk - \int_{K_0}^{\infty} \frac{(S_T-k)^+}{k^2}\,dk\right] e^{-r(T-t)}$$

We take the expectation under $\mathbb{Q}$:

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}\left[\log\left(\frac{S_T}{S_t}\right) e^{-r(T-t)}\right] = {}& \log\left(\frac{K_0}{S_t}\right) e^{-r(T-t)} + \frac{1}{K_0}\mathbb{E}_t^{\mathbb{Q}}\left[(S_T-K_0)e^{-r(T-t)}\right] \\ & - \int_0^{K_0} \frac{1}{k^2}\mathbb{E}_t^{\mathbb{Q}}\left[(k-S_T)^+ e^{-r(T-t)}\right] dk \\ & - \int_{K_0}^{\infty} \frac{1}{k^2}\mathbb{E}_t^{\mathbb{Q}}\left[(S_T-k)^+ e^{-r(T-t)}\right] dk\end{aligned}$$

Now we use the identities:

$$\mathbb{E}_t^{\mathbb{Q}}[S_T] = F_t(T), \quad e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}[(S_T-k)^+] = Call_t(T,k), \quad e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}[(k-S_T)^+] = Put_t(T,k)$$

where $\mathrm{Put}_t(T,k)$ and $\mathrm{Call}_t(T,k)$ denote the prices at time t of European put and call options, respectively, with maturity T and strike k, written on the SPX index.
Finally, we multiply both sides by $e^{r(T-t)}$:

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}\left[\log\left(\frac{S_T}{S_t}\right)\right] = {}& \Bigg[\log\left(\frac{K_0}{S_t}\right) + \frac{1}{K_0}(F_t(T)-K_0) \\ & - \int_0^{K_0} \frac{Put_t(T,k)e^{r(T-t)}}{k^2}\,dk - \int_{K_0}^{\infty} \frac{Call_t(T,k)e^{r(T-t)}}{k^2}\,dk\Bigg]\end{aligned}$$

to obtain:

$$\boxed{\mathbb{E}_t^{\mathbb{Q}}\left[\log\left(\frac{S_T}{S_t}\right)\right] = \log\left(\frac{K_0}{S_t}\right) - \left(1 - \frac{F_t(T)}{K_0}\right) - P_t(T, K_0)} \tag{5}$$

with:

$$P_t(T, K_0) := \int_0^{K_0} e^{r(T-t)}\,\mathrm{Put}_t(T, k)\,\frac{dk}{k^2} + \int_{K_0}^{\infty} e^{r(T-t)}\,\mathrm{Call}_t(T, k)\,\frac{dk}{k^2}$$

3.4 VIX Approximation and Economic Interpretation

From Part 3.1 equation (3) with $x = \frac{1}{2}$:

$$\frac{1}{2}\mathbb{E}_t^{\mathbb{Q}}[\bar{V}_{t,T}] = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \frac{dS_u}{S_u} - \log\left(\frac{S_T}{S_t}\right)\right]$$

From Part 3.2 equation (4):

$$\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \frac{dS_u}{S_u}\right] = (r - \delta)(T - t)$$

From Part 3.3 equation (5):

$$\mathbb{E}_t^{\mathbb{Q}}\left[\log\left(\frac{S_T}{S_t}\right)\right] = \log\left(\frac{K_0}{S_t}\right) - \left(1 - \frac{F_t(T)}{K_0}\right) - P_t(T, K_0)$$

Putting it all together:

$$\frac{1}{2}\mathbb{E}_t^{\mathbb{Q}}[\bar{V}_{t,T}] = P_t(T, K_0) + \log\left(\frac{F_t(T)}{K_0}\right) + \left(1 - \frac{F_t(T)}{K_0}\right)$$

Now, using the second-order Taylor expansion where $log(x) \approx (x-1) - \frac{(x-1)^2}{2}$:

$$\log\left(\frac{F_t(T)}{K_0}\right) \approx \left(\frac{F_t(T)}{K_0} - 1\right) - \frac{1}{2}\left(\frac{F_t(T)}{K_0} - 1\right)^2$$

Thus:

$$\boxed{\frac{1}{2}\mathbb{E}_t^{\mathbb{Q}}[\bar{V}_{t,T}] \approx P_t(T, K_0) - \frac{1}{2}\left(1 - \frac{F_t(T)}{K_0}\right)^2}$$

This matches the form of the VIX formula.

Interpretation

The CBOE's VIX formula provides a model-free estimate of risk-neutral expected variance. It approximates $E_t^{\mathbb{Q}}\left[\frac{1}{\eta}\int_t^{t+\eta} V_u\,du\right]$, with $\eta = \frac{30}{252}$, using a weighted sum of out-of-the-money (OTM) option prices across strikes, where each option is weighted by $\frac{\Delta K_i}{K_i^2}$.

Unlike parametric models, the VIX extracts volatility expectations directly from market prices. Our derivation shows it replicates a log contract whose value reflects expected future quadratic variation under the risk-neutral measure $\mathbb{Q}$. This avoids assumptions on volatility dynamics while maintaining theoretical soundness.

The VIX reflects risk-neutral expectations, incorporating both anticipated future volatility and the volatility risk premium. This dual content makes it a valuable market sentiment indicator.

Advantages

- The VIX is forward-looking and reacts instantly to new information, unlike realized variance, which is backward-looking and lags.
- It aggregates market expectations across a full range of strikes, offering robustness against mispricing and liquidity distortions.
- Unlike Black-Scholes implied volatility, it avoids assumptions of constant volatility and log-normal returns.

Assumptions

• The asset price follows a continuous-time stochastic process with stochastic volatility (but no jumps).
• Markets must be arbitrage-free and complete for risk-neutral valuation to hold.
• A continuum of liquid OTM options is assumed, though in practice this may require interpolation.
• The reference strike K_0 should be close to the forward price $F_t(T)$ for Taylor approximations to hold.
• Option prices must reflect risk-neutral expectations without distortion from microstructure effects or strike-dependent risk premiums.

4 Futures pricing

4.1 Affine Representation of VIX in Terms of Terminal Volatility

We aim to show that $\left(\frac{\text{VIX}_T}{100}\right)^2 = \frac{1}{\eta}(a + bV_T)$ for some constants a and b depending on the model parameters $\lambda, \theta, \xi, \rho, \eta$.
By assumption, the squared VIX is defined as the average expected variance over the interval $[T, T+\eta]$ under the risk-neutral measure:

$$\left(\frac{\text{VIX}_T}{100}\right)^2 = \frac{1}{\eta}\mathbb{E}_T^{\mathbb{Q}}\left[\int_T^{T+\eta} V_u \, du\right]$$

The variance process V_u follows the stochastic differential equation:

$$dV_u = \lambda(\theta - V_u)\, du + \xi\rho\sqrt{V_u}\, dB_u^{\mathbb{Q}} + \xi\sqrt{1-\rho^2}\sqrt{V_u}\, dZ_u^{\mathbb{Q}}$$

where $B^{\mathbb{Q}}$ and $Z^{\mathbb{Q}}$ are independent Brownian motions under the risk-neutral measure.
The integral form of the solution to this SDE is given by:

$$V_u = V_T + \int_T^u \lambda(\theta - V_s)\, ds + \varepsilon\rho \int_T^u \sqrt{V_s}\, dB_s^{\mathbb{Q}} + \varepsilon\sqrt{1-\rho^2}\int_T^u \sqrt{V_s}\, dZ_s^{\mathbb{Q}}$$

Taking expectations conditional on $\mathcal{F}_T$, and noting that the Brownian integrals have zero mean under the risk-neutral measure, we obtain:

$$\mathbb{E}_T^{\mathbb{Q}}[V_u] = V_T + \int_T^u \lambda(\theta - \mathbb{E}_T^{\mathbb{Q}}[V_s])\, ds$$

Defining $\mu(u) = \mathbb{E}_T^{\mathbb{Q}}[V_u]$, we arrive at the deterministic ODE:

$$\frac{d\mu(u)}{du} = \lambda(\theta - \mu(u)), \quad \mu(T) = V_T$$

whose solution is:

$$\mu(u) = (V_T - \theta)e^{-\lambda(u-T)} + \theta \tag{6}$$

We now plug this into the definition of the squared VIX:

$$\left(\frac{\text{VIX}_T}{100}\right)^2 = \frac{1}{\eta}\int_T^{T+\eta} \mu(s)\, ds$$

Computing the integral:

$$\begin{aligned}
\left(\frac{\text{VIX}_T}{100}\right)^2 &= \frac{1}{\eta}\int_T^{T+\eta} \left((V_T - \theta)e^{-\lambda(s-T)} + \theta\right)\, ds \\
&= \frac{1}{\eta}\left[\theta\eta + (V_T - \theta)\int_0^{\eta} e^{-\lambda u}\, du\right] \\
&= \frac{1}{\eta}\left[\theta\eta + \frac{1 - e^{-\lambda\eta}}{\lambda}(V_T - \theta)\right]
\end{aligned}$$

This can be rewritten in affine form:

$$\boxed{\left(\frac{\text{VIX}_T}{100}\right)^2 = \frac{1}{\eta}\left[\theta\left(\eta - \frac{1-e^{-\lambda\eta}}{\lambda}\right) + \frac{1-e^{-\lambda\eta}}{\lambda}V_T\right]} \tag{7}$$

Thus, we have identified:

$$a = \theta\left(\eta - \frac{1-e^{-\lambda\eta}}{\lambda}\right), \quad b = \frac{1-e^{-\lambda\eta}}{\lambda}$$

4.2 Laplace Transform of Terminal Volatility and Affine Expectation

We aim to show that $-\log \mathbb{E}_t^{\mathbb{Q}}[e^{-sV_T}] = c(T-t;s) + d(T-t;s)V_t$, where the functions c and d depend on time to maturity and the Laplace parameter $s > 0$, and are determined by the model parameters $\lambda, \theta, \xi, \rho$.

First note that:

$$\begin{aligned} -log(\mathbb{E}_t^Q[e^{-s\cdot V_T}]) &= c(T-t;s) + d(T-t;s)\cdot V_t \quad \text{with } (s>0) \\ \mathbb{E}_t^Q[e^{-s\cdot V_T}] &= e^{-c(T-t;s)-d(T-t;s)\cdot V_t} \end{aligned} \tag{8}$$

Let $f(T-t;V_t)$ be such that $\mathbb{E}_t^Q[e^{-s\cdot V_T}] = f(T-t;V_t)$. By ito's lemma we have :

$$\begin{aligned} df(T-t;V_t) &= \frac{\partial f}{\partial t}\,dt + \frac{\partial f}{\partial V_t}\,dV_t + \frac{1}{2}\frac{\partial^2 f}{\partial V_t^2}\,(dV_t)^2 \\ &= \frac{\partial f}{\partial t}\,dt + \frac{\partial f}{\partial V_t}\left(\lambda(\theta - V_t)\,dt + \xi\rho\sqrt{V_t}\,dB_t^{\mathbb{Q}} + \xi\sqrt{|1-\rho^2|\cdot V_t}\,dZ_t^{\mathbb{Q}}\right) \\ &\quad + \frac{1}{2}\frac{\partial^2 f}{\partial V_t^2}\left(\xi^2\rho^2 V_t\,dt + \xi^2|1-\rho^2|\cdot V_t\,dt\right) \quad \text{since } B_t^{\mathbb{Q}} \perp Z_t^{\mathbb{Q}} \\ &= \frac{\partial f}{\partial t}\,dt + \frac{\partial f}{\partial V_t}\left(\lambda(\theta - V_t)\,dt + \xi\rho\sqrt{V_t}\,dB_t^{\mathbb{Q}} + \xi\sqrt{|1-\rho^2|\cdot V_t}\,dZ_t^{\mathbb{Q}}\right) \\ &\quad + \frac{1}{2}\frac{\partial^2 f}{\partial V_t^2}\cdot\xi^2 V_t\,dt \quad \text{with } \rho \in [-1,1] \end{aligned}$$

We have that $f(T-t;V_t)$ is a martingale so it has 0 drift, we can derive a PDE by setting the drift term to 0.

$$\begin{aligned} 0 &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial V_t}\cdot\lambda(\theta - V_t) + \frac{1}{2}\frac{\partial^2 f}{\partial V_t^2}\cdot\xi^2 V_t \\ &= f(T-t;V_t)\left(\frac{\partial c}{\partial t}(T-t;s) + \frac{\partial d}{\partial t}(T-t;s)\cdot V_t\right) - \lambda(\theta - V_t)\cdot f(T-t;V_t)\cdot d(T-t;s) \\ &\quad + \frac{1}{2}\xi^2 V_t\cdot d(T-t;s)^2\cdot f(T-t;V_t) \end{aligned}$$

Then, we can remove $f(T-t;V_t)$:

$$\begin{aligned} 0 &= \frac{\partial c}{\partial t}(T-t;s) + \frac{\partial d}{\partial t}(T-t;s)\cdot V_t - \lambda(\theta - V_t)\cdot d(T-t;s) + \frac{1}{2}\xi^2 V_t\cdot d(T-t;s)^2 \\ &= V_t\left(\frac{1}{2}\xi^2 d(T-t;s)^2 + \lambda d(T-t;s) + \frac{\partial d}{\partial t}(T-t;s)\right) - \lambda\theta d(T-t;s) + \frac{\partial c}{\partial t}(T-t;s) \end{aligned}$$

We can now identify two ODEs by grouping the terms:

$$\begin{cases} \dfrac{\partial d}{\partial t}(T-t;s) = -\lambda d(T-t;s) - \dfrac{1}{2}\xi^2 d(T-t;s)^2, \\ \dfrac{\partial c}{\partial t}(T-t;s) = \lambda\theta d(T-t;s). \end{cases}$$

To determine the initial conditions, note that:

$$f(t; V_t) = \mathbb{E}_T^{\mathbb{Q}}[e^{-sV_T}] = e^{-sV_T} \quad \Rightarrow \quad c(0, s) = 0, \quad d(0, s) = s.$$

Solving these ODEs using WolframAlpha, we obtain:

$$d(T - t; s) = -\frac{2\lambda s e^{\lambda(T-t)}}{\xi^2 s(e^{\lambda(T-t)} - 1) - 2\lambda},$$

$$c(T - t; s) = \frac{2\theta\lambda}{\xi^2}\left(\log\left(2\lambda - \xi^2 s(e^{\lambda(T-t)} - 1)\right) - \log(2\lambda)\right).$$

This completes the derivation of the affine transform formula for the Laplace transform of V_T, which we can summarize as:

$$\boxed{-\log \mathbb{E}_t^{\mathbb{Q}}[e^{-sV_T}] = \frac{2\theta\lambda}{\xi^2}\left(\log\left(2\lambda - \xi^2 s(e^{\lambda(T-t)} - 1)\right) - \log(2\lambda)\right) - \frac{2\lambda s e^{\lambda(T-t)}}{\xi^2 s(e^{\lambda(T-t)} - 1) - 2\lambda} V_t}$$

4.3 Affine Expressions for Futures Prices

We aim to that the variance futures price and the VIX futures price are respectively given by:

$$f_t^{\mathrm{VA}}(T) = \frac{10{,}000}{T - t_0}\int_{t_0}^{t} V_u \, du + a^*(T - t) + b^*(T - t)V_t, \qquad f_t^{\mathrm{VIX}}(T) = \frac{100}{\sqrt{\eta}}\mathbb{E}_t^{\mathbb{Q}}\left[\sqrt{a' + b'V_T}\right],$$

where $a^*(\cdot), b^*(\cdot)$ and a', b' are functions of the model parameters.
The futures price f_{it} of an asset S_i for delivery at time T is given by:

$$f_{it} = \mathbb{E}_t^{\mathbb{Q}}[S_{iT}] = \mathbb{E}_t^{\mathbb{Q}}[f_{iT}].$$

In our case, this gives:

$$\begin{aligned} f_t^{\mathrm{VA}}(T) = \mathbb{E}_t^{\mathbb{Q}}[f_{iT}] &= \mathbb{E}_t^{\mathbb{Q}}\left[\frac{10{,}000}{T - t_0}\int_{t_0}^{T} V_u \, du\right] \\ &= \frac{10{,}000}{T - t_0}\mathbb{E}_t^{\mathbb{Q}}\left[\int_{t_0}^{t} V_u \, du + \int_{t}^{T} V_u \, du\right] \\ &= \frac{10{,}000}{T - t_0}\left(\int_{t_0}^{t} V_u \, du + \int_{t}^{T} \mathbb{E}_t^{\mathbb{Q}}[V_u] \, du\right) \quad \text{(Fubini)} \end{aligned} \tag{9}$$

From Part 4.1 equation (6), we have:

$$\mathbb{E}_t^{\mathbb{Q}}[V_u] = (V_t - \theta)e^{-\lambda(u-t)} + \theta, \quad \text{for } u \geq t.$$

Plugging this into the expression:

$$\begin{aligned} f_t^{\mathrm{VA}}(T) &= \frac{10{,}000}{T - t_0}\left(\int_{t_0}^{t} V_u \, du + \int_{t}^{T} \left[(V_t - \theta)e^{-\lambda(u-t)} + \theta\right] du\right) \\ &= \frac{10{,}000}{T - t_0}\left(\int_{t_0}^{t} V_u \, du + V_t \cdot \int_{t}^{T} e^{-\lambda(u-t)} \, du - \theta \cdot \int_{t}^{T} e^{-\lambda(u-t)} \, du + \theta(T - t)\right) \\ &= \boxed{\frac{10{,}000}{T - t_0}\left(\int_{t_0}^{t} V_u \, du + V_t \cdot \frac{1 - e^{-\lambda(T-t)}}{\lambda} + \theta\left((T - t) - \frac{1 - e^{-\lambda(T-t)}}{\lambda}\right)\right)} \end{aligned}$$

which gives the desired affine form in V_t with:

$$a^*(T - t) = \theta\left((T - t) - \frac{1 - e^{-\lambda(T-t)}}{\lambda}\right), \quad b^*(T - t) = \frac{1 - e^{-\lambda(T-t)}}{\lambda}$$

We now turn to the VIX futures. Recall that $f_T^{\text{VIX}}(T) = \text{VIX}_T$ and:

$$\left(\frac{\text{VIX}_T}{100}\right)^2 = \frac{1}{\eta}(a + bV_T), \quad \text{from Part 3.1 equation (7)}$$
$$\implies \text{VIX}_T = \frac{100}{\sqrt{\eta}}\sqrt{a + bV_T}.$$

Using the same argument as for variance futures, we compute:

$$\begin{aligned} f_t^{\text{VIX}}(T) &= \mathbb{E}_t^{\mathbb{Q}}[f_T^{\text{VIX}}(T)] \\ &= \boxed{\frac{100}{\sqrt{\eta}}\mathbb{E}_t^{\mathbb{Q}}\left[\sqrt{a + bV_T}\right]} \end{aligned}$$

Therefore, we identify $a' = a$ and $b' = b$, as in Part 3.1.

4.4 Integral Representation of the VIX Futures Price

We begin with a known Gaussian integral representation:

$$\sqrt{\frac{\pi}{x}} = \int_0^\infty e^{-sx}\,\frac{ds}{\sqrt{s}}$$

Then we compute:

$$\begin{aligned} \int_0^x \sqrt{\frac{\pi}{z}}\,dz &= \int_0^x \left(\int_0^\infty e^{-sz}\,\frac{ds}{\sqrt{s}}\right) dz \\ &= \int_0^\infty \left(\int_0^x e^{-sz}\,dz\right)\frac{ds}{\sqrt{s}} \quad \text{(Fubini)} \\ &= \int_0^\infty \frac{1}{\sqrt{s}}\left[\frac{1 - e^{-sx}}{s}\right] ds \\ &= 2\sqrt{x}\sqrt{\pi} \end{aligned}$$

We then represent $\sqrt{x}$ as:

$$\sqrt{x} = \frac{1}{2\sqrt{\pi}}\int_0^\infty \left(1 - e^{-xs}\right)\frac{ds}{s^{3/2}}$$

Now consider :

$$\begin{aligned} f_t^{\text{VIX}}(T) &= \frac{100}{\sqrt{\eta}}\,\mathbb{E}_t^{\mathbb{Q}}\left[\sqrt{a + bV_T}\right] \\ &= \frac{100}{\sqrt{\eta}}\,\mathbb{E}_t^{\mathbb{Q}}\left[\frac{1}{2\sqrt{\pi}}\int_0^\infty \left(1 - e^{-s(a+bV_T)}\right)\frac{ds}{s^{3/2}}\right] \\ &= \frac{50}{\sqrt{\pi\eta}}\int_0^\infty \left(1 - \mathbb{E}_t^{\mathbb{Q}}\left[e^{-s(a+bV_T)}\right]\right)\frac{ds}{s^{3/2}} \end{aligned}$$

From Part 4.2 equation (8) we have :

$$e^{-sa}\,\mathbb{E}_t^{\mathbb{Q}}\left[e^{-sbV_T}\right] = e^{-sa}\,e^{-c(T-t,sb)-d(T-t,sb)V_t}$$

Since $b = \frac{1-e^{-\lambda\eta}}{\lambda}$ and $\lambda, \eta > 0$, we have $s \cdot b > 0$ as required for the equality. Futhermore:

$$\begin{aligned} &\log\left(\mathbb{E}_t^{\mathbb{Q}}\left[e^{-sa-sbV_T}\right]\right) = -sa - c(T-t, sb) - d(T-t, sb)V_t \\ \implies &\log\left(\mathbb{E}_t^{\mathbb{Q}}\left[e^{-sa-sbV_T}\right]\right) = -l(s, T-t, V_t) \\ \implies &\mathbb{E}_t^{\mathbb{Q}}\left[e^{-sa-sbV_T}\right] = e^{-l(s,T-t,V_t)} \end{aligned}$$

With $l(s, T-t, V_t) = sa + c(T-t, sb) + d(T-t, sb)V_t$, leading to:

$$f_t^{\text{VIX}}(T) = \frac{50}{\sqrt{\pi\eta}}\int_0^\infty \left(1 - e^{-l(s,T-t,V_t)}\right)\frac{ds}{s^{3/2}}$$

4.5 Arbitrage Relation Between Spot VIX and Variance Futures

From Part 4.3, equation (9), we have:

$$f_t^{\mathrm{VA}}(t+\eta) = \frac{10{,}000}{t+\eta-t_0}\,\mathbb{E}_t^{\mathbb{Q}}\left[\int_{t_0}^{t+\eta} V_u\,du\right]$$

And:

$$\left(\frac{\mathrm{VIX}_t}{100}\right)^2 = \frac{1}{\eta}\,\mathbb{E}_t^{\mathbb{Q}}\left[\int_{t}^{t+\eta} V_u\,du\right]$$

$$\implies \mathrm{VIX}_t^2 = \frac{10,000}{\eta}\,\mathbb{E}_t^{\mathbb{Q}}\left[\int_{t}^{t+\eta} V_u\,du\right]$$

For a variance future contract $f_t^{VA}(t+\eta)$ that just been listed, namely, $t_0 = t$, we have:

$$\mathrm{VIX}_t^2 = f_t^{VA}(t+\eta)$$

We have that:

$$\left(\frac{\mathrm{VIX}_t}{100}\right)^2 \eta = 2\sum_i \left[\left(\frac{\Delta K_i}{K_i^2}\right) e^{r(T-t)}\mathcal{O}_t(T,K_i) - \left(\frac{F_t(T)}{K_i}-1\right)^2\right]$$

$$\implies \mathrm{VIX}_t^2 = \frac{20{,}000}{\eta}\left[\sum_i \left(\frac{\Delta K_i}{K_i^2}\right) e^{r(T-t)}\mathcal{O}_t(T,K_i) - \left(\frac{F_t(T)}{K_i}-1\right)^2\right]$$

Thus, when we refer to trading VIX_t^2 (shorting or longing), we specifically mean trading the replicating portfolio expressed above.

Now consider the following two cases:

- **Case 1:** If $\mathrm{VIX}_t^2 > f_t^{\mathrm{VA}}(t+\eta)$, then we can:
  - Short 1 unit of VIX_t^2
  - Long 1 unit of $f_t^{\mathrm{VA}}(t+\eta)$
  - Invest the difference $\left(\mathrm{VIX}_t^2 - f_t^{\mathrm{VA}}(t+\eta)\right) > 0$ at the risk-free rate
- **Case 2:** If $\mathrm{VIX}_t^2 < f_t^{\mathrm{VA}}(t+\eta)$, then we can:
  - Long 1 unit of VIX_t^2
  - Short 1 unit of $f_t^{\mathrm{VA}}(t+\eta)$
  - Invest the difference $\left(f_t^{\mathrm{VA}}(t+\eta) - \mathrm{VIX}_t^2\right) > 0$ at the risk-free rate

In both cases it is a zero cost portfolio and since we are suppose to have $\mathrm{VIX}_t^2 = f_t^{VA}(t+\eta)$, the terminal value will be strictly positive as we invested the difference at the risk free rate

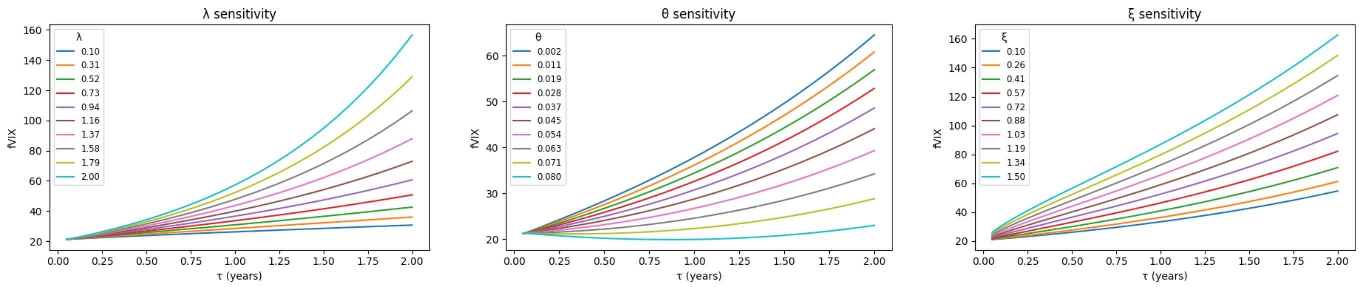
4.6 Numerical Computation of VIX Futures Prices

We compute the price of VIX futures as a function of the time to maturity, the current squared volatility V_t and the parameters (λ, θ, ξ) of the squared volatility process. Figure 2 illustrates the sensitivity of the VIX futures curve to variations in λ, θ, and ξ, while holding the spot variance fixed at $V_t = 0.044$. The parameters used for the base case are $\eta = \frac{30}{252}$, $\quad \lambda = 1.0, \quad \theta = 0.004, \quad \xi = 0.3$. The squared volatility of the SPX evolves according to:

$$dV_t = \lambda(\theta - V_t)\,dt + \xi\rho\sqrt{V_t}\,dB_t^{\mathbb{Q}} + \xi\sqrt{|1-\rho^2|}\sqrt{V_t}\,dZ_t^{\mathbb{Q}},$$

where $\lambda, \theta, \xi > 0$ and $\rho \in [-1, 1]$ are constants.

- **Effect of λ (mean reversion speed):** increasing λ steepens the curve significantly for large maturities. Although λ pulls V_t toward θ, it amplifies the convexity term more strongly, leading to higher VIX futures prices.
- **Effect of θ (long run variance):** increasing θ lowers the curve. Since the spot variance v_t is fixed at 0.044 (above θ), raising θ reduces the drift gap $(\theta - v_t)$, diminishing the convexity correction.
- **Effect of ξ (volatility diffusion coefficient):** increasing ξ shifts the curve upward almost linearly with τ. Greater variance volatility increases dispersion, this raises the expected square-root variance and thus the VIX futures price.

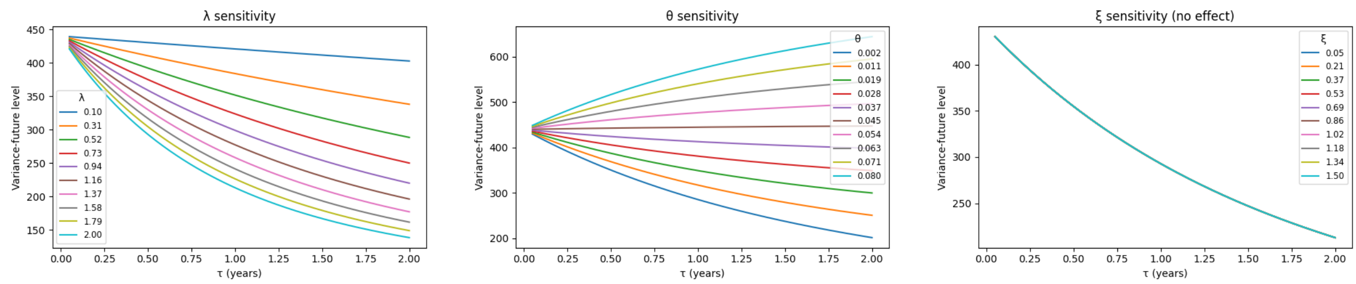


(a) λ sensitivity (b) θ sensitivity (c) ξ sensitivity

Figure 2: Sensitivity of the VIX futures price to the model parameters λ, θ, and ξ.

4.7 Numerical Computation of Variance Futures Prices

We compute the price of variance futures as a function of the time to maturity, the accrued variance $\int_{t_0}^{t}(100\sqrt{V_u})^2\,du$, the current squared volatility V_t and the parameters (λ, θ, ξ) of the squared volatility process. Figure 3 illustrates the sensitivity of the variance futures curve to variations in λ, θ, and ξ, while holding the spot variance fixed at $V_t = 0.044$. The parameters used for the base case are $\eta = \frac{30}{252}$, $\quad \lambda = 1, \quad \theta = 0.004, \quad$ accrued variance $= 0, \quad T - t_0 = T - t, \xi = 0.3$.



(a) λ sensitivity (b) θ sensitivity (c) ξ sensitivity

Figure 3: Sensitivity of the VA futures price to the model parameters λ, θ, and ξ.

Remember that:

$$f_t^{\text{VA}}(T) = \frac{10,000}{T - t_0}\left(\int_{t_0}^{t} V_u\,du + V_t \cdot \left(\frac{1 - e^{-\lambda(T-t)}}{\lambda}\right) + \theta \cdot \left((T - t) - \frac{1 - e^{-\lambda(T-t)}}{\lambda}\right)\right)$$

where $10,000\int_{t_0}^{t} V_u\,du$ is the accrued variance.

- **Effect of λ (mean reversion speed).** Larger λ lowers the curve. The weight on today's variance is $\frac{1-e^{-\lambda\tau}}{\lambda}\,v_t$, which decreases with λ, since $v_t > \theta$, its influence fades faster, pulling expected future variance down.
- **Effect of θ (long run variance).** The term $\theta\left(\tau - \frac{1-e^{-\lambda\tau}}{\lambda}\right)$ is linear in θ, increasing θ shifts the curve upward, nearly proportionally with maturity τ in the case when $\theta > V_t$. When $\theta < V_t$, it shifts the curve downward. This is because dV_t becomes negative when $\theta < V_t$.
- **Effect of ξ (volatility diffusion coefficient).** The pricing formula is affine in v_t and θ, with no diffusion term, so ξ does not enter: all curves lie exactly on top of each other.

4.8 Calibration of Volatility Model Parameters

By minimizing the root mean squared error (RMSE) in reconstructing the prices of f_t^{VA} and f_t^{VIX}, we obtained the following optimal parameters:

$$V_t = 0.0498, \quad \lambda = 1.2996, \quad \theta = 0.07054, \quad \xi = 0.2598.$$

The initial values for the optimization were selected via a grid search, and the estimation were carried out using the L-BFGS-B algorithm, we selected the initial parameters yielding the lowest RMSE.

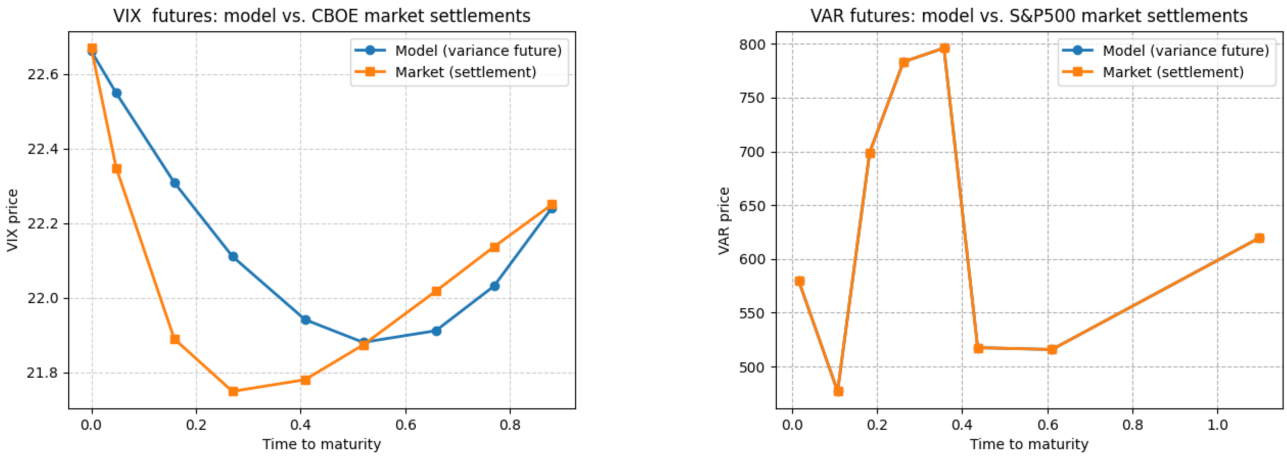


Figure 4: Models (VA or VIX) versus the Market after calibration

The fit for f_t^{VA} is nearly perfect, while the fit for f_t^{VIX} exhibits a rightward shift. This may suggest either the need for more data to improve the calibration of the VIX or a limitation in the numerical approximation of the integral involved in its computation, which can introduce additional reconstruction error.
The estimated parameters indicate a market environment characterized by moderately high current squared volatility (V_t), strong mean reversion (λ), a low long-run variance level (θ, with $V_t > \theta$), and a high volatility diffusion coefficient (ξ). These collectively reflect realistic volatility dynamics but also contribute to the difficulty of accurately replicating the VIX, due to the sensitivity of the model to integration accuracy.

4.9 Data Requirements for Calibrating Correlation

We have that

$$dV_u = \lambda(\theta - V_t)\,dt + \xi\rho\sqrt{V_t}\,dB_t^{\mathbb{Q}} + \xi\sqrt{1-\rho^2}\sqrt{V_t}\,dZ_t^{\mathbb{Q}}$$

and that

$$d\log S_s = (r-\delta)\,ds - \frac{1}{2}V_s\,ds + \sqrt{V_s}\,dB_s^{\mathbb{Q}},$$

In order to obtain the parameter ρ, we compute the correlation between a time series of the SPX level and a time series of V_t. We have direct access to the index level S_t. To build a matching series for V_t, we repeat the calibration procedure of the previous question on each business day, obtaining a sequence $\{V_{t_i}\}_{i=0}^{n}$. Using the data we collected, we can compute the correlation between the rates of change. Because ρ measures the instantaneous co movement of the random shocks in the model we must compare the changes in the two series.

$$r_i = \log S_{t_i} - \log S_{t_{i-1}}, \qquad \Delta V_i = V_{t_i} - V_{t_{i-1}}, \qquad i = 1, \ldots, n,$$

And the pearson correlation would be: $\rho = \dfrac{\sum_{i=1}^{n} r_i\,\Delta V_i}{\sqrt{\sum_{i=1}^{n} r_i^2}\,\sqrt{\sum_{i=1}^{n}(\Delta V_i)^2}}.$

Doing this will provide a discrete time estimate of the continuous time correlation parameter ρ.

4.10 Replication of an SPX Call Using VIX and Variance Futures

The implied volatility gives

$$\sigma_t(T,K) = \alpha(t,T,V_t) + \beta(t,T,V_t)\log\frac{K}{S_t} + \gamma(t,T,V_t)\left(\log\frac{K}{S_t}\right)^2$$

When the SPX call is at-the-money (ATM), $\log \frac{K}{S_0} = 0$ and hence the implied volatility simplifies to

$$\sigma_0(1, K) = \alpha(0, 1, V_0) = \alpha_0$$

We want a portfolio of SPX futures and variance futures matching those of the SPX call. To do so, we need to create a self-financing hedge that must cancel the option's sensitivities, in our case, there are: the index level S_t and the squared volatility V_t.
The Black–Scholes formula with $T = 1$ and ATM gives that for a SPX European call we have:

$$C_0 = e^{-\delta} S_0 \Phi(d_+) - e^{-r} K \Phi(d_-)$$

where

$$d_+ = \frac{(r - \delta + \frac{1}{2}\alpha_0^2)}{\alpha_0}, \quad d_- = d_+ - \alpha_0$$

This implies that

$$\Delta_0 = \frac{\partial C_0}{\partial S_0} = e^{-\delta}\Phi(d_+),$$

so we need to hold $e^{-\delta}\Phi(\frac{\alpha_0}{2} + \frac{(r-\delta)}{\alpha_0})$ SPX futures to neutralise the S_t sensitivity.
Now, to hedge the sensitivity to V_t, we have:

$$\frac{\partial C_0}{\partial V_0} = \frac{\partial C}{\partial \alpha_0}\frac{\partial \alpha_0}{\partial V_0}$$

Moroever, we have:

$$\text{Vega}_0 = \frac{\partial C_0}{\partial \alpha_0} = e^{-\delta} S_0 \varphi(d_+)\left(\frac{\delta - r}{\alpha_0^2} + \frac{1}{2}\right) - e^{-r} K \varphi(d_-)\left(\frac{\delta - r}{\alpha_0^2} - \frac{1}{2}\right)$$

So :

$$\frac{\partial C_0}{\partial V_0} = Vega_0 \frac{\partial \alpha_0}{\partial V_0}$$

But our exposure is V_t, and we need to hold future variance contract, hence we should inversely scale by the contribution of the variance future to the ∂V_t derivative. For a 1 year variance future we have:

$$f_0^{\mathrm{VA}}(1) = \frac{10000}{1 - t_0}\left(\int_{t_0}^{0} V_u \, du + V_0 \cdot \left(\frac{1 - e^{-\lambda(1-0)}}{\lambda}\right) + \theta \cdot \left((1 - 0) - \frac{1 - e^{-\lambda(1-0)}}{\lambda}\right)\right)$$

so one contract moves by

$$\frac{\partial f_0^{\mathrm{VA}}(1)}{\partial V_0} = 10\,000\left(\frac{1 - e^{-\lambda}}{\lambda}\right)$$

The replicating portfolio must cancel the exposure to its two sensitivites S_t and V_t. Hence, the hedging portfolio is composed of

$$n_S = \Delta_0 = e^{-\delta}\Phi\left(\frac{\alpha_0}{2} + \frac{(r - \delta)}{\alpha_0}\right), \qquad n_{\mathrm{VA}} = \frac{\frac{\partial C}{\partial V_0}}{\frac{\partial f^{\mathrm{VA}}}{\partial V_0}} = \frac{\lambda \cdot \text{Vega}_0 \frac{\partial \alpha_0}{\partial V_0}}{10\,000\big(1 - e^{-\lambda}\big)}.$$

Therefore, the replicating portfolio at time t is:

$$\boxed{e^{-\delta}\Phi\left(\frac{\alpha_0}{2} + \frac{(r - \delta)}{\alpha_0}\right) \text{ in SPX futures } + \frac{\lambda \cdot \text{Vega}_0 \frac{\partial \alpha_0}{\partial V_0}}{10\,000\big(1 - e^{-\lambda}\big)} \text{ in variance futures}}$$

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