3.6.4 Derivation of Taylor polynomials from first principles

Suppose we do not know the Taylor series for a given function $f(x)$ but wish to derive the first few Taylor polynomial approximations to $f(x)$ near $x = 0$. How do we find $T_0(x)$, $T_1(x)$, $T_2(x)$, …, $T_n(x)$? Determining these objects requires finding the numbers a_0, a_1, a_2, …. In order to do this we need to know not only the value of $f(x)$ at $x = 0$, namely $f(0)$, but also the values of the first n derivatives of $f(x)$ at $x = 0$. That is, we need to be given the values of $f(0)$, $f'(0)$, $f''(0) \equiv f^{(2)}(0)$, $f^{(3)}(0)$, …, $f^{(n)}(0)$. We will build each of the $T_i(x)$, $0 \le i \le n$, so that its function value and derivative at $x = 0$ match up to (and include) $f^{(i)}(0)$.

$f(x) \approx T_n(x)$

Solution:
We write

$$T_n(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n \tag{3.4}$$

where a_0, a_1, a_2, …, a_n are undetermined constants.

$x=0$, L.H.S $f(0)$
R.H.S $T_n(0) = a_0$

To find the constant a_0:
Put $x = 0$ in Equation (3.4). Therefore $T_n(0) = a_0 + 0 + 0 + 0 + \cdots + 0 = a_0$. We next insist that $T_n(0) = f(0)$. Hence $a_0 = f(0)$.

$T'_n(x) \approx f'(x)$ L.H.S $\underline{x=0}$

To find the constant a_1:
Differentiate Equation (3.4) with respect to x and the set $T_n^{(1)}(0) = f^{(1)}(0)$. ↳ $T'_n(0) = a_1$

$\frac{d}{dt} 2x$
$= 2$

$x=0$

$$T_n^{(1)}(x) = \frac{\mathrm{d}T_n}{\mathrm{d}x} = 0 + a_1 + \boxed{2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1}} \tag{3.5}$$

R.H.S: $f'(0)$

Therefore $T_n^{(1)}(0) = 0 + a_1 + 2a_2 0 + 3a_3 0^2 + \cdots + n a_n 0^{n-1} = a_1$. By insisting that $T_n^{(1)}(0) = f^{(1)}(0)$ we get $a_1 = f^{(1)}(0)$.

$T_n(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

To find the constant a_2:
Differentiate Equation (3.5) with respect to x and then set $T_n^{(2)}(0) = f^{(2)}(0)$. $T_n(x) \approx f(x)$

$T'_n(x) = a_1 + 2a_2 x + 3a_3 x^2$

$$T_n^{(2)}(x) = \frac{\mathrm{d}^2 T_n}{\mathrm{d}x^2} = 0 + 2 \times 1 a_2 x^0 + 3 \times 2 a_3 x^1 + \cdots + n(n-1) a_n x^{n-2} \tag{3.6}$$

$T^{(2)}_n(x) \approx f^2(x)$

$T^{(2)}_n(x) = 2a_2 + 6a_3 x$ $\boxed{x=0}$

$x=0$

Therefore $T_n^{(2)}(0) = 2 \times a_2 + 3 \times 2a_3 0 + \cdots + n(n-1) a_n 0^{n-2} = 2a_2$. By insisting that $T_n^{(2)}(0) = f^{(2)}(0)$ we get $a_2 = \frac{1}{2!} f^{(2)}(0)$.

$2a_2 \approx f^{(2)}(0)$

$a_2 = \frac{f^2(0)}{2}$

To find the constant a_3:
Differentiate Equation (3.6) with respect to x and then set $T_n^{(3)}(0) = f^{(3)}(0)$.

$$T_n^{(3)}(x) = \frac{\mathrm{d}^3 T_n}{\mathrm{d}x^3} = 0 + 3 \times 2 \times 1 a_3 x^0 + \cdots + n(n-1)(n-2) a_n x^{n-3} \tag{3.7}$$

Therefore $T_n^{(3)}(0) = 3 \times 2 \times 1a_3 + \cdots + n(n-1)(n-2)a_n 0^{n-3} = 3 \times 2 \times 1a_3$. By insisting that $T_n^{(3)}(0) = f^{(3)}(0)$ we get $a_3 = \frac{1}{3!} f^{(3)}(0)$.

Repeating this process n times, we find $a_n = \frac{1}{n!} f^{(n)}(0)$. Lastly, we substitute these a_i values into Equation (3.4), to obtain the Taylor polynomial of degree n centred at $x = 0$:

$$\begin{aligned} T_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \\ &= f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \cdots + \frac{f^{(n)}(0)}{n!} x^n. \end{aligned}$$

Knowing that $f(x) = \lim_{n \to \infty} T_n(x)$, this gives the Taylor series of $f(x)$ centred at $x = 0$ as

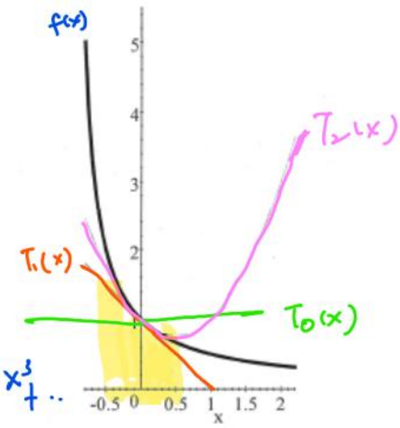
$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \cdots. \end{aligned}$$

$\boxed{\text{At } x=0}$ Given $f(x) \approx T_n(x)$

$f(x) \simeq T_n(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f^{(2)}(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \cdots + \frac{f^{(n)}(0)}{n!} x^n$

Example 3.28. (i) Derive the first four Taylor polynomials $T_0(x)$, $T_1(x)$, $T_2(x)$, $T_3(x)$ for the function $f(x) = \frac{1}{1+x}$ centred at $x = 0$.

Function	Value at $x = 0$
$f(x) = \frac{1}{1+x} = (1+x)^{-1}$	$f(0) = 1$
$f'(x) = -\frac{1}{(1+x)^2} = -(1+x)^{-2}$	$f'(0) = -1$
$f^{(2)}(x) = 2(1+x)^{-3}$	$f^{(2)}(0) = 2$
$f^{(3)}(x) = -6(1+x)^{-4}$	$f^{(3)}(0) = -6$



at $x=0$

$T_n(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \cdots$

$f(x) \simeq T_n(x) = 1 + \frac{(-1)}{1!} x + \frac{2}{2!} x^2 + \frac{(-6)}{3!} x^3 + \cdots$

$\frac{1}{1+x} \simeq 1 - x + x^2 - x^3 + \cdots$

$T_0(x) = 1$

$T_1(x) = 1 - x$

$T_2(x) = 1 - x + x^2$

$T_3(x) = 1 - x + x^2 - x^3$

(iii) $\dfrac{1}{1+3x} \approx 1-(3x)+(3x)^2-(3x)^3+\cdots$

$\dfrac{1}{1+3x} \approx 1-3x+9x^2-27x^3+\cdots$

(ii) Sketch $f(x)$, $T_1(x)$ and $T_2(x)$.

above the graph

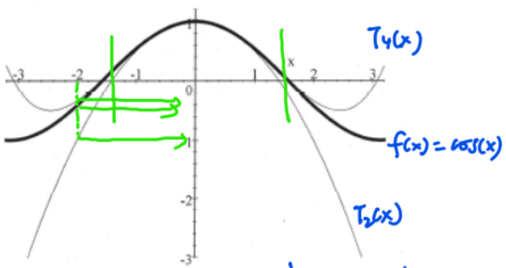
(iii) Deduce the Taylor polynomial of degree three for the function $g(x) = \dfrac{1}{1+3x}$ centred at $x = 0$.

Example 3.29. (i) Derive the Taylor polynomials $T_0(x)$, $T_2(x)$, $T_4(x)$ for the function $y = \cos x$ centred at $x = 0$. → power series

(ii) Using Mathematica (or otherwise) plot $y = \cos x$, as well as $T_0(x)$, $T_2(x)$, and $T_4(x)$ for the domain $-\pi \le x \le \pi$.

(iii) Deduce the Taylor polynomial of degree four for $y = \cos 3x$ centred at $x = 0$.

Function	Value at $x = 0$
$f(x) = \cos x$	$f(0) = \cos(0) = 1$
$f'(x) = -\sin x$	$f'(0) = 0$
$f^{(2)}(x) = -\cos x$	$f^{(2)}(0) = -1$
$f^{(3)}(x) = \sin x$	$f^{(3)}(0) = 0$
$f^{(4)}(x) = \cos x$	$f^{(4)}(0) = 1$



$\boxed{\text{At } x=0}$

$\cos(x) \approx T_n(x) = 1 + \frac{0}{1!}x + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$

$\cos(x) \approx 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots$

$T_0(x) = 1$

$T_2(x) = 1 - \frac{1}{2}x^2$

$T_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

$f(x) = \frac{1}{1+x}$

at $x=0$, $\frac{1}{1+x} \approx 1 - x + x^2 - x^3$

Example 3.30. Is it possible to have two different power series for the one function?

$f(x) = \frac{1}{1+x}$

$f'(x) = -(1+x)^{-2}$

$f^{(2)}(x) = 2(1+x)^{-3}$

$f^{(3)}(x) = -6(1+x)^{-4}$

$x=1$

$f(1) = \frac{1}{2}$

$f'(1) = -\frac{1}{4}$

$f^{(2)}(1) = \frac{1}{4}$

$f^{(3)}(1) = -\frac{3}{8}$

At $x = 1$

$\frac{1}{1+x} \approx \frac{1}{2} + \frac{(-\frac{1}{4})}{1!}(x-1) + \frac{(\frac{1}{4})}{2!}(x-1)^2 + \frac{(-\frac{3}{8})}{3!}(x-1)^3 + \cdots$

3.6.5 Taylor series centred at $x \neq 0$

In the previous section, we have been considering our Taylor polynomials and Taylor series centred at $x = 0$. This means that the Taylor polynomials $T_1(x), T_2(x), \ldots, T_n(x)$ of a function $f(x)$ will serve as good approximations to $f(x)$ at values near to $x = 0$. For example, the degree two Taylor polynomial $T_2(0.1)$ will serve as a good approximation of $f(0.1)$. On the other hand, $T_2(100)$ will serve as a lousy approximation to $f(100)$. Roughly speaking, this is because $T_n(x)$ is built up from information of the function $f(x)$ at $x = 0$ (namely, its derivatives at $x = 0$). So although you can evaluate $T_2(100)$, $T_2(x)$ does have a good idea about what is going on with $f(x)$ at $x = 100$. To circumvent this issue you could perhaps look at a higher degree Taylor polynomial, maybe $T_{50}(100)$. Now depending on the chosen function $f(x)$, this may be a good approximation to $f(100)$. But this requires us to calculate 50 derivatives! What if instead of centring our Taylor polynomials at $x = 0$, we can try centring it at some other number, say $x = c$, which is near our evaluation point? This leads us to a general Taylor polynomial centred at $x = c$

At $\boxed{x = c}$ $\quad T_n(x) \simeq f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f^{(2)}(c)}{2!}(x-c)^2 + \ldots$

$+ \ldots \frac{f^{(n)}(c)}{n!}(x-c)^n$

$$
\begin{aligned}
T_n(x; c) &= \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k \\
&= f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n.
\end{aligned}
$$

Hence, $T_n(x; 0) \equiv T_n(x)$. If $c = 0$

Taking the limit as $n \to \infty$ of $T_n(x; c)$, we get the Taylor series of f centred at $x = c$ as

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \\
&= f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots.
\end{aligned}
$$

Note that $f(x)$ is given by its Taylor series regardless of the centring value c (as long as the convergence occurs!). So, it's not particularly useful to look at Taylor series centred at values other than $x = 0$, unless $x = 0$ does not allow for convergence. Taylor polynomials centred at $x = c$ on the other hand are very useful. They allow us to obtain good approximations to f near $x = c$, and we usually only require a low order Taylor polynomial (often 2 or 3 suffices!).

$f(\frac{\pi}{2}) = 0$

$f'(\frac{\pi}{2}) = -1$

$f^{(2)}(\frac{\pi}{2}) = 0$

$f^{(3)}(\frac{\pi}{2}) = 1$

$f^{(4)}(\frac{\pi}{2}) = 0$

$f(x) = \cos x$

$f'(x) = -\sin x$

$f^2(x) = -\cos x$

$f^{(3)}(x) = \sin x$

$f^{(4)}(x) = \cos x$

Example 3.31. Derive the third degree Taylor polynomial of $f(x) = \cos(x)$, centred at $x = \pi/2$. That is, find $T_3(x; \pi/2)$. Use this to estimate $f(\frac{\pi}{2} + 0.1)$.

$\cos(x) \simeq 0 + \frac{-1}{1!}(x - \frac{\pi}{2}) + \frac{0}{2!}(x - \frac{\pi}{2})^2 + \frac{1}{3!}(x - \frac{\pi}{2})^3$

$\cos(x) \simeq T_3(x) = -(x - \frac{\pi}{2}) + \frac{1}{6}(x - \frac{\pi}{2})^3$

$\cos(\frac{\pi}{2} + 0.1) \simeq -(0.1) + \frac{1}{6}(0.1)^3 \simeq -0.09983$

Actual

$\cos(\frac{\pi}{2} + 0.1)$

$\simeq -0.09983$

$\Rightarrow$ good approximation $\Rightarrow$ cubic approximation

3.6.6 Cubic splines interpolation

Power series (and Taylor series) provide us with a method of approximating values of a function at some particular point. When we construct Taylor polynomials we get a higher level of accuracy when we use a higher degree polynomial. Suppose we are asked to evaluate a function at a particular point $x = c$ (for example finding the zeros of a function, or the intersection point of two functions) but we cannot use algebraic methods (such as the quadratic formula) to solve such a problem. What if we are not even given the function? What can we do? Lucky for us there are methods available that will give a good **approximation** of the solution.

$\frac{1}{1+x} = \underbrace{1 - x + x^2}.$

Algebraic methods will give us exact solutions, but the algebraic methods may not be simple to use. $f(x) = ax^2 + bx + c$

Numerical methods will give us an approximation, but are much easier to use.

Polynomial interpolation

Suppose we are given a set of data points $(x_i, f(x_i))$ (note that we do not have the function $f(x)$ explicitly given to us) and we want to build a function that *approximates* $f(x)$ with as much continuity as we can get. What do we do? We will introduce a method of **interpolation** to do this. We are essentially going to find a curve which best fits the data given. In science and engineering, numerical methods often involve curve fitting of experimental data.

Polynomial interpolation is simply a method of estimating the values of a function, between known data points. Thus linear interpolation would use two data points, quadratic interpolation would use three data points, and so on.

As an example, suppose you are asked to evaluate a function $f(x)$ at $x = 3.4$ but all you are given is the following table of data points.

3.4

x	0.000	1.200	2.400	3.600	4.000	6.000	7.000
$f(x)$	0.000	0.932	0.675	−0.443	−0.757	−0.279	0.657

Since $x = 3.4$ is not in the table, the best we can do is find an estimate of $f(3.4)$. We could construct a straight line built on the two points either side of $x = 3.4$ namely $(2.400, 0.675)$ and $(3.600, -0.443)$. Or we could build a quadratic based on any three points (that cover $x = 3.4$). If we were really keen we could build a cubic by selecting four points around $x = 3.4$. All we are doing is simply using a set of points near the target point ($x = 3.4$) to build a polynomial. Then we estimate $f(x)$ by evaluating the polynomial at the target point. This process is called **polynomial interpolation**.

This method will give us a *unique* polynomial for our approximation of the function $f(x)$. This is fine but we know that our accuracy is likely to decline as we get further away from our target point. What else can we do?

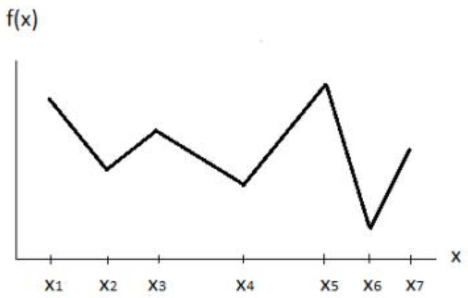
Piecewise polynomial interpolation: Cubic splines

Instead of trying to find *one* polynomial to fit our data points, what if we take sections of the data, and fit polynomials to each section, ensuring that the overall piecewise function is continuous. We would also like differentiability, but let us not be too picky for now.

The simplest polynomial to use is a *linear approximation*. This will produce a path that consists of line segments that pass through the points of the data set. The resulting **linear spline** function can be written as a piecewise function. Unfortunately though, we do not usually have continuity of the first derivatives at the data set points.

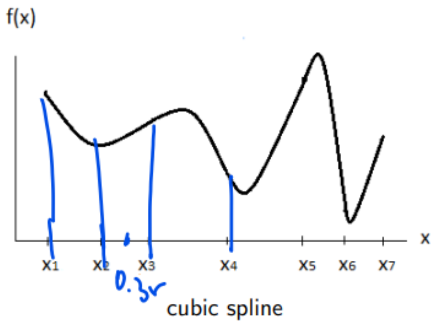
$f(x) = 2x+3.$

$f'(x) = 2$



linear spline

This technique can be easily extended to higher order polynomials. If we take piecewise *quadratic* polynomials, we can get continuity of the first derivatives, but the second derivatives will have discontinuities. If we take piecewise *cubic* polynomials, then we can make (with some work) both the first and second derivatives continuous.



cubic spline

$y'(0.35) \simeq \tilde{y}(x)$ appro

Suppose we are given a simple dataset and we are asked to estimate the derivative at say $x = 0.35$ How do we proceed? Here is one approach.

Construct, by whatever means, a smooth approximation $\tilde{y}(x)$ to $y(x)$ near $x = 0.35$. Then put $y'(0.35) \approx \tilde{y}'(0.35)$.

What we want is a method which

- Produces a unique approximation,
- Is continuous over the domain and
- Has, at least, a continuous first derivative over the domain.

Let us say we have a set of $n+1$ data points (x_i, y_i), $i = 0, 1, 2, \ldots, n$ and we wish to build an approximation $\tilde{y}(x)$ which has as much continuity as we can get.

Between each pair of points we will construct a cubic. Let $\tilde{y}_i(x)$ be the cubic function for the interval $x_i \le x \le x_{i+1}$, for $i = 0, 1, 2, \ldots, n-1$. We demand that the following conditions are met

- Interpolation condition

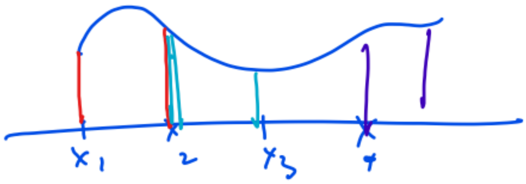
$$y_i = \tilde{y}_i(x_i) \tag{1}$$

- Continuity of the function

$$\tilde{y}_{i-1}(x_i) = y_i \tag{2}$$

- Continuity of the first derivative

$$\tilde{y}'_{i-1}(x_i) = \tilde{y}'_i(x_i) \tag{3}$$



- Continuity of the second derivative

$$\tilde{y}''_{i-1}(x_i) = \tilde{y}''_i(x_i) \tag{4}$$

Can we solve this system of equations? We need to balance the number of unknowns against the number of equations. We have $n+1$ data points and thus n cubics to compute. Each cubic ($f(x) = ax^3 + bx^2 + cx + d$) has 4 coefficients, thus we have $4n$ unknowns. And how many equations? From the above we count n equations in each of (1) and (2), and $n-1$ equations in each of (3) and (4). A total of $4n-2$ equations for $4n$ unknowns. We see that we will have to provide *two* extra pieces of information. For now let us press on, and see what comes up.

4 data points → 4 pair of coordinates.

We start by putting

$i=1$ $\tilde{y}_1(x) = y_1 +$
$i=2$ $\tilde{y}_2(x) = y_2$
$i=3$ $\tilde{y}_3(x) = y_3$

$$\tilde{y}_i(x) = y_i + a_i(x - x_i) + b_i(x - x_i)^2 + c_i(x - x_i)^3 \tag{5}$$

$\tilde{y}_i' = y_i' + a_1 + 2b_i(x-x_i) + 3c_i(x-x_i)^2$

which automatically satisfies equation (1). For the moment suppose we happen to know all of the second derivatives $\tilde{y}''_i(x)$. We then have $\tilde{y}''_i(x) = 2b_i + 6c_i(x - x_i)$ and evaluating this at $x = x_i$ leads to

$\tilde{y}_i'(x) = y_1 +$

$\tilde{y}_i''(x) = y_i'' + 2b_i + 6c_i(x-x_i)$

$$\boxed{b_i = y''_i/2,} \tag{6}$$

where we have introduced the shorthand notation

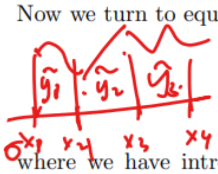
$\tilde{y}_i' = y_i' + a_i + 2b_i(x-x_i) + 3c_i(x-x_i)^2$
$\tilde{y}_i' = y_i' + 2b_i + 6c_i(x-x_i)$ → x_{i+1}

$y_1'' = y_4'' = 0$ $\quad$ $i=1$, $c_i = 0$ $\quad$ $y_0'' = y_3'' = 0$

$$y''_i = \begin{cases} \tilde{y}''_i(x_i), & i = 0, 1, 2, \ldots, n-1, \\ \tilde{y}''_{n-1}(x_n), & i = n. \end{cases}$$

$\tilde{y}_i''$

Now we turn to equation (4) $y''_{i+1} = y''_i + 6c_i(x_{i+1} - x_i)$ which gives



$$c_i = (y''_{i+1} - y''_i)/(6h_i) \tag{7}$$

where we have introduced $h_i = x_{i+1} - x_i$. Next we compute the a_i by applying equation (2),

$\tilde{y}_1 = y_1 + 3(x-1) + \frac{1}{2}(x-1)^2 + 5(x-1)^3 \quad 0 \le x \le 1$

$\tilde{y}_2 = 0 + \frac{1}{4}(x-3) - \frac{2}{3}(x-3)^2 - \frac{1}{3}(\quad 1 < x < 3.$

$$y_{i+1} = y_i + a_i h_i + \frac{1}{6}(y''_{i+1} + 2y''_i)h_i^2 \tag{8}$$

and so

$$a_i = \frac{y_{i+1} - y_i}{h_i} - \frac{1}{6}h_i(y''_{i+1} + 2y''_i). \tag{9}$$

It appears that we have completely determined each of the cubics, though we are yet to use (3), continuity in the first derivative. But remember that we don't yet know the values of y''_i. Thus equation (3) will be used to compute the y''_i. Using our values for a_i, b_i and c_i we find (after much fiddling) that equation (3) is

$$6\left(\frac{y_{i+1} - y_i}{h_i} - \frac{y_i - y_{i-1}}{h_{i-1}}\right) = h_i y''_{i+1} + 2(h_i + h_{i-1})y''_i + h_{i-1}y''_{i-1}. \tag{10}$$

The only unknowns in this equation are the y''_i of which there are $n+1$. But there are only $n-1$ equations. Thus we must supply two extra pieces of information.

The simplest choice is to set $y''_0 = y''_n = 0$. Then we have a tridiagonal system of equations[2] to solve for y''_i. That's as far as we need push the algebra – we can simply now use technology (such as Matlab, Mathematica, Wolfram Alpha...) to solve the tridiagonal system.

The recipe

- Solve equation (10) for y''_i,
- Compute all of the a_i from equation (9),
- Compute all of the b_i from equation (6),
- Compute all of the c_i from equation (7) and finally
- Assemble all of the cubics using equation (5).

Our job is done. We have computed the **cubic spline** for the our set of data points.

Example 3.32. Let us say we are given the set of data points in the following table. Find the cubic spline that best fits this data.

[2] Often a system of equations will give a coefficient matrix of a special structure. A tridiagonal system of equations is one such that the coefficient matrix has zero entries everywhere except for in the main diagonal and in the diagonals above and below the main diagonal.

i	0	1	2	3
x_i	-2	-1	1	3
$f(x_i)$	3	0	2	1

We are going to use equation 5 to give three cubics, $\tilde{y}_0(x)$, $\tilde{y}_1(x)$ and $\tilde{y}_2(x)$. Recall

$$\tilde{y}_i(x) = y_i + a_i(x - x_i) + b_i(x - x_i)^2 + c_i(x - x_i)^3 \tag{5}$$

From the data points we have $x_0 = -2$, $x_1 = -1$, $x_2 = 1$, $x_3 = 3$, $y_0 = 3$, $y_1 = 0$, $y_2 = 2$ and $y_3 = 1$.

We also know that $y_0'' = y_3'' = 0$.

Putting this information into equation 10 we obtain the following two equations

$$\begin{array}{lrcl} \text{When } i = 1 & 24 & = & 6y_1'' + 2y_2'' \\ \text{When } i = 2 & -9 & = & 2y_1'' + 8y_2'' \end{array}$$

Solving this system of equations we find that $y_1'' = \dfrac{105}{22}$ and $y_2'' = -\dfrac{51}{22}$.

The use of equation 9 will give us $a_0 = -\dfrac{167}{44}$, $a_1 = -\dfrac{31}{22}$ and $a_2 = \dfrac{23}{22}$.

Equation 6 gives $b_0 = 0$, $b_1 = \dfrac{105}{44}$ and $b_2 = -\dfrac{51}{44}$.

Equation 7 gives $c_0 = \dfrac{35}{44}$, $c_1 = -\dfrac{13}{22}$ and $c_2 = \dfrac{17}{88}$.

Now using equation 5 we produce the following three cubic polynomials, which determine the cubic spline:

$$\tilde{y}_0(x) = 3 - \frac{167}{44}(x+2) + \frac{35}{44}(x+2)^3 \text{ for } -2 \le x < -1$$

$$\tilde{y}_1(x) = -\frac{31}{22}(x+1) + \frac{105}{44}(x+1)^2 - \frac{13}{22}(x+1)^3 \text{ for } -1 \le x < 1$$

$$\tilde{y}_2(x) = 2 + \frac{23}{22}(x-1) - \frac{51}{44}(x-1)^2 + \frac{17}{88}(x-1)^3 \text{ for } 1 \le x \le 3$$

Example 3.33. For the above example, check that the four conditions (1, 2, 3 and 4) are met.

Example 3.34. Compute the cubic spline that passes through the following $f(x_i)$ data set points

$y_1'' = y_4'' = 0$

i	x_i	y_i	$h_i = x_{i+1} - x_i$
1	0	0	$h_1 = x_2 - x_1 = 1$
2	1	0.5	$h_2 = x_3 - x_2 = 1$
3	2	2	$h_3 = x_4 - x_3 = 1$
4	3	1.5	

x	0	1	2	3
$f(x)$	0	0.5	2	1.5

Eq 10 $\quad 6\left(\frac{y_{i+1}-y_i}{h_i} - \frac{y_i - y_{i-1}}{h_{i-1}}\right) h_i y''_{i+1} + 2(h_i + h_{i-1}) y_i'' + h_{i-1} y''_{i-1}$

$\boxed{i=2}$ $\quad 6\left(\frac{y_3 - y_2}{h_2} - \frac{y_2 - y_1}{h_1}\right) = h_2 y_3'' + 2(h_2 + h_1) y_2'' + h_1 y_1''$

$$6\left(\frac{2-0.5}{1} - \frac{0.5-0}{1}\right) = (1) y_3'' + 2(1+1) y_2'' + h_1 (0)$$

$$6 = y_3'' + 4y_2'' \quad \text{— ①}$$

$\boxed{i=3}$ $\quad -12 = 4y_3'' + y_2'' \quad$ — ②

solve ① & ② $\quad y_2'' = \frac{12}{5}, \; y_3'' = -\frac{18}{5}$

Eq 9 $\quad a_i = \frac{y_{i+1} - y_i}{h_i} - \frac{1}{6} h_i (y''_{i+1} + 2y_i'')$

$\boxed{i=1}$ $\quad a_1 = \frac{0.5}{1} - \frac{1}{6}(1)\left(\frac{12}{5} + 2(0)\right) = \frac{1}{10}$

$\boxed{i=2}$ $\quad a_2 = \frac{3}{2} - \frac{1}{6}(1)\left(-\frac{18}{5} + 2\left(\frac{12}{5}\right)\right) = \frac{13}{10}$

$\boxed{i=3}$ $\quad a_3 = -\frac{1}{2} - \frac{1}{6}(1)\left(0 + 2\left(-\frac{18}{5}\right)\right) = \frac{7}{10}$

Eq 6 $\quad b_i = \frac{y_i''}{2}$

$\boxed{i=1}$ $\quad b_1 = \frac{y_1''}{2} = \frac{0}{2} = 0$

$\boxed{i=2}$ $\quad b_2 = \frac{y_2''}{2} = \frac{6}{5}$

$\boxed{i=3}$ $\quad b_3 = \frac{y_3''}{2} = -\frac{9}{5}$

Eq 7 $\quad c_i = \frac{y''_{i+1} - y_i''}{6h_i}$

$\boxed{i=1}$ $\; c_1 = 2/5 \qquad \boxed{i=2}$ $\; c_2 = -1 \qquad \boxed{i=3}$ $\; c_3 = 3/5$

Cubic Spline function $\boxed{EQ5}$ $\quad \tilde{y}_i = y_i + a_i(x - x_i) + b_i(x - x_i)^2 + c_i(x - x_i)^3$

$\tilde{y}_1 = 0 + \frac{1}{10}(x-0) + 0(x-0)^2 + \frac{2}{5}(x-0)^3 \iff \tilde{y}_1 = \frac{1}{10}x + \frac{3}{5}x^3$

$\tilde{y}_2 = \frac{1}{2} + \frac{13}{10}(x-1) + \frac{6}{5}(x-1)^2 + (-1)(x-1)^3 \iff \tilde{y}_2 = \frac{1}{2} + \frac{13}{10}(x-1) + \frac{6}{5}(x-1)^2 - (x-1)^3$

$\tilde{y}_3 = 2 + \frac{7}{10}(x-2) + \left(-\frac{9}{5}\right)(x-2)^2 + \left(\frac{3}{5}\right)(x-2)^3$

$\iff \tilde{y}_3 = 2 + \frac{7}{10}(x-2) - \frac{9}{5}(x-2)^2 + \frac{3}{5}(x-2)^3$