# Week 10 Tutorial Sheet

(To be completed during the Week 10 tutorial class)

**Objectives:** The tutorials, in general, give practice in problem solving, in analysis of algorithms and data structures, and in mathematics and logic useful in the above.

**Instructions to the class:** You should actively participate in the class. The preparation problems are not assessed. We strongly recommend that everyone tries to solve the preparation problems before the tutorial as you will benefit the most from the tutorial if you come properly prepared. However, you can still attend the tutorial if you have not solved those problems beforehand.

**Instructions to Tutors:** The purpose of the tutorials is not to solve the practical exercises! The purpose is to check answers, and to discuss particular sticking points, not to simply make answers available.

**Supplementary problems:** The supplementary problems provide additional practice for you to complete after your tutorial, or as pre-exam revision. Problems that are marked as **(Advanced)** difficulty are beyond the difficulty that you would be expected to complete in the exam, but are nonetheless useful practice problems as they will teach you skills and concepts that you can apply to other problems.

## Tutorial Problems

**Problem 1.** **(Preparation)** Consider the following circulation with demands problem presented in Figure 1. Does it have a feasible solution?

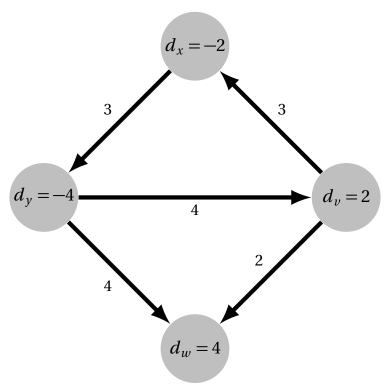



Figure 1: An instance of the circulation with demands problem. The demand is indicated in each vertex, and the capacity in each edge.

**Problem 2.** Consider the following circulation with demands and lower bounds problem presented in Figure 2. Does it have a feasible solution?

**Problem 3.** Consider the following circulation with demands and lower bounds problem presented in Figure 3. Does it have a feasible solution?

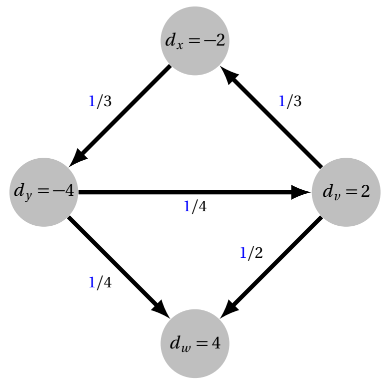



Figure 2: An instance of the circulation with demands and lower bounds problem. The demand is indicated in each vertex, and in each edge its capacity is in black and its lower bound in blue.

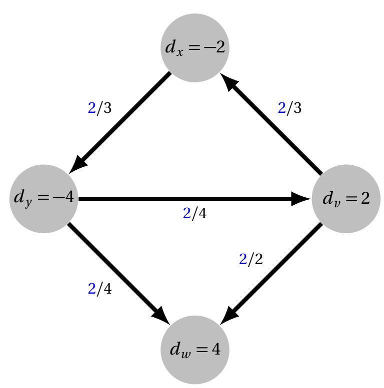



Figure 3: An instance of the circulation with demands and lower bounds problem. The demand is indicated in each vertex, and in each edge its capacity is in black and its lower bound in blue.

**Problem 4.** Recall the concept of a directed acyclic graph (DAG). A *path cover* of a DAG is a set of paths that include every vertex **exactly once** (multiple paths can not touch the same vertex). Note that path covers can include paths of length 0 (i.e., paths that contain a single vertex but no edge at all). A minimum path cover is a path cover consisting of as few paths as possible. Devise an efficient algorithm for finding a minimum path cover of a DAG. [Hint: Use bipartite matching]

**Problem 5.** You are in charge of projects at a large company and need to decide for this year, which projects the company will undertake. Each project has a profit value associated with it. Some projects are worth positive profit, meaning you earn money from completing them. Some projects are worth negative profit, meaning they cost more money than they make. Projects have prerequisites with other projects, meaning that a project can only be completed once all of its prerequisites have been completed. The goal is to determine which projects to complete in order to make the maximum amount of profit, while ensuring that all prerequisite relationships are satisfied. This problem can be solved by reducing it to a minimum cut problem as follows:

- Create a flow network with a source vertex $s$ and a sink vertex $t$.
- Create a vertex for each project.
- For each project $x$ with positive profit $p$, add a directed edge from $s$ to $x$ whose weight is $p$.
- For each project $x$ with negative profit $-p$, add a directed edge from $x$ to $t$ whose weight is $p$.
- For each project $x$ that has $y$ as a prerequisite, add a directed edge from $x$ to $y$ with weight $\infty$.

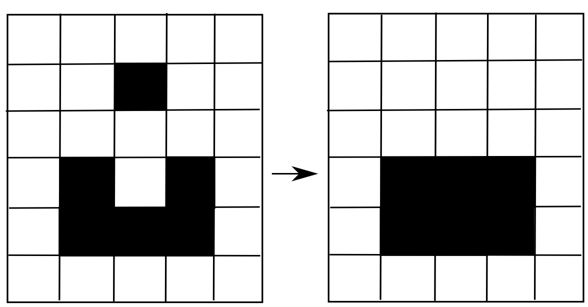
The minimum cut of this network can be used to determine the optimal set of projects to complete.

(a) Explain what the components of the minimum $s-t$ cut correspond to in the optimal solution.

(b) What is the purpose of the infinite capacity edges?

(c) Explain how to compute maximum profit from the capacity of the minimum $s-t$ cut.

**Problem 6.** You are the owner of a plot of land that can be described as an $n \times m$ grid of unit-square-sized cells. Each cell of land is either filled in, or a hole. The government has decided to crack down on safety regulations and requires you to fence off the holes in the ground. Each unit of fencing will cost you $\$c_{\text{fence}}$. In order to reduce the amount of fencing required, you have the option to fill in some of the holes, or even dig some additional holes. It will cost you $\$c_{\text{dig}}$ to dig a new hole, or $\$c_{\text{fill}}$ to fill in an existing hole. For safety, there are no holes on the boundary cells of your land, and you are not permitted to make any. For example, in the following case (where black represents a hole), with $c_{\text{fence}} = 10$, $c_{\text{dig}} = 10$, and $c_{\text{fill}} = 30$, the cost to fence the land initially would be \$160. The optimal solution is to dig out the middle cell and fill in the topmost hole, yielding a total cost of \$140.



Describe how to determine the minimum cost to make your land safe by reducing it to a minimum cut problem.

**Problem 7.** A useful application of maximum flow to people interested in sports is the *baseball elimination* problem. Consider a season of baseball in which some games have already been played, and the schedule for all of the remaining games is known. We wish to determine whether a particular team can possibly end up with the most wins or at least tied for the most wins. For example, consider the following stats.

| | Wins | Games Left |
|---|---|---|
| Team 1 | 30 | 5 |
| Team 2 | 28 | 10 |
| Team 3 | 26 | 8 |
| Team 4 | 20 | 9 |

It is trivial to determine that Team 4 has no chance of winning, since even if they win all 9 of their remaining games, they will be at least one game behind Team 1. However, things get more interesting if Team 4 can win enough games to reach the current top score.

| | Wins | Games Left |
|---|---|---|
| Team 1 | 30 | 5 |
| Team 2 | 28 | 10 |
| Team 3 | 29 | 8 |
| Team 4 | 20 | 11 |

In this case, Team 4 can reach 31 wins, but it doesn't matter since the other teams have enough games left that one of them must reach 32 wins. We can determine the answer with certainty if we know not just the number of games remaining, but the exact teams that will play in each of them. A complete schedule consists of the number of wins of each team, the number of games remaining, and for each remaining game, which team they will be playing against. An example schedule might look like the following.

| | | Games Remaining | | | | |
|---|---|---|---|---|---|---|
| | Wins | Total | vs T1 | vs T2 | vs T3 | vs T4 |
| Team 1 | 29 | 5 | 0 | 2 | 1 | 2 |
| Team 2 | 28 | 10 | 2 | 0 | 4 | 4 |
| Team 3 | 28 | 8 | 1 | 4 | 0 | 3 |
| Team 4 | 25 | 9 | 2 | 4 | 3 | 0 |

Describe an algorithm for determining whether a given team can possibly end up with the most wins or at least tied for the most wins. Your algorithm should make use of maximum flow.

**Problem 8.** You work on a company that provides essential services and needs to have one employee on duty each Saturday and Sunday. You got the results of a Doodle showing, for each employee $e_i$, which weekend days they are available to work during the next couple of weekends. You also know the maximum amount $m_i$ of weekend days that each employee $e_i$ is allowed to work during that time. And for each employee $e_i$, your team has agreed that it should work at least $\ell_i$ of those days to achieve a fair distribution of the workload.

(a) How can you figure out if there is an allocation that has exactly one employee on duty each Saturday and each Sunday, and that meets the constraints above?

(b) Suppose that stronger employment regulations are been approved, and for each weekend, each employee will only be allowed to work at most one day. How can you solve this version of the problem?

## Supplementary Problems

**Problem 9.** Implement the algorithm based on Ford-Fulkerson to solve the problem of circulation with demands and lower bounds.