$y \xrightarrow{\text{Differentiation}} \frac{dy}{dx}$

$\xleftarrow{}$ Anti-derivative (integration)

Notation

$\int \boxed{}\, dx$

function / expression

let $\frac{d}{dx} F(x) = f(x)$

$\longleftarrow \int f(x) = F(x) + c$, c: constant

(a) $\frac{d}{dx} x^2 = 2x$

(b) $\frac{d}{dx} (x^2 - 30) = 2x$

(c) $\frac{d}{dx} (x^2 + \frac{1}{5}) = 2x$

(a) $\int 2x\, dx = x^2$?

(b) $\int 2x\, dx = x^2 - 30$

(c) $\int 2x\, dx = x^2 + \frac{1}{5}$

conclusion

$\int 2x\, dx = x^2 + c$

Chapter 4

Integration

$\boxed{\text{Indefinite Integral}} \rightarrow +c$

4.1 Fundamental theorem of calculus

4.1.1 Revision

Computing the **indefinite integral** $I = \int f(x)dx$ is no different from finding a function $F(x)$ such that $\frac{dF}{dx} = f(x)$. Thus $\int \frac{dF}{dx}dx = F(x)$. The function $F(x)$ is called an **anti-derivative** of $f(x)$.

You should recall some of the basic integrals.

Recall $\frac{d}{dx} x^n = n x^{n-1}$

$$\int k dx = kx + C, \text{ where } C \in \mathbb{R}$$

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$$

$$\int \sin(x)dx = -\cos(x) + C$$

$$\int \cos(x)dx = \sin(x) + C$$

$$\int e^x dx = e^x + C$$

$$\int \frac{1}{x}dx = \ln|x| + C$$

$\frac{d}{dx} \cos(x) = -\sin(x)$

$\frac{d}{dx} \sin(x) = \cos(x)$

$\frac{d}{dx} e^x = e^x$

$\frac{d}{dx} \log_e x = \frac{1}{x}$

Proof:

$\frac{d}{dx} x^{n+1} = (n+1)x^n$

$\left(\frac{1}{n+1}\right) \frac{d}{dx} x^{n+1} = x^n$

Take integration to both side, wrt x

$\int \left[\left(\frac{1}{n+1}\right) \cancel{\frac{d}{dx}} x^{n+1}\right] \cancel{dx} = \int x^n\, dx$

$\frac{x^{n+1}}{n+1} = \int x^n\, dx$

Recall also the properties of indefinite integrals:

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\int [f(x) - g(x)]\, dx = \int f(x)dx - \int g(x)dx$$

$$\int kf(x)dx = k \int f(x)dx \text{ for any constant } k$$

There are also a few tricks we can use to find $F(x)$, such as *integration by substitution* and *integration by parts*.

Integration by substitution

If $I = \int f(x)dx$ looks nasty, try changing the variable of integration. That is, put $u = u(x)$ for some chosen function $u(x)$, then invert the function to find $x = x(u)$ and substitute into the integral.

Hint:
Let u = hidden function
(inside brackets)

$$I = \int f(x)dx = \int f(x(u))\frac{dx}{du}du$$

If we have chosen well, then this second integral will be easy to do.

Example 4.1. Find $\int 4x \cos(x^2 + 5)dx$ $\longrightarrow \int [\cos(u)][2du]$

Let $u = x^2+5$

$\frac{dy}{dx} = 2x$

$du = 2x\, dx$

$(\times 2)$ $2du = 4x\, dx$

$= 2\int \cos(u)\, du$

$= 2\sin(u) + c$

$\int 4x \cos(x^3+5)\, dx = 2\sin(x^2+5) + c$

Integration by parts

This is a very powerful technique based on the **product rule** for derivatives. Recall that

$$\frac{d(fg)}{dx} = g\frac{df}{dx} + f\frac{dg}{dx}$$

Now integrate both sides

$$\cancel{\int} \frac{\cancel{d}(fg)}{\cancel{dx}}\cancel{dx} = \int g\frac{df}{dx}dx + \int f\frac{dg}{dx}dx$$

But integration is the inverse of differentiation, thus we have

$$fg = \int g\frac{df}{dx}dx + \int f\frac{dg}{dx}dx$$

which we can re-arrange to

$$\boxed{\int f\frac{dg}{dx}dx = fg - \int g\frac{df}{dx}dx}$$

Thus we have converted one integral into another. The hope is that the second integral is easier than the first. This will depend on the choices we make for f and $\frac{dg}{dx}$.

Example 4.2. Find $\int xe^x dx$.

① ②

Solution: We have to split the integrand xe^x into two pieces, f and $\frac{dg}{dx}$.

If we choose $f(x) = x$ and $\frac{dg}{dx} = e^x$ then $\frac{df}{dx} = 1$ and $g(x) = e^x$.

Then

Let $u = x$ $\quad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = 1 \quad v = e^x$

$$\begin{aligned}\int xe^x dx &= fg - \int g\frac{df}{dx}dx \\ &= xe^x - \int 1\cdot e^x dx \\ &= xe^x - e^x + C\end{aligned}$$

$\int xe^x dx = \int (xe^x - e^x)\, dx$

Example 4.3. Find $\int x\cos(x)dx$.

let $u = dx$ $\quad \frac{dv}{dx} = \cos(ux)dx$

$v =$

Solution: Choose $f(x) = x$ and $\frac{dg}{dx} = \cos(x)$ then $\frac{df}{dx} = 1$ and $g(x) = \sin(x)$.

Then

$$\begin{aligned}\int x\cos(x)dx &= fg - \int g\frac{df}{dx}dx \\ &= x\sin(x) - \int 1\cdot \sin(x)dx \\ &= x\sin(x) + \cos(x) + C\end{aligned}$$

Example 4.4. Find $\int x \sin(x) dx$

Let $u = \cancel{\sin(x)}$ $\quad \frac{dv}{dx} = x$

$u = x$ $\quad \frac{dv}{dx} = \sin x$

$\frac{du}{dx} = 1$ $\quad v = -\cos x$

$\int x \sin(x) = -x\cos x + \int \cos x \, dx$

$= -x \cos x + \sin x + c$

Concern about ⇒ order of the integration

Differentiation Compares file.

① LIATE.

4.1.2 Fundamental Theorem of Calculus

The **Fundamental Theorem of Calculus** states that:

If $f(x)$ is a continuous function on the interval $[a, b]$ and there is a function $F(x)$ such that $F'(x) = f(x)$, then

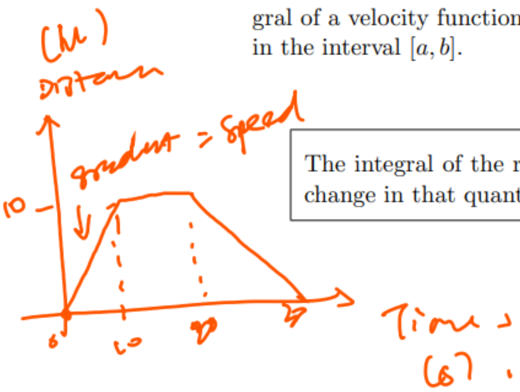
$$\int_a^b f(x)dx = F(b) - F(a)$$

Indefinite integral

$\int f(x) \, dx = F(x) + c$

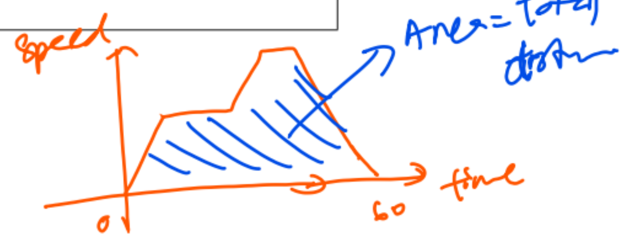
Note that $\int_a^b f(x)dx$ is known as the **definite integral** from a to b as we are integrating the function $f(x)$ between the values $x = a$ and $x = b$.

> Can we interpret this theorem in some physical way? Of course! Let $s(t)$ be a continuous function which gives the **position** of a moving object at time t where t is in the interval $[a, b]$. We know that $s'(t)$ gives the **velocity** of the object at time t, and we want to know what is the meaning of $\int_a^b s'(t)dt$.
>
> Recall that distance = velocity × time. Thus for any small interval Δt in $[a, b]$ we have $s'(t) \times \Delta t \approx$ distance travelled in Δt. Adding each successive calculation of the distance travelled for the small intervals of time Δt from $t = a$ to $t = b$ will give us (approximately) the **total distance travelled** over the interval $[a, b]$.
>
> Integrating the velocity function $s'(t)$ over the interval $[a, b]$ will then give us the **total distance travelled** over the interval $[a, b]$. Thus the definite integral of a velocity function can be interpreted as the total distance travelled in the interval $[a, b]$.



The integral of the rate of change of any quantity gives the total change in that quantity.

$\frac{\text{speed}}{\text{time}}$ = acceleration



4.2 Area under the curve

When $f(x)$ is a positive function and $a < b$ then the definite integral

$$\int_a^b f(x)dx$$

gives the area between the graph of the function $f(x)$ and the x - axis. In other words

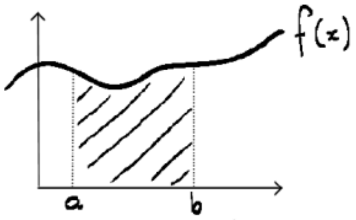
$$\int_a^b f(x)dx = A$$

$\int_a^b f(x)\,dx = [F(x)]_a^b$ or $F(x)\Big|_a^b$

$= F(b) - F(a)$

Properties of Definite Integrals

* $\int_a^b f(x)dx = -\int_b^a f(x)dx$



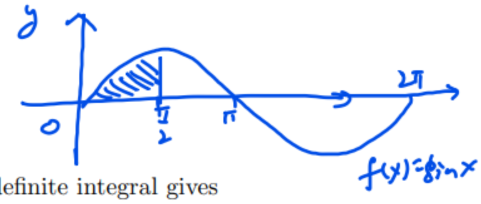
recommended to sketch graph for confirmation

Example 4.5. Find the area between the graph of $y = \sin x$ and the x - axis, between $x = 0$ and $x = \frac{\pi}{2}$.

$A = \int_0^{\frac{\pi}{2}} (\sin x)\,dx = [-\cos x]_0^{\pi/2}$

$= -\cos(\frac{\pi}{2}) - (-\cos(0))$

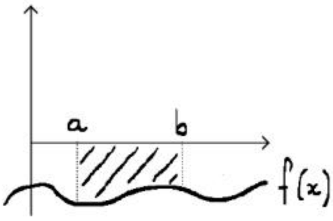
$= 1$ units

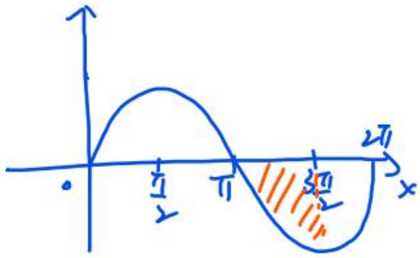


When $f(x)$ is a negative function and $a < b$ then the definite integral gives the negative of the area between the graph of the function $f(x)$ and the x - axis.

area is below axis

$$\int_a^b f(x)dx = -A$$





Example 4.6. Find the area between the graph of $y = \sin x$ and the x - axis, between $x = \pi$ and $x = \frac{3\pi}{2}$.

$A = \int_{\pi}^{3\pi/2} (\sin x)\, dx$

$= \left[-\cos x\right]_{\pi}^{3\pi/2}$

$A = -\cos\left(\frac{3\pi}{2}\right) - (-\cos(\pi))$

$A = -1$ {area below x-axis}

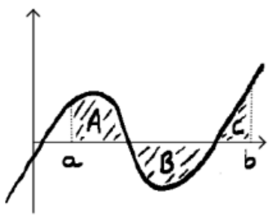
Area is 1 units

When $f(x)$ is positive for some values of x in the interval $[a, b]$ and negative for other values in the interval $[a, b]$ then the definite integral gives the sum of the areas above the x - axis and subtracts the areas below the x - axis.

In other words

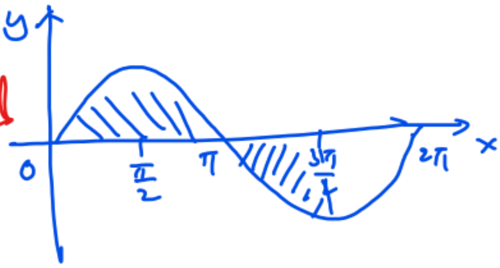
$$\int_a^b f(x)dx = A - B + C$$

Not one piece



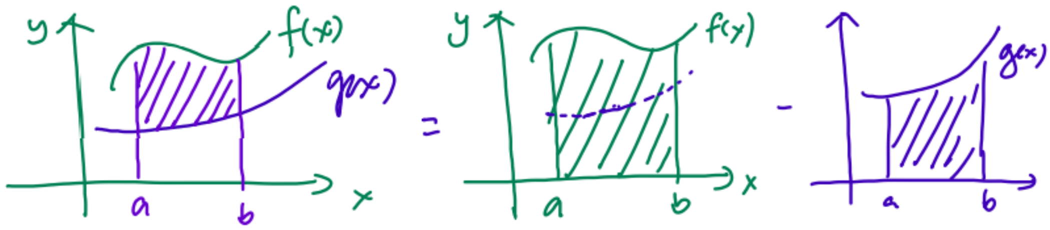
Example 4.7. Find the area between the graph of $y = \sin x$ and the x - axis, between $x = 0$ and $x = \frac{3\pi}{2}$.

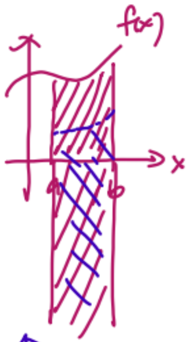
$A = \int_0^{\frac{3\pi}{2}} (\sin x)\, dx =$ Not getting Answer



$A = \int_0^{\pi} (\sin x)\, dx + \left|\int_{\pi}^{3\pi/2} (\sin x)\, dx\right|$

$A = \left[-\cos x\right]_0^{\pi} + \left|\left[-\cos x\right]_{\pi}^{3\pi/2}\right| = 1 + 1 + |(-1)| = 3$ units^2





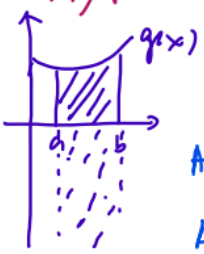
Area between two curves. Given two continuous functions $f(x)$ and $g(x)$ where $f(x) \geq g(x)$ for all x in the interval $[a, b]$, the area of the region bounded by the curves $y = f(x)$ and $y = g(x)$, and the lines $x = a$ and $x = b$ is given by the definite integral

top - bottom

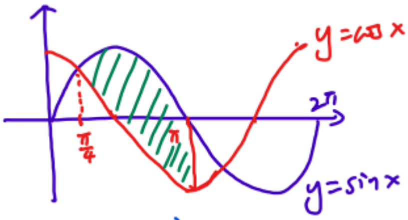
$$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

This is true regardless of whether the functions are positive, negative, or a combination of both. Can you see why?

Example 4.8. Find the area between the graphs of $y = \sin x$ and $y = \cos x$ between $x = \frac{\pi}{4}$ and $x = \pi$.



$A = \int_{\frac{\pi}{4}}^{\pi} (\sin x - \cos x)\,dx$



$A = \Big[-\cos x - \sin x\Big]_{\frac{\pi}{4}}^{\pi}$

$A = -\cos(\pi) - \sin(\pi) - \left(-\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right)$

$A = 1 - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = (1 + \sqrt{2})$ units^2 (exact form)

Look carefully at the next example.

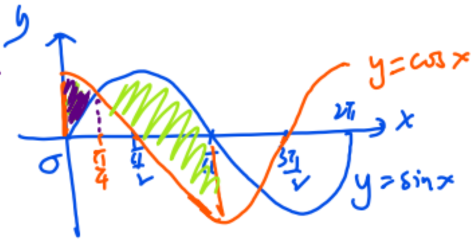
Example 4.9. Find the area between the graphs of $y = \sin x$ and $y = \cos x$ between $x = 0$ and $x = \pi$.

$A = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,dx + \int_{\frac{\pi}{4}}^{\pi} (\sin x - \cos x)\,dx$

$= \Big[\sin x + \cos x\Big]_0^{\frac{\pi}{4}} + 1 + \sqrt{2}$

$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) + (0 + 1) + 1 + \sqrt{2}$

$= \sqrt{2} + 2 + \sqrt{2} = (2 + 2\sqrt{2})$ units^2



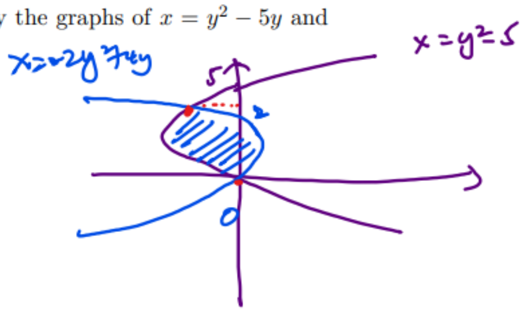
Example 4.10. Find the area bounded by the graphs of $x = y^2 - 5y$ and $x = -2y^2 + 4y$.

POI $y^2 - 5y = -2y^2 + 4y$

$3y^2 - 9y = 0$

$3y(y - 3) = 0$

$y = 0, \quad y = 3$



$A = \int_0^3 (-2y^2 + 9y) - (y^2 - 5y)\,dx = \int_0^3 (3y^2 + 5y)\,dx$

$A = \left[-y^3 + \frac{9}{3y^2}\right]_0^3 = \frac{27}{2}$ units^2

4.3 Trapezoidal rule

Sometimes it may not be all that simple to integrate a function. (As an example, try finding the anti-derivative of e^{-x^2}.) When we encounter situations such as this we can again turn to **numerical methods of approximation** to help us out, avoiding the need to integrate the function. One such method is the **Trapezoidal rule** which (as its name suggests) uses the area of the trapezium to approximate the area under the graph of a function $f(x)$. Recall the area of a trapezium is given by

$$A = \frac{1}{2}(m+n)w$$

where m and n are lengths of the parallel sides of the trapezium, and w is the distance between the parallel lengths (i.e. the width).

If the interval is $[a, b]$ then $w = b - a$. If the interval $[a, b]$ is divided into n equal sub-intervals, then each sub-interval has width $w_i = \dfrac{b-a}{n} = \Delta n$ and the successive heights $(m+n)$ of the parallel sides are given by $f(a) + f(a + \Delta n)$; $f(a+\Delta n) + f(a + 2\Delta n)$; …, $f(a + (n-1)\Delta n) + f(b)$.

The sum of the areas of each of the trapezoids created by each sub-interval can then be stated as:

$$A = \frac{1}{2}\frac{b-a}{n}\Bigg(f(a) + 2f(a+\Delta n) + \ldots + 2f(a + (n-1)\Delta n) + f(b)\Bigg).$$

Altering our notation slightly gives

$A = \frac{1}{2}h(a+b)$

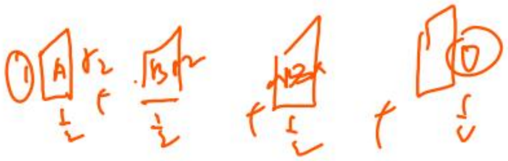
$$A = \sum_{i=0}^{n-1} \frac{b-a}{2n}\Bigg(f(x_i) + f(x_i + \Delta n)\Bigg).$$

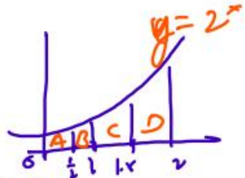
Note that when $i = 0$, $x_0 = a$ and thus $f(x_0) = f(a)$, and when $i = n-1$, $f(x_{n-1} + \Delta n) = f(b)$.

Thus the sum of the areas of each of the trapezoids created by each sub-interval can be stated as:

Trapezium Rule

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}\left(f(a) + f(b) + 2\sum_{i=1}^{n-1} f(x_i)\right)$$





approximately area:

Example 4.11. Use the Trapezoidal rule with $n = 4$ to find an approximate value of $\int_0^2 2^x dx$.

4 pieces of rects

Solution

In the interval $[0, 2]$ when $n = 4$ we have four trapezoids each of width $\frac{1}{2}$. The endpoints of our interval are $a = 0$ and $b = 2$ thus $f(a) = f(0) = 2^0 = 1$ and $f(b) = f(2) = 2^2 = 4$. Note that $\dfrac{b-a}{2n} = \dfrac{2}{2 \times 4} = \dfrac{1}{4}$. Thus

$$\begin{aligned}
\int_0^2 2^x dx &\approx \frac{b-a}{2n}\left(f(a) + f(b) + 2\sum_{i=1}^{n-1} f(x_i)\right) \\
&= \frac{1}{4}\left(1 + 4 + 2\sum_{i=1}^{3} 2^{x_i}\right) \\
&= \frac{1}{4}\left(1 + 4 + 2(2^{1/2} + 2^1 + 2^{3/2})\right) \\
&= \frac{1}{4}\left(5 + 2(\sqrt{2} + 2 + 2\sqrt{2})\right) \\
&= \frac{1}{4}(9 + 6\sqrt{2})
\end{aligned}$$

Example 4.12. Use the Trapezoidal rule with $n = 5$ to find an approximate value of

$$\int_0^\pi \sqrt{\sin x}\, dx$$