LECTURE 16: RELATIONS

A binary relation R on a set A consists of A and a set of ordered pairs from $A \times A$.

↳ when (a,b) is in this set, we write aRb

* If (x,y) is in the set
  - ↳ write ⇒ xRy
  - ↳ say ⇒ x is R-related to y

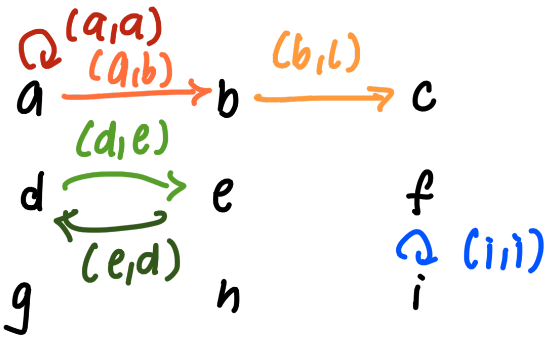
* If (x,y) is NOT in the set
  - ↳ write → $x \not R y$

ORDER MATTERS!

↳ can be xRy but $y \not R x$

EXAMPLES:

Let R be the relation on $\{a,b,c,d,e,f,g,h,i\}$ given by the set $\{(a,a),(a,b),(b,c),(d,e),(e,d),(i,i)\}$



bRc → YES
eRf → NO
aRc → NO
dRe → YES
iRi → YES

a is NOT directly related to c like:

OUN HAVE, so $a \not R c$

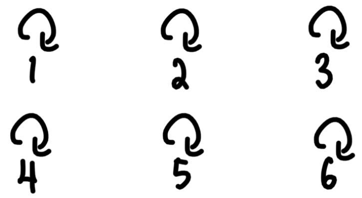
* xRy if and only if $y = f(x)$

↳ any function $f: x \to y$ can be viewed as a relation R on $X \cup Y$
↳ union

MORE EXAMPLES : EQUALITY ON $\mathbb{R}$

GIVE THE SET OF ORDERED PAIRS FOR THE RELATION "="
ON $\{1,2,3,4,5,6\}$ AND DRAW AN ARROW DIAGRAM FOR IT.

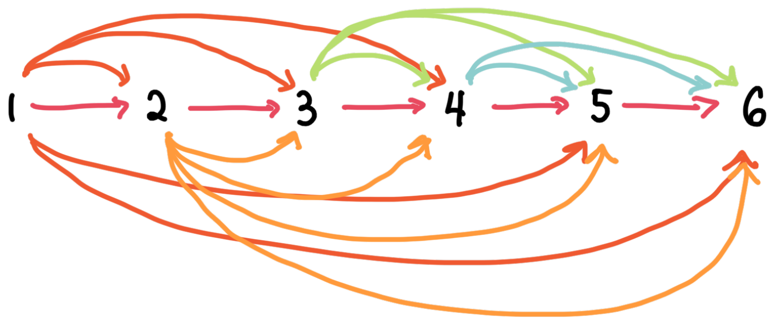
$\{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$



MORE EXAMPLES : < RELATION ON $\mathbb{R}$

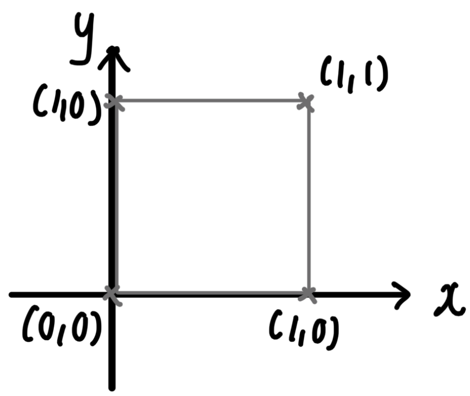
GIVE THE SET OF ORDERED PAIRS FOR THE RELATION "<"
ON $\{1,2,3,4,5,6\}$ AND DRAW AN ARROW DIAGRAM FOR IT.

$\{(1,2), (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6),$
$(3,4), (3,5), (3,6), (4,5), (4,6), (5,6)\}$



MORE EXAMPLES :

USE LOGIC SYMBOLS AND THE $\leq$ RELATION TO WRITE A
RELATION BETWEEN REAL NUMBERS X AND Y WHICH SAYS
THAT THE POINT (x,y) LIES IN THE SQUARE WITH CORNERS
$(0,0)$, $(1,0)$, $(0,1)$ AND $(1,1)$



$\therefore$ ANSWER :
$(0 \leq x) \wedge (x \leq 1) \wedge (0 \leq y) \wedge (y \leq 1)$

MORE EXAMPLES: SUBSET RELATION $\subseteq$

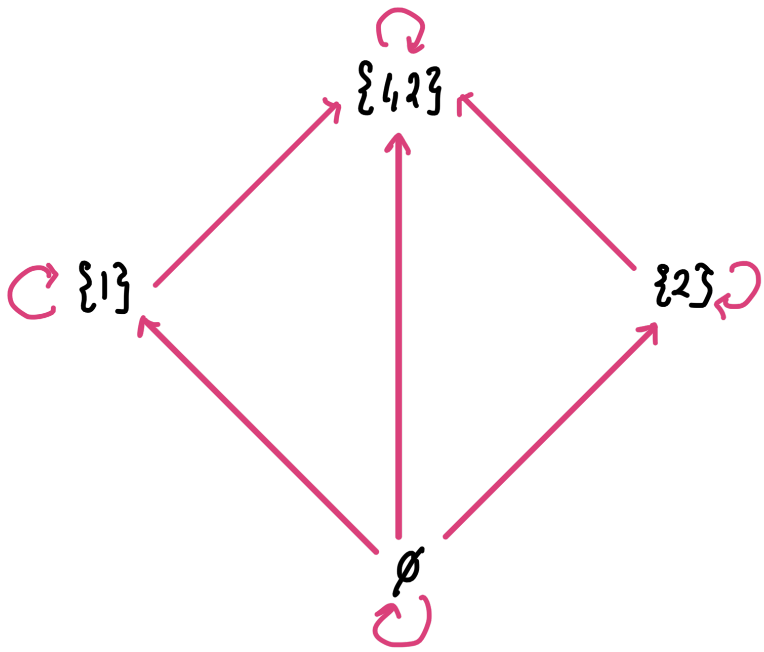
↳ CONSISTS OF ORDERED PAIRS OF SETS (A,B) SUCH THAT $A \subseteq B$

↳ A AND B MUST BOTH BE SUBSETS OF SOME UNIVERSAL SET U

GIVE THE SET OF ORDERED PAIRS FOR THE RELATION "$\subseteq$" ON $P(\{1,2\})$ AND DRAW AN ARROW DIAGRAM FOR IT.

$$P(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$$

$$\{(\emptyset,\emptyset), (\emptyset,\{1\}), (\emptyset,\{2\}), (\emptyset,\{1,2\}), (\{1\},\{1\}), (\{1\},\{1,2\}), (\{2\},\{2\}), (\{2\},\{1,2\}), (\{1,2\},\{1,2\})\}$$



MORE EXAMPLES: CONGRUENCE MODULO n

↳ FIXED n, CONGRUENCE MODULO n = BINARY RELATION

fixed integer $n \geq 2$

↳ CONSIST ALL ORDERED PAIRS OF INTEGERS (a,b) SUCH THAT n DIVIDES $a-b$

* $a \equiv b \pmod{n}$ MEANS THAT a AND b HAVE SAME REMAINDER WHEN DIVIDED BY n

* DEFINITION: $a \equiv b \pmod{n}$ if n divides $a-b$

Q: WHICH INTEGERS ARE CONGRUENT TO 1 MODULO 7 ?

INTEGERS IN THE SET $\{\dots, -20, -13, -6, 1, 8, 15, 22, \dots\}$
THIS IS THE SET $\{7k+1 : k \in \mathbb{Z}\}$

Q: WHICH INTEGERS ARE CONGRUENT TO 2 MODULO 5 ?

INTEGERS IN THE SET $\{\dots, -13, -8, -3, 2, 7, 12, 17, \dots\}$
THIS IS THE SET $\{5k+2 : k \in \mathbb{Z}\}$

Numbers with the same parity (even or odd) are congruent modulo ???2.

Decimal numbers ending in the same digit are congruent modulo ???10.

The time (in hours) can be thought of as a number modulo ???24.

An angle (in degrees) can be thought of as a number modulo ???360.

16.3 Properties of congruence

As the symbol $\equiv$ suggests, congruence mod n is a lot like equality. Numbers a and b which are congruent mod n are not necessarily equal, but they are "equal up to multiples of n," because they have equal remainders when divided by n.

Because congruence is like equality, congruence $a \equiv b \pmod{n}$ behave a lot like equations. In particular, they have the following three properties.

1. Reflexive property.

$$a \equiv a \pmod{n}$$

for any number a.

2. Symmetric property.

$$a \equiv b \pmod{n} \Rightarrow b \equiv a \pmod{n}$$

for any numbers a and b.

3. Transitive property.

$$a \equiv b \pmod{n} \text{ and } b \equiv c \pmod{n} \Rightarrow$$
$$a \equiv c \pmod{n}$$

for any numbers a, b and c.

These properties are clear if one remembers that $a \equiv b \pmod{n}$ means a and b have the same remainder on division by n.

Question Let R be the binary relation on $\mathbb{Z}$ defined by xRy if and only if $x \equiv y \pmod{3}$). Roughly, what would an arrow diagram for R look like?