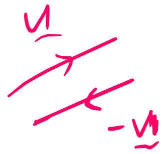
Problem Set One: Linear Algebra - Vectors, Dot Product, Cross Product

Vectors

$\underset{\sim}{u} + (-\underset{\sim}{v})$

$\underset{\sim}{u} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$



1. Consider the vectors $\underset{\sim}{u} = (2, -2, 3)$ and $\underset{\sim}{v} = (3, 2, -2)$.

a. Compute the following:

$\underset{\sim}{u} + \underset{\sim}{v}$

$\underset{\sim}{u} + 2\underset{\sim}{v}$

$\underset{\sim}{u} - \underset{\sim}{v}$

$\underset{\sim}{v} - \underset{\sim}{u}$

$$\underset{\sim}{u} + \underset{\sim}{v} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$$

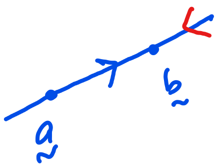
$$\underset{\sim}{u} + 2\underset{\sim}{v} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 6 \\ 4 \\ -4 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ -1 \end{pmatrix}$$

$$\underset{\sim}{u} - \underset{\sim}{v} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ 5 \end{pmatrix}$$

$$\underset{\sim}{v} - \underset{\sim}{u} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -5 \end{pmatrix}$$

NOTE: $\underset{\sim}{u} - \underset{\sim}{v} = -(\underset{\sim}{v} - \underset{\sim}{u})$

$\underset{\sim}{r} = \underset{\sim}{a} + \underset{\sim}{d}t$



$y = mx + c$

b. Show that $2\underset{\sim}{u} + 2\underset{\sim}{v} = 2(\underset{\sim}{u} + \underset{\sim}{v})$.

c.i. Find the length of $\underset{\sim}{u}$.

ii. Find the length of $\underset{\sim}{v}$.

iii. Find a unit vector in the direction of $\underset{\sim}{u}$.

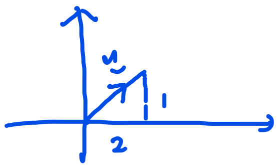
$$\begin{pmatrix} 4 \\ -4 \\ 6 \end{pmatrix} + \begin{pmatrix} 6 \\ 4 \\ -4 \end{pmatrix}$$

b) L.H.S $\quad 2\underset{\sim}{u} + 2\underset{\sim}{v} = 2\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \\ 2 \end{pmatrix}$

R.H.S $\quad 2(\underset{\sim}{u} + \underset{\sim}{v}) = 2\begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \\ 2 \end{pmatrix}$

Therefore $2\underset{\sim}{u} + 2\underset{\sim}{v} = 2(\underset{\sim}{u} + \underset{\sim}{v})$

$\underset{\sim}{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$



$|\underset{\sim}{u}| = \sqrt{2^2 + 1^2} = \sqrt{5}$ units

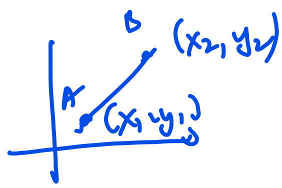
c) (i) $\underset{\sim}{u} = (a, b, c) \quad |\underset{\sim}{u}| = \sqrt{a^2 + b^2 + c^2}$

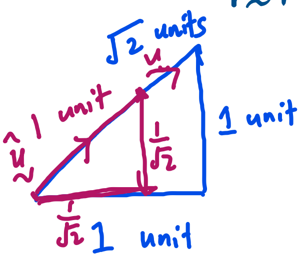
$\underset{\sim}{u} = \sqrt{2^2 + (-2)^2 + 3^2} = \sqrt{17}$ units

(ii) $\underset{\sim}{v} = \sqrt{3^2 + 2^2 + (-2)^2} = \sqrt{17}$ units

(iii) unit vector of $\underset{\sim}{u}$, $\hat{\underset{\sim}{u}} = \dfrac{\underset{\sim}{u}}{|\underset{\sim}{u}|}$

$$\hat{\underset{\sim}{u}} = \frac{1}{\sqrt{17}} \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$$

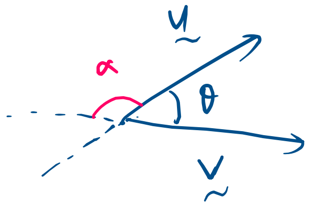




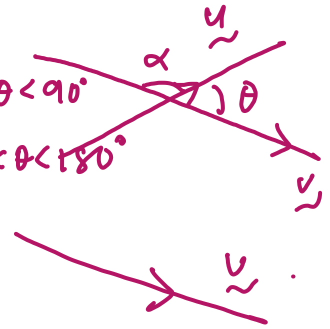
$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$\underset{\sim}{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad |\underset{\sim}{u}| = \sqrt{2}$

$\hat{\underset{\sim}{u}} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \quad |\hat{\underset{\sim}{u}}| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$



$\underset{\sim}{u}\cdot\underset{\sim}{v}$	Results
+ve	acute angle $0<\theta<90^\circ$
-ve	obtuse angle $90^\circ<\theta<180^\circ$
0	90° $\perp$



Vector Dot Product → scalar

2. For each of the pairs of vectors given below, calculate the vector dot product and the angle θ between the vectors.

a. $\underset{\sim}{v} = (3, 2, -2)$ and $\underset{\sim}{w} = (1, -2, -1)$

b. $\underset{\sim}{v} = (0, -1, 4)$ and $\underset{\sim}{w} = (4, 2, -2)$

c. $\underset{\sim}{v} = (2, 0, 2)$ and $\underset{\sim}{w} = (-3, -2, 0)$

What does it mean (geometrically) when the dot product is negative?

$$\boxed{\cos\theta = \frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|}}$$

$$\theta = \cos^{-1}\left(\frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|}\right)$$

(a) $\underset{\sim}{v}\cdot\underset{\sim}{w}$

$= \begin{pmatrix}3\\2\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\-1\end{pmatrix}$

$= (3\times1) + (2\times-2) + (-2\times-1)$

$= 3 + (-4) + (2)$

$= 1$ → (+ve) acute angle.

$\cos\theta = \dfrac{1}{\sqrt{17}\sqrt{6}}$

$\theta = \cos^{-1}\left(\dfrac{1}{\sqrt{17}\sqrt{6}}\right)$

$\theta = 84.3^\circ$

(b) $\underset{\sim}{v}\cdot\underset{\sim}{w}$

$= \begin{pmatrix}0\\-1\\4\end{pmatrix}\cdot\begin{pmatrix}4\\2\\-2\end{pmatrix}$

$= 0 + (-2) + (-8)$

$= -10$ → (-ve) obtuse angle

$\cos\theta = \dfrac{-10}{\sqrt{17}\sqrt{24}}$

$\theta = \cos^{-1}\left(\dfrac{-10}{\sqrt{17}\sqrt{24}}\right)$

$\theta = 119.7^\circ$

(c) $\underset{\sim}{v}\cdot\underset{\sim}{w}$

$= \begin{pmatrix}2\\0\\2\end{pmatrix}\cdot\begin{pmatrix}-3\\-2\\0\end{pmatrix} = -6$

$\cos\theta = \dfrac{-6}{\sqrt{8}\sqrt{13}}$

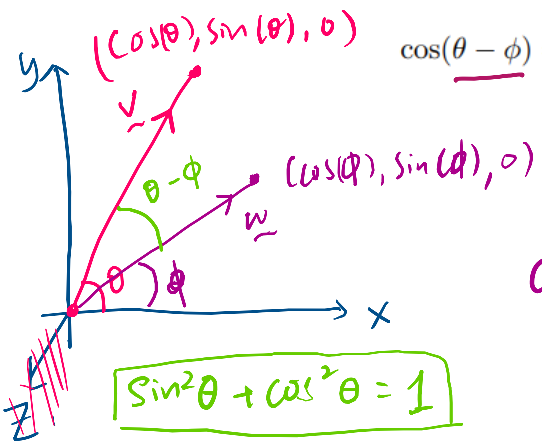
$\theta = \cos^{-1}\left(\dfrac{-6}{\sqrt{8}\sqrt{13}}\right)$

$\theta \approx 126^\circ$

When dot product is negative, angle between the vectors $\underset{\sim}{u}$ & $\underset{\sim}{v}$ is obtuse angle

3. Given the two vectors $\underset{\sim}{v} = (\cos(\theta), \sin(\theta), 0)$ and $\underset{\sim}{w} = (\cos(\phi), \sin(\phi), 0)$, use the dot product to derive the trigonometric identity

$$\cos(\theta - \phi) = \cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi).$$



$\boxed{\sin^2\theta + \cos^2\theta = 1}$

Find angle between vector $\underset{\sim}{v}$ & $\underset{\sim}{w}$, $(\theta - \phi)$

$\cos(\theta-\phi) = \dfrac{\underset{\sim}{v}\cdot\underset{\sim}{w}}{|\underset{\sim}{v}||\underset{\sim}{w}|}$

$\cos(\theta-\phi) = \dfrac{\cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi) + 0}{\underbrace{\sqrt{\cos^2(\theta)+\sin^2(\theta)}}_{1}\ \underbrace{\sqrt{\cos^2(\phi)+\sin^2(\phi)}}_{1}}$

$\cos(\theta-\phi) = \cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi)$

4. Use the dot product to determine which of the following vectors are perpendicular to one another: $\underset{\sim}{u} = (3, 2, -2), \underset{\sim}{v} = (1, 2, -2), \underset{\sim}{w} = (2, -1, 2)$.

$\underset{\sim}{u} \cdot \underset{\sim}{v} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = 3 + 4 + 4 = 11$ acute angle

$\underset{\sim}{u} \cdot \underset{\sim}{w} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = 6 - 2 - 4 = 0$ ✓, perpendicular / (90°) orthogonal

$\underset{\sim}{v} \cdot \underset{\sim}{w} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = 2 - 2 - 4 = -4$ obtuse angle

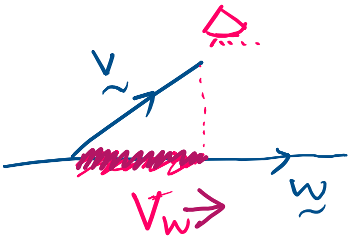
5. Find the scalar projection of the vector $\underset{\sim}{v}$ in the direction of $\underset{\sim}{w}$ (ie find v_w) for the following pairs of vectors:

a. $\underset{\sim}{v} = (3, 2, -2)$, $\underset{\sim}{w} = (1, -2, -1)$
b. $\underset{\sim}{v} = (0, -1, 4)$, $\underset{\sim}{w} = (4, 2, -2)$
c. $\underset{\sim}{v} = (2, 0, 2)$, $\underset{\sim}{w} = (-3, -2, 0)$

} same as Q2

(a) 1
(b) −10
(c) −6

What does it mean (geometrically) when the scalar projection is negative?



$\Rightarrow$ $\boxed{V_w = \frac{\underset{\sim}{v} \cdot \underset{\sim}{w}}{|\underset{\sim}{w}|}}$

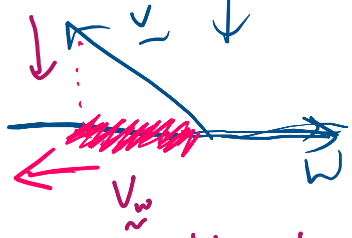
$\underset{\sim}{V_w} \rightarrow$ $\quad \underset{\sim}{V_w} = \frac{1}{2}\underset{\sim}{w}$

$\longrightarrow \underset{\sim}{w}$

−ve sign means the shadow is in opposite direction of $\underset{\sim}{w}$

(a) $V_w = \frac{1}{\sqrt{6}}$ units

(b) $V_w = \frac{-10}{\sqrt{16 + 4 + 4}} = \frac{-10}{\sqrt{24}}$, $\therefore V_w = \frac{10}{\sqrt{24}}$ units



(c) $V_w = \frac{-6}{\sqrt{13}}$, $\therefore V_w = \frac{6}{\sqrt{13}}$ units

$\underset{\sim}{V_w} = -\frac{1}{2}\underset{\sim}{w}$

6. Find the vector projection of the vector $\underset{\sim}{v}$ in the direction of $\underset{\sim}{w}$ (ie find $\underset{\sim}{v}_w$) for the pairs of vectors in Question 5($a - c$).

vector projection $\quad \underset{\sim}{V_w} = (V_w)\,\hat{\underset{\sim}{w}} = \left(\frac{\underset{\sim}{v} \cdot \underset{\sim}{w}}{|\underset{\sim}{w}|^2}\right)\underset{\sim}{w}$

unit vector

(a) $\underset{\sim}{V_w} = \frac{1}{(\sqrt{6})^2}\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

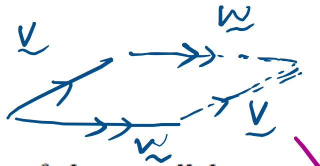
(b) $\underset{\sim}{V_w} = \frac{-10}{(\sqrt{24})^2}\begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} = \frac{-10}{24}\begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} = \frac{10}{24}\begin{pmatrix} -4 \\ -2 \\ 2 \end{pmatrix}$

take note

(c) $\underset{\sim}{V_w} = \frac{-6}{(\sqrt{13})^2}\begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix} = -\frac{6}{13}\begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix} = \frac{6}{13}\begin{pmatrix} +3 \\ +2 \\ 0 \end{pmatrix}$

☆ mistakes made in the sign!

Vector Cross Product → vector



7. For each pair of vectors in Question 5$(a - c)$:

Calculate the vector cross product, $\underset{\sim}{v} \times \underset{\sim}{w}$, and find the area of the parallelogram spanned by the two vectors $\underset{\sim}{v}$ and $\underset{\sim}{w}$.

What can you say about vectors $\underset{\sim}{v}$ and $\underset{\sim}{w}$ if $|\underset{\sim}{v} \times \underset{\sim}{w}| = 0$?

$$\begin{array}{c|ccc} & i & j & k \\ \hline v & 3 & 2 & -2 \\ w & 1 & -2 & -1 \end{array}$$

(a) $$\underset{\sim}{v} \times \underset{\sim}{w} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2-(4) \\ -[-3-(-2)] \\ -6-2 \end{pmatrix} = \begin{pmatrix} -6 \\ 1 \\ -8 \end{pmatrix}$$

$$\underset{\sim}{v} \times \underset{\sim}{w} = \begin{pmatrix} \searrow - \nearrow \\ \nearrow - \searrow \\ \searrow - \nearrow \end{pmatrix}$$

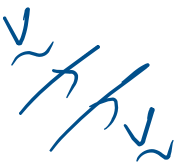
Area ▱ $= |\underset{\sim}{v} \times \underset{\sim}{w}| = \sqrt{36+1+64} = \sqrt{101}$ units^2

(b) $$\underset{\sim}{v} \times \underset{\sim}{w} = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix} \times \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ 16 \\ 4 \end{pmatrix}$$ → negative the result

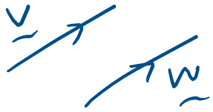
Area ▱ $|\underset{\sim}{v} \times \underset{\sim}{w}| = \sqrt{36+256+16} = \sqrt{308}$ units^2

(c) $$\underset{\sim}{v} \times \underset{\sim}{w} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \\ -4 \end{pmatrix}$$

Area ▱ $|\underset{\sim}{v} \times \underset{\sim}{w}| = \sqrt{16+36+16} = \sqrt{68}$ units^2

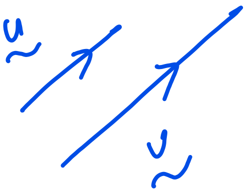


$\underset{\sim}{v} \times \underset{\sim}{v} = \underset{\sim}{0}$



$\underset{\sim}{v} \times \underset{\sim}{w} = \underset{\sim}{0}$

$|\underset{\sim}{v} \times \underset{\sim}{w}| = 0$



$\underset{\sim}{v} \times \underset{\sim}{w} = \underset{\sim}{0}$ $\underset{\sim}{v}$ & $\underset{\sim}{w}$ are parallel ?

8. For the vectors $\underset{\sim}{v} = (3, 2, -1)$ and $\underset{\sim}{w} = (2, 1, 3)$, verify that

a. $\underset{\sim}{v} \times \underset{\sim}{w} = -\underset{\sim}{w} \times \underset{\sim}{v}$.

b. $\underset{\sim}{v} \times \underset{\sim}{v} = 0$.

(a) $$\underset{\sim}{v} \times \underset{\sim}{w} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ -11 \\ -1 \end{pmatrix}$$

$$-\underset{\sim}{w} \times \underset{\sim}{v} = \begin{pmatrix} -2 \\ -1 \\ -3 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ -11 \\ -1 \end{pmatrix}$$

(b) $$\underset{\sim}{v} \times \underset{\sim}{v} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

9. For the vectors $\underset{\sim}{u} = (1, -2, -1)$, $\underset{\sim}{v} = (3, 2, -1)$ and $\underset{\sim}{w} = (2, 1, 3)$, find

a. $(\underset{\sim}{u} \times \underset{\sim}{v}).\underset{\sim}{w}$ and $(\underset{\sim}{u} \times \underset{\sim}{w}).\underset{\sim}{v}$.

b. $\underset{\sim}{u} \times (\underset{\sim}{v} + \underset{\sim}{w})$ and $(\underset{\sim}{u} \times \underset{\sim}{v}) + (\underset{\sim}{u} \times \underset{\sim}{w})$.

What do you notice? Do you think this is true for all vectors $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$?

(a) $$(\underset{\sim}{u} \times \underset{\sim}{v}) \cdot \underset{\sim}{w} = \left[\begin{pmatrix}1\\-2\\-1\end{pmatrix} \times \begin{pmatrix}3\\2\\-1\end{pmatrix}\right] \cdot \begin{pmatrix}2\\1\\3\end{pmatrix}$$

$$= \begin{pmatrix}4\\-2\\8\end{pmatrix} \cdot \begin{pmatrix}2\\1\\3\end{pmatrix}$$

$$= 8 - 2 + 24$$

$$= 30$$

$$(\underset{\sim}{u} \times \underset{\sim}{w}) \cdot \underset{\sim}{v} = \left[\begin{pmatrix}1\\-2\\-1\end{pmatrix} \times \begin{pmatrix}2\\1\\3\end{pmatrix}\right] \cdot \begin{pmatrix}3\\2\\-1\end{pmatrix} = \begin{pmatrix}-5\\-5\\5\end{pmatrix} \cdot \begin{pmatrix}3\\2\\-1\end{pmatrix}$$

$$= -15 - 10 - 5$$

$$= -30$$

$$(\underset{\sim}{u} \times \underset{\sim}{v}) \cdot \underset{\sim}{w} = -(\underset{\sim}{u} \times \underset{\sim}{w}) \cdot \underset{\sim}{v}$$

(b) $$\underset{\sim}{u} \times (\underset{\sim}{v} + \underset{\sim}{w}) = \begin{pmatrix}1\\-2\\-1\end{pmatrix} \times \begin{pmatrix}5\\3\\2\end{pmatrix} = \begin{pmatrix}-1\\-7\\13\end{pmatrix}$$

$$(\underset{\sim}{u} \times \underset{\sim}{v}) + (\underset{\sim}{u} \times \underset{\sim}{w}) = \begin{pmatrix}4\\-2\\8\end{pmatrix} + \begin{pmatrix}-5\\-5\\-5\end{pmatrix} = \begin{pmatrix}-1\\-7\\13\end{pmatrix}$$

$$\underset{\sim}{u} \times (\underset{\sim}{v} + \underset{\sim}{w}) = (\underset{\sim}{u} \times \underset{\sim}{v}) + (\underset{\sim}{u} \times \underset{\sim}{w})$$

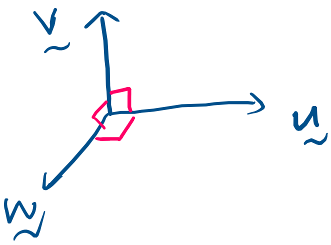
All these statements are true for all vectors $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$

10. For the vectors $\underset{\sim}{u} = (1, 0, 0)$, $\underset{\sim}{v} = (0, 1, 0)$ and $\underset{\sim}{w} = (0, 0, 1)$, find $\underset{\sim}{u} \times \underset{\sim}{v}$.
Without performing any calculations, what would you expect $\underset{\sim}{v} \times \underset{\sim}{w}$ to be?
What about $\underset{\sim}{u} \times \underset{\sim}{w}$?

$$\underset{\sim}{u} \times \underset{\sim}{v} = \begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} \iff \underset{\sim}{u} \times \underset{\sim}{v} = \underset{\sim}{w}$$

$$\underset{\sim}{v} \times \underset{\sim}{w} = \underset{\sim}{u}$$

$$\underset{\sim}{u} \times \underset{\sim}{w} = \underset{\sim}{v}$$



$(\underset{\sim}{u} \times \underset{\sim}{w}) \cdot \underset{\sim}{u} = \underset{\sim}{0}$

All are perpendicular!