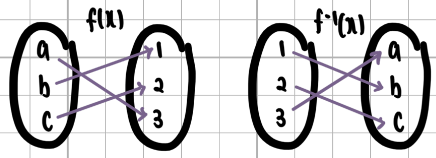
LECTURE 13 : FUNCTIONS

* f(x) can be written as $f: x \mapsto$
* Domain of $f(x)$ = Range of $g^{-1}(x)$

- one-to-one
  ↳ Each x value maps to one distinct y value
  ↳ Eg. $f(x) = 3x-1$
- many-to-one
  ↳ some f(x) values generated by more than 1 x value
  ↳ Eg. $f(x) = x^2 - 2x + 3$
  (Domain = x values ; range = y values)
- one-to-many DO NOT EXIST

- Inverse functions
  ↳ only one-to-one functions have inverses
  ↳



↳ Eg. $f(x) = 5x+3$
let x be y
let f(x) be x
$x = 5y+3$
$\frac{x-3}{5} = y$
$\therefore f^{-1}(x) = \frac{x-3}{5}$

13.1 Defining functions via sets

A function f consists of a *domain* X, a *codomain* Y, and a set of ordered pairs from $X \times Y$ which has exactly one ordered pair (x, y) for each $x \in X$.

When (a, b) is in this set we write $f(a) = b$.

The set of y values occurring in these pairs is the *image* of f.

" f is a function with domain X and codomain Y " is shorten to $f: X \to Y$

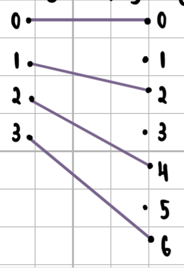
Function consists of :
- Domain X
- Codomain Y
- set of ordered pairs from X×Y
  ↳ 1 ordered pair (x,y) for each $x \in Y$
  ↳ set of y values is called image of function
  ↳ image always subset of codomain but may not be equal
    ↳ if equal, the function is onto

Eg. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x$
∴ The set of ordered pairs defining f is $\{(x, 2x) : x \in \mathbb{R}\}$

Eg. What set of ordered pairs does $f: \{0,1,2,3\} \to \mathbb{N}$ defined by $f(x) = x^2$ correspond to?
$\therefore \{(0,0), (1,1), (2,4), (3,9)\}$

Arrow diagrams

Eg. Let $f: \{0,1,2,3\} \to \{0,1,2,3,4,5,6\}$ be defined by $f(x) = 2x$

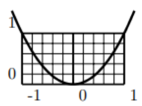


∴ The image of f is $\{0,2,4,6\}$
so f is not onto

1. The squaring function square$(x) = x^2$ with domain $\mathbb{R}$, codomain $\mathbb{R}$, and pairs

$$\{(x, x^2) \ : \ x \in \mathbb{R}\},$$

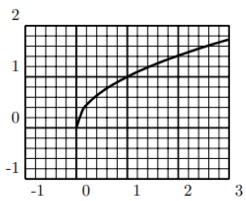
which form what we usually call the *plot* of the squaring function.



The image of this function (the set of y values) is the set $\mathbb{R}^{\geqslant 0}$ of real numbers $\geqslant 0$.

2. The square root function sqrt$(x) = \sqrt{x}$ with domain $\mathbb{R}^{\geqslant 0}$, codomain $\mathbb{R}$, and pairs

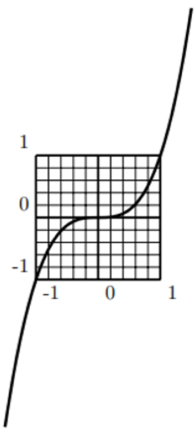
$$\{(x, \sqrt{x}) \ : \ x \in \mathbb{R} \text{ and } x \geqslant 0\}.$$



The image of this function (the set of y values) is the set $\mathbb{R}^{\geqslant 0}$.

3. The cubing function cube$(x) = x^3$ with domain $\mathbb{R}$, codomain $\mathbb{R}$, and pairs

$$\{(x, x^3) \ : \ x \in \mathbb{R}\},$$



The image of this function is the whole of the codomain $\mathbb{R}$, so it is onto.

* WHICH OF THE FOLLOWING RULES DEFINE FUNCTIONS ?

- For each non-empty set S of natural numbers, let $f(S)$ be the least member of S.

  Yes.

- For each set X of real numbers between 0 and 1, let $g(X)$ be the least member of X.

  No - $g(\{x : x \in \mathbb{R} \text{ and } \frac{1}{2} < x < 1\})$ is not defined.

- For each circle C in the (x, y) plane, let $h(C)$ be the minimum distance from C to the x axis.

  Yes.

- For a pair A, B of sets of real numbers let $s(A, B)$ be the smallest set which has both A and B as subsets.

  Yes (depending on your interpretation of "smallest").
  $s(A, B) = A \cup B$.

- For a pair A, B of sets of real numbers let $t(A, B)$ be the largest set which is a subset of both A and B.

  Yes (depending on your interpretation of "largest").
  $t(A, B) = A \cap B$.

13.2 Arrow notation

If f is a function with domain A and codomain B we write

$$f : A \to B,$$

and we say that f is from A to B.

For example, we could define

$$\text{square} : \mathbb{R} \to \mathbb{R}.$$

We could also define

$$\text{square} : \mathbb{R} \to \mathbb{R}^{\geqslant 0}.$$

Likewise, we could define

$$\text{cube} : \mathbb{R} \to \mathbb{R}.$$

However we could not define

$$\text{cube} : \mathbb{R} \to \mathbb{R}^{\geqslant 0},$$

because for some $x \in \mathbb{R}$, cube(x) is negative. For example, cube$(-1) = -1$.

Eg. Which of the following functions can be defined on the whole of IR, so that the function values also lie in IR
(In other words, which can be IR → IR functions?)

x^2 Yes.

$\frac{1}{x}$ No - undefined for $x = 0$.

$\log(x)$ No - undefined for $x \leq 0$ (because $e^x > 0$ for all $x \in \mathbb{R}$)

$\sqrt{x}$ No - undefined for $x < 0$.

$\sqrt[3]{x}$ Yes.

13.3 One-to-one functions

A function $f : X \to Y$ is *one-to-one* if for each y in the image of f there is only one $x \in X$ such that $f(x) = y$.

13.4 Proving a function is one-to-one

There is an equivalent way of phrasing the definition of one-to-one: a function $f : X \to Y$ is one-to-one when, for all $x_1, x_2 \in X$,

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

This can be useful for proving that some functions are or are not one-to-one.

Eg. The function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2+1$ is not one-to-one because $f(-1)=2$ and $f(1)=2$ and so $f(-1)= f(1)$

Eg. The function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 6x+2$ is one-to-one because

$$f(x_1) = f(x_2)$$
$$6x_1+2 = 6x_2+2$$
$$6x_1 = 6x_2$$
$$x_1 = x_2$$

To show that a function $f : X \to Y$ is one-to-one we must show that, for all $x_1, x_2 \in X$,

if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

To show that a function $f : X \to Y$ is *not* one-to-one we must show that there exist $x_1, x_2 \in X$ such that

$f(x_1) = f(x_2)$ and $x_1 \neq x_2$.

Example Show $g : \mathbb{N} \to \mathbb{Z}$ defined by $g(x) = (x + 6)^2 + 1$ is one-to-one.

Suppose that

$$\begin{array}{lrcl} & g(x_1) & = & g(x_2) \quad \text{for some } x_1, x_2 \in \mathbb{N} \\ \text{Then} & (x_1+6)^2+1 & = & (x_2+6)^2+1. \\ \text{So} & (x_1+6)^2 & = & (x_2+6)^2. \\ \text{So} & x_1+6 & = & x_2+6. \end{array}$$

(Two positive integers with equal squares are equal.)

$$\text{So} \quad x_1 = x_2.$$

This shows that g is one-to-one.

LECTURE 14 : EXAMPLE OF FUNCTIONS

14.1 Functions of several variables

- Define a function sum : IR × IR → IR by sum(x,y) = x+y
- Domain = IR × IR , Codomain in IR
- Inputs = ordered pairs (x,y) of real numbers

} Function of 2 variables x and y

- Define function binomial : IR × IR × N → IR by binomial (a,b,n) = $(a+b)^n$
- Inputs = ordered tiples (x,y,n) ⇒ x and y are real numbers, n = natural number

} Function of 3 variables

Eg. What are the ordered pairs which define the function sum : {1,2} × {1,2} → N defined by sum(x,y) = x+y ?

∴ We have sum ((1,1))=2 , sum ((1,2))=3, sum ((2,1)) =3 and sum ((2,2))=4 ⇒ ((x,y), x+y)

so { ((1,1),2) , ((1,2),3) , ((2,1),3), ((2,2),4) }

* multivariable function abbreviation f((x,y)) to f(x,y)

* Greatest common decimal (GCD) : Domain ⇒ $\mathbb{Z} \times \mathbb{Z} - \{(0,0)\}$

Codomain ⇒ N

* Reciprocal : Domain ⇒ IR − {0}

Codomain ⇒ IR − {0}

14.2 Sequence

- infinite sequence of numbers like $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}\ldots$ viewed as the function $f: \mathbb{N} \to \mathbb{R}$ defined by $f(n) = 2^{-n}$
  - ↳ input f = natural numbers ($\mathbb{N}$)
  - ↳ output = real numbers ($\mathbb{R}$)

- infinite sequence of numbers like $a_0, a_1, a_2, a_3\ldots$ viewed as the function $g(n) = a_n$ from $\mathbb{N}$ to some set containing the values a_n

Eg. For each of the following sequences, find a function f such that the sequence is $f(0), f(1), f(2)\ldots$

(a) $1, 1/2, 1/3, 1/4, 1/5\ldots$

pattern: denominator add by 1 every sequence $\therefore f: \mathbb{N} \to \mathbb{Q},\ f(n) = \frac{1}{n+1}$

(b) $5, 1, -3, -7, -11, -15\ldots$

pattern: minus 4 every sequence $\therefore f: \mathbb{N} \to \mathbb{Z},\ f(n) = 5-4n$

(c) $4, 12, 36, 108, 324, 972\ldots$

pattern: multiply by 4 every sequence $\therefore f: \mathbb{N} \to \mathbb{Z},\ f(n) = 4(3^n)$

14.3 Characteristic function

- subset of $\mathbb{N} = \{0,1,2,3\ldots\}$ can be represented by its characteristic function
- eg. set of squares represented by function $\chi: \mathbb{N} \to \{0,1\}$ defined by: $\chi(n) = \begin{cases} 1 & \text{if } n \text{ is a square} \\ 0 & \text{if } n \text{ is not a square} \end{cases}$

$\therefore$ sequence would be 110100001000000100000000100000000001000...

- HENCE, any set or property of natural numbers is represented by a function $\chi: \mathbb{N} \to \{0,1\}$

Eg. If A and B are subsets of $\mathbb{N}$ with characteristic function $\chi_A(n)$ and $\chi_B(n)$, then what set does the function $\chi_A(n)\chi_B(n)$ represent?

If $n \in A$ and $n \in B$ then $\chi_A(n)\chi_B(n) = 1 \times 1 = 1$.

If $n \in A$ and $n \notin B$ then $\chi_A(n)\chi_B(n) = 1 \times 0 = 0$.

If $n \notin A$ and $n \in B$ then $\chi_A(n)\chi_B(n) = 0 \times 1 = 0$.

If $n \notin A$ and $n \notin B$ then $\chi_A(n)\chi_B(n) = 0 \times 0 = 0$.

So $\chi_A(n)\chi_B(n)$ is the characteristic function of $A \cap B$

Question Let d be a positive integer. If $\chi_d : \mathbb{N} \to \{0,1\}$ is a function defined by

$$\chi_d(x) = \begin{cases} 1, & \text{if } x \text{ divides } d; \\ 0, & \text{if } x \text{ does not divide } d. \end{cases}$$

then what is $1\chi_d(1) + 2\chi_d(2) + 3\chi_d(3) + \cdots + d\chi_d(d)$?

Answer
The sum of the positive divisors of d

Question How many functions are there with domain $\{1,2,3,4\}$ and codomain $\{-1,0,1\}$?

Answer
The domain has 4 elements. (There are 4 possible inputs.)
For each input, we can decide if it's mapped to -1 or 0 or 1.
We can do this in $\underbrace{3 \times 3 \times \cdots \times 3}_{4} = 3^4 = 81$ ways.

Question How many functions are there with domain X and codomain Y?

Answer
The domain has $|X|$ elements. (There are $|X|$ possible inputs.)
For each input, we have $|Y|$ options for where it's mapped to.
We can do this in $\underbrace{|Y| \times |Y| \times \cdots \times |Y|}_{|X|} = |Y|^{|X|}$ ways.

codomain$^{\text{domain}}$

14.4 Boolean functions

The connectives $\wedge, \vee$ and $\neg$ are functions of variables whose values come from the set $\mathbb{B} = \{\mathsf{T}, \mathsf{F}\}$ of Boolean values (named after George Boole).

$\neg$ is a function of one variable, so

$$\neg : \mathbb{B} \to \mathbb{B}$$

and it is completely defined by giving its values on T and F, namely

$$\neg\mathsf{T} = \mathsf{F} \quad \text{and} \quad \neg\mathsf{F} = \mathsf{T}.$$

This is what we previously did by giving the truth table of $\neg$.

$\wedge$ and $\vee$ are functions of two variables, so

$$\wedge : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$$

and

$$\vee : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$$

They are completely defined by giving their values on the pairs $\{\mathsf{T},\mathsf{T}\}, \{\mathsf{T},\mathsf{F}\}, \{\mathsf{F},\mathsf{T}\}, \{\mathsf{F},\mathsf{F}\}$ in $\mathbb{B} \times \mathbb{B}$, which is what their truth tables do.

Eg. Hamming distance:

Let B_n be the set of all binary strings of length n.

Hamming distance is a function $h : B_n \times B_n \to \mathbb{N}$ defined by $h(s,t)$ equals the number of places in which s and t disagree.

For example,
$h(000, 101) = 2$,
$h(011, 010) = 1$,
$h(10111, 01000) = 5$.

LECTURE 15

Complicated functions are often built from simple parts. For example, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (x^2+1)^3$ is computed by doing the following steps in succession:

- square,
- add 1,
- cube.

We say that $f(x) = (x^2+1)^3$ is the composite of the functions (from $\mathbb{R}$ to $\mathbb{R}$)

- square(x)=x^2,
- successor(x)=$x+1$,
- cube(x)=x^3.

15.1 Notation for composite function

f(x) = cube (successor (square (x)))

same as

f = cube ∘ successor ∘ square

* f(x) = g(h(x))
same as f = g∘h
↳ f is the composite of g & h

Warning: Remember that $g \circ h$ means "do h first, then g." $g \circ h$ is usually different from $h \circ g$.

Example.

$$\text{square}(\text{successor}(x)) = (x+1)^2 = x^2+2x+1$$
$$\text{successor}(\text{square}(x)) = x^2+1$$

MORE EXAMPLES :

Question 15.2 Write the following as composites of square(x), sqrt(x), successor(x) and cube(x).

(Assume that all of these have domain and codomain $\{x : x \in \mathbb{R} \text{ and } x \geq 0\}$.)

$$\begin{aligned}
\sqrt{1+x^3} &= \text{sqrt}(\text{successor}(\text{cube}(x))) = \text{sqrt} \circ \text{successor} \circ \text{cube}(x) \\
x^{\frac{3}{2}} &= \text{sqrt}((\text{cube}(x)) = \text{sqrt} \circ \text{cube}(x) \\
(1+x)^3 &= \text{cube}(\text{successor}(x)) = \text{cube} \circ \text{successor}(x) \\
(1+x^3)^2 &= \text{square}(\text{successor}(\text{cube}(x))) = \text{square} \circ \text{successor} \circ \text{cube}(x)
\end{aligned}$$

Note Composition of functions is associative: $(f \circ g) \circ h = f \circ (g \circ h)$. So we don't bother with the brackets.

15.2 Conditions for compositions

* Composition functions don't always exists

Example. If reciprocal : $\mathbb{R} - \{0\} \to \mathbb{R}$ is defined by reciprocal$(x) = \frac{1}{x}$ and predecessor : $\mathbb{R} \to \mathbb{R}$ is defined by predecessor$(x) = x - 1$, then reciprocal ∘ predecessor does not exist, because predecessor$(1) = 0$ is not a legal input for reciprocal.

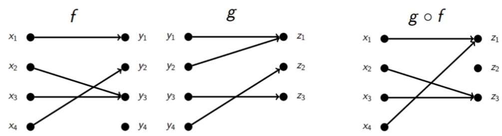
To avoid this problem, we demand that the codomain of h be equal to the domain of g for $g \circ h$ to exist. This ensures that each output of h will be a legal input for g.

Let $h : A \to B$ and $g : C \to D$ be functions. Then $g \circ h : A \to D$ exists if and only if $B = C$.

Let g: C→D and h: A→B be functions
- g∘h exists if and only if C=B
- if it exists, g∘h : A→D and is definded by g∘h(x) = g(h(x))

Examples :

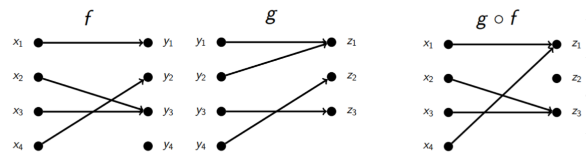
Question Let $f : A \to B$ and $g : C \to D$ be the functions pictured below.



Does $g \circ f$ exist?

codomain$(f) = \{y_1, y_2, y_3, y_4\}$ and domain$(g) = \{y_1, y_2, y_3, y_4\}$.
So $g \circ f$ does exist because codomain(f) = domain(g).
$g \circ f : A \to D$

Question Let $f : A \to B$ and $g : C \to D$ be the functions pictured below.



Does $g \circ f$ exist?

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So $g \circ f$ does exist because codomain(f) = domain(g).
$g \circ f : A \to D$

15.3 The identity function

On each set A the function i_A : A→A defined by $i_A(x) = x$, is called the identity function (on A)

15.4 Inverse functions

Functions $f : A \to A$ and $g : A \to A$ are said to be inverses (of each other) if

$$f \circ g = g \circ f = i_A.$$

Example. square and sqrt are inverses of each other on the set $\mathbb{R}^{\geqslant 0}$ of reals $\geqslant 0$.

$$\text{sqrt}(\text{square}(x)) = x \text{ and square}(\text{sqrt}(x)) = x.$$

In fact, this is exactly what sqrt is supposed to do – reverse the process of squaring. However, this works only if we restrict the domain to $\mathbb{R}^{\geqslant 0}$. On $\mathbb{R}$ we do not have sqrt(square(x)) = x because, for example,

$$\text{sqrt}(\text{square}(-1)) = \text{sqrt}(1) = 1.$$

This problem arises whenever we seek an inverse for a function which is not one-to-one. The squaring function on $\mathbb{R}$ sends both 1 and -1 to 1, but we want a single value 1 for sqrt(1). Thus we have to restrict the squaring function to $\mathbb{R}^{\geqslant 0}$.

15.5 Conditions for inverse

A function f can have an inverse without its domain and codomain being equal.

> The inverse of a function $f : A \to B$ is a function $f^{-1} : B \to A$ such that
> $$f^{-1} \circ f = i_A \quad \text{and} \quad f \circ f^{-1} = i_B.$$

Note that $f^{-1} \circ f$ and $f \circ f^{-1}$ are both identity functions but they have different domains.

Not every function has an inverse, but we can neatly classify the ones that do.

> Let $f : A \to B$ be a function. Then $f^{-1} : B \to A$ exists if and only if f is one-to-one and onto.

Let $f : A \to B$.

The function $f^{-1} : B \to A$ exists if and only if f is one-to-one and onto.

(Remember onto means image$(f) = B$.)

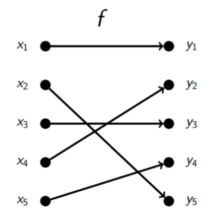
If it exists, $f^{-1} : B \to A$ is defined by $f^{-1}(y)$ equals the unique $x \in A$ such that $f(x) = y$.

We have $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

Note f^{-1} is just a notation for "the inverse function of f".
It is *not* an exponential.

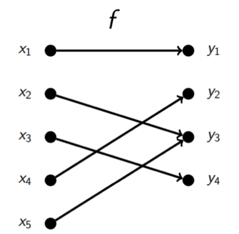
Examples:

Question Let $f : A \to B$ be the function pictured below.



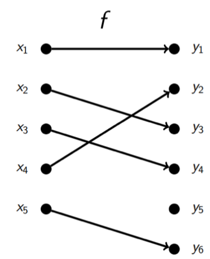
Does f^{-1} exist? Yes.

Question Let $f : A \to B$ be the function pictured below.



Does f^{-1} exist? No.
f is not one-to-one.

Question Let $f : A \to B$ be the function pictured below.



Does f^{-1} exist? No.
f is not onto.

Question 15.4:

Question 15.4 What feature do

$\neg : \mathbb{B} \to \mathbb{B}$ defined by $\neg(x) = \neg x$;

$f(x) : \mathbb{R} - \{0\} \to \mathbb{R} - \{0\}$ defined by $f(x) = \frac{1}{x}$; and

$g(x) : \mathbb{R} - \{1\} \to \mathbb{R} - \{1\}$ defined by $g(x) = \frac{x}{x-1}$;

have in common?

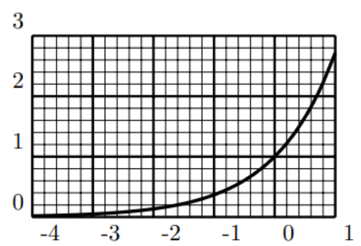
They are their own inverses.

Question Let $f : \{x : x \text{ is a Monash student}\} \to \mathbb{N}$ be the function defined by $f(x)$ equals the ID number of x. Does f^{-1} exist?

Answer
No. f is not onto. (E.g. there is no student with ID number 10^{200}.)

Example: e^x and log

Consider $f : \mathbb{R} \to \mathbb{R}^{\geqslant 0} - \{0\}$ defined by $f(x) = e^x$. We know that e^x is one-to-one (e.g. because it is strictly increasing), and onto. So it has an inverse f^{-1} on $\mathbb{R}^{\geqslant 0} - \{0\}$.



Plot of $y = e^x$.

In fact, $f^{-1} = \log(y)$ where
$$\log : \mathbb{R}^{\geqslant 0} - \{0\} \to \mathbb{R}.$$

Now
$$e^{\log x} = x \quad \text{and} \quad \log(e^x) = x,$$

so $e^{\log x}$ and $\log(e^x)$ are both identity functions, but they have different domains.

The domain of $e^{\log x}$ is $\mathbb{R}^{\geqslant 0} - \{0\}$ (note log is defined only for reals > 0). The domain of $\log(e^x)$ is $\mathbb{R}$.