**MAT1830 - Discrete Mathematics for Computer Science**
**Tutorial Sheet #5 and Additional Practice Questions**

## Tutorial Questions

1. Let $X = \{a, b, c, d\}$ and $Y = \{a, b, e, f, g, h\}$.

   (a) Make up and draw the arrow diagram of a function $q : X \to Y$ whose image contains 3 elements, or explain why no such function exists.

   (b) Make up and draw the arrow diagram of a function $r : Y \to X$ which is one-to-one, or explain why no such function exists.

   (c) Make up and draw the arrow diagram of a function $s : Y \to X$ such that $s(a) = a$, $s(b) = a$, $s(e) = a$ and the image of $s$ is $X$, or explain why no such function exists.

2. Let $p$, $q$ and $s$ be the following functions.

   $p : \mathbb{Z} \to \mathbb{Z}$ defined by $p(x) = \begin{cases} 1, & \text{if } x \text{ is even;} \\ -1, & \text{if } x \text{ is odd.} \end{cases}$

   $q : \mathbb{Z} \to \mathbb{R}$ given by the set $\{(x, x^2 - \frac{1}{2}) : x \in \mathbb{Z}\}$

   $s : \{x : x \in \mathbb{R} \text{ and } x \geq 0\} \to \mathbb{R}$ defined by $s(x) = \sqrt{x} + 2$

   $t : \mathbb{Z} \to \mathbb{N} \times \{-1, 1\}$ defined by $t(x) = (y, z)$ where $y = x^2$ and $z = \begin{cases} 1, & \text{if } x \geq 0; \\ -1, & \text{if } x < 0. \end{cases}$

   (a) What are the images of $p$, $q$, $s$ and $t$? Which of them are onto?

   (b) Which of $p$, $q$, $s$ and $t$ are one-to-one?

   (c) Do $p$ and $q$ have inverse functions? If they do, give a formula for the function and give its domain, codomain, and image.

3. Let $p$, $q$, $s$ and $t$ be the functions defined in Question 2. State whether each of the following compositions exists and, if they do, give a formula for the function and give its domain, codomain, and image.

   (a) $p \circ q$

   (b) $q \circ t$

   (c) $q \circ p$

4. Let $D$ be the set of all dogs.

   (a) Which of the following subsets of $D \times D$ correspond to functions from $D$ to $D$? Why or why not?

      i. $\{(x, y) : y \text{ is the mother of } x\}$
      ii. $\{(x, y) : y \text{ is the brother of } x\}$
      iii. $\{(x, y) : y \text{ is the eldest dog in the same litter as } x\}$
      iv. $\{(x, y) : y \text{ is the eldest daughter of } x\}$

   (b) Let $m : D \to D$ and $e : D \to D$ be the functions corresponding to i and iii in part (a). What are the following?

      i. $m \circ m(\text{Rover})$
      ii. $e \circ m(\text{Rover})$
      iii. $m \circ e(\text{Rover})$

(See over for practice questions.)

**Practice Questions**

1. Let $A = \{1, 2\}$ and $B = \{10, 11, 12\}$. Write down the set of ordered pairs which corresponds to the function $f : A \times B \to \mathbb{N}$ defined by $f((a, b)) = a^2 b$.

2. Which of the following rules correspond to functions. If they do, are those functions one-to one?

   (a) For each banana $b$, let $\ell(b)$ be the length of $b$ in centimetres (rounded down).

   (b) For each circle $C$ in the $(x, y)$-plane whose centre is the origin, let $r(C)$ be the radius of $C$.

   (c) For each latitude-longitude pair $(x, y)$, let $C((x, y))$ be the set of all coffee shops within one kilometer of $(x, y)$.

   (d) For each set of integers $X$, let $m(X)$ be the smallest integer in $X$.

3. Let $P$ be the set of all propositional logic sentences. Let $d : P \to \mathbb{R}$ be a function defined by $d(\psi) = \frac{c}{2^n}$ where $c$ is the number of interpretations under which $\psi$ is true and $n$ is the number of different variables in $\psi$.

   (a) What is $d(p \wedge q)$? What is $d(p \wedge (q \vee r))$? (where $p$, $q$ and $r$ are variables)

   (b) What can we say about $\psi$ if $d(\psi) = 1$?

   (c) Can you think of a way to think of $d$ as a probability?

   (d) What do you think the image of $d$ is?

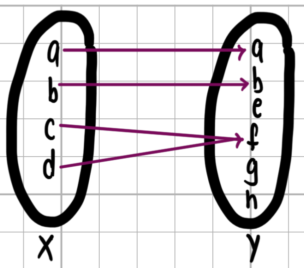
   (e) Can you prove your answer to (d)?

4. (a) Find a function $f : \mathbb{N} \to \mathbb{Z}$ which is one-to-one and has image $\mathbb{Z}$.

   (b) Do you think your answer to (a) means that there is the "same number" of natural numbers as integers?

   (c) Do you think your answer to (a) means that the natural numbers are "the same as the integers, just in disguise"?

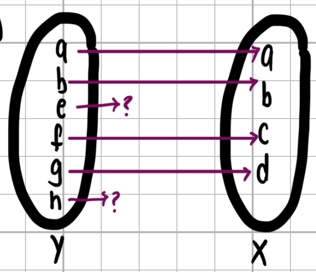
1)(a)




Function :
- everybody in domain must map out
- everybody in domain can only map once

* have 4 elements but want 3 image
  ∴ don't want onto
* question didn't specify need one-to-one function
* can many-to-one but cannot one-to-many

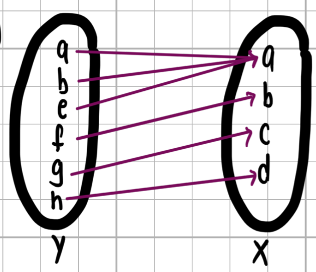
(b)




* one-to-one only
* cannot many-to-one
* have 6 element but only 4 element in X

∴ HENCE, cannot be done

(c)




* want onto ⇒ every element in X to be mapped
* no need one-to-one function due to:
  s(a) = a , s(b) = a, s(e) = a

2) Let $p$, $q$ and $s$ be the following functions.

$p : \mathbb{Z} \to \mathbb{Z}$ defined by $p(x) = \begin{cases} 1, & \text{if } x \text{ is even;} \\ -1, & \text{if } x \text{ is odd.} \end{cases}$

$q : \mathbb{Z} \to \mathbb{R}$ given by the set $\{(x, x^2 - \frac{1}{2}) : x \in \mathbb{Z}\}$

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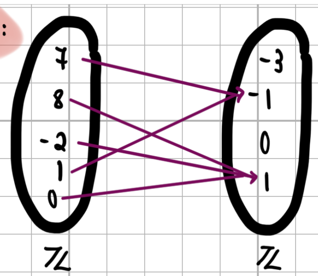
" what you give the function is even then it'll give you 1 "

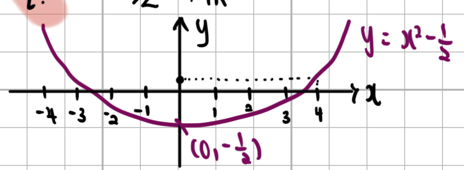
p:




* NOT onto
* NOT one-to-one

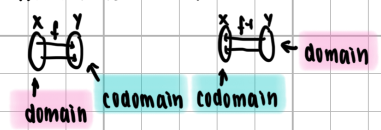
q:

$\mathbb{Z} \to \mathbb{R}$

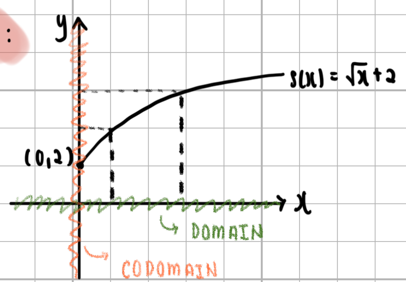



y-axis = output value
x-axis = input value

* NOT onto
* NOT one-to-one
* IF function is one-to-one & onto, it will have inverses
  ↳ inverse function elements domain is co-domain...

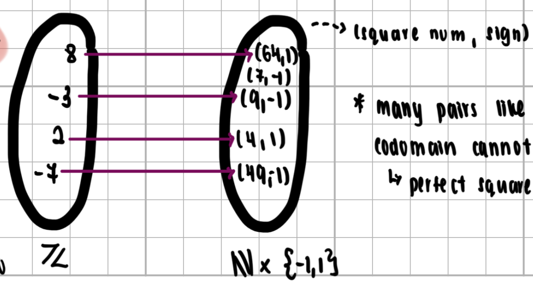



s:




* image < codomain , NOT onto
* √ one-to-one
* NOT INVERSE due to not onto
  ↳ to be inverse need both one-to-one & onto

t:




* many pairs like (7,-1) in codomain cannot be paired
  ↳ perfect square problem

(b) Let A be the set of all non-empty subsets of {1, 2, . . . , 10} and let f and g be the following functions.

f : A → Z defined by f (X) = a − b, where a is the largest element of X and b is the smallest element of X.

g : A → A defined by g(X) = X ∪ {1, 2}.

(i) Write down f ({2, 3, 6}). Write down g({2, 7, 10}).

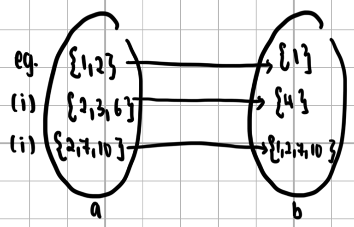
(ii) Is $f$ one-to-one? Is $g$ one-to-one?

(iii) What is the range of $f$?

(iv) Does $f \circ g$ exist? If it does exist, write down $f \circ g(\{9\})$.

(v) Does $g \circ f$ exist? Explain. If it does exist, write down $g \circ f(\{9\})$.

{1,2,3,4,5,6,7,8,9,10}




(ii) f is not one-to-one

g is one-to-one

(iii) f range (1—9)

(iv)