2.5 Matrix Inverse

Suppose we have a system of equations

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

and that we write in the matrix form

$X = \frac{B}{A}$ (Algebra)

$$AX = B$$

Can we find another matrix, call it A^{-1}, such that

$2 \times 2^{-1} : 2 \times \frac{1}{2} = 1$

$$A^{-1}A = I = \text{the identity matrix}$$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

If so, then we have

$$A^{-1}AX = A^{-1}B \qquad \Rightarrow \qquad X = A^{-1}B$$

Thus we have found the solution of the original system of equations.

For a 2×2 matrix it is easy to verify that

determinant

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$\det(A)$ or $|A|$

Note that not all matrices will have an inverse. For example, if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\searrow - \nearrow$ ① Given matrix A must square matrix

swap

negate

then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

② $\det(A) \neq 0$

and for this to be possible we must have $ad - bc \neq 0$. $\Rightarrow$ matrix A has inverse

In later lectures we will see some different methods for computing the inverse A^{-1} for other (square) matrices larger than 2×2.

matrix A is invertible

$\searrow - \nearrow$

Example 2.13. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $A^{-1} = \frac{1}{-2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$

$\det(A) = (1 \times 4) - (2 \times 3)$
$= 4 - 6$
$= -2$

Check $AA^{-1} = I$

$\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

Properties 1

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

$(A^T)^{-1} = \frac{1}{4-6}\begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$

$A^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$

$(A^{-1})^T = \begin{pmatrix} -2 & 3/2 \\ 1 & -1/2 \end{pmatrix}$

Properties 2

$(A^{-1})^{-1} = \frac{1}{1-\frac{3}{2}}\begin{pmatrix} -\frac{1}{2} & -1 \\ -\frac{3}{2} & -2 \end{pmatrix} = -2\begin{pmatrix} -\frac{1}{2} & -1 \\ -\frac{3}{2} & -2 \end{pmatrix}$

$(A^{-1})^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = A$

Properties of inverses

- A square matrix has at most one inverse.

  Proof: If B_1 and B_2 are both inverses of A, then $AB_1 = B_1A = I$ and $AB_2 = B_2A = I$. So $B_1 = B_1I = B_1(AB_2) = (B_1A)B_2 = IB_2 = B_2$. So the inverse of A is unique.
- If A is invertible, then so is A^T and $(A^T)^{-1} = (A^{-1})^T$.
- If A is invertible, then so is A^{-1} and $(A^{-1})^{-1} = A$.
- If A and B are invertible matrices of the same size, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

$(AB)^T = B^TA^T$

$\boxed{AB \neq BA}$

- $(A_1A_2 \ldots A_m)^{-1} = A_m^{-1} \ldots A_2^{-1}A_1^{-1}$.
- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc \neq 0$, then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
- **Cancellation laws:** If A is invertible, then
  - $AB = AC$ implies $B = C$ (just multiply on the left by A^{-1})
  - $BA = CA$ implies $B = C$ (just multiply on the right by A^{-1}).
  - $BAC = DAE$ does **not** imply $BC = DE$.

A^{-1}

$B_\wedge C$

$Ax = B$

$AA^{-1} = A^{-1}A = I$

$A^{-1}Ax = A^{-1}B$

$AA^{-1}x = BA^{-1}$

- **Solving systems:** Let A be $n \times n$ and invertible. Then the linear system $A\underset{\sim}{x} = \underset{\sim}{b}$ always has exactly one solution, namely $\underset{\sim}{x} = A^{-1}\underset{\sim}{b}$.
- **Rank test:** An $n \times n$ matrix A is invertible if and only if it has **full rank** $r = n$.

NOTE

$A^{-1}Ax = BA^{-1}$

$\Rightarrow$ incorrect

2.6 Matrix Transpose

If A is a matrix of size $m \times n$ then the **transpose** A^T of A is the $n \times m$ matrix defined by $A^T(j, i) = A(i, j)$.

Consider the matrix $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \end{bmatrix}$. The transpose of A, namely A^T is simple to find. Row 1 of matrix A becomes column 1 of matrix A^T, and row 2 of matrix A becomes column 2 of matrix A^T. Thus $A^T = \begin{bmatrix} 2 & 0 \\ 3 & 1 \\ 4 & 5 \end{bmatrix}$.

2×3

3×2

Example 2.14. Let $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 \end{bmatrix}$.

Find B^T and C^T.

$$B^T = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{pmatrix} \qquad C^T = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Note the following:

1 • $(A^T)^T = A$

2 • $(cA)^T = cA^T$

3 • $(A+B)^T = A^T + B^T$

4 • $(AB)^T = B^T A^T$ *

Example 2.15. Verify the above using matrices $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

① $(A^T)^T = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}^T = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} = A$

② $(2A)^T = 2A^T$

L.H.S $\begin{pmatrix} 4 & 6 \\ 8 & 10 \end{pmatrix}^T = \begin{pmatrix} 4 & 8 \\ 6 & 10 \end{pmatrix}$

R.H.S $2\begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 6 & 10 \end{pmatrix}$

③ $(A+B)^T = \begin{pmatrix} 3 & 5 \\ 7 & 9 \end{pmatrix}^T = \begin{pmatrix} 3 & 7 \\ 5 & 9 \end{pmatrix}$

$A^T + B^T = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 5 & 9 \end{pmatrix}$

④ $(AB)^T = \left[\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right]^T = \begin{pmatrix} 11 & 16 \\ 19 & 28 \end{pmatrix}^T = \begin{pmatrix} 11 & 19 \\ 16 & 28 \end{pmatrix}$

$B^T A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 11 & 19 \\ 16 & 28 \end{pmatrix}$

2.7 Determinants

The determinant function **det** is a function that assigns to each $n \times n$ matrix A a number **det** A called the **determinant** of A. The function is defined as follows:

- If $n = 1$: $A = [a]$ and we define $\det A := a$.
- If $n = 2$: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and we define $\det A := ad - bc$.

$\searrow - \nearrow$

- If $n > 2$: It gets a bit complicated now, but it is not too bad. Firstly create a sub-matrix S_{ij} of A by deleting the i^{th} row and the j^{th} column.

  Then define

$$\det A := a_{11} \det S_{11} - a_{12} \det S_{12} + a_{13} \det S_{13} - \cdots \pm a_{1n} \det S_{1n}$$

The quantity $\det S_{ij}$ is called the *minor of entry* a_{ij} and is denoted M_{ij}. The number $(-1)^{i+j} M_{ij}$ is called the *cofactor of entry* a_{ij}. Thus to compute $\det A$ you have to compute a *chain of determinants* from $(n-1) \times (n-1)$ determinants all the way down to 2×2 determinants.

This method of defining and evaluating $\det A$ is called **Laplace's expansion along the first row**. We can, in fact, use any row (or any column) to calculate $\det A$.

We often write $\det A = |A|$.

$$\det(A) = \sum_{i} \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(S_{ij})$$

Example 2.16. Compute the determinant of

(fix row)

fix row $i=1$
column $j=1,2,3$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 3 & 4 & 5 \\ 6 & 0 & 9 \end{bmatrix}$$

fix last row $i=3$

$\det(A) = (-1)^{3+1}(6)\begin{vmatrix} 7 & 2 \\ 4 & 5 \end{vmatrix} + (-1)^{3+3}(9)\begin{vmatrix} 1 & 7 \\ 3 & 4 \end{vmatrix}$

$\boxed{i=1, j=1}$ $\quad i=1, j=2 \quad i=1, j=3$

$\det(A) = (-1)^{1+1}(1)\begin{vmatrix} 4 & 5 \\ 0 & 9 \end{vmatrix} + (-1)^{1+2}(7)\begin{vmatrix} 3 & 5 \\ 6 & 9 \end{vmatrix} + (-1)^{1+3}(2)\begin{vmatrix} 3 & 4 \\ 6 & 0 \end{vmatrix}$

$= 36 + (-1)(7)(-3) + (2)(-24)$

$= 36 + 21 - 48 = 9$

$= (6)(27) + (9)(-17)$

$= 9$

When we expand the determinant about any row or column, we must observe the following pattern of $\pm$ signs (these correspond to the $(-1)^{i+j}$ in C_{ij} - check!).

$$\begin{bmatrix} + & - & + & - & + & - & \cdots \\ - & + & - & + & - & + & \cdots \\ + & - & + & - & + & - & \cdots \\ - & + & - & + & - & + & \cdots \end{bmatrix}$$

This is best seen in an example.

Example 2.17. By expanding about the second ~~row~~ column compute the determinant of

fix column #2 $j=2$
row $i=1,2,3$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 3 & 4 & 5 \\ 6 & 0 & 9 \end{bmatrix}$$

$i=1, j=2 \quad i=2, j=2 \quad i=3, j=2$

$\det(A) = (-1)^3(7)\begin{vmatrix} 3 & 5 \\ 6 & 9 \end{vmatrix} + (-1)^4(4)\begin{vmatrix} 1 & 2 \\ 6 & 9 \end{vmatrix} + (-1)^5(0)\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix}$

$= (-1)^3(7)(-3) + (4)(-3) = 21 - 12 = 9$

$A = \begin{pmatrix} 1 & 7 & 2 \\ 3 & 4 & 5 \\ 6 & 0 & 9 \end{pmatrix}$ 3x3

Short-cut Method

$\det(A)$

$= \begin{vmatrix} 1 & 7 & 2 & 1 & 7 \\ 3 & 4 & 5 & 3 & 4 \\ 6 & 0 & 9 & 6 & 0 \end{vmatrix}$ ↘ − ↗

$=(36+210+0)-(48+0+189)$

$= 9$

Example 2.18. Compute the determinant of

fix row 2, $i=2$

$j=1,2,3$

$$A = \begin{bmatrix} 1 & 2 & 7 \\ 0 & 0 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$\det(A) = 0 + 0 + (-1)^{2+3}(3)\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}$

$= 0$

2.7.1 Properties of determinants

- If we interchange two rows (or two columns) of A the resulting matrix has determinant equal to $-\det A$.
- If we add a multiple of one row to another row (similarly for columns), the resulting matrix has determinant equal to $\det A$.
- If we multiply a row or column of A by a scalar α, the resulting matrix has determinant equal to $\alpha(\det A)$.
- If A has a row or column of zeros, then $\det A = 0$.
- If two rows (or columns) of A are identical, then $\det A = 0$.

shown in example

- For any fixed $i = 1, \ldots, n$ we have $\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$.
- For any fixed $j = 1, \ldots, n$ we have $\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$.

Determinant test: An $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.

2.7.2 Vector cross product using determinants

The rule for a vector cross product can be conveniently expressed as a determinant. Thus if $\underset{\sim}{v} = v_x\underset{\sim}{i} + v_y\underset{\sim}{j} + v_z\underset{\sim}{k}$ and $\underset{\sim}{w} = w_x\underset{\sim}{i} + w_y\underset{\sim}{j} + w_z\underset{\sim}{k}$ then

$$\underset{\sim}{v} \times \underset{\sim}{w} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

2.7.3 Cramer's rule

Recall that if a linear system $A\underset{\sim}{x} = \underset{\sim}{b}$ has a unique solution, then $\underset{\sim}{x} = A^{-1}\underset{\sim}{b}$ is this solution. If we substitute the formula for the inverse A^{-1} from the previous section (using det S_{ji}) into the product $A^{-1}\underset{\sim}{b}$ we arrive at **Cramer's rule** for solving the linear system $A\underset{\sim}{x} = \underset{\sim}{b}$.

Cramer's rule: Let $A\underset{\sim}{x} = \underset{\sim}{b}$ be a linear system with a unique solution. This means that A is a square matrix with non-zero determinant. Let A_i be the matrix that results from A by replacing the ith column of A by $\underset{\sim}{b}$. Then

$$x_i = \frac{det A_i}{det A}$$

Examples of Cramer's rule will be given in tutorials.

2.8 Obtaining inverses using Gauss-Jordan elimination

The most efficient method for computing the inverse of a matrix is by **Gauss-Jordan elimination** which we have met earlier.

- Use row-operations to reduce A to the identity matrix.
- Apply exactly the same row-operations to a matrix set initially to the identity.
- The final matrix is the inverse of A.

We usually record this process in a large augmented matrix.

$[A|I]$

- Start with $[A|I]$.
- Apply row operations to obtain $[I|A^{-1}]$
- Read off A^{-1}, the inverse of A.

check $AA^{-1} = I$

$[I|A^{-1}]$

Recall that some texts use the term *Gaussian elimination* to refer to reducing a matrix to its *echelon* form, and the term *Jordan elimination* to refer to reducing a matrix to its *reduced echelon* form. In this manner, the **Gauss-Jordan algorithm** can be described diagrammatically as follows:

$$[A|I] \xrightarrow[G.A]{} [U|*] \xrightarrow[J.A]{} [I|B] \quad \text{where } B = A^{-1}.$$

In words, provided A has rank n:

- augment A by the identity matrix;
- perform the Gaussian algorithm to bring A to echelon form U and $[A|I]$ to $[U|*]$;
- perform the Jordan algorithm to bring U to reduced echelon form I and $[U|*]$ to $[I|B]$ (in other words use elementary row operations to make the pivots all 1s and to produce zeros above the pivots);
- then $B = A^{-1}$.

Example 2.19. Use the Gauss-Jordan algorithm to invert the following matrix A.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 4 & 4 & 0 & 0 & 1 \end{array}\right]$$

$A = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & 1 \\ 1 & 4 & 4 \end{pmatrix}$

$$\xrightarrow{R_3 \leftarrow R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 3 & 1 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_3 \leftarrow 3R_3 - 2R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 3 & -2 \end{array}\right) \xrightarrow[R_1 \leftarrow R_1 - 3R_3]{R_2 \leftarrow R_2 - R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -5 & -9 & 6 \\ 0 & 2 & 0 & -2 & -2 & 2 \\ 0 & 0 & 1 & 2 & 3 & -2 \end{array}\right)$$

$$\xrightarrow[R_2 \leftarrow \frac{1}{2} R_2]{R_1 \leftarrow R_1 - \frac{1}{2} R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & -8 & 5 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 2 & 3 & -2 \end{array}\right) \qquad A^{-1} = \begin{pmatrix} -4 & -8 & 5 \\ -1 & -1 & 1 \\ 2 & 3 & -2 \end{pmatrix}$$

Gauss Jordan Algorithm is to help us to get inverse of matrix A only?

Example 2.20. Solve the linear system

$$\begin{array}{ccccccc} x & + & y & + & 3z & = & 2 \\ & & 2y & + & z & = & 0 \\ x & + & 4y & + & 4z & = & 1 \end{array}$$

Convert this linear system into matrix form

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & 1 \\ 1 & 4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \qquad AX = B, \quad X = A^{-1}B$$

From Example 2.19 $A^{-1} = \begin{pmatrix} -4 & -8 & 5 \\ -1 & -1 & 1 \\ 2 & 3 & -2 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 & -8 & 5 \\ -1 & -1 & 1 \\ 2 & 3 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix}$$

2.8.1 Inverse - another method

Here is another way to compute the inverse of a matrix.

- Select the i^{th} row and j^{th} column of A.
- Compute $(-1)^{i+j}\frac{\det S_{ij}}{\det A}$
- Store this entry at a_{ji} (row j and column i) in the inverse matrix.
- Repeat for all other entries in A.

That is, if

$$A = \left[\quad a_{ij} \quad \right]$$

then

$$A^{-1} = \frac{1}{\det A}\left[\quad (-1)^{i+j}\det S_{ji} \quad \right]$$

This method works but it is rather tedious.