Chapter 2

Matrices

eg.

$$\begin{pmatrix} 1 & -2 \\ 3 & -4 \\ 2 & 5 \end{pmatrix}$$

3x2

3 by 2 ⇒ 3 rows

size ⇒ 3x2 ⇒ 2 columns

2.1 Introduction - notation and operations

read: m by n

An $m \times n$ **matrix** A is a rectangular array of numbers consisting of m rows and n columns. We say A is of **size** $m \times n$.

We use capital letters to represent matrices, for example:

a_{ij}

eg. $a_{31} = 2$

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

3x3 2x2

Entries within a matrix are denoted by subscripted lower case letters. For the matrix A above we have $a_{11} = 3$, $a_{12} = 2$, $a_{13} = -1$, $a_{21} = 1$, $a_{22} = -1$, $a_{23} = 1$ and so forth. Here, A is a 3×3 matrix

read: row then column

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

where a_{ij} = the entry in row i and column j of A.

An $m \times n$ matrix can be represented similarly. Note that m denotes number of **rows** in the matrix, and n denotes the number of **columns**.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

For brevity we sometimes write $A = [a_{ij}]$. This also reminds us that A is a **matrix** with **elements** a_{ij}.

A **square** matrix is an $n \times n$ matrix.

↳ #rows = #columns

2x2

3x3

4x4

Add / subtract

2.1.1 Operations on matrices

- *Equality*:

$$A = B$$

only when all entries in A equal those in B.

- *Addition*: Normal addition of corresponding elements. For example:

$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Same sizes

$$\underset{2\times 3}{\begin{bmatrix} 1 & 2 & 7 \\ 2 & -1 & 3 \end{bmatrix}} + \underset{2\times 3}{\begin{bmatrix} 3 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix}} = \underset{2\times 3}{\begin{bmatrix} 4 & 3 & 11 \\ 3 & 1 & 4 \end{bmatrix}}$$

- *Multiplication by a number*: $\lambda A = \lambda$ times each entry of A. For example:

$$5 \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 15 \\ 20 & 5 \end{bmatrix}$$

Check for multiplication

# of column(s) for 1st matrix = # of row(s) for 2nd matrix

- *Multiplication of matrices*:

$ux = \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 6 & 2 \end{pmatrix}$

row #1 × column #1

⇒ just one entry

$$\begin{bmatrix} & & \cdots & & \\ & & \cdots & & \\ a & b & c & d & \cdots \\ & & \cdots & & \\ & & \cdots & & \end{bmatrix} \begin{bmatrix} \cdots & e & \cdots \\ \cdots & f & \cdots \\ \cdots & g & \cdots \\ \cdots & h & \cdots \\ \cdots & \vdots & \cdots \end{bmatrix} = \begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & i & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \vdots & \cdots \end{bmatrix}$$

$$\boxed{i = a \cdot e + b \cdot f + c \cdot g + d \cdot h + \ldots}$$

$\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 6 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1+6 & -1+2 \\ 2+24 & -2+8 \end{pmatrix}$

$= \begin{pmatrix} 7 & 1 \\ 26 & 6 \end{pmatrix}$

Note that we can only multiply matrices that fit together. That is, if A and B are a pair of matrices, then in order that AB make sense, we must have the number of columns of A equal to the number of rows of B. We also say that matrices A and B are **compatible** for multiplication if A has size $m \times n$ and B has size $m \times r$. The product AB is then a matrix of size $n \times r$.

- *Transpose*: Flip rows and columns, denoted by $[\cdots]^T$. For example:

$$\begin{bmatrix} 1 & 2 & 7 \\ 0 & 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 7 & 4 \end{bmatrix}$$

Example 2.1. Does the following make sense?

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 0 & 2 \\ 4 & 1 \end{bmatrix}$$

2×2 3×2

×

$\begin{pmatrix} 1 & 7 \\ 0 & 2 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$

3×2 2×2

✓

$\begin{pmatrix} 1 & 7 \\ 6 & 2 \\ 4 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$

$= \begin{pmatrix} 2+28 & 3+7 \\ 0+8 & 0+2 \\ 8+4 & 12+1 \end{pmatrix} = \begin{pmatrix} 30 & 10 \\ 8 & 2 \\ 12 & 13 \end{pmatrix}$

$AB \neq BA$

2.1.2 Some special matrices

- *The Identity matrix*: (square matrix)

eg.

$$\begin{pmatrix} 5 & 5 \\ 2 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

For any square matrix A we have $IA = AI = A$.

- *The Zero matrix*: A matrix whose entries are all zeroes.
- *Symmetric matrices*: Any matrix A for which $A = A^T$. Transport — row into column
- *Skew-symmetric matrices*: Any matrix A for which $A = -A^T$. Sometimes also called anti-symmetric.

Symmetric

$$A: \begin{pmatrix} 2 & 6 & 4 \\ 6 & -1 & 0 \\ 4 & 0 & -3 \end{pmatrix}$$

$$AT = \begin{pmatrix} 2 & 6 & 4 \\ 6 & -1 & 0 \\ 4 & 0 & -3 \end{pmatrix}$$

Skew-Symmetric

$$A = \begin{pmatrix} 0 & -1 & 3 & 6 \\ 1 & 0 & 2 & -5 \\ -3 & -2 & 0 & 0 \\ -6 & 5 & -4 & \end{pmatrix}$$

$$-AT = \begin{pmatrix} 0 & 1 & -3 & 6 \\ -1 & 0 & -2 & 5 \\ 3 & 2 & 0 & 4 \\ 6 & -5 & -5 & 0 \end{pmatrix}$$

2.1.3 Properties of matrices

- $AB \neq BA$
- $(AB)C = A(BC)$
- $(A^T)^T = A$
- $(AB)^T = B^T A^T$

Example 2.2. Given $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 7 \\ 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$ verify the above four properties.

$$AB = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 7 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 20 \\ 4 & 30 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 7 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 30 & 10 \\ 8 & 2 \end{pmatrix}$$

$$AB \neq BA$$

$$(AB)C = A(BC)$$

$$\begin{pmatrix} 2 & 20 \\ 4 & 30 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 64 & 2 \\ 98 & 4 \end{pmatrix}$$

$$A(BC) = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 23 & 1 \\ 6 & 0 \end{pmatrix} = \begin{pmatrix} 64 & 2 \\ 98 & 4 \end{pmatrix}$$

$$(AB)C = A(BC)$$

$$(A^T) = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}^T = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$$

$$(A^T)^T = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$$

$$(A^T)^T = A$$

$$(AB)^T = B^T A^T$$

$$AB^T = \begin{pmatrix} 2 & 20 \\ 4 & 30 \end{pmatrix}^T = \begin{pmatrix} 2 & 4 \\ 20 & 30 \end{pmatrix}$$

$$B^T A^T = \begin{pmatrix} 1 & 0 \\ 7 & 2 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 20 & 30 \end{pmatrix}$$

$$(ABC)^T = C^T B^T A^T$$

Algebra

$2x = 9$ | $x = 9 \times 2^{-1}$

$x = \frac{9}{2}$

Algebra

$2 \times \frac{1}{2} = 1$

$2(2^{-1}) = 1$

AB

$A+B$

$A-B$

$\frac{A}{B} = \frac{\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}}{\begin{pmatrix} 1 & -1 \\ 0 & 5 \end{pmatrix}}$?

$A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -1 \\ 0 & 5 \end{pmatrix}$

Given matrix A. Then exist A^{-1} iff

(1) A must be square eg. 2×2, 3×3

(2) $\det(A) = |A| \neq 0$

Check $AA^{-1} = I$

2.1.4 Inverses of square matrices

A square matrix A is called **invertible** if there is a matrix B such that $AB = BA = I$ (where I is the identity matrix). We call B the **inverse** of A and write $B = A^{-1}$.

In later lectures we will see how to compute the inverse of a square matrix.

2.2 Gaussian Elimination

In previous lectures we introduced systems of linear equations, and briefly looked at how to solve these. The most efficient method for solving systems of linear equations is by using **Gaussian elimination**. This is essentially the row reduction that we have already encountered, but with a few extra steps. We will walk through this method using a typical example.

$$\begin{array}{rcrcrcll} 2x & + & 3y & + & z & = & 10 & \quad (1) \\ x & + & 2y & + & 2z & = & 10 & \quad (2) \\ 4x & + & 8y & + & 11z & = & 49 & \quad (3) \end{array}$$

$$\begin{array}{rcrcrcll} 2x & + & 3y & + & z & = & 10 & \quad (1) \\ & & y & + & 3z & = & 10 & \quad (2)' \leftarrow 2(2) - (1) \\ & & 2y & + & 9z & = & 29 & \quad (3)' \leftarrow (3) - 2(1) \end{array}$$

$$\begin{array}{rcrcrcll} 2x & + & 3y & + & z & = & 10 & \quad (1) \\ & & y & + & 3z & = & 10 & \quad (2)' \\ & & & & 3z & = & 9 & \quad (3)'' \leftarrow (3)' - 2(2)' \end{array}$$

Previously we would then solve this system using back-substitution, $z = 3$, $y = 1$, $x = 2$.

Note how we record the next set of row-operations on each equation. This makes it much easier for someone else to see what you are doing and it also helps you track down any arithmetic errors.

In this example we found

Example 2.3

Back substitution

$$\begin{array}{rcrcrcll} 2x & + & 3y & + & z & = & 10 & \quad (1) \\ & & y & + & 3z & = & 10 & \quad (2)' \\ & & & & 3z & = & 9 & \quad (3)'' \end{array}$$

$(3)'' \rightarrow z = 3$

$(2)' \rightarrow y + 3(3) = 10$, $y = 1$

Why stop there? We can apply more row-operations to eliminate terms *above* the diagonal. This does *not* involve back-substitution. This method

(1) $2x = 4$

$x = 2$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

$$\begin{aligned} 2x + 3y + z &= 10 \quad -(1) \\ y + 3z &= 10 \quad -(2)' \\ 3z &= 9 \quad -(3)'' \end{aligned}$$

$$\begin{aligned} 6x + 9y &= 21 \quad (1)' \leftarrow (1)\times 3 - (3)'' \\ y &= 1 \quad -(2)'' \leftarrow (2)' - (3)'' \\ 3z &= 9 \quad -(3)'' \end{aligned}$$

$$\begin{aligned} 6x &= 12 \quad (1)'' - (1)' - (2)''\times 9 \\ y &= 1 \quad -(2)'' \\ 3z &= 9 \quad -(3)'' \end{aligned}$$

$$\begin{aligned} x &= 2 \\ y &= 1 \\ z &= 3 \end{aligned}$$

of row reduction is known as **Gaussian elimination**.[1]

Example 2.3. Continue from the previous example and use row-operations to eliminate the terms above the diagonal. Hence solve the system of equations.

2.2.1 Gaussian elimination strategy

1. Use row-operations to eliminate elements below the diagonal.
2. Use row-operations to eliminate elements above the diagonal.
3. If possible, re-scale each equation so that each diagonal element $= 1$.
4. The right hand side is now the solution of the system of equations.

If you stop after step one you are doing Gaussian elimination with back-substitution (this is usually the easier option).

2.2.2 Exceptions

Here are some examples where problems arise.

Example 2.4. A zero on the diagonal

$$\begin{array}{rcrcrcrcrl} 2x & + & y & + & 2z & + & w & = & 2 & \quad (1) \\ 2x & + & y & - & z & + & 2w & = & 1 & \quad (2) \\ x & - & 2y & + & z & - & w & = & -2 & \quad (3) \\ x & + & 3y & - & z & + & 2w & = & 2 & \quad (4) \end{array}$$

$$\begin{array}{rcrcrcrcrl} 2x & + & y & + & 2z & + & w & = & 2 & \quad (1) \\ & & 0y & - & 3z & + & w & = & -1 & \quad (2)' \leftarrow (2) - (1) \\ & - & 5y & + & 0z & - & 3w & = & -6 & \quad (3)' \leftarrow 2(3) - (1) \\ & + & 5y & - & 4z & + & 3w & = & 2 & \quad (4)' \leftarrow 2(4) - (1) \end{array}$$

[1]In some texts, using row operations to eliminate terms below the diagonal only is known as Gaussian elimination, whereas using row operations to eliminate terms below and above the diagonal is known as Gauss-Jordan elimination.

The zero on the diagonal on the second equation is a serious problem, it means we can not use that row to eliminate the elements below the diagonal term. Hence we swap the second row with any other lower row so that we get a non-zero term on the diagonal. Then we proceed as usual.

$$
\begin{array}{rcrcrcrcrl}
2x & + & y & + & 2z & + & w & = & 2 & \quad (1) \\
 & - & 5y & + & 0z & - & 3w & = & -6 & \quad (2)'' \leftarrow (3)' \\
 & & 0y & - & 3z & + & w & = & -1 & \quad (3)'' \leftarrow (2)' \\
 & + & 5y & - & 4z & + & 3w & = & 2 & \quad (4)' \leftarrow 2(4) - (1)
\end{array}
$$

The result is $w = 2$, $z = 1$, $y = 0$ and $x = -1$.

Example 2.5. A consistent and under-determined system

Suppose we start with three equations and we wind up with

$$
\begin{array}{rcrcrcrl}
2x & + & 3y & - & z & = & 1 & \quad (1) \\
 & - & 5y & + & 5z & = & -1 & \quad (2)' \\
 & & & & 0z & = & 0 & \quad (3)''
\end{array}
$$

The last equation tells us nothing! We can't solve it for any of x, y and z. We really only have 2 equations, not 3. That is 2 equations for 3 unknowns. This is an **under-determined** system.

We solve the system by choosing any number for one of the unknowns. Say we put $z = \lambda$ where λ is any number (our choice). Then we can leap back into the equations and use back-substitution.

The result is a **one-parameter family of solutions**

$$x = \frac{1}{5} - \lambda, \qquad y = \frac{1}{5} + \lambda, \qquad z = \lambda$$

y - let $y = \lambda$

$(2)' \Rightarrow z = \frac{5\lambda - 1}{5}$

Since we found a solution we say that the system is **consistent**.

Example 2.6. An inconsistent system

Had we started with

$$
\begin{array}{rcrcrcrl}
2x & + & 3y & - & z & = & 1 & \quad (1) \\
x & - & y & + & 2z & = & 0 & \quad (2) \\
3x & + & 2y & + & z & = & 0 & \quad (3)
\end{array}
$$

we would have arrived at

$$\begin{array}{rcrcrcrl} 2x & + & 3y & - & z & = & 1 & \quad (1) \\ & - & 5y & + & 5z & = & -1 & \quad (2)' \\ & & & & 0z & = & -2 & \quad (3)'' \Leftarrow \text{Not logic} \end{array}$$

This last equation makes no sense as there are no **finite** values for z such that $0z = -2$ and thus we say that this system is **inconsistent** and that the system has **no solution**.

2.3 Systems of equations using matrices

Consider the system of equations

$$\begin{array}{rcrcrcr} 3x & + & 2y & - & z & = & -1 \\ x & - & y & + & z & = & 4 \\ 2x & + & y & - & z & = & -1 \end{array}$$

We can rewrite this system using matrix notation. The **coefficients** of our equations form a 3×3 matrix A

$$A = \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \end{array} \quad \text{coefficient matrix}$$

The variables (x, y and z) can be written as a 3×1 matrix (also known as a **column vector**) X

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and the right hand side can also be written as a column vector B

$$B = \begin{bmatrix} \cancel{3} \\ \cancel{1} \\ \cancel{6} \end{bmatrix} \begin{array}{r} -1 \\ 4 \\ -1 \end{array}$$

Thus our system of equations $AX = B$ becomes

$AA^{-1} = I$

Why need inverse?

$$\begin{bmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cancel{3} \\ \cancel{1} \\ \cancel{6} \end{bmatrix} \begin{array}{r} -1 \\ 4 \\ -1 \end{array}$$

$AX = B$

$A^{-1}AX = A^{-1}B$

$IX = A^{-1}B$

$X = A^{-1}B$

Example 2.7. Write the system of equations

$$\begin{array}{rcrcrcr} 3x & + & 2y & - & z & = & -1 \\ x & - & y & + & z & = & 4 \\ 2x & + & y & - & z & = & -1 \end{array}$$

in matrix notation.

$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}$

$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$B = \begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix}$$

$AX = B$

2.3.1 The augmented matrix

Consider the system of equations:

$$\begin{array}{rcrcrcrl} 2x & + & 3y & + & z & = & 10 & (1) \\ x & + & 2y & + & 2z & = & 10 & (2) \\ 4x & + & 8y & + & 11z & = & 49 & (3) \end{array}$$

Coefficient Matrix

$A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \\ 4 & 8 & 11 \end{pmatrix}$

The **augmented matrix** of A is the matrix **augmented** by the column vector $\underline{b}$.

$b = \begin{pmatrix} 10 \\ 10 \\ 49 \end{pmatrix}$

Gaussian Elimination

① obtain Augmented matrix $[A|b]$

② Perform row-operation till echelon form - triangular (bottom entries are zero)

③ Back-substitution to solve the system eqn.

$$[A|\underline{b}] = \left[\begin{array}{ccc|c} 2 & 3 & 1 & 10 \\ 1 & 2 & 2 & 10 \\ 4 & 8 & 11 & 49 \end{array}\right]$$

Previously we used Gaussian elimination to solve systems of linear equations, where we labelled our equations (1), (2), (3) and so forth. It is much more efficient to set up our system using matrices, and then perform Gaussian elimination on the augmented matrix. Gaussian elimination (using matrices) consists of bringing the augmented matrix to **echelon form** using **elementary row operations**. This allows us to then solve a much simpler system of equations.

→ Staircase form

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 10 \\ 1 & 2 & 2 & 10 \\ 4 & 8 & 11 & 49 \end{array}\right) \xrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow 2R_2 - R_1} \left(\begin{array}{ccc|c} 2 & 3 & 1 & 10 \\ 0 & 1 & 3 & 10 \\ 0 & 2 & 9 & 29 \end{array}\right) \xrightarrow{R_3 \leftarrow R_3 - 2R_1} \left(\begin{array}{ccc|c} 2 & 3 & 1 & 10 \\ 0 & 1 & 3 & 10 \\ 0 & 0 & 3 & 9 \end{array}\right)$$

echelon form!

Back-substitution

$R_3 \rightarrow 3z = 9 \iff z = 3$

$R_2 \rightarrow y + 3z = 10$
$y = 1$

$R_1 \rightarrow 2x + 3y + z = 10$
$x = 2$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

we want to go until reduced echelon form:

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 10 \\ 0 & 1 & 3 & 10 \\ 0 & 0 & 3 & 9 \end{array}\right) \xrightarrow[R_2 \leftarrow R_2 - R_3,\ R_1 \leftarrow R_1 - \frac{1}{3}R_3]{R_3 \leftarrow \frac{1}{3}R_3} \left(\begin{array}{ccc|c} 2 & 3 & 0 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{R_1 \leftarrow R_1 - 3R_2} \left(\begin{array}{ccc|c} 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right) -$$

$$\xrightarrow{R_1 \leftarrow \frac{1}{2}R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right) \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

2.4 Row echelon form

A matrix is in **row echelon** form if it satisfies the following two conditions:

- If there are any zero rows, they are at the bottom of the matrix.
- The first non-zero entry in each non-zero row (called the **leading entry** or **pivot**) is to the right of the pivots in the rows above it.

A matrix is in **reduced echelon** form if it also satisfies:

- Each pivot entry is equal to 1.
- Each pivot is the only non-zero entry in its column.

Example 2.8. Write down three matrices in echelon form and circle the pivots.

$$\begin{array}{ccc} x & y & z \end{array} \quad \begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix} \qquad \begin{array}{ccc} x & y & z \end{array} \quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{ccc} x & y & z \end{array} \quad \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

pivots = 3   # pivots = 2   # pivots = 1

Leading variables: x, y, z   LV: x, z   LV: x

Free variables: None.   FV: y   FV: y, z

Example 2.9. Write down three matrices in reduced echelon form.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

pivots = 3   # pivots = 3   # pivots = 2

The variables corresponding to the columns containing pivots are called the **leading variables**. The variables corresponding to the columns that do not contain pivots are called **free variables**. Free variables are not restricted by the linear equations - they can take arbitrary values, and we often denote these by Greek letters (such as α, β and so forth). The leading variables are then expressed in therms of the free variables.

(LV)   (FV)

LV + # FV = Total number of variables

Example 2.10. For the following linear systems
(a) Write down the augmented matrix and bring it to echelon form.
(b) Identify the free variables and the leading variables.
(c) Write down the solution(s) if any exist.

(d) Give a geometric interpretation of your results.

(i)

(a) $\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -2 & 4 \end{array}\right) \xrightarrow{R_2 \leftarrow R_1 - R_2} \begin{array}{ccc} x & y & \\ \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 3 & -3 \end{array}\right) \end{array}$

$$x + y = 1$$
$$x - 2y = 4$$

(d) These two lines has intersection at $(2,-1)$

$\mathrm{rank}(A) = 2$

(b) Leading variable: x & y
Free variables: None

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

(c) Back-sub $\Rightarrow$ $R_2 \to y = -1$
$R_1 \to x = 2$

(ii)

(a) $\left(\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & -1 & 2 \end{array}\right) \xrightarrow{R_2 \leftarrow R_1 - R_2} \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & -1 \end{array}\right)$

$$x - y = 1$$
$$x - y = 2$$

$\Rightarrow$ inconsistent (No sol'n

(b) Leading variable: x
Free variable: y
$\mathrm{rank}(A) = 1$

(C) Back-sub.
$R_2 \to$ illegical situation

two lines parallel

(iii)

(a) $\left(\begin{array}{cc|c} 1 & -1 & 1 \\ 3 & -3 & 3 \end{array}\right) \xrightarrow{R_1 \leftarrow 3R_1 - R_2} \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right)$

$$x - y = 1$$
$$3x - 3y = 3$$

$\to$ $\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}$ $x - y = 1$

(b) Leading variable: X
Free variable: y
$\mathrm{rank}(A) = 1$

(c) $R_2 \to$ No working
$R_1 \to x - y = 1$
Let $y = \lambda$

$R_1 \to x = 1 + \lambda$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1+\lambda \\ \lambda \end{pmatrix}$ infinite sol'n.

Echelon Form

(A) No solution
$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & \neq 0 \end{array}\right)$

(B) unique solution
$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & 4 & 4 \end{array}\right)$
cannot be zero; Any real number

(C) infinite sol'n (bring parameters) μ, λ, γ
$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Example 2.11. Consider the linear system

(a) $\left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 2 & 6 & 9 & 5 & 1 \\ -1 & -3 & 3 & 0 & k \end{array}\right)$

$$x_1 + 3x_2 + 3x_3 + 2x_4 = 1$$
$$2x_1 + 6x_2 + 9x_3 + 5x_4 = 1$$
$$-x_1 - 3x_2 + 3x_3 = k$$

(C)(i) No sol'n, $k+3 \neq 0 \Leftrightarrow k \neq -3$
(ii) Infinite sol'n $k+3 = 0 \Leftrightarrow k = -3$
(iii) one sol'n, impossible

(b) LV: x_1 & x_3; FV: x_2 & x_4

(d) Infinite sol'n, $k = -3$
$\left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

(a) Write down the augmented matrix and bring it to echelon form.
(b) Identify the free variables and the leading variables.
(c) For what values of the number k does the system have (i) no solution, (ii) infinitely many solutions, (iii) exactly one solution?
(d) When a solution or solutions exist, find them.

$\to$ Back substitution
Let $x_2 = \alpha$, $x_4 = \beta$
$R_2 \to 3x_3 + \beta = -1$
$x_3 = \frac{-1-\beta}{3}$
$R_1 \to x_1 + 3\alpha - 1 - \beta + 2\beta = 1$
$x_1 = 2 - 3\alpha - \beta$

$R_2 \leftarrow R_2 - 2R_1$
$R_3 \leftarrow R_1 + R_3$

$\left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 3 & 1 & -1 \\ 0 & 0 & 6 & 2 & k+1 \end{array}\right) \xrightarrow{R_3 \leftarrow R_3 - 2R_2} \left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 0 & k+3 \end{array}\right)$

$\mathrm{rank}(A) = 2$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2-3\alpha-\beta \\ \alpha \\ -\frac{1}{3}-\frac{1}{3}\beta \\ \beta \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -\frac{1}{3} \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \end{pmatrix}$$

2.4.1 Rank

Rank of the coefficient A

The **rank** of a matrix is the number of non-zero rows (also the number of pivots) in its row echelon form. The rank of a matrix is denoted by rank(A).

rank(A) = # pivot(s)

The rank of a matrix gives us information about the solutions of the associated linear system.

Importantly, if the number of rows in the augmented matrix is equal to the rank of the matrix, then the system of linear equations has a **unique** solution.

A linear system of m equations in n variables will give an $m \times n$ matrix A. Once we have reduced the matrix to echelon form, and found the **rank** $= r$ of the reduced matrix (let's call the reduced matrix U) we can deduce the following informative properties:

Properties

Maximal rank(A) = total # of variable(s) $\Rightarrow$ unique sol'n

rank(A) $< n$

1. Number of variables $= n$
2. Number of leading variables $= r$
3. Number of free variables $= n - r$
4. $r \leq m$ (because there is at most one pivot in each of the m rows of U).
5. $r \leq n$ (because there is at most one pivot in each of the n columns of U).
6. If $r = n$ there are no free variables and there will be either no solution or one solution.
7. If $r < n$ there is at least one free variable and there will be either no solution or infinitely many solutions.
8. If there are more variables than equations, that is $n > m$, then $r < n$ and so there will be either no solution or infinitely many solutions.

Example 2.12. What is the rank of each of the matrices in the previous examples?

$$\begin{aligned} 2x + y - z &= 2 \\ x - 3y + z &= 5 \\ x + y - 3z &= -1 \end{aligned}$$

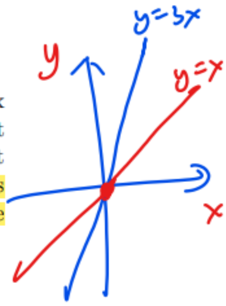
$$\begin{aligned} 2x + y - z &= 0 \\ x - 3y + z &= 0 \\ x + y - 3z &= 0 \end{aligned}$$

$y = x$

$y = 3x$

2.4.2 Homogeneous systems

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & -3 & 0 \end{array}\right)$$



A **homogeneous system** is one of the form $A\underset{\sim}{x} = \underset{\sim}{0}$. The augmented matrix is therefore $[A|\underset{\sim}{0}]$ and its echelon form is $[U|\underset{\sim}{0}]$. The last non-zero row cannot be $[0\,0\,\ldots\,0\,d]$, $d \neq 0$, so a homogeneous system is never inconsistent. In fact $\underset{\sim}{x} = \underset{\sim}{0}$ is always a solution. Geometrically, the lines, planes or hyperplanes represented by the equations in a homogeneous system all pass through the origin.

2.4.3 Summary

When we reduce a matrix to echelon form, we do so by performing **elementary row operations**. On a matrix, these operations are

- Interchange two rows (which we denote by $R_i \leftrightarrow R_j$).
- Multiply one row by a non-zero number ($cR_i \rightarrow R_i$).
- Add a multiple of one row to another row ($R_i + cR_j \rightarrow R_i$).

Every matrix can be brought to echelon form by a sequence of elementary row operations using **Gaussian elimination**. This is sometimes given as an algorithm:

A **linear system** of equations $A\underset{\sim}{x} = \underset{\sim}{b}$, or $AX = B$, can be written in the general form:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}$$

where A is the $m \times n$ **matrix of coefficients**, $\underset{\sim}{x}$ (or X) is the $n \times 1$ matrix (or column vector) of variables, and $\underset{\sim}{b}$ (or B) is the $m \times 1$ matrix (or column vector) of constant terms.

The **augmented matrix** is the matrix

$$[A|\underset{\sim}{b}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

In order to solve a general linear system $A\underset{\sim}{x} = \underset{\sim}{b}$ we:

1. Bring the augmented matrix to echelon form: $[A|\underset{\sim}{b}] \to [U|\underset{\sim}{c}]$. Since each elementary row operation is reversible, the two systems $A\underset{\sim}{x} = \underset{\sim}{b}$ and $U\underset{\sim}{x} = \underset{\sim}{c}$ have exactly the same solutions.

2. Solve the triangular system $U\underset{\sim}{x} = \underset{\sim}{c}$ by **back-substitution**.

For a **general linear system** $A\underset{\sim}{x} = \underset{\sim}{b}$ or its corresponding triangular form $U\underset{\sim}{x} = \underset{\sim}{c}$ there are **three** possibilities.

1. There is **no solution** - this happens when the last non-zero row of $[U|\underset{\sim}{c}]$ is $[0\,0 \dots 0 \; d\,]$ with $d \neq 0$, in which case the equations are **inconsistent**.

2. There are **infinitely many solutions** - this happens when the equations are **consistent** and there is at least one free variable.

3. There is **exactly one solution** - this happens when the equations are **consistent** and there are no free variables.