$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$ $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix}$

$\underset{\sim}{r} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$ $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix}$

Problem Set Two: Linear Algebra - Lines and Planes, Systems of Equations

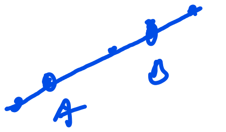
$y = mx + c$

Lines and Planes

position vector   direction vector

$\underset{\sim}{r} = \underset{\sim}{a} + t\underset{\sim}{b}$

1. Consider the points $(1, 2, -1)$ and $(2, 0, 3)$.



a. Find a vector equation of the line through these points in parametric form.

b. Find the distance between this line and the point $(1, 0, 1)$.

(a) $\underset{\sim}{b} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$

Vector eqn of line (vector form)

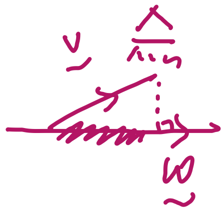
$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \underset{\sim}{r} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$

Parametric form

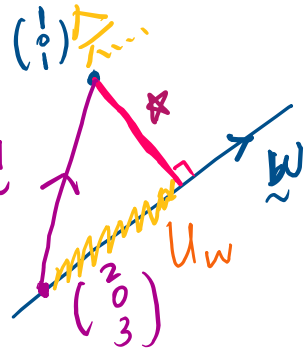
$x(t) = 2 + t$

$y(t) = -2t$

$z(t) = 3 + 4t$



(b) $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$



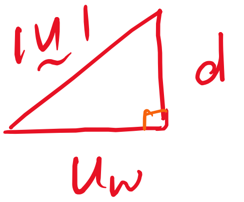
Form a vector $\underset{\sim}{u}$ by connecting the two points

$\underset{\sim}{u} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$\underset{\sim}{w} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$ So we can have scalar projection $\underset{\sim}{u}$ onto $\underset{\sim}{w}$

$$U_w = \frac{\underset{\sim}{u} \cdot \underset{\sim}{w}}{|\underset{\sim}{w}|} = \frac{1 + 0 + 8}{\sqrt{21}} = \frac{9}{\sqrt{21}}$$

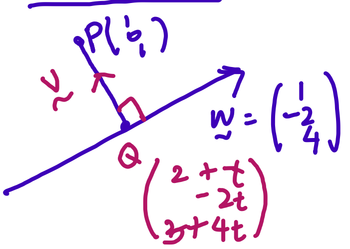
Pythagoras' Theorem -



$$d = \sqrt{|\underset{\sim}{u}|^2 - (U_d)^2}$$

$$d = \sqrt{(\sqrt{5})^2 - \left(\frac{9}{\sqrt{21}}\right)^2} = \frac{2\sqrt{14}}{7} \text{ units}$$

Method 2



$\underset{\sim}{v} = \begin{pmatrix} 1+t \\ -2t \\ 2+4t \end{pmatrix}$

$\underset{\sim}{v} \cdot \underset{\sim}{w} = 0$

$\begin{pmatrix} 1+t \\ -2t \\ 2+4t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} = 0$

$1 + t + 4t + 8 + 16t = 0$

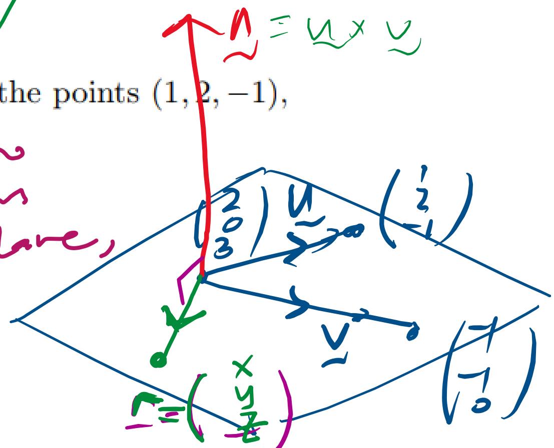
$21t = -9$

$t = -\frac{3}{7}$ $|\underset{\sim}{v}| = |\overrightarrow{PQ}|$

$$|\underset{\sim}{v}| = \sqrt{\left(\frac{4}{7}\right)^2 + \left(\frac{6}{7}\right)^2 + \left(\frac{2}{7}\right)^2}$$

$$= \frac{2\sqrt{14}}{7} \text{ units}$$

	i	j	k
u	-1	2	-4
v	-3	-1	-3



2. Find an equation of the plane that passes through the points $(1, 2, -1)$, $(2, 0, 3)$ and $(-1, -1, 0)$. First we need two vectors on the plane,

$$\underset{\sim}{u} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix} \checkmark$$

$$\underset{\sim}{v} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ -3 \end{pmatrix} \checkmark$$

we need two vector to form the normal vector, $\underset{\sim}{n} = \underset{\sim}{u} \times \underset{\sim}{v} = \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix} \times \begin{pmatrix} -3 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} -10 \\ 9 \\ 7 \end{pmatrix}$ / $\begin{pmatrix} 10 \\ -9 \\ -7 \end{pmatrix}$

any one of the point given

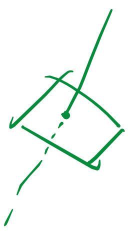
$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\left[\boxed{r} - \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \right] \cdot \underset{\sim}{n} = 0$

$$\begin{pmatrix} x-2 \\ y-0 \\ z-3 \end{pmatrix} \cdot \begin{pmatrix} -10 \\ 9 \\ 7 \end{pmatrix} = 0 \rightarrow \text{vector form}$$

$-10x + 20 + 9y + 7z - 21 = 0$ ✓ cartesian form

$10x - 9y - 7z = -1$ / $-10x + 9y + 7z = 1$ (same)

3. Consider a plane defined by the equation $3x + 4y - z = 2$ and a line defined by the following vector equation (in parametric form)



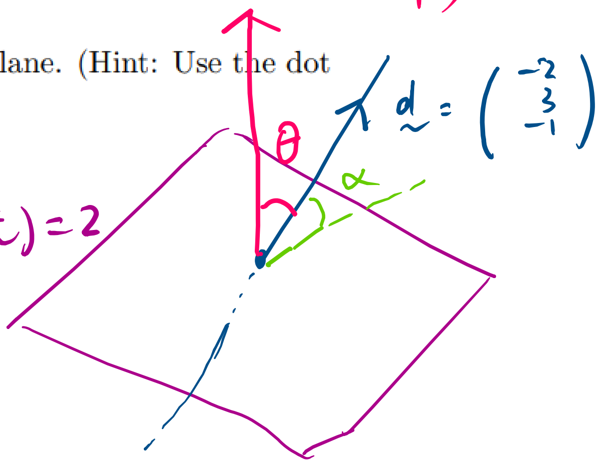
$$x(t) = 2 - 2t, \qquad y(t) = -1 + 3t, \qquad z(t) = -t. \qquad \underset{\sim}{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$$

a. Find the point where the line intersects the plane. (Hint: Substitute the parametric form into the equation of the plane.)

$\underset{\sim}{n} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$

b. Find a normal vector to the plane. ✓

c. Find the angle at which the line intersects the plane. (Hint: Use the dot product.)



$3x + 4y - z = 2$

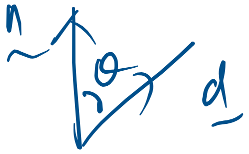
(a) $3(2-2t) + 4(-1+3t) - (-t) = 2$

$6 - 6t - 4 + 12t + t = 2$

$t = 0$ ✓

POI $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

(b) normal vector plane, $\underset{\sim}{n} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$



$$\cos\theta = \frac{\underset{\sim}{n}\cdot\underset{\sim}{d}}{|\underset{\sim}{n}||\underset{\sim}{d}|}$$

$$= \frac{-6+12+1}{\sqrt{26}\sqrt{14}}$$

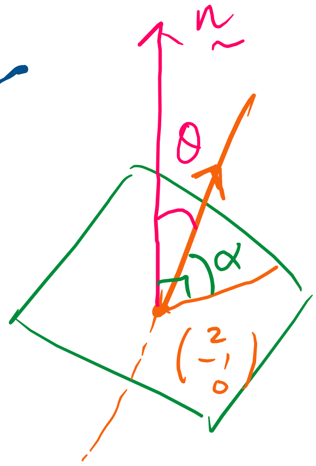
✓ $\theta = \cos^{-1}\left(\frac{7}{\sqrt{26}\sqrt{14}}\right)$

$\theta = 68.5°$

Angle required

$\alpha = 90° - 68.5$

$\alpha = 21.5$



4. Find the minimum distance between the two lines defined by

$$x(t) = 1 + t, \qquad y(t) = 1 - 3t, \qquad z(t) = -2 + 2t$$

$\underset{\sim}{r_1} = \begin{pmatrix}1\\1\\-2\end{pmatrix} + t\begin{pmatrix}1\\-3\\2\end{pmatrix}$

and

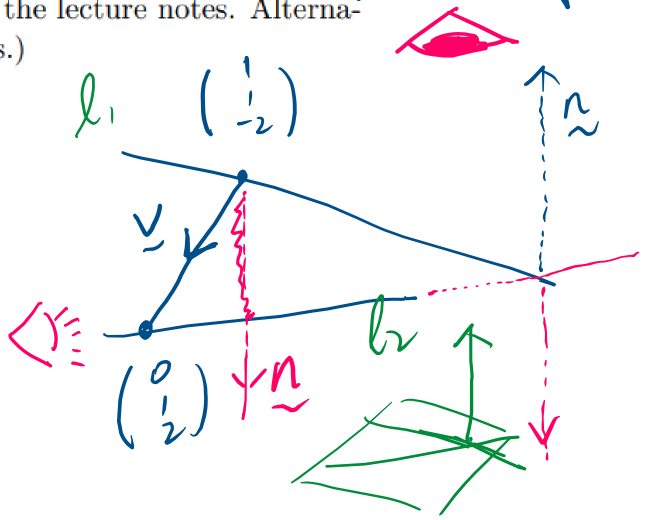
$$x(s) = 3s, \qquad y(s) = 1 - 2s, \qquad z(s) = 2 - s$$

$\underset{\sim}{r_2} = \begin{pmatrix}0\\1\\2\end{pmatrix} + s\begin{pmatrix}3\\-2\\-1\end{pmatrix}$

(Hint: Use scalar projection as demonstrated in the lecture notes. Alternatively, first define the lines within parallel planes.)

connecting two points on the two lines

$\underset{\sim}{v} = \begin{pmatrix}0\\1\\2\end{pmatrix} - \begin{pmatrix}1\\1\\-2\end{pmatrix} = \begin{pmatrix}-1\\0\\4\end{pmatrix}$ ✓



$\underset{\sim}{n} = \underset{\sim}{d_1} \times \underset{\sim}{d_2} = \begin{pmatrix}1\\-3\\2\end{pmatrix} \times \begin{pmatrix}3\\-2\\-1\end{pmatrix}$

scalar project $\underset{\sim}{v}$ on $\underset{\sim}{n}$

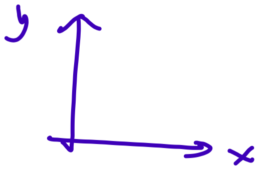
$\underset{\sim}{n} = \begin{pmatrix}7\\7\\7\end{pmatrix} = 7\begin{pmatrix}1\\1\\1\end{pmatrix}$

distance $= V_n$

$\underset{\sim}{n} = \begin{pmatrix}1\\1\\1\end{pmatrix}$

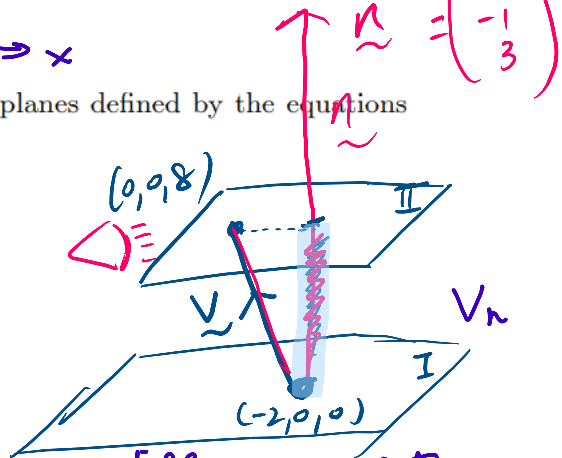
$$= \frac{\underset{\sim}{v}\cdot\underset{\sim}{n}}{|\underset{\sim}{n}|} = \frac{-1+0+4}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3} \text{ units}$$

$y = 2x+1$



5. Find the distance between the parallel planes defined by the equations $2x - y + 3z = -4$ and $2x - y + 3z = 24$.

I II



We need to find the random point on each plane.

Random substitution

plane I $2x - y + 3z = -4$

set $y = z = 0$, $x = -2$ $(-2,0,0)$

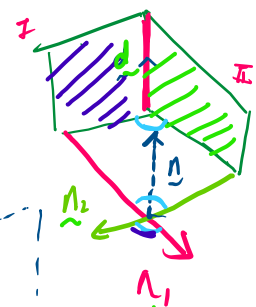
plane II $2x - y + 3z = 24$

set $x = y = 0$, $3z = 24$, $z = 8$ $(0,0,8)$

may have different points. ⇒ form a vector connect these two point, so that we can do scalar projection on $\underset{\sim}{n}$

$$\underset{\sim}{V} = \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 8 \end{pmatrix}$$

$$V_n = \frac{\underset{\sim}{v} \cdot \underset{\sim}{n}}{|\underset{\sim}{n}|} = \frac{4 + 0 + 24}{\sqrt{14}} = \frac{28}{\sqrt{14}} \text{ unit}$$



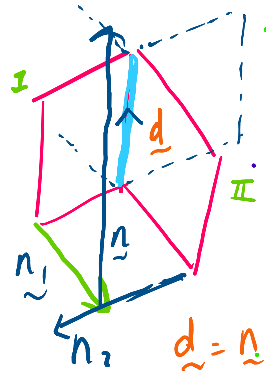
6. Consider two planes defined by the equations $3x + 4y - z = 2$ and $-2x + y + 2z = 6$.

(a) Find where the planes intersect the x, y and z axes.

(b) Find normal (ie perpendicular) vectors for the planes.

(c) Find an equation of the line defined by the intersection of these planes. (Hint: Use the normal vectors to define the direction of the line.)

(d) Find the angle between these two planes.



$\underset{\sim}{d} = \underset{\sim}{n} = \underset{\sim}{n_1} \times \underset{\sim}{n_2}$

	$3x+4y-z=2$	$-2x+y+2z=6$
(a) x-intercept $(y=z=0)$	$(\frac{2}{3},0,0)$	$(-3,0,0)$
y-intercept $(x=z=0)$	$(0,\frac{1}{2},0)$	$(0,6,0)$
z-intercept $(x=y=0)$	$(0,0,-2)$	$(0,0,3)$
(b) normal	$\underset{\sim}{n_1} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$	$\underset{\sim}{n_2} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$

$$\underset{\sim}{d} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \\ 11 \end{pmatrix}$$

we need a point on the line

$$\begin{cases} 3x+4y-z=2 & -① \\ -2x+y+2z=6 & -② \end{cases}$$

$① \times 2 + ② \Rightarrow 4x + 9y = 10$

Choose $y = 2$, $4x + 18 = 10$

$x = -2$

Subs $x=-2, y=2$ into ①

$z = 0$ ⇒ $(-2,2,0)$

⇒ may be different point

(c) line of intersection for two planes

$$\underset{\sim}{r} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 9 \\ -4 \\ 11 \end{pmatrix} t$$ / $4x + 9y = 10$ ✓

(d) $\cos\theta = \dfrac{n_1 \cdot n_2}{|n_1||n_2|} \Leftrightarrow \theta = \cos^{-1}\left(\dfrac{-6+4-2}{\sqrt{26}\sqrt{9}}\right) = 105.2^\circ$

$= \cos^{-1}\left(\dfrac{-4}{3\sqrt{26}}\right)$

Systems of Equations

7. Solve each of the following linear systems using elementary row operations and back substitution.

(a)

$$\begin{cases} x + y = 5 & -(1) \\ 2x + 3y = 1 & -(2) \end{cases}$$

$(1)\times 2 - (2) \Rightarrow$

$$\begin{array}{r} 2x + 2y = 10 \\ -\ \ 2x + 3y = 1 \\ \hline -y = 9 \end{array}$$

$$\begin{cases} x + y = 5 & -(1) \\ -y = 9 & -(2)' \end{cases} \leftarrow (1)\times 2 - (2)$$

* Back substitution $\Rightarrow$ (2)' $\Rightarrow$ $y = -9$ ✓

Subs $y = -9$ into (1) $\quad x + (-9) = 5$

$x = 14$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ -9 \end{pmatrix}$$

(b)

$$\begin{cases} x + 2y - z = 6 & -(1) \\ 2x + 5y - z = 13 & -(2) \\ x + 3y - 3z = 4 & -(3) \end{cases}$$

$$\begin{cases} x + 2y - z = 6 & -(1) \\ y + z = 1 & -(2)' \leftarrow (2) - (1)\times 2 \\ y - 2z = -2 & -(3)' \leftarrow (3) - (1) \end{cases}$$

$$\begin{cases} x + 2y - z = 6 & -(1) \\ y + z = 1 & -(2)' \\ 3z = 3 & -(3)'' \leftarrow (2)' - (3)' \end{cases}$$

Back Subs

(3)'' $\Rightarrow z = 1$

Subs $z = 1$ into (2)' $\Rightarrow y = 0$

Subs $y = 0$, $z = 1$ into (1)

$x = 7$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 1 \end{pmatrix}$$

8. Find all possible solutions for the following (under-determined) system of equations

$$\begin{aligned} x + 2y - z &= 6 \\ 2x + 5y - z &= 13 \\ 2x + 4y - 2z &= 12 \end{aligned}$$

$\times 2 \Rightarrow$

$$\begin{cases} x + 2y - z = 6 & -(1) \\ 2x + 5y - z = 13 & -(2) \end{cases}$$

(Hint: You have two equations but three unknowns. You will need to introduce one free parameter.)

$$\begin{cases} x + 2y - z = 6 & -(1) \\ y + z = 1 & -(2)' \leftarrow (2) - (1)\times 2 \end{cases}$$

free parameter

Back subs. from (2)'. Let $z = \alpha$, $y = 1 - \alpha$

(1) $\Rightarrow$ $x + 2 - 2\alpha - \alpha = 6$

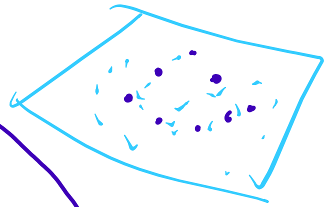
$x = 4 + 3\alpha$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 + 3\alpha \\ 1 - \alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

eqn line vector form

9. Find all possible solutions for the system (yes, this system has only one equation) of equations

$$x + 2y - z = 6 \longrightarrow \text{A plane}$$



(Hint: You have one equation but three unknowns. You will need to introduce two free parameters.)

let $\boxed{\begin{matrix} y = \alpha \\ z = \beta \end{matrix}}$ $\Rightarrow$ $x + 2\alpha - \beta = 6$

$x = 6 - 2\alpha + \beta$

In parametric form.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 - 2\alpha + \beta \\ \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

10. Consider two planes defined by $3x+4y-z = 2$ and $-2x+y+2z = 6$. Find the intersection of the planes using linear systems (ie solve the equations for x, y and z, without taking the cross product of the vectors normal to each plane). $\rightarrow$ Q6

$3x + 4y - z = 2$ — (1)

$-2x + y + 2z = 6$ — (2)

$(1) \times 2 \Rightarrow 6x + 8y - 2z = 4$

$(2) \times 3 \Rightarrow -6x + 3y + 6z = 18$ (+)

$11y + 4z = 22$

$3x + 4y - z = 2$ — (1)

$11y + 4z = 22$ — (2)' $\leftarrow$ (1) × 2 + (2) × 3

Back Subs (2)', let $z = t$ ✓

$11y = 22 - 4t$

$y = 2 - \frac{4}{11}t$ ✓

Subs ~~z~~ & y into (1):

$(1) \Rightarrow 3x + 4(2 - \frac{4}{11}t) - t = 2$

$3x + 8 - \frac{16}{11}t - t = 2$ $\qquad \underset{\sim}{v} = 2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$3x = -6 + \frac{27}{11}t$ $\qquad \underset{\sim}{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$x = -2 + \frac{9}{11}t$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 + \frac{9}{11}t \\ 2 - \frac{4}{11}t \\ t \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} + \frac{t}{11}\begin{pmatrix} 9 \\ -4 \\ 11 \end{pmatrix}$$

Q6 $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 9 \\ -4 \\ 11 \end{pmatrix}$