MAT1830 - Discrete Mathematics for Computer Science
Tutorial Sheet #9 and Additional Practice Questions

Tutorial Questions

1. Captain Kirk fires photon torpedoes, one at a time, at a Romulan ship until he hits it. Each torpedo has a 10% chance of hitting the ship (independent of any other torpedoes). Let X be the number of torpedos that *miss* the ship.

   (a) What's the name for the probability distribution of X?

   (b) What is the probability that $X \leq 2$?

   (c) Give an explanation for your why your answer to (b) is what it is. (Without referring to formulas.)

   (d) What would be the (likely) effect on X if Mr Spock fires the torpedoes instead? Spock is a much better shot than Kirk.

2. Write down the next four values of each of the following recursive sequences.

   (a) $r_0 = 3, \qquad r_n = 2r_{n-1} - 1 \quad$ for all integers $n \geq 1$.

   (b) $t_0 = 1, t_1 = 1, t_2 = -2, \qquad t_n = t_{n-1}t_{n-3} \quad$ for all integers $n \geq 3$.

3. Rewrite the following expressions without using $\sum$ or $\prod$.

   (a) $\displaystyle\sum_{i=6}^{10} \frac{1}{2i+1}$ (b) $\displaystyle\prod_{i=4}^{6} \left(\frac{x^i}{2i} + i\right)$ (c) $\displaystyle\sum_{i=0}^{3} \frac{(-1)^i}{(2i+1)!} x^{2i+1}$

4. Aperture Labs has a research division of 400 scientists, each of whom has a 2.5% chance of making a major breakthrough each year. Let Y be the number major breakthroughs made by the Aperture research division in a given year.

   (a) What's the name for the probability distribution of Y?

   (b) What is $\mathrm{E}[Y]$? What is $\mathrm{Var}[Y]$?

   (c) What's the probability that exactly $\mathrm{E}[Y]$ breakthroughs are made?

   (d) Aperture's annual report confidently predicts that between 5 and 20 major breakthroughs will be made this year. Write an expression for the probability of this occurring.

   Umbrella Corp has a research division of 25 scientists, each of whom has a 40% chance of making a major breakthrough each year. Let Z be the number major breakthroughs made by the Umbrella research division in a given year.

   (e) What is $\mathrm{E}[Z]$? What is $\mathrm{Var}[Z]$?

   (f) Which row in the following table would you guess belongs to each company? Why?

company	Number of breakthroughs in year									
	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018
?	8	7	16	13	14	10	10	9	7	6
?	13	10	9	7	8	11	10	12	7	10

Practice Questions

1. Write down the next four values of each of the following recursive sequences.

   (a) $s_0 = 0,\ s_1 = 2, \quad s_n = 2s_{n-1} - 3s_{n-2}$ for $n \geq 2$.

   (b) $u_0 = 2, \quad u_n = 3u_{n-1} + n$ for $n \geq 1$.

2. Consider the following pseudo code of a function "foo".

   function foo(x) (input: a positive integer)
       if $x = 0$ **then**
           return 1
       else
           return $x \times \text{foo}(x-1)$
       end if
   end function

   (a) $x = 4 \quad 4 \times foo(4-1) = 4 \times foo(3)$
   $= 4foo(3)$

   (a) What will foo return when given input 4?

   (b) What is the recurrence relation corresponding to foo?

   (c) What function of x does foo calculate?

3. The Fibonacci sequence is defined recursively by

   $$t_0 = 0,\ t_1 = 1, \quad t_k = t_{k-1} + t_{k-2} \text{ for } k \geq 2.$$

   Use strong induction to prove that, for all $n \geq 0$,

   $$t_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.$$

4. (a) Let P be a Poisson random variable with $\lambda = 10$. What is $E(P)$? What is $\mathrm{Var}(P)$?

   (b) Pick a value of k between 0 and 20 and calculate $\Pr(P) = k$.

   (c) Now let B_1, B_2, B_3 be binomial random variables with $(n,p) = (100, \frac{1}{10}), (1000, \frac{1}{100}), (10000, \frac{1}{1000})$. What are the expected values and variances of these variables?

   (d) For the same value of k you used in (b), calculate $\Pr(B_1) = k$, $\Pr(B_2) = k$ and $\Pr(B_3) = k$.

   (e) Is something going on here? If so what and why?

WEEK 10 NOTES (FROM PASS)

* Discrete Uniform Distribution
↳ choose 1 of a set of consecutive integers
↳ $Pr(X=k) = \frac{1}{b-a+1}$
↳ $E(X) = (a+b)/2$
↳ $var(X) = \frac{(b-a+1)^2-1}{k}$

* Bernuolli Distribution
↳ single process with success probability p and fail otherwise
↳ $Pr(X=k) \begin{cases} p & \text{for } k=1 \\ 1-p & \text{for } k=0 \end{cases}$
↳ $E(X) = p$
↳ $var(X) = p(1-p)$

* Geometric Distribution
↳ k fallows before a success
↳ $Pr(X=k) = p(1-p)^k$
↳ $E(X) = (1-p)/p$
↳ $var(X) = (1-p)/p^2$

* Binomial Distribution
↳ exactly k successes
↳ $Pr(X=k) = ({}^nC_k)\, p^k (1-p)^{n-k}$
↳ $E(X) = np$
↳ $var(X) = np(1-p)$

* Possan Distribution
↳ know average of λ events occur per time period, count probability that k events occurred in a time period
↳ $Pr(X=k) = (\lambda^k e^{-\lambda})/k!$
↳ $E(X) = \lambda$
↳ $var(X) = \lambda$

** Σ = sigma (sum)
↳ eg. $\sum_{k+1}^{n} k = 1+2+\ldots+n$

** π = pi (product)
↳ eg. $\prod_{k+1}^{n} k = 1 \times 2 \times \ldots \times n$

** arithmetic progression ⟶ $a_n = a_1 + (n-1)d$, $s_n = {}^n/_2(a_1+a_n)$
geometric progression ⟶ $a_n = a_1(r^{n-1})$, $s_n = (a_1 - a_1(r^n))/(1-r)$

Tutorial sheet #9

Tutorial Questions

3) (a) $\sum_{i=6}^{10} \frac{1}{2i+1} = \frac{6}{2(6)+1} + \frac{7}{2(6)+1} + \frac{8}{2(6)+1} + \frac{9}{2(6)+1} + \frac{10}{2(6)+1}$

(b) $\prod_{i=4}^{6} \left(\frac{x^i}{2i} + i\right) = \frac{4}{(4^4/2(4))+4} \times \frac{5}{(5^4/2(4))+4} \times \frac{6}{(6^4/2(4))+4}$

(c) $\sum_{i=0}^{3} \frac{(-1)^i}{(2i+1)!}(x^{2i+1}) = \frac{(-1)^0}{(2(0)+1)!}(0^{2(0)+1}) + \frac{(-1)^0}{(2(0)+1)!}(1^{2(0)+1}) + \frac{(-1)^0}{(2(0)+1)!}(2^{2(0)+1})$
$+ \frac{(-1)^0}{(2(0)+1)!}(3^{2(0)+1})$

2) next 4 values for...

(a) $r_0 = 3;\ r_n = 2r_{n-1} - 1$ for all integers $n \geq 1$

$3 = 2(r)_{0-1} - 1$	$n:1 \Rightarrow$
$3 = 2r_{-1} - 1$	$n:2 \Rightarrow$
$4/2 = r_{-1}$	$n:3 \Rightarrow$
$2 = r_{-1}$	$n:4 \Rightarrow$

(b) $t_0 = 1,\ t_1 = 1,\ t_2 = -2;\ t_n = t_{n-1} t_{n-3}$ for all integers $n \geq 3$

$1 = t_{0-1} t_{0-3}$	$n:1 \Rightarrow$
$1 = t_{-1} t_{-3}$	$n:2 \Rightarrow$
$1 = t_{-4}$	$n:3 \Rightarrow$
	$n:4 \Rightarrow$

Start of class: (13 may 2022, Friday)

* uniform distribution eg. rolling fair dice (equal probability of getting a side)
* bernuolli distibution eg. flipping a coin (heads or tails, success or failure)
* binomial distibution eg. fixed num of coins flipped
  ↳ multiple bernuolli distribution (total number of successes)
* geometric distribution eg. "doesn't give up until succeeds"
* poissan distribution eg. arrival of babies in maturnity ward "random"
  ↳ keyword: completely random

→ X tracks the number of failures
= number of failures until the first success
= $Pr(X=x) = \underbrace{(1-p)^x}_{\text{fails } x \text{ times}} \underbrace{p}_{\text{before success}}$

Y tracks number of failures + 1st success
= number of failures to the first success
= X → excludes first success
  Y → includes first success
= $Pr(Y=y) = (1-p)^{y-1}p$

* poissan allows you to calculate the probability of a random event happening
  ↳ give quantitative estimate of a random event
  ↳ λ = average number

2. Rewrite the following expressions using $\sum$ or $\prod$ notation.

(a) $(x-1)(x-4)(x-9)(x-16)\cdots(x-64)$

(b) $\frac{1}{4^4}+\frac{1}{6^5}+\frac{1}{8^6}+\frac{1}{10^7}+\cdots+\frac{1}{22^{15}}$

$\frac{1}{4^4}+\frac{1}{6^5}+\frac{1}{8^6}+\frac{1}{10^7}+\cdots+\frac{1}{22^{15}}$

3. For each integer $n \geq 1$, let t_n be the number of strings of n letters that can be produced by concatenating (running together) copies of the strings "a", "bc" and "cb".

For example, $t_1 = 1$ ("a" is the only possible string) and $t_2 = 3$ ("aa", "bc" and "cb" are the possible strings).

(a) Find t_3 and t_4.

(b) Find a recurrence for t_n that holds for all $n \geq 3$. Explain why your recurrence gives t_n.

Question 1 : (a) geometric distribution

↳ try-and-try until fired the ship

(b) probability $X \leq 2$?

$Pr(P=k) = p(1-p)^k$

"probability that Krik have to shoot at least 3 times or more until it hits ?"

10% chance on hitting the ship ⇒ $p=0.1$

$Pr(X \leq 2) = Pr(X=0) + Pr(X=1) + Pr(X=2)$

$= p + (1-p)(p) + (1-p)^2(p)$

$= 0.1 + (1-0.1)(0.1) + (1-0.1)(1-0.1)(0.1)$

$= 0.1 + (0.9)(0.1) + (0.9)(0.9)(0.1)$

$= 0.1 + 0.09 + 0.081$

$= 0.19 + 0.081$

$= 0.271$ #

$P(X \leq 2) = 1-(1-p)^3$

Question 4: (a) Binomial distribution

(b) * 400 scientists

* 2.5% probability to make major breakthrough per year

* $E(Y) = np$

* $Var(Y) = np(1-p)$

(f) varience range

"measure of the spread of expected values"

$E(Y) = (400)(0.025)$

$= 10$

↳ 10 major breakthrough per year

$Var(Y) = 10(1-0.025)$

$= 10(0.975)$

$= 9.75$

(c) " probability that 10 breakthrough per year"

$Pr(Y=10) = {}^{400}C_{10}\ (0.025)^{10}(0.975)^{390}$

(d) " expression for probability that 5 to 20 major breakthrough per year "

$Pr(5 \leq Y \leq 20) = \sum_{y=5}^{400} {}^{400}C_y\ (0.025)^y\ (0.975)^{400-y}$

(e) umbrella corps ⇒ 25 scientist, 40% major breakthrough per year

Z = major breakthrough per year by umbrella corps

Find $E(Z)$, $Var(Z)$

$Z \sim Bin(25, 0.4)$

$E(Z) = 25 \times 0.4$

$= 10$

$Var(Z) = 6$

2. Rewrite the following more compactly using $\sum$ or $\prod$ notation.

(i) $(2x-1)(5x-4)(8x-9)(11x-16)(14x-25)(17x-36)(20x-49)$ [2]

(ii) $\frac{2}{3-0}+\frac{4}{4+5}+\frac{8}{5-10}+\frac{16}{6+15}+\frac{32}{7-20}+\frac{64}{8+25}$ [2]

[Answer only required]

3. Astronomers estimate that the average number of asteroids that hit the Earth's surface each day is 17.

(i) Of the distributions that we studied in Lecture 25, which is likely to be the best model for the number of asteroids that hit the Earth's surface each day? Assume for subsequent parts that this is the distribution. [1]

(ii) What is the probability that no asteroids hit the Earth's surface tomorrow? [1]

(iii) What is the probability that no asteroids hit the Earth's surface within the next two days. [3]

4. Let $T(n)$ be the number of ways of writing n as a sum of positive integers $a_1+a_2+\cdots+a_k$, where $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_k \leqslant a_1+1$. For example, $T(5)=5$, since 5 can be written as 5, $2+3$, $1+2+2$, $1+1+1+2$, and $1+1+1+1+1$.

from tutorial sheet #9

Question 2: (a) inhomogenuous → $r_n = 2r_{n-1} - 1$; $r_0 = 3$; for all integers $n \geq 1$

"next term is 2 times the previous term −1"

$r_0 = 3$, $r_1 = 6-1 = 5$, $r_2 = 10-1 = 9$, $r_3 = 18-1 = 17$, $r_4 = 34-1 = 33$

may ask : part (ii) give close form formula OR tell me the n formula directly

$r_0 = 3 = 2+1$ what we may do is... $= 2^0+1$ and get... $r_n = 2^{n+1}+1$

$r_1 = 5 = 4+1$ $= 2^2+1$

$r_2 = 9 = 8+1$ $= 2^3+1$ but... that may be wrong...

$r_3 = 17 = 16+1$ $= 2^4+1$

$r_4 = 33 = 32+1$ $= 2^5+1$

2)(i) $(2x-1)(5x-4)(8x-9)(11x-16)(14x-25)(17x-36)(20x-49)$

multiplication = π notation

general term = $(a_i x - b_i)$ → a_i & b_i = some term

* when $i = 1$, I want get 1st term $(2x-1)$

** create a table :

i	ai	bi
1	2	1
2	5	4
3	8	9
4	11	16
5	14	25
6	17	36
7	20	49

* $a_i = 2 + (i-1)3$

$a_i = 3i - 1$

* $b_i = i^2$

HENCE : $\prod_{i=1}^{i=7} (3i-1)x - i^2$

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3)(a) poissan distubution (because it's completely random)

(b) λ value is for a day = 17 hits Earth's surface each day

$pr(x=0) = (17^0 e^{-17})/0!$

(c) λ value is for 2 days = 17 (2) hits Earth's surface ~~each~~ 2 days

= 34 hits Earth's surface ~~ach~~ 2 days

$pr(x=0) = (34^0 e^{-34})/0!$

2)(i) $\prod_{i=1}^{i=7} (3i-1)x - i^2$

(ii)

i	ai	bi	ci
1	2	3	0
2	4	4	5
3	8	5	10
4	16	6	15
5	32	7	20
6	64	8	25

$\left.\frac{ai}{bi-ci}\right\}$ general term

$ai = 2^{n+1}$

$bi = i+2$

$ci = 5(i-1)$

$$\frac{2^{n+1}}{(i+2)-(5(i-1))}$$

$$\sum_{i=1}^{i=6} \frac{2^{n+1}}{(i+2)-(5(i-1))}$$