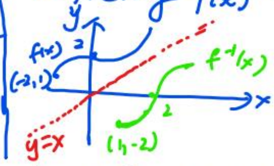
Inverse function $f^{-1}(x)$

→ Given $f(x)$ must be one-to-one

→ Swap of x & y variables

origin	inverse
(x, y)	(y, x)

(3) Graph $f(x)$ & $f^{-1}(x)$ is by reflecting $f(x)$ in the line $y=x$



(4) POI of $f(x)$ & $f^{-1}(x)$ is always on the line $y=x$

3.3 Differentiating inverse, circular and exponential functions

3.3.1 Inverse functions and their derivatives

The **inverse** of a function f is the function that reverses the operation done by f. The inverse function is denoted by f^{-1}. It satisfies the relation

$$y = f(x) \Leftrightarrow x = f^{-1}(y).$$

Here $\Leftrightarrow$ means 'implies in both directions'. Since x is normally chosen as the independent variable of a function and as x is always plotted on the horizontal axis in the xy-plane, the graph of the inverse function of $y = f(x)$ is defined by the relation

$$y = f^{-1}(x) \Leftrightarrow x = f(y).$$

In practice, to obtain the inverse function f^{-1} to a given function $y = f(x)$, we

- solve this equation to obtain x in terms of y
- interchange the labels x and y to give $y = f^{-1}(x)$

$y = f(x)$ | $f^{-1}(x) = x$ ← make x as subject

Note that $f\big(f^{-1}(x)\big) = x = f^{-1}\big(f(x)\big)$.

It is possible to plot the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same diagram. In this case, the graph of $y = f^{-1}(x)$ is the mirror image of the graph of $y = f(x)$ in the line $y = x$.

Example 3.7. Find the inverse function of $y = f(x) = \frac{1}{5}(4x - 3)$. A sketch of $f(x)$ is given below. Note that $y = x$ is the thin line shown in the diagram. Sketch $f^{-1}(x)$ on the same axis.

Step I: Let $y = f(x)$

Step II: Swap x & y

Step III: Make y the subject

Step IV: Assign $f^{-1}(x) = y$ (from Step III)

(1) $y = \frac{1}{5}(4x-3)$

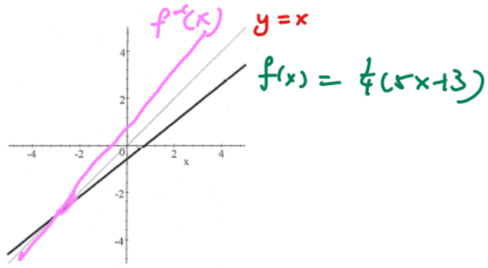
(2) $x = \frac{1}{5}(4y-3)$

(3) $4y - 3 = 5x$

$y = \frac{5x+3}{4} = \frac{1}{4}(5y+3)$

$y = \frac{5}{4}x + \frac{3}{4}$

(4) $f^{-1}(x) = \frac{5}{4}x + \frac{3}{4}$

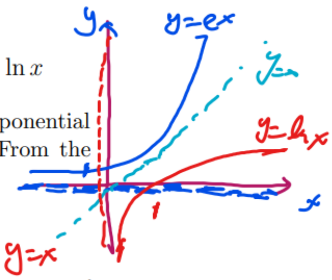


Relationship of exponential & logarithmic function

$y = a^x \iff x = \log_a y$ (equivalent function)

Only base e to study $\log_e x$

$\ln x$



3.3.2 Exponential and logarithmic functions: e^x and $\ln x$

A very important example of a function and its inverse are the exponential function $y = e^x$ and the natural logarithm function $y = \ln x$. From the definition of the inverse function we have

$$y = f(x) = e^x \Leftrightarrow x = \ln y$$

Now re-labelling x and y, we obtain $f^{-1}(x) = \ln x$ as the inverse function of $f(x) = e^x$. We note from the definition $y = e^x \Leftrightarrow x = \ln y$, that

- $\ln(e^x) = \ln(y) = x$
- $e^{\ln y} = e^x = y$

This explicitly demonstrates the inverse behaviour of e^x and $\ln x$.

As an illustration, since $e^0 = 1$, we have $\ln 1 = \ln e^0 = 0$. This means that as the point $(0, 1)$ lies on the graph of $y = e^x$, the point $(1, 0)$ must lie on the graph of $y = \ln x$. This feature is seen in the graphs of $y = e^x$ and $y = \ln x$ given below.

In general, for any function $f(x)$ since $b = f(a) \Leftrightarrow a = f^{-1}(b)$, it follows that if the point (a, b) lies on the graph of $y = f(x)$, then (b, a) is a point on the graph of $y = f^{-1}(x)$.

Index Law $a^n > 0$

① $a^0 = 1$

② $a^m \cdot a^n = a^{m+n}$

③ $\frac{a^m}{a^n} = a^{m-n}$

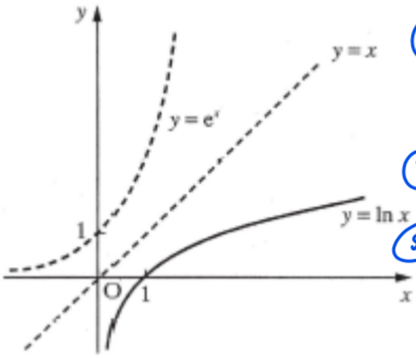
④ $a^{\frac{1}{2}} = \sqrt{a}$

eg. $a^{3/2} = \sqrt{a^3}$

$a^{\frac{1}{q}} = \sqrt[q]{a}$

⑤ $(ab)^m = a^m b^m$

⑥ $(\frac{a}{b})^m = \frac{a^m}{b^m}$



Logarithm Law

① $\log_a a = 1$

② $\log_a 1 = 0$

③ $\log_a (mn) = \log_a m + \log_a n$

④ $\log_a (\frac{m}{n}) = \log_a m - \log_a n$

⑤ $\log_a m^p = p \log_a m$

Sometimes we will need to restrict the domain of a function in order to find its inverse.

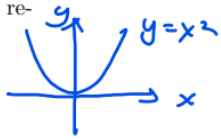
Example 3.8. Find the inverse function $f^{-1}(x)$ of $f(x) = x^2$ by first restricting the domain to $[0, \infty)$.

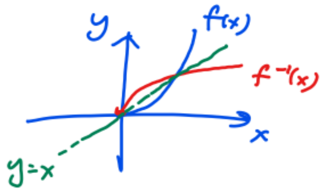
Let $y = x^2$

$x = y^2$

$y = \pm\sqrt{x}$

$f^{-1}(x) = \sqrt{x}$





Derivative rule for inverse functions

If $y = f^{-1}(x) \Leftrightarrow x = f(y)$, then $\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{f'(y)}$

Derivative of f(x)

$y = f(x) = \frac{1}{5}(4x-3)$

$f'(x) = \frac{4}{5}$

$\frac{dy}{dx} = \frac{4}{5}$

Example 3.9. Find the derivative of the function $f(x)$ and its inverse $f^{-1}(x)$ for $f(x) = \frac{1}{5}(4x-3)$. Check that the answers satisfy the derivative rule for inverse functions.

Inverse $f^{-1}(x)$

① $y = \frac{1}{5}(4x-3)$

② $x = \frac{1}{5}(4y-3)$

③ $5x = 4y - 3$

$y = \frac{5}{4}x + \frac{3}{4}$

$f^{-1}(x) = \frac{5}{4}x + \frac{3}{4}$

Inverse $y = \frac{1}{5}(4x-3)$

make x the subject $x = \frac{5}{4}y + \frac{3}{4}$

$\frac{dx}{dy}$ $\frac{5}{4}$

$\frac{d}{dx} = \frac{1}{\frac{dx}{dy}} = 1/(\frac{5}{4}) = \frac{4}{5}$

Example 3.10. Show that $\dfrac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \dfrac{1}{x}$ given that e^x is the inverse function of $\ln x$ and $\dfrac{\mathrm{d}}{\mathrm{d}x}(e^x) = e^x$.

Solution: Put $y = \ln x$. This implies $x = e^y$.

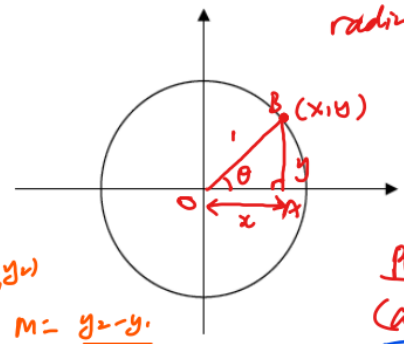
Therefore $\dfrac{\mathrm{d}}{\mathrm{d}y}(x) = \dfrac{\mathrm{d}}{\mathrm{d}y}(e^y) = e^y$ and hence $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\mathrm{d}x/\mathrm{d}y} = \dfrac{1}{e^y} = \dfrac{1}{x}$.

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{e^y}$$

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

3.3.3 Derivatives of circular functions

Circular (or **trigonometric**) functions $\sin x$, $\cos x$, $\tan x$, etc arise in problems involving functions that are periodic and repetitive, such as those that describe the orbit of a planet about its parent star. Here we acquaint ourselves with the derivatives of such functions.



radius = 1 unit

$x = \cos\theta$

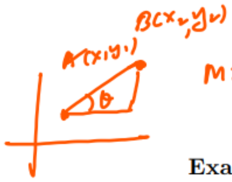
$y = \sin\theta$

$\frac{y}{x} = \tan\theta = \frac{\sin\theta}{\cos\theta}$

Pythagoras Theorem

$(\cos x)^2 + (\sin x)^2 = 1^2$

$\boxed{\cos^2 x + \sin^2 x = 1}$



$m = \frac{y_2 - y_1}{x_2 - x_1}$

$\tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$

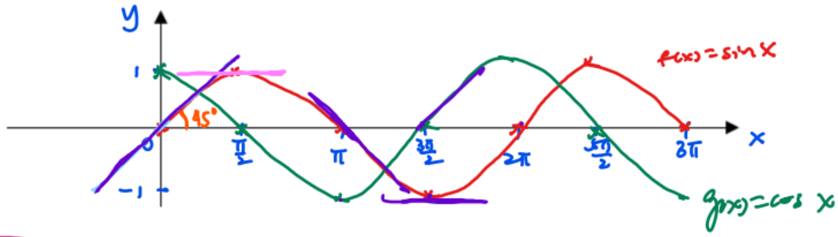
$\Rightarrow m = \tan\theta$

Example 3.11. Sketch the graphs of $f(x) = \sin x$ and $g(x) = \cos x$ on the same diagram for the interval $0 \le x \le 3\pi$. Use the tangent line method to estimate the values of $f'(x)$ at the points $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \dots$ on the same diagram. Do these values seem to match the curve $g(x)$?

$m = \tan\theta$

$m = \tan 45°$

$m = 1$



checking gradient (slope)

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
f'(x)	1	0	-1	0	1
g(x)	1	0	-1	0	1

$f'(x) = g(x)$

$\frac{d}{dx}\sin(x) = \cos(x)$

$\frac{d}{dx}\sin(x) = \cos(x)$

$\frac{d}{dx}\tan(x) = \sec^2(x)$

$\frac{d}{dx}\cos(x) = -\sin(x)$

Before examining the derivatives of circular functions in more detail, we consider two basic inverse circular functions $\sin^{-1} x$ and $\tan^{-1} x$.

Note that the alternative notation for inverse circular functions is: $\sin^{-1} x = \arcsin x$, $\cos^{-1} x = \arccos x$ and $\tan^{-1} x = \arctan x$.

Example 3.12. The graphs of $y = \sin x$ and $y = \tan x$ are shown below by the heavy curves for the restricted domains $[-\frac{\pi}{2}, \frac{\pi}{2}]$, i.e. $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and $(-\frac{\pi}{2} < x < \frac{\pi}{2})$, respectively.

Sketch the inverse functions $\sin^{-1} x$ and $\tan^{-1} x$ using mirror reflection across the line $y = x$.

The domain of $\sin^{-1} x$ is

The range of $\sin^{-1} x$ is

The domain of $\tan^{-1} x$ is

The range of $\tan^{-1} x$ is

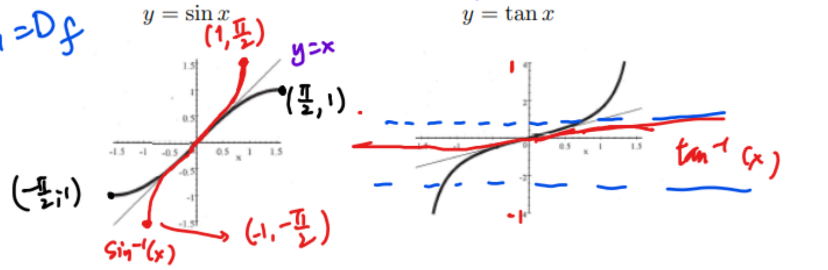
	Domain	Range
$\sin(x)$	$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$	$-1 \le y \le 1$
$\sin^{-1}(x)$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

	Domain	Range
$\tan(x)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$(-\infty, \infty)$
$\tan^{-1}(x)$	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$

$D_{f^{-1}} = R_f$

$R_{f^{-1}} = D_f$

$y = \sin x$ $\qquad$ $y = \tan x$



Note: The reason for the use of a restricted domain in specifying inverse circular functions is that if all of the graph of $y = \sin x$ or $\tan x$ were naively reflected across the line $y = x$, there would be more than one choice (in fact there would be an infinite number of choices) for the ordinate (y) value of the inverse function. We saw this at work in a previous example. **A function may have only a single value for each x in its domain.** We also note that $\tan\left(\frac{\pi}{2}\right)$ and $\tan\left(-\frac{\pi}{2}\right)$ are not defined ($\pm\infty$).

The values of the derivatives of the six basic circular (i.e. trigonometric) functions are shown in the table below, along with the derivatives of the three main inverse circular functions $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$. Also listed are the derivatives of the basic exponential and logarithm functions.

All subject to x

If No longer exist with x (having expression involved with x, such that $x \neq 1$) apply chain Rule

Table of the derivatives of the basic functions of calculus

Original function f	**Derivative function** f'
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x \equiv 1 + \tan^2 x$
$\operatorname{cosec} x \equiv 1/\sin x$	$-\operatorname{cosec} x \cdot \cot x$
$\sec x \equiv 1/\cos x$	$\sec x \cdot \tan x$
$\cot x \equiv 1/\tan x$	$-\operatorname{cosec}^2 x$
$\sin^{-1} x$ domain: $-1 \le x \le 1$ (i.e. $\lvert x\rvert \le 1$)	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$ domain: $-1 \le x \le 1$ (i.e. $\lvert x\rvert \le 1$)	$-\dfrac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$ domain: $-\infty < x < \infty$	$\dfrac{1}{1+x^2}$
e^x	e^x
$\ln x$ domain: $x > 0$	$\dfrac{1}{x}$

Example 3.13. Find $\dfrac{dy}{dx}$ when y is given by

$\frac{d}{dx}\sin x = \cos x$

(a) $\sin(2x+3)$

$\frac{d}{dx}\sin\boxed{(2x+3)} = 2\cos(2x+3)$

Take NOTE!

$\frac{d}{dx}\sin(2x+3) = \cos(2x+3)(2)$ please use bracket

WRONG! $\cancel{= \cos(4x+6)}$

(b) $x^2 \cos x$ $\quad \frac{d}{dx} x^2 \cos x$

Product Rule

$$= 2x \cos x + x^2 (-\sin x)$$
$$= 2x \cos x - x^2 \sin x$$

(c) $x \tan(2x + 1)$

Product Rule

$$\frac{d}{dx} x \tan(2x+1)$$
$$= (1) \tan(2x+1) + 2x \sec^2(2x+1)$$
$$= \tan(2x+1) + 2x \sec^2(2x+1)$$

(d) $\tan^{-1} x^2$

Chain Rule

$$\frac{d}{dx} \tan^{-1}(x^2) = \frac{2x}{1+(x^2)^2} = \frac{2x}{1+x^4}$$

(e) Prove the differentiation formula $\frac{\mathrm{d}}{\mathrm{d}x}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$.

Let $y = \sin^{-1}(x)$

$\sin y = x$

$x = \sin y$

$\frac{dx}{dy} = \cos y$

Differentiation

$\frac{dy}{dx} = 1 / \frac{dx}{dy}$

$\frac{dy}{dx} = \frac{1}{\cos y}$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}}$

Recall Identity

$\cos^2 y + \sin^2 y = 1$

$\cos^2 y = 1 - \sin^2 y$

$\cos y = \pm\sqrt{1-\sin^2 y}$

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

(f) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \arcsin(e^{2x})$

$y = \sin^{-1}(e^{2x})$ $\quad$ ① $\sin^{-1}$ $\quad$ ② e^{2x} $\quad$ ③ $2x$

Chain Rule

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(e^{2x})^2}} (2e^{2x}) = \frac{2e^{2x}}{\sqrt{1-e^{4x}}}$$

(g) Prove the differentiation formula $\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$.

let $y = \cos^{-1}x$

$\cos y = x$

$x = \cos y$

$\frac{dx}{dy} = -\sin y$

$\frac{dy}{dx} = 1/\frac{dx}{dy}$

$\frac{dy}{dx} = \frac{1}{-\sin y}$

$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\cos^2 y}}$

$\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$

$\boxed{\begin{array}{l}\cos^2 y + \sin^2 y = 1 \\ \sin y = \pm\sqrt{1-\cos^2 y}\end{array}}$

(h) Find $\frac{\mathrm{d}s}{\mathrm{d}t}$ when $s = \ln(\tan(2t))$

Chain Rule

① $\ln(\ )$ ② $\tan(\ )$ ③ $2t$

$\frac{ds}{dt} = \frac{1}{\tan(2t)}\left(2\sec^2(2t)\right) = \frac{2\sec^2(2t)}{\tan(2t)}$

(i) Find $\frac{\mathrm{d}g}{\mathrm{d}t}$ when $g = \sqrt{t}\sin^{-1}(t^2)$ $= t^{\frac{1}{2}}\sin^{-1}(t^2)$

$\sin^{-1}(x) \neq \frac{1}{\sin(x)}$

Product Rule

$\frac{dg}{dt} = \frac{1}{2}t^{-\frac{1}{2}}\left[\sin^{-1}(t^2)\right] + t^{\frac{1}{2}}\left[\frac{2t}{1-(t^2)^2}\right]$

$= \frac{\sin^{-1}(t^2)}{2\sqrt{t}} + \frac{2t\sqrt{t}}{\sqrt{1-t^4}}$

(j) Find $\frac{\mathrm{d}g}{\mathrm{d}x}$ when $g = \frac{5x^3+3x}{(x^2+3)^2}$

Quotient Rule

$\frac{dg}{dx} = \frac{(15x^2+3)(x^2+3)^2 - (5x^3+3x)\left[2(2x)(x^2+3)\right]}{((x^2+3)^2)^2}$

$\frac{dg}{dx} = \frac{(15x^2+3)(x^2+3)^2 - 4x(x^2+3)(5x^3+3x)}{(x^2+3)^4}$