LECTURE 17 : EQUIVALENCE RELATIONS

* EQUALITY & CONGRUENCE mod n (fixed n) ARE EXAMPLES OF EQUIVALENCE RELATIONS

* EQUIVALENCE RELATION R ON SET A IS A BINARY RELATION WITH...

1. REFLEXIVITY : aRa FOR ALL $a \in A$
   ↳ REFLECTION FOR ALL a ELEMENTS
2. SYMMETRY : $aRb \Rightarrow bRa$ FOR ALL $a, b \in A$
   ↳ $a \sim b \xrightarrow{?} b \sim a$
3. TRANSITIVITY : aRb and bRc $\Rightarrow$ aRc FOR ALL $a, b, c \in A$
   ↳ $a \sim b \wedge b \sim c \xrightarrow{?} a \sim c$

REFLEXIVITY + TRANSITIVITY + SYMMETRY = EQUIVALENCE RELATION

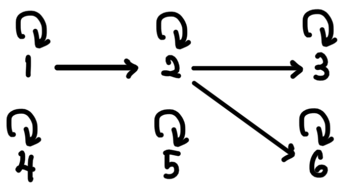
REFLEXIVITY

TO PROVE R IS REFLEXIVE, SHOW THAT...
↳ FOR ALL $x \in A$, xRx

* PICTURE :

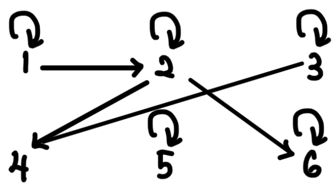
TO PROVE R IS NOT REFLEXIVE, SHOW THAT...
↳ THERE IS AN $x \in A$ SUCH THAT $x \not R x$

LET R BE THE RELATION ON A PICTURED BELOW. IS R REFLECTIVE ?



ANS : YES FOR ALL $x \in A$

LET S BE THE RELATION ON A PICTURED BELOW. IS S REFLECTIVE ?

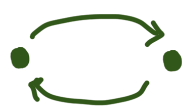


ANS : NO, $4 \not S 4$

SYMMETRY

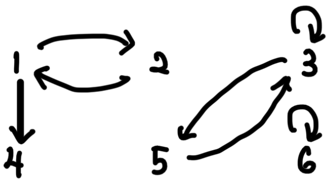
TO PROVE R IS SYMMETRIC, SHOW THAT...
↳ FOR ALL $x, y \in A$, IF xRy then yRx

* PICTURE :



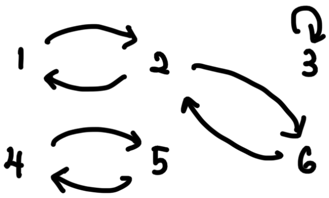
TO PROVE R IS NOT SYMMETRIC, SHOW THAT...
↳ THERE ARE SOME $x, y \in A$ SUCH THAT xRy BUT $y \not R x$

LET R BE THE RELATION ON A PICTURED BELOW. IS R SYMMETRIC ?



ANS: NO, $1R4$ BUT $4 \not{R} 1$

LET S BE THE RELATION ON A PICTURED BELOW. IS S SYMMETRIC ?



ANS: YES, FOR ALL $x, y \in A$ IF xSy THEN ySx

TRANSITIVITY

TO PROVE R IS TRANSITIVE, SHOW THAT...

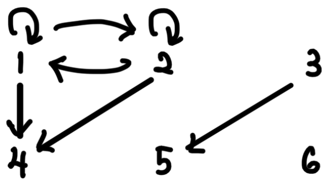
↳ FOR ALL $x, y, z \in A$, IF xRy AND yRz THEN xRz

TO PROVE R IS NOT TRANSITIVE, SHOW THAT...

↳ THERE ARE SOME $x, y, z \in A$ SUCH THAT xRy and yRz BUT $x \not{R} z$

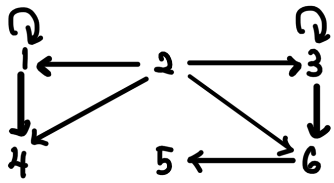
* PICTURE :

LET R BE THE RELATION ON A PICTURED BELOW. IS R TRANSITIVE ?



ANS: YES. FOR ALL $x, y, z \in A$, IF xRy and yRz then xRz

LET S BE THE RELATION ON A PICTURED BELOW. IS S TRANSITIVE ?



ANS: NO, $3S6$ AND $6S5$ BUT $3 \not{S} 5$

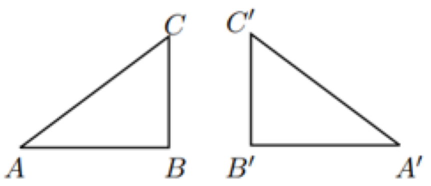
17.1 Other equivalence relations

1. Equivalence of fractions.

   Two fractions are equivalent if they reduce to the same fraction when the numerator and denominator of each are divided by their gcd. E.g. $\frac{2}{4}$ and $\frac{3}{6}$ are equivalent because both reduce to $\frac{1}{2}$.

2. Congruence of triangles.

   Triangles ABC and $A'B'C'$ are congruent if $AB = A'B'$, $BC = B'C'$ and $CA = C'A'$. E.g. the following triangles are congruent.

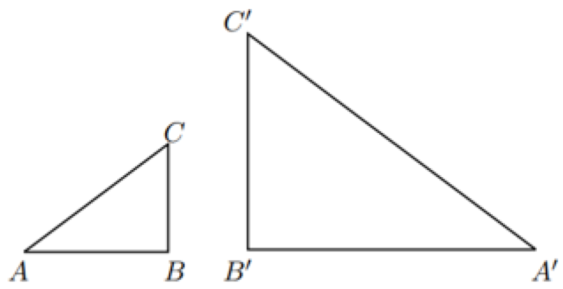


3. Similarity of triangles.

   Triangles ABC and $A'B'C'$ are similar if

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}.$$

E.g the following triangles are similar



4. Parallelism of lines.

   The relation $L \| M$ (L is parallel to M) is an equivalence relation.

Remark

In all these cases the relation is an equivalence because it says that objects are the *same* in some respect.

1. Equivalent fractions have the same reduced form.

2. Congruent triangles have the same side lengths.

3. Similar triangles have the same shape.

4. Parallel lines have the same direction.

Sameness is always reflexive (a is the same as a), symmetric (if a is the same as b, then b is the same as a) and transitive (if a is the same as b and b is the same as c, then a is the same as c).

MORE EXAMPLES:

Flux Exercise (LQMTZZ)

Which of the following relations are equivalence relations on $\mathbb{Z}$?

(1) R defined by xRy if and only if $|x| = |y|$
(2) S defined by xSy if and only if $x^3 - y^3 = 1$
(3) T defined by xTy if and only if x divides y
(4) U defined by xUy if and only if 5 divides $x - y$

A. Just (1) and (3)
B. Just (1)
C. Just (1) and (4)
D. Just (1), (2) and (4)

Answer

(2) is not reflexive. E.g. $1 \not S 1$ because $1^3 - 1^3 \neq 1$.
So (2) is not an equivalence relation.
(3) is not symmetric. E.g. $3T6$ but $6 \not T 3$ (3 divides 6 but 6 doesn't divide 3).
So (3) is not an equivalence relation.
(1) and (4) are equivalence relations (details on next two slides).
So C.

MORE DETAILED WORKINGS:

(1) R defined on $\mathbb{Z}$ by xRy if and only if $|x| = |y|$

Reflexive: Yes. $|a| = |a|$ for all $a \in \mathbb{Z}$.
Symmetric: Yes. If $|a| = |b|$, then $|b| = |a|$ for all $a, b \in \mathbb{Z}$.
Transitive: Yes. If $|a| = |b|$ and $|b| = |c|$, then $|a| = |c|$ for all $a, b, c \in \mathbb{Z}$.
So it is an equivalence relation.

(2) S defined on $\mathbb{Z}$ by xSy if and only if $x^3 - y^3 = 1$

Reflexive: No. $1^3 - 1^3 \neq 1$ so $1 \not S 1$.
Symmetric: No. $1^3 - 0^3 = 1$ but $0^3 - 1^3 \neq 1$, so $1S0$ but $0 \not S 1$.
Transitive: No. $1^3 - 0^3 = 1$ and $0^3 - (-1)^3 = 1$ but $1^3 - (-1)^3 \neq 1$, so $1S0$ and $0S(-1)$ but $1 \not S (-1)$.
So it is not an equivalence relation.

(3) T defined on $\mathbb{Z}$ by xTy if and only if x divides y

Reflexive: Yes. a divides a for all $a \in \mathbb{Z}$.
Symmetric: No. 3 divides 6 but 6 does not divide 3, so $3T6$ but $6 \not T 3$.
Transitive: Yes. If a divides b and b divides c, then a divides c for all $a, b, c \in \mathbb{Z}$.
So it is not an equivalence relation.

(4) U defined on $\mathbb{Z}$ by xUy if and only if 5 divides $x - y$

Yes. This relation is the same as $x \equiv y \pmod 5$ and we know that's an equivalence relation.

WHAT IS THE SAME ABOUT THE EQUIVALENT OBJECTS FOR THE EQUIVALENCE RELATIONS (1) AND (4)?

(1) R DEFINED ON $\mathbb{Z}$ BY xRy IF AND ONLY IF $|x| = |y|$
↳ X AND Y HAVE THE SAME MAGNITUDE

(2) U DEFINED ON $\mathbb{Z}$ BY xUy IF AND ONLY IF 5 DIVIDES x-y
↳ X AND Y HAVE THE SAME REMAINDER WHEN DIVIDED BY 5

MORE EXAMPLES:

Question What are the equivalence classes of the equivalence relations (1) and (4)?

(1) R defined on $\mathbb{Z}$ by xRy if and only if $|x| = |y|$

$\{0\}, \{1, -1\}, \{2, -2\}, \{3, -3\} \ldots$

(4) U defined on $\mathbb{Z}$ by xUy if and only if 5 divides $x - y$

$\{\ldots, -10, -5, 0, 5, 10, \ldots\}$,
$\{\ldots, -9, -4, 1, 6, 11, \ldots\}$,
$\{\ldots, -8, -3, 2, 7, 12, \ldots\}$,
$\{\ldots, -7, -2, 3, 8, 13, \ldots\}$,
$\{\ldots, -6, -1, 4, 9, 14, \ldots\}$

17.2 Equivalence classes

Conversely, we can show that if R is a reflexive, symmetric and transitive relation then aRb says that a and b are the same in some respect: *they have the same R-equivalence class.*

If R is an equivalence relation we define the *R-equivalence class* of a to be

$$[a] = \{s : sRa\}.$$

Thus $[a]$ consists of all the elements related to a. It can also be defined as $\{s : aRs\}$, because sRa if and only if aRs, by symmetry of R.

Examples

- The parallel equivalence class of a line L consists of all lines parallel to L.
- The equivalence class of 1 for congruence mod 2 is the set of all odd numbers.

17.4 Partitions and equivalence classes

A *partition* of a set S is a set of subsets of S such that each element of S is in exactly one of the subsets.

Using what we showed in the last section, we have the following.

If R is an equivalence relation on a set A, then the equivalence classes of R form a partition of A. Two elements of A are related if and only if they are in the same equivalence class.

Example. Let R be the relation on $\mathbb{Z}$ defined by aRb if and only if $a \equiv b \pmod 3$. The three equivalence classes of R are

$$\{x : x \equiv 0 \pmod 3\} = \{3k : k \in \mathbb{Z}\}$$
$$\{x : x \equiv 1 \pmod 3\} = \{3k+1 : k \in \mathbb{Z}\}$$
$$\{x : x \equiv 2 \pmod 3\} = \{3k+2 : k \in \mathbb{Z}\}.$$

These partition the set $\mathbb{Z}$.

17.3 Equivalence class properties

Claim. *If two elements are related by an equivalence relation R on a set A, their equivalence classes are equal.*

Proof. Suppose $a, b \in A$ and aRb. Now

$$\begin{aligned} s \in [a] &\Rightarrow sRa \text{ by definition of } [a] \\ &\Rightarrow sRb \text{ by transitivity of } R \\ &\quad\;\; \text{since } sRa \text{ and } aRb \\ &\Rightarrow s \in [b] \text{ by definition of } [b]. \end{aligned}$$

Thus all elements of $[a]$ belong to $[b]$. Similarly, all elements of $[b]$ belong to $[a]$, hence $[a] = [b]$.

Claim. *If R is an equivalence relation on a set A, each element of A belongs to exactly one equivalence class.*

Proof. Suppose $a, b, c \in A$, and $c \in [a] \cap [b]$.

$$\begin{aligned} & c \in [a] \text{ and } c \in [b] \\ \Rightarrow\; & cRa \text{ and } cRb \\ & \text{by definition of } [a] \text{ and } [b] \\ \Rightarrow\; & aRc \text{ and } cRb \text{ by symmetry} \\ \Rightarrow\; & aRb \text{ by transitivity} \\ \Rightarrow\; & [a] = [b] \\ & \text{by the previous claim.} \end{aligned}$$

Examples of partitions

$\{\{1, 6\}, \{2\}, \{3, 4, 5\}\}$ is a partition of $\{1, 2, 3, 4, 5, 6\}$

$\{\{x : x \in \mathbb{Z} \text{ and } x \text{ is even}\}, \{x : x \in \mathbb{Z} \text{ and } x \text{ is odd}\}\}$ is a partition of $\mathbb{Z}$.