Question 19 **(8 marks)**

Prove that the HAMILTONIAN CIRCUIT problem is NP-complete, by reduction from HAMILTONIAN PATH. You may assume that HAMILTONIAN PATH is NP-complete.

Definitions:

A **Hamiltonian path** in a graph G is a path that includes every vertex of G. All the vertices on the path must be distinct.

A **Hamiltonian circuit** in a graph G is a circuit that includes every vertex of G. All the vertices on the circuit must be distinct.

HAMILTONIAN PATH
Input: Graph G.
Question: Does G have a Hamiltonian path?

HAMILTONIAN CIRCUIT
Input: Graph G.
Question: Does G have a Hamiltonian circuit?

HAMILTONIAN CIRCUIT belongs to NP:

Given a graph G, let the certificate be a Hamiltonian circuit of G. This can be verified in polynomial time, by checking that the circuit is indeed a circuit and that it visits each vertex exactly once.

Polynomial-time reduction from HAMILTONIAN PATH to HAMILTONIAN CIRCUIT:

Given a graph G (which may or may not have a Hamiltonian path), construct a new graph H by adding a new vertex v and joining it, by n new edges, to every vertex of G. (Here, n denotes the number of vertices of G.)

We show that G has a Hamiltonian path if and only if H has a Hamiltonian circuit.

Suppose G has a Hamiltonian path. Call its end vertices u and w. Then a Hamiltonian circuit of H can be obtained by adding the new vertex v, and the edges uv and wv, to the Hamiltonian path. So H has a Hamiltonian circuit.

Conversely, suppose H has a Hamiltonian circuit C. This circuit must include v, and two edges incident with v. Let u and w be the two vertices of G that are incident with v in C. (So C includes the edges uv and wv as well as the vertex v.) The rest of C must constitute a Hamiltonian path between u and w in G. So G has a Hamiltonian path.

This completes the proof that G has a Hamiltonian path if and only if H has a Hamiltonian circuit.

It remains to observe that the construction of H from G can be done in polynomial time.

Therefore the construction of H from G is a polynomial-time reduction from HAMILTONIAN PATH to HAMILTONIAN CIRCUIT.

Since HAMILTONIAN PATH is NP-complete, and it's polynomial-time reducible to HAMILTONIAN CIRCUIT, and HAMILTONIAN CIRCUIT is in NP, we conclude that HAMILTONIAN CIRCUIT is NP-complete.

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Question 19 **(8 marks)**

Prove that the GRAPH 4-COLOURABILITY problem is NP-complete, by reduction from GRAPH 3-COLOURABILITY. You may assume that GRAPH 3-COLOURABILITY is NP-complete.

Definitions:

For any positive integer k, a k**-colouring** in a graph G is an assignment of "colours" from the set $\{1, 2, \ldots, k\}$ to the vertices of G such that (a) each vertex gets exactly one colour from the set, and (b) adjacent vertices get different colours.

GRAPH 3-COLOURABILITY
Input: Graph G.
Question: Does G have a 3-colouring?

GRAPH 4-COLOURABILITY
Input: Graph G.
Question: Does G have a 4-colouring?

GRAPH 4-COLOURABILITY is in NP, since it has a polynomial-time verifier: the certificate is a 4-colouring, and this can be checked in polynomial time: for each edge, check that its endpoints are differently coloured, and check that the total number of colours used is at most 4.

We now prove that GRAPH 3-COLOURABILITY is polynomial-time reducible to GRAPH 4-COLOURABILITY. The reduction works as follows.

> Input: graph G.
> Create a new vertex v, and join it to every vertex of G. Denote the resulting graph by $G + v$.
> Output $G + v$.

This reduction is polynomial-time computable, since we just have to add one new vertex, and the number of new edges is at most the number of vertices in G.

If G is 3-colourable, then we can give a new colour to v, different to the three colours used on G. This gives a 4-colouring of $G + v$, since each new edge is properly coloured. (For each new edge, one of its endpoints is the new colour, which cannot appear on the other endpoint.) So we have shown that, if G is 3-colourable, then $G + v$ is 4-colourable.

Now suppose $G + v$ has a 4-colouring. We will show that G is 3-colourable.
Consider v. Since it is adjacent to every vertex of G, the colour given to v (in the 4-colouring) cannot appear on any vertex of G. Once that colour is excluded, there are only three colours left, so we must have a 3-colouring of G. So we have shown that, if $G + v$ is 4-colourable,

then G is 3-colourable.

We therefore have: G is 3-colourable if and only if $G + v$ is 4-colourable. This, together with its polynomial-time computability, shows that the reduction given above is a polynomial-time reduction from GRAPH 3-COLOURABILITY to GRAPH 4-COLOURABILITY. Since GRAPH 3-COLOURABILITY is known to be NP-complete, and since we showed at the start that GRAPH 4-COLOURABILITY is in NP, we conclude that GRAPH 4-COLOURABILITY is NP-complete.

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Question 18 **(8 marks)**

Prove that the problem LONG PATH is NP-complete, using reduction from HAMILTONIAN PATH. You may assume that HAMILTONIAN PATH is NP-complete.

Definitions:

HAMILTONIAN PATH
Input: Graph G.
Question: Does G have a path that contains every vertex?

LONG PATH
Input: Graph G, with an even number of vertices.
Question: Does G contain a path of length $\geq n/2$?

In each of these definitions, n is the number of vertices in the graph G.

LONG PATH is in NP, because it has a polynomial-time verifier. Given an input graph G on n vertices, the certificate is a path of length $\geq n/2$ in G. The verifier just goes along the path, checking that the successive vertices are indeed adjacent, that no vertex is repeated, and that there are $\geq n/2$ edges altogether. This takes polynomial time.

We now give a polynomial-time reduction from HAMILTONIAN PATH to LONG PATH.

Input: graph G.
Create $n-2$ new vertices, $v_1, v_2, \ldots, v_{n-2}$, that are distinct from all vertices of G.
Add these new vertices to G, with no new edges, making a new graph of $2n-2$ vertices.
Output this new graph.

The time taken by this algorithm is dominated by the creation of $n-2$ new vertices, so it runs in polynomial time.

Let H be the new graph constructed from G by this algorithm. We show that G has a Hamiltonian path if and only if H has a "long path", i.e., a path of length $\geq |V(H)|/2$.

Suppose G has a Hamiltonian path P. This has length $n-1$. This path is still present in H. The number of vertices of H is given by

$$|V(H)| = n + (n-2) = 2n-2 = 2(n-1).$$

The length of P is half this amount. So H has a path of length $\geq |V(H)|/2$. So $H \in$ LONG PATH.

Now suppose $H \in$ LONG PATH. So H has a path of length $|V(H)|/2$, which is $2(n-1)/2$, which is $n-1$. This path cannot include the new isolated vertices, so it must be in G. A path of length $n-1$ in G must be a Hamiltonian path. So G has a Hamiltonian path.