Problem sheet 1 : [vectors, planes]

4) $\underset{\sim}{u} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$ $\quad \underset{\sim}{u} \times \underset{\sim}{v} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix}$

$\underset{\sim}{v} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ $\quad \underset{\sim}{u} \times \underset{\sim}{w} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ -4 \end{pmatrix}$

$\underset{\sim}{w} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ $\quad \underset{\sim}{v} \times \underset{\sim}{w} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix}$

5) Find v_w

(a) $\underset{\sim}{v} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$ & $\underset{\sim}{w} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

$$v_w = \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} \frac{1}{\sqrt{1+4+1}}$$
$$= \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} \frac{1}{\sqrt{6}}$$

(b) $\underset{\sim}{v} = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$ & $\underset{\sim}{w} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$

$$v_w = \begin{pmatrix} 0 \\ -2 \\ -8 \end{pmatrix} \frac{1}{\sqrt{16+4+4}}$$
$$= \begin{pmatrix} 0 \\ -2 \\ -8 \end{pmatrix} \frac{1}{\sqrt{24}}$$

(c) $\underset{\sim}{v} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ & $\underset{\sim}{w} = \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix}$

$$v_w = \begin{pmatrix} -6 \\ 0 \\ 0 \end{pmatrix} \frac{1}{\sqrt{9+4}}$$
$$= \begin{pmatrix} -6 \\ 8 \end{pmatrix} \frac{1}{\sqrt{13}}$$

6) (a) $\underset{\sim}{v} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$ & $\underset{\sim}{w} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

$$v_w = \left[\begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} \frac{1}{6} \right] \times \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

(b) $\underset{\sim}{v} = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$ & $\underset{\sim}{w} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$

$$v_w = \left[\begin{pmatrix} 0 \\ -2 \\ -8 \end{pmatrix} \frac{1}{24} \right] \times \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$$

(c) $\underset{\sim}{v} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ & $\underset{\sim}{w} = \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix}$

$$v_w = \left[\begin{pmatrix} -6 \\ 0 \\ 0 \end{pmatrix} \frac{1}{13} \right] \times \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix}$$

7) (a) $\underset{\sim}{v} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$ & $\underset{\sim}{w} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2-4 \\ -2+3 \\ -6-2 \end{pmatrix} = \begin{pmatrix} -6 \\ 1 \\ -8 \end{pmatrix}$$

(b) $\underset{\sim}{v} = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$ & $\underset{\sim}{w} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$

$$\begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -2-8 \\ 16-0 \\ 0+4 \end{pmatrix} = \begin{pmatrix} -10 \\ 16 \\ 4 \end{pmatrix}$$

(c) $\underset{\sim}{v} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ & $\underset{\sim}{w} = \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0+4 \\ -6-0 \\ -4+0 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \\ -4 \end{pmatrix}$$

8) $\underset{\sim}{v} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ & $\underset{\sim}{w} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

(a) $\underset{\sim}{v} \times \underset{\sim}{w} = -\underset{\sim}{w} \times \underset{\sim}{v}$

$$\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix} \qquad -\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -3 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -6 \\ -2 \\ 3 \end{pmatrix}$$

(b) $\underset{\sim}{v} \times \underset{\sim}{v} = 0$

$$\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \\ 1 \end{pmatrix} \neq 0$$

9) $\underset{\sim}{u} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ $\quad \underset{\sim}{v} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ $\quad \underset{\sim}{w} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

(a) $$\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2-2 \\ -3+1 \\ 2+6 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 8 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 24 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -6+1 \\ -2-3 \\ 1+4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -5 \\ -5 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -15 \\ -10 \\ -5 \end{pmatrix}$$

10) $\underset{\sim}{u} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $\underset{\sim}{v} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $\underset{\sim}{w} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$$\underset{\sim}{u} \times \underset{\sim}{v} = \begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0-0\\0-0\\1-0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$\underset{\sim}{v} \times \underset{\sim}{w}$ would be $\begin{pmatrix}1\\0\\0\end{pmatrix}$

$\underset{\sim}{u} \times \underset{\sim}{w}$ would be $\begin{pmatrix}0\\1\\0\end{pmatrix}$

Problem sheet 2:

1) points ⇒ $\begin{pmatrix}1\\2\\-1\end{pmatrix}$ & $\begin{pmatrix}2\\0\\3\end{pmatrix}$

(a) vector equation (parametric form)

$$\begin{pmatrix}2\\0\\3\end{pmatrix} - \begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}1\\-2\\4\end{pmatrix}$$

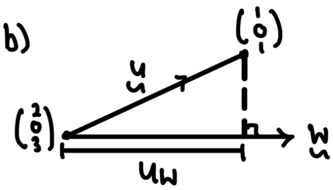
$$\left.\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\0\\3\end{pmatrix} + \begin{pmatrix}1\\-2\\4\end{pmatrix}t \quad\right\} \text{vector form}$$

parametric form: $x(t) = 2+t$

$y(t) = -2t$

$z(t) = 3+4t$

(b)



$\underset{\sim}{w} = \begin{pmatrix}1\\-2\\4\end{pmatrix}$

$$\underset{\sim}{u} \longrightarrow \begin{pmatrix}2\\0\\3\end{pmatrix} - \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\2\end{pmatrix}$$

Distance between line & point:

scalar projection $\rightsquigarrow v_w = \dfrac{v \cdot w}{|w|}$

$$u_w = \begin{pmatrix}1\\0\\2\end{pmatrix}\begin{pmatrix}1\\-2\\4\end{pmatrix} \div \sqrt{1^2+2^2+4^2}$$

$$= \begin{pmatrix}1\\0\\8\end{pmatrix} \div \sqrt{1+4+16}$$

$$= \begin{pmatrix}1\\0\\8\end{pmatrix} \div \sqrt{21}$$

$$= \begin{pmatrix}1\\0\\8\end{pmatrix} \frac{1}{\sqrt{21}}$$

2) Equation of plane

points ⇒ $\begin{pmatrix}1\\2\\-1\end{pmatrix}, \begin{pmatrix}2\\0\\3\end{pmatrix}, \begin{pmatrix}-1\\-1\\0\end{pmatrix}$

equation ⇒ $ax+by+cz=d$

① get difference of vectors/points

② obtain normal vector by cross product

③ pick a point, multiple with equation

$\begin{pmatrix}1\\2\\-1\end{pmatrix} \longrightarrow a+2b-c=d$

$\begin{pmatrix}2\\0\\3\end{pmatrix} \longrightarrow 2a+3c=d$

$\begin{pmatrix}-1\\-1\\0\end{pmatrix} \longrightarrow -a-b=d$

$\underset{\sim}{u} = \begin{pmatrix}1\\2\\-1\end{pmatrix} - \begin{pmatrix}2\\0\\3\end{pmatrix} = \begin{pmatrix}-1\\2\\-4\end{pmatrix}$

$\underset{\sim}{v} = \begin{pmatrix}-1\\-1\\0\end{pmatrix} - \begin{pmatrix}2\\0\\3\end{pmatrix} = \begin{pmatrix}-3\\-1\\-3\end{pmatrix}$

normal vector, $\underset{\sim}{n} = \begin{pmatrix}-1\\2\\-4\end{pmatrix} \times \begin{pmatrix}-3\\-1\\-3\end{pmatrix} = \begin{pmatrix}-6-4\\12-3\\1+6\end{pmatrix} = \begin{pmatrix}-10\\9\\7\end{pmatrix}$

$$\begin{pmatrix}x-2\\y-0\\z-3\end{pmatrix} \cdot \begin{pmatrix}-10\\9\\7\end{pmatrix} = 0$$

$$(20-10x) + (9y) + (7z-21) = 0$$

$$-10x + 9y + 7z = 1 \quad \#$$

→ equation!

Extra from 2) Equation of plane

points = $\begin{pmatrix}1\\2\\7\end{pmatrix}, \begin{pmatrix}2\\3\\4\end{pmatrix}, \begin{pmatrix}2\\1\\8\end{pmatrix}$

$\begin{pmatrix}2\\3\\4\end{pmatrix}-\begin{pmatrix}1\\2\\7\end{pmatrix}=\begin{pmatrix}1\\1\\-3\end{pmatrix}$

$\begin{pmatrix}2\\3\\4\end{pmatrix}-\begin{pmatrix}2\\1\\8\end{pmatrix}=\begin{pmatrix}0\\2\\-4\end{pmatrix}$

normal vector = $\begin{pmatrix}1\\1\\-3\end{pmatrix}\times\begin{pmatrix}0\\2\\-4\end{pmatrix}=\begin{pmatrix}-4+6\\0+4\\2-0\end{pmatrix}$

$= \begin{pmatrix}2\\4\\2\end{pmatrix}$

$\begin{pmatrix}x-2\\y-3\\z-4\end{pmatrix}\cdot\begin{pmatrix}2\\4\\2\end{pmatrix}=0$

$(2x-4)+(4y-12)+(2z-8)=0$

$2x+4y+2z=24$

$x+2y+z=12$ #

3) plane : $3x+4y-z=2$

line : $x(t)=2-2t$

$y(t)=-1+3t$

$z(t)=-t$

(a) line intersect plane

$3(2-2t)+4(-1+3t)-(-t)=2$

$6-6t-4+12t+t=2$

$2-6t+12t+t=2$

$7t=0$

$t=0$

$x(0)=2-2(0)=2$

$y(0)=-1+3(0)=-1$

$z(0)=-(0)=0$

$\left.\right\}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\-1\\0\end{pmatrix}$ #

(b) Normal vector to plane

plane : $3x+4y-z=2$

$\hookrightarrow \underset{\sim}{n}=(3,4,-1)$

(c) line-plane intersection's angle

$\cos(\theta)=\dfrac{\underset{\sim}{n}\cdot\underset{\sim}{d}}{|\underset{\sim}{n}||\underset{\sim}{d}|}$

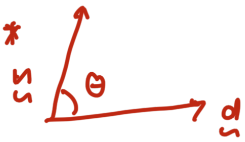
$=\dfrac{-6+12+1}{\sqrt{26}\sqrt{14}}$

$\theta=\cos^{-1}\left(\dfrac{-6+12+1}{\sqrt{26}\sqrt{14}}\right)$

$\theta=68.5°$

angle $=90°-68.5°$

$=21.5°$



$\underset{\sim}{n}=\begin{pmatrix}3\\4\\-1\end{pmatrix}$ $\underset{\sim}{d}=\begin{pmatrix}2\\-1\\0\end{pmatrix}$

4) min distance between 2 lines

line 1 : $x(t)=1+t$, $y(t)=1-3t$, $z(t)=-2+2t$

line 2 : $x(s)=3s$, $y(s)=1-2s$, $z(s)=2-s$

TIPS ⟶ scalar projection

line 1 ⇒ $\begin{pmatrix}1\\1\\-2\end{pmatrix}+\begin{pmatrix}1\\-3\\2\end{pmatrix}t$

line 2 ⇒ $\begin{pmatrix}0\\1\\2\end{pmatrix}+\begin{pmatrix}3\\-2\\-1\end{pmatrix}s$

$\begin{pmatrix}1\\1\\-2\end{pmatrix}-\begin{pmatrix}0\\1\\2\end{pmatrix}=\begin{pmatrix}1\\0\\-4\end{pmatrix}$

$\hookrightarrow \underset{\sim}{v}$

normal vector, $\underset{\sim}{n} = \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 3+4 \\ 6+1 \\ -2+9 \end{pmatrix}$

$= \begin{pmatrix} 7 \\ 7 \\ 7 \end{pmatrix} \simeq 7\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

scalar projection, $v_n = \dfrac{\begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{\sqrt{1^2+1^2+1^2}}$

$= \dfrac{\begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}}{\sqrt{3}}$

$= \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$

$= \dfrac{1+0-4}{\sqrt{3}}$

$= 3/\sqrt{3}$ units #

5) Distance between parallel planes

plane 1 $\rightarrow 2x-y+3z=-4$

plane 2 $\rightarrow 2x-y+3z=24$

TIPS = Random substitution

plane 1 vector = $(2,-1,3)$

plane 2 vector = $(2,-1,3)$ } same vector = prove is parallel

plane 1, let $x=y=0$

$2(0)-(0)+3z=-4$

$3z=-4$

$z=-4/3$

plane 2, let $y=z=0$

$2x-(0)+3(0)=24$

$2x=24$

$x=12$

$(0,0,-4/3)$ $(12,0,0)$

$\begin{pmatrix} 0 \\ 0 \\ -4/3 \end{pmatrix} - \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -12 \\ 0 \\ -4/3 \end{pmatrix}$ $n = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

$\underset{\sim}{v_n} = \dfrac{\begin{pmatrix} -12 \\ 0 \\ -4/3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}}{\sqrt{2^2+(-1)^2+3^2}} = \dfrac{-24+0-4}{\sqrt{4+1+9}} = \dfrac{-28}{\sqrt{14}}$ units

6) plane 1 $\rightarrow$ $3x+4y-z=2$
plane 2 $\rightarrow$ $-2x+y+2z=6$

$$\begin{pmatrix}3\\4\\-1\end{pmatrix}-\begin{pmatrix}-2\\1\\2\end{pmatrix}=\begin{pmatrix}5\\3\\-3\end{pmatrix}$$

(a) planes intersection
plane 1 vector $\rightarrow$ $(3,4,-1)$
plane 2 vector $\rightarrow$ $(-2,1,2)$

x-intercept $(y=z=0)$
↳ plane 1 : $3x+4(0)-(0)=2 \rightsquigarrow 3x=2 \rightsquigarrow x=2/3$
↳ plane 2 : $-2x+(0)+2(0)=6 \rightsquigarrow -2x=6 \rightsquigarrow x=-3$

y-intercept $(x=z=0)$
↳ plane 1 : $3(0)+4y-(0)=2 \rightsquigarrow 4y=2 \rightsquigarrow y=1/2$
↳ plane 2 : $-2(0)+y+2(0)=6 \rightsquigarrow y=6$

z-intercept $(x=y=0)$
↳ plane 1 : $3(0)+4(0)-z=2 \rightsquigarrow -z=2 \rightsquigarrow z=-2$
↳ plane 2 : $-2(0)+(0)+2z=6 \rightsquigarrow 2z=6 \rightsquigarrow z=3$

(b) plane 1 normal $\rightsquigarrow$ $(3,4,-1)$
plane 2 normal $\rightsquigarrow$ $(-2,1,2)$

(c) equation of line that define intersection of plane
TIPS $\rightsquigarrow$ Normal vector define line direction

plane 1 normal $\rightsquigarrow$ $(3,4,-1)$
plane 2 normal $\rightsquigarrow$ $(-2,1,2)$

$$\boxed{\begin{pmatrix}3\\4\\-1\end{pmatrix}\times\begin{pmatrix}-2\\1\\2\end{pmatrix}=\begin{pmatrix}8+1\\2-6\\3+8\end{pmatrix}=\begin{pmatrix}9\\-4\\11\end{pmatrix} \Big\}\ \text{direction of line}}$$

① $3x+4y-z=2$ ② $-2x+y+2z=6$

consider ①
$2(3x+4y-z=2)$
$6x+8y-2z=4$
$6x+8y-4=2z$

① - ②
$-2x+y+6x+8y-4=6$
$4x+9y=10$ #

(d) $\cos(\theta) = \dfrac{\underset{\sim}{n_1} \cdot \underset{\sim}{n_2}}{|\underset{\sim}{n_1}||\underset{\sim}{n_2}|}$

$\underset{\sim}{n_1}$ = normal plane 1 = $(3, 4, -1)$
$\underset{\sim}{n_2}$ = normal plane 2 = $(-2, 1, 2)$

$$\cos(\theta) = \frac{\begin{pmatrix}3\\4\\1\end{pmatrix} \cdot \begin{pmatrix}-2\\1\\2\end{pmatrix}}{\sqrt{3^2+4^2+1}\sqrt{(-2)^2+1+2^2}}$$

$$\cos(\theta) = \frac{-6+4-2}{\sqrt{9+16+1}\sqrt{4+144}}$$

$$\cos(\theta) = \frac{-4}{\sqrt{26}\sqrt{9}}$$

$$\theta = \cos^{-1}\left(\frac{-4}{\sqrt{26}\sqrt{9}}\right)$$

$$\theta = 105.2^\circ$$

7) Elementary row operation & back substitution

(a) ① $x+y=5$
② $2x+3y=1$

back substitution :

consider ①
$x = 5-y$ — ①

sub ① - ②
$2(5-y)+3y=1$
$10-2y+3y=1$
$y=-9$

$x = 5-(-9)$
$x = 45$

(b) $x+2y-z=6$ ①
$2x+5y-z=13$ ②
$x+3y-3z=4$ ③

Elementary row operation :

① - ②
$(x-2x)+(2y-5y)+(-z+z) = 6-13$
$-x-3y=-7$
$7 = x+3y$ — ④

③ - 3①
$(x-3x)+(3y-6y)+(-3z+3z) = 4-18$
$-2x-3y=-14$
$14 = 2x+3y$ — ⑤

back substitution :

⑤ - ④
$14 = 2(7-3y)+3y$
$14 = 14-6y+3y$
$14 = 14-3y$
$0 = -3y$
$y = 0$

$x = 7-3y$
$x = 7-3(0)$
$x = 7$

$x+2y-6 = z$
$7+2(0)-6 = z$
$z = 7-6$
$z = 1$

$\therefore x=7,\ y=0,\ z=1$

8) $x + 2y - z = 6$ ①
$2x + 5y - z = 13$ ②
$2x + 4y - 2z = 12$ ③

② - ①
$(2x - x) + (5y - 2y) + (-z + z) = 13 - 6$
$x + 3y = 7$

③ = ① × 2
$2(x + 2y - z = 6) \Rightarrow 2x + 4y - 2z = 12$

③ - 2②
$(2x - 4x) + (4y - 10y) + (-2z + 2z) = 12 - 2(13)$
$-2x - 6y = 12 - 26$
$-2x - 6y = -14$
$14 = 2x + 6y$

④ $x + 3y = 7$
⑤ $14 = 2x + 6y$

④ - ⑤
$14 = 2(7 - 3y) + 6y$
$14 = 14 - 6y + 6y$
$0 = 0$ ... ?

since ① and ③ is the same, free variable is needed.

① $x + 2y - z = 6$
② $2x + 5y - z = 13$

let $z = \beta$,
$x + 2y - \beta = 6$ ①
$2x + 5y - \beta = 13$ ②

② - 2①
$(2x - 2x) + (5y - 4y) + (-\beta + 2\beta) = 13 - 12$
$y + \beta = 1$
$y = 1 - \beta$

$x + 2(1 - \beta) - \beta = 6$
$x + 2 - 2\beta - \beta = 6$
$x - 3\beta = 4$
$x = 4 + 3\beta$ $\qquad z = \beta$

HENCE,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 + 3\beta \\ 1 - \beta \\ \beta \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \#$$

9) $x + 2y - z = 6$
1 equation, 3 unknowns = introduce 2 free variables

let $x = \alpha$, $y = \beta$: $\alpha + 2\beta - z = 6$
$z = \alpha + 2\beta - 6$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \alpha + 2\beta - 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -6 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \#$$

10) Find planes intersection

plane 1 $\rightsquigarrow$ $3x + 4y - z = 2$ ①

plane 2 $\rightsquigarrow$ $-2x + y + 2z = 6$ ②

② + 2①

$(-2x + 6x) + (y + 8y) + (2z - 2z) = 6 - 4$

$4x + 9y = 2$

2 equation, 3 unknowns $\Rightarrow$ introduce 1 free variable

let $y = \beta$,

$4x + 9(\beta) = 2$

$4x = 2 - 9\beta$

$x = 1/2 - 9\beta/4$

$\therefore x = 1/2 - 9\beta/4$

$y = \beta$

$z = -1/2 - 11\beta/4$

$3x + 4y - z = 2$

$3x + 4y - 2 = z$

$3(1/2 - 9\beta/4) + 4\beta - 2 = z$

$3/2 - 27\beta/4 + 4\beta - 2 = z$

$-1/2 - 27\beta/4 + 16\beta/4 = z$

$-1/2 - 11\beta/4 = z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/2 - 9\beta/4 \\ \beta \\ -1/2 - 11\beta/4 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix} + \beta \begin{pmatrix} -9/4 \\ 1 \\ -11/4 \end{pmatrix} \quad \#$$

Problem set 3: [matrix]

1) (a) $$2\begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & -8 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -1 & -9 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1\times2 + 1\times3 & 1\times(-1) + 1\times1 \\ 1\times2 + (-4)\times3 & 1\times(-1) + (-4)\times1 \end{bmatrix} = \begin{bmatrix} 2+3 & -1+1 \\ 2-12 & -1-4 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -10 & -5 \end{bmatrix}$$

(c) $$\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 2\times1 + -1\times1 & 2\times1 + -1\times-4 \\ 3\times1 + 1\times1 & 3\times1 + 1\times-4 \end{bmatrix} = \begin{bmatrix} 2-1 & 2+4 \\ 3+1 & 3-4 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 4 & -1 \end{bmatrix}$$

(d) $$\begin{bmatrix} 1 & 1 & 3 \\ 1 & -4 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2+3+3 & -1+1+6 \\ 2-12+2 & -1-4+4 \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ -8 & -1 \end{bmatrix}$$

2) (row, column)

(a) $\begin{bmatrix} 2 & -1 \\ 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}$

$(3\times2)(2\times2)$

possible

(b) $\begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 1 \\ 1 & 2 \end{bmatrix}$

$(2\times2)(3\times2)$

not possible

(c) $\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}\begin{bmatrix} 2 & 3 & 7 \end{bmatrix}$

$(3\times1)(1\times3)$

possible

(d) $\begin{bmatrix} 2 & 3 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$

$(1\times3)(3\times1)$

not possible

$\Rightarrow$ scalar matrix

3) (a) $x+y=5$

$2x+3y=1$

$$\begin{bmatrix} x \\ y \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}}_{\text{coefficient matrix}} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

(b) $x+2y-z=6$

$2x+5y-z=13$

$x+3y-3z=4$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ 1 & 3 & -3 \end{bmatrix}}_{\text{coefficient matrix}} = \begin{bmatrix} 6 \\ 13 \\ 14 \end{bmatrix}$$

4) gaussian elimination

$x+2y-z=6$

$2x+5y-z=13$

$x+3y-3z=4$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 2 & 5 & -1 & 13 \\ 1 & 3 & -3 & 4 \end{array}\right]$$

$R3 = R3 - R1$ $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 2 & 5 & -1 & 13 \\ 0 & 1 & -2 & -2 \end{array}\right]$

$R2 = R2 - 2R1$ $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & -2 & -2 \end{array}\right]$

$R3 = R3 - R2$ $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -3 & -3 \end{array}\right]$

$R3 = R3/-3$ $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$

$R2 = R2 - R3$ $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$

$R1 = R1 + R3$ $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$

$R1 = R1 - 2R2$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$ $\longrightarrow$ Final answer: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 1 \end{pmatrix}$

5) $x+y=5$

$2x+3y=1$

(a) coefficient matrix : $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

augmented matrix : $\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 2 & 3 & 1 \end{array}\right]$

(b) gaussian elimination

augmented matrix → row echelon form } bottom Δ is zero

$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 2 & 3 & 1 \end{array}\right]$

$R2 = R2 - 2R1$ $\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & -9 \end{array}\right]$

(c) reduced row echelon form (diagonal line is 1, other all 0)

$R1 = R1 - R2$ $\left[\begin{array}{cc|c} 1 & 0 & 14 \\ 0 & 1 & -9 \end{array}\right]$

(d) coefficient matrix rank

rank (A) = # total variables = 2

(e) leading variables, free variables

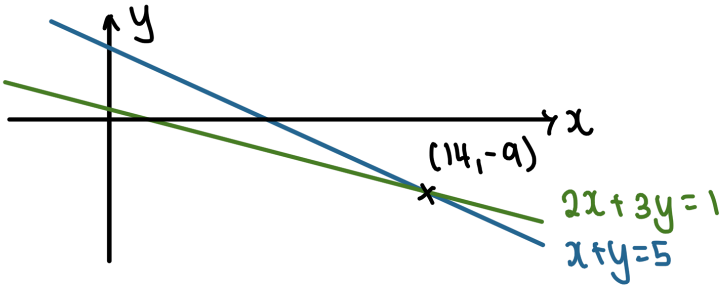
leading variables : x, y

free variables : none

(f) $\left[\begin{array}{cc|c} 1 & 0 & 14 \\ 0 & 1 & -9 \end{array}\right]$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ -9 \end{pmatrix}$ $\rightarrow$ $x = 14$, $y = -9$

(g) sketch geometrical interpretation



6) $3x - y + 2z = 3$

$x + 2y - z = 2$

$2x - 3y + az = b$

$(a, b \in \mathbb{R})$

when reach echelon form

(a) unique sol'n

△ cannot be zero

(b) No sol'n

△ must be zero, ☆ cannot zero

(c) Infinite sol'n

△ & ☆ must be zero.

6. Consider the following system of linear equations

$$3x - y + 2z = 3$$
$$x + 2y - z = 2$$
$$2x - 3y + az = b$$

Find conditions on a and b $(a, b \in R)$ such that the system has

$\left(\begin{array}{ccc|c} 3 & -1 & 2 & 3 \\ 1 & 2 & -1 & 2 \\ 2 & -3 & a & b \end{array}\right) \xrightarrow[R_3 \leftarrow R_3 - 2R_2]{R_2 \leftarrow 3R_2 - R_1} \left(\begin{array}{ccc|c} 3 & -1 & 2 & 3 \\ 0 & 7 & -5 & 3 \\ 0 & -7 & a+2 & b-4 \end{array}\right) \xrightarrow{R_3 \leftarrow R_2 + R_3} \left(\begin{array}{ccc|c} 3 & -1 & 2 & 3 \\ 0 & 7 & -5 & 3 \\ 0 & 0 & a-3 & b-1 \end{array}\right)$ echelon form

a. No solution.

b. One solution.

c. Infinitely many solutions.

(a) No sol'n, Last row ⇒ 0 0 0 ≠0

$a - 3 = 0$ and $b - 1 \neq 0$

$a = 3$ and $b \neq 1$

(b) one sol'n Last row ⇒ 0 0 ≠0 Anything

$a - 3 \neq 0$ $b - 1 \in \mathbb{R}$

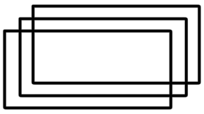
$a \neq 3$ $b \in \mathbb{R}$

(c) Infinite Sol'n Last row ⇒ 0 0 0 0

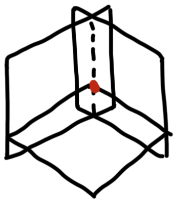
$a - 3 = 0$ and $b - 1 = 0$

$a = 3$ and $b = 1$

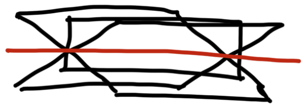
7) No intersection



Intersection at a point



Intersection in a common line



8) has the value equal to the total number of variables in the system would result in a unique solution

9) $x + 2y + 2z + 3w = 3$
$2x + 4y + 4z + 7w = 5$

$$\begin{array}{c} \left(\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 3 \\ 2 & 4 & 4 & 7 & 5 \end{array}\right) \\ \begin{array}{cccc} x & y & z & w \end{array} \end{array}$$

leading variables : x, w
free variables : y, z

echelon form : $R_2 = 2R_1 - R_2$ $\left(\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 3 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right)$

2 equation, 4 unknowns = require 2 free parameters

let $y = \beta$, $z = \alpha$

$x + 2\beta + 2\alpha + 3w = 3$
$x = 3 - 2\beta + 2\alpha + 3w$
$y = \beta$
$z = \alpha$
$w = -1$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2\beta + 2\alpha \\ \beta \\ \alpha \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \#$$

10) $x + y + z = 0$
$-2x + 5y + 2z = 0$
$-7x + 7y + z = 0$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 0 \\ -7 & 7 & 1 & 0 \end{array}\right)$$

$R_2 = R_2 + 2R_1$ $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 0 \\ -7 & 7 & 1 & 0 \end{array}\right)$

$R_3 = R_3 + 7R_1$ $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 0 \\ 0 & 14 & 8 & 0 \end{array}\right)$

$R3 = R3 - 2R2$ $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\exists$ infinite solution

let $z = \beta$ (free variable)

$y = -4/7\,\beta$ $x = -3/7\,\beta$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3/7 \\ -4/7 \\ 1 \end{pmatrix}\beta + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \#$$

Problem set 4:

1) $A = \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 3 \\ 1 & 3 \\ 1 & 2 \end{bmatrix}$

$$\boxed{A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \text{ when } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}}$$

$$A^{-1} = \frac{1}{-4-1}\begin{pmatrix} -4 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{-5}\begin{pmatrix} -4 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 4/5 & 1/5 \\ 1/5 & -1/5 \end{pmatrix}$$

(a) $$AA^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} 4/5 & 1/5 \\ 1/5 & -1/5 \end{pmatrix} = \begin{pmatrix} 4/5 + 1/5 & 1/5 - 1/5 \\ 4/5 - 4/5 & 1/5 + 4/5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(b) $$A^{-1}A = \begin{pmatrix} 4/5 & 1/5 \\ 1/5 & -1/5 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} = \begin{pmatrix} 4/5 + 1/5 & 4/5 - 4/5 \\ 1/5 - 1/5 & 1/5 + 4/5 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(c) $$(A^{-1})^{-1} = \begin{pmatrix} 4/5 & 1/5 \\ 1/5 & -1/5 \end{pmatrix}^{-1}$$

$$= \frac{1}{(4/5)(-1/5) - (1/5)(1/5)}\begin{pmatrix} -1/5 & -1/5 \\ -1/5 & 4/5 \end{pmatrix}$$

$$= \frac{1}{-4/25 - 1/25}\begin{pmatrix} -1/5 & -1/5 \\ -1/5 & 4/5 \end{pmatrix}$$

$$= -\frac{1}{5/25}\begin{pmatrix} -1/5 & -1/5 \\ -1/5 & 4/5 \end{pmatrix}$$

$$= -\frac{1}{1/5}\begin{pmatrix} -1/5 & -1/5 \\ -1/5 & 4/5 \end{pmatrix}$$

$$= -5\begin{pmatrix} -1/5 & -1/5 \\ -1/5 & 4/5 \end{pmatrix} \simeq \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}$$

which is equivalent to A.

HENCE, $(A^{-1})^{-1} = A$ #

2) $B = \begin{pmatrix} 2 & 5 \\ 6 & 15 \end{pmatrix}$ $\boxed{A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \text{ when } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}}$

$$B^{-1} = \frac{1}{2(15)-5(6)}\begin{pmatrix} 15 & -5 \\ -6 & 2 \end{pmatrix}$$

$$= \frac{1}{30-30}\begin{pmatrix} 15 & -5 \\ -6 & 2 \end{pmatrix}$$

= NA because $1/30-30 = 1/0$ HENCE since B^{-1} don't exist, B is not invertible

3) $B = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$ Find B^{-1}, B^2, $(B^2)^{-1}$

$$B^{-1} = \frac{1}{2(4)-9}\begin{pmatrix} 4 & -3 \\ -3 & 2 \end{pmatrix}$$

$$B^{-1} = \frac{1}{-1}\begin{pmatrix} 4 & -3 \\ -3 & 2 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix}$$

$$B^2 = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$$

$$B^2 = \begin{pmatrix} 2\times2+3\times3 & 2\times3+3\times4 \\ 3\times2+4\times3 & 3\times3+4\times4 \end{pmatrix}$$

$$B^2 = \begin{pmatrix} 4+9 & 6+12 \\ 6+12 & 9+16 \end{pmatrix}$$

$$B^2 = \begin{pmatrix} 13 & 18 \\ 18 & 25 \end{pmatrix}$$

$$(B^2)^{-1} = \frac{1}{13(25)-18(18)}\begin{pmatrix} 25 & -18 \\ -18 & 13 \end{pmatrix}$$

$$= \frac{1}{325-324}\begin{pmatrix} 25 & -18 \\ -18 & 13 \end{pmatrix}$$

$$= \begin{pmatrix} 25 & -18 \\ -18 & 13 \end{pmatrix} \#$$

$(B^2)^{-1} = B^{-1}B^{-1}$ #

4) matrix transpose

(a) $A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 1 & 3 \\ 1 & 2 & -1 \end{pmatrix}$ Find A^T & $(A^T)^T$.

① ②

$$A^T = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ -1 & 3 & -1 \end{pmatrix}$$

Basically, look vertically then write horizontally

$$(A^T)^T = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ -1 & 3 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ -1 & 3 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2\times2+1\times3+1\times-1 & 2\times1+1\times1+3\times1 & 2\times1+1\times2+1\times-1 \\ 3\times2+1\times3+2\times-1 & 3\times1+1\times1+2\times3 & 3\times1+1\times2+2\times-1 \\ -1\times2+3\times3+-1\times-1 & -1\times1+3\times1+-1\times3 & -1\times1+3\times2+-1\times-1 \end{pmatrix}$$

$$= \begin{pmatrix} 4+3-1 & 2+1+3 & 2+2-1 \\ 6+3-2 & 3+1+6 & 3+2-2 \\ -2+9+1 & -1+3-3 & -1+6+1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 6 & 3 \\ 7 & 10 & 3 \\ 10 & -1 & 7 \end{pmatrix}$$

(b) $B = \begin{bmatrix} 2 & 3 \\ 1 & 3 \\ 1 & 2 \end{bmatrix}$ Find B^T & B^TB.

$B^T = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 3 & 2 \end{bmatrix}$

* $BB^T \neq B^TB$

$$B^TB = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 3 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2\times2+1\times1+1\times1 & 3\times2+3\times1+2\times1 \\ 3\times2+3\times1+2\times1 & 3\times3+3\times3+2\times2 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 11 \\ 11 & 22 \end{pmatrix}$$

5) $A = \begin{bmatrix} 4 & -2 \\ 1 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$

Find A^T, B^T, A^TB^T

$A^T = \begin{bmatrix} 4 & 1 \\ -2 & 5 \end{bmatrix}$ $B^T = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$

* $(AB)^T = B^TA^T$

$(AB)^T \neq A^TB^T$

$$A^TB^T = \begin{bmatrix} 4 & 1 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{pmatrix} 4\times2+1\times-1 & 4\times2+1\times-1 \\ -2\times2+5\times-1 & -2\times2+5\times1 \end{pmatrix}$$

$$= \begin{pmatrix} 8-1 & 8-1 \\ -4-5 & -4+5 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 7 \\ -9 & 1 \end{pmatrix}$$

6) $A = \begin{bmatrix} 4 & -2 \\ 1 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$

(a) $(A+B)^T = \begin{pmatrix} 6 & -3 \\ 3 & 6 \end{pmatrix}^T$

$= \begin{pmatrix} 6 & 3 \\ -3 & 6 \end{pmatrix}$

(b) $A^T + B^T = \begin{pmatrix} 4 & 1 \\ -2 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 6 & 3 \\ -3 & 6 \end{pmatrix}$

(c) $A + A^T$ & $B + B^T$

$A + A^T = \begin{pmatrix} 4 & -2 \\ 1 & 5 \end{pmatrix} + \begin{pmatrix} 4 & 1 \\ -2 & 5 \end{pmatrix}$

$= \begin{pmatrix} 8 & -1 \\ -1 & 10 \end{pmatrix}$

$B + B^T = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$

(d) $(A + A^T)^T = \begin{pmatrix} 8 & -1 \\ -1 & 10 \end{pmatrix}^T$

$= \begin{pmatrix} 8 & -1 \\ -1 & 10 \end{pmatrix}$

$(B+B^T)^T = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}^T$

$= \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$

(e) $(A^T)^{-1} = \begin{pmatrix} 4 & 1 \\ -2 & 5 \end{pmatrix}^{-1}$

$= \frac{1}{22}\begin{pmatrix} 5 & -1 \\ 2 & 4 \end{pmatrix}$

(f) $(A^{-1})^T = \left[\frac{1}{22}\begin{pmatrix} 5 & 2 \\ -1 & 4 \end{pmatrix}\right]^T$

$= \begin{pmatrix} 5/22 & 2/22 \\ -1/22 & 4/22 \end{pmatrix}^T$

$= \begin{pmatrix} 2/11 & -1/11 \\ 1/22 & 5/22 \end{pmatrix}$

$B^{-1} = \frac{1}{2+2}\begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix}$

$= \frac{1}{4}\begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1/4 & 1/4 \\ -1/2 & 1/2 \end{pmatrix}$

(g) $(AB)^{-1} = \begin{pmatrix} 4\times2+-2\times2 & 4\times-1+-2\times1 \\ 1\times2+5\times2 & 1\times-1+5\times1 \end{pmatrix}^{-1}$

$= \begin{pmatrix} 8-4 & -4-2 \\ 2+10 & -1+5 \end{pmatrix}^{-1}$

$= \begin{bmatrix} 4 & -6 \\ 12 & 4 \end{bmatrix}^{-1}$

$= \frac{1}{88}\begin{pmatrix} 4 & 6 \\ -12 & 4 \end{pmatrix} \simeq \begin{pmatrix} 4/88 & 6/88 \\ -12/88 & 4/88 \end{pmatrix}$

(h) $B^{-1}A^{-1} = \begin{pmatrix} 5/22 & 2/22 \\ -1/22 & 4/22 \end{pmatrix}\begin{pmatrix} 1/4 & 1/4 \\ -1/2 & 1/2 \end{pmatrix}$

$= \begin{pmatrix} 5/22\times1/4+2/22\times-1/2 & 5/22\times1/4+2/22\times-1/2 \\ -1/22\times1/4+4/22\times-1/2 & -1/22\times1/4+4/22\times1/2 \end{pmatrix}$

7) matrix determinant

$A = \begin{bmatrix} 2 & -1 \\ 1 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 3 \\ 1 & 2 & -1 \end{bmatrix}$

(a) determinant (det)

$\det(A) = (2\times 7) - (-1\times 1)$
$= 14 + 1$
$= 15$

$\det(B) = (2\times 1) - (3\times 1)$
$= 2 - 3$
$= -1$

$\det(C) = 2\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} - 3\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} + -1\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$
$= 2\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} - 3\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$
$= 2(-1-6) - 3(-1-3) - (2-1)$
$= 2(-7) - 3(-4) - 1$
$= -14 + 12 - 1$
$= -3$

when $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$,
det formula $= ad - bc$

(b) $AB = \begin{pmatrix} 2 & -1 \\ 1 & 7 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$
$= \begin{pmatrix} 2\times 2 + 1\times -1 & 2\times 3 + 1\times -1 \\ 1\times 2 + 7\times 1 & 1\times 3 + 7\times 1 \end{pmatrix}$
$= \begin{pmatrix} 4-1 & 6-1 \\ 2+7 & 3+7 \end{pmatrix}$
$= \begin{pmatrix} 3 & 5 \\ 9 & 10 \end{pmatrix}$

$\det(AB) = (3\times 10) - (5\times 9)$
$= 30 - 45$
$= -15$

(c) $BA = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 7 \end{pmatrix}$
$= \begin{pmatrix} 2\times 2 + 3\times 1 & 2\times -1 + 3\times 7 \\ 1\times 2 + 1\times 1 & 1\times -1 + 1\times 7 \end{pmatrix}$
$= \begin{pmatrix} 4+3 & -2+21 \\ 2+1 & -1+7 \end{pmatrix}$
$= \begin{pmatrix} 7 & 19 \\ 3 & 6 \end{pmatrix}$

$\det(BA) = (7\times 6) - (3\times 19)$
$= 42 - 57$
$= -15$

(d) $\det(A)\det(B) = (-15)(-15)$
$= 225$

(e) $A^T = \begin{pmatrix} 2 & 1 \\ -1 & 7 \end{pmatrix}$
$\det(A^T) = (2\times 7) - (1\times -1)$
$= 14 + 1$
$= 15$

(f) $A^{-1} = \frac{1}{15}\begin{pmatrix} 7 & 1 \\ -1 & 2 \end{pmatrix}$
$A^{-1} = \begin{pmatrix} 7/15 & 1/15 \\ -1/15 & 2/15 \end{pmatrix}$
$\det(A^{-1}) = (-1/15)(1/15) - (7/15)(2/15)$
$= -1/225 - 14/225$
$= -15/225$
$= -1/15$

(g) $\det(A)\det(A^{-1}) = (-1/15)(-15)$
$= 1$

8) coefficient matrix

(a) $\left(\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 1 & 3 & 0 & 7 \\ 2 & 5 & -1 & 13 \end{array}\right)$

coefficient matrix: $\begin{pmatrix} 1 & 2 & -1 \\ 1 & 3 & 0 \\ 2 & 5 & -1 \end{pmatrix}$

determinant: $(-3-5+0) - (-6+0-2) = -8-(-8)$
$= 0 \Rightarrow C^{-1}$ don't exist

(b) $(1 \;\; 2 \;\; -1 \mid 1 \;\; 6)$

coefficient matrix: $(1 \;\; 2 \;\; -1)$
determinant: does not exist

$C = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

9) $A = \begin{bmatrix} 2 & -1 \\ 1 & 7 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$

(a) $C = \left(\begin{array}{ccc|ccc} 2 & 3 & -1 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 1 & 0 \\ 1 & 2 & -1 & 0 & 0 & 1 \end{array}\right)$

$R1 = R1 - R2$ $\left(\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & -1 & 0 \\ 1 & 1 & 3 & 0 & 1 & 0 \\ 1 & 2 & -1 & 0 & 0 & 1 \end{array}\right)$

$R3 = R3 - R1$ $\left(\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & -1 & 0 \\ 1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 3 & -1 & 1 & 1 \end{array}\right)$

$R3 = R3/3$ $\left(\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & -1 & 0 \\ 1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/3 & 1/3 & 1/3 \end{array}\right)$

$R2 = R2 - R1$ $\left(\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & -1 & 0 \\ 0 & -1 & 3 & -1 & 2 & 0 \\ 0 & 0 & 1 & -1/3 & 1/3 & 1/3 \end{array}\right)$

$R2 = R2 \times -1$ $\left(\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & -1 & 0 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 0 & 0 & 1 & -1/3 & 1/3 & 1/3 \end{array}\right)$

$R1 = R1 - 2R2$ $\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 3 & 0 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 0 & 0 & 1 & -1/3 & 1/3 & 1/3 \end{array}\right)$

$R1 = R1 - 2R3$ $\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/3 & 1/3 & -2/3 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 0 & 0 & 1 & -1/3 & 1/3 & 1/3 \end{array}\right)$

$R2 = R2 + 3R3$ $\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/3 & 1/3 & -2/3 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1/3 & 1/3 & 1/3 \end{array}\right)$

$C^{-1} = \begin{pmatrix} -1/3 & 1/3 & -2/3 \\ 0 & -1 & 1 \\ -1/3 & 1/3 & 1/3 \end{pmatrix}$

$\begin{pmatrix} 2 & 3 & -1 \\ 1 & 1 & 3 \\ 1 & 2 & -1 \end{pmatrix}\begin{pmatrix} -1/3 & 1/3 & -2/3 \\ 0 & -1 & 1 \\ -1/3 & 1/3 & 1/3 \end{pmatrix} = I$?

(b) $\det(C) = -3$

$a_{ji} = (-1)^{i+j} \dfrac{\det S_{ij}}{-3}$

pairs :

i	1	1	1
j	1	2	3

$\begin{pmatrix} 2 & 3 & -1 \\ 1 & 1 & 3 \\ 1 & 2 & -1 \end{pmatrix}$

Tut #4 Q9(b) $C = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 1 & 3 \\ 1 & 2 & -1 \end{pmatrix}$

Q7(a)

$\det(C) = (-2-2+9) - (-1+12-3)$
$= 5 - 8 = -3$ ✓

$C^{-1} = \begin{pmatrix} \frac{7}{3} & -\frac{1}{3} & -\frac{10}{3} \\ -\frac{4}{3} & \frac{1}{3} & \frac{7}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$ Q9(a)

$C = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 1 & 3 \\ 1 & 2 & -1 \end{pmatrix}$ ✓

a_{ji}

$\boxed{i=1, j=1}$ ✓

$a_{11} = \dfrac{(-1)^2 \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix}}{-3} = \dfrac{7}{3}$

$\boxed{i=1, j=2}$

$\boxed{a_{21}} = \dfrac{(-1)^3 \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix}}{-3} = -\dfrac{4}{3}$

$\boxed{i=1, j=3}$

$a_{31} = \dfrac{(-1)^4 \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}}{-3} = -\dfrac{1}{3}$

2.8.1 Inverse - another method

Here is another way to compute the inverse of a matrix.

- Select the i^{th} row and j^{th} column of A.
- Compute $(-1)^{i+j} \frac{\det S_{ij}}{\det A}$ = a_{ji}
- Store this entry at a_{ji} (row j and column i) in the inverse matrix.
- Repeat for all other entries in A.

That is, if

$$A = [\quad a_{ij} \quad]$$

then

$$A^{-1} = \frac{1}{\det A}[\quad (-1)^{i+j} \det S_{ji} \quad]$$

This method works but it is rather tedious.

$\boxed{i=2, j=1}$

$a_{12} = \dfrac{(-1)^3 \begin{vmatrix} 3 & -1 \\ 2 & -1 \end{vmatrix}}{-3} = -\dfrac{1}{3}$

$\boxed{i=3, j=1}$

$a_{13} = \dfrac{(-1)^4 \begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix}}{-3} = \dfrac{-10}{3}$

$\boxed{i=2, j=2}$

$a_{22} = \dfrac{(-1)^4 \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix}}{-3} = \dfrac{1}{3}$

$\boxed{i=3, j=2}$

$a_{23} = \dfrac{(-1)^5 \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix}}{-3} = \dfrac{7}{3}$

$\boxed{i=2, j=3}$

$a_{32} = \dfrac{(-1)^5 \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}}{-3} = \dfrac{1}{3}$

$\boxed{i=3, j=3}$

$a_{33} = \dfrac{(-1)^6 \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}}{-3} = \dfrac{1}{3}$

10) Cramer's rule

$$2x + 3y - z = 4$$
$$x + y + 3z = 1$$
$$x + 2y - z = 3$$

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 1 & 1 & 3 & 1 \\ 1 & 2 & -1 & 3 \end{array}\right)$$

$\det C = -3$

$$\boxed{\text{formula: } \frac{\det A_i}{\det A}}$$

$$x_i = \frac{\begin{pmatrix} 2 & 3 & -1 \\ 1 & 1 & 3 \\ 1 & 2 & -1 \end{pmatrix}}{-3}$$

$$= \frac{-3\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} + -1\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}}{-3}$$

$$= \frac{-3\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}}{-3}$$

$$= \frac{-3(-1-3) - (2-1)}{-3}$$

$$= \frac{-3(-4) - 1}{-3}$$

$$= -11/3$$

$$y_i = \frac{\begin{pmatrix} 2 & 3 & -1 \\ 1 & 1 & 3 \\ 1 & 2 & -1 \end{pmatrix}}{-3}$$

$$= \frac{2\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}}{-3}$$

$$= \frac{2(-1-6) - (2-1)}{-3}$$

$$= \frac{2(-7) - 1}{-3}$$

$$= \frac{-14-1}{-3}$$

$$= -15/-3 \simeq 5$$

$$z_i = \frac{\begin{pmatrix} 2 & 3 & -1 \\ 1 & 1 & 3 \\ 1 & 2 & -1 \end{pmatrix}}{-3}$$

$$= \frac{2(2-1) - 3(2-1)}{-3}$$

$$= \frac{2-3}{-3}$$

$$= -1/-3 \simeq 1/3$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -11/3 \\ 5 \\ 1/3 \end{pmatrix} \#$$

Problem set 5:

1) $$\frac{df}{dx} = \lim_{\Delta x \to 0} \left(\frac{f(x+\Delta x) - f(x)}{\Delta x} \right)$$

(a) $f(x) = x^3 + 2x$

$$f(\Delta x + x) = (\Delta x + x)^3 + 2(\Delta x + x)$$
$$= (\Delta x + x)(\Delta x + x)(\Delta x + x) + 2\Delta x + 2x$$
$$= (\Delta x^2 + 2x\Delta x + x^2)(\Delta x + x) + 2\Delta x + 2x$$
$$= \Delta x^3 + 2x\Delta x^2 + x^2\Delta x + x\Delta x^2 + 2x^2\Delta x + x^3 + 2\Delta x + 2x$$
$$= \Delta x^3 + x^3 + 3x\Delta x^2 + 3x^2\Delta x + 2\Delta x + 2x$$

$$f(h+x) \simeq h^3 + x^3 + 3xh^2 + 3x^2h + 2h + 2x$$

$$\frac{df}{dx} : \lim_{\Delta x \to 0} \left(\frac{f(x+\Delta x) - f(x)}{\Delta x} \right)$$

$$\frac{df}{dx} = \lim_{h \to 0} \left(\frac{h^3 + \cancel{x^3} + 3xh^2 + 3x^2h + 2h + \cancel{2x} - \cancel{x^3} - \cancel{2x}}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{h^3 + 3xh^2 + 3x^2h + 2h}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{\cancel{h}(h^2 + 3xh + 3x^2 + 2}{\cancel{h}} \right)$$

$$= \lim_{h \to 0} (h^2 + 3xh + 3x^2 + 2)$$

$$= (0)^2 + 3x(0) + 3x^2 + 2$$

$$= 3x^2 + 2 \ \#$$

$$\boxed{\text{Checking} : \frac{dy}{dx}(x^3 + 2x) = 3x^2 + 2}$$

(b) $f(x) = 1/x^2$

$f(x+h) = 1/(x+h)^2$

$$\frac{df}{dx} : \lim_{\Delta x \to 0} \left(\frac{f(x+\Delta x) - f(x)}{\Delta x} \right)$$

$$= \lim_{h \to 0} \left(\frac{1/(x^2 + 2xh + h^2) - 1/x^2}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{\dfrac{x^2 - (x^2 + 2xh + h^2)}{x^2(x^2 + 2xh + h^2)}}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{\dfrac{x^2 - x^2 - 2xh - h^2}{x^2(x^2 + 2xh + h^2)}}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{\dfrac{-2xh - h^2}{x^2(x^2 + 2xh + h^2)}}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{\dfrac{h(-2x - h)}{x^2(x^2 + 2xh + h^2)}}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{h(-2x - h)}{x^2(x^2 + 2xh + h^2)} \div h \right)$$

$$= \lim_{h \to 0} \left(\frac{\cancel{h}(-2x - h)}{x^2(x^2 + 2xh + h^2)} \times \frac{1}{\cancel{h}} \right)$$

$$= \lim_{h \to 0} \left(\frac{-2x - h}{x^2(x^2 + 2xh + h^2)} \right)$$

$$= \lim_{h \to 0} \left(\frac{-2x - h}{x^4 + 2x^3h + x^2h^2} \right)$$

$$= \frac{-2x - 0}{x^4 + 2x^3(0) + x^2(0)^2} \longrightarrow = \frac{-2x}{x^4} = -\frac{2}{x^3} \ \#$$

$$\boxed{\begin{aligned} \text{Checking} : &\ \frac{dy}{dx}\left(\frac{1}{x^2}\right) dx \\ &= \frac{dy}{dx}(x^{-2})\, dx \\ &= -2x^{-3} \\ &= -2/x^3 \end{aligned}}$$

2) (a) $f(x) = 5x^2 - 2x + 1$

$f'(x) = 10x - 2$

(b) $f'(x) = (4x-2)^7$

$= 7(4x-2)^6(4)$

$= 28(4x-2)^6$

(c) $f'(x) = (3x^2 - x + 1)^3$

$= 3(3x^2 - x + 1)^2(6x - 1)$

$= (18x - 3)(3x^2 - x + 1)^2$

(d) $f'(x) = (2x+4)^7 + (3x-2)^5$

$= 7(2x+4)^6(2) + 5(3x-2)^4(3)$

$= 14(2x+4)^6 + 15(3x-2)^4$

3) curve $\rightsquigarrow$ $y = x^2 + 6$

point $\rightsquigarrow$ $(x, y) = (3, 15)$

gradient of curve @ $(3, 15)$

= differentiate the curve

$= \frac{dy}{dx}(x^2 + 6)\,dx$

$= 2x$

$= 2(3)$

$= 6$

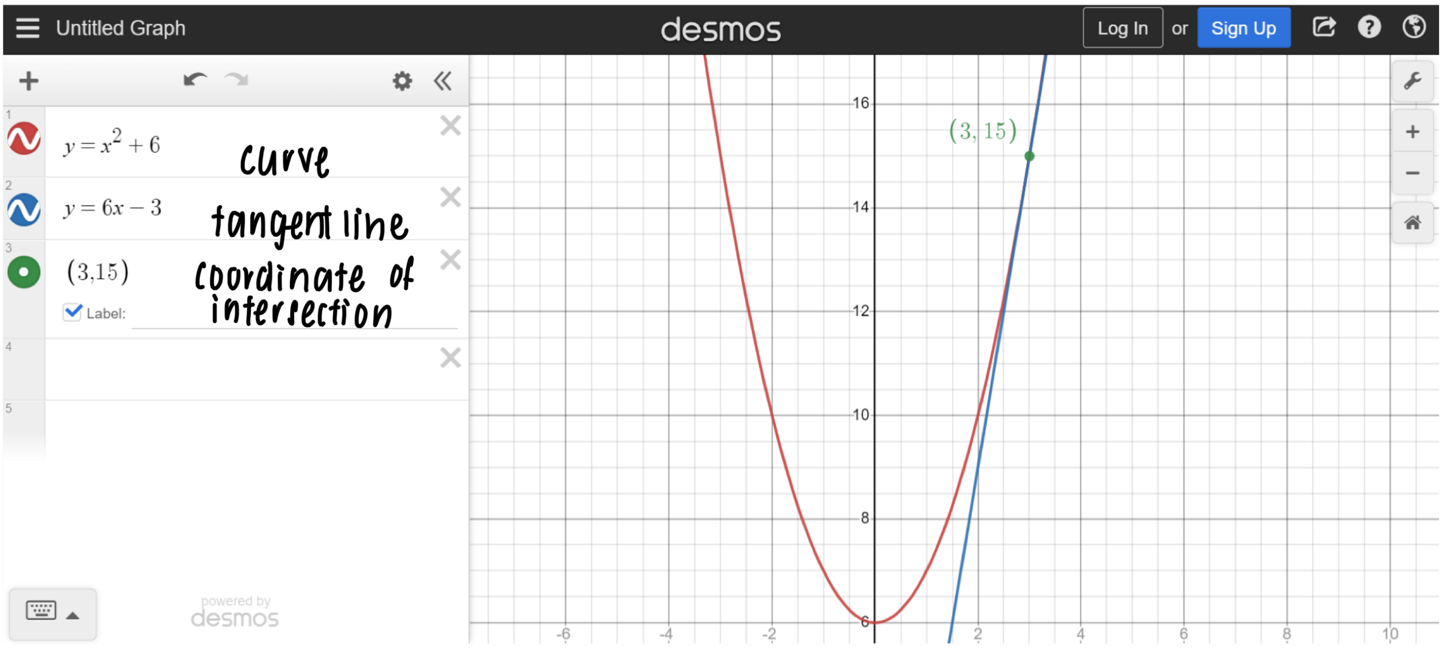
tangent line $\rightsquigarrow$ $y = mx + c$

$(15) = (6)(3) + c$

$15 = 18 + c$

$-3 = c$

$\therefore$ $y = 6x - 3$ is the equation of tangent line @ $(3, 15)$



4) Product rule, Quotient rule, Chain rule

(a) $f(x) = (2x+4)^7(3x-2)^5$

$\rightsquigarrow$ Product rule, Chain rule

$u = (2x+4)^7$

$du/dx = 7(2x+4)^6(2)$

$= 14(2x+4)^6$

$v = (3x-2)^5$

$dv/dx = 5(3x-2)^4(3)$

$= 15(3x-2)^4$

$f'(x) = 14(2x+4)^6(3x-2)^5 + 15(3x-2)^4(2x+4)^7$

(b) $f(x) = (5x^2 - 2x + 1)(4x-2)^7$

$\rightsquigarrow$ Product rule, Chain rule

$u = 5x^2 - 2x + 1$

$du/dx = 10x - 2$

$v = (4x-2)^7$

$dv/dx = 7(4x-2)^6(4)$

$= 28(4x-2)^6$

$f'(x) = (10x-2)(4x-2)^7$

$+ 28(4x-2)^6(5x^2 - 2x + 1)$

(c) $f(x) = \sqrt{1+2x^2} = (1+2x^2)^{1/2}$

$\rightsquigarrow$ Chain rule

$f'(x) = \frac{1}{2}(1+2x^2)^{-1/2}(4x)$

$= 2x(1+2x^2)^{-1/2}$

(d) $f(x) = 1/(1+x^2) = (1+x^2)^{-1}$

$\rightsquigarrow$ Chain rule

$f'(x) = -1(1+x^2)(2x)$

$= -2x(1+x^2)$

(e) $f(x) = x/(1+x^2)$

$\rightsquigarrow$ quotient rule

$u = x \qquad v = 1+x^2$

$du/dx = 1 \qquad dv/dx = 2x$

$$f'(x) = \frac{(1+x^2) - 2x^2}{(1+x^2)^2}$$

5) $f(x) = (4x-2)^7$ @ $x=0$

gradient of slope $= dy/dx\,(4x-2)^7\,dx$

$= 7(4x-2)^6(4)$

$= 28(4x-2)^6$

$= 28(4(0)-2)^6$

$= 28(-2)^6$

$= 1792$

$f(x) = (4(0)-2)^7$

$= -128$

equation of tangent line $\rightsquigarrow$ $y = mx+c$

$-128 = 1792(0)+C$

$C = -128$

$y = 1792x - 128$ #

6) (a) $y = (x-3)/(x-2)$

$u = x-3 \qquad v = x-2$

$du/dx = 1 \qquad dv/dx = 1$

$$\frac{dy}{dx} = \frac{(x-2)-(x-3)}{(x-2)^2}$$

(b) $y = 1/x^{1/2} = x^{-1/2}$

$$\frac{dy}{dx} = -\frac{1}{2}x^{-3/2}$$

(c) $$y = \left(\frac{1}{x^{1/2}} + x^{1/2}\right)^2$$

$$\frac{dy}{dx} = 2\left(\frac{1}{x^{1/2}} + x^{1/2}\right)\left(-\frac{1}{2}x^{-3/2} + \frac{1}{2}x^{-1/2}\right)$$

(d) $$y = \frac{x}{(x^2-1)^{1/2}}$$

$u = x \qquad v = (x^2-1)^{1/2}$

$du/dx = 1 \qquad dv/dx = 1/2\,(x^2-1)^{-1/2}(2x)$

$= x(x^2-1)^{-1/2}$

$$\frac{dy}{dx} = \frac{(x^2-1)^{1/2} - x^2(x^2-1)^{-1/2}}{(x^2-1)}$$

7) $y = \frac{x-3}{x-2}$ @ point $x=0$

$u = x-3 \qquad v = x-2$

$du/dx = 1 \qquad dv/dx = 1$

$$\frac{dy}{dx} = \frac{(x-2)-(x-3)}{(x-2)^2}$$

$$= \frac{1}{x^2-4x+4}$$

$$= \frac{1}{(0)^2-4(0)+4}$$

$= 1/4$ $\longrightarrow$ gradient of slope

$y = \frac{0-3}{0-2}$

$= -3/-2$

$= 3/2$

$y = mx+c$

$3/2 = 1/4\,(0)+c$

$c = 3/2$

equation of tangent :

$y = x/4 + 3/2$ #

8) $f(x) = 5x^2 - 2x + 1$

Find c when $f'(c) = 0$; Find $f(c)$; Find $(c, f(c))$

$f'(x) = 10x - 2$

$0 = 10x - 2$

$2 = 10x$

$1/5 = x$

$c = x = 1/5$

$(c, f(c)) = (1/5, 4/5)$

$= (0.2, 0.8)$

$f(1/5) = 5(1/5)^2 - 2(1/5) + 1$

$= 5/25 - 2/5 + 1$

$= 1/5 - 2/5 + 1$

$= -1/5 + 1$

$= 4/5$

$f(c) = f(1/5) = 4/5$

$\therefore$ $c = 1/5$

$f(1) = 4/5$

$(c, f(c)) = (1/5, 4/5)$

$= (0.2, 0.8)$

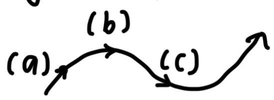
9) $f(x) = x^3 - 2x$ x-points : $-2, 0, 2$

$f'(x) = 3x^2 - 2$

$f''(x) = 6x$

x	-2	0	2
y	-4	0	4
m	10	-2	10
nature	↗	↘	↗

rough shape :



(a) increasing

(b) constant

(c) decreasing

$f(-2) = (-2)^3 - 2(-2)$

$= -8 + 4$

$= -4$

$f(0) = (0)^3 - 2(0)$

$= 0$

$f(2) = (2)^3 - 2(2)$

$= 8 - 4$

$= 4$

$f'(-2) = 3(-2)^2 - 2$

$= 3(4) - 2$

$= 10$

$f'(0) = 3(0)^2 - 2$

$= -2$

$f'(2) = 3(2)^2 - 2$

$= 12 - 2$

$= 10$

$f'(x) = 3x^2 - 2$

constant ↪ $f'(x) = 0$

$0 = 3x^2 - 2$

$2 = 3x^2$

$2/3 = x^2$

$\sqrt{2/3} = x$

↪ constant x

$y = (\sqrt{2/3})^3 - 2(\sqrt{2/3})$

↪ constant y

10) local maximum, local minimum @ $y = x^3 + 4x^2 + 5x + 2$

$\frac{dy}{dx} = 3x^2 + 8x + 5$

$0 = 3x^2 + 8x + 5$

$0 = (x+1)(3x+5)$

$x = 1$, $x = 5/3$

$y = (1)^3 + 4(1)^2 + 5(1) + 2$

$= 1 + 4 + 5 + 2 = 12 \rightarrow (1, 12)$

$y = (5/3)^3 + 4(5/3)^2 + 5(5/3) + 2$

$= 704/27 \rightarrow (5/3, 704/27)$

$\frac{d^2y}{dx^2} = 6x + 8$

when $x = 1$,

$6(1) + 8 = 7 + 8 = 15$

when $x = 5/3$

$6(5/3) + 8 = 10 + 8 = 18$

$\therefore$ HENCE,

$(1, 12) \longrightarrow$ local minimum

$(5/3, 704/27) \longrightarrow$ local maximum #

Problem set 6 :

$f(x) = \sin(x) \rightsquigarrow f'(x) = \cos(x)$	$f(x) = x^n \longrightarrow f'(x) = nx^{n-1}$
$f(x) = \cos(x) \rightsquigarrow f'(x) = -\sin(x)$	$f(x) = e^x \longrightarrow f'(x) = e^x$
$f(x) = \tan(x) \rightsquigarrow f'(x) = \sec^2(x)$	$f(x) = \ln(x) \longrightarrow f'(x) = 1/x$

1) Trigonometry Functions

(a) $f(x) = \sin(3x-2)$

$f'(x) = \cos(3x-2) \times 3$

$f'(x) = 3\cos(3x-2)$

(b) $g(x) = \cos^2(3x)$

$g(x) = (\cos(3x))^2$

$g'(x) = 2(\cos(3x)) \times -\sin(3x) \times 3$

$g'(x) = -6\cos(3x)\sin(3x)$

(c) $h(x) = x\sin(x)$

$u = x \qquad v = \sin(x)$

$du/dx = 1 \qquad dv/dx = \cos(x)$

$h'(x) = \sin(x) + x\cos(x)$

(d) $f(z) = \tan^3(z)$

$f(z) = [\tan(z)]^3$

$f'(z) = 3[\tan(z)] \times \sec^2(z)$

$f'(z) = 3\tan(z)\sec^2(z)$

2) Exponential Functions

(a) $f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

(b) $f(x) = e^{x^2+x}$

$f'(x) = (2x+1)e^{x^2+x}$

(c) $f(x) = (3x-2)e^{-x}$

$u = (3x-2) \qquad v = e^{-x}$

$du/dx = 3 \qquad dv/dx = -e^{-x}$

$f'(x) = 3e^{-x} - e^{-x}(3x-2)$

(d) $f(x) = e^x/(1+e^x)$

$u = e^x \qquad v = 1+e^x$

$du/dx = e^x \qquad dv/dx = e^x$

$$f'(x) = \frac{e^x(1+e^x) - e^{2x}}{(1+e^x)^2}$$

3) Logarithmic Functions

(a) $f(x) = \ln(3x-2)$

$f'(x) = 1/(3x-2)$

(b) $f(x) = \ln((x-3)/(x-2))$

$f'(x) = \frac{x-2}{x-3}$

(c) $h(x) = \frac{1}{x}\ln(x)$

$u = 1/x \qquad v = \ln(x)$

$du/dx = -x^{-2} \qquad dv/dx = 1/x$

$h'(x) = -x^{-2}\ln(x) + 1/x^2$

4) Inverse Functions

$f(x) = 2x^3 - 5$

Find $f'(x)$ and $f^{-1}(x)$

$f'(x) = 6x^2$

$$f^{-1}(x) = \frac{2x^{3+1}}{3+1} - \frac{5x}{1}$$

$f^{-1}(x) = 2/4\,(x^4) - 5x$

$f^{-1}(x) = \frac{1}{2}(x^4) - 5x$

5) $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$

Demonstrated in lecture

$\boxed{\sin^{-1}(x) \neq 1/\sin(x)}$

(e) Prove the differentiation formula $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$.

let $y = \sin^{-1}(x)$

$\sin y = x$

$x = \sin y$

$\frac{dx}{dy} = \cos y$

Differentiation

$\frac{dy}{dx} = 1/\frac{dx}{dy}$

$\frac{dy}{dx} = \frac{1}{\cos y}$

Recall Identity

$\cos^2 y + \sin^2 y = 1$

$\cos^2 y = 1 - \sin^2 y$

$\cos y = \pm\sqrt{1-\sin^2 y}$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}}$

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

6) $\frac{d}{dx}\tan^{-1}(x)\ dx$

let $y = \tan^{-1}(x)$

$x = \tan(y)$

$\frac{dx}{dy} = \sec^2(y)$

$\sec^2(x) \equiv 1+\tan^2(x)$

$\frac{dx}{dy} = 1+\tan^2(y)$

$x = \tan(y)$

$x^2 = \tan^2(y)$

$\frac{dx}{dy} = 1+x^2$

$\frac{dy}{dx} = \frac{1}{dx/dy}$

$\frac{dy}{dx} = \frac{1}{1+x^2}$ #

7) (a) $f(x) = \sin^{-1}(x/2)$

** $y = \sin^{-1}(x/2)$

$dy/dx = 1/\sqrt{1-x^2}$

$f'(x) = 1/\sqrt{(1-(x/2)^2)}$

$f'(x) = \frac{1}{\sqrt{1-(x/2)(x/2)}}$

$f'(x) = \frac{1}{\sqrt{1-x^2/4}}$

(b) $f(x) = \tan^{-1}(3x)$

$f'(x) = 1/(1+(3x)^2)$

$f'(x) = \frac{1}{1+9x^2}$

(c) $f(x) = \sqrt{(1+x^2)}\tan^{-1}(x)$

$u = (1+x^2)^{1/2}$ $\quad v = \tan^{-1}(x)$

$du/dx = 1/2(1+x^2)^{-1/2}(2x)$ $\quad dv/dx = 1/(1+x^2)$

$= x(1+x^2)^{-1/2}$

$f'(x) = x(1+x^2)^{-1/2}\tan^{-1}(x) +$
$(1+x^2)^{1/2}(1/(1+x^2))$

8) Higher order derivatives

Find d^2y/dx^2

(a) $y = x^3+3x^2-5x+1$

$dy/dx = 3x^2+6x-5$

$d^2y/dx^2 = 6x+6$

(b) $y = (1+x^2)^{1/2}$

$dy/dx = 1/2\ (1+x^2)^{-1/2}(2x)$

$= x(1+x^2)^{-1/2}$

$u = x$ $\quad v = (1+x^2)^{-1/2}$

$du/dx = 1$ $\quad dv/dx = -\frac{1}{2}(1+x^2)^{-3/2}(2x)$

$= -x(1+x^2)^{-3/2}$

$d^2y/dx^2 = (1+x^2)^{-1/2} - x^2(1+x^2)^{-3/2}$

9) $y = \ln(x^2+x+1)$

$dy/dx = 1/(x^2+x+1) \times (2x+1)$

$\frac{dy}{dx} = \frac{2x+1}{x^2+x+1}$

$u = 2x+1$ $\quad v = x^2+x+1$

$du/dx = 2$ $\quad dv/dx = 2x+1$

$\frac{d^2y}{dx^2} = \frac{2(x^2+x+1)-(2x+1)^2}{(x^2+x+1)^2}$

10) (a) $f(x) = e^{3x}$

$f'(x) = 3e^{3x}$

$f''(x) = 9e^{3x}$

$f'''(x) = 27e^{3x}$

$\vdots$

$f^n(x) = 3^n e^{3x}$

(b) $f(x) = \ln(x+2)$

$f'(x) = 1/(x+2)$

$u = 1$ $\quad v = x+2$

$du/dx = 0$ $\quad dv/dx = 1$

$f''(x) = -1/(x+2)^2$

$u = -1$ $\quad v = (x+2)^2$

$du/dx = 0$ $\quad dv/dx = 2(x+2)$

$f'''(x) = -2(x+2)/(x+2)^4$

Problem set 7 :

1) Parametric Curves

t	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$\frac{3\pi}{2}$	2π
x	2	$\sqrt{3}$	$\sqrt{2}$	1	0	-2	0	2
y	0	1	$\sqrt{2}$	$\sqrt{3}$	2	0	-2	0

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

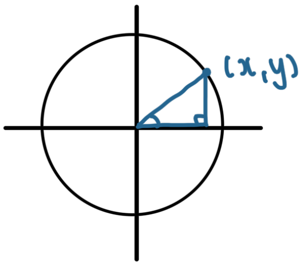
$$\downarrow$$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1^2$$

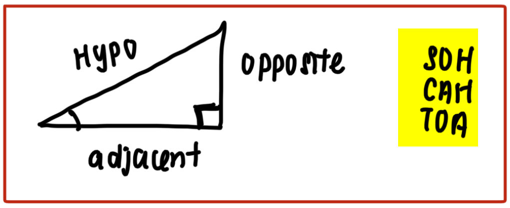
$\cos^2(x) + \sin^2(x) = 1$

$\cos(x) = x/2$ $\quad$ $\sin(x) = y/3$

$x = 2\cos(x)$ $\quad$ $y = 3\sin(x)$



$\cos^2(x) + \sin^2(x) = 1$



2) not doable

↳ look at teacher's applied worksheet

3) sub x, y, z to each parametric function

4) Parametric Differentiation

dy/dx @ $t = 0$, $-\pi < t < \pi$

(a) $x(t) = \sin(t)$, $y(t) = \cos(t)\sin(t)$

$u = \cos(t)$ $\quad$ $v = \sin(t)$

$du/dt = -\sin(t)$ $\quad$ $dv/dt = \cos(t)$

$dy/dt = -\sin(t)\sin(t) + \cos^2(t)$

$dy/dt = -\sin^2(t) + \cos^2(t)$

$dx(t)/dt = \cos(t)$

$$\frac{dy(t)}{dx(t)} = \frac{\cos^2(t) - \sin^2(t)}{\cos(t)}$$

(b) $x(t) = 2(t - \cos(t))$, $y(t) = \sin(t)$

$dx(t)/dt = 2 + \sin(t)$ $\quad$ $dy(t)/dt = \cos(t)$

$$\frac{dy(t)}{dx(t)} = \frac{\cos(t)}{2 + \sin(t)}$$

(c) $x(t) = 3t^2 - t$, $y(t) = 5t - 3$

$dx(t)/dt = 6t - 1$ $\quad$ $dy(t)/dt = 5$

$$\frac{dy(t)}{dx(t)} = \frac{5}{6t-1}$$

(d) $x(t) = 5t - 3$, $y(t) = 3t^2 - t$

$dx(t)/dt = 5$ $\quad$ $dy(t)/dt = 6t - 1$

$$\frac{dy(t)}{dx(t)} = \frac{6t-1}{5}$$

5) $x = t\cos(t)$ $\quad$ $y = t\sin(t)$

$u = t$ $\quad$ $v = \cos(t)$ $\quad$ $u = t$ $\quad$ $v = \sin(t)$

$du/dt = 1$ $\quad$ $dv/dt = -\sin(t)$ $\quad$ $du/dt = 1$ $\quad$ $dv/dt = \cos(t)$

$dx/dt = \cos(t) - t\sin(t)$ $\quad$ $dy/dt = \sin(t) + t\cos(t)$

$$\frac{dy}{dx} = \frac{\sin(t) + t\cos(t)}{\cos(t) - t\sin(t)}$$

6) curve A :→ $x(v) = 3$, $y(v) = 4$, $z(v) = v$ for $0 < v < 2$

curve B :→ $x(u) = 3 + \sin(u)$, $y(u) = 4 - u$, $z(u) = 1 - u$ for $-1 < u < 1$

if curve A intersect curve B, x, y, z values must be the same

curve A = curve B	curve A = curve B	curve A = curve B
$x(v) = x(u)$	$4 = 4 - u$	$v = 1 - u$
$3 = 3 + \sin(u)$	$u = 4 - 4$	$u = 1 - v$ OR IF $u = 0$
$0 = \sin(u)$	$u = 0$	$v = 1 - 0$
$u = 0$		$v = 1$

$(u, v) = (0, 1)$

Curve A : $x(v) = 3$; $y(v) = 4$; $z(v) = v = 1$

↳ coordinate : 3 4 1 → $(3, 4, 1)$

curve B : $x(u) = 3 + \sin(0) = 3$; $y(u) = 4 - 0 = 4$; $z(u) = 1 - 0 = 1$

↳ coordinate : 3 4 1 → $(3, 4, 1)$

plane tangent to curve A, curve B

curve A	gradient vector (d/dv)
$x(v) = 3$	0
$y(v) = 4$	0
$z(v) = v$	v

curve B	gradient vector (d/du)
$x(u) = 3 + \sin(u)$	$\cos(u)$
$y(u) = 4 - u$	-1
$z(u) = 1 - u$	-1

$v = 1$

$\cos(u)$ sub $u = 0$
$= \cos(0) = 1$

$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

intersection : $\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ tangent plane :→ $\underset{\sim}{p} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ #

7) $x(t) = \cos(t)$ $y(t) = \sin(t)$ for $0 \le t \le \pi$

(a) $dx(t)/dt = -\sin(t)$

$dy(t)/dt = \cos(t)$

(b) $x^2 + y^2 = 1$

$y^2 = 1 - x^2$ OR $x^2 = 1 - y^2$

$y = \sqrt{1 - x^2}$ $\quad x = \sqrt{1 - y^2}$ (NA)

$\frac{dy}{dx} = \frac{1}{2}(1 - x^2)^{-1/2}(-2x)$ ↳ need dy/dx not dx/dy

$dy/dx = -x(1 - x^2)^{-1/2}$

(c) $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt}$

$= -x / \sqrt{1 - x^2}$

$= -\cos(t) / \sqrt{1 - \cos(t)^2}$

$= -\cos(t) / \sqrt{\sin^2(t)}$

$= -\cos(t) / \sin(t)$ $\quad 1 - \cos^2(t) = \sin^2(t)$

$= dy/dt \; / \; dx/dt$

8) Equation of tangent @ $t = \pi/3$ $\quad$ $y - y_1 = m(x - x_1)$

$x(t) = 4\sin(t) \qquad y(t) = 2\cos(t)$

$dx(t)/dt = 4\cos(t) \qquad dy(t)/dt = -2\sin(t)$

$dx(\pi/3)/dt = 2(1/2) \qquad dy(\pi/3)/dt = -2(\sqrt{3}/2)$

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$$

$= -2\sin(t) / 4\cos(t)$

$= -2\sin(\pi/3) / 4\cos(\pi/3)$

$= -2(\sqrt{3}/2) / 4(1/2)$

$= -\sqrt{3}/2 \rightsquigarrow$ gradient, m

coordinates : $x(\pi/3) = 4\sin(\pi/3)$

$= 4(\sqrt{3}/2)$

$= 2\sqrt{3}$

$y(\pi/3) = 2\cos(\pi/3)$

$= 2(1/2)$

$= 1$

$(2\sqrt{3}, 1)$

$y - y_1 = m(x - x_1)$

$y - 1 = -\sqrt{3}/2(x - 2\sqrt{3})$

$y = -\sqrt{3}x/2 + (2\sqrt{3})(\sqrt{3}/2) + 1$

$y = -\sqrt{3}x/2 + 3 + 1$

$y = -\sqrt{3}x/2 + 4$

9) $x(t) = \dfrac{1-t^2}{1+t^2}$; $y(t) = \dfrac{2t}{1+t^2}$

(a) $x^2 + y^2 = 1$

$$\left(\frac{1-t^2}{1+t^2}\right)^2 + \left(\frac{2t}{1+t^2}\right)^2 = \left(\frac{1-t^2}{1+t^2}\right)\left(\frac{1-t^2}{1+t^2}\right) + \left(\frac{2t}{1+t^2}\right)\left(\frac{2t}{1+t^2}\right)$$

$$= \frac{(1-t^2)^2}{(1+t^2)^2} + \frac{4t^2}{(1+t^2)^2}$$

$$= \frac{(1-t^2)(1-t^2)}{(1+t^2)(1+t^2)} + \frac{4t^2}{(1+t^2)(1+t^2)}$$

$$= \frac{1-2t^2+t^4}{1+2t^2+t^4} + \frac{4t^2}{1+2t^2+t^4}$$

$$= \frac{1+2t^2+t^4}{1+2t^2+t^4}$$

$= 1$ (shown)

(b) point $(-1, 0)$ unusualness

$-1 = \dfrac{1-t^2}{1+t^2} \qquad 0 = \dfrac{2t}{1+t^2}$

$-1 - t^2 = 1 - t^2 \qquad 0(1+t^2) = 2t$

$-2 = -t^2 + t^2 \qquad 0 = 2t$

$-2 = 0 \qquad t = 0$

(NA) $\qquad \therefore$ uniquely defined

10) Find tangent vector when position vector $\rightsquigarrow \underset{\sim}{r}(t) = (x(t), y(t), z(t))$

(a) (i) $x(t) = 3t^2$
$y(t) = 4t$
$z(t) = 5$
$\underset{\sim}{r}(t) = (3t^2, 4t, 5)$
$dx(t)/dt = 6t$
$dy(t)/dt = 4$
$dz(t)/dt = 0$
tangent vector
$= (6t, 4, 0)$

(ii) $x(t) = 3\cos(t)$
$y(t) = 2\sin(t)$
$z(t) = t$
$\underset{\sim}{r}(t) = (3\cos(t), 2\sin(t), t)$
$dx(t)/dt = -3\sin(t)$
$dy(t)/dt = 2\cos(t)$
$dz(t)/dt = 1$
tangent vector
$= (-3\sin(t), 2\cos(t), 1)$

(b) position vector $\rightsquigarrow \underset{\sim}{r}(t) = (x(t), y(t), z(t)) \rightsquigarrow$ for some curve
$1 = \underset{\sim}{r}' \cdot \underset{\sim}{r}'$ for all t

RHS use product rule:
$0 = \underset{\sim}{r}'' \cdot r' + \underset{\sim}{r}'' \cdot r'$
$0 = 2\underset{\sim}{r}'' \cdot r'$
$\underset{\sim}{r}'' \cdot \underset{\sim}{r}' = 0 \Leftrightarrow$ orthogonal
↳ normal

$\therefore r''$ is the normal to the curve

Problem set 8:

1) $f(x) = e^{-x}$

$f'(x) = -e^{-x}$	when $x=0$	$f'(x) = -1$	$a_1 = -1/1! = -1$
$f''(x) = e^{-x}$	when $x=0$	$f''(x) = 1$	$a_2 = 1/2! = 1/2$
$f'''(x) = -e^{-x}$	when $x=0$	$f'''(x) = -1$	$a_3 = -1/3! = -1/6$
$f''''(x) = e^{x}$	when $x=0$	$f''''(x) = 1$	$a_4 = 1/4! = 1/24$

partial sum: $-1 + 1/2 - 1/6 + 1/24$

2) (a) $f(x) = \ln(1-x) \longrightarrow$ logarithmic function
(b) $f(x) = \sin(x) \longrightarrow$ trigonometric function
(c) $f(x) = e^{-x}\sin(x) \longrightarrow$ product rule

3) $f(x) = \sin^{-1}(x)$; $x=0 \longrightarrow$ trigonometric function + quotient rule

4) (a) $f(x) = \cos(x)$ } trigonometric function
(b) $f(x) = \sin(2x)$
(c) $f(x) = e^{x}$ } logarithmic function
(d) $f(x) = \arctan(x) = \tan^{-1}(x)$
$f'(x) = 1/(1+x^2) = (1+x^2)^{-1}$
$f''(x) = -2x(1+x^2)^{-2}$
$f'''(x) = -2(1+x^2)^{-2} + 8x^2(1+x^2)^{-3}$
$f''''(x) = 8x(1+x^2)^{-3} + 16x(1+x^2)^{-3} - 48x^3(1+x^2)^{-4}$ ....

5) (a) $f(x) = \cos(x) \sin(2x)$

(b) $f(x) = e^{-x^2}$

(c) $f(x) = \arctan(\arctan(x))$

$f(x) = \tan^{-1}(\tan^{-1}(x))$

6)

6)(a) $f(x) = e^x$ when $a = 1$

$f'(x) = e^x$ $\quad x = a = 1$ $\quad f'(1) = e^1 = e$ **1st differentiation**

$f''(x) = e^x$ $\quad x = a = 1$ $\quad f''(1) = e^1 = e$ **2nd differentiation**

$f'''(x) = e^x$ $\quad x = a = 1$ $\quad f'''(1) = e^1 = e$ **3rd differentiation**

$f''''(x) = e^x$ $\quad x = a = 1$ $\quad f''''(1) = e^1 = e$ **4th differentiation**

$T_1(x) = e \longrightarrow T_1(x) = e$

$T_2(x) = e/1!\,(x-1) \longrightarrow T_2(x) = e(x-1)$

$T_3(x) = e/2!\,(x-1)^2 \longrightarrow T_3(x) = \frac{e}{2}(x-1)^2$

$T_4(x) = e/3!\,(x-1)^3 \longrightarrow T_4(x) = \frac{e}{6}(x-1)^3$

$Tn(x) = e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3$

(b) $f(x) = e^x$ when $a = -1$

$f'(x) = e^x$ $\quad x = a = -1$ $\quad f'(-1) = e^{-1} = 1/e$ **1st differentiation**

$f''(x) = e^x$ $\quad x = a = -1$ $\quad f''(-1) = e^{-1} = 1/e$ **2nd differentiation**

$f'''(x) = e^x$ $\quad x = a = -1$ $\quad f'''(-1) = e^{-1} = 1/e$ **3rd differentiation**

$f''''(x) = e^x$ $\quad x = a = -1$ $\quad f''''(-1) = e^{-1} = 1/e$ **4th differentiation**

$T_1(x) = 1/e \longrightarrow T_1(x) = 1/e$

$T_2(x) = [(1/e)/1!](x+1) \longrightarrow T_2(x) = (1/e)(x+1)$

$T_3(x) = [(1/e)/2!](x+1)^2 \longrightarrow T_3(x) = (1/2e)(x+1)^2$

$T_4(x) = [(1/e)/3!](x+1)^3 \longrightarrow T_4(x) = (1/6e)(x+1)^3$

$Tn(x) = 1/e + (1/e)(x+1) + (1/2e)(x+1)^2 + (1/6e)(x+1)^3$

$Tn(x) = 1/e + (x+1)/e + (x+1)^2/2e + (x+1)^3/6e$

7)

7) $x = 0$ for $\log(1+x)$, $\log(1-x)$

$f(x) = \log(1+x/1-x)$ $\qquad u = 1+x \quad v = 1-x$

$\qquad du/dx = 1 \quad dv/dx = -1$

quotient rule, logarithmic function

$f'(x) = (1/\ln(10))(1-x/1+x)((1-x)-(-1-x)/(1-x)^2)$

$f'(x) = (1/\ln(10))(1-x/1+x)(2/(1-x)^2)$

$f'(x) = (1-x/(1+x)(\ln(10)))(2/(1-x)^2)$

$f'(x) = 2-2x/(\ln(10)+x\ln(10))(1-x)^2$

$$f'(x) = \frac{2-2x}{[\ln(10)+x\ln(10)][(1-x)^2]}$$

$$f'(x) = \frac{2\cancel{(1-x)}}{(\ln(10)+x\ln(10))(1-x)^{\cancel{2}}}$$

$$f'(x) = \frac{2}{\ln(10)+x\ln(10)-x\ln(10)-x^2\ln(10)}$$

$$f'(x) = \frac{2}{\ln(10)-x^2\ln(10)}$$ #

First differentiation

logarithmic function differentiation :

- $d/dx\,(\log_a(x))\,dx = (1/\ln(a))(1/x)(x')$ differentiate x

- eg. $f(x) = \log_7(2x)$

$f'(x) = (1/\ln(7))(1/2x)(2)$

$f'(x) = (1/\ln(7))(1/x)$

$f'(x) = 1/x\ln(7)$

Linear Approximation

8) $f(x) = (1+x)^{1/2}$ @ $x=0$

Find approximation for $f(x)$ when $x=1$

$f'(x) = 1/2(1+x)^{-1/2}(1)$

$f'(0) = 1/2(1)^{-1/2}$ $\qquad$ $f'(1) = 1/2(2)^{-1/2}$

$f'(0) = 1/2$ $\qquad$ $f'(1) = 0.3535533906$

$n=1$, $x=0$

$T_1(x) = (1+x)^{1/2} \cong 1 + x((1/2)/1!)$

$(1+x)^{1/2} \cong 1 + x/2$

$n=1$, $x=1$

RHS $= 1 + x/2$ $\qquad$ LHS $= (1+x)^{1/2}$

$= 1 + 1/2$ $\qquad$ $= (1+1)^{1/2}$

$= 3/2$ $\qquad$ $= \sqrt{2}$

$\therefore 3/2 \neq \sqrt{2}$ with 3s.f. so it is not reasonable

9) $f(x) = \sin(x)$ @ $x=0$

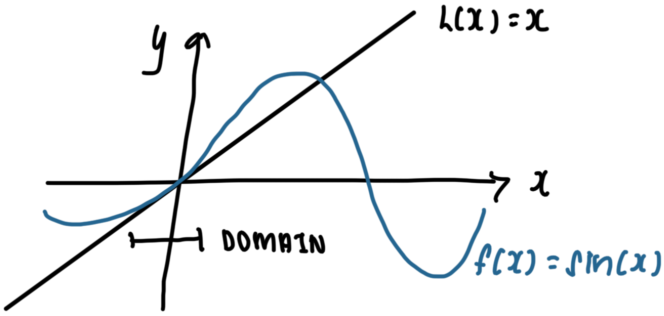
$f'(x) = \cos(x)$

$f(0) = \sin(0)$ $\qquad$ $f'(0) = \cos(0)$

$f(0) = 0$ $\qquad$ $f'(0) = 1$

$L(x) = 0 + 1/1!\,(x)$

$L(x) = x$



$\therefore$ $L(x)$ is reasonable only in specific domain

Cubic spline

⑩ $6\left(\dfrac{y_{i+1}-y_i}{h_i} - \dfrac{y_i - y_{i-1}}{h_{i-1}}\right) = h_i y''_{i+1} + 2(h_i + h_{i-1})y''_i + h_{i-1}y''_{i-1}$

⑨ $a_i = \dfrac{y_{i+1}-y_i}{h_i} - \dfrac{1}{6} h_i (y''_i+1 + 2y''_i)$

⑥ $b_i = y''_i/2$

⑦ $c_i = \dfrac{y_{i+1}-y''_i}{6h_i}$

⑤ $\tilde{y}_i(x) = y_i + a_i(x-x_i) + b_i(x-x_i)^2 + c_i(x-x_i)^3$

10) equation $\rightsquigarrow$ $f(x) = x + 1/x$

x-points $\rightsquigarrow$ $x_1 = 1/2$

$x_2 = 1$

$x_3 = 3/2$

$x_4 = 2$

Problem set 9 :

1) integration by substitution

(a) $f(x) = 2x\cos(x^2+2)$

$u = x^2+2$

$du/dx = 2x$

$du = 2x\,dx$

$\int 2x\cos(x^2+2)\,dx$

$= \int \cos(x^2+2)\,du$

$= \int \cos(u)\,du$

$= \sin(u) + C$

$= \sin(x^2+2) + C$

HENCE, $\int 2x\cos(x^2+2)\,dx = \sin(x^2+2)+C$

(b) $f(x) = 3x\sin(x^2+2)$

$u = x^2+2$

$du/dx = 2x$

$[\,du = 2x\,dx\,]$ multiply $3/2 \rightsquigarrow (3/2)\,du = 3x\,dx$

$\int 3x\sin(x^2+2)\,dx$

$= \int \sin(u)\,3/2\,du$

$= 3/2\int \sin(u)\,du$

$= 3/2(-\cos(u)) + C$

$= -3/2\cos(u) + C$

$= -3/2\cos(x^2+2) + C$

(c) $f(x) = 4x^2(x^3-5)^{1/2}$

$u = x^3-5$

$du/dx = 3x^2$

$(du = 3x^2\,dx) \times 4/3 \longrightarrow 4/3\,du = 4x^2\,dx$

$\int 4x^2(x^3-5)^{1/2}\,dx$

$= \int u^{1/2}\,du$

$= u^{3/2}/3/2 + C$

$= 2u^{3/2}/3 + C$

$= \dfrac{2(x^3-5)^{3/2}}{3} + C$

(d) $f(x) = \cos(x)\,e^{2\sin(x)}$

$u = \sin(x)$

$du/dx = \cos(x) \longrightarrow du = \cos(x)\,dx$

$\int \cos(x)\,e^{2\sin(x)}\,dx$

$= \int e^{2u}\,du$

$= \frac{1}{2}e^{2u} + C$

$= \dfrac{e^{2\sin(x)}}{2} + C$

2) integration by parts

(a) $\int x\cos(x)\,dx$
$= x\sin(x) - \int \sin(x)(1)\,dx$
$= x\sin(x) - \int \sin(x)\,dx$
$= x\sin(x) + \cos(x) + C$

(b) $\int xe^{-x}\,dx$
$= -xe^{-x} - \int -e^{-x}(1)\,dx$
$= -xe^{-x} + \int e^{-x}\,dx$
$= -xe^{-x} - e^{-x} + C$

(c) $\int y(y+1)^{1/2}\,dy$
$= y\left(\frac{(y+1)^{3/2}}{3/2}\right) - \int \left(\frac{(y+1)^{3/2}}{3/2}\right)(1)\,dy$
$= \frac{2y(y+1)^{3/2}}{3} - \int \frac{2(y+1)^{3/2}}{3}\,dy$
$= \frac{2}{3}y(y+1)^{3/2} - \frac{2}{3}\int (y+1)^{3/2}\,dy$
$= \frac{2}{3}y(y+1)^{3/2} - \frac{2}{3}\left(\frac{(y+1)^{5/2}}{5/2}\right)\,dy$
$= \frac{2}{3}y(y+1)^{3/2} - \frac{4}{15}(y+1)^{5/2} + C$

(d) $\int x^2\ln(x)\,dx$
$f = \ln(x)$ $\quad dg/dx = x^2$
$g = \int x^2 = x^3/3$

$\int x^2\ln(x)\,dx$
$= \ln(x)(x^3/3) - \int x^3/3\,(1/x)\,dx$
$= \ln(x)(x^3/3) - \frac{1}{3}\int x^3/x\,dx$
$= \ln(x)(x^3/3) - \frac{1}{3}\int x^2\,dx$
$= \frac{x^3\ln(x)}{3} - \frac{1}{3}\left(\frac{x^3}{3}\right) + C$
$= \frac{x^3\ln(x)}{3} - \frac{x^3}{9} + C$
$= \frac{3x^3\ln(x) - x^3}{9} + C$

$$\boxed{\int f\frac{dg}{dx}\,dx = fg - \int g\frac{df}{dx}\,dx}$$

LATE
↳ determine who differentiate, who integrate
↳ Log, Algebra, Trigo, Exponential
* Higher priority → differentiation
* Lower priority → integration
** High ⟶ Low
L A T E

3) $\int \sin^2(x)\,dx$ & $\int \cos^2(x)\,dx$
$\int \sin(x)\sin(x)\,dx$
$f = \sin(x)$ $\quad dg/dx = \sin(x)$
$g = \int \sin(x)\,dx$
$g = -\cos(x)$

$\int \sin^2(x)\,dx$
$= \sin(x)(-\cos(x)) - \int -\cos(x)(\cos(x))\,dx$
$= -\cos(x)\sin(x) - \int -\cos^2(x)\,dx$
$= -\cos(x)\sin(x) + \int \cos^2(x)\,dx$
$\cos^2(x) + \sin^2(x) = 1$
$= -\cos(x)\sin(x) + \int 1 - \sin^2(x)\,dx$
$= -\cos(x)\sin(x) + \int 1 - \int \sin^2(x)\,dx$
$= -\cos(x)\sin(x) + x - \int \sin^2(x)\,dx$
$2\int \sin^2(x)\,dx = -\cos(x)\sin(x) + x + C$
$\int \sin^2(x)\,dx = [(-\cos(x)\sin(x) + x)/2] + C$

$\int \cos^2(x)\,dx = \int (\cos(x))(\cos(x))\,dx$
$f = \cos(x)$ $\quad dg/dx = \cos(x)$
$df/dx = -\sin(x)$ $\quad g = \int \cos(x)\,dx$
$g = \sin(x)\,dx$
$\int \cos^2(x)\,dx = \sin(x)(\cos(x)) - \int \sin(x)(-\sin(x))\,dx$
$\int \cos^2(x)\,dx = \sin(x)\cos(x) + \int \sin^2(x)\,dx$
$\int \cos^2(x)\,dx = \sin(x)\cos(x) + \int 1 - \cos^2(x)\,dx$
$2\int \cos^2(x)\,dx = \sin(x)\cos(x) + x + C$
$\int \cos^2(x)\,dx = \frac{\sin(x)\cos(x) + x}{2} + C$

4) $\int e^x \sin(x)\,dx$ & $\int e^x \cos(x)\,dx$

LATE ⟶ LOG, ALGEBRA, TRIGO, EXPONENTIAL

$\int e^x \sin(x)\,dx$

$u = \sin(x)$ $\quad dv/dx = e^x$

$du/dx = \cos(x)$ $\quad v = e^x$

$\int e^x \sin(x)\,dx = e^x \sin(x) - \int e^x \cos(x)\,dx$

BUT! T and E don't matter much.

$u = e^x$ $\quad dv/dx = \sin(x)$

$du/dx = e^x$ $\quad v = -\cos(x)$

$\int e^x \sin(x)\,dx = -e^x \cos(x) - \int e^x \sin(x)\,dx$

$2\int e^x \sin(x)\,dx = -e^x \cos(x) + C$

$\int e^x \sin(x)\,dx = -\frac{1}{2} e^x \cos(x) + C$

$\int e^x \cos(x)\,dx$

Same for $\int e^x \cos(x)\,dx$!

5) (a) $\int \cos(x) \sin(x)\, e^{\cos(x)}\,dx$

$u = \cos(x)$

$du/dx = -\sin(x)$

$du = -\sin(x)\,dx \longrightarrow -du = \sin(x)\,dx$

$\int -ue^u\,du = -\int ue^u\,du$

$f = u$ $\quad g = e^u$

$df/du = 1$ $\quad dg/du = e^u$

$-\int ue^u\,du = ue^u - \int e^u\,du$

$-\int ue^u\,du = ue^u - e^u$

$\int ue^u\,du = e^u - ue^u + C$

$\int \cos(x) e^{\cos(x)}\,dx = e^{\cos(x)} - \cos(x) e^{\cos(x)} + C$

(b) $\int e^{2x} \cos(e^x)\,dx$

(c) $\int \pi e^{\sqrt{x}}\,dx$

6) $\cancel{\int \frac{1}{x}\,dx} = 1 + \cancel{\int \frac{1}{x}\,dx} + C$

$0 = 1 + C$

$C = -1$

7) Area under curve

$y = \sin^2(x)$ $\quad y = \cos^2(x)$ $\quad$ for $0 \leq x \leq \pi/2$

$\sin^2(x) = \cos^2(x)$

$\sin(x)\sin(x) = \cos(x)\cos(x)$

$\tan(x) = \sin(x)/\cos(x)$

$\tan^2(x) = 1$

$\tan(x) = \sqrt{1}$

$x = \tan^{-1}(\sqrt{1})$

$x = \pi/4$

↳ intersection x

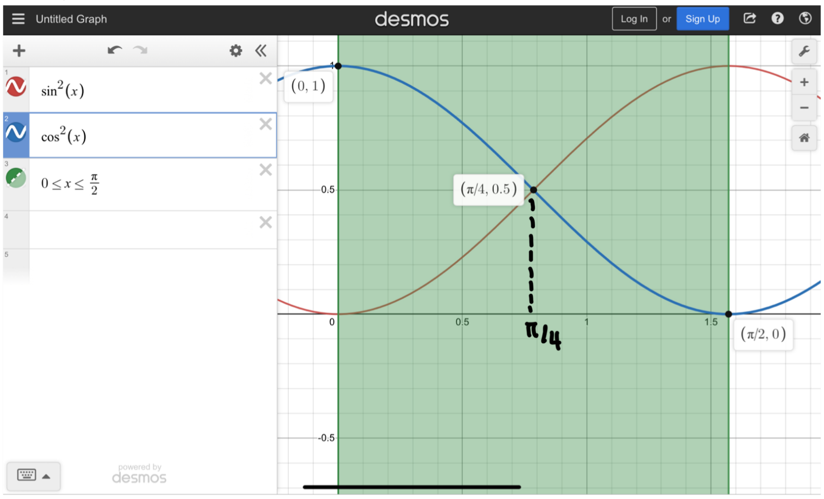
↱ intersection y

$y = \sin^2(\pi/4)$

$y = 1/2$

intersection coordinate

↳ $(\pi/4, 1/2)$



$0 \to \pi/4 \rightsquigarrow \cos^2(x) - \sin^2(x)$

$\pi/4 \to \pi/2 \rightsquigarrow \sin^2(x) - \cos(x)$

area between curve = $\int_0^{\pi/4} \cos^2(x) - \sin^2(x) \, dx - \int_{\pi/4}^{\pi/2} \sin^2(x) - \cos^2(x) \, dx$

8) Trapezoidal rule

$A = \frac{1}{2}(m+n)w$

(a) $\int_0^4 x^2 \, dx$

width, $w = \frac{b-a}{n}$

(i) $n=1$

width, $w = 4$

$A = \frac{1}{2}(4)(0+16) = 32$ units²

(ii) $n=2$

width, $w = 2$

$A = \frac{1}{2}(2)(0+16+2(0+4)) = 24$ units²

(b) define integral $\int_0^4 x^2 \, dx$

$\int_0^4 x^2 \, dx$

$= [x^3/3]_0^4$

$= 4^3/3 - 0^3/3$

$= 64/3$

Higher nth value, more similar to define integral

9) $\int_0^{\pi} x\cos(x) \, dx$; $n=4$

width $= \pi - 0/4 = \pi/4$

$\frac{1}{2}\left(-\pi - 0 + 2 \sum_{i=1}^{4-1} (x_i \cos(x_i))\right)$

$= \frac{1}{2}(-\pi + 2(1\cos(1) + 2\cos(2) + 3\cos(3)))$

$= \frac{1}{2}(-\pi + 2\cos(1) + 4\cos(2) + 6\cos(3))$

$= -\pi/2 + \cos(1) + 2\cos(2) + 3\cos(3)$

$= -4.832765184$

10) $\int_0^1 \frac{4}{1+x^2} \, dx$ with n = your chosen value

Problem set 10:

1) Functions of several variable

(a) $f(x,y) = x^2 + y^2 - 2$

(b) $f(x,y) = 2 - x^2 - y^2$

(c) $f(x,y) = (x+2)^2 + (y-1)^2$

(d) $f(x,y) = (x+2)^2/4 + (y-1)^2/9$