2 modulo 8 = 2 mod 8

1) modulo method :

① divide dividend by divisor

$\frac{2}{8} = 0.25$

② quotient multiply by divisor

$0 \times 8 = 0$

③ dividend subtract ans ②

$2 - 0 = 2$

2) modulus method

① Find HCM of divisor ≤ dividend

multiplies of 8 = 0, 8, 16, 24, 32

↳ ≤ 2 is 0

② dividend substract ① ans

$2 - 0 = 2$

* If $a \equiv b \pmod{n}$ and d divides a, b, n then :

$$\left[\frac{a}{d} \equiv \frac{b}{d} \left(\bmod \frac{n}{d}\right) \right]$$

$$\left[ax \equiv 1 \pmod{n} \right]$$

mathematical induction (lecture 9)

weak induction :

① Base step

$n^2 + 3n$ for all integers $n \geq 1$

when $n = 1$, $(1)^2 + 3(1) = 1 + 3$

$= 4$

4 is even so P(1) is TRUE

② Inductive step

n 1 2 3 4 ...

P(n) T T T T

} expect to be TRUE for all cases

Assuming that P(K) = TRUE , P(K+1) must be proven to be true too.

$P(k) \longrightarrow P(k+1)$

if $k = 0$, $P(1) \equiv T$ } base step

$P(1) \rightarrow P(2)$

$$\begin{aligned} P(1+1) = P(2) &= (k+1)^2 + 3(k+1) \\ &= (1+1)^2 + 3(1+1) \\ &= (2)^2 + 3(2) \\ &= 4 + 6 \\ &= 10 \end{aligned}$$

⟶ 10 is even so P(2) is TRUE

③ Conclusion

$P(1) \equiv$ TRUE , $P(2) \equiv$ TRUE so $P(1) \rightarrow P(2)$

And if $P(1) \rightarrow P(2)$ then $P(2) \rightarrow P(3)$ is possible so

$\therefore P(n) \equiv T$ for all integers $n \geq 1$

* Generally,

Base step ⇒ base on question

Inductive step ⇒ (k) & (k+1) cases

Conclusion ⇒ since <base step>, <IS>, <conclusion from question>

Strong induction:

generally: <Base step>

since $P(1) \wedge P(2) \equiv$ TRUE

<Inductive step>

$k=2$: $P(1) \wedge P(2) \rightarrow P(3)$

$k=3$: $P(1) \wedge P(2) \wedge P(3) \rightarrow P(4)$

$k=4$: $P(1) \wedge P(2) \wedge P(3) \wedge P(4) \rightarrow P(5)$

$\vdots$

$P(n) \equiv$ TRUE

$P(n)$: Every a_n is even for $n \geq 0$,
where $a_0 = 2$, $a_1 = 6$, $a_n = a_{n-1} + a_{n-2}$

① Base step

Given $a_0 = 2$ & $2 =$ even, then $P(0) \equiv$ TRUE

Given $a_1 = 6$ & $6 =$ even, then $P(1) \equiv$ TRUE

② Induction step

Assuming $P(0) \wedge P(1) \wedge P(2) \wedge \ldots \wedge P(k) \rightarrow P(k+1)$

Assuming $P(n)$ is TRUE for all $n \leq k$, $n \in \mathbb{Z}$, $k \in \mathbb{Z}$, $k > 1$

WHTP $P(k+1)$ is TRUE

is $a_k, a_{k-1}, a_{k-2}, a_{k-3} \cdots a_n$ all even

WHTP $a_{k+1} =$ even

$$
\begin{aligned}
\text{Now, } a_{k+1} &= a_k + a_{k-1} && \text{from recursive relationship} \\
&= 2m + 2N && P(k) \wedge P(k-1) \text{ from assumption} \\
&= 2[m+N] && m, N \in \mathbb{Z} \\
a_{k+1} &= 2W && W = m + N \in \mathbb{Z}
\end{aligned}
$$

$\Rightarrow P(k+1) \equiv T$

③ Conclusion

since $P(0) \wedge P(1) \equiv T$ <Base step>

$P(0) \wedge P(1) \rightarrow P(2)$ <Inductive step, $k=1$>

$P(0) \wedge P(1) \wedge P(2) \rightarrow P(3)$ <Inductive step, $k=2$>

$\vdots$

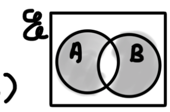
$P(n) \equiv T$ for all $n \geq 0$

sets (lecture 11)

universal set = local set that only includes elements under consideration
eg. $\mathbb{N} = \{0,1,2,3\}$

symbol	name	meaning /examples
$\{\}$ $\emptyset$	empty set	no elements in set
$A \cup B$	union	elements belong to set A OR set B
$A \cap B$	intersection	elements belong to both set A and set B
$\subseteq$	subset	$\{7,14\} \subseteq \{7,14,21,28\}$
$P(C)$	power set	$C = \{7,14,21\}$ $P(C) = \{\{\}, \{7\}, \{14\}, \{21\}, \{7,14\}, \{7,21\}, \{14,21\}, \{7,14,21\}\}$
$\in$	element of	$D = \{7,14,21,28\}$; $7 \in D$
$A \times B$	product	$\{3,5\} \times \{7,8\} = \{\{3,7\}, \{5,7\}, \{3,8\}, \{5,8\}\}$
(A,B)	ordered pair	collection of 2 elements eg. $(7,8)$
$\mathbb{Q}$	rational numbers set	$\mathbb{Q} = \{x \mid x = a/b, a, b \in \mathbb{Z}\}$
$\mathbb{Z}$	integer numbers set	$\mathbb{Z} = \{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$
$\mathbb{C}$	complex numbers set	$C = \{z \mid z = a+bi, -\infty < a < \infty, \ldots\}$
$\mathbb{R}$	real numbers set	$R = \{x \mid -\infty < x < \infty\}$

* $\mathbb{N}$ = natural numbers $\{0,1,2,3\ldots\}$



symmetric difference : $A \Delta B = (A-B) \cup (B-A) = (A \cup B) - (A \cap B)$

eg. $\{1,2,3\} \Delta \{3\}$
$= \{1,2,3\} - \{3\} \cup \{3\} - \{1,2,3\}$
$= \{1,2\} \cup \{1,2\}$
$= \{1,2\}$

set operations + logic operations

$x \in A \cup B$ if and only if $(x \in A) \vee (x \in B)$

$x \in A \cap B$ if and only if $(x \in A) \wedge (x \in B)$

$x \in A - B$ if and only if $(x \in A) \wedge (x \notin B)$

$x \in A \triangle B$ if and only if $(x \in A) \underline{\vee} (x \in B)$

exclusive OR 

* $\wedge \Rightarrow \mathbb{B} \times \mathbb{B} \to \mathbb{B}$
  $\vee \Rightarrow \mathbb{B} \times \mathbb{B} \to \mathbb{B}$

using logic to solve $\overline{A \cup B} = \bar{A} \cap \bar{B}$

$$
\begin{aligned}
x \in \overline{A \cup B} &= \neg(x \in A \cup B) \\
&= \neg((x \in A) \vee (x \in B)) \\
&= \neg(x \in A) \wedge \neg(x \in B) \\
&= (x \in \bar{A}) \wedge (x \in \bar{B}) \\
&= x \in \bar{A} \cap \bar{B}
\end{aligned}
$$

A	B	C
0	0	0
0	1	1
1	0	1
1	1	0

Ordered pairs (a,b) * order is significant

$\{a,b\} = \{b,a\}$ BUT $(a,b) \neq (b,a)$

proving a function is one-to-one (lecture 13)

Q: $\mathbb{N} \to \mathbb{Z}$ defined by $g(x) = (x+6)^2 + 1$ is one-to-one

assuming $g(x_1) = g(x_2)$ for some $x_1, x_2 \in \mathbb{N}$

then...

$$
\begin{aligned}
(x_1+6)^2 + 1 &= (x_2+6)^2 + 1 \\
(x_1+6)^2 &= (x_2+6)^2 \\
x_1 + 6 &= x_2 + 6 \\
x_1 &= x_2
\end{aligned}
$$

HENCE, g is indeed one-to-one

sequence (lecture 14)

* GCD → domain : $\mathbb{Z} \times \mathbb{Z} - \{0,0\}$
  → codomain : $\mathbb{N}$

infinite number sequence → $f: \mathbb{N} \to \mathbb{R}$ defined by $f(n) = 2^{-n}$

→ input : natural num; output : real numbers

Q: How many functions are there with domain $\{1,2,3,4\}$ and codomain $\{-1,0,1\}$

$\text{codomain}^{\text{domain}}$ & codomain = 3 elements & domain = 4 elements

HENCE, $3^4 = 81$ ways #

Composite functions (lecture 15)

* notation : $f(x)$ = cube (successor (square (x)))

↳ cube ∘ successor ∘ square

** Basically... $f(x) = g(h(x))$ same as $f: g \circ h$

↳ f is the composite of g & h

↳ composite function don't always exists

Inverse functions ⇝ $f: A \to A$ and $g: A \to A$

* $i_A(x) = x$ as... $f \circ g = g \circ f = i_A$

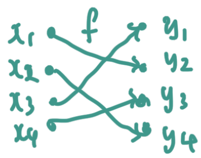
** inverse only exist for functions if functions is one-to-one and onto

in other words... all right dot must connect left dot like:

if $f: A \to B$

$f^{-1}: B \to A$

onto → image (f) = B



lecture 16 & 17 on another file

lecture 19

* ordered selections without repetition : $\dfrac{n!}{(n-r)!}$ from a set of r elements $(0 \leq r \leq n)$

eg. n = number of people

r = gold, silver, bronze medal

* unordered selections without repetition : $\dfrac{n!}{r!(n-r)!}$ ⇒ written as $\binom{n}{r}$

from a set of r elements $(0 \leq r \leq n)$

* combination of r elements from a set S

= subset of S with r elements

eg. pick a team of 4 from a group of 9

$$\binom{9}{4} = \frac{9!}{4!(9-4)!} = \frac{9 \times 8 \times 7 \times 6 \cdots}{4 \times 3 \times 2 \times 1 \; \cancel{(5)!}}$$

$$= \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1}$$

$$= \frac{9 \times \cancel{8} \times 7 \times 6}{\cancel{8} \times 3}$$

$$= \frac{9 \times 7 \times 6}{3}$$

$$= 378/3 = 126 \#$$

* ordered selections with repetition : n^r

* unordered selections with repetition : $\dfrac{(n+r-1)!}{r!(n-1)!}$

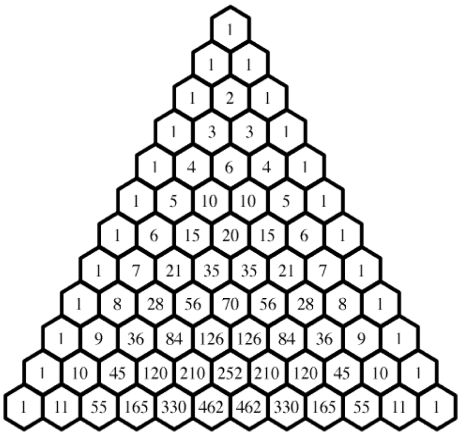
** MULTISET : A SET THAT ALLOWS ELEMENTS TO APPEAR MORE THAN ONCE

** PIGEONHOLE PRINCIPLE : if n items are placed in m containers, then at least one container has at least $\lceil \frac{n}{m} \rceil$ items.
eg. 21 tasks divided to 4 processors, the busiest processor have 6 tasks to complete instead of 5

Lecture 20

Pascal triangle : binomial coefficients

$$
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\
\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6} \\
\vdots \qquad \vdots \qquad \vdots
\end{array}
$$



* patterns : $\binom{n}{r} = \binom{n}{n-r}$ for all $0 \leq r \leq n$

** HENCE, $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$ for all $1 \leq r \leq n$

$\binom{n-1}{r}$ combination of r elements of X that does not contain x

$\binom{n-1}{r-1}$ combination of r elements of X that do contain x

* choose zero things = $\binom{0}{0} = 1$ (top part of Pascal's Triangle)

$(a+b)^0$	1	2^0	1	$(a-b)^0$
$(a+b)^1$	$1+1$	2^1	$1-1$	$(a-b)^1$
$(a+b)^2$	$1+2+1$	2^2	$1-2+1$	$(a-b)^2$
$(a+b)^3$	$1+3+3+1$	2^3	$1-3+3-1$	$(a-b)^3$
$(a+b)^4$	$1+4+6+4+1$	2^4	$1-4+6-4+1$	$(a-b)^4$
$(a+b)^5$	$1+5+10+10+5+1$	2^5	$1-5+10-10+5-1$	$(a-b)^5$
$(a+b)^6$	$1+6+15+20+15+6+1$	2^6	$1-6+15-20+15-6+1$	$(a-b)^6$
$(a+b)^7$	$1+7+21+35+35+21+7+1$	2^7	$1-7+21-35+35-21+7-1$	$(a-b)^7$

BINOMIAL THEORUM + PASCAL'S TRIANGLE
COEFFICIENTS

$P(X=r) \Rightarrow {}^nC_r (p^r)(q^{n-r})$
$\Rightarrow \binom{n}{r}(p^r)(q^{n-r})$

Inclusion & Exclusion (?)
↳ $|A \cup B| = |A| + |B| - |A \cap B|$
↳ if $|A \cup B \cup C|$ then $= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Lecture 21

- $Pr(A)$ = sum of the probability of outcomes in A

eg. $Pr(A) = \dfrac{\text{number of outcomes in A}}{\text{total number of outcomes}}$

- $A \cup B \longrightarrow$ A OR B
- $A \cap B \longrightarrow$ A AND B
- $\bar{A} \longrightarrow$ NOT A : (sample set − A)

* $Pr(A \cap B)$ where A and B are 2 events $\leq \min(Pr(A), Pr(B))$

** $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$

eg. if $A = \{1,2\}$ & $B = \{2,3,4\}$ in dice rolling scenario:

$Pr(A) = 2/6$, $Pr(B) = 3/6$ (6 cus dice has 6 sides)

$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$ * $A \cap B = \{2\}$

$= 2/6 + 3/6 - 1/6$ HENCE, $Pr(A \cap B) = 1/6$

$= 4/6$

$= 2/3$ #

* mutually exclusive : $Pr(A \cup B) = Pr(A) + Pr(B)$

↳ can't occur together , Hence, $Pr(A \cap B) = 0$

* independant events (don't affect each other)

↳ 2 events are independant if $Pr(A \cap B) = Pr(A) * Pr(B)$

Lecture 22

↳ conditional probability

- $Pr(A \cap B) = Pr(A|B) * P(B)$ } implies : $Pr(A|B) = Pr(A \cap B) / Pr(B)$

↳ ** $P(A) + P(A') = 1$ **

↳ $P(A|B)$ & $P(B|A)$ table :

		P(A|B)	P(B|A)
normal event	:	$P(A \cap B) / P(B)$	$P(B \cap A) / P(A)$
mutually exclusive	:	0	0
independant event	:	$P(A)$	$P(B)$

↳ ** Bayes' Theorum (for event A & B) :

$$Pr(A|B) = \frac{Pr(B|A) * Pr(A)}{\underbrace{Pr(B|A) * Pr(A) + Pr(B|\bar{A}) * Pr(\bar{A})}_{\text{basically... } Pr(B)}}$$

↳ calculate conditional probability of event A given event B when $Pr(A)$, $Pr(B|A)$, $Pr(B|\bar{A})$ is provided

Lecture 23

↳ 2 random variable (X,Y) are independant if
$$Pr(X=x \wedge Y=y) = Pr(X=x) * Pr(Y=y)$$

↳ eg.

x	-1	0	1
$Pr(X=x)$	1/6	1/3	1/2

then the distributions of $X+1, 2X$ and X^2 are

y	0	1	2
$\Pr(X+1=y)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

y	−2	0	2
$\Pr(2X=y)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

y	0	1
$\Pr(X^2=y)$	$\frac{1}{3}$	$\frac{2}{3}$

eg. Let X and Y be independent random variables with distributions

x	0	2
$\Pr(X=x)$	$\frac{1}{4}$	$\frac{3}{4}$

y	0	1	2	3
$\Pr(Y=y)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

What is $\Pr(X+Y=3)$?

A. 1/4
B. 1/16
C. 3/16
D. 3/4

3 → $0+3 = (1/4)(1/4) = 1/16$
3 → $2+1 = (3/4)(1/4) = 3/16$

$1/16 + 3/16 = 4/16 = 1/4$ #

Lecture 24

* expected value → $E(X) = \sum x_i p_i$
* varience → $Var(X) = \sum [p_i(x_i - E(x))^2]$

↳ $E[X+Y] = E[X] + E[Y]$

distribution table :

x	x_1	x_2	x_3
$P(X=x)$	p_1	p_2	p_3

hence, $E(x) = (x_1p_1) + (x_2p_2) + (x_3p_3)$

$Var(x) = p_1(x_1 - E(x))^2 + p_2(x_2 - E(x))^2 + p_3(x_3 - E(x))^3$

Lecture 25

• uniform distribution → $E(X) = (a+b)/2$
→ $Var(X) = [(a+b-1)^2 - 1]/12$
↳ but can simplify to $(a+b)^2/12$ imo
→ $Pr(X=k) = 1/(b-a+1)$

↓ all outcomes are equally likely
eg. fair dice

• bernuolli distribution → $E(X) = p$
→ $Var(X) = p(1-p)$
→ $Pr(X=k) = \begin{cases} p & \text{for } k=1 \\ 1-p & \text{for } k=0 \end{cases}$

↓ either success or failure, no in-between

• geometric distribution → $E(X) = (1-p)/p$
→ $Var(X) = (1-p)/p^2$
→ $Pr(X=k) = p(1-p)^k$

↓ k follows before a success

- binomial distribution → $E(X) = np$
  → $var(X) = np(1-p)$
  → $Pr(X=k) = ({}^{n}C_{k})(p^{k})(1-p)^{n-k}$

  exactly k successes

- poisan distribution → $E(X) = \lambda$
  → $var(X) = \lambda$
  → $Pr(X=k) = (\lambda^{k} e^{-\lambda})/k!$

  know average of λ events occur per time period, count probability that k events occurred in a time period

- Σ = sigma (sum) → $\sum_{k=1}^{n} k = 1+2+\dots+n$
- π = pi (multiplication) → $\prod_{k=1}^{n} k = 1\times 2\times\dots\times n$

- arithmetic progression → $a_n = a_1+(n-1)d$, $S_n = n/2(a_1+a_n)$
  geometric progression → $a_n = a_1(r^{n-1})$, $S_n = (a_1-a_1(r^n))/(1-r)$

Lecture 26

Example. $F(0)+F(1)+\cdots+F(n) = F(n+2)-1$.

Proof *Base step.* $F(0) = 0 = F(2)-1$, because $F(2)=1$.

Induction step. We have to show

$$F(0)+F(1)+\cdots+F(k) = F(k+2)-1$$
$$\Rightarrow F(0)+F(1)+\cdots+F(k+1) = F(k+3)-1.$$

Well,

$$F(0)+F(1)+\cdots+F(k) = F(k+2)-1$$
$$\Rightarrow F(0)+F(1)+\cdots+F(k+1) = F(k+2)+F(k+1)-1,$$

by adding $F(k+1)$ to both sides

$$\Rightarrow F(0)+F(1)+\cdots+F(k+1) = F(k+3)-1$$

since $F(k+2)+F(k+1) = F(k+3)$ by the Fibonacci recurrence relation

This completes the induction.

Definition The Fibonacci sequence $F(0), F(1), F(2), \dots$ is defined by $F(0)=0$, $F(1)=1$ and $F(i) = F(i-1)+F(i-2)$ for all integers $i \geq 2$.

Example Prove that $F(0)+F(1)+\cdots+F(n) = F(n+2)-1$ for all integers $n \geq 0$.

Solution Let $P(n)$ be the statement "$F(0)+F(1)+\cdots+F(n) = F(n+2)-1$".

Base step. $F(0)=0$ and $F(2)-1 = 1-1 = 0$, so $P(0)$ is true.

Induction step. Suppose that $P(k)$ is true for some integer $k \geq 0$. This means that $F(0)+F(1)+\cdots+F(k) = F(k+2)-1$.

We want to prove that $P(k+1)$ is true. We want to show that $F(0)+F(1)+\cdots+F(k+1) = F(k+3)-1$.

$$\begin{aligned} F(0)+F(1)+\cdots F(k+1) &= (F(0)+F(1)+\cdots+F(k))+F(k+1) \\ &= F(k+2)-1+F(k+1) && \text{(by } P(k)) \\ &= F(k+3)-1 && \text{(by definition of } F(k+3)) \end{aligned}$$

So $P(k+1)$ is true.

This proves that $P(n)$ is true for each integer $n \geq 0$.

Lecture 27

$1+2+3+\cdots+n$ is written $\sum_{k=1}^{n} k$,

$1+a+a^2+\cdots+a^n$ is written $\sum_{k=0}^{n} a^k$.

Σ is capital sigma, standing for "sum."

$1\times 2\times 3\times\cdots\times n$ is written $\prod_{k=1}^{n} k$.

Π is capital pi, standing for "product."

products : $n! = \prod_{i=1}^{n} i$

addition : $1+a+a^2+\dots+a^n = \sum_{i=0}^{n} a^i$

first sum : $\sum_{i=1}^{n} i^2$

more products : $\prod_{i=7}^{9} (x-i) = (x-7)(x-8)(x-9)$

$\prod_{i=2}^{5} \binom{6}{i} = \binom{6}{2}\binom{6}{3}\binom{6}{4}\binom{6}{5}$

Lecture 28

- **ARITHMETIC SEQUENCE :** $t_n = a + (n-1)d$
  - ↳ initial value : $t_1 = a$
  - ↳ recurrance relation : $t_{k+1} = t_k + d$
- **GEOMETRIC SEQUENCE :** $t_n = ar^{n-1}$
  - ↳ initial value : $t_1 = a$
  - ↳ recurrance relation : $t_{k+1} = rt_k$
- **Example** Let $t_0, t_1, t_2, \ldots$ be the sequence defined by $t_0 = 1,\ t_1 = 2$ and $t_{k+1} = 2t_k - t_{k-1}$.

Solution Let $P(n)$ be the statement "$t_n = n + 1$".

Base steps.
$t_0 = 1$ and $0 + 1 = 1$, so $P(0)$ is true.
$t_1 = 2$ and $1 + 1 = 2$, so $P(1)$ is true.

Induction step. Suppose that $P(0), \ldots, P(k)$ are true for some integer $k \geq 1$. We want to prove that $P(k+1)$ is true. We want to show that $t_{k+1} = k + 2$.

$$\begin{aligned} t_{k+1} &= 2t_k - t_{k-1} \\ &= 2(k+1) - k && \text{(by } P(k) \text{ and } P(k-1)\text{)} \\ &= k + 1 \end{aligned}$$

So $P(k+1)$ is true.

This proves that $P(n)$ is true for each integer $n \geq 0$.

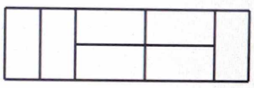
- **HOMOGENEOUS**

eg. $t_n = cf(n)$ OR $t_n = 2^n$ } have constant c, eg. for cf(n) is c

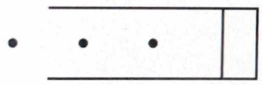
- **IN-HOMOGENEOUS**

eg. $t_{k+1} = t_k + 3$ OR $t_k = t_{k+1} + 7$ } have a term other than t_k terms

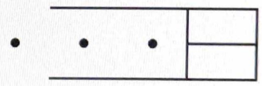
28.2 Let T_n be the number of ways of tiling a $2 \times n$ strip with 2×1 tiles (which may be rotated so they are 1×2). Find T_n for $n = 1, 2, 3, 4$. Find a recurrence relation for T_n.

E.g.  is one option counted by T_7.

$T_1 = 1,\ T_2 = 2,\ T_3 = 3,\ T_4 = 5.$
How do we find a recurrence relation for T_n?
Consider what happens at the right hand end:

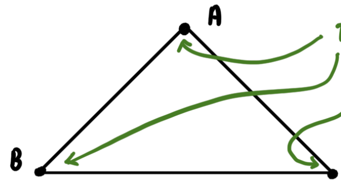


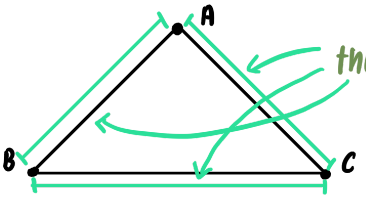
or



In the first case removing the right most tile leaves us with a tiling of a $2 \times (n-1)$ strip.
In the second case removing the two right most tiles leaves us with a tiling of a $2 \times (n-2)$ strip.
Since these possibilities do not overlap, and between them include every option, we see that $T_n = T_{n-1} + T_{n-2}$.

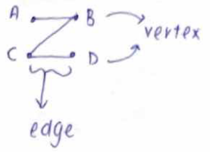
Lecture 29





HENCE, vertex set : {A, B, C}
edges set : {{A, B}, {B, C}, {C, A}} → can be written as {AB, BC, CA}

- SIMPLE GRAPH = X multiple edges between same vertice / edges
- MULTIGRAPH = √ multiple edges between same vertice / edges

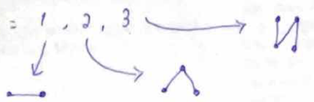


vertex = {A, B, C, D}
edge = {AB, BC, CD}

complete graphs => every pair of vertices join by an edge

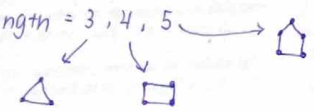
length of path = number of edge ≠ number of vertices

path of length = 1, 2, 3



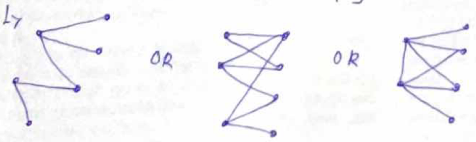
vertex = $\{v_1, \dots, v_{e+1}\}$
edge = $\{v_1 v_2, \dots v_e v_{e+1}\}$

cycle of length = 3, 4, 5



vertex = $\{v_1 \dots v_{e+1}\}$
edge = $\{v_1 v_2, v_2 v_3, \dots v_{e-1} v_e, v_e v_1\}$

bipartite
↳ √ renamed vertices
↳ edge join vertex $\{u_1, u_2, \dots u_i\}$
↳ vertex in $\{v_1, v_2, \dots v_j\}$
↳

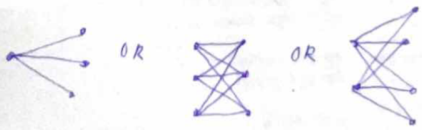


- all paths are bipartite

$\boxed{\text{BUT}}$

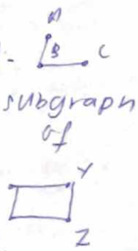
- cycles of odd lengths are not bipartite
- cycles of even lengths are bipartites

complete bipartite
↳ if every vertex is join by an edge to another vertex
↳



subgraphs
↳ subgraphs of graph G whose vertex set is a subset of vertex set of G and edge set is a ~~edge~~ subset of edge set of G

eg.

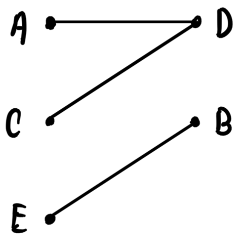


* graph = bipartite

if and only if
- X subgraph
- odd-length cycle

* connected graph with no subgraph = odd-length cycle = bipartite

* disconnected graph:

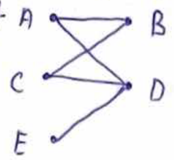


cus no path from D to E

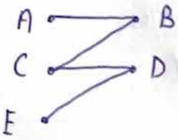
Lecture 30 — 36

path = no repeated vertices = don't pass through same nodes / dots
trail = no repeated edges = don't walk / use same line

bipartite graph
↳ we say it's bipartite if vertices can split to 2 sides and edges (lines) can only go opposite sides and not between vertices of same sides
↳ eg.



* complete bipartite = only 1 edge (line) connect the vertice from left to right
↳ eg.



↳ max num of edge of a bipartite graph with n vertices
- n = odd ⟶ $(n^2-1)/4$
- n = even ⟶ $n^2/4$

adjacency matrix
↳ 1 = V_i is adjacent to V_j
↳ (i,j) entry in kth power of adjacency matrix = num of walks of length k between V_i and V_j
↳ length of walks:
- odd ⟶ eg. 1 : $V_i \rightarrow V_j$
  3 : $V_i \rightleftarrows V_j$
- even ⟶ eg. 2 : $V_i \rightleftarrows V_j$
  4 : $V_i \rightleftarrows V_j$

$$\begin{array}{c|c} & V_1 \quad V_2 \quad V_3 \quad V_4 \ \} \text{end} \\ \begin{array}{c} V_1 \\ V_2 \\ V_3 \end{array} & \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \\ \text{start} & \end{array}$$

* acyclic = no cycle

conditions to check whether graphs exist:
- sum of degrees is even = 2 × (num of edge)
- properties: (simple graph)
  - ✗ loops, multiple edges
  - ✓ connected / disconnected
  - simple graph of n vertices, each vertex can only have n-1 degrees

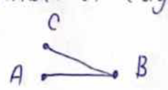
- tree properties:
  - tree with n vertices = (n-1) degrees
  - connected simple graph, acyclic
- spanning tree properties:
  - subgraph of tree graph, use every vertice
  - ✗ cycles, loops
  - remove 1 edge = disconnected
  - + 1 edge = cycle
  - n-1 degrees
  - connected all vertices without cycles, min edge

- spanning tree (minimum)
  1. start lowest weight edge
     ↳ connect 2 vertices with said edge
  2. Find 2nd lowest weight edge and find alternative to link both edges
     ↳ found? ignore the edge. Not found? draw edge between said vertice
  3. continue from low to high weight edge

Euler trail → no repeating edge, use all edge
→ trail + start * end different point

Closed Euler trail → no repeating edge, use all edge
→ trail + start * end same point
→ even degrees for of vertex → NOT ODD DEGREE VERTICES

* degree → number of edge of a vertex
→ eg.



2 degrees : B
1 degree : C, A

Algorithm to find graph's spanning tree: ① Breath-first algorithm (BFS)
② Depth-first algorithm (DFS)

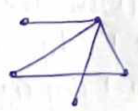
① BFS
- vertex-based technique (shortest path in graph)
- start : head vertex "parent"
  Then : add an edge to "child"
  ↳ all child present then "parent" dies
  Hence : "child" connect to next "child"
  End : last "child" connected

② DFS
- edge-based technique
- "got line, walk. no line, go backwards, continue"

EXAMPLES NEXT PAGE

Examples of "trees":

✓ tree



✗ tree, cus have cycle

✗ tree, cus disconnected

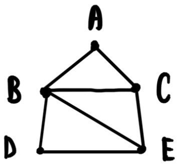
**MATRIX MULTIPLICATION:
(Just for reference)

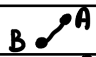
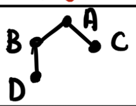
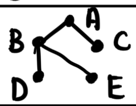
$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 3 & 2 \end{pmatrix} \times \begin{pmatrix} 0 & 3 & 1 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 7 & 9 & 3 \\ 3 & 5 & 2 \\ 8 & 4 & 3 \end{pmatrix}$$

method : $(1\times3)+(2\times0)+(3\times2) = 9$

* EXAMPLE OF BFS:

G:

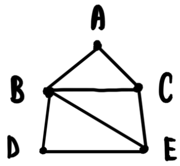


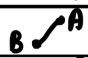
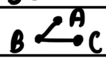
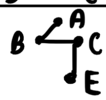
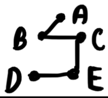
STEP	Q	T
1	A	• A
2	AB	
3	ABC	
4	BC	A OUT
5	BCD	
6	BCDE	
7	CDE	B OUT
8	DE	C OUT
9	E	

everything is connected to C is already present, which is A, B, E so C can come out

* EXAMPLE OF DFS:

G:



STEP	T
A	• A
AB	
ABC	
ABCE	
ABCED	
ABCE	D OUT
ABC	E OUT
AB	C OUT
A	B OUT