MAT1830 - Discrete Mathematics for Computer Science
Tutorial Sheet #4 and Additional Practice Questions

Tutorial Questions

1. (a) Let $a_1, a_2, a_3, \ldots$ be a sequence of integers defined by $a_1 = 1$, $a_2 = 3$, and $a_t = a_{t-1} + a_{t-2}$ for each integer $t \geq 3$. (So the sequence goes $1, 3, 4, 7, 11, 18, \ldots$) How would you (informally) convince your friend Dwayne that every term in this sequence will be positive? How would you phrase this argument more formally?

   * set × set = set of ordered pair *

   (b) Prove by induction that 3 divides $n^3 - 7n + 6$ for all integers $n \geq 0$.

$\mathbb{N}$: natural numbers ; $\mathbb{Q}$: rational numbers $\mathbb{Z}$: integers (can be +ve/-ve)

2. (a) Are the following true or false?

   $\subseteq$ = subset

   i. $\mathbb{N} \subseteq \mathbb{Q}$ TRUE

   * An empty set is a subset of any set

   ii. $2 \subseteq \mathbb{N}$ FALSE

   * set {} × set {} = set of ordered set → eg. $\{a,b,c\} \times \{d,e\} = \{(a,d),(a,e),(b,d),(b,e),(c,d),(c,e)\}$

   iii. $\{6, 7\} \subseteq \mathbb{N}$ TRUE

   2 itself is just a number, NOT A subset of $\mathbb{N}$ BUT if its $\{2\} \subseteq \mathbb{N}$

   iv. $\{3\} \in \mathbb{Q}$ FALSE, number 3 is a member but {3} is not a member of $\mathbb{Q}$

   v. $\mathbb{N} \subseteq \{x : x \in \mathbb{R}, x \geq 0\}$ TRUE

   $S = \{-1, 0, 1\}$

   vi. $\{\} \subseteq \mathbb{N}$ TRUE

   $P(S) = \{\{\}, \{-1\}, \{0\}, \{1\}, \{-1,0\}, \{-1,1\}, \{0,1\}, \{-1,0,1\}\}$

   $\in$ = member

   vii. $(a, d) \in \{a, b, c\} \times \{d, e\}$ TRUE

   (members of power set is member of original set)

   viii. $\mathbb{N} \times \mathbb{N} \subseteq \mathbb{Z} \times \mathbb{Z}$ TRUE

   $\mathbb{N} \times \mathbb{N} = \{(n,n),(n,n)\ldots,(n,n)\}$

   $\mathbb{Z} \times \mathbb{Z} = \{(2,9), (-2,-7), (1,3) \ldots (-2,7)\}$

   (b) Let $S = \{-1, 0, 1\}$. What is $\mathcal{P}(S)$?

   (c) Let $T = \{1, 2, \ldots, 10\}$. How many elements would $\mathcal{P}(T)$ have? Which of the following would be elements of $\mathcal{P}(T)$?

   $T = \{1,2,3,4,5,6,7,8,9,10\}$

   $P(T) = 2^{10}$ subsets

   * Every member of T is the subset of P(T) *

   empty set {

   i. $\emptyset$ YES

   ii. $\{1, 4, 6\}$ YES

   iii. 3 NO, 3 is a member of T

   iv. $\{2, 4, 12\}$ NO, 12 is not in mother set

   v. $\{1, 2, \ldots, 10\}$ YES

   * HOW MANY members in A×B

   A : n1 members B = n2 members A×B total members = n1 × n2

3. Let $A = \{1, 2\}$ and $B = \{-1, 0, 1\}$.

   * A − B = {2}
   * B − A = {-1, 0}
   * A Δ B = (A−B) ∪ (B−A) = {-1, 0, 2}

   (d) &

   = ANS

   (a) What is $A \cup B$? $\{1, 2, -1, 0\}$

   (b) What is $A \cap B$? $\{1\}$

   (c) What is $A \times B$?

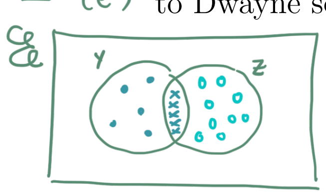
   (d) Is it true that, for all sets X, Y and Z, $(X \cup Y) \cap Z = X \cup (Y \cap Z)$? Why or why not?

   (e) Is it true that, for all sets Y and Z, $\mathcal{P}(Y) \cap \mathcal{P}(Z) = \mathcal{P}(Y \cap Z)$? Why or why not?

4. (a) Scrooge McDuck has created his own currency, duckbucks, which has only \$4 and \$7 notes. Using induction prove that, for any $n \geq 18$, n duckbucks can be made from \$4 and \$7 notes.

   (b) You're leaving Dwayne in charge of a store with a stack of \$4 and \$7 duckbuck notes. Assume all purchases will result in \$18 or more change. What (informal) instructions would you give to Dwayne so he can always make change for the customers?

(e)



P(Y) = { subset with x, subset with x, •, subset with • }

P(Z) = { subset with x, subset with x, o, subset with o }

P(Y) ∩ P(Z) = { subset with x }

(See over for practice questions.)

Practice Questions

1. Let U be some universal set and let P, Q and R be subsets of U. Let

$$S_1 = P \cup (\overline{Q \cap R})$$
$$S_2 = (P \cup (P \cap Q)) \cap \overline{R}.$$

   (a) Draw a Venn diagram to determine whether it is always the case that $S_1 = S_2$.

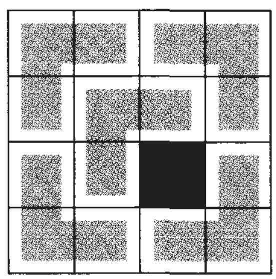
   (b) If this is not always the case, then does it ever happen? If so, when?

   (c) Let p, q and r be the propositions "$x \in P$", "$x \in Q$" and "$x \in R$". Find formulas in logic which mean "$x \in S_1$" and "$x \in S_2$". Find truth tables for these.

   (d) Do the truth tables contain the same information as you Venn diagrams? More? Less?

2. An L-tromino is like a domino, but made of three squares in the shape of an "L". Prove by induction that, for any n, if any one square is removed from a $2^n \times 2^n$ chess board, then the remaining squares can be completely covered by L-trominos.

   (The following picture shows an example on a 4×4 board.)



   Hint: A $2^{k+1} \times 2^{k+1}$ board can be split into four $2^k \times 2^k$ boards.

3. (a) How would you formally define intersections and unions of infinitely many sets?

   (b) Can you find a collection of infinitely many sets of integers $A_1, A_2, A_3, \ldots$ such that each set contains infinitely many integers, $A_{i+1} \subseteq A_i$ for $i = 1, 2, 3, \ldots$, and $A_1 \cap A_2 \cap A_3 \cap \cdots = \emptyset$?

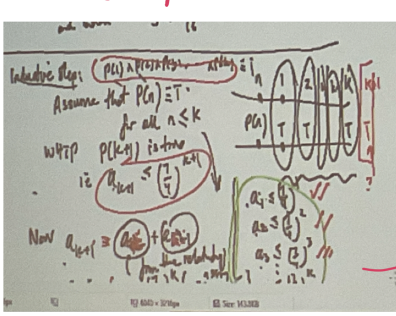
4. As in 1(a) on the previous page, define a sequence of integers $a_1, a_2, a_3, \ldots$ by $a_1 = 1$, $a_2 = 3$, and $a_{t+1} = a_t + a_{t-1}$ for each integer $t \geq 3$. Prove by strong induction that $a_n \leq (\frac{7}{4})^n$ for each integer $n \geq 1$.

$a_n \leq \left(\frac{7}{4}\right)^n$ for each integer $n \geq 1$

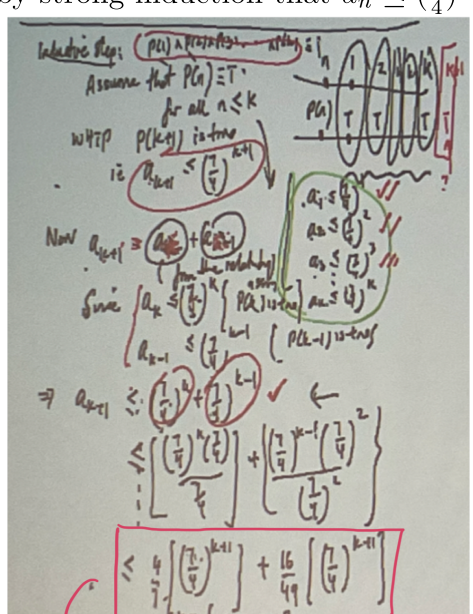
$n=1$
• $a_1 = 1$; $\left(\frac{7}{4}\right)^1 = \frac{7}{4}$ $P(1) \equiv T$

$n=2$
• $a_2 = 3$; $\left(\frac{7}{4}\right)^2 = \frac{49}{16}$ $P(2) \equiv T$

Inductive step :



continue...



(b) $\frac{n^3-7n+6}{3}$ for all integers $n \geq 0$.

Let $P(n)$ be $\frac{n^3-7n+6}{3}$ for all integers $n \geq 0$.

Base step $\Rightarrow$ $P(0) = \frac{(0)^3-7(0)+6}{3}$

$= 6/3$

$= 2$

$S = \{-1, 0, 1\}$

$P(S) = \{\{\}, \{-1\}, \{0\}, \{1\}, \{-1,0\}, \{-1,1\}, \{0,1\}, \{-1,0,1\}\}$

$P(P(S)) = P[P(S)]$

is... $\geq \leq \frac{4}{7}\left[\left(\frac{7}{4}\right)^{k+1}\right] + \frac{16}{49}\left[\left(\frac{7}{4}\right)^{k+1}\right]$

$\leq \left(\frac{7}{4}\right)^{k+1}\left[\frac{4}{7} + \frac{16}{49}\right]$

$\leq \left(\frac{7}{4}\right)^{k+1}\left[\frac{44}{49}\right] \rightarrow < 1$

A $\frac{44}{49}$

$A_{k+1} \leq \left(\frac{7}{4}\right)^{k+1}$

$P(k+1) \equiv T$

conclusion:

since $P(1) \wedge P(2)$

$P(1) \wedge P(2) \rightarrow P(3)$

$P(1) \wedge P(2) \wedge P(3) \rightarrow P(4)$

therefore...

$P(n)$ is true for $n \geq 1$