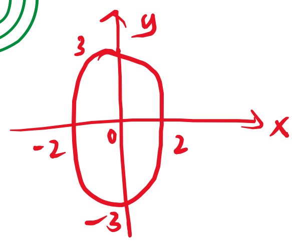
Problem Set Seven: Parametric Curves and Parametric Differentiation

Parametric Curves

Cartesian
elimate "t"

$\boxed{x^2+y^2=r^2}$

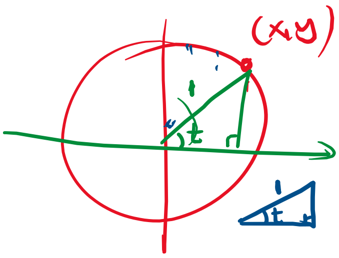
$\left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2=1$



1. Find a parametric form of the ellipse

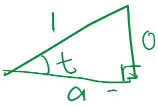
$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

That is, find functions $x(t)$, $y(t)$ for $0 \le t < c$ such that every point on the ellipse is described by the given functions for at least one value of t (and ideally, to each point there is exactly one value of t).



$\left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2=1$ ✓

$\cos^2(t)+\sin^2(t)=1$



$\Rightarrow$ $\frac{x}{2}=\cos(t)$

$x(t)=2\cos(t)$

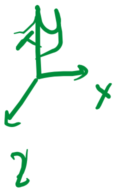
$\frac{y}{3}=\sin(t)$

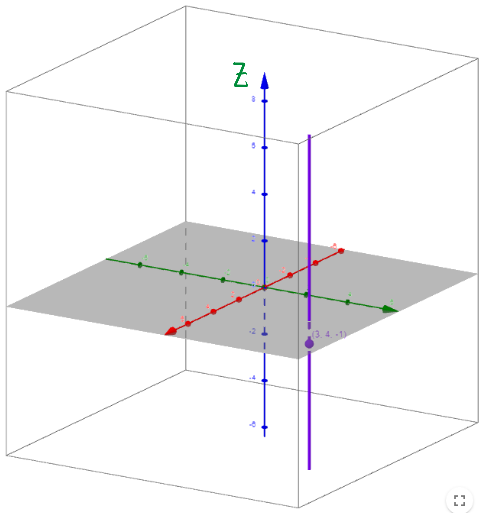
$y(t)=3\sin(t)$

$0 \le t < 2\pi$

3D Parametrization – GeoGebra

2. What do the following parametric curves represent? Sketch each one separately.





(a) $x(v)=3,\ y(v)=4,\ z(v)=v$ for $-\infty < v < \infty$

(b) $x(u)=5,\ y(u)=3u-2,\ z(u)=5u+1$ for $-1 < u < 1$

(c) $x(t)=3t^2,\ y(t)=4t,\ z(t)=5$ for $1 \le t \le 7$

(d) $x(t)=3\cos(t),\ y(t)=2\sin(t),\ z(t)=3t-1$ for $0 \le t < 2\pi$

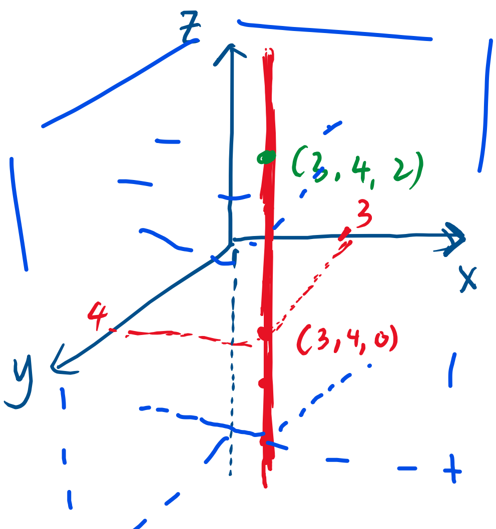
(e) $x(w)=w\sin(w),\ y(w)=w\cos(w),\ z(w)=w$ for $0 < w < \infty$

(f) $x(t)=\sin(t),\ y(t)=0,\ z(t)=t$ for $0 < t < 2\pi$

(a)

v	-2	-1	0	1	2
$x(v)$	3	3	3	3	3
$y(v)$	4	4	4	4	4
$z(v)$	-2	-1	0	1	2
Point	$(3,4,-2)$	$(3,4,-1)$	$(3,4,0)$	$(3,4,1)$	$(3,4,2)$

Line parallel to z-axis and passes through the point (3,4,0).

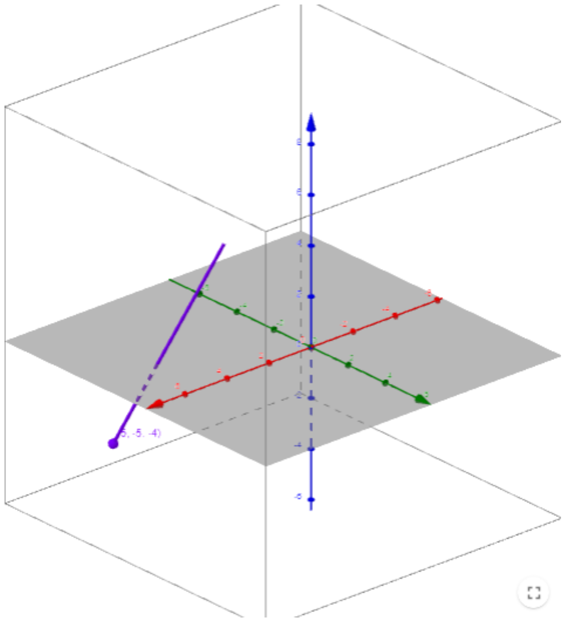


(b) straight line drawn from (5,-5,-4) to (5,1,6) exclusive.

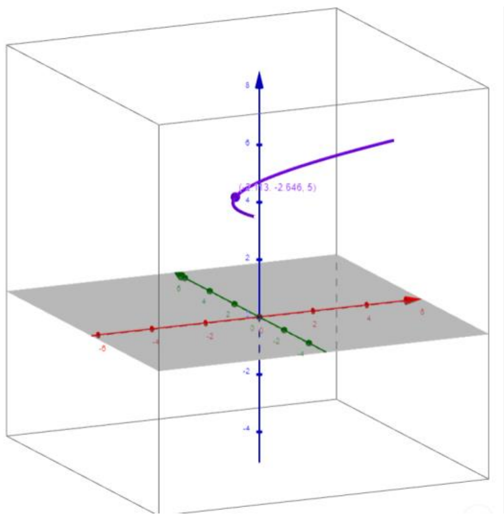
u	-1	1
x	5	5
y	1	-5
z	6	-4

end points

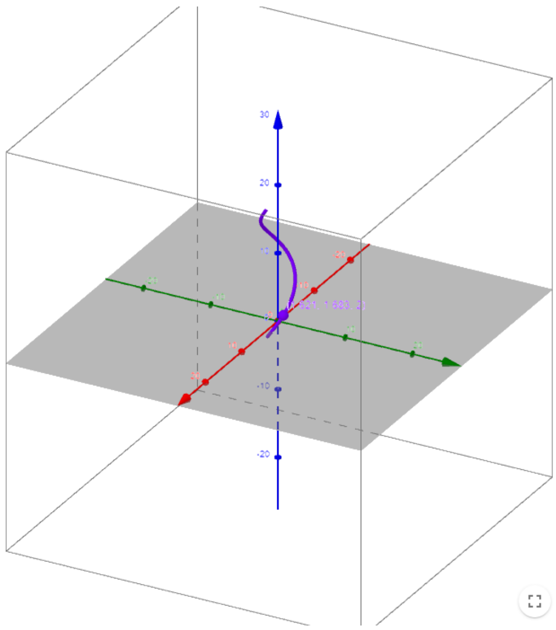
$-1 < t < 1$



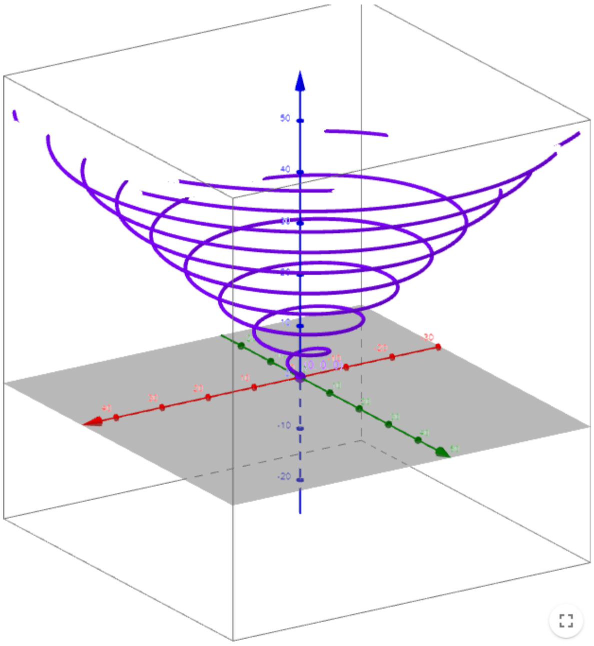
(c) Parabolic curve drawn from (3,4,5) to (147,28,5) inclusive.



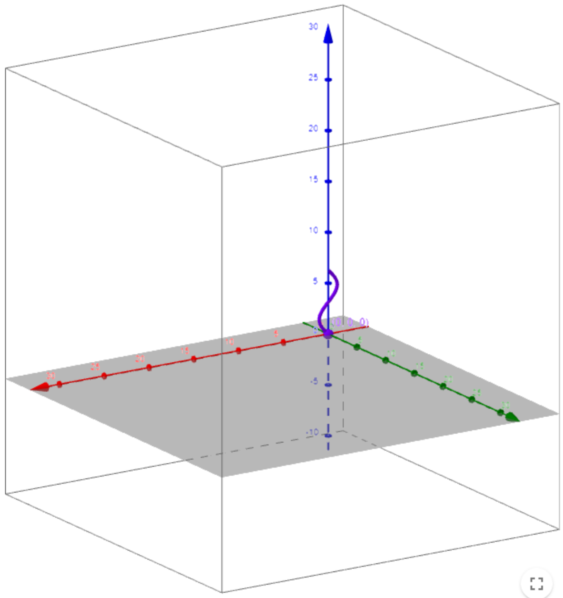
(d) This is an elliptical helix curve along the z-axis, drawn from (3,0,-1) to (3,0,$6\pi - 1$), where the ellipse is defined as $\frac{x^2}{9} + \frac{y^2}{4} = 1$.



(e) This is a conical helix (also known as circular spiral curve) begin at (0, 0, 0) and increasing out along the z-axis.



(f) Due to y is "zero", the actual axes involved indeed only x- and z-axes. It is just a simple 2D curve of sine function in the XZ-plane, drawn from (0,0,0) to (0, 0, 2π).



3. By finding the value of the parameter t, show that each of the following curves passes through the given point.

(a) $x(t) = 4t - 1$, $y(t) = 7t + 3$, $z(t) = t^2 - 1$ for the point $(-1, 3, -1)$

(b) $x(t) = 2\cos(t)$, $y(t) = 3\sin(t)$, $z(t) = 2 + t$ for the point $(2, 0, 2)$

Show that the point $(-1, 0, 1)$ is on neither of the above curves.

Substitute the values x, y, z from the given point into the parametric function

(a) $-1 = 4t - 1$, $t = 0$ | $3 = 7t + 3$, $t = 0$ | $-1 = t^2 - 1$, $t = 0$ $\Rightarrow$ $(-1, 3, -1)$ is on curve

$-1 = 4t - 1$, $t = 0$ | $0 = 7t + 3$, $t = -3/7$ different $\Rightarrow$ $(-1, 0, 1)$ is not on curve because we do not get same t-value.

(b) $2 = 2\cos(t)$, $\cos(t) = 1$, $t = \cos^{-1}(1) = 0$ | $0 = 3\sin t$, $t = \sin^{-1}(0)$, $t = 0$ | $2 = 2 + t$, $t = 0$ $\Rightarrow$ $(2, 0, 2)$ is on curve

$-1 = 2\cos(t)$, $t = \cos^{-1}(-\frac{1}{2})$, $t = \frac{2\pi}{3}$ | $0 = 3\sin(t)$, $t = \sin^{-1}(0)$, $t = 0$ $\Rightarrow$ $(-1, 0, 1)$ is NOT curve not same t-value

Parametric Differentiation

4. Compute the derivative $\frac{dy}{dx}$ for each of the following curves at $t = 0$. You can assume that each curve is defined over the interval $-\pi < t < \pi$.

$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt}$ $\boxed{\text{chain Rule}}$

(a) $x(t) = \sin(t)$, $y(t) = \cos(t)\sin(t)$

(b) $x(t) = 2(t - \cos(t))$, $y(t) = \sin(t)$

(c) $x(t) = 3t^2 - t$, $y(t) = 5t - 3$

(d) $x(t) = 5t - 3$, $y(t) = 3t^2 - t$

(a) $\frac{dx}{dt} = \cos t$ $\quad \frac{dy}{dt} = \cos^2(t) - \sin^2(t)$

$\frac{dy}{dx} = \frac{\cos^2(t) - \sin^2(t)}{\cos(t)}$

$\left.\frac{dy}{dx}\right|_{t=0} = \frac{1-0}{1} = 1$ (gradient)

(b) $\frac{dx}{dt} = 2(1 + \sin(t))$ $\quad \frac{dy}{dx} = \frac{\cos(t)}{2(1+\sin(t))}$

$\frac{dy}{dt} = \cos(t)$ $\quad \left.\frac{dy}{dx}\right|_{t=0} = \frac{1}{2(1)} = \frac{1}{2}$

(c) $\frac{dx}{dt} = 6t - 1$

$\frac{dy}{dt} = 5$

$\frac{dy}{dx} = \frac{5}{6t-1}$

$\left.\frac{dy}{dx}\right|_{t=0} = \frac{5}{-1} = -5$

(d) $\frac{dx}{dt} = 5$ $\quad \frac{dy}{dt} = 6t - 1$

$\frac{dy}{dx} = \frac{6t-1}{5}$

$\left.\frac{dy}{dx}\right|_{t=0} = -\frac{1}{5}$

5. The equations $x = t\cos(t)$ and $y = t\sin(t)$ are the parametric equations for a spiral curve known as the Spiral of Archimedes.[1] Find $\frac{dy}{dx}$ in terms of t. Can you find the Cartesian form of this spiral? (You need to find a relation between x and y that does not involve t.)

proved during lecture

$$\boxed{\begin{aligned}\frac{dx}{dt} &= \cos(t) - t\sin(t)\\ \frac{dy}{dt} &= \sin(t) + t\cos(t)\end{aligned}} \rightarrow \frac{dy}{dx} = \frac{\sin(t) + t\cos(t)}{\cos(t) - t\sin(t)}$$

(in term of t)

$x^2 = t^2\cos^2(t)$ — ①

$y^2 = t^2\sin^2(t)$ — ②

$\Rightarrow$ ① + ② $x^2 + y^2 = t^2(\cos^2(t) + \sin^2(t))$

$x^2 + y^2 = t^2$

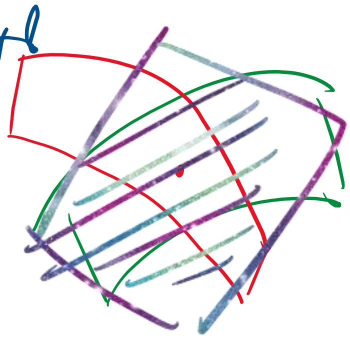
$\frac{y}{x} = \frac{\cancel{t}\sin(t)}{\cancel{t}\cos(t)} = \tan(t)$

$t = \tan^{-1}\left(\frac{y}{x}\right)$

$x^2 + y^2 = \left[\tan^{-1}\left(\frac{y}{x}\right)\right]^2$

$y = \sqrt{x^2 - \left[\tan^{-1}\left(\frac{y}{x}\right)\right]^2}$

$\frac{dy}{dx}$ = difficult



6. ~~Show~~ Find that the following pair of curves intersect.

Curve A: $x(v) = 3$, $y(v) = 4$, $z(v) = v$, for $0 < v < 2$.
Curve B: $x(u) = 3 + \sin(u)$, $y(u) = 4 - u$, $z(u) = 1 - u$, for $-1 < u < 1$.

Hence construct a plane that is tangent to both curves at the point of intersection.

$\begin{pmatrix}3\\4\\v\end{pmatrix} = \begin{pmatrix}3+\sin(u)\\4-u\\1-u\end{pmatrix}$ have POI

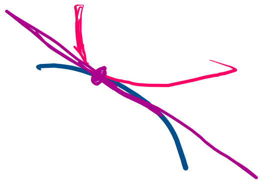
All the components (x, y, z) must be same, if they intersect.

$x(v) = x(u)$ $y(v) = y(u)$ $z(v) = z(u)$

$3 = 3 + \sin(u)$ $4 = 4 - u$ $v = 1 - u$

$\sin(u) = 0$ $u = 0$ $v = 1$

$u = 0$



$(u, v) = (0, 1)$ subs into curve (A) & (B) to see if same coordinates

curve (A), $v = 1$, $x = 3$, $y = 4$, $z = 1$ $\Rightarrow (3, 4, 1)$

curve (B), $u = 0$, $x = 3$, $y = 4$, $z = 1$ $\Rightarrow (3, 4, 1)$

POI $(3, 4, 1)$

tangent plane

position $\underset{\sim}{r} = \underset{\sim}{a} + \underset{\sim}{b}t$ direction vector (tangent vector) → line eqn

In vector

plane eqn $ax + by + cz = d$ → gradient vector curve A

$$\underset{\sim}{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + t\begin{pmatrix} p \\ q \\ r \end{pmatrix} + s\begin{pmatrix} l \\ m \\ n \end{pmatrix}$$

gradient vector curve B

tangent vector
gradient vector Curve A

$(u,v) = (0,1)$

gradient vector for Curve B

at POI $= \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$, $(u,v) = (0,1)$

Curve A

$x(v) = 3$
$y(v) = 4$
$z(v) = v$

$x'(v) = 0$
$y'(v) = 0$
$z'(v) = 1$

$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$x'(u) = \cos(u) \Rightarrow x'(0) = 1$
$y'(u) = -1$
$z'(u) = -1$

$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

Curve B
$x(u) = 3 + \sin(u)$
$y(u) = 4 - u$
$z(u) = 1 - u$

Tangent plane

$$\underset{\sim}{r} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

7. Consider the following parametric equations for the half circle:

$$x(t) = \cos(t) \qquad y(t) = \sin(t) \qquad \text{for } 0 \le t \le \pi$$

(a) Find $\dfrac{dy}{dt}$ and $\dfrac{dx}{dt}$.

(b) Rearrange $x^2 + y^2 = 1$ (for non negative values of y only) and find $\dfrac{dy}{dx}$.

(c) Show that $\dfrac{dy}{dx} = \dfrac{dy}{dt} \Big/ \dfrac{dx}{dt}$.

$y^2 = 1 - x^2$
$y = \pm\sqrt{1-x^2}$
$y = \sqrt{1-x^2}$

(a) $\dfrac{dy}{dt} = \cos(t)$, $\dfrac{dx}{dt} = -\sin(t)$

(b) $y = \sqrt{1-x^2} = (1-x^2)^{\frac{1}{2}}$ [make y the subject]

$\dfrac{dy}{dx} = \dfrac{-x}{\sqrt{1-x^2}}$

(c) $\dfrac{dy}{dt} \Big/ \dfrac{dx}{dt} = \dfrac{\cos(t)}{-\sin(t)}$

From question [$x = \cos(t)$]

From part (b) $\Rightarrow \dfrac{dy}{dx} = \dfrac{-x}{\sqrt{1-x^2}}$

$1 - \cos^2\theta = \sin^2\theta$
$1 - \sin^2\theta = \cos^2\theta$

$\dfrac{dy}{dx} = \dfrac{-\cos(t)}{\sqrt{1-\cos^2(t)}} = \dfrac{-\cos(t)}{\sqrt{\sin^2(t)}} = \dfrac{-\cos(t)}{\sin(t)} = \dfrac{dy}{dt} \Big/ \dfrac{dx}{dt}$

Therefore $\dfrac{dy}{dx} = \dfrac{dy}{dt} \Big/ \dfrac{dx}{dt}$

$y = mx + c$

8. Find the equation of the tangent line to

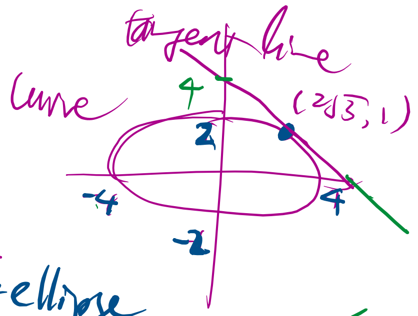
$$x(t) = 4\sin(t) \qquad y(t) = 2\cos(t) \qquad \text{for } 0 \le t \le 2\pi \qquad \text{at } t = \frac{\pi}{3}.$$

$\frac{dx}{dt} = -4\cos t$ $\quad \frac{x}{4} = \sin(t) \quad \frac{y}{2} = \cos(t)$

$\frac{dy}{dt} = 2\sin t$

Sketch both the curve and the tangent line.

$\left(\frac{x}{4}\right)^2 = \sin^2(t) \quad \left(\frac{y}{2}\right)^2 = \cos^2(t) \qquad \left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \rightarrow$ ellipse



$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ $\quad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

[1] The Spiral of Archimedes was studied by Archimedes himself, around 225 BC. Notably this spiral can be used to trisect an angle, and square the circle (both of which are classical mathematics problems from early Greek times, along with the problem of doubling the cube). In polar form, the spiral has the equation $r = a\theta$.

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{2\sin t}{-4\cos t}, \quad \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{3}} = -\frac{\sqrt{3}}{2} \, (M_t)$$

$\boxed{t = \frac{\pi}{3}}$ $\quad x\left(\frac{\pi}{3}\right) = 2\sqrt{3} \quad y\left(\frac{\pi}{3}\right) = 1 \rightarrow (2\sqrt{3}, 1)$ point

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -\frac{\sqrt{3}}{2}(x - 2\sqrt{3})$$

$$\boxed{y = -\frac{\sqrt{3}}{2}x + 4}$$

9. Consider the parametric equations:

$\boxed{\begin{array}{l} x(t) = \cos(t) \\ y(t) = \sin(t) \end{array}}$

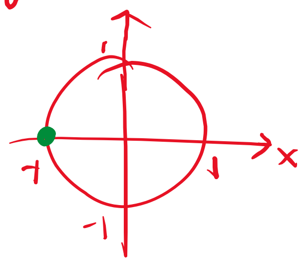
$$\boxed{\begin{aligned} x(t) &= \frac{1 - t^2}{1 + t^2} \\ y(t) &= \frac{2t}{1 + t^2} \end{aligned}}$$

$\boxed{\begin{array}{l} x(t) = \cos(t) \\ y(t) = \sin(t) \end{array}}$

$x^2 + y^2 = 4 \Rightarrow x(t) = 2\cos(t)$, $y(t) = 2\sin(t)$

(a) Show that this is equivalent to the equation of a circle $x^2 + y^2 = 1$.

(b) What is unusual about the point $(-1, 0)$ using the above parametrisation?



(a) $x^2 = \frac{(1-t^2)^2}{(1+t^2)^2} = \frac{1 - 2t^2 + t^4}{(1+t^2)^2} \quad \Bigg| \quad y^2 = \frac{4t^2}{(1+t^2)^2}$

L.H.S $\quad x^2 + y^2 = \frac{1 - 2t^2 + t^4}{(1+t^2)^2} + \frac{4t^2}{(1+t^2)^2} = \frac{(1+t^2)^2}{1 + 2t^2 + t^4}$

$x^2 + y^2 = \frac{(1+t^2)^2}{(1+t^2)^2} = 1$ = R.H.S

(b) $-1 = \frac{1 - t^2}{1 + t^2}$ $\quad \Bigg| \quad 0 = \frac{2t}{1+t^2}$ $\Rightarrow$ it should be unique defined

$-1 - \cancel{t^2} = 1 - \cancel{t^2}$ $\quad \Bigg| \quad 2t = 0$

$-1 = 1$?? Not logic $\quad \Bigg| \quad t = 0$

10a. Find the tangent vector to each of the following curves given by the position vector $\underset{\sim}{r}(t) = (x(t), y(t), z(t))$ where:

i. $x(t) = 3t^2$ $\quad y(t) = 4t$ $\quad z(t) = 5$ $\Longrightarrow V(t) = \begin{pmatrix} 3t^2 \\ 4t \\ 5 \end{pmatrix}$ $\quad V'(t)$

ii. $x(t) = 3\cos(t)$ $\quad y(t) = 2\sin(t)$ $\quad z(t) = t$ $\Rightarrow U(t) = \begin{pmatrix} 3\cos t \\ 2\sin t \\ t \end{pmatrix}$

10b. Let $\underset{\sim}{r}(t) = (x(t), y(t), z(t))$ be the position vector along some curve for $-\infty < t < \infty$. Suppose t has been chosen so that $1 = \underset{\sim}{r}' \cdot \underset{\sim}{r}'$ for all t. Show that $\boxed{0 = \underset{\sim}{r}' \cdot \underset{\sim}{r}''}$

What does this tell you about the relationship of the vector $\underset{\sim}{r}''$ to the curve? (Hint: Use the Product rule.)

(a) (i) $V'(t) = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 6t \\ 4 \\ 0 \end{pmatrix} \leftarrow$ tangent vector

(ii) $U'(t) = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} -3\sin(t) \\ 2\cos(t) \\ 1 \end{pmatrix}$

(b) $\underset{\sim}{r}(t) = (x(t), y(t), z(t))$

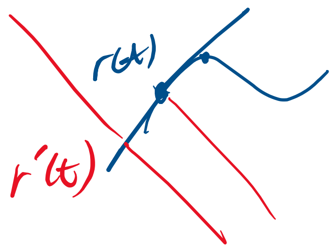
$\underset{\sim}{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$

Given that

$1 = \underset{\sim}{r}' \cdot \underset{\sim}{r}'$

$r(t)$

$r'(t)$



RHS Use product Rule

$0 = \underbrace{\underset{\sim}{r}'' \cdot \underset{\sim}{r}' + \underset{\sim}{r}' \cdot \underset{\sim}{r}''}_{\text{Same}}$

$0 = 2\underset{\sim}{r}'' \cdot \underset{\sim}{r}'$

$\boxed{\underset{\sim}{r}'' \cdot \underset{\sim}{r}' = 0} \Leftrightarrow$ Orthogonal

normal