MAT1830 : PRACTICE QUESTIONS ON INDUCTION

1. Prove using induction that $\sum_{i=3}^{i=n} (2i) = (n+3)(n-2)$ for all integers n > 3.

2. Prove by induction that $1 \cdot 2^2 + 2 \cdot 3^2 + 3 \cdot 4^2 + \ldots$ to n terms is given by $\frac{1}{12} n(n+1)(n+2)(3n+5)$

3. **P**rove by induction $3^{4n-2} + 17^n + 22$ is divisible by 16 for every positive integer *n*.

4. Prove that, if the statement $n! > 2n + 2^n$ is true for $n = k$, where k is a positive integer, then it is true for $n = k + 1$.

   Hence, find the set of values of *n* for which the statement is true.

5. The sequence of real numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 = 1$ and $u_{n+1} = \dfrac{5u_n + 4}{u_n + 2}$ for all $n \geq 1$. Prove by induction that $u_n < 4$ for all $n \geq 1$.

6. Let r0, r1, r2, …. be a recursive sequence defined by

   r0 = 3; r1 = 2 and $r_n = (18n) r_{n-1} + (12) r_{n-2}$ for all integers n ≥ 2.

   Prove using strong induction that 2^n divides r_n for all integers n ≥ 0.

7. If the sequence u_1, u_2, u_3, ….. is defined by $u_1 = 1$, $u_2 = 2$ and $u_{r+2} + 4u_r = 4u_{r+1}$,

   Prove by induction that $u_n = 2^{n-1}$

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8. Prove by induction that the number of steps to complete the disk movements in the tower ofHanoi problem is $(2^n - 1)$ for a system of n disks, for all n≥1

1) $\sum_{i=3}^{i=n} (2i) = (n+3)(n-2)$ for all integers $n \geq 3$

4) $n! > 2n + 2^n$ is true for $n = k$ where k is a positive integer, then if is TRUE for $n = k+1$

↳ asking for inductive step without providing base step

Inductive step: Assume $k! > 2k + 2^k$, WHTP: $(k+1)! > 2(k+1) + 2^{k+1}$

We know that $(k+1)! = (k+1)(k!)$

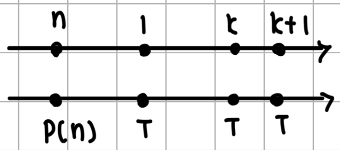
$(k+1)! > (k+1)(2k + 2^k)$

$(k+1)! > 2k(k+1) + 2^k(k+1)$

* We know that $(k+1) > 2$ as long as $k > 1$

and... We know that $2(k+1)k > 2(k+1)$ as long as $k > 1$

HENCE, $(k+1)! > 2(k+1) + 2^{k+1}$ as long as $k > 1$



base step:

n	n!	$2n+2^n$	TRUE / FALSE
2	2	8	FALSE
3	6	14	FALSE
4	24	24	FALSE
5	120	42	TRUE

HENCE, $n \geq 5$ is TRUE #

5) Let $P(n)$: $u_n < 4$ for the sequence where $u_{n+1} = (5u_n + 4)/(u_n + 2)$ for all $n \geq 1$; $u_1 = 1$

Base step: when $n = 1$, $u_1 = 1$
indeed $1 < 4$; hence, $u_1 < 4$
HENCE, $P(n) \equiv$ TRUE

Inductive step: assume $P(k) \equiv T$ ie $u_k < 4$
WHTP that $P(k+1) \equiv T$ where $u_{k+1} < 4$

we know $u_{k+1} = (5u_k + 4)/(u_k + 2)$
endgame: show $u_{k+1} < 4$

$$u_{k+1} = \frac{a[u_k + 2] + b}{(u_k + 2)}$$

$$u_{k+1} = \frac{5(u_k + 2) - 6}{(u_k + 2)} \Rightarrow \left[5 - \frac{6}{(u_k + 2)}\right]$$

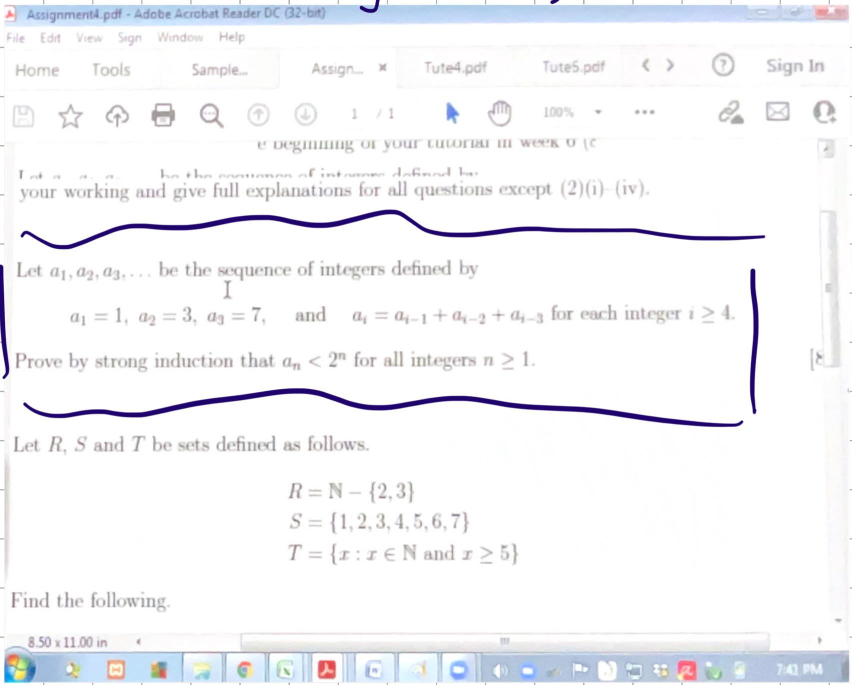
since $u_k < 4$ (from assumption) $\longrightarrow$ $u_{k+2} < 6$
and $6/(u_{k+2})$ where $u_{k+2} < 6$ and will be ≥ 1
and $u_{k+1} = 5 - 6/(u_k + 2)$ must be < 4

so in conclusion : $u_{k+1} < 4$ is indeed TRUE

* another method $\Rightarrow$ $u_{k+1} - 4 = 5u_k + 4$

Conclusion: Since $P(1) \equiv T$
$P(1) \to P(2)$ when $k = 1$
$P(2) \to P(3)$ when $k = 2$
Hence, $P(n) \equiv T$ for all $n \geq 1$

Extra Question : (strong induction)



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your working and give full explanations for all questions except (2)(i)–(iv).

Let $a_1, a_2, a_3, \dots$ be the sequence of integers defined by

$$a_1 = 1,\ a_2 = 3,\ a_3 = 7, \quad \text{and} \quad a_i = a_{i-1} + a_{i-2} + a_{i-3} \text{ for each integer } i \geq 4.$$

Prove by strong induction that $a_n < 2^n$ for all integers $n \geq 1$.

Let R, S and T be sets defined as follows.

$$R = \mathbb{N} - \{2, 3\}$$
$$S = \{1, 2, 3, 4, 5, 6, 7\}$$
$$T = \{x : x \in \mathbb{N} \text{ and } x \geq 5\}$$

Find the following.

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$a_1=1, a_2=3, a_3=7$ and $a_i = a_{i-1} + a_{i-2} + a_{i-3}$ for each integer $i \geq 4$
prove by induction that $a_n < 2^n$ for all integer $n \geq 1$

Let $P(n)$: $a^n < 2^n$ for the recursive sequence
$a_i = a_{i-1} + a_{i-2} + a_{i-3},\ i \geq 4$
where $a_1 = 1,\ a_2 = 3,\ a_3 = 7$
for all $n \geq 1$

Base step: when $n=1$, $a_1 = 1$ (given) and $2^n\big|_{n=1} = 2$
Indeed $1 < 2 \Rightarrow P(1) \equiv T$

When $n=2$, $a_2 = 3$ (given) and $2^n\big|_{n=2} = 4$
Indeed $3 < 4 \Rightarrow P(2) \equiv T$

When $n=3$, $a_3 = 7$ (given) and $2^n\big|_{n=3} = 8$
Indeed $7 < 8 \Rightarrow P(3) \equiv T$

Inductive step: Assume $P(n) \equiv T$ for all $n \leq k$ ie $a_k < 2^k, a_{k-1} < 2^{k-1}, \dots, a_1 < 2^1$
We have to prove $P(k+1) \equiv T$ ie WHTP $a_{k+1} < 2^{k+1}$

Now $a_{k+1} = a_k + a_{k-1} + a_{k-2}$
$< 2^k + 2^{k-1} + 2^{k-2}$ (from the inductive assumption)

$< \left(\frac{2^{k+1}}{2}\right) + \left(\frac{2^{k+1}}{2^2}\right) + \frac{2^{k+1}}{2^3}$

→ like :

* $2^k = (2^{k+1})/2$

* $2^{k-1} = (2^{k+1})/2^2$

* $2^{k-2} = (2^{k+1})/2^3$
$= 2^{k+1-3}$
$= 2^{k-2}$

$< 2^{k+1}\left\{\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right\}$

ie $\boxed{a_{k+1} < 2^{k+1}\left\{\frac{7}{8}\right\}}$

ie $\boxed{a_{k+1} < 2^{k+1}(1)}$ {obviously since $\frac{7}{8} < 1$

$\therefore P(k+1) \equiv T$

Conclusion: Since $P(1) \wedge P(2) \wedge P(3) \equiv T$ (BS)
$\therefore$ $P(1) \wedge P(2) \wedge P(3) \rightarrow P(4)$ {IS, k
$P(1) \wedge P(2) \wedge P(3) \wedge P(4) \rightarrow P(5)$
⋮

therefore, $P(n)$ is TRUE for all $n \geq 1$

Conclusion: Since $P(1) \wedge P(2) \wedge P(3) \equiv T$ (BS)
$P(1) \wedge P(2) \wedge P(3) \rightarrow P(4)$

(2) (a) Determine whether the propositions $p \to (q \vee \neg r)$ and $(p \wedge \neg q) \to \neg r$ are logically equivalent using either a truth table or laws of logic.

(b) Let A, B and C be sets. If a is the proposition "$x \in A$", b is the proposition "$x \in B$" and ...

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② $p \to (q \vee \neg r) \stackrel{?}{\equiv} (p \wedge \neg q) \to \neg r$

LHS: $p \to (q \vee \neg r)$

$\equiv \neg p \vee (q \vee \neg r)$ {Implication Law}

$\equiv (\neg p \vee q) \vee \neg r$ {Associative}

$\equiv (\cancel{\neg}\ \neg p \vee [\neg\neg q]) \vee \neg r$

$= \neg(p \wedge \neg q)$ {De mogan}

(c) $\neg[\forall x \exists y\, \neg P(x,y)] \equiv \exists x \forall y\, \neg\neg P(x,y) \equiv \exists x \forall y\, P(x,y)$

(c) Consider the statement $\forall x \exists y \neg P(x,y)$. Write down a negation of the statement that does not use the symbol $\neg$. [2]

(d) Under the interpretation where x and y are in $\mathbb{R} - \{0\}$ and $P(x,y)$ is "$xy \geq 0$", is the original statement in (c) true or is its negation true? [5]

(e) Is the statement $(\exists x(P(x) \vee Q(x))) \to ((\exists x P(x)) \vee (\exists x Q(x)))$ valid? If it is, explain why. If it isn't, give an interpretation under which it is false.

(d) Statement: $\forall x \exists y\, \neg P(x,y)$; Negation: $\exists x \forall y\, P(x,y)$

Interpretation: $x, y \in \mathbb{R} - \{0\}$

$P(x,y) : xy \geq 0$