Binomial Distributions

- Back to Discrete Probability Distributions
- Exercise for Binomial Distributions

If a Bernoulli trial is performed repeatedly, with the probability of success in a trial occurring with constant probability, i.e. the trials are independent and considering the number of success is called a **binomial distribution**

An event or trial is repeated n times, and in each trial only two outcomes, A and A' then in each trial the probability of event A occurring is p and $P(A') = q = 1 - p$.

If the discrete random variable X is binomially distributed with parameters n and p, then it can be written as

X ~ b(n,p) , X ~ B(n,p) or X ~ bi(n,p), X ~ Bi(n,p)

If the n trials, the probability of A occurring x times $(x \le n)$, is:

$$P(X=x) = {}^nC_x p^x (1-p)^{n-x} \quad \text{or} \quad P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$x = 0, 1, 2, 3, \ldots, n$$

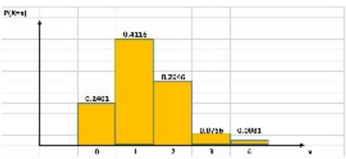
Example 1

If n is 4 and p is 0.3 then q = 0.7, written as X ~ Bi(4,0.3). The distribution is shown below using CAS calculator.

x	0	1	2	3	4
$P(X=x)$	0.2401	0.4116	0.2646	0.0756	0.0081

$P(X=2) = {}^4C_2 \times (0.3)^2 \times (0.7)^2$
$= \text{Binom PDF} = 0.2646$ ✓

Binomial distribution



$${}^nC_r = \frac{n!}{(n-r)!}$$

$${}^3C_2 = \frac{3!}{(1)!} = \frac{3 \times 2 \times 1}{} = 1$$

① Bernoulli dist (flip a coin once)

$\left\langle \begin{matrix} H \\ T \end{matrix} \right\}$

x	0	1
$P(X=x)$	$1-p$	p

a) 1 trial
b) 2 outcomes (mutually exclusive)
c) success $(x=1) \rightarrow P(X=1) = p$
d) failure $(x=0) \rightarrow P(X=0) = (1-p)/q$
e) $E(X) = p$
f) $Var(X) = p(1-p)$
g) $SD(X) = |\sqrt{Var(X)}| = |\sqrt{p(1-p)}|$

$0.5 \times 0.5 \times 0.5$

HHH TTTT ... 20
H TTT H

0.5 H — 0.5 H, 0.5 T
0.5 T — 0.5 H, T

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② Binomial dist (flipping a coin 20) → H (success) → $P(X=3)$

5
a) more than 1 trial
b) no of trials → "n"
c) each trial will have 2 mutually exclusive outcome
d) the prob of sucess (p) & failure (1-p) for each trial will be SAME
e) each trial are INDEPENDENT to each other
g) notation (1m) ✓
↳ $X \sim B(n,p)$
↳ $X \sim Bi(n,p)$
↳ $X \sim Bin(n,p)$ ✓
h) formula $X \sim B(20, 0.5)$ ✓
i) $P(X=3) = {}^nC_x \times p^x \times (1-p)^{n-x}$ → formula sheet
(2m) ✓ $= {}^{20}C_3 \times (0.5)^3 \times (0.5)^{17}$ (scientific calcu / CAS)
✓ $= 0.001087$

$\boxed{\text{CAS}}$ → ii) $P(X=3) = \text{BinomPDF}$ → menu → 5 → 5 → A → n = 20, p = 0.5, x = 3 } "ok"
(1m)

$\boxed{\text{CAS}}$ → iii) $P(X \ge 4) = P(X=4) + P(X=5) \ldots + P(X=20)$
$= \text{BinomCDF}$ → menu → 5 → 5 → B → n = 20, p = 0.5, lower bound = 4, upper bound = 20 } "ok" = 0.9987

inequality
$(\ge, \le, >, <)$

i) $E(X) = np$
j) $Var(X) = npq = np(1-p)$
k) $SD(X) = |\sqrt{npq}|$

Example 2 → Binomial → no of trial, no of outcomes

A normal fair die is rolled ten times. → n = 10
Determine the probability of obtaining exactly — getting a six (1/6) ✓, not getting a six (5/6)

$X \sim Bi(10, 1/6)$ ✓

(a) three sixes

$P(X=3) = \text{Binom PDF} = 0.155$ ✓

15.5% → 3 times no 6 on the upper most face

(b) five sixes

$P(X=5) = \text{BinomPDF} = 0.013$ ✓

1.3%

Answers: 0.155, 0.013

Example 3 → Binomial — 0.6 goal, 0.4 no goal

Suppose that each time a particular soccer player takes a penalty kick the probability of scoring a goal is 0.6.
Determine the probability that if the player takes eight penalty kicks they will score → n = 8

(a) exactly five times

$X \sim Bi(8, 0.6)$

$P(X=5) = \text{BinomPDF} = 0.2787$ ✓

(b) at least five times goal

$P(X \ge 5) = P(X=5) + P(X=6) + P(X=7) + P(X=8)$

$P(X \ge 5) = \text{BinomCDF} = 0.5941$ ✓

Answers: 0.279, 0.594

$E(X) = \sum x_i p_i$
$Var(X) = \sum (x_i - \mu)^2 \cdot p_i$
$SD(X) = |\sqrt{Var(X)}|$

Mean and variance for binomial distribution with parameters n and p

$E(X) = \sum x_i p_i$
$Var(X) = \sum (x_i - \mu)^2 \cdot p_i$
$SD(X) = |\sqrt{Var(X)}|$

Mean and variance for binomial distribution with parameters n and p

Binomial

Expected value = $E(X)$ = Mean = $\mu = np$

Variance = $\sigma^2 = npq = np(1-p)$

given in formula sheet

$SD(X) = |\sigma| = |\sqrt{Var(X)}| = |\sqrt{np(1-p)}|$

Example 1

Find the mean and standard deviation of a binomial distribution with n, the number of trials, equal to 12 and p, the probability of success in each trial equal to 0.25.

$X \sim B(12, 0.25)$
$E(X) = np = 12(0.25) = 3$
$SD(X) = |\sqrt{npq}| = |\sqrt{12(0.25)(0.75)}| = 1.5$
$Var(X) = (1.5)^2 = 2.25$

Answers: 3, 1.5

CAS → solve({ $E(X)$, $Var(X)$ }, n, p) (2m)

Example 2

A binomial distribution has a mean of 9.6 and a standard deviation of 2.4. Find n, the number of trials and p, the probability of success in each trial.

$E(X) = 9.6$
$SD(X) = 2.4$
n, p

$E(X) = np = 9.6$
$n = \frac{9.6}{p}$ — ① $= \frac{9.6}{0.4} = 24$

$Var(X) = np(1-p) = (2.4)^2$
$\left(\frac{9.6}{p}\right)(p)(1-p) = 5.76$
$9.6 - 9.6p = 5.76$
$p = \frac{5.76 - 9.6}{-9.6} = 0.4$

Answers: 24, 0.4

$n = 10$, $P(H) = P(T) = 0.5$
$P(X=2) = {}^{10}C_2(0.5)^2(0.5)^8$

$X \sim B(n, p)$

Graphs of binomial distribution

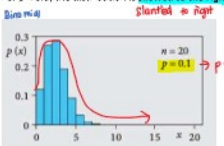
Shape of the binomial distribution
* will be tested in the form explaination in WACE (1m - 3m)

① $X \sim B(n,p)$
a) p → < 0.5 → +ve skew / skewing to right
= 0.5 → no skew / symmetrical
> 0.5 → -ve skew/skewing to left

b) no of trials, n
* when 'n' increases, the binomial distribution will tend to become more SYMMETRICAL/ACCURATE
↳ $p = 0.1 < 0.5$ → $\boxed{+\text{skew}}$ → symmetrical
$n = 20$ ↳ n↑ $n = 200$

For p < 0.5, the distribution is *skewed to the right* (a *positive skew*).
slanted to right

Binomial

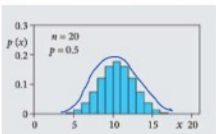


no skewing

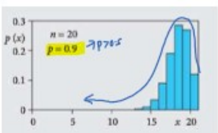
evenly distributed

For p = 0.5 the distribution is *symmetrical*.

Binomial to normal
p = 0.5



For p > 0.5, the distribution is *skewed to the left* (a *negative skew*).



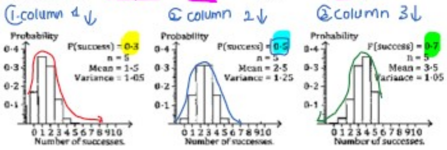
Example 1

The graphs for the binomial distributions with

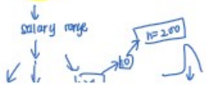
+ve skew: p = 0.3, symm: p = 0.5 and -ve skew: p = 0.7

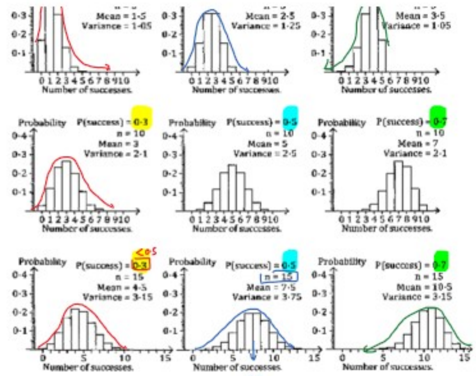
are shown below for n = 5, n = 10 and n = 15

① column 1 ↓ ② column 2 ↓ ③ column 3 ↓

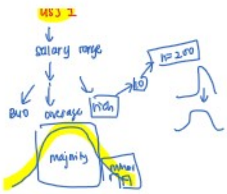


population of people in USJ 1
salary range





7·5



Note

**
- the symmetrical nature of the graphs for which P(success) = 0.5
- the skewed nature of the graphs for which P(success) ≠ 0.5 → p>0·5 → p<0·5
- the graphs of P(success) = k and P(success) = 1-k are mirror images

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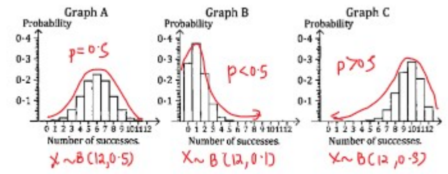
- the graphs for P(success) ≠ 0.5 appear to move towards a more symmetrical distribution as n increases

skewed graph → symmetrical → when 'n'↑

Example 2

The three graphs below show binomial distributions for
n = 12 and P(success) = 0.1, 0.5 and 0.8

Which graphs has which P(success) value?

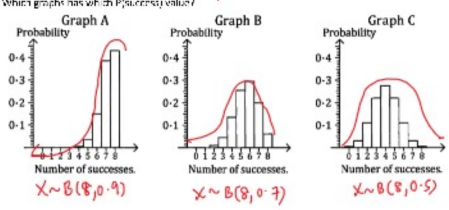


Answers: B(0.1), A(0.5), C(0.8)

Example 3

The three graphs below show binomial distributions for → left move
n = 8 and P(success) = 0.5, 0.7 and 0.9
symm p>0·5 → -ve skew → left

Which graphs has which P(success) value?



Answers: C(0.5), B(0.7), A(0.9)