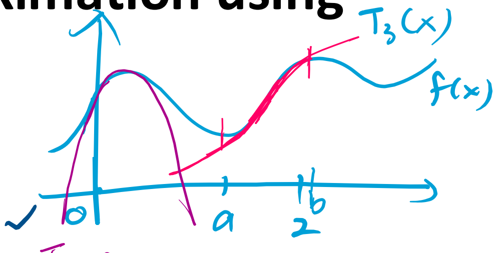
Problem Set Eight: Function Approximation using Taylor Series and Cubic Spline



$T_n(x)$
$x=0$

at $x = 0$:

$$T_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n.$$

at $\boxed{x=a}$

$$T_n(x) = f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor Series

1. Calculate $f^{(1)}$, $f^{(2)}$, $f^{(3)}$ and $f^{(4)}$ for the function $f(x) = e^{-x}$. Now calculate the values of each of these derivatives at $x = 0$ and calculate $a_n = \frac{f^{(n)}(0)}{n!}$ to construct the first five partial sums of the Taylor series, $T_0(x)$, $T_1(x)$, $T_2(x)$, $T_3(x)$ and $T_4(x)$.

$f(x) = e^{-x}$ $\quad f(0) = 1$

$f^{(1)}(x) = -e^{-x}$ $\quad f^{(1)}(0) = -1$

$f^{(2)}(x) = e^{-x}$ $\quad f^{(2)}(0) = 1$

$f^{(3)}(x) = -e^{-x}$ $\quad f^{(3)}(0) = -1$

$f^{(4)}(x) = e^{-x}$ $\quad f^{(4)}(0) = 1$

$T_0(x) = 1$

$T_1(x) = 1 - x$

$T_2(x) = 1 - x + \frac{1}{2}x^2$

$T_3(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3$

$T_4(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4$

$a_1 = \frac{f'(0)}{1!}$

$a_2 = \frac{f^{(2)}(0)}{2!}$

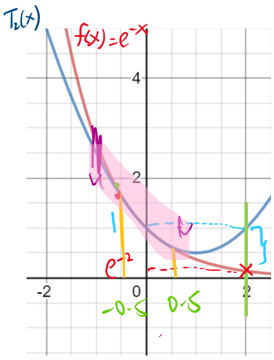
$$T_n(x) = 1 + \frac{-1}{1!}x + \frac{1}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

$$e^{-x} \approx 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots$$

This Function Approximation is suitable apply to value of x around $x=0$, e.g. $-0.5 < x < 0.5$

$\frac{(-1)^{n-1}}{(n-1)!}x^{n-1}$

$n \geq 1$



$x=2$ $\quad f(2) = e^{-2} = 0.1353$

$x=2, T_2(2) = 1$

$T_n(x) = f(2) + \frac{f'(2)}{1!}(x-2)$

$T_2(x)$ $\quad T_n(x)$ at $x=0$

big gap.

$T_n(x) \approx e^{-x}$ at $x=2$ is not accurate

2. Construct a Taylor series for each of the following functions, centred at $x = 0$.

(a) $f(x) = \ln(1-x)$

(b) $f(x) = \sin(x)$

(c) $f(x) = e^{-x}\sin(x)$

Question 2

$x=0$

(a) $f(x) = \ln(1-x)$ $\qquad f(0) = 0$

$f^{(1)}(x) = \dfrac{-1}{1-x} = -(1-x)^{-1}$ $\qquad f^{(1)}(0) = -1$

$f^{(2)}(x) = -(1-x)^{-2}$ $\qquad f^{(2)}(0) = -1$

$f^{(3)}(x) = -2(1-x)^{-3}$ $\qquad f^{(3)}(0) = -2$

$f^{(4)}(x) = -6(1-x)^{-4}$ $\qquad f^{(4)}(0) = -6$

$$T_n(x) = 0 + \frac{-1}{1!}x + \frac{-1}{2!}x^2 + \frac{-2}{3!}x^3 + \frac{-6}{4!}x^4 + \cdots$$

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots + \frac{-1}{n}x^n,$$

$n \geq 1$

(b) $f(x) = \sin x$ $\qquad f(0) = 0$

$f'(x) = \cos x$ $\qquad f^{(1)}(0) = 1$ ✓

$f^{(2)}(x) = -\sin x$ $\qquad f^{(2)}(0) = 0$

$f^{(3)}(x) = -\cos x$ $\qquad f^{(3)}(0) = -1$ ✓

$f^{(4)}(x) = \sin x$ $\qquad f^{(4)}(0) = 0$

$f^{(5)}(x) = \cos x$ $\qquad f^{(5)}(0) = 1$

$$T_n(x) = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$$

$$\sin(x) \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

$$\sin(x) \approx x - \frac{1}{6}x^3 + \frac{1}{120}x^5 = T_5(x)$$

$n \geq 0.$

2. Construct a Taylor series for each of the following functions, centred at $x = 0$.

(a) $f(x) = \ln(1 - x)$

(b) $f(x) = \sin(x)$

(c) $f(x) = e^{-x}\sin(x)$

(c) $f(x) = e^{-x}\sin(x)$ $\quad f(0) = 0$

$f^{(1)}(x) = -e^{-x}\sin(x) + e^{-x}\cos(x)$

$= e^{-x}(-\sin(x) + \cos(x))$ $\quad f^{(1)}(0) = 1$

$f^{(2)}(x) = -e^{-x}(-\sin(x) + \cos(x)) + e^{-x}(-\cos(x) - \sin(x))$ $\quad f^{(2)}(0) = -2$

$= -2e^{-x}\cos(x)$

$f^{(3)}(x) = 2e^{-x}\cos(x) + 2e^{-x}\sin(x)$ $\quad f^{(3)}(0) = 2$

$= 2e^{-x}(\cos(x) + \sin(x))$

$f^{(4)}(x) = -2e^{-x}(\cos(x) + \sin(x)) + 2e^{-x}(-\sin(x) + \cos(x))$ $\quad f^{(4)}(0) = 0$

$= -4e^{-x}\sin(x)$

$T_n(x) = 0 + \frac{1}{1!}x + \frac{-2}{2!}x^2 + \frac{2}{3!}x^3 + \frac{0}{4!}x^4 + \cdots$

$e^{-x}\sin(x) \approx x - x^2 + \frac{1}{3}x^3 + \cdots$

3. Construct the Taylor series (up to $T_3(x)$ is sufficient) for the function $f(x) = \sin^{-1}(x)$, centred at $x = 0$.

$f(x) = \sin^{-1}(x)$ $\quad f(0) = 0$

$f^{(1)}(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}$ $\quad f^{(1)}(0) = 1$

$f^{(2)}(x) = -\frac{1}{2}(1-x^2)^{-3/2}(-2x)$ $\quad f^{(2)}(0) = 0$

$= x(1-x^2)^{-3/2}$

$f^{(3)}(x) = (1-x^2)^{-3/2} + (x)(-\frac{3}{2})(1-x^2)^{-5/2}(-2x)$ $\quad f^{(3)}(0) = 1$

$= (1-x^2)^{-3/2} + 3x^2(1-x^2)^{-5/2}$

$T_n(x) = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$

$\sin^{-1}(x) \approx \underbrace{\frac{1}{1!}x + \frac{1}{3!}x^3}_{} + \underbrace{\frac{1}{5!}x^5 + \frac{1}{7!}x^7}_{} + \cdots + \frac{1}{(2n+1)!}x^{2n+1}, n \geq 0$

4. Find the first four non-zero terms of the Taylor series (about $x = 0$) for each of the following functions

(a) $f(x) = \cos(x)$

(b) $f(x) = \sin(2x)$

(c) $f(x) = e^x$

(d) $f(x) = \arctan(x)$

(a) $f(x) = \cos(x)$ $\quad f(0) = 1$

$f^{(1)}(x) = -\sin(x)$ $\quad f^{(1)}(0) = 0$

$f^{(2)}(x) = -\cos(x)$ $\quad f^{(2)}(0) = -1$

$f^{(3)}(x) = \sin(x)$ $\quad f^{(3)}(0) = 0$

check! $f^{(4)}(x) = \cos(x)$ $\quad f^{(4)}(0) = 1$

$f^{(5)}(x) = -\sin(x)$ $\quad f^{(5)}(0) = 0$

$f^{(6)}(x) = -\cos(x)$ $\quad f^{(6)}(0) = -1$

$$T_n(x) = 1 + \frac{-1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{-1}{6!}x^6 + \dots$$

$$\cos(x) \approx 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots$$

(b) $f(x) = \sin(2x)$ $\quad f(0) = 0$

$f^{(1)}(x) = 2\cos(2x)$ $\quad f^{(1)}(0) = 2$

$f^{(2)}(x) = -4\sin(2x)$ $\quad f^{(2)}(0) = 0$

$f^{(3)}(x) = -8\cos(2x)$ $\quad f^{(3)}(0) = -8$

$f^{(4)}(x) = 16\sin(2x)$ $\quad f^{(4)}(0) = 0$

$f^{(5)}(x) = 32\cos(2x)$ $\quad f^{(5)}(0) = 32$

$f^{(6)}(x) = -64\sin(2x)$ $\quad f^{(6)}(0) = 0$

$f^{(7)}(x) = -128\cos(x)$ $\quad f^{(7)}(0) = -128$

$$T_n(x) = \frac{2}{1!}x + \frac{-8}{3!}x^3 + \frac{32}{5!}x^5 + \frac{-128}{7!}x^7 + \dots$$

$$\sin(2x) \approx 2\left[x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \dots\right]$$

(c) $f(x) = e^x$ $\quad f(0) = 1$ $\quad T_n(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$

$f^{(1)}(x) = e^x$ $\quad f^{(1)}(0) = 1$ $\quad \boxed{x=0}$ $\quad e^x \approx 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$

$f^{(2)}(x) = e^x$ $\quad f^{(2)}(0) = 1$

$f^{(3)}(x) = e^x$ $\quad f^{(3)}(0) = 1$

(4)

(d) $f(x) = \tan^{-1}(x)$ $\qquad f(0) = 0$ $\quad x=0$

$f^{(1)}(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$ $\qquad f^{(1)}(0) = 1$

$f^{(2)}(x) = -2x(1+x^2)^{-2}$ $\qquad f^{(2)}(0) = 0$

$(-2x)[-2(1+x^2)^{-3}(2x)]$

$f^{(3)}(x) = -2(1+x^2)^{-2} + 8x^2(1+x^2)^{-3}$ $\qquad f^{(3)}(0) = -2$

$f^{(4)}(x) = 8x(1+x^2)^{-3} + 16x(1+x^2)^{-3} - 48x^3(1+x^2)^{-4}$

$= 24x(1+x^2)^{-3} - 48x^3(1+x^2)^{-4}$ $\qquad f^{(4)}(0) = 0$

$f^{(5)}(x) = 24(1+x^2)^{-3} - 144x^2(1+x^2)^{-4} - 144x^2(1+x^2)^{-4} + 384x^4(1+x^2)^{-5}$

$= 24(1+x^2)^{-3} - 288x^2(1+x^2)^{-4} + 384x^4(1+x^2)^{-5}$ $\qquad f^{(5)}(0) = 24$

$f^{(8)}(x) = 11520x(\quad)$

$f^{(6)}(x) = -144x(1+x^2)^{-4} - 576x(1+x^2)^{-4} + \cdots\cdots$ $\qquad f^{(6)}(0) = 0$

$= -720x(1+x^2)^{-4} + \cdots\cdots$

$f^{(8)}(0) = 0$

$f^{(7)}(x) = -720(1+x^2)^{-4} + 5760x^2(1+x^2)^{-5} + \cdots$ $\qquad f^{(7)}(0) = -720$

$f^{(9)}(x) = 11520(\quad)$

$T_n(x) = \frac{1}{1!}x + \frac{-2}{3!}x^3 + \frac{24}{5!}x^5 + \frac{-720}{7!}x^7 + \cdots\cdots$

$\tan^{-1}(x) \approx x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots \qquad (-1)^n \frac{x^{2n+1}}{(2n+1)},\ n \geq 0$

5. Use the results of the previous question to obtain the first two non-zero terms of the Taylor series (about $x = 0$) for the following functions

(a) $f(x) = \cos(x)\sin(2x)$ $\qquad$ Q4 (a) (b) results

(b) $f(x) = e^{-x^2}$

(c) $f(x) = \arctan(\arctan(x))$

$f(x) = \frac{e^x}{1+x} \Rightarrow \boxed{\text{power series}}$

(a)

$f(x) = \cos(x)\sin(2x)$

$\cos(x)\sin(2x) \approx \left[1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots\right]\left[2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots\right]$

$= 2x - \frac{4}{3}x^3 - x^3 + \cdots\cdots$

$\cos(x)\sin(2x) \approx 2x - \frac{7}{3}x^3$

Q5(b) $f(x)=e^{-x^2}$

From 4(c) $e^{x} \approx 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$

$$e^{-x^2} \approx 1 + (-x^2) + \frac{1}{2!}(-x^2)^2 + \frac{1}{3!}(-x^2)^3 + \cdots$$

$$e^{-x^2} \approx 1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6$$

Q5 (c) $f(x) = \tan^{-1}(\tan^{-1}(x))$ Q4(d) $\tan^{-1}(x) \approx x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$

$$f(x) = \tan^{-1}\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots\right)$$

$$T_n(x) = \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7\right) - \frac{1}{3}\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7\right)^3 + \frac{1}{5}(\quad)^5$$

$$T_n(x) = \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7\right) - \frac{1}{3}x^3 + \cdots$$

expand $(x^3 + \cdots\cdots)$ power is too high so ignore

$$\tan^{-1}(\tan^{-1}(x)) \approx x - \frac{2}{3}x^3 + \boxed{\ }x^5 + \cdots$$

6. Compute the Taylor polynomial T_n, about the given point, for each of the following functions:

(a) $f(x) = e^x$, about $a = 1$.

(b) $f(x) = e^x$, about $a = -1$.

$\boxed{x=0}$ $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots + \frac{1}{n!}x^n + \ldots$

$f(x)=e^x$

$f^{(1)}(x)=e^x$

$f^{(2)}(x)=e^x$

$f^{(3)}(x)=e^x$

(a) $x=1$

$f(1)=e$

$f^{(1)}(1)=e$

$f^{(2)}(1)=e$

$f^{(3)}(1)=e$

$\boxed{x=1}$

$$e^x \approx e + \frac{e}{1!}(x-1) + \frac{e}{2!}(x-1)^2 + \frac{e}{3!}(x-1)^3 + \cdots$$

$$e^x \approx e\left[1 + (x-1) + \frac{1}{2!}(x-1)^2 + \frac{1}{3!}(x-1)^3 + \cdots\right]$$

(b) $x=-1$

$f(-1)=e^{-1}$

$f^{(1)}(-1)=e^{-1}$

$f^{(2)}(-1)=e^{-1}$

$f^{(3)}(-1)=e^{-1}$

$\boxed{x=-1}$

$$e^x \approx e^{-1} + \frac{e^{-1}}{1!}(x+1) + \frac{e^{-1}}{2!}(x+1)^2 + \frac{e^{-1}}{3!}(x+1)^3 + \cdots$$

$$e^x \approx e^{-1}\left[1 + (x+1) + \frac{1}{2!}(x+1)^2 + \frac{1}{3!}(x+1)^3 + \cdots\right]$$

7. Compute the Taylor series, around $x = 0$, for $\log(1+x)$ and $\log(1-x)$. $\log_e(1+x)$

Hence obtain a Taylor series for $f(x) = \log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x)$

power series table

$\log_e(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \cdots$

Derive $\log_e(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 - \frac{1}{6}x^6 + \cdots$

$\log\left(\frac{1+x}{1-x}\right) = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \cdots$

$= 2\left[x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \cdots + \frac{1}{2n+1}x^{2n+1}\right]$

$f(x) = \log(1+x)$

$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$

$f^{(2)}(x) = -(1+x)^{-2}$

$f'''(x) = +2(1+x)^{-3}$

Linear Approximation $L(x) = T_1(x)$

8. Write down the linear approximation to $f(x) = \sqrt{1+x}$ at $x = 0$. Use this to find an approximation for $f(x)$ when $x = 1$. Is this a reasonable approximation for $\sqrt{2}$? Explain.

$x=0$

$f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}}$ $f(0) = 1$

$f^{(1)}(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}} = \frac{1}{2\sqrt{1+x}}$ $f^{(1)}(0) = \frac{1}{2}$

$T_1(x) = \sqrt{1+x} \approx 1 + \frac{\frac{1}{2}}{1!}x$ $\boxed{L(x) = 1 + \frac{1}{2}x}$

$\sqrt{1+x} \approx 1 + \frac{1}{2}x$ (at $x = 0$)

when $x=1$, L.H.S $\sqrt{1+1} = \sqrt{2} \approx 1.4142\ldots$

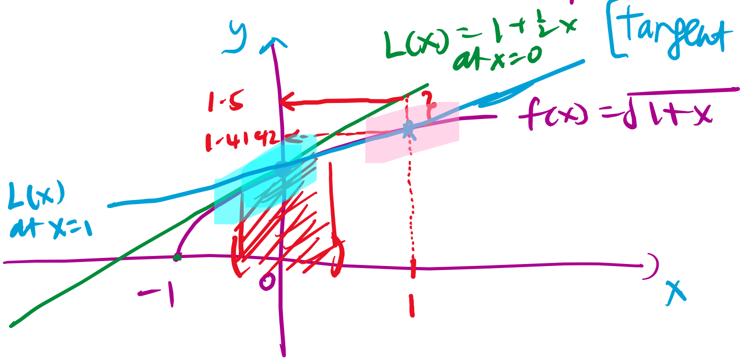
≈ 1.41 (2dp)

R.H.S $1 + \frac{1}{2}(1) = 1.5$

≈ 1.50 (2dp)   Difference $= 0.09$

Not reasonable. Difference is not small enough correct to 2 dp. at $x=1$

[tangent line at $x=0$ for $f(x)$]



Additional information $f(x) = \sqrt{1+x}$

At $x=1$: $L(x) = T_1(x) = f(1) + \frac{f^{(1)}(1)}{1!}(x-1)$

$x = 1$: $f(1) = \sqrt{2}$, $f^{(1)}(1) = \frac{1}{2\sqrt{2}}$

$L(x) = T_1(x) = \sqrt{2} + \frac{1}{2\sqrt{2}}(x-1)$

at $x=1$: $\sqrt{1+x} \approx \sqrt{2} + \frac{\sqrt{2}}{4}x - \frac{\sqrt{2}}{4}$

$T(x) = \frac{3}{4}\sqrt{2} + \frac{\sqrt{2}}{4}x$ $\Rightarrow$ estimate $x=1$. $\sqrt{2} = \frac{3}{4}\sqrt{2} + \frac{\sqrt{2}}{4} = \sqrt{2}$

9. Write down the linear approximation to $f(x) = \sin(x)$ at $x = 0$. Sketch the graphs of $f(x)$ and $L(x)$ (the linear approximation) on the same set of axes. Is $L(x)$ a reasonable approximation for $f(x) = \sin(x)$? Explain.

$f(x) = \sin(x)$

$f^{(1)}(x) = \cos(x)$

$x = 0$

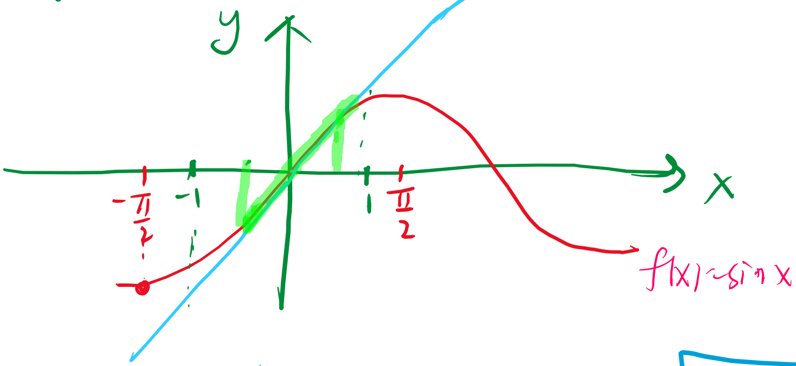
$f(0) = 0$

$f^{(1)}(0) = 1$

$L(x) = 0 + \frac{1}{1!}x$

$\boxed{L(x) = x}$

$\sin(x) \approx x$



L(x) is reasonable for only specific Domain eg. $[-\frac{1}{2}, \frac{1}{2}]$ only example $[-\frac{1}{4}, \frac{1}{4}]$

Cubic Splines

10a. Find the cubic spline approximation for the function $f(x) = x + \frac{1}{x}$, using the points on the graph of $f(x)$ corresponding to $x_1 = \frac{1}{2}$, $x_2 = 1$, $x_3 = \frac{3}{2}$ and $x_4 = 2$.

10b. Check that the following conditions are met for the three cubic equations found above:

i. Interpolation condition: $y_i = \tilde{y}_i(x_i)$.

ii. Continuity of the function: $\tilde{y}_{i-1}(x_i) = \tilde{y}_i(x_i)$.

iii. Continuity of the first derivative: $\tilde{y}'_{i-1}(x_i) = \tilde{y}'_i(x_i)$.

iv. Continuity of the second derivative: $\tilde{y}''_{i-1}(x_i) = \tilde{y}''_i(x_i)$.

#10 (a) $f(x) = x + \frac{1}{x}$ $y_1'' = y_4'' = 0$ ✓

i	x_i	$y_i = f(x_i)$	$h_i = x_{i+1} - x_i$
1	$\frac{1}{2}$	$5/2$	$h_1 = x_2 - x_1 = \frac{1}{2}$
2	1	2	$h_2 = x_3 - x_2 = 1/2$
3	$3/2$	$13/6$	$h_3 = x_4 - x_3 = 1/2$
4	2	$5/2$	

$a_1 = \frac{-4}{3}$ $b_1 = 0$ $c_1 = 4/3$

$a_2 = -\frac{1}{3}$ $b_2 = 2$ $c_2 = -4/3$

$a_3 = 2/3$ $b_3 = 0$ $c_3 = 0.$

Eqn (10) $i=2$, $6\left(\frac{\frac{13}{6} - 2}{1/2} - \frac{2 - 5/2}{1/2}\right) = \frac{1}{2}y_3'' + 2(1)y_2'' + \frac{1}{2}y_1''$

$$16 = y_3'' + 4y_2'' \quad — (1)$$

$i=3$, $6\left(\frac{5/2 - 13/6}{1/2} - \frac{13/6 - 2}{1/2}\right) = \frac{1}{2}y_4'' + 2(1)y_3'' + \frac{1}{2}y_2''$

$$4 = 4y_3'' + y_2'' \quad — (2)$$

Solve (1) & (2) $\Rightarrow$ $y_2'' = 4$, $y_3'' = 0$

$\Rightarrow$ We need to find a_i, b_i & c_i for $i = 1, 2, 3$.

Eqn (9)

$i=1$, $a_1 = \frac{2 - 5/2}{1/2} - \frac{1}{12}(4+0) = -4/3$

$i=2$, $a_2 = \frac{13/6 - 2}{1/2} - \frac{1}{2}(0+8) = -1/3$

$i=3$, $a_3 = \frac{5/2 - 13/6}{1/2} - \frac{1}{2}(0+0) = 2/3$

Eqn (8) $b_i = \frac{y_i''}{2}$

$i=1$, $b_1 = \frac{0}{2} = 0$

$i=2$, $b_2 = \frac{4}{2} = 2$

$i=3$, $b_3 = \frac{0}{2} = 0$

Eq (7) $i=1$, $c_1 = \frac{4-0}{3} = 4/3$

$i=2$, $c_2 = \frac{0-4}{3} = -4/3$.

$i=3$, $c_3 = \frac{0-0}{3} = 0$

Putting those coefficients, three cubic splines:

$$\tilde{y}_1(x) = \frac{5}{2} - \frac{4}{3}\left(x-\frac{1}{2}\right) + 0\left(x-\frac{1}{2}\right)^2 + \frac{4}{3}\left(x-\frac{1}{2}\right)^3 \qquad \frac{1}{2} \le x \le 1$$

$$\tilde{y}_2(x) = 2 - \frac{1}{3}(x-1) + 2(x-1)^2 - \frac{4}{3}(x-1)^3 \qquad 1 \le x \le \frac{3}{2}$$

$$\tilde{y}_3(x) = \frac{13}{6} + \frac{2}{3}\left(x-\frac{3}{2}\right) + 0\left(x-\frac{3}{2}\right)^2 + 0\left(x-\frac{3}{2}\right)^3 \qquad \frac{3}{2} \le x \le 2$$

10(b) (i) interpolation condition: $y_i = \hat{y}_i(x_i)$

i	x_i	$y_i = f(x_i)$
1	$\frac{1}{2}$	$5/2$
2	1	2
3	$3/2$	$13/6$
4	2	$5/2$

$\tilde{y}_1(x_1) = \tilde{y}_1(\frac{1}{2}) = \frac{5}{2} = y_1$

$\hat{y}_2(x_2) = \hat{y}_2(1) = 2 = y_2$ Hence the interpolation condition is met.

$\tilde{y}_3(x_3) = \tilde{y}_3(\frac{3}{2}) = \frac{13}{6} = y_3$

(ii) Continuity of the function $\tilde{y}_{i-1}(x_i) = \hat{y}_i(x_i)$

for $i=2$, $\tilde{y}_1(x_2) = \hat{y}_1(1) = \frac{5}{2} - \frac{2}{3} + \frac{1}{6} = 2 = \hat{y}_2(x_2)$

for $i=3$, $\hat{y}_2(x_3) = \hat{y}_2(\frac{3}{2}) = 2 - \frac{1}{6} + \frac{1}{2} - \frac{1}{6} = \frac{13}{6} = \hat{y}_3(x_3)$

Hence, the continuity of function $\tilde{y}_{i-1}(x_i) = \hat{y}_i(x_i)$ is met

(iii) Continuity of the first derivative $\hat{y}'_{i-1}(x_i) = \tilde{y}'_i(x_i)$

$\tilde{y}_1' = -\frac{4}{3} + 4(x-\frac{1}{2})^2 \qquad \tilde{y}_2' = -\frac{1}{3} + 4(x-1) - 4(x-1)^2 \qquad \tilde{y}_3' = \frac{2}{3}$

$i=2$ $\tilde{y}_1'(x_2) = \tilde{y}_1'(1) = -\frac{4}{3} + 4(\frac{1}{4}) = -\frac{1}{3}$

$\tilde{y}_2'(x_2) = \hat{y}_2'(1) = -\frac{1}{3}$. $\Rightarrow$ met!

$i=3$ $\tilde{y}_2'(x_3) = \tilde{y}_2'(\frac{3}{2}) = -\frac{1}{3} + 4(\frac{1}{2}) - 4(\frac{1}{4}) = \frac{2}{3}$

$\tilde{y}_3'(x_3) = \tilde{y}_3'(\frac{3}{2}) = \frac{2}{3}$

(iv) Continuity for 2nd derivative $y''_{i-1}(x_i) = y''_i(x_i)$ $\Rightarrow$ met!

$\tilde{y}_1'' = 8(x-\frac{1}{2})$

$\tilde{y}_2'' = 4 - 8(x-1)$

$\tilde{y}_3'' = 0$

$i=2$

$\tilde{y}_1''(x_2) = \tilde{y}_1''(1) = 4$

$y_2''(x_2) = y_2''(1) = 4$

$i=3$

$\tilde{y}_2''(x_3) = \tilde{y}_2''(\frac{3}{2}) = 4 - 8(\frac{3}{2} - 1) = 0$

$\tilde{y}_3''(x_3) = \tilde{y}_3''(\frac{3}{2}) = 0$