Monash University
Faculty of Information Technology
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FIT2014
Solutions for Exercises 2
Quantifiers, Games, Proofs

1.

(a) $\mathsf{CrossesToMove}(P) \ \wedge\ \exists X : \mathsf{CrossesWins}(\,\mathsf{ResultingPosition}(P, X)\,)$
or $\exists X : \mathsf{CrossesToMove}(P) \ \wedge\ \mathsf{CrossesWins}(\,\mathsf{ResultingPosition}(P, X)\,)$

(b) $\mathsf{NoughtsToMove}(P) \ \wedge\ \exists X : \mathsf{NoughtsWins}(\,\mathsf{ResultingPosition}(P, X)\,)$
or $\exists X : \mathsf{NoughtsToMove}(P) \ \wedge\ \mathsf{NoughtsWins}(\,\mathsf{ResultingPosition}(P, X)\,)$

(c) The statement (a) is False, because it is not Crosses' turn. The statement (b) is True, because Noughts can play in the top left cell to complete a line.

(d)

$$\begin{aligned}\forall P \ : \ (\ &(\mathsf{CrossesToMove}(P) \ \Rightarrow\ \forall X \neg \mathsf{NoughtsWins}(\mathsf{ResultingPosition}(P, X)))\\ &\wedge (\mathsf{NoughtsToMove}(P) \ \Rightarrow\ \forall X \neg \mathsf{CrossesWins}(\mathsf{ResultingPosition}(P, X)))\ )\end{aligned}$$

(e) $\mathsf{CrossesToMove}(P) \ \wedge\ \exists X\ \forall Y\ \exists Z : \mathsf{CrossesWins}(\,\mathsf{ResultingPosition}(P, X, Y, Z)\,)$

(f) $\neg\,\mathsf{CrossesToMove}(P) \ \vee\ \forall X\ \exists Y\ \forall Z : \neg\,\mathsf{CrossesWins}(\,\mathsf{ResultingPosition}(P, X, Y, Z)\,)$

(g)

$$\exists X_1\, \forall X_2\, \exists X_3\, \forall X_4\, \exists X_5\, \forall X_6\, \exists X_7\, \forall X_8\, \exists X_9 \ : \ \mathsf{CrossesWins}(\,\mathsf{ResultingPosition}(P_0, X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9)\,)$$

(h)

$$\forall X_1\, \exists X_2\, \forall X_3\, \exists X_4\, \forall X_5\, \exists X_6\, \forall X_7\, \exists X_8\, \forall X_9 \ : \ \mathsf{NoughtsWins}(\,\mathsf{ResultingPosition}(P_0, X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9)\,)$$

The following is also correct (why?):[1]

$$\forall X_1\, \exists X_2\, \forall X_3\, \exists X_4\, \forall X_5\, \exists X_6\, \forall X_7\, \exists X_8 \ : \ \mathsf{NoughtsWins}(\,\mathsf{ResultingPosition}(P_0, X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)\,)$$

(i)

$$\begin{aligned}&(\,\exists X_1\, \forall X_2\, \exists X_3\, \forall X_4\, \exists X_5\, \forall X_6\, \exists X_7\, \forall X_8\, \exists X_9 \ : \ \neg\,\mathsf{NoughtsWins}(\,\mathsf{ResultingPosition}(P_0, X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9)\,)\,)\\ \wedge\ &(\,\forall X_1\, \exists X_2\, \forall X_3\, \exists X_4\, \forall X_5\, \exists X_6\, \forall X_7\, \exists X_8\, \forall X_9 \ : \ \neg\,\mathsf{CrossesWins}(\,\mathsf{ResultingPosition}(P_0, X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9)\,)\,)\end{aligned}$$

In fact, using some basic properties of the game, it's possible to show that the second part of the above solution is sufficient:[2]

$$\forall X_1\, \exists X_2\, \forall X_3\, \exists X_4\, \forall X_5\, \exists X_6\, \forall X_7\, \exists X_8\, \forall X_9 \ : \ \neg\,\mathsf{CrossesWins}(\,\mathsf{ResultingPosition}(P_0, X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9)\,)$$

[1] Thanks to FIT2014 tutor Zhi Hao Tan for spotting an error in an earlier version of (h)&(i) and helping correct it.
[2] Thanks to FIT2014 tutor Nathan Companez for spotting and correcting an error in an earlier version of this note to the solution of part (i).

Why is that?

(j)

$$\exists X_1\,\exists X_2\,\exists X_3\,\exists X_4\,\exists X_5\,\exists X_6\,\exists X_7\,\exists X_8\,\exists X_9 \;:\; \mathsf{NoughtsWins}(\,\mathsf{ResultingPosition}(P_0, X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9)\,)$$

In fact, if Noughts wins at all, then it must win on the eighth move (i.e., Noughts's fourth move), because the last move (X_9) is by Crosses and that move cannot undo a line of Noughts already formed. So an alternative solution is:[3]

$$\exists X_1\,\exists X_2\,\exists X_3\,\exists X_4\,\exists X_5\,\exists X_6\,\exists X_7\,\exists X_8 \;:\; \mathsf{NoughtsWins}(\,\mathsf{ResultingPosition}(P_0, X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)\,)$$

Furthermore, for Noughts to win from P_0, it is sufficient to create a line of three Noughts while no line of three crosses is created, and also every line of three Noughts is part of a position which has three Crosses such that those Crosses are *not* in a line. Therefore, it is possible for Noughts to win if and only if it is possible for Noughts to win within six moves (i.e., three moves each). Therefore another correct solution is:

$$\exists X_1\;\exists X_2\;\exists X_3\;\exists X_4\;\exists X_5\;\exists X_6 \;:\; \mathsf{NoughtsWins}(\,\mathsf{ResultingPosition}(P_0, X_1, X_2, X_3, X_4, X_5, X_6)\,)$$

The justification for six-move solution makes use of some more detailed properties of Noughts and Crosses. The justification for the eight-move solution only uses the property that it is never a *disadvantage* to move in this game. (In other words, it would never be advantageous to *pass*, if that were allowed.) The nine-move solution uses no specific properties of the game at all, other than that it finishes in at most nine moves. It was all we were looking for in this question, as the focus is on predicate logic, quantifiers, and the relationship between quantifiers and assertions about moves in games.

2. Assume, by way of contradiction, that there exists a nonempty hereditary language that does not contain the empty string. Let L be such a language.

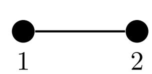
Since L is nonempty, it contains at least one string, and therefore it contains a shortest string. Let x be a shortest string in L. Since L does not contain the empty string (by assumption), x cannot be empty. So it has length ≥ 1, and therefore contains some letters.

Since L is hereditary, some string x^- obtained from x by deleting one letter of x must also belong to L. But this gives a member of L which is shorter than the shortest possible string in L. This is a contradiction. So our assumption, that there exists a nonempty hereditary language that does not contain the empty string, must be wrong. Therefore every nonempty hereditary language contains the empty string.

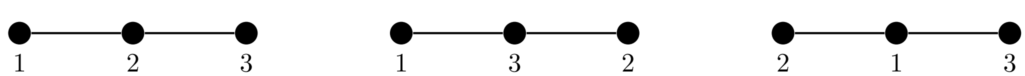
3.

(a)

The only tree on one vertex is: 

The only tree on two vertices is: 

There are three trees on three vertices:



[3] Thanks to FIT2014 tutor Nhan Bao Ho for this observation and the next one.

(b) Inductive basis: when $n = 1$, we have $t_1 = 1$ and $(n-1)! = (1-1)! = 0! = 1$, so the claim holds.

(c) Given a tree T on n vertices, we can form a tree on $n+1$ vertices by taking a new vertex labelled $n+1$ and joining it by an edge to any one of the vertices of T. Since T has n vertices, this can be done in n different ways.

Note that the new vertex becomes a leaf in the new tree.

(d) No. Two different trees on n vertices cannot give the same tree on $n+1$ vertices by this construction, because the tree on n vertices is always preserved within the larger tree, so that the two larger trees must be different too.

(e) Since each tree on n vertices gives rise to n different trees on $n+1$ vertices, and since no tree on $n+1$ vertices can arise from two different trees on n vertices, we have

$$t_{n+1} \geq n\,t_n\,.$$

(f)

$$\begin{aligned} t_{n+1} &\geq n\,t_n \qquad \text{(by part (e) above)} \\ &\geq n\,(n-1)! \qquad \text{(by the Inductive Hypothesis)} \\ &= n!\,. \end{aligned}$$

So the claim holds for n.

This completes the Inductive Step (i.e., going from n to $n+1$).

(g) So, by the Principle of Mathematical Induction, it is true <u>for all n</u> that $t_n \geq (n-1)!$.

In fact, the number of trees is much larger than this! The exact formula for the number of labelled trees on n vertices is n^{n-2}. This is known as **Cayley's formula** after the English mathematician Arthur Cayley (1821–1895). There is a great variety of beautiful proofs of this important formula.

4. Base case: $n = 1$:

The tree with one vertex has zero edges, which is $1 - 1$, so the claim is true for $n = 1$.

Inductive step:

Let $k \geq 1$.

Assume that any tree with k vertices has $k - 1$ edges.

Let T be any tree with $k + 1$ vertices.

Now, every tree with ≥ 2 vertices has a leaf, and removing any leaf from a tree gives another tree with one fewer vertex and one fewer edge.

So, remove a leaf from T. Let T^- be the smaller tree so obtained. It has k vertices, so we can apply the Inductive Hypothesis to it. This tells us that T^- has $k-1$ edges. Since we only deleted one edge when we deleted the leaf, this implies that T has $(k-1)+1$ edges, i.e., k edges. This is one fewer than its number of vertices. This completes the inductive step.

Therefore, by Mathematical Induction, it is true that, for all n, every tree on n vertices has $n-1$ edges.

5.

Base case: $n = 3$:

$3! = 6$, while $(3-1)^3 = 2^3 = 8$, so the inequality is true for $n = 3$.

Inductive Step:

Suppose that $n! \leq (n-1)^n$ is true for a particular number n, where $n \geq 3$.

Let's look at what happens at $n+1$.

$$\begin{aligned}
(n+1)! &= (n+1)\cdot n! \qquad \text{(to express it in terms of a smaller case)} \\
&\le (n+1)\cdot(n-1)^n \qquad \text{(by the Inductive Hypothesis, i.e., } n! \le (n-1)^n\text{)} \\
&= (n+1)(n-1)(n-1)^{n-1} \qquad \text{(a slight rearrangement } \ldots \\
&\qquad\qquad \ldots \text{we are hoping to get } n+1 \text{ factors, all } \le n \ldots\text{)} \\
&= (n^2-1)(n-1)^{n-1} \qquad \text{(a little high-school algebra } \ldots\text{)} \\
&< n^2(n-1)^{n-1} \qquad \text{(and now we \textit{do} have } n+1 \text{ factors, all } \le n\text{)} \\
&< n^{n+1} \\
&= ((n+1)-1)^{n+1}.
\end{aligned}$$

This last line is just to make it clear that the expression is of the required form.

So, by Mathematical Induction, it is true <u>for all $n \ge 3$</u> that $n! \le (n-1)^n$.

Supplementary exercises

6. Inductive basis ($n = 1$): P_1 is just x_1, so if $x_1 = \mathrm{F}$, then P_1 is immediately False. So the statement holds in this case.

Inductive step:

Let $k \ge 1$. Assume that, if $x_1 = \mathrm{F}$, then P_k is False (the Inductive Hypothesis).

Now, if $x_1 = \mathrm{F}$, then we have

$$\begin{aligned}
P_{k+1} &= P_k \wedge x_{k+1} \qquad \text{(as noted in the question)} \\
&= \mathrm{F} \wedge x_{k+1} \qquad \text{(by the Inductive Hypothesis)} \\
&= \mathrm{F},
\end{aligned}$$

which completes the inductive step.

Therefore the statement holds for all n, by Mathematical Induction.

7.

(a) k-th odd number $= 2k-1$

(b) Inductive basis: when $k = 1$, the sum of the first k odd numbers is just the first odd number, 1, which equals 1^2, so it equals k^2.

(c)

$$\begin{aligned}
&\text{Sum of the first } k+1 \text{ odd numbers} \\
&= 1 + 3 + \cdots + ((k+1)\text{-th odd number}) \\
&= (1 + 3 + \cdots + (k\text{-th odd number})) + ((k+1)\text{-th odd number}) \\
&= (\text{sum of the first } k \text{ odd numbers}) + ((k+1)\text{-th odd number}) \\
&= (\text{sum of the first } k \text{ odd numbers}) + 2(k+1) - 1 \\
&\qquad\qquad \text{(using our formula from part (a))}
\end{aligned}$$

(d) Continuing from above,

$$\begin{aligned} &\dots \\ &= \quad k^2 + 2(k+1) - 1 \qquad\qquad \text{(by the Inductive Hypothesis)} \end{aligned}$$

(e) Continuing from above,

$$\begin{aligned} &\dots \\ &= \quad k^2 + 2k + 1 \\ &= \quad (k+1)^2. \end{aligned}$$

This completes the Inductive Step (i.e., going from k to $k+1$).

(f) So, by the Principle of Mathematical Induction, it is true <u>for all k</u> that the sum of the first k odd numbers is k^2.

8. (a)

(b)

Inductive basis:

Suppose $n \leq 100$. Applying `wc` to standard output of this length gives a one-line standard output stating the numbers of lines, words and characters, with the number of characters being n. This output has 1 line, 3 (or 4?) words and some small number of characters consisting of the digit 1, the digit 3, a couple of digits (at most) for n, and some number of spaces (say, 21 altogether, but the analysis is much the same if this number is different). Applying `wc` again gives one line with these numbers in it: 1, 3, 25, again alongside 21 spaces. Another application of `wc` gives the same result. So the claim is true for $n \leq 99$.

Inductive step:

Now suppose the claim is true when the file/string has $\leq k$ characters, where $k \geq 100$. Suppose we are given a file/string of $k+1$ characters. Applying `wc` gives a one-line standard output, giving numbers of lines, words and characters as l, w and $k+1$, respectively, set out something like

$$l \qquad w \qquad k+1$$

For any number x, write digits(x) for the number of digits in x. Our one-line output has some number s of spaces, say $s = 21$; the number of non-space characters is

$$\text{digits}(l) + \text{digits}(w) + \text{digits}(k+1).$$

Now, $l \leq k+1$ and $w \leq k+1$.[4] Therefore the number of characters in the one-line output above is $\leq s + 3 \cdot \text{digits}(k+1)$. We claim this is $\leq k$ if k is large enough.

To see this, first try $k = 100$. Then

$$\begin{aligned} s + 3 \cdot \text{digits}(k+1) &= 21 + 3 \cdot \text{digits}(100+1) \\ &= 21 + 3 \cdot \text{digits}(101) \\ &= 21 + 3 \cdot 3 \\ &= 30, \end{aligned}$$

which is indeed $\leq k$. (Only minor changes are needed here if the actual value of s is not 21; it will not be *much* different from 21.) Now, whenever k increases by 1, digits($k+1$) either stays the same or increases by 1. But it only increases rarely, when $k+1$ becomes 100, then when it becomes 1000, and so on. So, given that $s + 3 \cdot \text{digits}(k+1) \leq k$ for $k = 100$ (and by a good margin), this inequality continues to hold for all higher k.[5]

[4] In fact, $w \leq (k+1)/2$, since every consecutive pair of words must have at least one space between them. But we don't need this better upper bound on w.

[5] Thanks to FIT2014 tutor Nathan Companez for part of this argument.

Since this is now $\leq k$, the Inductive Hypothesis tells us that some further applications of `wc` will eventually give constant output.

Therefore the result follows for all n, by the Principle of Mathematical Induction.

9.

(i) Inductive basis: $H_1 = 1 = \log_e e > \log_e 2$, using the fact that $e > 2$.

(ii) Our inductive hypothesis is that $H_n \geq \log_e(n+1)$ is true for n. We need to use this to show that n can be replaced by $n+1$ in this inequality, i.e., $H_{n+1} \geq \log_e((n+1)+1)$.

$$\begin{aligned}
H_{n+1} &= 1 + \frac{1}{2} + \cdots + \frac{1}{n+1} && \text{First, we need to relate this to } H_n. \\
&= \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) + \frac{1}{n+1} \\
&= H_n + \frac{1}{n+1} && \text{We've now expressed } H_{n+1} \text{ in terms of } H_n\,. \\
& && \text{So we can apply the Inductive Hypothesis.} \\
&\geq \log_e(n+1) + \frac{1}{n+1} && \text{by the Inductive Hypothesis} \\
&\geq \log_e(n+1) + \log_e\left(1 + \frac{1}{n+1}\right) && \text{using } \log_e(1+x) \leq x, \text{ with } x = \tfrac{1}{n+1} \\
&= \log_e(n+1) + \log_e\left(\frac{n+2}{n+1}\right) \\
&= \log_e\left((n+1) \cdot \frac{n+2}{n+1}\right) \\
&= \log_e(n+2) \\
&= \log_e((n+1)+1).
\end{aligned}$$

So, we've shown that, if the claimed inequality holds for n, then it holds for $n+1$.

(iii) By the Principle of Mathematical Induction, the claimed inequality must hold for all n.

Further properties of the harmonic numbers, and their applications in computer science, may be found in:

- Donald E. Knuth, *The Art of Computer Programming. Volume 1: Fundamental Algorithms. Third Edition.* Addison-Wesley, Reading, Ma., USA, 1997. See Section 1.2.7, pp. 75–79.

Comments:

The n-th harmonic number H_n is close to $\log_e n$, in fact they differ by < 1. The amount by which they differ is denoted by γ and is known as *Euler's constant.* Its value is $0.57721566\ldots$. It is not yet known whether or not this number is rational.

To see why H_n should be so closely related to $\log_e n$, we compare the *sum* H_n to the *integral* $\int_1^{n+1} \frac{1}{x}\,dx$. This is the area under the curve $y = \frac{1}{x}$ between the vertical lines $x = 1$ and $x = n+1$. Roughly speaking,

$$\sum_{i=1}^{n} \frac{1}{i} \approx \int_1^{n+1} \frac{1}{x}\,dx.$$

Now, the derivative of $1/x$ is $\log_e(x)$, so the integral is

$$\int_1^{n+1} \frac{1}{x}\,dx = \log_e(n+1) - \log_e 1 = \log_e(n+1).$$

So

$$\sum_{i=1}^{n} \frac{1}{i} \;\approx\; \log_e(n+1).$$

10.

Inductive basis:

When $n = 1$, the formula gives

$$\begin{aligned}
F_1 &= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right) \\
&= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}-1+\sqrt{5}}{2}\right) \\
&= \frac{1}{\sqrt{5}} \cdot \frac{2\sqrt{5}}{2} \\
&= 1.
\end{aligned}$$

When $n = 2$, the formula gives

$$\begin{aligned}
F_2 &= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right) \\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right) + 1 - \left(\left(\frac{1-\sqrt{5}}{2}\right) + 1\right)\right) \\
&\quad \text{(using the fact that, for } x = (1 \pm \sqrt{5})/2\text{, we know } x^2 - x - 1 = 0\text{, i.e., } x^2 = x + 1\text{)} \\
&= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} + 1 - 1\right) \\
&= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}-1+\sqrt{5}}{2}\right) \\
&= \frac{1}{\sqrt{5}} \cdot \frac{2\sqrt{5}}{2} \\
&= 1.
\end{aligned}$$

Since we have dealt with the cases $n \le 2$, we may now assume $n \ge 3$.

Inductive step:

Suppose that, *for all* $m < n$, the formula holds for F_m. This is our inductive hypothesis. (It is a stronger type of inductive hypothesis than just assuming the formula holds for F_m when $m = n - 1$. But that's ok.)

For convenience, write

$$x = \frac{1+\sqrt{5}}{2}, \qquad y = \frac{1-\sqrt{5}}{2}.$$

Now consider F_n.

$$\begin{aligned}
F_n &= F_{n-1} + F_{n-2} \\
&\quad \text{(using the definition of } F_n \text{ for } n \ge 3\text{)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{5}}\left(x^{n-1}-y^{n-1}\right)+\frac{1}{\sqrt{5}}\left(x^{n-2}-y^{n-2}\right) \\
&\quad \text{(by our inductive hypothesis, applied twice: once with } m=n-1\text{, and once with } m=n-2\text{)} \\
&= \frac{1}{\sqrt{5}}\left(x^{n-1}-y^{n-1}+x^{n-2}-y^{n-2}\right) \\
&= \frac{1}{\sqrt{5}}\left(x^{n-1}+x^{n-2}-(y^{n-1}+y^{n-2})\right) \\
&= \frac{1}{\sqrt{5}}\left(x^{n-2}(x+1)-y^{n-2}(y+1)\right) \\
&= \frac{1}{\sqrt{5}}\left(x^{n-2}x^2-y^{n-2}y^2\right) \\
&\quad \text{(using } x^2=x+1 \text{ and } y^2=y+1\text{)} \\
&= \frac{1}{\sqrt{5}}\left(x^n-y^n\right).
\end{aligned}
$$

So the formula holds for F_n as well.

In summary of the inductive step: we showed that, if the formula for F_m holds when $m < n$, then it holds for $m = n$.

Conclusion: by the Principle of Mathematical Induction, it follows that the formula is true for all values of n.

This exercise illustrates that, although it may be hard to discover the correct formula for some quantity (…surely, the expression for F_n comes as a surprise when you first meet it, and you may wonder how anyone came up with it!), *once you have the formula*, induction is a powerful tool for proving it. In research, we often *discover* a “fact” by an unpredictable process of creative exploration, and then *prove* it by a different technique — often, by induction.

This exercise has also established the connection between the famous Fibonacci numbers and the equally famous “Golden Ratio”, $\varphi = (1+\sqrt{5})/2$.

Extra exercise to give more insight into this connection, for the curious or mathematically inclined: use the above result to prove that the ratio between two successive Fibonacci numbers tends to φ as $n \to \infty$.

Among the many applications of Fibonacci numbers in computer science is the fact that the Euclidean algorithm, for computing GCD, takes longest when the two input numbers are successive Fibonacci numbers.