Chapter 3

Calculus

Reference books

1. G. James, *Modern Engineering Mathematics*, 4th edition, Prentice Hall, 2007.
2. J. Stewart, *Calculus Early Transcendentals*, 7th edition, Cengage Learning, 2012.

3.1 Differentiation

3.1.1 Rate of change

Differentiation is the mathematical method that we use to study the **rate of change** of physical quantities.

Speed $= \frac{\text{Distance}}{\text{Time}}$

Let us look at this with an example. Consider a point P that is moving with a *constant speed* v along a straight line. Let s be the distance moved by the point after time t. The distance moved after time t is given by the formula $s = vt$. If Δt denotes a finite change in time t, the corresponding change of distance is given by $\Delta s = v\Delta t$.

The **rate of change** of s with t is then simply

V: volume

t: time

h: height

$$\frac{\text{change of } s}{\text{change in } t} = \frac{\Delta s}{\Delta t} = v = \text{ average speed over the time interval } \Delta t.$$

* Rate of change in volume over time, $\frac{dV}{dt}$

* Rate of change in volume over height change, $\frac{dV}{dh}$

Suppose now that the speed of P *varies* with time. By making Δt become very small, i.e. taking the limit as $\Delta t \to 0$, we define the **derivative** of s with respect to t at time t as the rate of change of s with respect to t, as $t \to 0$.

If we let $\mathrm{d}s$ and $\mathrm{d}t$ be the **infinitesimal changes** in s and t, then we can write:

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \text{ instantaneous speed of } P \text{ at time } t.$$

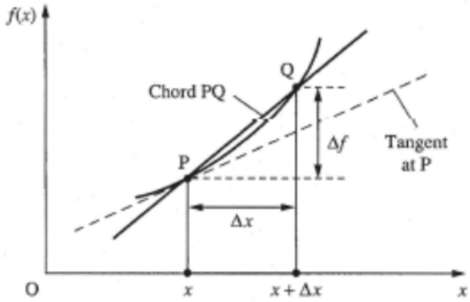
3.1.2 Definition of the derivative $f'(x)$ and the slope of a tangent line

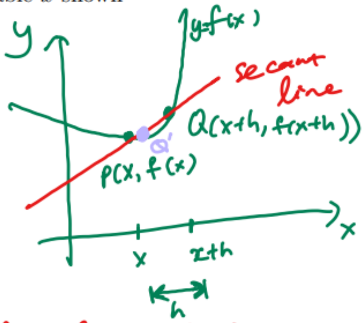
Consider the graph of the function $y = f(x)$ of the single variable x shown in the plot below.

Recall

Gradient (slope)

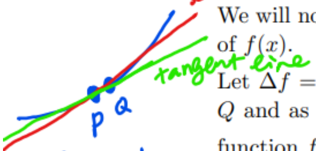
$M = \frac{RISE}{RUN} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$





$M_{PQ} = \frac{f(x+h) - f(x)}{(x+h) - x}$

$M_{PQ} = \frac{f(x+h) - f(x)}{h}$



We will now compare the *average rate of change* of $f(x)$ with the *derivative* of $f(x)$.

Let $\Delta f = f(x + \Delta x) - f(x)$ be the change in f as we go from point P to Q and as x changes from x to $x + \Delta x$. The *average rate of change* of the function $f(x)$ on the interval Δx is $\frac{\Delta f}{\Delta x}$. This is the slope of the chord PQ.

gradient to the curve at P $\approx M_{PQ}$

The **derivative** of $f(x)$ at the point x is defined as

$$\frac{\mathrm{d}f}{\mathrm{d}x} = f'(x) = \lim_{\Delta x \to 0} \left(\frac{\Delta f}{\Delta x} \right) = \lim_{\Delta x \to 0} \left(\frac{f(x + \Delta x) - f(x)}{\Delta x} \right).$$

Let $f'(x)$ = gradient to curve at P (tangent line of curve at P)

The derivative is the **slope** (or **gradient**) of the local tangent line to the curve $y = f(x)$ at the point P. That is, $f'(x) = \tan\theta$, where θ is the angle between the two dashed lines.

$f'(x) \approx \frac{f(x+h) - f(x)}{h}$

By consider taking limit of h tends to zero,

$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$

1st principle of derivative

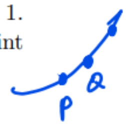
$$\boxed{\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1}$$

The derivative $f'(x)$ is thus the **instantaneous rate of change** of f with respect to x at the point P.

$\frac{d}{dx}x^n = nx^{n-1}$

$f(x) = 25x - 5x^2$ $\quad \boxed{f'(x) = 25 - 10x}$

$f'(x) \Rightarrow$ gradient function



Example 3.1. Use the definition of the derivative to obtain **from first principles** the value of $f'(x)$ for the function $f(x) = 25x - 5x^2$ at $x = 1$. Find the equation of the tangent line to the graph of $y = f(x)$ at the point $(1, 20)$ in the xy-plane.

Solution:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x\to 0}\left(\frac{f(x+\Delta x) - f(x)}{\Delta x}\right)$$

$f(x+\Delta x) = 25(x+\Delta x) - 5(x+\Delta x)^2$

$= \cancel{25x} + 25(\Delta x) - \cancel{5x^2} - 10x(\Delta x) - 5(\Delta x)^2$

$f(x) = \cancel{25x} - \cancel{5x^2}$

$f(x+\Delta x) - f(x) = \Delta x\,[25 - 10x - 5(\Delta x)]$

$$\frac{df}{dx} = \lim_{\Delta x\to 0}\left(\frac{\cancel{\Delta x}\,(25 - 10x - 5(\Delta x))}{\cancel{\Delta x}}\right)$$

$$\frac{df}{dx} = \lim_{\Delta x\to 0}\,[25 - 10x - 5(\Delta x)]$$

$$\frac{df}{dx} = 25 - 10x$$

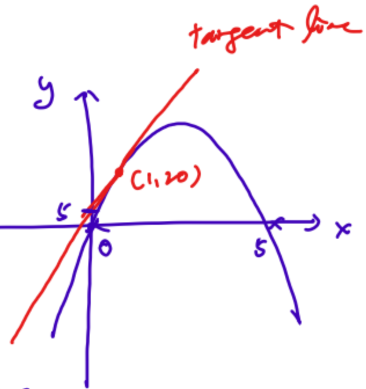
$f(x) = 25x - 5x^2$ at $(1, 20)$

$= 5x(5 - x)$

$f'(x) = 25 - 10x$ (gradient function)

$f'(1) = m_t$ (gradient of tangent line)

$m_t = 15$



<u>Eqn of tangent line</u>

$y - y_1 = m(x - x_1)$ / $y = mx + c$

$y - 20 = 15(x - 1)$

$y = 15x + 5$

3.1.3 Techniques of differentiation - rules

Most mathematical functions are readily differentiable without the need to resort to the first principles definition. It is simply a matter of applying one or more of the **rules of differentiation** which are collected in the following table. It is assumed that c and n are constants.

Notation

$y \to \frac{dy}{dx}$

$f(x) \to f'(x)$

$\frac{d}{dx}(\quad)$

$f(x) = 2x^0$

$f'(x) = 2(0x^{-1})$

Description	Function	Derivative
Constant $y = 2$	$f(x) = c$	$f'(x) = 0$
Power of x	$f(x) = x^n$	$f'(x) = nx^{n-1}$
Multiplication by a constant c	$cf(x)$	$\frac{\mathrm{d}}{\mathrm{d}x}(cf(x)) = cf'(x)$
Sum (or difference) of functions	$f(x) \pm g(x)$	$f'(x) \pm g'(x)$
Product of functions	$f(x)g(x)$	$f(x)g'(x) + g(x)f'(x)$
Quotient of functions	$\frac{f(x)}{g(x)}$	$\frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

Chain rule for composite functions

If $u = g(x)$ and $y = f(u)$ so that $y = f(g(x))$ then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = f'(u)g'(x)$$

Example 3.2. (a) Find the derivative of $f(x) = x^3 + 2x^2 - 5x - 6$ with respect to x.

power rule + constant

$$f'(x) = 3x^2 + 2(2x^1) - 5(1x^0) - 0$$

$$f'(x) = 3x^2 + 4x - 5$$

product rule

(b) Find the derivative of $y = (x^5 + 6x^2 + 2)(x^3 - x + 1)$ with respect to x.

$$\frac{dy}{dx} = (x^5+6x^2+2)\frac{d}{dx}(x^3-x+1) + (x^3-x+1)\frac{d}{dx}(x^5+6x^2+2)$$

$$\frac{dy}{dx} = (x^5+6x^2+2)(3x^2-1) + (x^3-x+1)(5x^4+12x)$$

$$\frac{dy}{dx} = 3x^7 - x^5 + 18x^4 - 6x^2 + 6x^2 - 2 + 5x^7 + 12x^4 - 5x^5 - 12x^2 + 5x^4 + 12x$$

$$= 8x^7 - x^5 + 30x^4 - 12x^2 + 12x - 2$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Quotient Rule

(c) Find the derivative of $f(x) = \frac{x^2+1}{x^2-1}$ with respect to x.

$$f'(x) = \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2}$$

$$f'(x) = \frac{\cancel{2x^3} - 2x - \cancel{2x^3} - 2x}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

(d) Use the chain rule to find $\frac{dy}{dx}$ when ① $\frac{d}{dx}[\ ]^5$ ② $\frac{d}{dx}(2x+3)$

(i) $y = (\underbrace{2x+3}_{②})^5$ ①

$y = (X)^5$

$$\frac{dy}{dx} = \underbrace{5(2x+3)^4}\cdot(2)$$

$$= 10(2x+3)^4$$

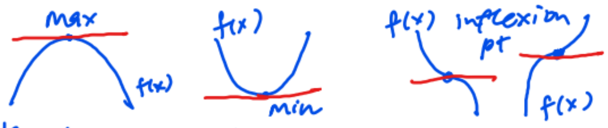
(ii) $y = \sqrt{(3x^2+1)}$

$$y = (3x^2+1)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(3x^2+1)^{-\frac{1}{2}}\cdot(6x)$$

$$= 3x(3x^2+1)^{-\frac{1}{2}}$$

$$= \frac{3x}{\sqrt{3x^2+1}}$$



Slope/Gradient = 0 $[f'(x) = 0] \Rightarrow$ stationary point

Stationary point / Turning point

① Find $\frac{dy}{dx}$ or $f'(x)$

② set $\frac{dy}{dx} = 0$ / $f'(x) = 0$

$\Rightarrow$ solve for x-value(s)

critical value eg. $x = a$

$f'(x) \rightarrow$ gradient function

3.2 Maximum and minimum of functions

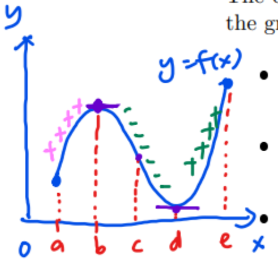
③ Coordinates $(a, f(a))$

④ Determine the nature of turning pt.

method ①

* 1st Derivative Test

* 2nd Derivative Test



The derivative of a function $f(x)$ tells us important information regarding the graph of $y = f(x)$.

- If $f'(x) > 0$ on the interval $[a, b]$ then the function $f(x)$ is **increasing** on that interval.
- If $f'(x) < 0$ on the interval $[a, b]$ then the function $f(x)$ is **decreasing** on that interval.
- If $f'(x) = 0$ on the interval $[a, b]$ then the function $f(x)$ is **constant** on that interval.

m > 0 | m < 0 | m = 0 | m - undefined

$(a,b) \rightarrow f'(x) > 0$

$(b,d) \rightarrow f'(x) < 0$

$(d,e) \rightarrow f'(x) > 0$

$\left.\begin{matrix} x = b \\ x = c \\ x = d \end{matrix}\right\} f'(x) = 0$

> A function $f(x)$ has a **local maximum** at $x = c$ if $f(x) \leq f(c)$ for values of x in some open interval containing c. (Note an 'open' interval is an interval NOT including the end points, i.e. the interval (a, b) rather than $[a, b]$.)
>
> A function $f(x)$ has a **local minimum** at $x = c$ if $f(x) \geq f(c)$ for values of x in some open interval containing c.

Note that the interval endpoints cannot correspond to a local maximum or a local minimum. Can you see why this is so? No increasing / decreasing. $\frac{dy}{dx} = 0$

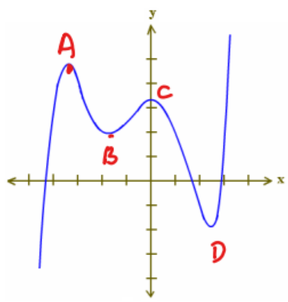
Example 3.3. Identify the local maxima and minima on the graph of the function below.

local maximum

A, C

local minimum

B, D



How do we find the local maxima and minima?

Local maxima and local minima occur where the derivative of the function is zero. They can also occur where the derivative does not exist (consider the function $f(x) = |x|$ at the point $x = 0$. For the function $f(x)$ we define the **extrema**, or **critical points**, as the points $x = c$ such that:

- $f'(c) = 0$, or
- $f'(c)$ does not exist.

It is important to note that $f'(c) = 0$ does not imply that the function $f(x)$ must have a local maximum or minimum at $x = c$. Consider the function $f(x) = x^3$ at $x = 0$ to explore this further. Thus having $f'(c) = 0$ is only a *necessary* requirement, rather than a *sufficient* requirement for the existence of local maxima or minima.

We can also note the following with regards to the graph of $f(x)$.

- At a point on the graph of the function $f(x)$ corresponding to a local maximum, the function changes from increasing to decreasing.
- At a point on the graph of the function $f(x)$ corresponding to a local minimum, the function changes from decreasing to increasing.

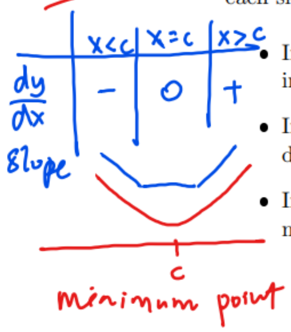
Note that the tangent line (if it exists) is *horizontal* at the point $x = c$ corresponding to either a local maximum or a local minimum.

Set $f'(x) = 0$ solve for x. , $x = c$

The First Derivative Test Critical point $(c, f(c))$

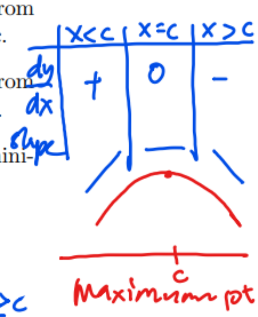
Using this test is simple. All we do is look at the sign of the derivative at each side of the critical point c:

Case I

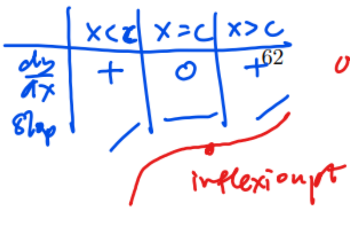


- If $f'(x)$ changes from positive to negative (i.e. $f(x)$ changes from increasing to decreasing) then $f(x)$ has a local maximum at $x = c$.
- If $f'(x)$ changes from negative to positive (i.e. $f(x)$ changes from decreasing to increasing) then $f(x)$ has a local minimum at $x = c$.
- If $f'(x)$ does not change, then $f(c)$ is neither a maximum nor a minimum value for $f(x)$.

Case II



Case III



or

In summary to find the **local extrema** we

- Find all critical points.
- For each critical point, decide whether it corresponds to a local maximum or minimum (or neither) using the First Derivative Test.

Example 3.4. Find the local extrema for the function $f(x) = x^3 - 5x^2 - 8x + 7$ over the interval $\mathbb{R}$.

Solution: First we find the critical points by differentiating the function and solving for x when $f'(x) = 0$.

$f'(x) = 3x^2 - 10x - 8 = 0$

$(3x+2)(x-4) = 0$

$x = -2/3 \;,\; x = 4$

Next we inspect the critical points found above.

when $x = -2/3$

$y = f(-2/3) = \frac{256}{27}$ $\left(-\frac{2}{3}, \frac{256}{27}\right)$

When $x = 4$

$y = f(4) = -41$

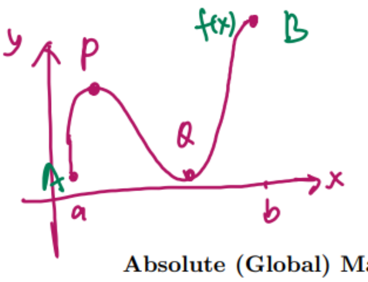
$(4, -41)$

Using the First Derivative Test there is a local maximum at $x = -2/3$ and a local minimum at $x = 4$

$f'(x)$: $(x=-1)$ $+$ 0 at $-2/3$; $x=0$ $-$; 0 at 4; $x=5$ $+$

$+$ 0 $-$ 0 $+$ at $-2/3$, 4: max, min

The corresponding values of the function at the local extrema are $f(4) = -41$ min and $f(-2/3) = \frac{256}{27}$



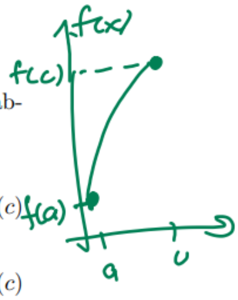
$P \rightarrow$ local max

$Q \rightarrow$ local min.

Absolute maxima.

$B \rightarrow$ Absolute max

$Q \rightarrow$ Absolute min



Absolute (Global) Maximum and Minimum

Since we have been talking about local extrema, we must also mention absolute (global) extrema.

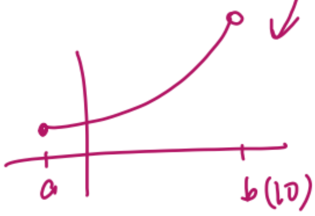
- A function $f(x)$ has an **absolute minimum** at $x = c$ if $f(x) \geq f(c)$ for all x in the domain $[a, b]$ for $a \leq c \leq b$.
- A function $f(x)$ has an **absolute maximum** at $x = c$ if $f(x) \leq f(c)$ for all x in the domain $[a, b]$ for $a \leq c \leq b$.

The **Extreme Value Theorem** states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ then $f(x)$ obtains an absolute maximum and an absolute minimum at some points in the interval.

Note that the interval $[a, b]$ must be a **closed** interval. Why is this necessary?

Example 3.5. What happens with the Extreme Value Theorem if a function $f(x)$ is not continuous?

$f'(b) \Rightarrow$ undefined.



To find the **absolute extrema** of a continuous function on a **closed** interval:

1. Find the values of the function at all critical points in the interval.

2. Find the values of the function at the end points of the interval.

3. Compare all of these for maximum / minimum.

To find the **absolute extrema** of a continuous function over an **open** interval (including $\mathbb{R}$):

1. Find the values of the function at all critical points in the interval.

2. Find the limit of the function as x approaches the endpoints of the interval (or $\pm\infty$).

3. Compare all of these for maximum / minimum.

Example 3.6. Find the absolute extrema for the function $f(x) = e^{-x^2}$. $= \frac{1}{e^{x^2}}$

Given $f'(x) = -2x\, e^{-x^2}$

Find critical x-value, $f'(x) = 0$

$e^{-x^2} \neq 0$, $-2x = 0$

$x = 0$

$y = e^0 = 1$

critical pt $(0, 1)$

$f'(x)$ + 0 −

0

$(0,1)$ is local maximum

open interval $(-\infty, \infty)$

$x \to -\infty$, $f(x) \to 0$
$x \to +\infty$, $f(x) \to 0$ } valid exists

$\Rightarrow$ continuous.

$f(0) > f(x) \quad \forall x \in \mathbb{R}$

$\Rightarrow$ Absolute max $(0, 1)$.