Problem Set Ten: Functions of Several Variables, Partial Derivatives, Tangent Planes

Functions of Several Variables

1. Sketch the graphs of the following functions of several variables:

(a) $f(x, y) = x^2 + y^2 - 2$

(b) $f(x, y) = 2 - x^2 - y^2$

(c) $f(x, y) = (x + 2)^2 + (y - 1)^2$

(d) $f(x, y) = \dfrac{(x+2)^2}{4} + \dfrac{(y-1)^2}{9}$

(a) $c: f(x,y) \rightarrow c: x^2+y^2-2$

$z = f(x,y)$

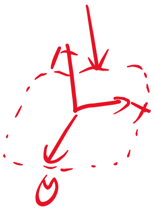
$x^2+y^2 = c+2$

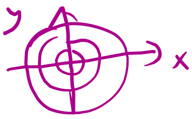
contour curve

$c \geq -2$

$c=0, \; x^2+y^2=2 \Rightarrow r=\sqrt{2}$

$c=2, \; x^2+y^2=4 \Rightarrow r=2$





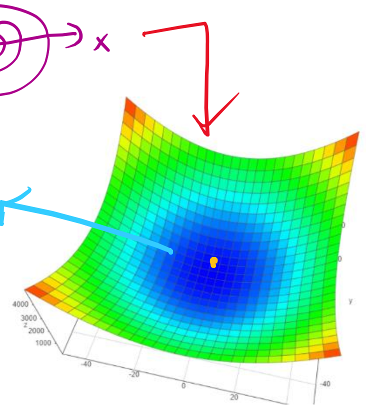
Level curve $f(x,y) = x^2+y^2-2$

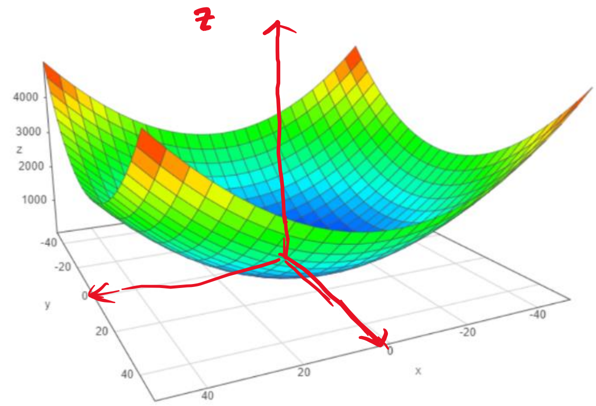
If $y=c$, $f(x) = x^2 + (c^2-2)$

If $x=c$, $f(y) = y^2 + (c^2-2)$

surface is parabolic bowl with base resting at $z=-2$, $(0,0,-2)$

$(0,0,-2)$





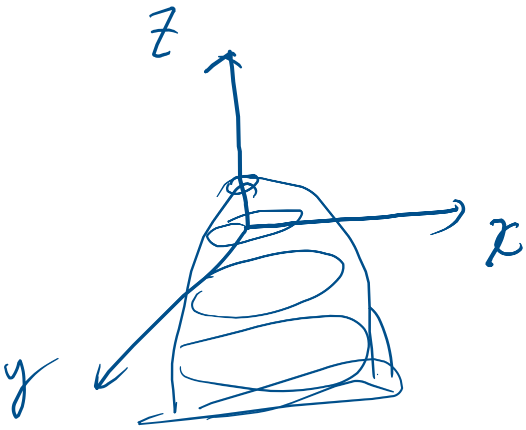
(b) Surface is parabolic bowl hanging at $(0,0,2)$

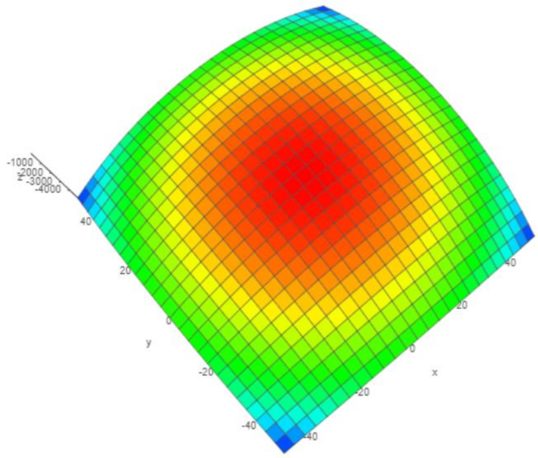
contour curve, $c = 2-x^2-y^2$

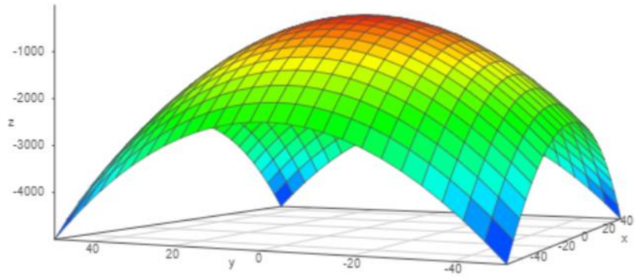
$x^2+y^2 = 2-c$ $\quad c \leq 2$

$c=0$, $r=\sqrt{2}$

$c=-2$, $r=2$







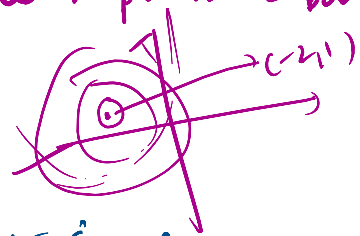
(c) $f(x,y) = (x+2)^2 + (y-1)^2$

$f(x,y) = x^2+y^2 \Rightarrow$ centre at $(0,0,0)$

Surface is parabolic bowl with base resting at $(-2,1,0)$

$x+2=0$
$x=-2$

$y-1=0$
$y=1$



Contour curve

$f(x,y)=c$

$(x+2)^2+(y-1)^2=c$

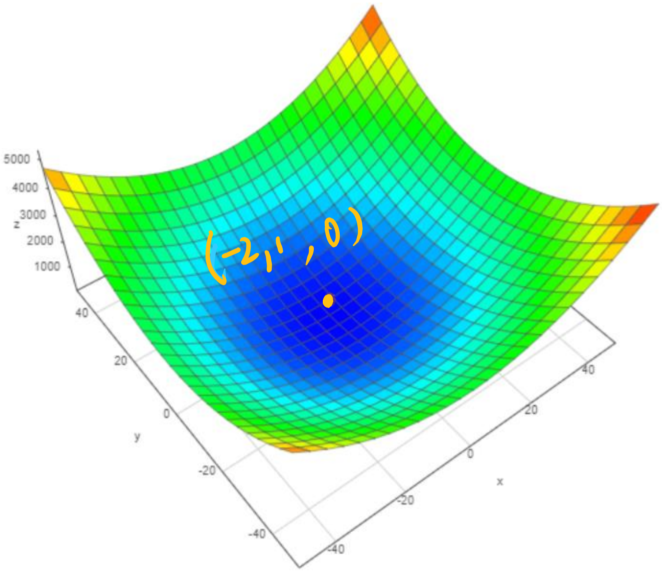
$c=0$, $r=0$

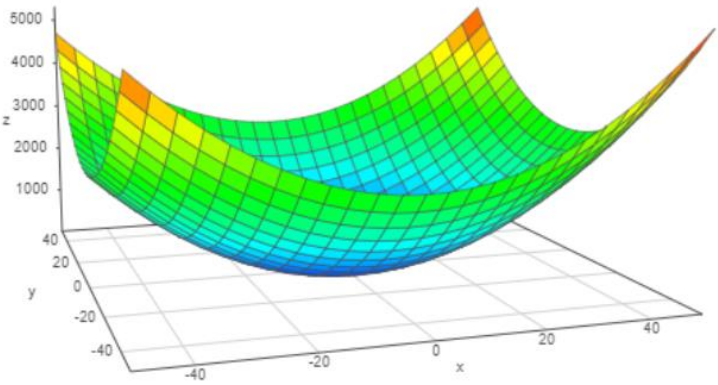
$c=4$, $r=2$ $\Rightarrow$ centre at $(-2,1)$

Level curve

* let $y=c$, $f(x)=(x+2)^2+(c-1)^2$ parabolic curve with turning point $(-2,(c-1)^2)$

* let $x=c$, $f(y)=(y-1)^2+(c+2)^2$ parabola vertex at $(1,(c+2)^2)$





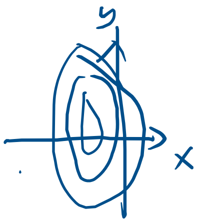
(d) $f(x,y) = \frac{(x+2)^2}{4} + \frac{(y-1)^2}{9}$

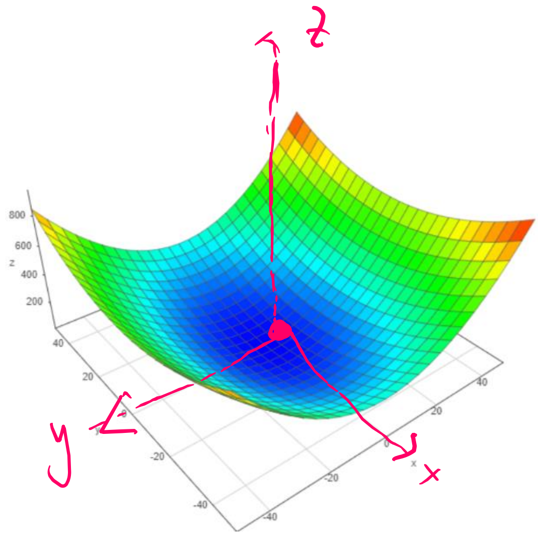
Contour curve

Let $f(x,y) = c$

$$\frac{(x+2)^2}{4} + \frac{(y-1)^2}{9} = c$$

$\Rightarrow$ ellipse with centre $(-2, 1)$





Level curve : parabolas dilated by a factor 4 and 9 in x- & y-axis direction respectively.

2. Identify the surface given by the following equations. Remember you are trying to combine the three equations into one equation involving x, y and z but not u and v.

Eliminate "v"

$$x = 4u + 3v + 5 \quad (1) \qquad y = 2u + v + 1 \quad (2) \qquad z = u + v + 1 \quad (3)$$

(2) − (3) $\Rightarrow$ $y - z = u$

(1) − (2) × 3 $\Rightarrow$ $x = 4u + \cancel{3v} + 5$

(−) $3y = 6u + \cancel{3v} + 3$

$x - 3y = -2u + 2$

$x - 3y = -2(y - z) + 2$

$x - y - 2z = 2$ $\Rightarrow$ plane equation

Partial Derivatives

3. For the function $f(u,v) = uv(1 - u^2 - v^2)$, evaluate $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$.

$\frac{\partial f}{\partial u} = f_u$

$f(u,v) = uv - u^3v - uv^3$ ✓

$$\frac{\partial f}{\partial u} = v - 3u^2v - v^3 \qquad \frac{\partial f}{\partial v} = u - u^3 - 3uv^2$$

4. Evaluate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for each of the following functions:

f_x

(a) $f(x,y) = \cos(x)\cos(y)$

(b) $f(x,y) = \sin(xy)$

(c) $f(x,y) = \frac{\log(1+x)}{\log(1+y)}$

(d) $f(x,y) = \frac{x+y}{x-y}$

(e) $f(x,y) = e^x y$

(a) $\frac{\partial f}{\partial x} = -\sin(x)\cos(y)$ $\qquad \frac{\partial f}{\partial y} = -\cos(x)\sin(y)$

$\frac{1}{\log(1+y)}$

quotient rule

(b) $\frac{\partial f}{\partial x} = y\cos(xy)$ $\qquad \frac{\partial f}{\partial y} = x\cos(xy)$

(c) $\frac{\partial f}{\partial x} = \frac{1}{\log(1+y)}\left[\frac{1}{1+x}\right] = \frac{1}{(1+x)\log(1+y)}$

$\frac{\partial f}{\partial y} = [\log(1+x)]\left[0 - \frac{\frac{1}{1+y}}{[\log(1+y)]^2}\right] = \frac{-\log(1+x)}{(1+y)[\log(1+y)]^2}$

$f = \frac{x+y}{x-y}$

(d) $\frac{\partial f}{\partial x} = \frac{(x-y)(1) - (x+y)(1)}{(x-y)^2} = \frac{-2y}{(x-y)^2}$ $\quad \frac{\partial f}{\partial y} = \frac{(x-y)(1) - (x+y)(-1)}{(x-y)^2} = \frac{2x}{(x-y)^2}$

(e) $f(x,y) = e^x y$

$\frac{\partial f}{\partial x} = e^x y \qquad \frac{\partial f}{\partial y} = e^x$

5. If $f(x, y) = 2x^4 - 3x^3y^2 + 2x^2y$, find $f_x(1, 2)$ and $f_y(1, 2)$. What do these two values represent?

$f_x = 8x^3 - 9x^2y^2 + 4xy$ $\quad$ | $\quad f_y = -6x^3y + 2x^2$

$f_x(1,2) = 8(1)^3 - 9(1)^2(2)^2 + 4(1)(2)$ $\quad$ | $\quad f_y(1,2) = -6(1)^3(2) + 2(1)^2$

$= -20$ $\qquad\qquad = -10$

These values represent slopes

f_x slope corresponds to x-axis 3 (yz-plane)

f_y - - - - - - - - - - y-axis 3. (xz-plane)

6. If $f(x, y, z) = \sin(xy)\cos(yz^2)$, find f_x, f_y and f_z.

$f_x = y\cos(xy)\cos(yz^2)$

$f_y = x\cos(xy)\cos(yz^2) + \sin(xy)[-z^2\sin(yz^2)]$ → Product Rule

$= x\cos(xy)\cos(yz^2) - z^2\sin(xy)\sin(yz^2)$ simplified

$f_z = -2yz\sin(xy)\sin(yz^2)$

Tangent Planes

Tangent line $\quad y - y_1 = m(x - x_1)$

(a,b) $\quad y = y_1 + m(x - x_1)$

for $f(x,y)$ $\quad y = f(a) + f'(a)(x-a)$

$f(x_1)$

Tangent plane

$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

7. If $f(x, y) = \frac{x^2}{4} + \frac{y^2}{9}$, find f_x and f_y. Use this to find the equation of the tangent plane at the point $(-2, 3)$. Sketch the function and the tangent plane.

$f_x = \frac{1}{2}x$ $\quad$ | $\quad f_y = \frac{2}{9}y$ $\quad$ | $\quad (a,b,c)$

$f_x(-2,3) = -1$ $\quad$ | $\quad f_y(-2,3) = \frac{2}{3}$ $\quad$ | $\quad f(-2,3) = 2$ $\quad (-2,3,2)$

eqn of tangent plane:

$z = 2 - (x+2) + \frac{2}{3}(y-3) \iff x - \frac{2}{3}y + z = -2 \iff [ax + by + cz = d]$ form

8. Compute the tangent plane approximation for each of the following functions at the stated point.

tangent plane for f(x) is also linear approx for f(x)

(a) $f(x,y) = 2x + 3y$ at the point $(1,2)$

(b) $g(x,y) = 4 - x^2 - y^2$ at the point $(1,1)$

(c) $h(x,y) = \sin(x)\cos(y)$ at the point $(\frac{\pi}{4}, \frac{\pi}{4})$

(d) $q(x,y,z) = \ln(x^2 + y^2 + z^2)$ at the point $(1,0,1)$

(e) $r(x,y) = -xye^{-x^2-y^2}$ at the point $(1,1)$

(f) $u(x,y,z) = \sqrt{1 - x^2 - y^2 - z^2}$ at the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

Q8. (a) $f(x,y) = 2x + 3y$

$f_x = 2$ $\quad f_y = 3$

$f_x(1,2) = 2$ $\quad f_y(1,2) = 3$

$f(1,2) = 2 + 6 = 8$

Equ tangent plane

$z = 8 + 2(x-1) + 3(y-2)$

$2x + 3y - z = 0$

(b) $g(x,y) = 4 - x^2 - y^2 \rightarrow g(1,1) = 2$

$g_x = -2x$ $\quad g_y = -2y$

$g_x(1,1) = -2$ $\quad g_y(1,1) = -2$

Eqn of tangent plane

$z = 2 - 2(x-1) - 2(y-1)$

$2x + 2y + z = 6$

$g(x,y) = 4 - x^2 - y^2 \approx 2 - 2(x-1) - 2(y-1)$ at $(1,1,2)$

(c) $h(x,y) = \sin(x)\cos(y)$ $\quad h(\frac{\pi}{4}, \frac{\pi}{4}) = \frac{1}{2} \rightarrow$ Find (a,b,c)

$\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

$\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

$h_x = \cos(x)\cos(y)$ $\quad h_y = -\sin(x)\sin(y)$

$h_x(\frac{\pi}{4}, \frac{\pi}{4}) = \frac{1}{2}$ ✓ $\quad h_y(\frac{\pi}{4}, \frac{\pi}{4}) = -\frac{1}{2}$ ✓

$(\frac{\pi}{4}, \frac{\pi}{4}, \frac{1}{2})$

Eqn of tangent $\quad z = \frac{1}{2} + \frac{1}{2}(x - \frac{\pi}{4}) - \frac{1}{2}(y - \frac{\pi}{4})$

$\frac{1}{2}x - \frac{1}{2}y - z = -\frac{1}{2}$

$y = f(x)$ tangent line at (a,b) $\quad y - b = f'(a)(x-a)$

$z = f(x,y)$ tangent plane at (a,b,c) $\quad z - c = f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$W = f(x,y,z)$ tangent plane at (a,b,c,d) $\quad w - d = f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$

(d) $g(x,y,z) = \ln(x^2+y^2+z^2)$ $\quad * f(1,0,1) = \ln(1+0+1) = \ln 2 \rightarrow (1,0,1,\ln 2)$

$\boxed{\text{point}}$ $x\ y\ z\ ,w$ $\quad g(x,y,z)$

$g_x = \frac{2x}{x^2+y^2+z^2}$ $\quad g_x(1,0,1) = 1$

$g_y = \frac{2y}{x^2+y^2+z^2}$ $\quad g_y(1,0,1) = 0$

$g_z = \frac{2z}{x^2+y^2+z^2}$ $\quad g_z(1,0,1) = 1$

Eqn of tangent plane:

$$w = \ln 2 + \overset{g_x}{(1)}(x-1) + \overset{g_y}{(0)}(y-0) + \overset{g_z}{(1)}(z-1)$$

$$x + z - w = 2 - \ln 2$$

Tangent plane

$$w = f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$$

$\boxed{x=a}$

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$$

$$T_n(x,y) = f(a,b) + \frac{f_x'(a,b)}{1!}(x-a) + \frac{f_y'(a,b)}{1!}(y-b)$$

$$T_n(x,y,z) =$$

(e) $r(x,y) = -xy\,e^{-x^2-y^2}$ $\quad r(1,1) = -(1)(1)e^{-1-1} = -e^{-2}$

$r_x = -ye^{-x^2-y^2} - xy(-2x)e^{-x^2-y^2}$

$r_x = (-1+2x^2)ye^{-x^2-y^2}$

$r_x(1,1) = e^{-2}$

$r_y = -xe^{-x^2-y^2} + 2xy^2e^{-x^2-y^2}$

$r_y = (-1+2y^2)xe^{-x^2-y^2}$

$r_y(1,1) = e^{-2}$

Eqn tangent $\quad z = -e^{-2} + e^{-2}(x-1) + e^{-2}(y-1)$

(f) $u(x,y,z) = \sqrt{1-x^2-y^2-z^2} = (1-x^2-y^2-z^2)^{\frac{1}{2}}$ $\quad f(\frac{1}{2},\frac{1}{2},\frac{1}{2}) = \frac{1}{2}$

$u_x = \frac{-x}{\sqrt{1-x^2-y^2-z^2}}$ $\quad u_x(\frac{1}{2},\frac{1}{2},\frac{1}{2}) = -1$ $\quad u_y(\frac{1}{2},\frac{1}{2},\frac{1}{2}) = -1$ $\quad u_z = (\frac{1}{2},\frac{1}{2},\frac{1}{2}) = -1$

$u_y = \frac{-y}{\sqrt{1-x^2-y^2-z^2}}$

$u_z = \frac{-z}{\sqrt{1-x^2-y^2-z^2}}$

Eqn tangent plane

$$w = \tfrac{1}{2} - (x-\tfrac{1}{2}) - (y-\tfrac{1}{2}) - (z-\tfrac{1}{2})$$

$$x + y + z + w = 2$$

9. Use the tangent plane approximations found in the previous question to estimate the function at the stated points. Compare your estimate with that given by a calculator using the function itself.

(a) $f(x, y)$ at the point $(1.1, 1.9)$

(b) $g(x, y)$ at the point $(0.1, 1.1)$

(c) $h(x, y)$ at the point $(\frac{3\pi}{16}, \frac{5\pi}{16})$

(d) $q(x, y, z)$ at the point $(0.8, 0.1, 0.9)$

(e) $r(x, y)$ at the point $(0.9, 1.1)$

(f) $u(x, y, z)$ at the point $(0.6, 0.4, 0.6)$

Tangent plane $\Rightarrow$ Linear approximation to function, $T_1(x)$

8(d) Tangent plane at $(1,0,1)$] suitable to use for approx at the point in Q9(d)

9(d) Approx $\Rightarrow$ at $(0.8, 0.1, 0.9)$

Recall 8(d) $q(x,y,z) = \ln(x^2+y^2+z^2) \approx \ln 2 + (x-1) + (z-1)$ at $(1,0,1)$

$q(0.8, 0.1, 0.9) = \ln(0.8^2+0.1^2+0.9^2)$

RHS $\approx \ln 2 + (0.8-1) + (0.9-1)$

$= \ln 2 - 0.3 \approx 0.3931 \approx 0.4$

≈ 0.393 (3dp)

≈ 0.39 (2dp)

Actual

LHS $\ln(0.8^2+0.1^2+0.9^2) \approx 0.3784 \approx 0.4$

≈ 0.378 (3dp)

≈ 0.38 (2dp)

(a) $f(x,y) = 2x+3y \qquad (1.1, 1.9)$

$2x+3y \approx 8 + 2(x-1) + 3(y-2) \Rightarrow$ Linear approx.

At $(1.1, 1.9)$ $\qquad 2(1.1)+3(1.9) \approx 8 + 2(0.1) + 3(-0.1)$

$= 8 + 0.2 - 0.3$

$f(1.1, 1.9) \approx 7.9$

(b) $g(x,y) = 4 - x^2 - y^2 \approx 2 - 2(x-1) - 2(y-1)$

$g(0.1, 1.1) \approx 2 - 2(0.9) - 2(0.1)$

$= 3.6$

(c) $h(x,y) = \sin(x)\cos(y) \approx \frac{1}{2} + \frac{1}{2}(x - \frac{\pi}{4}) - \frac{1}{2}(y - \frac{\pi}{4})$

part (c) tangent plane approx $(\frac{\pi}{4}, \frac{\pi}{4})$

$h(x,y) \approx \frac{1}{2} + \frac{1}{2}(x - \frac{\pi}{4}) - \frac{1}{2}(y - \frac{\pi}{4})$

$h(\frac{3\pi}{16}, \frac{5\pi}{16}) \approx \frac{1}{2} + \frac{1}{2}(-\frac{\pi}{16}) - \frac{1}{2}(\frac{\pi}{16})$

$= \frac{1}{2} - \frac{\pi}{16}$

8(d) $g(x,y,z) \approx \ln 2 + (1)(x-1) + 0(y-0) + 1(z-1)$ $\quad (1,0,1,\ln 2)$

At $(0.8, 0.1, 0.9)$

(d) $g(x,y,z) = \ln(x^2+y^2+z^2) \approx \ln 2 + (x-1) + (z-1)$

$g(0.8, 0.1, 0.9) \approx \ln 2 + (-0.2) + (-0.1) = \ln 2 - 0.3$

(e) $r(x,y) = -xye^{-x^2-y^2} \approx -e^{-2} + e^{-2}(x-1) + e^{-2}(y-1)$

$r(0.9, 1.1) \approx -e^{-2} + e^{-2}(-0.1) + e^{-2}(0.1) = -e^{-2}$

(f) $u(x,y,z) = \sqrt{1-x^2-y^2-z^2} \approx \frac{1}{2} - (x-\frac{1}{2}) - (y-\frac{1}{2}) - (z-\frac{1}{2})$

$u(0.6, 0.4, 0.6) \approx \frac{1}{2} - (0.1) + (0.1) - (0.1) = 0.4$

10. Derive the linear approximation $T_1(x,y,z)$ for the function $f(x,y,z) = x^2 + y^2 + z^2$ at the point $(1,1,1)$. What does this function, and its linear approximation, represent?

$f_x = 2x \qquad f_y = 2y \qquad f_z = 2z \qquad f(1,1,1) = 3$

$\boxed{\text{At } (1,1,1)}$ $\qquad (1,1,1,3)$

$f_x = 2 \qquad f_y = 2 \qquad f_z = 2$

Tangent plane $\simeq T_1(x,y,z) = $ linear approx for $f(x,y,z)$ at (a,b,c,d) — $L(x,y,z)$

$\hookrightarrow w = 3 + 2(x-1) + 2(y-1) + 2(z-1) \rightarrow$ ~~tangent~~ plane

$T_1(x,y,z) = 3 + 2(x-1) + 2(y-1) + 2(z-1)$

$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots +$

$T_n(x,y,z) = f(a,b,c) + \frac{f_x(a,b,c)}{1!}(x-a) + \frac{f_y(a,b,c)}{1!}(y-b) + \frac{f_z(a,b,c)}{1!}(z-c)$

$\Downarrow \qquad + \frac{f_{xx}(a,b,c)}{2!}(x-a)^2 + \cdots\cdots$

3 variables

$f_{xy} \quad f_{yx} \quad f_{xz} \quad f_{zx}$

$f_{yy} \quad f_{zz} \quad f_{yz} \quad f_{zy}$