MAT1841 2022 [2]

Assignment 1 Marking Scheme

50 possible marks

(this assignment contributes 10% of your final mark)

Q1)

$\underset{\sim}{r_1}(t) = (-3,2,2) + t(4,0,1)$

$\underset{\sim}{r_2}(s) = (12,0,-9) + s(-4,3,0)$ 2

a) a normal vector will be perpendicular to both $\underset{\sim}{r_1}(t)$ & $\underset{\sim}{r_2}(s)$

$$\underset{\sim}{n} = (4,0,1) \times (-4,3,0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 0 & 1 \\ -4 & 3 & 0 \end{vmatrix}$$

you don't need use the determinant. you can use the formula directly! 1

$$= \hat{i}(0\cdot 0 - 3\cdot 1) - \hat{j}(4\cdot 0 - -4\cdot 1) + \hat{k}(4\cdot 3 - -4\cdot 0)$$

$$= -3\hat{i} - 4\hat{j} + 12\hat{k}$$

(any scalar multiple of this is correct) 1

b) scalar projection for minimum distance

create a random vector from $r_1(t)$ to $r_2(s)$ 1

I choose $t = s = 0$, $\underset{\sim}{r_1}(0) - \underset{\sim}{r_2}(0) = (-15,2,11) = \underset{\sim}{v}$

make projection of $\underset{\sim}{v}$ onto $\underset{\sim}{n}$ $= \dfrac{\underset{\sim}{v}\cdot\underset{\sim}{n}}{|\underset{\sim}{n}|}$ 2

$$\frac{(-15,2,11)\cdot(-3,-4,12)}{\sqrt{(-3,-4,12)\cdot(-3,-4,12)}} = \frac{45 - 8 + 132}{\sqrt{9+16+144}} = \frac{169}{\sqrt{169}} = 13$$ 2

~~note: it's easy for this to~~

note, depending on choices made, this could come out negative, but since we are asking for distance we only consider the ~~positive~~ absolute value.

c) find the endpoints
there are many ways to do this

one method is to make a vector from $\underset{\sim}{r}_1(t)$ to $\underset{\sim}{r}_2(s)$ to be parallel to $\underset{\sim}{n}$ (from part a) 2 marks for any valid method

$$\underset{\sim}{r}_1(t) - \underset{\sim}{r}_2(s) = \underset{\sim}{u} = [(-3,2,2) + t(4,0,1)] - [(12,0,-9) + s(-4,3,0)]$$

$$\underset{\sim}{u} = (-15+4t+4s,\ 2-3s,\ 11+t)$$ 1

to be parallel $\underset{\sim}{u} \times \underset{\sim}{n} = \underset{\sim}{0}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -15+4t+4s & 2-3s & 11+t \\ -3 & -4 & 12 \end{vmatrix}$$ 1

$$\underset{\sim}{u} \times \underset{\sim}{n} = \hat{i}\left[12(2-3s) - (-4)(11+t)\right] - \hat{j}\left[12(-15+4t+4s) - (-3)(11+t)\right] + \hat{k}\left[(-4)(-15+4t+4s) - (-3)(2-3s)\right]$$

$$= (4t-36s+68)\hat{i} + (-51t-48s+147)\hat{j} + (-16t-25s+66)\hat{k}$$ 1

3 equations 2 unknowns solve

1) $4t - 36s + 68 = 0 \Rightarrow t - 9s = -17$

2) $-51t - 48s + 147 = 0$

3) $-16t - 25s + 66 = 0 \Rightarrow 16t + 25s = 66$ 1

$$\begin{aligned} 16t + 25s &= 66 \\ 16t - 144s &= -272 \\ \hline 169s &= +338 \end{aligned}$$

$s = 2$ now solve for t

$t - 9(2) = -17$

$t + 18 = -17 \Rightarrow t = 1$

min distance when $\boxed{t = 1 \ \& \ s = 2}$ 2

check with remaining equation

$-51(1) - 48(2) + 147 = 0$ ✓

$-51 - 96 + 147 = 0$ ✓

not necessary

15 marks possible

Q2| . $P_1: \quad 2x + z = 7$
$P_2: \quad x - y + 2z = 6$

normal to P_1, $\underset{\sim}{n_1} = (2,0,1)$
normal to P_2, $\underset{\sim}{n_2} = (+1,-1,2)$

the intersecting line will have the orientation on $\underset{\sim}{n_1} \times \underset{\sim}{n_2}$

$\underset{\sim}{n_1} \times \underset{\sim}{n_2} = (1,-3,-2)$ 2

find any intersecting point.
I choose $x = 0$ so $z = 7$ from P1
for this to work for P2, $0 - y + 2(7) = 6$
$-y = -8 \Rightarrow y = 8$

a point on both planes is $(0,8,7)$ 2

an equation of the intersecting line is
$\underset{\sim}{r_1}(t) = (0,8,7) + t(1,-3,-2)$ 1
this is not unique.

5 marks possible

Q3|| $P_1: x - 4y + 8z = -15$
$P_2: x - 4y + 8z = 66$
$\underset{\sim}{n} = (1,-4,8)$ 1

create a vector between P_1 & P_2
and make the scalar projection onto $\underset{\sim}{n}$
define $\underset{\sim}{u} = (-15,0,0) - (66,0,0) = (-81,0,0)$ 1

scalar projection $\frac{\underset{\sim}{u} \cdot \underset{\sim}{n}}{|\underset{\sim}{n}|} = \frac{(-81,0,0)\cdot(1,-4,8)}{\sqrt{1+16+64}} = \frac{-81}{9} = -9$ 1

min dist is $\underline{9}$ 2

5 marks possible

Q4) $M = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 3 \\ -1 & -1 & 0 \end{bmatrix}$

a) $\det M = 1\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} - 0\begin{vmatrix} -2 & 3 \\ -1 & 0 \end{vmatrix} + (-2)\begin{vmatrix} -2 & 1 \\ -1 & -1 \end{vmatrix}$ 1

$= +3 - 0 + -2(2+1) = +3-6 = -3$ 2

b) calculate M^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -2 & 1 & 3 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$ 2

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & -1 & -2 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} R_2' \leftarrow R_2 + 2R_1 \\ R_3' \leftarrow R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & -3 & 3 & 1 & 1 \end{array}\right] R_3'' \leftarrow R_3' + R_2'$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1/3 & -1/3 \end{array}\right] R_3''' \leftarrow -\frac{1}{3}R_3''$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2/3 & -2/3 \\ 0 & 1 & 0 & 1 & 2/3 & -1/3 \\ 0 & 0 & 1 & -1 & -1/3 & -1/3 \end{array}\right] \begin{array}{l} R_1'' \leftarrow R_1 + 2R_3''' \\ R_2''' \leftarrow R_2'' + R_3''' \end{array}$$ 5

$$M^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & -2 \\ 3 & 2 & -1 \\ -3 & -1 & -1 \end{bmatrix}$$ 2

Q4 c)

Solve $MX = B$

where $B = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$

$$X = M^{-1}B \quad = \frac{1}{3}\begin{bmatrix} -3 & -2 & -2 \\ 3 & 2 & -1 \\ -3 & -1 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 6 \\ -3 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$ 1 2

15 marks possible

Q5) $M = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 3 \\ 0 & 1 & -1 \end{bmatrix}$

a) calculate det M

$$\det M = 1\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} - 0\begin{vmatrix} -2 & 3 \\ 0 & -1 \end{vmatrix} + (-2)\begin{vmatrix} -2 & 1 \\ 0 & 1 \end{vmatrix}$$ 1

$$= -4 - 0 + 4 = 0$$

b)

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -2 \\ -2 & 1 & 3 & 1 \\ 0 & 1 & -1 & b \end{array}\right]$$

$MX = B$ or $[M \mid B]$ 1

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 1 & -1 & b \end{array}\right] \quad R_2' \leftarrow R_2 + 2R_1$$

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$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & b+3 \end{array}\right]$$

$b = -3$ for infinitely many solutions 1

c) assume $b = -3$ choose $\underline{z = t}$ ① 1 $\Rightarrow y - t = -3$

$\Rightarrow y = t - 3$ ②

& $x - 2t = -2$

$\therefore x = 2t - 2$ ③

2

Solution $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ 1

(not unique)

10 marks possible