

$$\det(A) = ad - bc \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Problem Set Four: Matrices - Inverse, Transpose and Determinant

Matrix Inverse

1. Compute the inverse of the following matrices, or state why the inverse does not exist.

$\downarrow - \uparrow$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 3 \\ 1 & 3 \\ 1 & 2 \end{bmatrix}$$

Matrix B

① Not square matrix
Matrix B has size 3×2

B has no inverse?

Verify that $A^{-1}A = I$ and $AA^{-1} = I$. What is $(A^{-1})^{-1}$?

Matrix A

① Square matrix, 2×2

$$\text{② } \det(A) = -4 - 1 = -5$$

$$\det(A) \neq 0$$

Therefore A^{-1} exists.

Matrix A is invertible.

$$A^{-1} = \frac{1}{-5} \begin{pmatrix} -4 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{1}{5} \end{pmatrix}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

2. Show that the matrix $B = \begin{bmatrix} 2 & 5 \\ 6 & 15 \end{bmatrix}$ is not invertible. What are the implications of this for the following linear system?

$$\begin{pmatrix} 2 & 5 \\ 6 & 15 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$$

$$\begin{cases} 2x + 5y = 7 \\ 6x + 15y = 8 \end{cases}$$

$\Rightarrow \det(B) = 0 \Rightarrow \text{No soln!}$
 B^{-1} does not exist!

$$\begin{cases} 6x + 15y = 21 \\ 6x + 15y = 8 \end{cases}$$

$\rightarrow B^{-1}$ does not exist! B is NOT invertible

$$\begin{matrix} \text{matrix form} \\ BX = C \end{matrix}$$

$$X = B^{-1}C \Rightarrow \text{Cannot proceed}$$

3. Calculate B^{-1} , B^2 and $(B^2)^{-1}$ for the matrix $B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$.

Verify that $(B^2)^{-1} = B^{-1}B^{-1}$.

$$B^{-1} = \frac{1}{8-9} \begin{pmatrix} 4 & -3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix}$$

$$B^2 = BB = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 13 & 18 \\ 18 & 25 \end{pmatrix}$$

$$\begin{aligned} B^{-1}B^{-1} &= \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 25 & -18 \\ -18 & 13 \end{pmatrix} \end{aligned}$$

$$(B^2)^{-1} = \frac{1}{(13 \times 25) - (18 \times 18)} \begin{pmatrix} 25 & -18 \\ -18 & 13 \end{pmatrix} = \begin{pmatrix} 25 & -18 \\ -18 & 13 \end{pmatrix}$$

$$(B^2)^{-1} = B^{-1}B^{-1}$$

Matrix Transpose

$\rightarrow \text{row} \leftrightarrow \text{column}$

$$(A^{-1})^T = A$$

$$A^T = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & -1 \end{pmatrix} \quad (A^{-1})^{-1} = A$$

4a. For the matrix $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 3 \\ 1 & 2 & -1 \end{bmatrix}$ compute A^T . What is $(A^T)^T$?

$$(A^T)^T = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \end{pmatrix}$$

4b. For the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & 3 \\ 1 & 2 \end{bmatrix}$ compute B^T . Also compute BB^T and

$$(B^T)^T = B$$

$$AA^{-1} = A^{-1}A$$

$B^T B$.

$$B^T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}$$

$$BB^T = \begin{pmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 13 & 11 & 8 \\ 11 & 10 & 7 \\ 8 & 7 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 11 \\ 11 & 22 \end{pmatrix} \quad \boxed{BB^T \neq B^T B} \quad AB \neq BA$$

$$AA^{-1} = A^{-1}A = I$$

5. Consider the following matrices

$$\begin{aligned} (ab)^2 &= a^2 b^2 \\ (ab)^2 &= b^2 a^2 \end{aligned} \quad A = \begin{pmatrix} 4 & -2 \\ 1 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix} \quad A^T = \begin{pmatrix} 4 & 1 \\ -2 & 5 \end{pmatrix} \quad B^T = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}$$

Compute A^T , B^T and AB . Verify that $(AB)^T = B^T A^T$.

$$AB = \begin{pmatrix} 4 & -2 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ 12 & 4 \end{pmatrix} \quad A^T B^T = \begin{pmatrix} 4 & 1 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 9 \\ -9 & 1 \end{pmatrix}$$

$$(AB)^{-1} = B^{-1} A^{-1} AB = \begin{pmatrix} 4 & -2 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ 12 & 4 \end{pmatrix} \quad B^T A^T = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ -6 & 4 \end{pmatrix}$$

$$(AB)^T = \begin{pmatrix} 4 & 12 \\ -6 & 4 \end{pmatrix}$$

$$(ABC)^T = C^T B^T A^T$$

$$(AB)^T = B^T A^T \quad \boxed{(AB)^T \neq A^T B^T}$$

6. For the matrices A and B from Question 5, compute the following:

$$(a) (A+B)^T$$

$$(a) (A+B)^T = \begin{pmatrix} 6 & -3 \\ 3 & 6 \end{pmatrix}^T = \begin{pmatrix} 6 & 3 \\ -3 & 6 \end{pmatrix} \quad (b) A^T + B^T = \begin{pmatrix} 6 & 3 \\ -3 & 6 \end{pmatrix}$$

$$(b) A^T + B^T$$

$$(A+B)^T = A^T + B^T$$

$$(c) A + A^T \text{ and } B + B^T$$

$$(c) A + A^T = \begin{pmatrix} 8 & -1 \\ -1 & 10 \end{pmatrix} \quad B + B^T = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix} \quad \boxed{\text{Symmetric Matrix}}$$

$$(d) (A+A^T)^T$$

$$(d) (A+A^T)^T = \begin{pmatrix} 8 & -1 \\ -1 & 10 \end{pmatrix}$$

$$(B+B^T)^T = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\begin{cases} (e) (A^T)^{-1} \\ (f) (A^{-1})^T \\ (g) (AB)^{-1} \\ (h) B^{-1} A^{-1} \end{cases}$$

$$(A+A^T)^T = (A+A^T)$$

$$B+B^T = (B+B^T)^T \quad \text{if}$$

$$(e) (A^T)^{-1} = \frac{1}{22} \begin{pmatrix} 5 & -1 \\ 2 & 4 \end{pmatrix}$$

$$(f) (A^{-1})^T = \left[\frac{1}{22} \begin{pmatrix} 5 & 2 \\ -1 & 4 \end{pmatrix} \right]^T = \begin{pmatrix} \frac{5}{22} & \frac{1}{22} \\ -\frac{1}{22} & \frac{4}{22} \end{pmatrix}^T$$

$$= \begin{pmatrix} \frac{5}{22} & -\frac{1}{22} \\ \frac{1}{22} & \frac{4}{22} \end{pmatrix}$$

$$(A^T)^{-1} = (A^{-1})^T$$

$$= \begin{pmatrix} \frac{5}{22} & -\frac{1}{22} \\ \frac{1}{22} & \frac{4}{22} \end{pmatrix}$$

$$(h) B^{-1} = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

$$(g) (AB)^{-1} = \begin{pmatrix} 4 & -4 \\ 12 & 4 \end{pmatrix}^{-1}$$

$$= \frac{1}{16+72} \begin{pmatrix} 4 & 6 \\ -12 & 4 \end{pmatrix} = \begin{pmatrix} \frac{4}{88} & \frac{6}{88} \\ -\frac{12}{88} & \frac{4}{88} \end{pmatrix} \quad \boxed{B^{-1} A^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix}}$$

$$(AB)^T = B^T A^T$$

$$= \begin{pmatrix} \frac{4}{88} & \frac{6}{88} \\ -\frac{12}{88} & \frac{4}{88} \end{pmatrix}$$

$$\det(C) = (-1)^{i+j} a_{ij} |S_{ij}|$$

$$= (-1)^{1+1} (2) \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} \quad [i=1, j=1]$$

$$= (-1)^{1+2} (3) \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} \quad [i=1, j=2]$$

$$= (-1)^{1+3} (-1) \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \quad [i=1, j=3]$$

Matrix Determinant

7. Consider the following matrices

$$AB = \left(\begin{array}{cc} & \\ & \end{array} \right) A = \begin{bmatrix} 2 & -1 \\ 1 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$C = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 1 & 3 \\ 1 & 2 & -1 \end{pmatrix}$$

$$(a) \det(A) = 14 - (-1) = 15$$

$$\det(B) = 2 - 3 = -1$$

Laplace's Expansion

Fix 1st column (optional to choose other)

$$\det(C) = (-1)^2 (2) \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} + (-1)^3 (3) \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + (-1)^4 (-1) \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$

$$\det(C) = 2(-7) + (-3)(-4) + (1)(1) = -3 \checkmark$$

a. Compute the determinant for each matrix.

b. Compute AB and find $\det(AB)$.

c. Compute BA and find $\det(BA)$.

d. What is $\det(A) \det(B)$? \checkmark

e. Compute A^T . What is $\det(A^T)$?

f. Compute A^{-1} . What is $\det(A^{-1})$?

g. What is $\det(A) \det(A^{-1})$?

$$(b) AB = \begin{pmatrix} 2 & -1 \\ 1 & 7 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 7 & 10 \end{pmatrix}$$

$$\det(AB) = \det(A) \times \det(B) = 15 \times -1$$

$$(c) BA = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 7 \end{pmatrix} = \begin{pmatrix} 7 & 19 \\ 3 & 6 \end{pmatrix} \quad \det(BA) = -15$$

$$\det(AB) = \det(BA) = \det(B) \det(A) = -1 \times 15 = -15$$

$$(d) \det(A) \det(B) = 15 \times -1 = -15$$

$$(e) A^T = \begin{pmatrix} 2 & 1 \\ -1 & 7 \end{pmatrix} \quad \det(A^T) = 14 - (-1) = 15 \quad \det(A) = \det(A^T)$$

$$(f) A^{-1} = \frac{1}{15} \begin{pmatrix} 7 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{7}{15} & \frac{1}{15} \\ -\frac{1}{15} & \frac{2}{15} \end{pmatrix} \quad \det(A^{-1}) = \frac{14}{15^2} + \frac{1}{15^2} = \frac{15}{15^2} = \frac{1}{15}$$

$$(g) \det(A) \det(A^{-1}) = 15 \times \frac{1}{15} = 1 \quad \det(AA^{-1}) = \det(I) = 1$$

8. Write down the coefficient matrix, C , for each of the under determined systems. What do you notice about the determinant of each?

$$a. \begin{vmatrix} 1 & 2 & -1 & 0 \\ 1 & 3 & 0 & 1 \\ 2 & 5 & -1 & 2 \\ 1 & 1 & 3 & 5 \end{vmatrix}$$

$$\begin{aligned} x + 2y - z &= 6 \\ x + 3y + 0z &= 7 \\ 2x + 5y - z &= 13 \end{aligned}$$

$$C = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 3 & 0 \\ 2 & 5 & -1 \end{pmatrix}$$

$$\det(C) = (-3 - 5 + 0) - (16 + 0 - 1) = -8 - (-8) = 0 \rightarrow \text{not able to solve vs. } 3 \text{ eqns}$$

\Rightarrow No soln.



b.

$$x + 2y - z = 6$$

$$C = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{3 \times 3}$$

$\det(C) = 0$

C^{-1} does not exist!

$$BX = C \quad \Leftrightarrow \quad X = B^{-1}C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$C = (1 \ 2 \ -1)$$

$$\det(A) = (-1)^4 (1) \begin{vmatrix} 1 & 3 & -4 \\ 2 & -1 & -2 \\ 3 & -2 & 4 \end{vmatrix} + (-1)^5 (1) \begin{vmatrix} 1 & 2 & 4 \\ 2 & 1 & -2 \\ 3 & -1 & 4 \end{vmatrix} + \dots$$

$$= (-18 - 16 - 4) - ()$$

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & -1 \\ 2 & 1 & -1 & -2 \\ 3 & -1 & -2 & 4 \end{pmatrix}$$

$$09(a) \quad \left[\begin{array}{ccc|ccc} 2 & 3 & -1 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\substack{R_2 \leftarrow R_1 - 2R_2 \\ R_3 \leftarrow R_2 - R_3}} \left[\begin{array}{ccc|ccc} 2 & 3 & -1 & 1 & 0 & 0 \\ 0 & 1 & -7 & 1 & -2 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 \end{array} \right]$$

$$\det(C) = -3 \neq 0$$

9. Consider the matrix C from Question 7.

a. Use the Gauss-Jordan method to find C^{-1} . Check your answer by showing $CC^{-1} = I$.

b. You have calculated $\det(C)$ in Question 7. Use this, and the formula

$$a_{ji} = (-1)^{i+j} \frac{\det S_{ij}}{\det C}$$

$$\xrightarrow{R_3 \leftarrow R_2 + R_3} \left[\begin{array}{ccc|ccc} 2 & 3 & -1 & 1 & 0 & 0 \\ 0 & 1 & -7 & 1 & -2 & 0 \\ 0 & 0 & -3 & 1 & -1 & -1 \end{array} \right]$$

$$\xrightarrow{R_3 \leftarrow -\frac{1}{3}R_3} \left[\begin{array}{ccc|ccc} 2 & 3 & -1 & 1 & 0 & 0 \\ 0 & 1 & -7 & 1 & -2 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array} \right]$$

to compute C^{-1} , where S_{ij} is the 2×2 minor matrix of C with column i and row j removed.

$$\xrightarrow{\substack{R_2 \leftarrow R_2 + 7R_3 \\ R_1 \leftarrow R_1 + R_3}} \left(\begin{array}{ccc|ccc} 2 & 3 & 0 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{4}{3} & \frac{1}{3} & \frac{7}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array} \right)$$

$$\xrightarrow{R_1 \leftarrow R_1 - 3R_2} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{14}{3} & -\frac{2}{3} & \frac{20}{3} \\ 0 & 1 & 0 & -4 & 1 & 7 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array} \right)$$

$$\xrightarrow{R_1 \leftarrow \frac{1}{2}R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{3} & -\frac{1}{3} & \frac{10}{3} \\ 0 & 1 & 0 & -4 & 1 & 7 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array} \right) \xrightarrow{\text{STAR}} C^{-1}$$

$$C^{-1} = \begin{pmatrix} \frac{7}{3} & -\frac{1}{3} & \frac{10}{3} \\ -4 & 1 & 7 \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

Find soln.

$$(C)(\vec{x}) = \vec{1}$$

$$(\vec{x}) = C^{-1} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \text{Sol'n.}$$

$$\underline{\text{check}} \quad (C^{-1}) = I ?$$

$$\left(\quad \quad \right) \left(\quad \quad \right) = I$$

Tut #4 Q9(b) $C = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & 3 \\ 1 & 2 & -1 \end{pmatrix}$

Q7(a)

$$\det(C) = (-2 - 2 + 9) - (-1 + 12 - 3) = 5 - 8 = -3 \checkmark$$

$a_{ji} = \frac{(-1)^{i+j} \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix}}{-3} = \frac{7}{3}$

$$C^{-1} = \begin{pmatrix} \frac{7}{3} & -\frac{1}{3} & -\frac{10}{3} \\ -\frac{4}{3} & \frac{1}{3} & \frac{7}{3} \\ -\frac{1}{3} & \frac{3}{3} & \frac{1}{3} \end{pmatrix} \text{ Q9(a)}$$

$$C = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & 3 \\ 1 & 2 & -1 \end{pmatrix} \checkmark$$

2.8.1 Inverse - another method

Here is another way to compute the inverse of a matrix.

$a_{11} = \frac{(-1)^1 \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix}}{-3} = -\frac{4}{3}$

- Select the i^{th} row and j^{th} column of A .

$\hat{a}_{11} = \frac{(-1)^1 \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}}{-3} = -\frac{1}{3}$

- Compute $(-1)^{i+j} \frac{\det S_{ij}}{\det A} = a_{ji}$

$a_{21} = \frac{(-1)^2 \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}}{-3} = -\frac{1}{3}$

That is, if

then

$$A = [a_{ij}]$$

$$A^{-1} = \frac{1}{\det A} [(-1)^{i+j} \det S_{ji}]$$

This method works but it is rather tedious.

$\hat{a}_{11} = \frac{(-1)^1 \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix}}{-3} = -\frac{4}{3}$

$\hat{a}_{11} = \frac{(-1)^1 \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}}{-3} = -\frac{10}{3}$

$\hat{a}_{12} = \frac{(-1)^2 \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}}{-3} = \frac{1}{3}$

$\hat{a}_{13} = \frac{(-1)^3 \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}}{-3} = \frac{7}{3}$

$\hat{a}_{21} = \frac{(-1)^3 \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix}}{-3} = \frac{1}{3}$

$\hat{a}_{22} = \frac{(-1)^4 \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix}}{-3} = \frac{1}{3}$

$\hat{a}_{23} = \frac{(-1)^5 \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix}}{-3} = \frac{7}{3}$

$\hat{a}_{31} = \frac{(-1)^5 \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}}{-3} = \frac{1}{3}$

$\hat{a}_{32} = \frac{(-1)^6 \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}}{-3} = \frac{1}{3}$

$\hat{a}_{33} = \frac{(-1)^6 \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}}{-3} = \frac{1}{3}$

10. Use Cramer's rule to solve the following system of equations:

$$\underline{b} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$

$$\begin{aligned} 2x + 3y - z &= 4 \\ x + y + 3z &= 1 \\ x + 2y - z &= 3 \end{aligned}$$

2.7.3 Cramer's rule

Recall that if a linear system $A\vec{x} = \underline{b}$ has a unique solution, then $\vec{x} = A^{-1}\underline{b}$ is this solution. If we substitute the formula for the inverse A^{-1} from the previous section (using $\det S_{ji}$) into the product $A^{-1}\underline{b}$ we arrive at Cramer's rule for solving the linear system $A\vec{x} = \underline{b}$.

Cramer's rule: Let $A\vec{x} = \underline{b}$ be a linear system with a unique solution. This means that A is a square matrix with non-zero determinant. Let A_i be the matrix that results from A by replacing the i th column of A by \underline{b} . Then

$$x_i = \frac{\det A_i}{\det A}$$

Examples of Cramer's rule will be given in [tutorials](#).

x_1
 $b-1$

x_2

x_3

$$y = \frac{\begin{vmatrix} 2 & 4 & -1 \\ 1 & 1 & 3 \\ 1 & 3 & -1 \end{vmatrix}}{-3} = \frac{(6-3+12) - (-1+18-4)}{-3} = \frac{-6}{-3} = 2$$

$$z = \frac{\begin{vmatrix} 2 & 3 & 4 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}}{-3} = \frac{(6+8+3) - (4+4+9)}{-3} = \frac{0}{-3} = 0$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

(07) Matrix C

$$C = \begin{pmatrix} x & y & z \\ 2 & 3 & -1 \\ 1 & 1 & 3 \\ 1 & 2 & -1 \end{pmatrix} \quad \det(C) = -3$$

$$X = \begin{vmatrix} 4 & 3 & -1 \\ 1 & 1 & 3 \\ 1 & 2 & -1 \end{vmatrix}$$

$$X = \frac{(-4+27-2) - (-3+24-3)}{-3}$$

$$x = \frac{3}{-3}$$

$$x = -1$$