Problem Set Five: Calculus – Differentiation

1. The definition of the derivative is given below.

$$\frac{\mathrm{d}f}{\mathrm{d}x} := \lim_{\Delta x \to 0}\left(\frac{f(x+\Delta x) - f(x)}{\Delta x}\right)$$

$\Delta x \to h$

$$f'(x) = \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$$

Use this definition (ie use **first principles**) to find $\frac{\mathrm{d}f}{\mathrm{d}x}$ when:

a. $f(x) = x^3 + 2x$

b. $f(x) = \frac{1}{x^2}$

power Rule

$$\frac{d}{dx} x^n = n x^{n-1}$$

$$f(x) = x^3 + 2x$$

$$f'(x) = 3x^2 + 2$$

(a) $f(x+h) = (x+h)^3 + 2(x+h)$

$$f(x+h) = x^3 + 3hx^2 + 3h^2x + h^3 + 2x + 2h$$

$$f(x) = x^3 + 2x$$

1st principles of derivative

$$f'(x) = \lim_{h \to 0}\left[\frac{\cancel{x^3} + 3hx^2 + 3h^2x + h^3 + \cancel{2x} + 2h - \cancel{x^3} - \cancel{2x}}{h}\right]$$

$$f'(x) = \lim_{h \to 0}\left[\frac{\cancel{h}(3x^2 + 3hx + h^2 + 2)}{\cancel{h}}\right]$$

$$f'(x) = 3x^2 + 2$$

(b) $f(x) = \frac{1}{x^2}$ $\qquad f(x+h) = \frac{1}{(x+h)^2}$

$$f(x) = \frac{1}{x^2}$$

$$f(x) = x^{-2}$$

$$f'(x) = -2x^{-3}$$

$$= -\frac{2}{x^3}$$

$$f'(x) = \lim_{h \to 0}\left[\frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}\right]$$

$$f'(x) = \lim_{h \to 0}\left[\frac{\frac{x^2 - (x+h)^2}{x^2(x+h)^2}}{h}\right]$$

$$f'(x) = \lim_{h \to 0}\left[\frac{\frac{\cancel{x^2} - \cancel{x^2} - 2hx - h^2}{x^2(x+h)^2}}{h}\right]$$

$$f'(x) = \lim_{h \to 0}\left[\frac{\frac{h(-2x-h)}{x^2(x+h)^2}}{h}\right] = \lim_{h \to 0}\left[\frac{\cancel{h}(-2x-h)}{x^2(x+h)^2} \times \frac{1}{\cancel{h}}\right]$$

$$f'(x) = \frac{-2x}{x^4}$$

$$f'(x) = -\frac{2}{x^3} = -2x^{-3}$$

$\frac{d}{dx} c = 0$, c: constant

$\frac{d}{dx} x^n = n x^{n-1}$, $n \in \mathbb{R}$

Chain Rule

$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

hidden

2. Differentiate the following polynomial functions:

(a) $f(x) = 5x^2 - 2x + 1$

(b) $f(x) = (4x - 2)^7$

(c) $f(x) = (3x^2 - x + 1)^3$

(d) $f(x) = (2x + 4)^7 + (3x - 2)^5$

(a) $f'(x) = 10x - 2$

(b) $f(x) = (4x-2)^7$ $\quad \frac{d}{dx}(4x-2)$

$f'(x) = 7(4x-2)^6 (4) = 28(4x-2)^6$

(c) $f(x) = (3x^2 - x + 1)^3$

$f'(x) = 3(3x^2-x+1)^2 (6x-1)$

(d) $f'(x) = 7(2x+4)^6(2) + 5(3x-2)^4(3)$

$= 14(2x+4)^6 + 15(3x-2)^4$

3. Find the slope of the curve $y = x^2 + 6$ at the point $(x, y) = (3, 15)$. Use this to find the equation of the tangent line at this point. Make a quick sketch of $y = x^2 + 6$ and the tangent, to support your answer.

$\frac{dy}{dx}$ / $f'(x)$

gradient function

$y = x^2 + 6$

$\frac{dy}{dx} = 2x$

At $(3,15)$

$\left.\frac{dy}{dx}\right|_{x=3} = 2(3) = 6 \ (m_t)$

Eqn of tangent

$y = mx + c$

$15 = 6(3) + c$

$c = -3$

$y = 6x - 3$

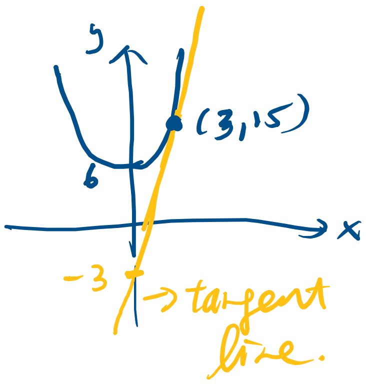
use $\boxed{\text{OR}}$

$(3,15)$, $m_t = 6$

$y - y_1 = m_t(x - x_1)$

$y - 15 = 6(x - 3)$

$y = 6x - 3$



4. State and use the appropriate rules of differentiation (ie Product rule, Quotient rule, Chain rule) to obtain $\frac{df}{dx}$ for the following functions:

$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$

(a) $f(x) = (2x + 4)^7(3x - 2)^5$

(b) $f(x) = (5x^2 - 2x + 1)(4x - 2)^7$

(c) $f(x) = \sqrt{1 + 2x^2}$

(d) $f(x) = \frac{1}{1 + x^2}$

(e) $f(x) = \frac{x}{1 + x^2}$

(a) Product Rule + chain Rule + Power Rule

let $u = (2x+4)^7$ $\qquad v = (3x-2)^5$

$\frac{du}{dx} = 7(2x+4)^6(2)$ $\qquad \frac{dv}{dx} = 5(3x-2)^4(3)$

$= 14(2x+4)^6$ $\qquad = 15(3x-2)^4$

$f'(x) = 14(2x+4)^6(3x-2)^5 + 15(3x-2)^4(2x+4)^7$

(b) $f(x) = (5x^2 - 2x + 1)(4x - 2)^7$

(c) $f(x) = \sqrt{1 + 2x^2}$

(d) $f(x) = \dfrac{1}{1 + x^2}$

(e) $f(x) = \dfrac{x}{1 + x^2}$

(b) Product Rule + Chain Rule + Power Rule

$f'(x) = (10x-2)(4x-2)^7 + (5x^2-2x+1)[7(4x-2)^6(4)]$

$f'(x) = (10x-2)(4x-2)^7 + 28(5x^2-2x+1)(4x-2)^6$

(c) Chain Rule + Power Rule

$f(x) = (1+2x^2)^{\frac{1}{2}}$

$f'(x) = \frac{1}{\cancel{2}}(1+2x^2)^{-\frac{1}{2}}(\overset{2}{\cancel{4}}x) = 2x(1+2x^2)^{-\frac{1}{2}} = \dfrac{2x}{\sqrt{1+2x^2}}$

(d) $f(x) = \dfrac{1}{1+x^2}$

Quotient Rule + Power Rule

$f'(x) = \dfrac{0(1+x^2)-(1)(2x)}{(1+x^2)^2}$

$f'(x) = \dfrac{-2x}{(1+x^2)^2}$

$\boxed{\text{OR}}$ $f(x) = (1+x^2)^{-1}$ Chain Rule + Power Rule

$f'(x) = -(1+x^2)^{-2}(2x)$

$f'(x) = \dfrac{-2x}{(1+x^2)^2}$

(e) $f(x) = \dfrac{x}{1+x^2}$ [Quotient Rule + Power Rule]

$f'(x) = \dfrac{(1)(1+x^2)-(x)(2x)}{(1+x^2)^2}$

$f'(x) = \dfrac{1+x^2-2x^2}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2}$

5. Find the gradient of the slope of the function $f(x) = (4x - 2)^7$ at the point $x = 0$. Use this to find the equation of the tangent line at this point.

$f'(x) = 7(4x-2)^6(4) = 28(4x-2)^6$

At $x=0$, $f'(0) = 28(-2)^6 = 1792\ (m_t)$

$x=0$

$y = f(0) = (-2)^7 = -128$

Eq of tangent, use $m_t = 1792$ & $(0, -128)$

$y - y_1 = m_t(x - x_1)$

$y + 128 = 1792(x)$

$y = 1792x - 128$

6. Find $\frac{dy}{dx}$ when:

(a) $y = \frac{x-3}{x-2}$

(b) $y = \frac{1}{\sqrt{x}}$

(c) $y = \left(\frac{1}{\sqrt{x}} + \sqrt{x}\right)^2$

(d) $y = \frac{x}{\sqrt{x^2-1}}$

(a) Quotient Rule

$$\frac{dy}{dx} = \frac{(1)(x-2) - (1)(x-3)}{(x-2)^2} = \frac{1}{(x-2)^2}$$

(b) $y = x^{-\frac{1}{2}}$ [Power Rule]

$$\frac{dy}{dx} = -\frac{1}{2}x^{-3/2}$$

5(c) $y = \left(\frac{1}{\sqrt{x}} + \sqrt{x}\right)^2 = \frac{1}{x} + 2 + x = x^{-1} + 2 + x$ [Power Rule]

$$\frac{dy}{dx} = -x^{-2} + 0 + 1 = 1 - \frac{1}{x^2}$$

[Quotient Rule + Power Rule + Chain Rule]

(d) $$\frac{dy}{dx} = \frac{(1)\sqrt{x^2-1} - (x)\left[\cancel{\tfrac{1}{2}}(x^2-1)^{-\frac{1}{2}}(\cancel{2}x)\right]}{x^2-1}$$ $\left[\sqrt{x^2-1} = (x^2-1)^{\frac{1}{2}}\right]$

$$\frac{dy}{dx} = \frac{\sqrt{x^2-1} - \frac{x^2}{\sqrt{x^2-1}}}{x^2-1}$$

$$\frac{dy}{dx} = \frac{\frac{\cancel{x^2}-1-\cancel{x^2}}{\sqrt{x^2-1}}}{x^2-1}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{x^2-1}} \times \frac{1}{x^2-1}$$

$$\frac{dy}{dx} = \frac{-1}{(x^2-1)^{3/2}} \Big/ -(x^2-1)^{-3/2}$$

Recall

$\left[a^m \times a^n = a^{m+n} \quad \sqrt{x^2-1} = (x^2-1)^{\frac{1}{2}}\right]$

Q6(a)

7. Find the slope of the tangent line for the function $y = \frac{x-3}{x-2}$ at the point $x = 0$. Use this to find the equation of the tangent line at this point. Make a sketch to help support your answer.

$$\frac{dy}{dx} = \frac{1}{(x-2)^2}$$

$\left.\frac{dy}{dx}\right|_{x=0} = \frac{1}{4}$ (m_t), $\quad x = 0$, $y = 3/2$ $\quad (0, 3/2)$

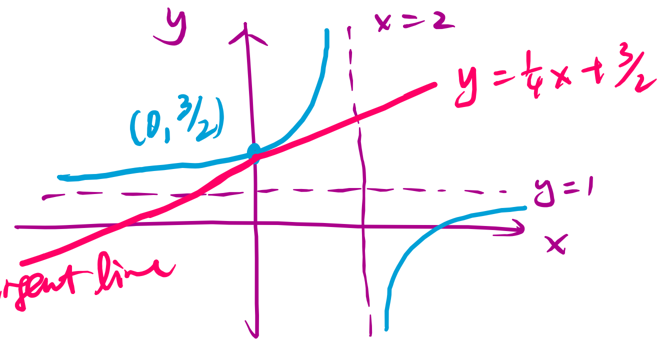
sketch graph

x						
y						

Eqn of tangent

$$y - 3/2 = \frac{1}{4}(x-0)$$

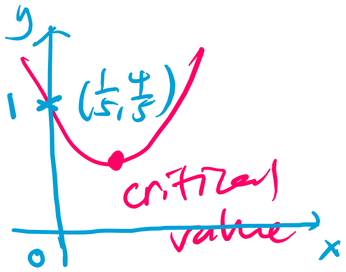
$$y = \frac{1}{4}x + 3/2$$



$$y = \frac{x-3}{x-2} = \frac{x-2}{x-2} + \frac{-1}{x-2}$$

$$= 1 - \frac{1}{x-2}$$

8. For the function $f(x) = 5x^2 - 2x + 1$, find the value of c such that $f'(c) = 0$. Find the value of $f(x)$ at c, ie find $f(c)$. What is the significance of the point $(c, f(c))$? Use this point to sketch the graph of $f(x)$.



$f'(x) = 10x - 2$

Given $f'(c) = 0$

$10c - 2 = 0$

$c = \frac{1}{5}$

when $c = \frac{1}{5}$

$f(x) = f(c) = f(\frac{1}{5})$

$= 5(\frac{1}{5})^2 - 2(\frac{1}{5}) + 1$

$= \frac{4}{5}$

$\boxed{\text{1st Derivative Test}}$

	$(x=0)$		$(x=1)$
Sign $f'(x)$	$-$	0	$+$
		$c=\frac{1}{5}$	

$\Rightarrow (\frac{1}{5}, \frac{4}{5})$ local minimum

9. Find the interval/s where the function $f(x) = x^3 - 2x$ is (a) increasing, (b) decreasing, and (c) constant.

$f'(x) = 3x^2 - 2$

$f'(x) = 0 \Rightarrow$ critical x-value(s)

$3x^2 - 2 = 0$

$x = \pm\sqrt{\frac{2}{3}}$

1st Derivative Test

Sign $f'(x)$	$+$	0	$-$	0	$+$
		$-\sqrt{\frac{2}{3}}$		$\sqrt{\frac{2}{3}}$	
slope	$+$	constant	$-$	constant	$+$

(a) Increasing

$x < -\sqrt{\frac{2}{3}}$ or $x > \sqrt{\frac{2}{3}}$

(b) Decreasing $-\sqrt{\frac{2}{3}} < x < \sqrt{\frac{2}{3}}$

(c) constant

$x = -\sqrt{\frac{2}{3}}$ or $+\sqrt{\frac{2}{3}}$

10. Identify the local maxima and minima for the function $y = x^3 + 4x^2 + 5x + 2$ over the interval R.

$\frac{dy}{dx} = 3x^2 + 8x + 5$

$\frac{dy}{dx} = 0$ (critical x-value(s))

$3x^2 + 8x + 5 = 0$

$(3x + 5)(x + 1) = 0$

$x = -\frac{5}{3}$ $\quad x = -1$

$y = \frac{4}{27}$ $\quad y = 0$

1st Derivative Test

Sign $\frac{dy}{dx}$	$+$	0	$-$	0	$+$
		$-\frac{5}{3}$		-1	
		local max		local min	
		$(-\frac{5}{3}, \frac{4}{27})$		$(-1, 0)$	