Monash University
Faculty of Information Technology
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FIT2014
Solutions for Exercises 1
Languages and Logic

1.

(a) $v_{0,1}$

(b) $\neg v_{0,2}$

(c) $v_{0,1} \wedge \neg v_{0,2} \wedge \neg v_{0,3} \wedge \neg v_{0,4}$

(d) $v_{5,1} \vee v_{5,2} \vee v_{5,3} \vee v_{5,4}$

(e) $\neg v_{5,1} \vee \neg v_{5,3}$

(f) $(\neg v_{5,1} \vee \neg v_{5,2}) \wedge (\neg v_{5,1} \vee \neg v_{5,3}) \wedge (\neg v_{5,1} \vee \neg v_{5,4}) \wedge (\neg v_{5,2} \vee \neg v_{5,3}) \wedge (\neg v_{5,2} \vee \neg v_{5,4}) \wedge (\neg v_{5,3} \vee \neg v_{5,4})$

An alternative but more complicated answer is:

$$\begin{array}{lll} (v_{5,1} \Rightarrow (\neg v_{5,2} \wedge \neg v_{5,3} \wedge \neg v_{5,4})) & \wedge & (v_{5,2} \Rightarrow (\neg v_{5,1} \wedge \neg v_{5,3} \wedge \neg v_{5,4})) \quad \wedge \\ (v_{5,3} \Rightarrow (\neg v_{5,1} \wedge \neg v_{5,2} \wedge \neg v_{5,4})) & \wedge & (v_{5,4} \Rightarrow (\neg v_{5,1} \wedge \neg v_{5,2} \wedge \neg v_{5,3})) \end{array}$$

This could be turned into CNF by first expressing each $v_{5,i} \Rightarrow (\cdots)$ as $\neg v_{5,i} \vee (\cdots)$ and then expanding each $\neg v_{5,i} \vee (\cdots)$ using the Distributive Law. This would eventually lead to the first expression given above.

Another alternative answer[1] is

$$(\neg v_{5,2} \wedge \neg v_{5,3} \wedge \neg v_{5,4}) \vee (\neg v_{5,1} \wedge \neg v_{5,3} \wedge \neg v_{5,4}) \vee (\neg v_{5,1} \wedge \neg v_{5,2} \wedge \neg v_{5,4}) \vee (\neg v_{5,1} \wedge \neg v_{5,2} \wedge \neg v_{5,3})$$

This is in DNF, which is further from CNF. It would be a lot of work, using the Distributive Law, to convert it to CNF. But it's still correct as an answer to (f), since CNF is not considered until (i).

(g) This is the conjunction of answers to (d) and (f):
$(v_{5,1} \vee v_{5,2} \vee v_{5,3} \vee v_{5,4}) \wedge$
$(\neg v_{5,1} \vee \neg v_{5,2}) \wedge (\neg v_{5,1} \vee \neg v_{5,3}) \wedge (\neg v_{5,1} \vee \neg v_{5,4}) \wedge (\neg v_{5,2} \vee \neg v_{5,3}) \wedge (\neg v_{5,2} \vee \neg v_{5,4}) \wedge (\neg v_{5,3} \vee \neg v_{5,4})$

(h) $v_{3,2} \Rightarrow (v_{4,1} \vee v_{4,3})$ or $\neg v_{3,2} \vee v_{4,1} \vee v_{4,3}$

(i)

single-literal clauses for time 0	as in (c)		4
At each time 1,2,3, you are in exactly one square.	3 chunks like (g) = 3×7	=	21
From one time to the next, you can only go left or right.	4×3 clauses like (h)	=	12
	TOTAL		37

[1]Thanks to FIT2014 tutor Raphael Morris for this solution.

Notes:

- Some of the expressions for the transitions are simpler than (h). If, at time t, you are at square 1, then the transition expression is just $v_{t,1} \Rightarrow v_{t+1,2}$, since you cannot move backwards. But it's still just one clause in the CNF expression.
- The expression from (c) insists that you are in square 1 at time 0, so for the transition to time 1 you can insist that $v_{1,2}$ is True. So, instead of the expression $v_{0,1} \Rightarrow v_{1,2}$, it's sufficient to just use the single-variable expression $v_{1,2}$. This cannot be done at later times, because the possible locations are not so constrained.

It can also be done with fewer clauses, using a more detailed analysis of exactly where you can be at each time.[2]

single-literal clauses for time 0	as in (c)	4
At each time 1, you are in square 2 and nowhere else.	like (c)	4
At each time 2, you're in exactly one of squares 1 & 3.		4
• two single literal clauses to exclude squares 2 & 4;	$(\neg v_{2,2}) \wedge (\neg v_{2,4})$	
• two two-literal clauses for		
<u>at least</u> one of squares 1 & 3,	$v_{2,1} \vee v_{2,3}$	
<u>at most</u> one of squares 1 & 3.	$\neg v_{2,1} \vee \neg v_{2,3}$	
At each time 3, you're in exactly one of squares 2 & 4.	similar to time 2	4
At time 2, if you're in square 4, you must go backwards.	$\neg v_{2,4} \vee v_{3,3}$	1
	TOTAL	17

2. We must show that, if $w \in$ PALINDROMES then $w \in \overline{\text{ODD-ODD}}$.

Suppose $w \in$ PALINDROMES. Then there is a string x such that *either* $w = x\overleftarrow{x}$ *or* $w = xy\overleftarrow{x}$, where $\overleftarrow{x}$ denotes the reverse of x and $y \in \{\text{a}, \text{b}\}$ is a single letter.

Suppose $w = x\overleftarrow{x}$. The numbers of a's and b's in $x\overleftarrow{x}$ are both even, since each is twice the number in x. So $w \in \overline{\text{ODD-ODD}}$.

Now suppose $w = xy\overleftarrow{x}$. Whichever letter in $\{\text{a}, \text{b}\}$ is *not* y must appear an even number of times in w, by the same argument we have used previously. So that letter does not appear an odd number of times in w. So $w \in \overline{\text{ODD-ODD}}$.

Alternative argument for the second case, pointed out by an FIT2014 student in 2013:

Now suppose $w = xy\overleftarrow{x}$. Since the length of w is odd, then it cannot have both an odd number of a's and an odd number of b's (else its length would be even). So $w \in \overline{\text{ODD-ODD}}$.

3. (a) We prove it by constructing the truth table of each. It can be convenient to do this in stages.

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$
F	F	F	F	F
F	F	T	F	F
F	T	F	F	F
F	T	T	T	T
T	F	F	F	T
T	F	T	F	T
T	T	F	F	T
T	T	T	T	T

P	Q	R	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$
F	F	F	F	F	F
F	F	T	F	T	F
F	T	F	T	F	F
F	T	T	T	T	T
T	F	F	T	T	T
T	F	T	T	T	T
T	T	F	T	T	T
T	T	T	T	T	T

[2]Thanks to FIT2014 tutors Thomas Hendrey and Rebecca Young for this solution.

The right-hand columns of each table are identical, so the two expressions (at the tops of those columns) are logically equivalent.

(b)

$$
\begin{aligned}
P \wedge (Q \vee R) &= \neg\neg P \wedge (\neg\neg Q \vee \neg\neg R) \\
&= \neg\neg P \wedge \neg(\neg Q \wedge \neg R) \qquad \text{(by one of De Morgan's Laws)} \\
&= \neg(\neg P \vee (\neg Q \wedge \neg R)) \qquad \text{(by the other of De Morgan's Laws)} \\
&= \neg((\neg P \vee \neg Q) \wedge (\neg P \vee \neg R)) \qquad \text{(by part (a) of this question)} \\
&= \neg(\neg(P \wedge Q) \wedge \neg(P \wedge R)) \qquad \text{(by De Morgan, twice)} \\
&= \neg\neg(P \wedge Q) \vee \neg\neg(P \wedge R) \qquad \text{(by De Morgan, one last time)} \\
&= (P \wedge Q) \vee (P \wedge R)
\end{aligned}
$$

Here we have used equality to stand for logical equivalence, which is normal.

4.

$$
\begin{aligned}
(P_1 \wedge \cdots \wedge P_n) \Rightarrow C &= \neg(P_1 \wedge \cdots \wedge P_n) \vee C \\
&= (\neg P_1 \vee \cdots \vee \neg P_n) \vee C,
\end{aligned}
$$

using De Morgan's Law.

Remark:

A disjunction of the form $\neg P_1 \vee \cdots \vee \neg P_n \vee C$, where $P_1, \ldots, P_n, C$ are each variables that can be True or False, is called a *Horn clause*. These play a big role in the theory of logic programming.

5.

(a) $\neg\mathsf{Judith} \vee \neg\mathsf{Margaret}$

(b) $(\mathsf{Judith} \vee \mathsf{Margaret}) \wedge (\neg\mathsf{Judith} \vee \neg\mathsf{Margaret})$

(c) $\mathsf{Judith} \vee \mathsf{Margaret} \vee \mathsf{Katherine}$

(d) $(\neg\mathsf{Judith} \vee \neg\mathsf{Margaret}) \wedge (\neg\mathsf{Judith} \vee \neg\mathsf{Katherine}) \wedge (\neg\mathsf{Margaret} \vee \neg\mathsf{Katherine})$

(e)

$(\mathsf{Judith} \vee \mathsf{Margaret} \vee \mathsf{Katherine}) \wedge (\neg\mathsf{Judith} \vee \neg\mathsf{Margaret}) \wedge (\neg\mathsf{Judith} \vee \neg\mathsf{Katherine}) \wedge (\neg\mathsf{Margaret} \vee \neg\mathsf{Katherine})$

(f) $(\mathsf{Judith} \vee \mathsf{Margaret}) \wedge (\mathsf{Judith} \vee \mathsf{Katherine}) \wedge (\mathsf{Margaret} \vee \mathsf{Katherine})$

(g) $\neg\mathsf{Judith} \vee \neg\mathsf{Margaret} \vee \neg\mathsf{Katherine}$

(h)

$(\neg\mathsf{Judith} \vee \neg\mathsf{Margaret} \vee \neg\mathsf{Katherine}) \wedge (\mathsf{Judith} \vee \mathsf{Margaret}) \wedge (\mathsf{Judith} \vee \mathsf{Katherine}) \wedge (\mathsf{Margaret} \vee \mathsf{Katherine})$

(i) $\mathsf{Judith} \wedge \mathsf{Margaret} \wedge \mathsf{Katherine}$

(j) $\neg\mathsf{Judith} \wedge \neg\mathsf{Margaret} \wedge \neg\mathsf{Katherine}$

6. Suggested approach:

1. Write the sentence as a conjunction of two smaller sentences.
2. In this case, you can write each of the two smaller sentences naturally using implication.
3. Then write each implication $A \Rightarrow B$ in the form $\neg A \vee B$.
4. Use Boolean algebra, as needed, to get the expression into CNF.

Following this approach:

1. Rewriting our original sentence as a conjunction of smaller sentences:

$$\begin{array}{c}\text{(If the graph is a tree, then it's bipartite and has a leaf)}\\ \wedge\ \text{(if it's not a tree, then it has a vertex of degree} \geq 2)\end{array}$$

2. The two smaller sentences can each be written as a logical implication:

if the graph is a tree, then it's bipartite and has a leaf	$\mathsf{Tree} \Rightarrow (\mathsf{Bipartite} \wedge \mathsf{Leaf})$
if it's not a tree, then it has a vertex of degree ≥ 2	$\neg\,\mathsf{Tree} \Rightarrow \mathsf{Internal}$

So we can rewrite our original sentence further:

$$(\mathsf{Tree} \Rightarrow (\mathsf{Bipartite} \wedge \mathsf{Leaf})) \wedge (\neg\,\mathsf{Tree} \Rightarrow \mathsf{Internal})$$

3. Writing each implication $A \Rightarrow B$ in the form $\neg A \vee B$ turns our sentence into

$$(\neg\,\mathsf{Tree} \vee (\mathsf{Bipartite} \wedge \mathsf{Leaf})) \wedge (\neg\,\neg\,\mathsf{Tree} \vee \mathsf{Internal})$$

4. We're nearly there. The first clause needs to be expanded using the Distributive Law. The second clause just needs a tiny bit of simplification: cancellation of the double negative.

$$(\neg\,\mathsf{Tree} \vee \mathsf{Bipartite}) \wedge (\neg\,\mathsf{Tree} \vee \mathsf{Leaf}) \wedge (\mathsf{Tree} \vee \mathsf{Internal})$$

7.

$$\begin{aligned}&(a \vee b) \wedge (\neg a \vee \neg b) \wedge \\ &(a \vee c \vee d) \wedge (\neg a \vee \neg c) \wedge (\neg a \vee \neg d) \wedge (\neg c \vee \neg d) \wedge \\ &(b \vee c \vee e) \wedge (\neg b \vee \neg c) \wedge (\neg b \vee \neg e) \wedge (\neg c \vee \neg e) \wedge \\ &(d \vee e) \wedge (\neg d \vee \neg e).\end{aligned}$$

8.

i. **taller(father(max), max)** $\wedge\neg$ **taller(father(max), father(claire))**

ii. $\exists X$ **taller(X, father(claire))**

iii.[3] $\forall X \exists Y\ \neg(X = Y) \wedge$ **taller(X, Y)**

iv. $\forall X$ **(taller(X, claire)** $\Rightarrow$ **taller(X, max))**

9.

(a) $\exists n\, \forall y : y \in L \ \Rightarrow\ |y| < n$
or
$\exists n\, \forall y \in L : |y| < n$

(b) $\forall n\, \exists y : y \in L \ \wedge\ |y| > n$
or
$\forall n\, \exists y \in L : |y| > n$

[3] Thanks to FIT2014 tutor Raphael Morris for a correction.

Supplementary exercises

10.

1. $\mathbf{S_A} = \mathbf{B} \wedge \neg\mathbf{C}$
2. $\mathbf{S_B} = \mathbf{A} \to \mathbf{C}$
3. $\mathbf{S_C} = \neg\mathbf{C} \wedge (\mathbf{A} \vee \mathbf{B})$
4. Yes, since
$$(\mathbf{S_A} \wedge \mathbf{S_B} \wedge \mathbf{S_C}) \to (\mathbf{B} \wedge \neg\mathbf{A} \wedge \neg\mathbf{C})$$
is a tautology.

11. If you apply the Distributive Law to the expression, then you obtain a large disjunction in which each disjunct contains:

- one literal from Harry, Ron, Hermione, Ginny (four options);
- one literal from ¬ Hagrid, Norberta (two options);
- either Fred ∧ George or ¬Fred ∧ ¬George (two options);
- and expanding the fourth "row" of the original expression gives:

$$\begin{array}{ll} & (\neg\text{Voldemort} \wedge \neg\text{Voldemort} \wedge \neg\text{Bellatrix}) \\ \vee & (\neg\text{Voldemort} \wedge \neg\text{Bellatrix} \wedge \neg\text{Dolores}) \\ \vee & (\neg\text{Voldemort} \wedge \neg\text{Dolores} \wedge \neg\text{Bellatrix}) \\ \vee & (\neg\text{Voldemort} \wedge \neg\text{Dolores} \wedge \neg\text{Dolores}) \\ \vee & (\neg\text{Bellatrix} \wedge \neg\text{Voldemort} \wedge \neg\text{Bellatrix}) \\ \vee & (\neg\text{Bellatrix} \wedge \neg\text{Bellatrix} \wedge \neg\text{Dolores}) \\ \vee & (\neg\text{Bellatrix} \wedge \neg\text{Dolores} \wedge \neg\text{Bellatrix}) \\ \vee & (\neg\text{Bellatrix} \wedge \neg\text{Dolores} \wedge \neg\text{Dolores}). \end{array}$$

But some of these simplify, and others are duplicates and can be omitted, leading to:

$$\begin{array}{ll} & (\neg\text{Voldemort} \wedge \neg\text{Bellatrix}) \\ \vee & (\neg\text{Voldemort} \wedge \neg\text{Bellatrix} \wedge \neg\text{Dolores}) \\ \vee & (\neg\text{Voldemort} \wedge \neg\text{Dolores}) \\ \vee & (\neg\text{Bellatrix} \wedge \neg\text{Dolores}). \end{array}$$

This in turn simplifies to:

$$\begin{array}{ll} & (\neg\text{Voldemort} \wedge \neg\text{Bellatrix}) \\ \vee & (\neg\text{Voldemort} \wedge \neg\text{Dolores}) \\ \vee & (\neg\text{Bellatrix} \wedge \neg\text{Dolores}). \end{array}$$

So there are three options.

Observe that each of the four parts of the CNF expression in the question contains variables that do not appear in any other part. It follows that the choices we make, when forming disjuncts using the Distributive Law, are independent. So the total number of disjuncts is $4 \times 2 \times 2 \times 3 = 48$. Each of these 48 disjuncts has six literals. It is clear that the DNF expression is much bigger than the CNF expression.

This answer is not unique, in the sense that there are other DNF expressions equivalent to the original CNF expression that have different sizes. One way to get such expressions is to not do all the simplifications mentioned above, so that the DNF expression obtained would be even larger.

12. (a) The Enrolment Rule specifies a set of conditions that must all hold. So we can begin by expressing it as a conjunction of three simpler conditions, each corresponding to one of the three parts of the Enrolment Rule:

$$\begin{aligned}
& (\text{at least one of FIT1045, FIT1048, FIT1051, FIT1053, ENG1003, ENG1013, FIT1040} \wedge \text{FIT1029} \\
& \quad \text{is required}) \\
\wedge\ & (\text{at least one of MAT1830, MTH1030, MTH1035, ENG1005 is required}) \\
\wedge\ & (\text{CSE2303 is prohibited})
\end{aligned}$$

Each of the three parts can itself be expressed as a *disjunction* of simpler propositions:

$$\begin{aligned}
& (\text{FIT1045} \vee \text{FIT1048} \vee \text{FIT1051} \vee \text{FIT1053} \vee \text{ENG1003} \vee \text{ENG1013} \vee (\text{FIT1040} \wedge \text{FIT1029})) \\
\wedge\ & (\text{MAT1830} \vee \text{MTH1030} \vee \text{MTH1035} \vee \text{ENG1005}) \\
\wedge\ & (\neg\, \text{CSE2303})
\end{aligned}$$

We're almost there! The second part here is ok, as it's a disjunction of four literals. The third part is also ok, since it's just a single literal (being the negation of CSE2303). But the first part is not ok for CNF yet. It's a disjunction, but it's not quite a disjunction *of literals*. Rather, it's a disjunction of six literals and another expression. Fortunately, we can expand this into a conjunction of two parts, using the Distributive Law. It currently has the form $A \vee (B \wedge C)$, where A is a disjunction of six literals and B and C are literals. The Distributive Law tells us that this is equivalent to $(A \vee B) \wedge (A \vee C)$, which is in CNF. So our whole expression is equivalent to

$$\begin{aligned}
& (\text{FIT1045} \vee \text{FIT1048} \vee \text{FIT1051} \vee \text{FIT1053} \vee \text{ENG1003} \vee \text{ENG1013} \vee \text{FIT1040}) \\
\wedge\ & (\text{FIT1045} \vee \text{FIT1048} \vee \text{FIT1051} \vee \text{FIT1053} \vee \text{ENG1003} \vee \text{ENG1013} \vee \text{FIT1029}) \\
\wedge\ & (\text{MAT1830} \vee \text{MTH1030} \vee \text{MTH1035} \vee \text{ENG1005}) \\
\wedge\ & (\neg\, \text{CSE2303})
\end{aligned}$$

This is now in CNF.

(b) The question only asked for three disjuncts, but to cover many of the possibilities, here's a complete DNF expression equivalent to the above CNF expression:

$$\begin{aligned}
& (\text{FIT1045} \wedge \text{MAT1830} \wedge \neg\, \text{CSE2303}) \\
\vee\ & (\text{FIT1045} \wedge \text{MTH1030} \wedge \neg\, \text{CSE2303}) \\
\vee\ & (\text{FIT1045} \wedge \text{MTH1035} \wedge \neg\, \text{CSE2303}) \\
\vee\ & (\text{FIT1045} \wedge \text{ENG1005} \wedge \neg\, \text{CSE2303}) \\
\vee\ & (\text{FIT1048} \wedge \text{MAT1830} \wedge \neg\, \text{CSE2303}) \\
\vee\ & (\text{FIT1048} \wedge \text{MTH1030} \wedge \neg\, \text{CSE2303}) \\
\vee\ & (\text{FIT1048} \wedge \text{MTH1035} \wedge \neg\, \text{CSE2303})
\end{aligned}$$

$$
\begin{aligned}
&\lor \quad (\mathrm{FIT1048} \land \mathrm{ENG1005} \land \neg\, \mathrm{CSE2303}) \\
&\lor \quad (\mathrm{FIT1051} \land \mathrm{MAT1830} \land \neg\, \mathrm{CSE2303}) \\
&\lor \quad (\mathrm{FIT1051} \land \mathrm{MTH1030} \land \neg\, \mathrm{CSE2303}) \\
&\lor \quad (\mathrm{FIT1051} \land \mathrm{MTH1035} \land \neg\, \mathrm{CSE2303}) \\
&\lor \quad (\mathrm{FIT1051} \land \mathrm{ENG1005} \land \neg\, \mathrm{CSE2303}) \\
&\lor \quad (\mathrm{FIT1053} \land \mathrm{MAT1830} \land \neg\, \mathrm{CSE2303}) \\
&\lor \quad (\mathrm{FIT1053} \land \mathrm{MTH1030} \land \neg\, \mathrm{CSE2303}) \\
&\lor \quad (\mathrm{FIT1053} \land \mathrm{MTH1035} \land \neg\, \mathrm{CSE2303}) \\
&\lor \quad (\mathrm{FIT1053} \land \mathrm{ENG1005} \land \neg\, \mathrm{CSE2303}) \\
&\lor \quad (\mathrm{ENG1003} \land \mathrm{MAT1830} \land \neg\, \mathrm{CSE2303}) \\
&\lor \quad (\mathrm{ENG1003} \land \mathrm{MTH1030} \land \neg\, \mathrm{CSE2303}) \\
&\lor \quad (\mathrm{ENG1003} \land \mathrm{MTH1035} \land \neg\, \mathrm{CSE2303}) \\
&\lor \quad (\mathrm{ENG1003} \land \mathrm{ENG1005} \land \neg\, \mathrm{CSE2303}) \\
&\lor \quad (\mathrm{ENG1013} \land \mathrm{MAT1830} \land \neg\, \mathrm{CSE2303}) \\
&\lor \quad (\mathrm{ENG1013} \land \mathrm{MTH1030} \land \neg\, \mathrm{CSE2303}) \\
&\lor \quad (\mathrm{ENG1013} \land \mathrm{MTH1035} \land \neg\, \mathrm{CSE2303}) \\
&\lor \quad (\mathrm{ENG1013} \land \mathrm{ENG1005} \land \neg\, \mathrm{CSE2303}) \\
&\lor \quad (\mathrm{FIT1040} \land \mathrm{FIT1029} \land \mathrm{MAT1830} \land \neg\, \mathrm{CSE2303}) \\
&\lor \quad (\mathrm{FIT1040} \land \mathrm{FIT1029} \land \mathrm{MTH1030} \land \neg\, \mathrm{CSE2303}) \\
&\lor \quad (\mathrm{FIT1040} \land \mathrm{FIT1029} \land \mathrm{MTH1035} \land \neg\, \mathrm{CSE2303}) \\
&\lor \quad (\mathrm{FIT1040} \land \mathrm{FIT1029} \land \mathrm{ENG1005} \land \neg\, \mathrm{CSE2303})
\end{aligned}
$$

This expression can be obtained from the CNF expression by applying the Distributive Law. Each disjunct is a conjunction of:

- one of the seven propositions in the list FIT1045, FIT1048, FIT1051, FIT1053, ENG1003, ENG1013, FIT1040 $\land$ FIT1029 (with FIT1040 $\land$ FIT1029 treated as a single proposition), AND
- one of the four propositions in the list MAT1830, MTH1030, MTH1035, ENG1005, AND
- the single proposition $\neg$ CSE2303.

The question only asks for three disjuncts, so you can give any three of the above disjuncts.

There are also some more complicated ways of writing DNF expressions equivalent to the CNF expression from (a).

(c) 28 disjuncts, if the above approach is used. There are more complicated expressions with more disjuncts. But there should not be any correct expressions with fewer than 28 disjuncts.

13.

$$
\begin{aligned}
&\quad (\mathsf{Leonard} \lor \mathsf{Cedric} \lor \mathsf{Arthur}) \ \land\ (\mathsf{Leonard} \lor \mathsf{Cedric} \lor \mathsf{Bill}) \\
&\land\ (\mathsf{Leonard} \lor \mathsf{Arthur} \lor \mathsf{Bill}) \ \land\ (\mathsf{Cedric} \lor \mathsf{Arthur} \lor \mathsf{Bill}) \\
&\land\ (\neg\mathsf{Leonard} \lor \neg\mathsf{Cedric} \lor \neg\mathsf{Arthur}) \ \land\ (\neg\mathsf{Leonard} \lor \neg\mathsf{Cedric} \lor \neg\mathsf{Bill}) \\
&\land\ (\neg\mathsf{Leonard} \lor \neg\mathsf{Arthur} \lor \neg\mathsf{Bill}) \ \land\ (\neg\mathsf{Cedric} \lor \neg\mathsf{Arthur} \lor \neg\mathsf{Bill})
\end{aligned}
$$

14.
(a) $L_{B,n} \vee L_{W,n} \vee L_{U,n}$

(b) If $L_{B,n}$ is true, then it's ok for vertex $n+1$ to be Black (since it then joins the black chain that includes vertex n, which must be ok as the position up to vertex n is legal). It could also be Uncoloured, since adding a new uncoloured vertex next to an existing vertex can never make a legal position illegal. But vertex $n+1$ cannot be White, as it is then in a chain of its own which has no Uncoloured neighbour.
Similarly, if $L_{W,n}$ is true, then vertex $n+1$ can be White or Uncoloured, but it cannot be Black.
Lastly, if $L_{U,n}$ is true, then vertex $n+1$ can be in any of the three states, since if it is coloured then it forms a chain of one vertex that already has an uncoloured neighbour, namely vertex n.

(c) If $A_{B,n}$ is true, then vertex $n+1$ can be Uncoloured, since that never hurts legality. But it cannot be Black or White. If it were Black, then it would join the Black chain that contains vertex n but does not yet have an uncoloured neighbour, so the position would remain almost legal but it wouldn't be legal. If vertex n were White, it would become a single-vertex chain with no uncoloured neighbour, so the position would be illegal. (Furthermore, the Black chain containing vertex n would not have an uncoloured neighbour, giving another reason for illegality, so the position is now not even *almost* legal.)
The same holds true for $A_{W,n}$: vertex $n+1$ can be Uncoloured, but not Black or White.
$A_{U,n}$ is impossible, since an almost legal position must have its final vertex coloured.

(d)
$L_{B,n+1}$ can be expressed as
$$(L_{B,n} \vee L_{U,n}) \wedge V_{B,n+1}.$$

$L_{W,n+1}$ can be expressed as
$$(L_{W,n} \vee L_{U,n}) \wedge V_{W,n+1}.$$

$L_{U,n+1}$ can be expressed as
$$(L_{B,n} \vee L_{W,n} \vee L_{U,n} \vee A_{B,n} \vee A_{W,n}) \wedge V_{U,n+1}.$$

$A_{B,n+1}$ can be expressed as
$$(L_{W,n} \vee A_{B,n}) \wedge V_{B,n+1}.$$

$A_{W,n+1}$ can be expressed as
$$(L_{B,n} \vee A_{W,n}) \wedge V_{W,n+1}.$$

15. (a)
$$\forall X \, \forall Y \;:\; (\mathbf{vertex}(X) \wedge \mathbf{vertex}(Y) \wedge \mathbf{edge}(X,Y) \Rightarrow (\mathbf{chosen}(X) \vee \mathbf{chosen}(Y))$$

(b)
$$\forall X \, \forall Y \;:\; (\mathbf{vertex}(X) \wedge \mathbf{vertex}(Y) \Rightarrow \mathbf{edge}(X,Y))$$

(c) Denote the set of edges of a graph G by $E(G)$. So the set of edges of $\overline{G}$ is denoted by $E(\overline{G})$.

We prove that, for any set U of vertices of G, this set U is a vertex cover of G if and only if $V \setminus U$ is a clique in $\overline{G}$.

U is a vertex cover of G

$\iff$ Every edge of G has at least one of its endpoints in U

$\iff$ For every pair of vertices u, v, if they're adjacent in G then $u \in U$ or $v \in U$

$\iff$ $\forall u \forall v \ : \ uv \in E(G) \Rightarrow (u \in U \vee v \in U)$

$\iff$ $\forall u \forall v \ : \ \neg (uv \in E(G)) \vee (u \in U \vee v \in U)$ (rewriting $P \Rightarrow Q$ as $\neg P \vee Q$)

$\iff$ $\forall u \forall v \ : \ uv \notin E(G)) \vee (u \in U \vee v \in U)$

$\iff$ $\forall u \forall v \ : \ uv \in E(\overline{G})) \vee (u \in U \vee v \in U)$ (using definition of $\overline{G}$)

$\iff$ $\forall u \forall v \ : \ uv \in E(\overline{G})) \vee \neg (\neg (u \in U) \wedge \neg (v \in U))$ (by de Morgan's Law)

$\iff$ $\forall u \forall v \ : \ uv \in E(\overline{G})) \vee \neg (u \notin U \wedge v \notin U)$

$\iff$ $\forall u \forall v \ : \ uv \in E(\overline{G})) \vee \neg (u \in V \setminus U \wedge v \in V \setminus U)$ (using definition of $V \setminus U$)

$\iff$ $\forall u \forall v \ : \ \neg (u \in V \setminus U \wedge v \in V \setminus U) \vee uv \in E(\overline{G}))$ (just re-ordering; unnecessary; purely cosmetic)

$\iff$ $\forall u \forall v \ : \ (u \in V \setminus U \wedge v \in V \setminus U) \Rightarrow uv \in E(\overline{G}))$ (rewriting $\neg A \vee B$ as $A \Rightarrow B$)

$\iff$ Every pair of vertices in $V \setminus U$ is adjacent in $\overline{G}$

$\iff$ $V \setminus U$ is a clique of $\overline{G}$

It follows that the mapping $U \mapsto V \setminus U$, which takes the *set* complement[4] of U within the entire vertex set V, is a bijection from vertex covers of size k in G to cliques of size $n-k$ in $\overline{G}$.

(d)

$$\text{size of smallest vertex cover in } G \quad = \quad n \ - \ \text{size of largest clique in } \overline{G}.$$

[4] *not* to be confused with the *graph* complement, $G \mapsto \overline{G}$.