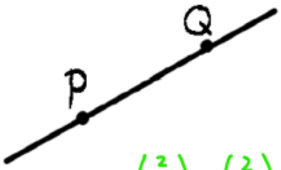
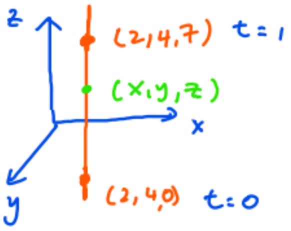
1.4 Lines in 3-dimensional space

Through any pair of distinct points we can always construct a straight line. These lines are normally drawn to be infinitely long in both directions.

$y = mx + c$

gradient ↑ y-intercept ↑





↓ position vector

Let $\underset{\sim}{a}$ = one of the on the line

$\underset{\sim}{v}$ = direction $= \begin{pmatrix}2\\4\\7\end{pmatrix} - \begin{pmatrix}2\\4\\0\end{pmatrix} = \begin{pmatrix}0\\0\\7\end{pmatrix}$

Example 1.14. Find all points on the line joining (2, 4, 0) and (2, 4, 7).

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\4\\0\end{pmatrix} + t\begin{pmatrix}0\\0\\7\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\4\\0\end{pmatrix} + t\begin{pmatrix}0\\0\\1\end{pmatrix}$$

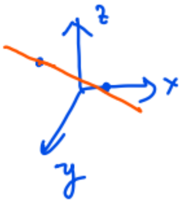
t=0

t=1

$\underset{\sim}{u} // \underset{\sim}{v}$

$\underset{\sim}{u} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$\underset{\sim}{v} = \begin{pmatrix}0\\0\\7\end{pmatrix} = 7\begin{pmatrix}0\\0\\1\end{pmatrix}$



Example 1.15. Find all points on the line joining (2, 0, 0) and (2, 4, 7).

$\underset{\sim}{v} = \begin{pmatrix}2\\4\\7\end{pmatrix} - \begin{pmatrix}2\\0\\0\end{pmatrix}$

$= \begin{pmatrix}0\\4\\7\end{pmatrix}$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\0\\0\end{pmatrix} + t\begin{pmatrix}0\\4\\7\end{pmatrix}$$

These equations for the line are all of the form

$$x(t) = a + pt\,, \qquad y(t) = b + qt\,, \qquad z(t) = c + rt$$

where t is a parameter (it selects each point on the line) and the numbers a, b, c, p, q, r are computed from the coordinates of two points on the line. (There are other ways to write an equation for a line.)

How do we compute a, b, c, p, q, r? It is a simple recipe.

- First put $t = 0$, then $x = a, y = b, z = c$. That is (a, b, c) are the coordinates of one point (such as P) on the line and so a, b, c are known.
- Next, put $t = 1$, then $x = a + p, y = b + q, z = c + r$. Take this to be the second point (such as Q) on the line, and thus solve for p, q, r.

A common interpretation is that (a, b, c) are the coordinates of one (any) point on the line and (p, q, r) are the components of a (any) vector parallel to the line.

let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\underset{\sim}{v} = \begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ 6 \end{pmatrix}$

replace $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix} + t \begin{pmatrix} -1 \\ 7 \\ 6 \end{pmatrix}$

Example 1.16. Find the equation of the line joining the two points (1, 7, 3) and (2, 0, −3).

parametric form of line

$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix} + t \begin{pmatrix} -1 \\ 7 \\ 6 \end{pmatrix}$

General form

line eqn (vector form)

$\underset{\sim}{r} = \underset{\sim}{a} + t\underset{\sim}{v}$

$x(t) = 1 - t$

$y(t) = 7 + 7t$

$z(t) = 3 + 6t$

Example 1.17. Show that a line may also be expressed as

$$\frac{x-a}{p} = \frac{y-b}{q} = \frac{z-c}{r}$$

provided $p \neq 0, q \neq 0$ and $r \neq 0$. This is known as the **Symmetric Form** of the equation for a straight line.

From parametric form, make t the subject

$t = -x + 1$ | $t = \dfrac{y-7}{7}$ | $t = \dfrac{z-3}{6}$

symmetric form

$\dfrac{-x+1}{1} = \dfrac{y-7}{7} = \dfrac{z-3}{6}$

Example 1.18. In some cases you may find a small problem with the form suggested in the previous example. What is that problem and how would you deal with it?

denominator $p, q, r \neq 0$

$\underset{\sim}{r} = \underset{\sim}{a} + t\underset{\sim}{v}$

If any of these unknowns are zero, then we have undefined situation.

Recall

solve eqns can get ⇒ 60 lines ()

Example 1.19. Determine if the line defined by the points (1, 0, 1) and (1, 2, 0) intersects with the line defined by the points (3, −1, 0) and (1, 2, 5).

$\underset{\sim}{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$

$\underset{\sim}{r}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+0t \\ 0-2t \\ 1+t \end{pmatrix}$

$\underset{\sim}{r}_2 = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix} = \begin{pmatrix} 3+2s \\ -1-3s \\ 0-5s \end{pmatrix}$

$\begin{cases} 1 = 3 + 2s & — ① \\ -2t = -1 - 3s & — ② \\ 1 + t = -5s & — ③ \end{cases}$

From ① $s = -1$

Subs $s = -1$ into ②

$t = -1$

check in ③

L.H.S ⇒ 0

R.H.S ⇒ 5

Not equal

No intersection!

Example 1.20. Is the line defined by the points (3, 7, −1) and (2, −2, 1) parallel to the line defined by the points (1, 4, −1) and (0, −5, 1).

only Need to check the direction vector.

line 1 $\underset{\sim}{v} = \begin{pmatrix} 3 \\ 7 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \\ -2 \end{pmatrix}$

line 2 $\underset{\sim}{u} = \begin{pmatrix} 1 \\ 9 \\ -2 \end{pmatrix} \Big/ \begin{pmatrix} -1 \\ -9 \\ 2 \end{pmatrix}$

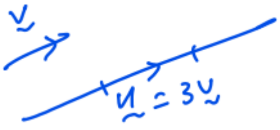
These lines are parallel

Example 1.21. Is the line defined by the points (3, 7, −1) and (2, −2, 1) parallel to the line defined by the points (1, 4, −1) and (−2, −23, 5).

Line 1 $\underset{\sim}{v} = \begin{pmatrix} 1 \\ 9 \\ -2 \end{pmatrix}$

$\underset{\sim}{u} = \begin{pmatrix} 3 \\ 27 \\ -6 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 9 \\ -2 \end{pmatrix}$

$\underset{\sim}{u} = 3\underset{\sim}{v}$



1.4.1 Vector equation of a line

The parametric equations of a line are

$$x(t) = a + pt \qquad y(t) = b + qt \qquad z(t) = c + rt$$

Note that

$$\begin{aligned} (a, b, c) &= \text{the vector to one point } (P) \text{ on the line} \\ (p, q, r) &= \text{the vector from the first point to} \\ & \text{the second point on the line } (P \text{ to } Q) \\ &= \text{a vector } \textit{parallel} \text{ to the line} \end{aligned}$$

Let's relabel these and put $\underset{\sim}{d} = (a, b, c)$, $\underset{\sim}{v} = (p, q, r)$ and $\underset{\sim}{r}(t) = (x(t), y(t), z(t))$, then

$$\text{position vector} \nearrow \qquad \underset{\sim}{r}(t) = \underset{\sim}{d} + t\underset{\sim}{v} \longrightarrow \text{direction}$$

This is known as the **vector equation of a line**.

Example 1.22. Write down the vector equation of the line that passes through the points $(1, 2, 7)$ and $(2, 3, 4)$.

$$\underset{\sim}{r}(t) = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$$

$$\underset{\sim}{r}(t) = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} \quad \text{or} \quad \underset{\sim}{r}(t) = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$$

$$\text{or} \quad \underset{\sim}{r}(t) = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$$

Example 1.23. Write down the vector equation of the line that passes through the points $(2, 3, 7)$ and $(4, 1, 2)$.

$$\underset{\sim}{r} = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} + t \begin{pmatrix} -2 \\ 2 \\ 5 \end{pmatrix}$$

Lines in R^3

The vector equation of a line L is determined using a point P on the line and a vector $\underset{\sim}{v}$ in the direction of the line. Let

$$\underset{\sim}{d} = (a, b, c)$$

be the position vector of P, and

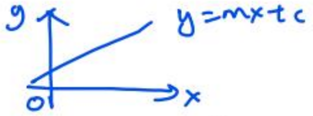
$$\underset{\sim}{v} = (p, q, r)$$

bet a vector parallel to the line, then a line is defined as all vectors which pass through the point P and are parallel to the vector $\underset{\sim}{v}$.

Thus the **vector** (or parametric) equation of the line L is given by

$$\underset{\sim}{r}(t) = \underset{\sim}{d} + t\underset{\sim}{v}$$

where t is a parameter. As t is varied all the points on L are traced out.



1.5 Planes in 3-dimensional space

$ax + by = c$

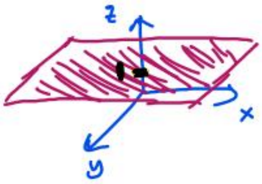
$ax + by + c = 0$

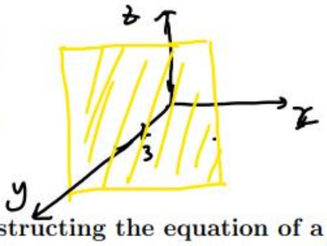
A plane in 3-dimensional space is a flat 2-dimensional surface. The standard equation for a plane in 3-d is

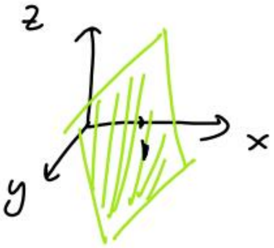
$$ax + by + cz = d$$

where a, b, c and d are some bunch of numbers that identify this plane from all other planes. (There are other ways to write an equation for a plane, as we shall see).

Example 1.24. Sketch each of the planes $z = 1$, $y = 3$ and $x = 1$.







1.5.1 Constructing the equation of a plane

A plane is *uniquely* determined by any three points (provided not all three points are contained on a line). Recall, that a line is fully determined by any pair of points on the line. $ax + by + cz = d$

We can find the equation of the plane that passes through the three points $(1, 0, 0)$, $(0, 3, 0)$ and $(0, 0, 2)$. To do this we need to compute a, b, c and d. We do this by substituting each point into the above equation,

$(1,0,0)$ 1st point $a \cdot 1 + b \cdot 0 + c \cdot 0 = d$ $\Leftrightarrow$ $a = d$

$(0,3,0)$ 2nd point $a \cdot 0 + b \cdot 3 + c \cdot 0 = d$ $\Leftrightarrow$ $3b = d$

$(0,0,2)$ 3rd point $a \cdot 0 + b \cdot 0 + c \cdot 2 = d$ $\Leftrightarrow$ $2c = d$

Now we have a slight problem, we are trying to compute four numbers, a, b, c, d but we only have three equations. We have to make an *arbitrary choice* for one of the four numbers a, b, c, d. Let's set $d = 6$. Then we find from the above that $a = 6$, $b = 2$ and $c = 3$. Thus the equation of the plane is

$$6x + 2y + 3z = 6$$

eg $d = 1$

$a = 1$

$b = \frac{1}{3}$

$c = \frac{1}{2}$

$x + \frac{1}{3}y + \frac{1}{2}z = 1$

$(\times 6)$ $6x + 2y + 3z = 6$

Example 1.25. What equation do you get if you chose $d = 1$ in the previous example? What happens if you chose $d = 0$?

$ax+by+cz=d$

$(-1,0,0) \rightarrow -a = d$

$(1,2,0) \rightarrow a+2b = d$

$(2,-1,5) \rightarrow 2a-b+5c=d$

Example 1.26. Find an equation of the plane that passes through the three points $(-1,0,0)$, $(1,2,0)$ and $(2,-1,5)$.

$d=1$, $a=-1$

$b=1$

$-2-1+5c=1$

$c=\frac{4}{5}$

$-x+y+\frac{4}{5}z=1$

$-5x+5y+4z=5$

$5x-5y-4z=-5$

1.5.2 Parametric equations for a plane

Recall that a line could be written in the parametric form

$$x(t) = a + pt$$

$$y(t) = b + qt$$

$$z(t) = c + rt$$

A line is one-dimensional so its points can be selected by a single parameter t.

However, a plane is two-dimensional and so we need two parameters (say u and v) to select each point. Thus it's no surprise that every plane can also be described by the following equations

$$\left.\begin{aligned} x(u,v) &= a + pu + lv \\ y(u,v) &= b + qu + mv \\ z(u,v) &= c + ru + nv \end{aligned}\right\} \text{parametric form plane eqn}$$

Now we have nine parameters a,b,c,p,q,r,l,m and n. These can be computed from the coordinates of three (distinct) points on the plane. For the first point put $(u,v)=(0,0)$, the second put $(u,v)=(1,0)$ and for the final point put $(u,v)=(0,1)$. Then solve for a through to n. 3rd point

Example 1.27. Find the parametric equations of the plane that passes through the three points $(-1,0,0)$, $(1,2,0)$ and $(2,-1,5)$.

$(-1,0,0)$

$(u,v)=(0,0)$

$-1=a$

$0=b$

$0=c$

$(1,2,0)$

$(u,v)=(1,0)$

$1=-1+p \Leftrightarrow p=2$

$2=0+q \Leftrightarrow q=2$

$0=0+r \Leftrightarrow r=0$

$(2,-1,5)$

$(u,v)=(0,1)$

$2=-1+l \Leftrightarrow l=3$

$-1=0+m \Leftrightarrow m=-1$

$5=0+n \Leftrightarrow n=5$

parametric form

Plane eqn:

$x(u,v)=-1+2u+3v$

$y(u,v)=0+2u-v$

$z(u,v)=0+0u+5v$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + u\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + v\begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} \Rightarrow \text{vector form}$$

Ex 1.26 $-5x+5y+4z=5$

Ex-1.27
$x= -1+2u+3v$
$y= 2u-v$
$z=5v$

Example 1.28. Show that the parametric equations found in the previous example describe exactly the same plane as found in Example 1.26 (Hint : substitute the answers from Example 1.27 into the equation found in Example 1.26).

$$L.H.S = -5(-1+2u+3v)+5(2u-v)+4(5v)$$
$$= 5-\cancel{10u}-\cancel{15v}+\cancel{10u}-5v+\cancel{20v}$$
$$= 5 = RHS$$

* **Example 1.29.** Find the parametric equations of the plane that passes through the three points $(-1, 2, 1)$, $(1, 2, 3)$ and $(2, -1, 5)$.

$(-1,2,1)$
$(u,v)=(0,0)$
$a=-1$
$b=2$
$c=1$

$(1,2,3)$
$(u,v)=(1,0)$
$1=-1+p \Rightarrow p=2$
$2=2+q \Rightarrow q=0$
$3=1+r \Rightarrow r=2$

$(2,-1,5)$
$(u,v=(0,1)$
$2=-1+l \Rightarrow l=3$
$-1=2+m \Rightarrow m=-3$
$5=1+n \Rightarrow n=4$

$x(u,v)= -1+2u+3v$
$y(u,v)= 2+0u-3v$
$z(u,v)= 1+2u+4v$

Example 1.30. Repeat the previous example but with points re-arranged as $(-1, 2, 1)$, $(2, -1, 5)$ and $(1, 2, 3)$. You will find that the parametric equations look different yet you know they describe the same plane. If you did not know this last fact, how would you prove that the two sets of parametric equations describe the same plane?

by substitution random value for u, v

$\underset{\sim}{n}=\begin{pmatrix}a\\b\\c\end{pmatrix}$ $\underset{\sim}{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ $\underset{\sim}{a}=\begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix}$

$\underset{\sim}{n}\cdot(\underset{\sim}{r}-\underset{\sim}{a})=0$ $\begin{pmatrix}1\\2\\5\end{pmatrix}$ $\underset{\sim}{a}=\begin{pmatrix}0\\-1\\4\end{pmatrix}$

$\begin{pmatrix}a\\b\\c\end{pmatrix}\cdot\begin{pmatrix}x-x_0\\y-y_0\\z-z_0\end{pmatrix}=0$

$\begin{pmatrix}1\\2\\5\end{pmatrix}\cdot\begin{pmatrix}x-0\\y+1\\z-4\end{pmatrix}=0$

1.5.3 Vector equation of a plane

The Cartesian equation for a plane is

$$\boxed{ax + by + cz = d}$$

for some numbers a, b, c and d. We will now re-express this in a vector form. Suppose we know one point on the plane, say $(x, y, z) = (x, y, z)_0$, then

$x+2y+2+5z-20=0$
$x+2y+5z=18$

$$ax_0 + by_0 + cz_0 = d$$
$$\Rightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

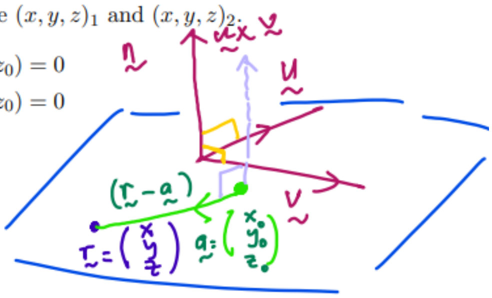
$\Rightarrow ax+by+cz = \text{constant}$

This is an equivalent form of the above equation.

Now suppose we have two more points on the plane $(x, y, z)_1$ and $(x, y, z)_2$. Then

$$a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0) = 0$$
$$a(x_2 - x_0) + b(y_2 - y_0) + c(z_2 - z_0) = 0$$

$\underset{\sim}{u}\cdot\underset{\sim}{v}=0$ (90°)



plane eqn in vector

$$(\underset{\sim}{r} - \underset{\sim}{a}) \cdot \underset{\sim}{n} = 0$$

$$\left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \right] \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = 0$$

Put $\Delta\underset{\sim}{x}_{10} = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$ and $\Delta\underset{\sim}{x}_{20} = (x_2 - x_0, y_2 - y_0, z_2 - z_0)$. Notice that both of these vectors lie in the plane and that

$$(a, b, c) \cdot \Delta\underset{\sim}{x}_{10} = (a, b, c) \cdot \Delta\underset{\sim}{x}_{20} = 0$$

What does this tell us? Simply that both vectors are orthogonal to the vector (a, b, c). Thus we must have that

$$(a, b, c) = \text{the normal vector to the plane}$$

Now let's put

$$\begin{aligned} \underset{\sim}{n} &= (a, b, c) &&= \text{the normal vector to the plane} \\ \underset{\sim}{d} &= (x_0, y_0, z_0) &&= \text{one (any) point on the plane} \\ \underset{\sim}{r} &= (x, y, z) &&= \text{a typical point on the plane} \end{aligned}$$

Then we have

$$\boxed{\underset{\sim}{n} \cdot (\underset{\sim}{r} - \underset{\sim}{d}) = 0}$$

This is the **vector equation of a plane**.

Example 1.31. Find the vector equation of the plane that contains the points $(1, 2, 7)$, $(2, 3, 4)$ and $(-1, 2, 1)$.

$$\underset{\sim}{u} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

$$\underset{\sim}{u} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$$

$$\underset{\sim}{v} = \begin{pmatrix} -3 \\ -1 \\ -3 \end{pmatrix}$$

$$\underset{\sim}{n} = \underset{\sim}{u} \times \underset{\sim}{v}$$

$$= \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} -3 \\ -1 \\ -3 \end{pmatrix}$$

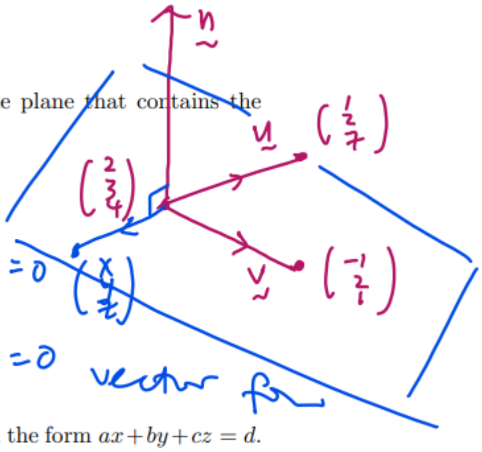
$$= \begin{pmatrix} 6 \\ -12 \\ -2 \end{pmatrix} = 2 \begin{pmatrix} 3 \\ -6 \\ -1 \end{pmatrix}$$

plane eqn

$$\left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right] \cdot \begin{pmatrix} 3 \\ -6 \\ -1 \end{pmatrix} = 0$$

$$\begin{pmatrix} x-2 \\ y-3 \\ z-4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -6 \\ -1 \end{pmatrix} = 0$$

vector form



Example 1.32. Re-express the previous result in the form $ax + by + cz = d$.

$$3x - 6 + (-6y + 18) + (-z + 4) = 0$$

$$3x - 6y - z = -16$$

Planes in R^3

The vector equation of a plane is determined using a point P on the plane and a direction $\underset{\sim}{n}$ (known as the **normal** direction) which is perpendicular to the plane. Then all vectors on the plane which pass through P are normal to $\underset{\sim}{n}$, i.e.

$$\underset{\sim}{n} \cdot (\underset{\sim}{r} - \underset{\sim}{d}) = 0$$

where $\underset{\sim}{r} = (x, y, z)$ is a typical point on the plane, and $\underset{\sim}{d} = (x_0, y_0, z_0)$ is a particular point (P) on the plane.

1.6 Systems of Linear Equations

1.6.1 Examples of Linear Systems

The central problem in linear algebra is to *solve systems of simultaneous linear equation.* A **system** of linear equations is a collection of equations which have the same set of variables. Let's look at some examples.

two variable

x & y

$$\begin{cases} 2x+y=4 \\ x-3y=1 \end{cases}$$

⇒ POI

Bags of coins

We have three bags with a mixture of gold, silver and copper coins. We are given the following information

Bag 1	contains	10 gold, 3 silver, 1 copper	and weighs 60g
Bag 2	contains	5 gold, 1 silver, 2 copper	and weighs 30g
Bag 3	contains	3 gold, 2 silver, 4 copper	and weighs 25g

The question is – What are the respective weights of the gold, silver and copper coins?

Let G, S and C denote the weight of each of the gold, silver and copper coins. Then we have the system of equations

$$\begin{array}{rcrcrcrl} 10G & + & 3S & + & C & = & 60 & -\textcircled{1} \\ 5G & + & S & + & 2C & = & 30 & -\textcircled{2} \\ 3G & + & 2S & + & 4C & = & 25 & -\textcircled{3} \end{array}$$

Decide one variable to be eliminated

Reduce to two variables

①&② ⇒ G, S variable

①&③ ⇒ G, S

↳ solve eqns

G = △

S = ☆

C = ⊗

Silly puzzles

John and Mary's ages add to 75 years. When John was half his present age he was twice as old as Mary. How old are they?

We have just two equations in our system:

$$\begin{array}{rcrcr} J & + & M & = & 75 \\ \frac{1}{2}J & - & 2M & = & 0 \end{array}$$

Intersections of planes

It is easy to imagine three planes in space. Is it possible that they share one point in common? Here are the equations for three such planes

$$\begin{array}{rcrcrcr} 3x & + & 7y & - & 2z & = & 0 \\ 6x & + & 16y & - & 3z & = & -1 \\ 3x & + & 9y & + & 3z & = & 3 \end{array}$$

Can we solve this system for (x, y, z)?

In all of the above examples we need to unscramble the set of **linear equations** to extract the unknowns (e.g. G, S, C etc).

To *solve* a system of linear equations is to find **solutions** to the sets of equations. In other words we find values that the variables can take such that each of the equations in the system is true.

$(2)-(1)\times 2$

$(3)-(1)$

$$\begin{array}{ll} 3x+7y-2z=0 & (1) \\ 2y+z=-1 & (2)' \\ 2y+5z=3 & (3)' \end{array}$$

1.6.2 A standard strategy

We start with the previous example

$$\begin{array}{rcrcrcrl} 3x & + & 7y & - & 2z & = & 0 & \qquad (1) \\ 6x & + & 16y & - & 3z & = & -1 & \qquad (2) \\ 3x & + & 9y & + & 3z & = & 3 & \qquad (3) \end{array}$$

Suppose by some process we were able to rearrange these equations into the following form

$$\begin{array}{rcrcrcrl} 3x & + & 7y & - & 2z & = & 0 & \qquad (1) \\ & & 2y & + & z & = & -1 & \qquad (2)' \\ & & & & 4z & = & 4 & \qquad (3)'' \end{array} \quad \rightarrow (3)'-(2)'$$

back-substitution

Then we could solve $(3)''$ for z

$$(3)'' \qquad \Rightarrow \qquad 4z = 4 \qquad \Rightarrow \qquad z = 1$$

and then substitute into $(2)'$ to solve for y

$$(2)' \qquad \Rightarrow \qquad 2y + 1 = -1 \qquad \Rightarrow \qquad y = -1$$

and substitute into (1) to solve for x

$$(1) \qquad \Rightarrow \qquad 3x - 7 - 2 = 0 \qquad \Rightarrow \qquad x = 3$$

How do we get the modified equations $(1), (2)'$ and $(3)''$?

The general method is to take suitable combinations of the equations so that we can eliminate various terms. This method is applied as many times as we need to turn the original equations into the simple form like $(1), (2)'$ and $(3)''$.

Let's start with the first pair of the original equations

$$\begin{array}{rcrcrcrl} 3x & + & 7y & - & 2z & = & 0 & \qquad (1) \\ 6x & + & 16y & - & 3z & = & -1 & \qquad (2) \end{array}$$

We can eliminate the $6x$ in equations (2) by *replacing* equation (2) with $(2) - 2(1)$,

$$\Rightarrow 0x + (16 - 14)y + (-3 + 4)z = -1 \qquad (2)'$$

$$\Rightarrow 2y + z = -1 \qquad (2)'$$

Likewise, for the $3x$ term in equation (3) we replace equation (3) with $(3) - (1)$,

$$\Rightarrow 2y + 5z = 3 \qquad (3)'$$

At this point our system of equations is

$$\begin{array}{rcrcrcrl} 3x & + & 7y & - & 2z & = & 0 & \qquad (1) \\ & & 2y & + & z & = & -1 & \qquad (2)' \\ & & 2y & + & 5z & = & 3 & \qquad (3)' \end{array}$$

The last step is to eliminate the $2y$ term in the last equation. We do this by replacing equation $(3)'$ with $(3)' - (2)'$

$$\Rightarrow \qquad 4z = 4 \qquad (3)''$$

So finally we arrive at the system of equations

$$\begin{array}{rcrcrcrl} 3x & + & 7y & - & 2z & = & 0 & \qquad (1) \\ & & & & 4z & = & 4 & \qquad (3)'' \end{array}$$

which, as before, we solve to find $z = 1$, $y = -1$ and $x = 3$.

The procedure we just went through is known as a reduction to **upper triangular form** and we used **elementary row operations** to do so. We then solved for the unknowns by **back substitution**.

This procedure is applicable to any system of linear equations (though beware, for some systems the back substitution method requires special care, we'll see examples later).

The general strategy is to eliminate all terms below the main diagonal, working column by column from left to right. More on this later!

1.6.3 Points, lines and planes - intersections

In previous lectures we saw how we could construct the equations for lines and planes. Now we can answer some simple questions.

How do we compute the intersection between a line and a plane? Can we be sure that they do intersect? And what about the intersection of a pair or more of planes?

The general approach to all of these questions is simply to write down equations for each of the lines and planes and then to search for a common point (i.e. a consistent solution to the system of equations).

Example 1.33. Is the point $(1, 2, 3)$ on the line $\underset{\sim}{r}(t) = (3, 4, 5) + ((2, 2, 2)t$?

Solution: We simply check if the following system of equations yields the same value for t.

$$\left.\begin{array}{lcl} 1 = 3 + 2t & \rightarrow & -2 = 2t \\ 2 = 4 + 2t & \rightarrow & -2 = 2t \\ 3 = 5 + 2t & \rightarrow & -2 = 2t \end{array}\right\} t = -1$$

Rearranging the top equation gives $t = -1$. In fact, each of these three equations gives $t = -1$ hence the point $(1, 2, 3)$ is on the line $\underset{\sim}{r}(t) = (3, 4, 5) + (2, 2, 2)t$.

Example 1.34. Is the point $(1, 2, 4)$ on the line $\underset{\sim}{r}(t) = (3, 4, 5) + (2, 2, 2)t$?

$1 = 3+2t \rightarrow t=-1$

$2 = 4+2t \rightarrow t=-1$

$4 = 5+2t \rightarrow t=-\frac{1}{2}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3+2t \\ 4+2t \\ 5+2t \end{pmatrix}$$

Not equal

(1,2,4) is NOT on line.

Example 1.35. Do the lines $\underset{\sim}{r}_1(t) = (1, 0, 0)+(1, 0, 0)t$ and $\underset{\sim}{r}_2(s) = (0, 0, 0)+(0, 1, 0)s$ intersect? If so, find the point of intersection.

Solution: To answer this question we simply solve the system of equations as follows.

$$\begin{aligned} 1 + 1t &= 0 + 0s \\ 0 + 0t &= 0 + 1s \\ 0 + 0t &= 0 + 0s \end{aligned}$$

This system of equations has the solution $t = -1$ and $s = 0$. Hence the two lines do intersect. To find the point of intersection, we put $t = -1$ into the first line (or $s = 0$ into the second line). This gives $\underset{\sim}{r}_1(-1) = (1, 0, 0) + (1, 0, 0)(-1) = (0, 0, 0)$. Hence the point of intersection of the two lines is the origin.

Example 1.36. Do the lines $\underset{\sim}{r}_1(t) = (1, 2, 3)+(1, 1, 2)t$ and $\underset{\sim}{r}_2(s) = (0, 0, 7)+(1, 1, 1)s$ intersect?

$$\overset{\underset{\sim}{r_1}}{\begin{pmatrix} 1+t \\ 2+t \\ 3+2t \end{pmatrix}} = \overset{\underset{\sim}{r_2}}{\begin{pmatrix} 0+s \\ 0+s \\ 1+7s \end{pmatrix}}$$

$1+t=s$ — ①

$2+t=s$ — ②

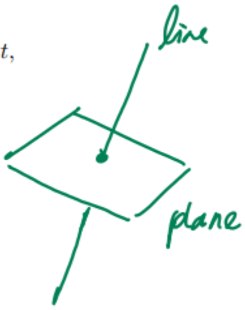
$3+2t=1+7s$

$2+2t=7s$ — ③

①, ② $\rightarrow$ Not logic

$\Rightarrow$ No intersection!

Example 1.37. Find the intersection of the line $x(t) = 1+3t$, $y(t) = 3-2t$, $z(t) = 1 - t$ with the plane $2x + 3y - 4z = 1$.



Subs x(t), y(t) & z(t) into plane eqn

$\Rightarrow$ solve for t

$2(1+3t)+3(3-2t)-4(1-t)=1$

$2+6t+9-6t-4+4t=1$

$4t=6$

$t=-3/2$

POI

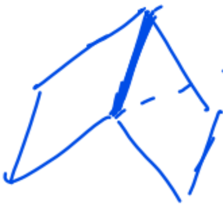
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -7/2 \\ 6 \\ 5/2 \end{pmatrix}$$

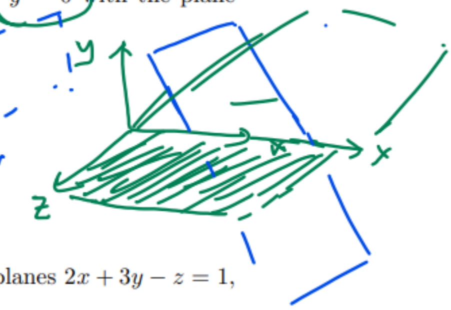
a line $\Uparrow$ $\Uparrow$ xz-plane

Example 1.38. Find the intersection of the plane $y = 0$ with the plane $2x + 3y - 4z = 1$.

$$\begin{cases} 2x+3y-4z=1 \\ y=0 \end{cases}$$

$2x-4z=1 \Rightarrow$ line left





Example 1.39. Find the intersection of the three planes $2x + 3y - z = 1$, $x - y = 2$ and $x = 1$

$$\begin{cases} 2x+3y-z=1 & (1) \\ x-y=2 & (2) \\ x=1 & (3) \end{cases}$$

Subs $x=1$ into (2)

$y=-1$

Subs $x=1$ & $y=-1$ into (1)

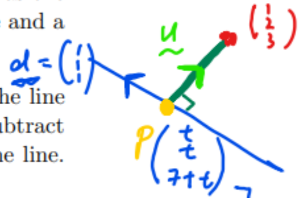
$z=-2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

1.6.4 Points, lines and planes - distances

Scalar projection

Now we are well equipped to be able to find the distances between points, lines and planes. There are various combinations we can have, such as the distance between a point and a plane, or the distance between a line and a plane.



Example 1.40. Find the distance between the point $(1, 2, 3)$ and the line given by the equation $\underset{\sim}{r}(t) = (0, 0, 7)+(1, 1, 1)t$. Solution: Firstly we subtract the position vector of the point $v = (1, 2, 3)$ from the equation of the line. This will give us a vector $\underset{\sim}{u}$ that is dependent on the parameter t.

$$\underset{\sim}{r} = \begin{pmatrix} 0+t \\ 0+t \\ 7+t \end{pmatrix}$$

$$\underset{\sim}{u} = \begin{pmatrix} t \\ t \\ 7+t \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} t-1 \\ t-2 \\ 4+t \end{pmatrix}$$

$$\begin{aligned} \underset{\sim}{u}(t) &= (0, 0, 7) + (1, 1, 1)t - (1, 2, 3) \\ &= (-1, -2, 4) + (1, 1, 1)t \\ &= (-1 + t, -2 + t, 4 + t) \end{aligned}$$

Think of the tail of this vector being fixed at the point $(1, 2, 3)$ and its tip running along the line as t changes. In order to find the shortest distance (note that when asked to find the *distance*, it is implied that this means find the *shortest distance*) we want to find the value of t for which the length of $\underset{\sim}{u}$ us as short as possible. We can also note that the shortest vector $\underset{\sim}{u}$ will

$\underset{\sim}{d} \cdot \underset{\sim}{u} = 0$ $(\perp)$

$\hookrightarrow$ help to t

$$\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}t-1\\t-2\\4+t\end{pmatrix}=0$$

$$t-1+t-2+4+t=0 \iff t=-\tfrac{1}{3}$$

be **perpendicular** to the direction of the line. This means that the **dot product** of the vectors (1, 1, 1) and $(-1+t, -2+t, 4+t)$ will be zero.

$\underset{\sim}{r}=\begin{pmatrix}0\\0\\7\end{pmatrix}+\begin{pmatrix}1\\1\\1\end{pmatrix}t$

$$(1,1,1)\cdot(-1+t,-2+t,4+t)=-1+t-2+t+4+t=1+3t=0$$

$|\underset{\sim}{u}| = \text{distance}$

Hence $t=-\frac{1}{3}$. Now put this value of t into the vector $\underset{\sim}{u}$ to give:

$$\underset{\sim}{u}(-\frac{1}{3})=(-1-\frac{1}{3},-2-\frac{1}{3},4-\frac{1}{3})=(-\frac{4}{3},-\frac{7}{3},\frac{11}{3})$$

$|u|=\sqrt{\left(-\frac{4}{3}\right)^2+\left(\frac{7}{3}\right)^2+\left(\frac{11}{3}\right)^2}$

$\underset{\sim}{v}=\begin{pmatrix}1\\2\\-4\end{pmatrix}$

Now simply calculate the **length** of $\underset{\sim}{u}(-\frac{1}{3})=(-\frac{4}{3},-\frac{7}{3},\frac{11}{3})$. This gives $|\underset{\sim}{u}|=$. If you plug $t=-\frac{1}{3}$ into the vector equation of the line, you get the coordinates of the point on the line that is closest to the point (1, 2, 3).

$=\dfrac{\sqrt{186}}{3}$ units

$$V_d=\frac{\underset{\sim}{v}\cdot\underset{\sim}{d}}{|\underset{\sim}{d}|}=\frac{1+2-4}{\sqrt{3}}=\frac{-1}{\sqrt{3}}$$

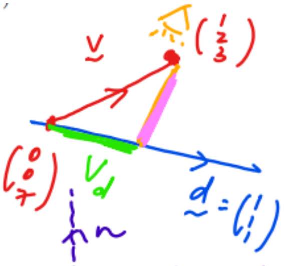
$V_d=\frac{1}{\sqrt{3}}$ units

$|\underset{\sim}{v}|=\sqrt{1^2+2^2+(-4)^2}$

$=\sqrt{21}$ units

$\text{distance}=\sqrt{|V|^2-(V_d)^2}$

$=\sqrt{21-\frac{1}{3}}=\sqrt{\frac{62}{3}}$ units



Example 1.41. Find the distance between the two lines

$$(1,2,3)+(1,1,2)t$$

and

$$(0,0,7)+(1,1,1)s$$

$\underset{\sim}{n}=\underset{\sim}{d}_1\times\underset{\sim}{d}_2$

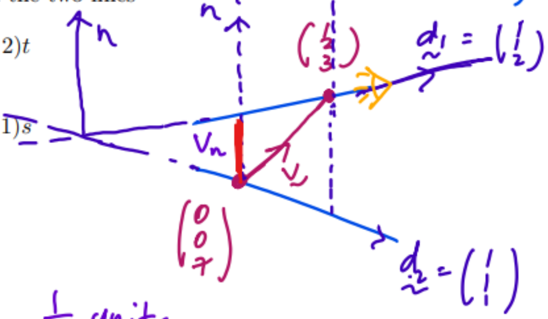
$=\begin{pmatrix}1\\1\\2\end{pmatrix}\times\begin{pmatrix}1\\1\\1\end{pmatrix}$

$=\begin{pmatrix}-1\\1\\0\end{pmatrix}$

$\underset{\sim}{v}:\begin{pmatrix}1\\2\\-4\end{pmatrix}$

$$V_n=\frac{\underset{\sim}{v}\cdot\underset{\sim}{n}}{|\underset{\sim}{n}|}$$

$$=\frac{-1+2}{\sqrt{2}}=\frac{1}{\sqrt{2}}\text{ units}$$



Parallel lines

If two lines are parallel, then it is easy to calculate the distance between them. Simply pick a point on one of the lines and calculate its distance from the other line, as per finding the distance between a point and a line. Remember that two lines are **parallel** if the direction vector of one line is a scalar multiple of the direction vector of the other line.

Example 1.42. Find the distance between the two lines

$\underset{\sim}{v} = \begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}0\\0\\7\end{pmatrix} = \begin{pmatrix}1\\2\\-4\end{pmatrix}$

$$(1, 2, 3) + (1, 1, 2)t$$

and

$$(0, 0, 7) + (2, 2, 4)s$$

$(0,0,7) + 2(1,1,2)s$

The yellow length is the shadow of $\underset{\sim}{V}$ onto $\underset{\sim}{d}$

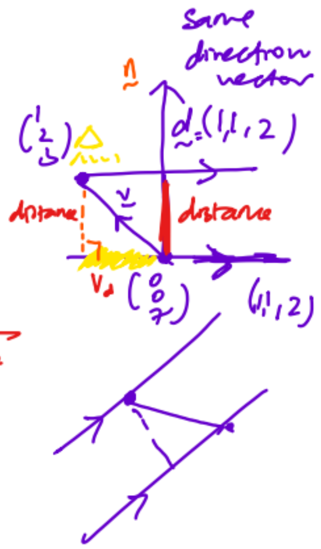
$V_d = \frac{\underset{\sim}{v} \cdot \underset{\sim}{d}}{|\underset{\sim}{d}|} = \frac{1+2-8}{\sqrt{6}} = \frac{-5}{\sqrt{6}}$

$V_d = \frac{5}{\sqrt{6}}$ units

$\text{distance} = \sqrt{|\underset{\sim}{v}|^2 - (V_d)^2}$

$= \sqrt{21 - \frac{25}{6}}$

$= \sqrt{\frac{101}{6}}$ units



Another way of finding the distance between two lines uses **scalar projection**. Using this method we find any vector that joins a point on one line to the other line, and then compute the scalar projection of this vector onto the vector orthogonal to both lines (it helps to draw a diagram).

Example 1.43. Find the distance between the point (2, 3, 4) and the plane given by the equation $x + 2y + 3z = 4$. $\Rightarrow \underset{\sim}{n} = \begin{pmatrix}1\\2\\3\end{pmatrix}$

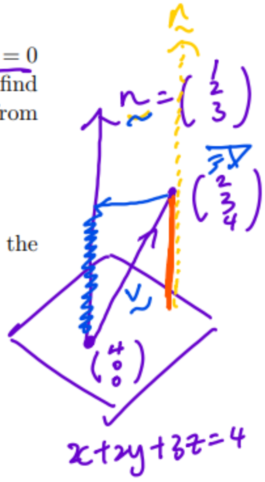
Solution: First of all we need to find a point on the plane. By setting $y = 0$ and $z = 0$ we find $x = 4$. Thus (4, 0, 0) is a point on the plane. Now we find the normal vector of the plane, $\underset{\sim}{n} = (1, 2, 3)$. We then form a vector $\underset{\sim}{v}$ from the point on the plane (4, 0, 0) to the given point (2, 3, 4). Thus

$$\underset{\sim}{v} = (2, 3, 4) - (4, 0, 0) = (-2, 3, 4)$$

Now find the scalar projection of $\underset{\sim}{v}$ onto the normal vector $\underset{\sim}{n}$. This is the shortest distance from the point (2, 3, 4) to the plane $x + 2y + 3z = 4$.

$V_n = \frac{\underset{\sim}{v} \cdot \underset{\sim}{n}}{|\underset{\sim}{n}|} = \frac{-2+6+12}{\sqrt{14}}$

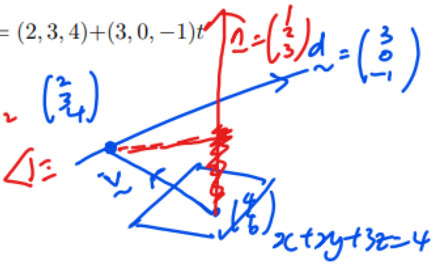
$= \frac{16}{\sqrt{14}}$ units



Example 1.44. Find the distance between the line $\underset{\sim}{r}(t) = (2, 3, 4)+(3, 0, -1)t$ and the plane plane $x + 2y + 3z = 4$.

$\underset{\sim}{v} = \begin{pmatrix}-2\\3\\4\end{pmatrix}$

$\underset{\sim}{V_n} = \frac{\underset{\sim}{v} \cdot \underset{\sim}{n}}{|\underset{\sim}{n}|} = \frac{-2+6+12}{\sqrt{14}} = \frac{16}{\sqrt{14}}$ units^2

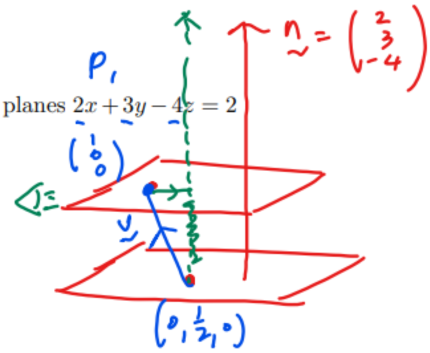


$$\boxed{\underset{\sim}{n}} \cdot (\underset{\sim}{r} - \underset{\sim}{a}) = 0$$

$$\boxed{\begin{pmatrix} a \\ b \\ c \end{pmatrix}} \left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}\right] = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz = k$$



Example 1.45. Find the distance between the two planes $2x+3y-4z=2$ and $4x+6y-8z=3$. P_2

$$2(2x+3y-4z)=3$$

$\underline{P_1}$ $(y=z=0)$, $x=1$ $\Rightarrow (1,0,0)$

$\underline{P_2}$ $(x=z=0)$. $y=\frac{1}{2}$ $\Rightarrow (0,\frac{1}{2},0)$

$\underset{\sim}{v} = (1, -\frac{1}{2}, 0)$ $\quad V_n = \dfrac{\underline{v} \cdot \underline{n}}{|\underset{\sim}{n}|} = \dfrac{2-\frac{3}{2}+0}{\sqrt{29}} = \dfrac{1}{2\sqrt{29}}$ units

1.6.5 Summary

- The equation $ax + by = c$ (or equivalently $a_1x_1 + a_2x_2 = b$ represents a **straight line** in 2-space.
- The equation $ax + by + cz = d$ (or equivalently $a_1x_1 + a_2x_2 + a_3x_3 = b$ represents a **plane** in 3-space.
- The equation $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$ represents a **hyperplane** in n-space.

The equation $a_1x_1+a_2x_2+a_3x_3+\cdots+a_nx_n = b$ is called a **linear equation** in the **variables** x_1, x_2, x_3, …, x_n with **coefficients** a_1, a_2, a_3, …, a_n and **constant term** b. In general we can study m linear equations in n variables:

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\
\vdots & & & & & & & & \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}$$

and call this a **system of linear equations**.

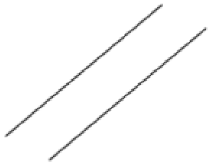
Every linear system satisfies one of the following:

- There is no solution
- There is exactly one solution
- There are infinitely many solutions

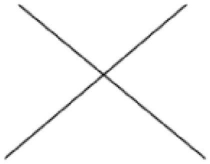
This seems obvious for the case of two or three variables if we view the equations geometrically. In this case a **solution** of a system of linear equations (of two or three variables) is a point in the intersection of the lines or planes represented by the equations.

When $n = 2$: Two lines may intersect at a single point (unique solution), or not at all (no solution), or they may even be the same line (infinite solutions).

No point of intersection (no solution)



One point of intersection (unique solution)



Infinite points of intersection (infinite solutions - intersection in the same line)

$x + 3y = 4$

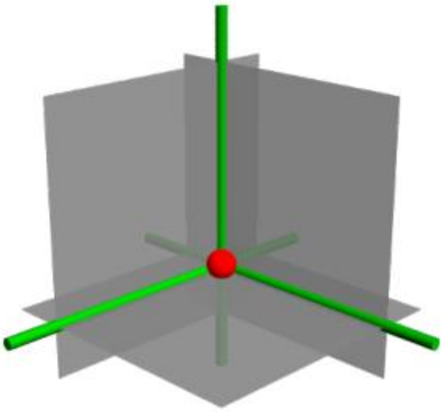
$2x + 6y = 8$ $\Big\} \times 2$

When $n = 3$: Three planes may intersect at a single point or along a common line or even not at all.

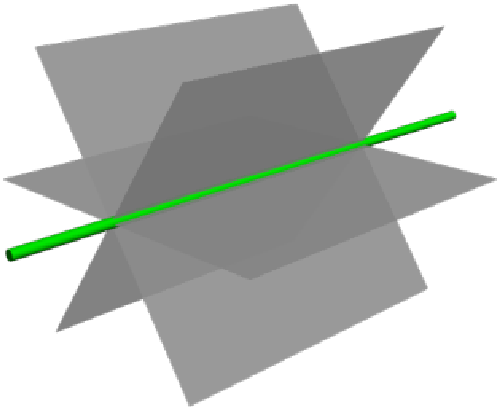
No point of intersection (no solution)

One point of intersection (unique solution)

One point of intersection (unique solution)



Intersection in a common line (infinite solutions)



Example 1.46. What other examples can you draw of intersecting planes?