Problem Set Nine: Integration, Area Under Curve, Trapezoidal Rule

Integration

1. Use a suitable substitution to integrate the following functions:

   (a) $f(x) = 2x\cos(x^2+2)$ $\int 2x\cos(x^2+2)\,dx$

   (b) $f(x) = 3x\sin(x^2+2)$

   (c) $f(x) = 4x^2\sqrt{x^3-5}$

   (d) $f(x) = \cos(x)e^{2\sin(x)}$

Check the above integrals are correct by differentiating each one.

$\frac{du}{dx}$

(a) Let $u = x^2+2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\int 2x\cos(x^2+2)\,dx$

$\int \cos(u)\,du = \sin(u)+c$

$\int 2x\cos(x^2+2)\,dx = \sin(x^2+2)+c$

(b) $\int 3x\sin(x^2+2)\,dx$

$3\int x\sin(x^2+2)\,dx$

let $u = x^2+2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$(\times\frac{3}{2})$

$\frac{3}{2}du = 3x\,dx$

$\frac{3}{2}\int \sin(u)\,du = -\frac{3}{2}\cos(u)+c$

$\int 3x\sin(x^2+2)\,dx = -\frac{3}{2}\cos(x^2+2)+c$

(c) $f(x) = 4x^2\sqrt{x^3-5} = 4x^2(x^3-5)^{\frac{1}{2}}$

let $u = x^3-5$

$du = 3x^2\,dx$

$\frac{4}{3}du = 4x^2\,dx$

$\int 4x^2(x^3-5)^{\frac{1}{2}}\,dx$

$\frac{4}{3}\int u^{\frac{1}{2}}\,du = \frac{4}{3}\left(\frac{u^{3/2}}{3/2}\right)+c$

$\int 4x^2(x^3-5)^{\frac{1}{2}}\,dx = \frac{8}{9}(x^3-5)^{3/2}+c$

(d) $f(x) = \cos(x) e^{2\sin(x)}$

Let $u = \sin x$

$du = \cos(x)\,dx$

$\int \cos(x)\, e^{2\sin(x)}\, dx$

$\int e^{2u}\, du = \frac{1}{2} e^{2u} + c$

$\int \cos(x) e^{2\sin(x)} = \frac{1}{2} e^{2\sin x} + c$

$\frac{d}{dx} a^x$

2. Evaluate each of the following using integration by parts. Recall that

$$\int f \frac{dg}{dx}\, dx = fg - \int g \frac{df}{dx}\, dx$$

"LIATE!"

① Choose different component

(a) $\int x \cos(x)\, dx$

(b) $\int xe^{-x}\, dx$

(c) $\int y\sqrt{y+1}\, dy$

(d) $\int x^2 \ln(x)\, dx$

$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$

Check the above integrals are correct by differentiating each one.

(a) Let $u = x$ $\quad \frac{dv}{dx} = \cos(x)$

$\frac{du}{dx} = 1$ $\quad V = \int \cos(x)\, dx = \sin(x)$

$\int x\cos(x)\, dx = x\sin(x) - \int \sin(x)\, dx$

$= x\sin(x) + \cos(x) + c$ ✓

(b) $\int xe^{-x}\, dx$

Let $u = x$ $\quad \frac{dv}{dx} = e^{-x}$

$\frac{du}{dx} = 1$ $\quad v = -e^{-x}$

$= -xe^{-x} + \int e^{-x}\, dx$

$= -xe^{-x} - e^{-x} + c$

index 2^x

L I A T E ← exp (e^x)

log

Algebra (polynomial)

trig

(c) $\int y\sqrt{y+1}\,dy = \int y(y+1)^{\frac{1}{2}}dy$

let $u=y$ $\quad \frac{dv}{dy}=(y+1)^{\frac{1}{2}}$

$\frac{du}{dy}=1 \quad v=\frac{2}{3}(y+1)^{3/2}$

$$\int y\sqrt{y+1}\,dy = \frac{2}{3}y(y+1)^{3/2} - \int \frac{2}{3}(y+1)^{3/2}dy$$

$$= \frac{2}{3}y(y+1)^{3/2} - \frac{2}{3}\left[\frac{2}{5}(y+1)^{5/2}\right]+c$$

$$= \frac{2}{3}y(y+1)^{3/2} - \frac{4}{15}(y+1)^{5/2}+c$$

(d) $\int x^2 \ln(x)\,dx = \frac{x^3}{3}\ln|x| - \int \frac{x^2}{3}dx$ "LATE"

[Let $u=\ln(x)$ $\quad \frac{dv}{dx}=x^2$

$\frac{du}{dx}=\frac{1}{x} \quad v=\frac{x^3}{3}$]

$$= \frac{x^3}{3}\ln(x) - \frac{1}{9}x^3+c$$

↓ Simplify

$$= \frac{x^3}{3}\left(\ln(x)-\frac{1}{3}\right)+c$$

3. Use integration by parts to find $\int \sin^2(x)\,dx$ and $\int \cos^2(x)\,dx$.

$$\int \sin^2(x)\,dx = \int [\sin(x)]^2 dx = \int [\sin(x)][\sin(x)]\,dx$$

Let $u=\sin(x)$ $\quad \frac{dv}{dx}=\sin(x)$

$\frac{du}{dx}=\cos(x) \quad v=-\cos(x)$

$[\sin^2\theta+\cos^2\theta=1]$

recall

$$\int \sin^2(x)\,dx = -\sin(x)\cos(x) + \int \cos^2(x)\,dx$$

$\cos^2\theta = 1-\sin^2\theta$

$\int \sin^2(x)\,dx = -\sin(x)\cos(x) + \int [1-\sin^2(x)]\,dx$

$\int \sin^2(x)\,dx = -\sin(x)\cos(x) + \int 1\,dx - \int \sin^2(x)\,dx$

$2\int \sin^2(x)\,dx = -\sin(x)\cos(x) + x$

* $\int \sin^2(x)\,dx = \frac{1}{2}[x - \sin(x)\cos(x)] + C$

$\int \cos(x)\cos(x)\,dx = \sin(x)\cos(x) + \int \sin^2(x)\,dx$

Let $u = \cos(x)$ $\quad \frac{dv}{dx} = \cos(x)$

$\frac{du}{dx} = -\sin(x)$ $\quad v = \sin(x)$

$= \sin(x)\cos(x) + \int 1\,dx - \int \cos^2(x)\,dx$

$2\int \cos^2(x)\,dx = \sin(x)\cos(x) + x$

* $\int \cos^2(x)\,dx = \frac{1}{2}[x + \sin(x)\cos(x)] + C$

NOTE: $\left[\begin{array}{l} \cos(2x) = 2\cos^2 x - 1 \\ \cos^2(x) = \frac{1}{2}(1+\cos(2x)) \end{array} \right]$ not necessary?

4. Use integration by parts twice to find $\int e^x \sin(x)\,dx$ and $\int e^x \cos(x)\,dx$.

LATER
x x ✓ ✓

Let $u = e^x$ $\quad \frac{dv}{dx} = \sin(x)$

$\frac{du}{dx} = e^x$ $\quad v = -\cos(x)$

$\int e^x \sin(x)\,dx = -e^x\cos(x) + \int e^x \cos(x)\,dx$

Let $u = e^x$ $\quad \frac{dv}{dx} = \cos(x)$

$\frac{du}{dx} = e^x$ $\quad v = \sin(x)$

$\int e^x \sin(x)\,dx = -e^x\cos(x) + e^x\sin(x) - \int e^x \sin(x)\,dx$

$2\int e^x \sin(x)\,dx = -e^x\cos(x) + e^x\sin(x)$

$\int e^x \sin(x)\,dx = \frac{1}{2} e^x [\sin(x) - \cos(x)] + C$

$\int e^x \cos(x)\,dx$

Let $u = e^x$ $\quad \frac{dv}{dx} = \cos(x)$

$\frac{du}{dx} = e^x$ $\quad v = \sin(x)$

$= e^x \sin(x) - \int e^x \sin(x)\,dx$

$u = e^x$ $\quad \frac{dv}{dx} = \sin(x)$

$\frac{du}{dx} = e^x$ $\quad v = -\cos(x)$

$= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\,dx$

$2\int e^x \cos(x)\,dx = e^x[\sin(x) + \cos(x)]$

$\int e^x \cos(x)\,dx = \frac{e^x}{2}[\sin(x) + \cos(x)] + C$

5. Use a substitution and integration by parts to evaluate each of the following:

(a) $\int \cos(x)\sin(x)e^{\cos(x)}\,dx$

(b) $\int e^{2x}\cos(e^x)\,dx$

(c) $\int \pi\, e^{\sqrt{x}}\,dx$

(a) Let $u = \cos(x)$

Substitution $\quad du = -\sin(x)\,dx$

$-du = \sin(x)\,dx$

$\int \cos(x)\sin(x)e^{\cos(x)}\,dx$

$\downarrow$

$-\int u e^u\,du$

By part, let $f = u$ $\quad \frac{dg}{du} = e^u$

$\frac{df}{du} = 1$ $\quad g = e^u$

$-\int u e^u\,du = -ue^u + \int e^u\,du$

$$-\int u e^u du = -ue^u + e^u + c$$

$$\int \cos(x)\sin(x)\, e^{\cos(x)} dx = -\cos(x) e^{\cos(x)} + e^{\cos(x)} + c$$

(b) $\int e^{2x} \cos(e^x) dx = \int e^x \cdot e^x \cos(e^x) dx$

Let $u = e^x$
$du = e^x dx$

$$\int u \cos(u)\, du = u\sin(u) - \int \sin(u) du$$
$$= u\sin(u) + \cos(u) + c$$

Let $f = u$ $\quad \frac{dg}{du} = \cos(u)$

$\frac{df}{du} = 1$ $\quad g = \sin(u)$

$$\int e^{2x} \cos(e^x) dx = e^x \sin(e^x) + \cos(e^x) + c$$

(c) $\int \pi e^{\sqrt{x}} dx = \pi \int e^{\sqrt{x}} dx \rightarrow \pi \int e^u (2u\, du)$

let $u = \sqrt{x} = x^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{2} x^{-\frac{1}{2}}$

$du = \frac{1}{2} x^{-\frac{1}{2}} dx$

Reuse

$2du = \frac{1}{\sqrt{x}} dx$

* $2du = \frac{1}{u} dx$

$2u\,du = dx$

$$= 2\pi \int u e^u du$$

By parts

Let $f = u$ $\quad \frac{dg}{du} = e^u$

$\frac{df}{du} = 1$ $\quad g = e^u$

$$2\pi \int u e^u du = 2\pi [ue^u - \int e^u du]$$
$$= 2\pi [ue^u - e^u] + c$$

$$\pi \int e^{\sqrt{x}} dx = 2\pi [\sqrt{x} e^{\sqrt{x}} - e^{\sqrt{x}}] + c$$

6. Spot the error in the following calculation.

$\int \frac{1}{x} dx = \ln|x| + c$

We wish to compute $\int \frac{1}{x} dx$. For this we will use integration by parts with $f = \frac{1}{x}$ and $g' = 1$. This gives us $f' = -\frac{1}{x^2}$ and $g = x$. Thus using $\int f\, g' = fg - \int g\, f'$ we find

$u = \frac{1}{x}$ $\quad \frac{dv}{dx} = 1$

$u' = -\frac{1}{x^2}$ $\quad v = x$

$$\int \frac{1}{x} dx = 1 + \int \frac{1}{x} dx$$

and rearranging this gives $\boxed{0 = 1.}$ $\Rightarrow$ Not logic!

x^{-1}

correction

$\int \frac{1}{x} dx = 1 + \int \frac{1}{x} dx + c$

Let $f = \frac{1}{x}$ $\quad g' = 1$

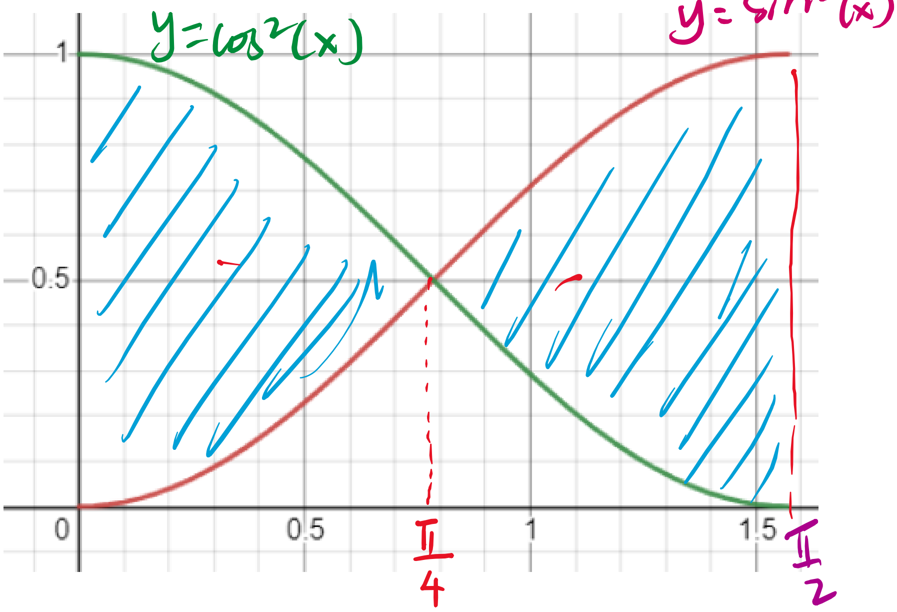
$\frac{df}{dx} = -\frac{1}{x^2}$ $\quad g = x$

$c = -1$ (logic b)

$\int_a^b (\text{Top} - \text{bottom})\, dx$

Area under the Curve

7. Find the area bounded by the graphs of $y = \sin^2(x)$ and $y = \cos^2(x)$ for $0 \le x \le \frac{\pi}{2}$ by using methods of integration.



POI

$\sin^2 x = \cos^2 x$

$\frac{\sin^2 x}{\cos^2 x} = 1$

$\tan^2 x = 1$

$\tan x = \pm 1$

$x = \tan^{-1}(1) = \frac{\pi}{4}$

$0 \le x \le \frac{\pi}{2}$

x

y

$\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$ ✓

$$A = \int_0^{\frac{\pi}{4}} \left[\cos^2(x) - \sin^2(x)\right] dx$$

$\sin^2 x = 1 - \cos^2 x$

$$A = \int_0^{\frac{\pi}{4}} \left[\cos^2(x) - 1 + \cos^2(x)\right] dx$$

$\int \cos^2(x)\, dx = \frac{1}{2}[x + \sin(x)\cos(x)] + c$

$$A = \int_0^{\frac{\pi}{4}} \left[2\cos^2(x) - 1\right] dx$$

Q3 Answer

$$A = \underbrace{\int_0^{\frac{\pi}{4}} 2\cos^2(x)\, dx}_{} - \underbrace{\int_0^{\frac{\pi}{4}} 1\, dx}_{}$$

$$A = \left[\cancel{x} + \sin(x)\cos(x)\right]_0^{\frac{\pi}{4}} - \left[\cancel{x}\right]_0^{\frac{\pi}{4}}$$

$$A = \left[\sin(x)\cos(x)\right]_0^{\frac{\pi}{4}}$$

$$A = \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - (0)(1) = \frac{1}{2} \text{ units}^2$$

Total shaded area $= 2 \times \frac{1}{2}$

$= 1$ units

Trapezoidal Rule

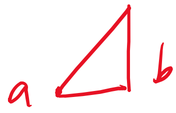
8a. Use the Trapezoidal rule to approximate $\int_0^4 x^2\,dx$ for the following:

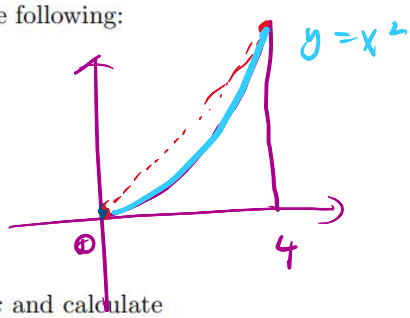
i. $n = 1$.

ii. $n = 2$.

iii. $n = 4$.

iv. $n = 8$.

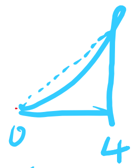


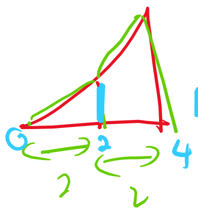


b. Compare each of these with the definite integral $\int_0^4 x^2\,dx$ and calculate the error in each case.

$A = \frac{1}{2}h(a+b)$

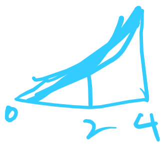
(a) $n=1 \Rightarrow A = \frac{1}{2}(4)(0+16) = 32$ units^2





$n=2 \Rightarrow A = \frac{1}{2}(2)[0+16+2(4)] = 24$ units^2

$\boxed{h = \frac{4-0}{2} = 2}$

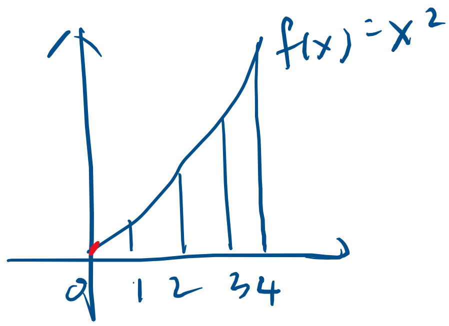


(a) (iii) $n=4$

4 sub-intervals

$f(x) = x^2$

Trapezoidal Rule



x	$f(x) = x^2$
0	0
1	1
2	4
3	9
4	16

$A \approx \frac{1}{2}(1)[0+16+2(14)]$

$A \approx 22$ units^2

$h = \frac{b-a}{n} = \frac{4-0}{4} = 1$

(iv) $n=8$

x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
$f(x) = x^2$	0	$\frac{1}{4}$	1	$\frac{9}{4}$	4	$\frac{25}{4}$	9	$\frac{49}{4}$	16

sum

$h = \frac{4-0}{8}$

$h = \frac{1}{2}$

$A \approx \frac{1}{2}\left(\frac{1}{2}\right)[0+16+2(\quad)]$

$= 21.5$ units^2

(b) $\int_0^4 x^2\,dx = \left[\frac{x^3}{3}\right]_0^4 = \frac{64}{3} \approx 21.33$ units^2

Error = actual − estimated

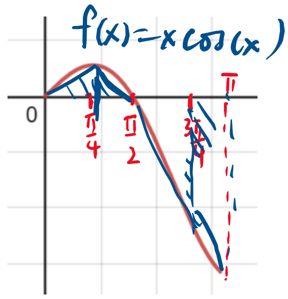
$n=1$, Error $= 32 - 21\frac{1}{3} = \frac{32}{3} \approx 10.67$

$n=2$, Error $= 24 - 21\frac{1}{3} = \frac{8}{3} \approx 2.67$

$n=4$, Error $= 22 - 21\frac{1}{3} \approx \frac{2}{3} \approx 0.67$

$n=8$, Error $= 21.5 - 21\frac{1}{3} \approx \frac{1}{6} \approx 0.167$

more sub-intervals involved, better area estimation in trapezoidal Rule



9. Use the Trapezoidal rule with $n = 4$ to find an approximate value of $\int_0^{\pi} x\cos(x)\,dx$. Use Question 2a to evaluate $\int x\cos(x)\,dx$ between $x = 0$ and $x = \pi$. How does this compare with your Trapezoidal approximation?

Trapezoidal Rule

$A \approx 2.4674$ units^2

$n=4$

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$f(x)$	0	$\frac{\sqrt{2}}{8}\pi$	0	$-\frac{3\sqrt{2}}{8}\pi$	$-\pi$

$A \approx \frac{1}{2}\left(\frac{\pi}{4}\right)\left[0 + |-\pi| + 2\left(\frac{\sqrt{2}}{8}\pi + 0 + \left|-\frac{3\sqrt{2}}{8}\pi\right|\right)\right]$

$= 2.4674$

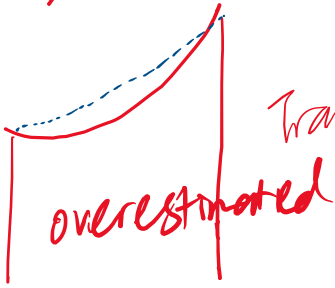
$A = \int_0^{\pi} x\cos(x)\,dx = \left[x\sin(x) + \cos(x)\right]_0^{\pi/2} + \left|\left[x\sin(x) + \cos(x)\right]_{\pi/2}^{\pi}\right|$

$A = \left(\frac{\pi}{2} - 1\right) + \left|(-1) - \left(\frac{\pi}{2} + 0\right)\right| = \frac{\pi}{2} - 1 + 1 + \frac{\pi}{2} = \pi$ exact units^2

$\int_0^{\pi} x\cos(x)\,dx = -2$ [This is not the area]

⇒ Estimation (over / under)

Trapezium Area



$n=8$ $A \simeq 3.138$

$n=16$ $A \simeq 3.140$

$n=20$ $A \simeq 3.141$

x	$f(x) = \frac{4}{1+x^2}$
0	4
1/4	64/17
1/2	16/5
3/4	64/25
1	2

10a. Choose an appropriate n and use the Trapezoidal rule to estimate the value of $\int_0^1 \frac{4}{1+x^2}\,dx$.

b. How is this a fair approximation for π?

(a) $A_T \simeq 3.13$ units^2 $\underline{(n=4)}$

(b) $\int_0^1 \frac{4}{1+x^2}\,dx = 4\int_0^1 \frac{1}{1+x^2}\,dx$

$h = \frac{b-a}{n}$

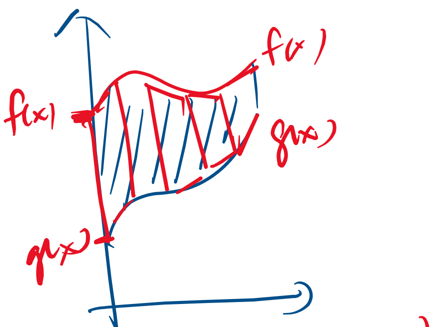
$h = \frac{1-0}{4} = \frac{1}{4}$

$$= 4\left[\tan^{-1}(x)\right]_0^1$$

$$= 4\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$

$$= 4\left[\left(\frac{\pi}{4}\right) - 0\right]$$

$$= \pi \approx 3.1412\ldots$$



$h(x) = f(x) - g(x)$

Trapezoidal Rule

If n is larger

it is considered fair approx. to π.