Question 12 **(7 marks)**

Recall that the **Fibonacci numbers**, F_n, are defined recursively as follows:

$$\begin{aligned} F_1 &= 1, \\ F_2 &= 1, \\ F_n &= F_{n-1} + F_{n-2} \qquad \text{for } n \geq 3. \end{aligned}$$

The first few numbers in the sequence are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

Note that the Fibonacci numbers are (after the first term) an *increasing* sequence of positive integers.

The language FIBONACCI is defined to be the set of all strings of the letter a whose length is a Fibonacci number. So,

$$\text{FIBONACCI} = \{ \, \mathtt{a}^{F_n} \ : \ n \in \mathbb{N} \, \}.$$

(a) Prove that the difference, $F_n - F_{n-1}$, between two consecutive Fibonacci numbers increases as n increases, i.e., $F_n - F_{n-1} > F_{n-1} - F_{n-2}$ for all $n \geq 5$.

$F_n - F_{n-1} = F_{n-2}$, and the Fibonacci numbers are increasing, so their differences are increasing too.

(b) Using (a), prove that the language FIBONACCI is not context-free.

Suppose (by way of contradiction) that FIBONACCI is context-free.

Then it has a CFG, in Chomsky Normal Form, with some number k of nonterminal symbols.

Let w be any word in the language FIBONACCI that has length $> 2^{k-1}$. Observe that it must be of the form $\mathtt{a}^{F_n}$ for some $F_n > 2^{k-1}$.

By the Pumping Lemma for CFLs, the word w can be partitioned into strings u, v, x, y, z (i.e., $w = uvxyz$) such that:

- v, y are not both empty,
- $|vxy| \leq 2^k$ . . . which we won't need . . . , and
- uv^ixy^iz is in FIBONACCI for all $i \geq 0$.

For convenience, write p for the combined length of v and y. So $p = |v| + |y|$. The first and second points above tell us that $1 \leq p \leq 2^k$.

Since $w = uvxyz$ is just a string of F_n a's, the string uv^2xy^2z is just a string of $F_n + p$

a's (since we have just taken w and repeated both v and y). More generally, the string uv^ixy^iz is just a string of $F_n + (i-1)p$ a's. The third point above means that a string of $F_n + (i-1)p$ a's must belong to the language FIBONACCI, for all i. So we have an infinite sequence of strings in FIBONACCI in which each string is exactly p letters longer than its predecessor, and this difference is positive since $p \geq 1$.

But we saw in part (a) that the difference between successive Fibonacci numbers is increasing. So, whatever p is, we see that for sufficiently large m, the difference $F_m - F_{m-1} > p$. This means that some of the numbers $F_n + (i-1)p$ cannot be Fibonacci numbers after all, since some of them must fall in the ever-increasing gaps between consecutive Fibonacci numbers. This means that (for large enough i) some of the strings of $F_n + (i-1)p$ a's cannot be in FIBONACCI after all. This is a contradiction.

So our original assumption, that FIBONACCI is context free, must be false. Hence FIBONACCI is not context-free.

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