$f(x) \longrightarrow f'(x) \longrightarrow f''(x) \longrightarrow f'''(x) \longrightarrow \cdots \longrightarrow f^{(n)}(x)$ e.g. $f^{(5)}(x)$

$y \longrightarrow \frac{dy}{dx} \longrightarrow \frac{d^2y}{dx^2} \longrightarrow \frac{d^3y}{dx^3} \longrightarrow \cdots \longrightarrow \frac{d^ny}{dx^n}$

3.4 Higher order derivatives

If $f(x)$ is a differentiable function, then its derivative $f'(x) = \dfrac{\mathrm{d}f}{\mathrm{d}x}$ is also a function and so may have a derivative itself. The derivative of a derivative is called the **second derivative** and is denoted by $f''(x)$. There are various ways it can be written:

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right) = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = f^{(2)}(x)$$

The second derivative, $f''(x)$ can be differentiated with respect to x to yield the **third derivative** $f'''(x) = \dfrac{\mathrm{d}^3 f}{\mathrm{d}x^3} = f^{(3)}(x)$. And so on!

In general, the $\mathbf{n^{th}}$ **derivative** of $f(x)$ is denoted by $\dfrac{\mathrm{d}^n f}{\mathrm{d}x^n}$ or $f^{(n)}(x)$.

Interpretation:

Earlier we used the first derivative to find local maxima and local minima. We can also use the *second derivative* at $x = c$ to find these.

- If $f''(c) > 0$ then the function has a local **minima** at c.
- If $f''(c) < 0$ then the function has a local **maxima** at c.
- If $f''(c) = 0$ then the test is inconclusive and we cannot determine if there is a local maxima or minima at c.

The second derivative $f''(x)$ also measures the *rate of change* of the first derivative $f'(x)$. As $f'(x)$ is the gradient or slope of the tangent line to the graph of $y = f(x)$ in the xy-plane, we see that:

(i) if $\dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2} > 0$ then $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ increases with increasing x and the graph of $y = f(x)$ is said to be locally **concave up**.

local min

(ii) if $\dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2} < 0$ then $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ decreases with decreasing x and the graph of $y = f(x)$ is said to be locally **concave down**.

local max

Example 3.14. If $f(x) = x^3 - 3x + 1$ find the first four derivatives $f'(x)$, $f''(x)$, $f^{(3)}(x)$ and $f^{(4)}(x)$. Determine for what values of x the curve is concave up and concave down. Also locate the **turning points** $(a, f(a))$ given where $f'(a) = 0$. Mark these features on the graph of $y = f(x)$ shown below.

$f'(x) = 3x^2 - 3$

$f''(x) = 6x$

$f^{(3)}(x) = 6$

$f^{(4)}(x) = 0$

Turning point

$f'(x) = 0$

$3x^2 - 3 = 0$

$x = \pm 1$

$x = 1$, $y = -1$

$x = -1$, $y = 3$

2nd derivative Test

For $(1, -1)$, $f''(1) = 6 > 0$

$\Rightarrow$ local min

For $(-1, 3)$, $f''(-1) = -6 < 0$

$\Rightarrow$ local max

curve up / down

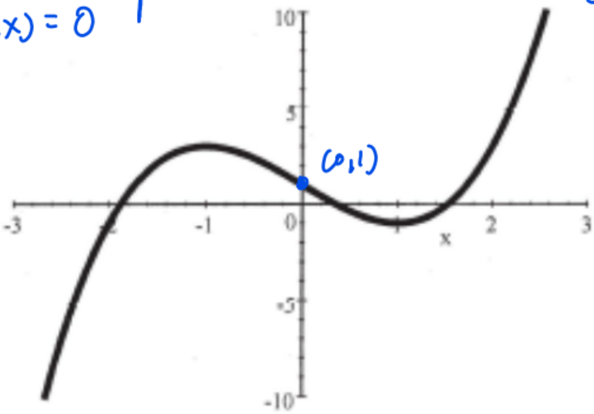
$f''(x) = 0$

$6x = 0$

$x = 0$

$y = 1$

$(0, 1)$



$x < 0$ $f(x)$ is concave down

$x > 0$ $f(x)$ is concave up

Example 3.15. Find the second derivative of the function $y = e^{-x} \sin 2x$.

[Product Rule + Chain Rule]

$$\frac{dy}{dx} = -e^{-x} \sin 2x + e^{-x} [2 \cos 2x] = e^{-x}(-\sin 2x + 2\cos 2x)$$

$$\frac{d^2y}{dx^2} = -e^{-x}(-\sin 2x + 2\cos 2x) + e^{-x}[-2\cos 2x - 4\sin 2x]$$

$$= e^{-x}[\sin 2x - 2\cos 2x - 2\cos 2x - 4\sin 2x]$$

$$= e^{-x}(-4\cos 2x - 3\sin 2x)$$

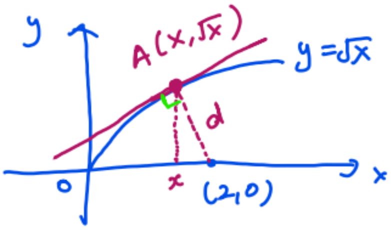
$$= -e^{-x}(4\cos 2x + 3\sin 2x)$$

Example 3.16. Find the second derivative of the function $y = \frac{\ln x}{x}$. (Quotient Rule)

$$\frac{dy}{dx} = \frac{(\frac{1}{x})(x) - (1)\ln x}{x^2} = \frac{1-\ln x}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{(-\frac{1}{x})(x^2) - (2x)(1-\ln x)}{(x^2)^2} = \frac{-x-2x+2x\ln x}{x^4} = \frac{-3x+2x\ln x}{x^4} = \frac{-3+2\ln x}{x^3}$$

Example application: Find the point on the graph of $\sqrt{x}$, $x \geq 0$ closest to $(2,0)$.



Let d : shortest distance from $(2,0)$ to curve

$$d = \sqrt{(x-2)^2 + (\sqrt{x} - 0)^2}$$

$$d = \sqrt{x^2-4x+4 + x} = (x^2-3x+4)^{\frac{1}{2}}$$

objective : d is minimum

$$\frac{dd}{dx} = \frac{1}{2}(x^2-3x+4)^{-\frac{1}{2}}(2x-3) = \frac{2x-3}{2\sqrt{x^2-3x+4}}$$

critical x-value, $\frac{dd}{dx} = 0$

Denominator $\neq 0$

$2x-3=0$

$x = 3/2$

$y = \sqrt{3/2}$

$(3/2, \sqrt{3/2})$

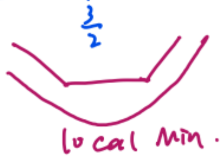
1st Derivative Test

$(x=0)$ $(x=2)$

$\frac{dd}{dx}$ -ve 0 +ve

$\frac{3}{2}$

slope



2nd Derivative Test

$$\frac{d^2d}{dx^2} = \frac{4\sqrt{x^2-3x+4} - (2x-3)\left[\frac{2x-3}{\sqrt{x^2-3x+4}}\right]}{4(x^2-3x+4)}$$

$$\left.\frac{d^2d}{dx^2}\right|_{x=3/2} = +ve > 0$$

$\Rightarrow$ local min

Closest point to $(2,0)$ on the curve is $(3/2, \sqrt{3/2})$

3.5 Parametric curves and differentiation

3.5.1 Parametric curves

The equation that describes a curve C in the Cartesian xy-plane can sometimes be very complicated. In that case it can be easier to introduce an independent parameter t, so that the coordinates x and y become functions of t. We explored this when we looked at the vector equations of lines and planes. That is, $x = f(t)$ and $y = g(t)$. The curve C is **parametrically** represented by

$$C = \{(x, y) : x = f(t) \quad y = g(t) \quad t_1 \leq t \leq t_2\}$$

As the independent parameter t goes from t_1 to t_2, the point $P = ((x(t), y(t))$ on the curve C moves from $P_1 = (x_1, y_1)$ to $P_2 = (x_2, y_2)$.

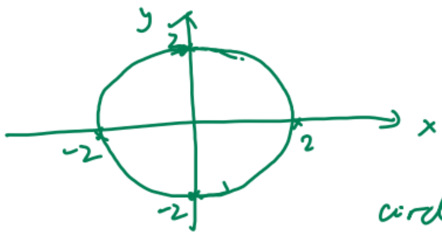
Example 3.17. What do the following parametric curves represent?

(a) $C = \{(x, y) : x = 2\cos(t), \quad y = 2\sin(t), \quad 0 \leq t \leq 2\pi\}$

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	2	$\sqrt{3}$	$\sqrt{2}$	1	0	-2	0	2
y	0	1	$\sqrt{2}$	$\sqrt{3}$	2	0	-2	0

$x = 2\cos(t)$ | $y = 2\sin(t)$

$x^2 = 4\cos^2(t)$ | $y^2 = 4\sin^2(t)$



$x^2 + y^2 = 4\cos^2(t) + 4\sin^2(t)$

$x^2 + y^2 = 4\left[\cos^2(t) + \sin^2(t)\right]$

$x^2 + y^2 = 4$

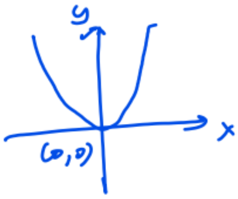
circle functions with centre $(0,0)$ & radius 2 units.

(b) $C = \{(x, y) : x = 2t, \quad y = 4t^2, \quad -\infty < t < \infty\}$

$x = 2t$ | $y = 4t^2$

$x^2 = 4t^2$

$y = x^2$



parabola curve that has vertex at $(0,0)$

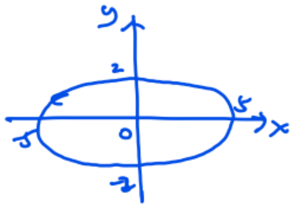
$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

(c) $C = \{(x, y) : x = 5\cos(t), \quad y = 2\sin(t), \quad 0 \le t \le 2\pi\}$

$x = 5\cos(t)$ | $y = 2\sin(t)$

$x^2 = 25\cos^2(t)$ | $y^2 = 4\sin^2(t)$

$\frac{x^2}{25} = \cos^2(t)$ | $\frac{y^2}{4} = \sin^2(t)$



$\frac{x^2}{25} + \frac{y^2}{4} = \cos^2(t) + \sin^2(t)$

$\frac{x^2}{25} + \frac{y^2}{4} = 1$

Ellipse with centre at $(0,0)$

$a = 5$ and $b = 2$

(d) $C = \{(x, y) : x = t\cos(t), \quad y = t\sin(t), \quad t > 0\}$

$x^2 = t^2\cos^2(t)$ $\qquad$ $y^2 = t^2\sin^2(t)$

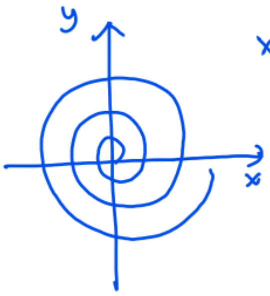
$\frac{y}{x} = \frac{t\sin(t)}{t\cos(t)}$

$x^2 + y^2 = t^2[\cos^2(t) + \sin^2(t)]$

$\frac{y}{x} = \tan(t)$

$x^2 + y^2 = t^2$

$t = \tan^{-1}\left(\frac{y}{x}\right)$



$x^2 + y^2 = \left[\tan^{-1}\left(\frac{y}{x}\right)\right]^2$

Spiral curve at $(0,0)$

3.5.2 Parametric differentiation

it is now a natural progression to ask what is the value of the *slope* $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point $\big((x(t), y(t)\big)$ on the curve. This is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \Big/ \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{g'(t)}{f'(t)} \quad \text{where } f'(t) = \frac{\mathrm{d}f}{\mathrm{d}t} \text{ and } g'(t) = \frac{\mathrm{d}g}{\mathrm{d}t}.$$

Chain Rule

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Example 3.18. Sketch the curve represented parametrically by:

$$C = \{(x, y) : x = a\cos t \quad y = a\sin t \quad 0 \le t \le 2\pi\}.$$

$x(t)$ $\downarrow$ $\frac{dx}{dt}$ $\quad \frac{dt}{dx} = 1 \Big/ \frac{dx}{dt}$ $\quad y(t)$ $\downarrow$ $\frac{dy}{dt}$

$\frac{dy}{dx} = \frac{dy}{\cancel{dt}} \times \frac{\cancel{dt}}{dx}$

$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}}$ $\Rightarrow$ in terms of t

Find the derivative function $\frac{\mathrm{d}y}{\mathrm{d}x}$. Find the equation of the tangent line to the curve at the point corresponding to $t = \frac{\pi}{4}$ and draw this on the sketch for the case $a = 2$.

$\boxed{a=2}$

$x = 2\cos(t)$

$\frac{dx}{dt} = -2\sin(t)$

Solution: Here $f(t) = a\cos t, \quad g(t) = a\sin t$. Therefore $x^2+y^2 = a^2(\cos^2 t + \sin^2 t) = a^2$. The curve C is thus a circle of radius a centred at the origin. As t goes from $0 \to 2\pi$, the circle is described once, in the positive direction, starting at the point $(2, 0)$. Now, $f'(t) = -a\sin t = -y, \quad g'(t) = a\cos t = x$ and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \Big/ \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{g'(t)}{f'(t)} = \frac{x}{-y} = -\frac{a\cos t}{-a\sin t} = -\cot t$$

$y = 2\sin(t)$

$\frac{dy}{dt} = 2\cos(t)$

At $t = \frac{\pi}{4}$, $x\left(\frac{\pi}{4}\right) = a\cos\left(\frac{\pi}{4}\right) = \frac{a}{\sqrt{2}}$, $y\left(\frac{\pi}{4}\right) = a\sin\left(\frac{\pi}{4}\right) = \frac{a}{\sqrt{2}}$, and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{a/\sqrt{2}}{a/\sqrt{2}} = -1.$$

$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$

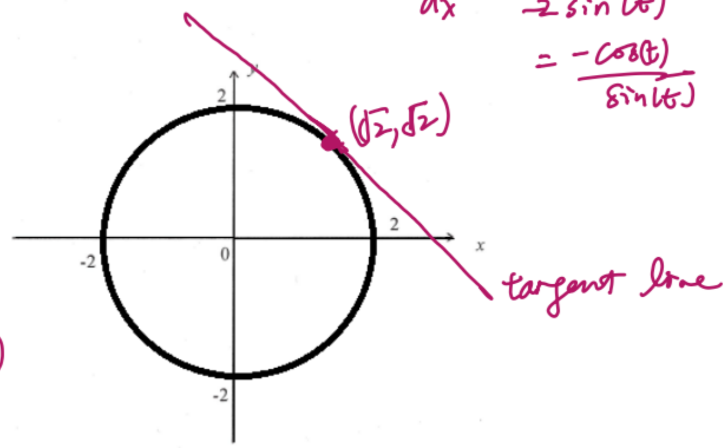
$\frac{dy}{dx} = \frac{2\cos(t)}{-2\sin(t)} = -\cot(t)$

$= \frac{-\cos(t)}{\sin(t)}$

$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = \frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = -1$

$x = 2\cos\left(\frac{\pi}{4}\right) = \sqrt{2}$

$y = 2\sin\left(\frac{\pi}{4}\right) = \sqrt{2}$



tangent line

$y - y_1 = m(x - x_1)$

$y - \sqrt{2} = -1(x - \sqrt{2})$

$y = -x + 2\sqrt{2}$

Tangent Line: The equation of a straight line of slope m which passes through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$

Now taking $a = 2$ and collecting values, we have:

$t = \frac{\pi}{4}$

$x_1 = \sqrt{2}$ $\quad y_1 = \sqrt{2}$ $\quad m = -1$

and thus the required answer is:

$$y = -x + 2\sqrt{2}$$

Example 3.19. Consider the curve represented parametrically by:

$$C = \{(x, y) : x = 1 + 3t^2, \quad y = 1 + 2t^3, \quad -\infty < t < \infty\}.$$

$\frac{dx}{dt} = 6t$

(i) Find $\frac{dy}{dx}$ as a function of t $\quad \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{6t^2}{6t} = t \quad \frac{dy}{dt} = 6t^2$

(ii) Evaluate $\frac{dy}{dx}$ at $t = 1$ and find the tangent line to the curve at this point.

$\left.\frac{dy}{dx}\right|_{t=1} = 1 \; (m_t)$

when $t = 1$, $x = 1 + 3(1)^2 = 4$

$y = 1 + 2(1)^3 = 3$

$(4, 3)$

Eqn of tangent is

$y - 3 = 1(x - 4)$

$y = x - 1$

Example 3.20. Find the equation of the tangent line to

$$x = 5\cos(t) \qquad y = 2\sin(t) \qquad \text{for } 0 \le t \le 2\pi \qquad \text{at } t = \frac{\pi}{4}.$$

$\frac{dx}{dt} = -5\sin(t) \quad \frac{dy}{dt} = 2\cos(t)$

$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{2\cos(t)}{-5\sin(t)}$

$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = \frac{2\cos(\frac{\pi}{4})}{-5\sin(\frac{\pi}{4})}$

$= \frac{2\left(\cancel{\frac{\sqrt{2}}{2}}\right)}{-5\left(\cancel{\frac{\sqrt{2}}{2}}\right)}$

$= \frac{2}{-5} \; (m_t)$

When $t = \frac{\pi}{4}$, $x = 5\cos(\frac{\pi}{4}) = \frac{5\sqrt{2}}{2}$

$y = 2\sin(\frac{\pi}{4}) = \sqrt{2}$

Eqn tangent is

$y - \sqrt{2} = -\frac{2}{5}\left(x - \frac{5}{2}\sqrt{2}\right)$

$y = -\frac{2}{5}x + 2\sqrt{2}$

3.6 Function approximations

We are now going to take a step in an interesting direction, and look at how to *approximate* a function by a number of different methods.

3.6.1 Introduction to power series

A **geometric sequence** a_n $(n = 0, 1, 2, 3, \ldots)$ is one in which the **ratio** of successive terms, namely a_{n+1}/a_n is a constant, say r. That is, $a_{n+1}/a_n = r$. Thus $a_1 = ra_0$, $a_2 = ra_1 = r^2 a_0$, and so on. We write:

Arithmetic
1, 3, 5, 7, ...

$$a_0,\ a_1,\ a_2,\ a_3,\ \ldots = a_0,\ ra_0,\ r^2a_0,\ r^3a_0,\ \ldots$$

Geometric
2, 4, 8, 16, ...

$$\underset{1st}{a},\ \underset{2nd}{ar},\ ar^2,\ ar^3,\ \ldots,\ \underset{n^{th}}{ar^{n-1}}$$

G. Sequence

A **finite geometric series** consists of the *sum* S_n of the first n terms of the geometric sequence. Setting the initial term $a_0 = a$, a constant, we have:

$a = 2$, $r = 3$

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \tag{i}$$

$\times r$ $\quad \times r$ $\quad \times r$ $\quad \times r$

We can easily find S_n. To do this we multiply both sides of Equation (i) by r to obtain:

t_{20}

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n \tag{ii}$$

ar^{n-1}

Subtracting Equation (ii) from (i) then yields:

$= (2)(3)^{20-1}$

$$(1 - r)S_n = a - ar^n$$

$= 2324522934$

Hence as long as $r \neq 1$, the sum of the first n terms of a geometric series is:

$$S_n = \sum_{k=0}^{n-1} ar^k = \frac{a(1 - r^n)}{1 - r} \tag{iii}$$

For the particular case of $r = 1$, we see from Equation (i) that $S_n = an$.

$$S_n = a + \left[\cancel{ar + ar^2 + \cdots + ar^{n-1}}\right]$$

$$(-)\ (\times r) \qquad rS_n = \left[\cancel{ar + ar^2 + \cdots + ar^{n-1}}\right] + ar^n$$

$$(1-r)\,S_n = a - ar^n$$

$$S_n = \frac{a(1-r^n)}{1-r}, \quad r \neq 1$$

$S_n = \frac{a(1-r^n)}{1-r}$

Example 3.21. Find the geometric series of the following sequence

$r = \frac{t_2}{t_1}$

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

$r = \frac{1}{2} / 1 = \frac{1}{2}$

(i) when $n = 3$, i.e. find S_3. $a=1$ $r=\frac{1}{2}$

$S_3 = 1 + \frac{1}{2} + \frac{1}{4} = 1.75$

$S_3 = \frac{1(1-(\frac{1}{2})^3)}{\frac{1}{2}} = \frac{7}{8} / \frac{1}{2} = \frac{7}{4} \approx 1.75$

(ii) when $n = 5$, i.e. find S_5.

$S_5 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$
$= \frac{31}{16} = 1.9375$

$S_5 = \frac{1(1-(\frac{1}{2})^5)}{\frac{1}{2}}$
$= \frac{31}{16}$

(iii) what happens as $n \to \infty$?

$n \to \infty$

$\boxed{S_\infty = \frac{a}{1-r}}$

$-1 < r < 1$ Convergent Series

$r \neq 0$ $\to$ getting smaller

$\underbrace{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots}$

$S_n = \frac{a(1-r^n)}{1-r}$

$(1-r^n) \longrightarrow 1$

$n = 30$

If r is small eg. $r = \frac{1}{2}$ $r^n = (\frac{1}{2})^{30} = 0.000000000009$

Example 3.22. A nervous investor is deciding whether to invest a sum of $\$P_0$ in a company that is advertising a high fixed interest rate of $I\%$ for the next N years. To allay all fear it is agreed that at the end of each year, he/she can withdraw all principal less the interest earned on that year. That interest is then used as principal for the next year's investment. The company is to pocket the interest on the last year of the plan as a penalty. What is the total value P_N of his/her asset at the end of the final year? Calculate P_N for the case where $P_0 = 100000$, $I = 25\%$ and $N = 10$.

Solution: At the end of the first year, the value of the investment is $P_1 = P_0 + iP_0$.

Year no.	Investment value at start of year	Investment value at end of year	Amount withdrawn
1	P_0	$P_0 + iP_0$	P_0
2	iP_0	$iP_0 + i^2P_0$	iP_0
3			
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$N-1$			
N	$i^{N-1}P_0$	$i^{N-1}P_0 + i^N P_0$	

Hence at the end of year N the investor has got back a total sum
$S_N = P_N = P_0 + iP_0 + i^2P_0 + \ldots + i^{N-1}P_0 =$

Substituting numerical values, the final value of the investment for the nervous investor is: $\$133,333.21$.

Final return for the bank is: $0.10

No of year	Amount invested	Amount at the end of the year	Amount withdraw
1	100000	125000	100000
2	25000	31250	25000
3	6250	7812.5	6250
4	1562.5	1953.125	1562.5
5	390.625	488.28125	390.625
6	97.65625	122.0703125	97.65625
7	24.4140625	30.51757813	24.4140625
8	6.103515625	7.629394531	6.103515625
9	1.525878906	1.907348633	1.525878906
10	0.381469727	0.476837158	0.381469727
	0.095367432		133333.2062

3.6.2 Power series

We have seen that a **finite geometric series** of n terms has the sum

Convergent series.

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \sum_{k=0}^{n-1} ar^k = \frac{a(1-r^n)}{1-r}.$$

$S_\infty = \frac{a}{1-r}$

Suppose we allow n to become very large. Then provided that $\boxed{-1 < r < 1}$, we have $r^n \to 0$ as $n \to \infty$. For example $\left(\frac{1}{2}\right)^{n\to\infty} = 0$. Now setting $a = 1$, $r = x$ and taking $n \to \infty$, it follows that

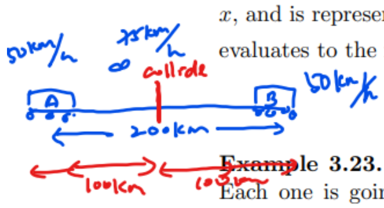
$r \neq 0$

$S_\infty = \frac{1}{1-x}$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^{n-1} + \cdots = \sum_{k=0}^{\infty} x^k. \tag{3.1}$$

The right hand side of Equation (3.1) is called a **power series** in the variable x, and is represented using a so-called infinite sum.[1] Here, this power series evaluates to the function $f(x) = \frac{1}{1-x}$.

$f(x) = \frac{1}{1+x}$



Example 3.23. Two trains 200 km apart are moving toward each other. Each one is going at a constant speed of 50 kilometres per hour. A fly starting on the front of one of them flies back and forth between them at a rate of 75 kilometres per hour (fast fly!). The fly does this until the two trains collide. What is the total distance the fly has flown?

Speed = $\frac{\text{Dm}}{\text{tm}}$

the tim needed is 2hou

75 x 2 = 150km

Power series

A general power series in the variable x has the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots \tag{3.2}$$

where a_0, a_1, a_2, ... are constants. Putting $a_0 = 1$, $a_1 = 1$, $a_2 = 1$, ... etc, we recover the geometric power series that is defined in Equation (3.1).

[1] A *series* is an object which allows us to give rigorous meaning to the concept of 'infinite sum'. To be precise, for a sequence $b_0, b_1, \ldots$, its series is defined as $\sum_{n=0}^{\infty} b_n = \lim_{n\to\infty} S_n$, where $S_n = \sum_{i=0}^{n} b_i$ is called its partial sum. A power series is just a special case of a series, namely take $b_n = a_n x^n$. In this course we will not cover series in detail, only power series.

A power series is actually a limit, and hence it may not converge for all $x \in (-\infty, \infty)$. However, if x is taken to be sufficiently small, namely $-R < x < R$, the power series will exist as a function of x, which we will call $f(x)$. The largest R for which this occurs is called the **radius of convergence**, and guarantees that the power series $f(x)$ exists for $-R < x < R$. In fact, it may even exist at $x = R$ or $x = -R$. This leads us to the idea of representing continuous functions of x by a power series. That is, we have

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots \tag{3.3}$$

where the domain of $f(x)$ is either $(-R, R), [-R, R), (-R, R]$ or $[-R, R]$.

$-R < x < R$ | $-R \leq x < R$

Table of Useful Power Series

Power series	**Domain**
$\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$	$-1 < x < 1$
$\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots$	$-1 < x < 1$
$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots$	$-\infty < x < \infty$
$\ln(1+x) \equiv \log_e(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$ $+(-1)^n \dfrac{x^{n+1}}{n+1} + \cdots$	$-1 < x \leq 1$
$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots + (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} + \cdots$	$-\infty < x < \infty$
$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + (-1)^n \dfrac{x^{2n}}{(2n)!} + \cdots$	$-\infty < x < \infty$

3.6.3 Taylor series

Taylor polynomials and linear approximation

We have just learnt that power series are functions. What if we in some sense asked the reverse question? Namely, can arbitrary functions be expressed as a power series? For instance, rather than defining a function through a power series, what if we already have some function $f(x)$ that we want to express as a power series? Can we do that? How? The answer is yes and such a power series representation of $f(x)$ is called its **Taylor series**.

Let's say we have a function $f(x)$ and let's assume it can be represented as a power series like Equation (3.3). Suppose we truncate the power series defined in Equation (3.3) after the first $(n+1)$ terms. That is, let us stop the series at the term $a_n x^n$. We then obtain an n^{th} degree polynomial in x. This polynomial, denoted by $T_n(x)$, is called the **Taylor polynomial** of degree n for the function $f(x)$ centred at $x = 0$. We have

$$T_n(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$$

Note that $T_n(x)$ is a finite polynomial, its domain includes all x. That is, $-\infty < x < \infty$. Now the Taylor series of f is then given by $\lim_{n\to\infty} T_n(x) = f(x)$. In other words, a function's Taylor series is precisely its power series representation. Please take a moment to understand the distinction between a power series and a Taylor series! $T_n(x)$

Example 3.24. Use the table of basic power series to find $T_0(x)$, $T_1(x)$, $T_2(x)$, $T_3(x)$ for e^x.

$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n$

Solution:

$T_0(x) = 1$ Zeroth

$T_1(x) = 1 + x$ Linear

$T_2(x) = 1 + x + \frac{1}{2}x^2$ Quadratic

$T_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$ Cubic

The first of these Taylor polynomials, namely $T_0(x) = 1$ simply matches the height of the graph of $y = f(x)$ at $x = 0$. It is sometimes called the **zeroth approximation** to $y = f(x)$ at $x = 0$. It can also be called the **zeroth Taylor polynomial** for f at $a = 0$.

The next Taylor polynomial, namely $T_1(x) = 1 + x$, is called the **linear approximation** to $f(x)$ at $a = 0$. The equation $y = T_1(x)$ is the equation of the **tangent line** to the graph $y = f(x)$ at $x = 0$.

The diagram below shows the graph of $y = e^x$ (thick curve) on the domain $-1.5 \le x \le 1.5$, along with the graphs of $y = T_0(x) = 1$ and the **linear approximation** function $y = T_1(x) = 1 + x$.

$y = e^0 = 1$

$x = 0$, $T_0 = 1$, $T_1 = 1$, $T_2 = 1$, $T_3 = 1$ — perfect match

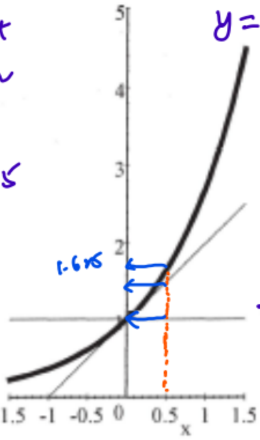
$x = 0.5$, $y = e^{0.5} \approx 1.65$

$T_0 = 1$

$T_1 = 1.5$

$T_2 = 1.625$

$T_3 = 1.649$



$T_2(x) = 1 + x + \frac{1}{2}x^2$

$T_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$

$e^x \approx 1 + x$

$-\cancel{0.5} < x < \cancel{0.5}$

$-0.2 < x < 0.2$

Better approx fnt for $y = e^x$ is $T_3(x)$

$-0.5 < x < 0.5$

$e^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$

Example 3.25. Use the power series table to find:

$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots +$

(i) $T_0(x)$, $T_1(x)$, $T_2(x)$, $T_3(x)$, for $f(x) = \ln(1 + x)$

$T_0(x) = 0$

$T_1(x) = x$

$T_2(x) = x - \frac{1}{2}x^2$

$T_3(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3$

(ii) $T_1(x)$, $T_3(x)$, $T_5(x)$, for $f(x) = \sin x$

$T_1(x) = x$ $\quad$ $T_1(x) = T_2(x)$

$T_3(x) = x - \frac{1}{6}x^3$

$T_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

(iii) $T_1(x)$, $T_3(x)$, $T_5(x)$, for $f(x) = \sin 2x$

$T_1(x) = 2x$

$T_3(x) = 2x - \frac{1}{6}(2x)^3 = 2x - \frac{4}{3}x^3$

$T_5(x) = 2x - \frac{4}{3}x^3 + \frac{1}{120}(2x)^5 = 2x - \frac{4}{3}x^3 + \frac{32}{15}x^5$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}$$

(iv) $T_0(x)$, $T_2(x)$, $T_6(x)$, for $f(x) = \cos 3x$

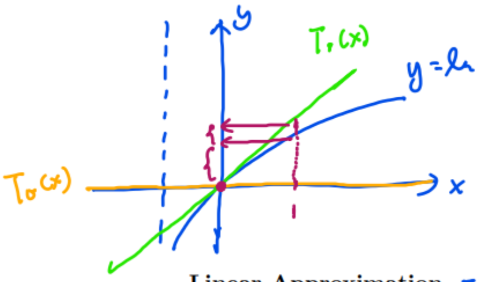
$T_0(x) = 1$

$T_2(x) = 1 - \frac{1}{2!}(3x)^2 = 1 - \frac{9}{2}x^2$

$T_6(x) = 1 - \frac{9}{2}x^2 + \frac{1}{4!}(3x)^4 - \frac{1}{6!}(3x)^6 = 1 - \frac{9}{2}x^2 + \frac{27}{8}x^3 - \frac{81}{80}x^6$

Example 3.26. For parts (i) and (ii) of the example above, draw sketches of the graphs of $y = f(x)$, $T_0(x)$ and $T_1(x)$.

(i) $y = \ln(1+x)$ $T_0(x) = 0$ $T_1(x) = x$



$\boxed{x = 0.1}$

$y = \ln(1.1) = 0.095$

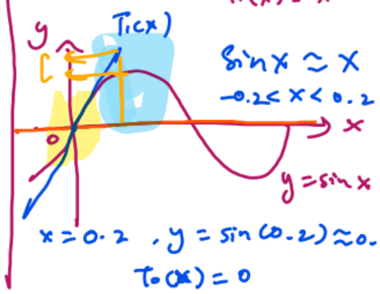
$T_0(x) = 0$

$T_1(x) = 0.1$

$T_n(x) = \cdots$

(ii) $y = \sin x$ $T_0(x) = 0$

$T_1(x) = x$



$\sin x \simeq x$

$-0.2 < x < 0.2$

$x = 0.2$, $y = \sin(0.2) \approx 0.2$

$T_0(x) = 0$

$T_1(x) = 0.2$

Linear Approximation $T_1(x)$

The Taylor polynomial of degree one ($y = T_1(x)$) is the **linear approximation** to $y = f(x)$ centred at $x = 0$. Clearly $T_1(x) = f(0) + f'(0)x$.

Example 3.27. (i) Find the linear approximation to

at $x=0$, $T_2(x) = f(0) + f'(0)x + \frac{f^{(2)}(0)}{2!}x^2$

$$y = f(x) = \frac{e^{3x}}{2+x}$$

centred at $x = 0$.

Solution:

$f(x) = \frac{e^{3x}}{2+x}$ $\qquad$ $f(0) = \frac{e^0}{2} = \frac{1}{2}$

$f'(x) = \frac{(3e^{3x})(2+x) - e^{3x}}{(2+x)^2}$ $\qquad$ $f'(0) = \frac{3(1)(2) - 1}{(2)^2} = \frac{5}{4}$

$$T_1(x) = f(0) + f'(0)\,x$$

$$\frac{e^{3x}}{2+x} \approx \frac{1}{2} + \frac{5}{4}x$$ around the value at $x=0$

$\boxed{\text{at } x=0}$

Hence $T_1(x) = \frac{1}{2} + \frac{5}{4}x$

(ii) Use the linear approximation to $f(x)$ to estimate the value of $f(0.1)$.

Actual value

$$f(x) = \frac{e^{3x}}{2+x}$$

$$f(0.1) = \frac{e^{0.3}}{2.1} \approx 0.6428 \approx 0.64 \text{ (2dp)}$$

linear approximation

$$T_1(x) = \frac{1}{2} + \frac{5}{4}x$$

$$T_1(0.1) = \frac{1}{2} + \frac{5}{4}(0.1) = 0.625 \approx 0.63 \text{ (2dp)}$$

linear approximation is consider good?

$\Rightarrow$

$\Rightarrow$ okay!