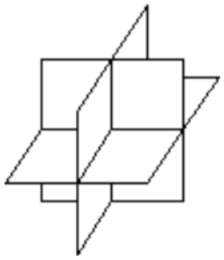
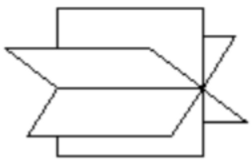
Lesson Four: The Intersection of Three Planes
There are EIGHT ways that three planes can intersect.
Method: Examine the normals to narrow down the number of possibilities.

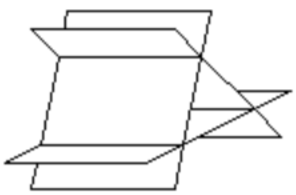
CASE I. NO NORMALS PARALLEL Solution is ...



-a point (Three planes intersect in a point.)



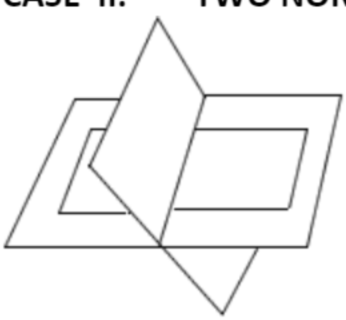
-a line (Three planes intersect in one unique line.)



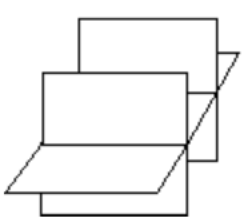
-no solution (Three planes intersect in three unique lines.)

For this case you must solve the system to see what the solution is.

CASE II. TWO NORMALS PARALLEL Solution is...

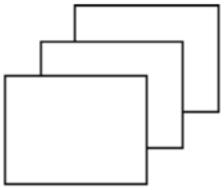


-a line (Two parallel/coincident planes and one non parallel plane.)



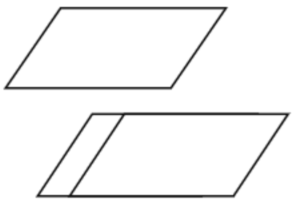
-no solution (Two parallel/non-coincident planes and one non parallel plane intersect in two unique lines.)

CASE III. ALL NORMALS PARALLEL Solution is...



-no solution (Three parallel non/coincident planes.)

-a plane (Three parallel/coincident planes.)



-no solution (Two parallel/coincident planes and one parallel/non-coincident plane.)

Example 1. Solve
$$\begin{aligned}\pi_1 &: x+2y+3z+4=0\\ \pi_2 &: x-y-3z-8=0\\ \pi_3 &: 2x+y+6z+14=0\end{aligned}$$

$$\begin{aligned}\vec{n}_1 &= (1,2,3)\\ \vec{n}_2 &= (1,-1,3)\\ \vec{n}_3 &= (2,1,6)\end{aligned} \quad \Rightarrow \text{ no normals are parallel}$$

Therefore we must solve the system to see what solution we have.
The reduced matrix is.....

$$\begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & 3 & 6 & -12 \\ 0 & 3 & 0 & 6 \end{bmatrix}$$ so the solution is a point..... $(x,y,z)=(1,2,-3)$

Example 2. Solve
$$\begin{aligned}\pi_1 &: x+y+2z=-2\\ \pi_2 &: 3x-y+14z=6\\ \pi_3 &: x+2y=-5\end{aligned}$$

$$\begin{aligned}\vec{n}_1 &= (1,1,2)\\ \vec{n}_2 &= (3,-1,14)\\ \vec{n}_3 &= (1,2,0)\end{aligned} \quad \Rightarrow \text{ no normals are parallel}$$

Therefore we must solve the system to see what solution we have.
The reduced matrix is.....

$\begin{bmatrix} 1 & 1 & 2 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so the solution is a line. Let z=t, a parameter and express x and y in terms of t.

The solution is the line $(x, y, z) = (-4t+1, -3+2t, t)$.

Example 3. Solve

$$\begin{aligned} \pi_1 &: x - y + 4z = 5 \\ \pi_2 &: 3x + y + z = -2 \\ \pi_3 &: 5x - y + 9z = 1 \end{aligned}$$

$\vec{n}_1 = (1,-1,4)$
$\vec{n}_2 = (3,1,1)$ $\Rightarrow$ no normals are parallel
$\vec{n}_3 = (5,-1,9)$

Therefore we must solve the system to see what solution we have.
The reduced matrix is.....

$\begin{bmatrix} 1 & -1 & 4 & 5 \\ 0 & -4 & 11 & 19 \\ 0 & 0 & 0 & -5 \end{bmatrix}$ so there is no solution. The three planes intersect in three unique lines.

Example 4. Solve

$$\begin{aligned} \pi_1 &: x - y + 2z = 1 \\ \pi_2 &: 2x - 2y + 4z = 2 \\ \pi_3 &: 3x + 2y - z = 4 \end{aligned}$$

$\vec{n}_1 = (1,-1,2)$
$\vec{n}_2 = (2,-2,4)$ $\Rightarrow$ $\vec{n}_2 = 2\vec{n}_1, but\ \vec{n}_2 \neq k\vec{n}_3; k \in \Re$
$\vec{n}_3 = (3,2,-1)$ $\quad \therefore \pi_2\ and\ \pi_1\ are\ parallel\ only.\ Check\ for\ coincidence.$
$D_2 = 2D_1. \therefore \pi_2\ and\ \pi_1\ are\ coincident\ too.$

Therefore we have two parallel/coincident planes and one non parallel plane intersecting in a unique line.

Let z=t, a parameter and solve for x and y in terms of t....to get.... $(x, y, z) = \left(\frac{-3}{5}t + \frac{6}{5}, \frac{7}{5}t + \frac{1}{5}, t\right)$.

Example 5. Solve

$$\begin{aligned} \pi_1 &: 2x + y + 3z = 4 \\ \pi_2 &: 2x + y + 3z = 7 \\ \pi_3 &: 3x - y + z = 10 \end{aligned}$$

$$\begin{aligned}
\vec{n}_1 &= (2,1,3) \\
\vec{n}_2 &= (2,1,3) \quad \Rightarrow \quad \vec{n}_1 = 1\vec{n}_2, but\ \vec{n}_1 \neq k\vec{n}_3;\ k \in \Re \\
\vec{n}_3 &= (3,-1,1) \qquad \therefore \pi_1\ and\ \pi_2\ are\ parallel\ only.\ Check\ for\ coincidence. \\
& \qquad D_1 \neq 1D_2.\ \therefore \pi_1\ and\ \pi_2\ are\ non-coincident.
\end{aligned}$$

Therefore there is no solution. Two parallel, non-coincident planes are intersected by a third in two unique lines.

Example 6. Solve
$$\begin{aligned}
\pi_1 &: x - y + 2z = 4 \\
\pi_2 &: 2x - 2y + 4z = 8 \\
\pi_3 &: -3x + 3y - 6z = -12
\end{aligned}$$

$$\begin{aligned}
\vec{n}_1 &= (1,-1,2) \\
\vec{n}_2 &= (2,-2,4) \quad \Rightarrow \quad \vec{n}_1 = \frac{1}{2}\vec{n}_2 = \frac{-1}{3}\vec{n}_3\ and\ D_1 = \frac{1}{2}D_2 = \frac{-1}{3}D_3 \\
\vec{n}_3 &= (-3,3,-6) \qquad \therefore The\ three\ planes\ are\ parallel\ and\ coincident.
\end{aligned}$$

Therefore the solution is the plane. Since all three equations are essentially the same, choose one and let z=t, y=s, t and s are parameters. Using π_1, solve for x in terms of s and t.

Then the solution is the plane given by
$$\begin{aligned}
x &= s - 2t + 4 \\
y &= s \\
z &= t
\end{aligned}$$

Example 7. Solve
$$\begin{aligned}
\pi_1 &: 2x - y + 5z = 4 \\
\pi_2 &: 2x - y + 5z = 7 \\
\pi_3 &: 4x - 2y + 10z = 9
\end{aligned}$$

$$\begin{aligned}
\vec{n}_1 &= (2,-1,5) \\
\vec{n}_2 &= (2,-1,5) \qquad \Rightarrow \vec{n}_1 = \vec{n}_2 = \frac{1}{2}\vec{n}_3\ but\ D_1 \neq D_2\ and\ D_1 \neq \frac{1}{2}D_3\ \ and\ D_2 \neq \frac{1}{2}D_3 \\
\vec{n}_3 &= (4,-2,10)
\end{aligned}$$

Therefore the three planes are parallel/non-coincident. So there is no solution.

Example 8.

Solve
$$\begin{aligned}
\pi_1 &: 3x + 4y + 5z = 6 \\
\pi_2 &: 6x + 8y + 10z = 13 \\
\pi_3 &: 12x + 16y + 20z = 26
\end{aligned}$$

$$\begin{aligned}
\vec{n}_1 &= (3,4,5) \\
\vec{n}_2 &= (6,8,10) \qquad \Rightarrow \vec{n}_3 = 4\vec{n}_1 = 2\vec{n}_2\ but\ D_3 \neq 4D_1\ and\ D_3 = 2D_2\ \ and\ 4D_1 \neq 2D_2 \\
\vec{n}_3 &= (12,16,20)
\end{aligned}$$

Therefore two planes are parallel/coincident (π_3 and π_2) and one plane is parall/non-coincident (π_1). So there is no solution.