Problem Set Eleven: Chain Rule, Directional Derivative, Stationary Points

Chain Rule

1. Given $f(x, y) = 2x^2 + 4y - 2$ and $x(s) = 3s$, $y(s) = 2s^2$, compute $\frac{df}{ds}$ by direct substitution (ie by first constructing $f(s)$) and also by the Chain rule.

① Direct Substitution $f(x,y) \longrightarrow f(s)$

$$f(s) = 2(3s)^2 + 4(2s^2) - 2$$

$$f(s) = 26s^2 - 2$$

$$\frac{df}{ds} = 52s$$

eg. $f(s) = \frac{\sqrt{e^x} \quad \tan^{-1}(e^x)}{\sqrt[3]{2x^2 - 3x}}$

$\frac{df}{ds}$? .... complicated

② Chain Rule $\rightarrow$ $\frac{df}{ds} = \left(\frac{\partial f}{\partial x} \times \frac{dx}{ds}\right) + \left(\frac{\partial f}{\partial y} \times \frac{dy}{ds}\right)$

$\frac{\partial f}{\partial x} = 4x \qquad \frac{\partial f}{\partial y} = 4$

$\frac{dx}{ds} = 3 \qquad \frac{dy}{ds} = 4s$

$$\frac{df}{ds} = (4x)(3) + (4)(4s)$$

$$\frac{df}{ds} = 4(3s)(3) + 16s$$

$$\frac{df}{ds} = 52s$$

2. Given $f(x, y, z) = 2x^2 + 4y + 3z^3 - 5$ and $x(s) = 3s$, $y(s) = 2s^2$, $z(s) = s^3$ compute $\frac{df}{ds}$ using the Chain rule.

$\frac{\partial f}{\partial x} = 4x \;\Big|\; \frac{\partial f}{\partial y} = 4 \;\Big|\; \frac{\partial f}{\partial z} = 9z^2 \;\Big|\; \frac{dx}{ds} = 3 \;\Big|\; \frac{dy}{ds} = 4s \;\Big|\; \frac{dz}{ds} = 3s^2$

Chain Rule

$$\frac{df}{ds} = \left(\frac{\partial f}{\partial x} \times \frac{dx}{ds}\right) + \left(\frac{\partial f}{\partial y} \times \frac{dy}{ds}\right) + \left(\frac{\partial f}{\partial z} \times \frac{dz}{ds}\right)$$

$$= 4(3s)(3) + (4)(4s) + 9(s^3)^2(3s^2)$$

$$\frac{df}{ds} = 52s + 27s^8$$

3. For the function $f(x,y) = y^2\sin(x)$ verify that

$$\frac{\partial}{dx}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$f_{xy} = f_{yx}$

$\frac{\partial f}{\partial x} = y^2\cos(x)$ $\qquad$ $\frac{\partial f}{\partial y} = 2y\sin(x)$

$$\left.\begin{aligned} \text{L.H.S}: \ \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial}{\partial x}(2y\sin(x)) = 2y\cos(x) \\ \text{R.H.S}: \ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial}{\partial y}(y^2\cos(x)) = 2y\cos(x) \end{aligned}\right\} \text{same}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \Rightarrow \text{Clairaut's Theorem}$$

4. Given $f(x,y) = 2xy$ and $x(r,\theta) = r\cos(\theta)$, $y(r,\theta) = r\sin(\theta)$, compute $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$.

$f(x,y) \rightarrow f(r,\theta) \rightarrow \frac{\partial f}{\partial r}, \frac{\partial f}{\partial \theta}$

$f(x,y) = 2xy$	$x(r,\theta) = r\cos(\theta)$	$y(r,\theta) = r\sin(\theta)$
$\checkmark\ \frac{\partial f}{\partial x} = 2y$	$\frac{\partial x}{\partial r} = \cos(\theta)$	$\frac{\partial y}{\partial r} = \sin(\theta)$
$\checkmark\ \frac{\partial f}{\partial y} = 2x$	$\frac{\partial x}{\partial \theta} = -r\sin(\theta)$	$\frac{\partial y}{\partial \theta} = r\cos(\theta)$

$$\frac{\partial f}{\partial r} = \left(\frac{\partial f}{\partial x} \times \frac{\partial x}{\partial r}\right) + \left(\frac{\partial f}{\partial y} \times \frac{\partial y}{\partial r}\right) \quad \text{(chain Rule)}$$

$$\frac{\partial f}{\partial r} = 2r\sin(\theta)\cos(\theta) + 2r\cos(\theta)\sin(\theta)$$

$$\frac{\partial f}{\partial r} = 4r\sin(\theta)\cos(\theta) = 2r\sin(2\theta)$$

$$\frac{\partial f}{\partial \theta} = \left(\frac{\partial f}{\partial x} \times \frac{\partial x}{\partial \theta}\right) + \left(\frac{\partial f}{\partial y} \times \frac{\partial y}{\partial \theta}\right) \quad \text{(chain Rule)}$$

$$= 2r\sin(\theta)(-r\sin(\theta)) + [2r\cos(\theta)][r\cos(\theta)]$$

$$= -2r^2\sin^2(\theta) + 2r^2\cos^2\theta$$

$$\frac{\partial f}{\partial \theta} = 2r^2(\cos^2(\theta) - \sin^2(\theta)) = 2r^2\cos(2\theta)$$

5. Let $f = f(x, y)$ be an arbitrary function of (x, y), and $x(r, \theta) = r\cos(\theta)$, $y(r, \theta) = r\sin(\theta)$.

a. Use the Chain rule to express $\dfrac{\partial f}{\partial r}$ and $\dfrac{\partial f}{\partial \theta}$ in terms of the first partial derivatives of $f(x, y)$.

$f(x,y)$ $\quad \swarrow \dfrac{\partial f}{\partial x} \quad \searrow \dfrac{\partial f}{\partial y}$

$x(r,\theta) = r\cos(\theta)$ $\qquad y = r\sin(\theta)$

$\dfrac{\partial x}{\partial r} = \cos(\theta)$ $\qquad \dfrac{\partial y}{\partial r} = \sin(\theta)$

$\dfrac{\partial x}{\partial \theta} = -r\sin(\theta)$ $\qquad \dfrac{\partial y}{\partial \theta} = r\cos(\theta)$

(a) $$\frac{\partial f}{\partial r} = \left(\frac{\partial f}{\partial x} \times \frac{\partial x}{\partial r}\right) + \left(\frac{\partial f}{\partial y} \times \frac{\partial y}{\partial r}\right) = \frac{\partial f}{\partial x}\cos(\theta) + \frac{\partial f}{\partial y}\sin(\theta)$$

$$\frac{\partial f}{\partial \theta} = \left(\frac{\partial f}{\partial x} \times \frac{\partial x}{\partial \theta}\right) + \left(\frac{\partial f}{\partial y} \times \frac{\partial y}{\partial \theta}\right) = -\left(\frac{\partial f}{\partial x}\right) r\sin(\theta) + \frac{\partial f}{\partial y} r\cos(\theta)$$

b. Use the Chain rule again to express $\dfrac{\partial^2 f}{\partial r^2}$ and $\dfrac{\partial^2 f}{\partial \theta^2}$ in terms of the second partial derivatives of $f(x, y)$.

$$\frac{\partial F}{\partial r} = \frac{\partial F}{\partial x} \times \frac{\partial x}{\partial r} + \frac{\partial F}{\partial y} \times \frac{\partial y}{\partial r}$$
$$= \frac{\partial}{\partial x}(F)$$

$$\frac{\partial^2 f}{\partial r^2} = \frac{\partial}{\partial r}\left(\overset{F}{\frac{\partial f}{\partial r}}\right) = \left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial r}\right)\right) \times \left(\frac{\partial x}{\partial r}\right) + \left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial r}\right)\right) \times \left(\frac{\partial y}{\partial r}\right) \leftarrow \text{Chain Rule}$$

From (a) ⇒ Recall $\dfrac{\partial f}{\partial r} = \dfrac{\partial f}{\partial x}\cos(\theta) + \dfrac{\partial f}{\partial y}\sin(\theta)$

$$\frac{\partial^2 f}{\partial r^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\cos(\theta) + \frac{\partial f}{\partial y}\sin(\theta)\right)[\cos(\theta)] + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\cos(\theta) + \frac{\partial f}{\partial y}\sin(\theta)\right)[\sin(\theta)]$$

$$= \frac{\partial^2 f}{\partial x^2}\cos^2(\theta) + \frac{\partial^2 f}{\partial x \partial y}\sin(\theta)\cos(\theta) + \frac{\partial^2 f}{\partial y \partial x}\cos(\theta)\sin(\theta) + \frac{\partial^2 f}{\partial y^2}\sin^2(\theta)$$

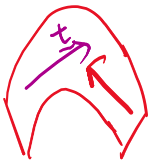
$$\frac{\partial^2 f}{\partial r^2} = \frac{\partial^2 f}{\partial x^2}\cos^2(\theta) + 2\frac{\partial^2 f}{\partial x \partial y}\sin(\theta)\cos(\theta) + \frac{\partial^2 f}{\partial y^2}\sin^2(\theta)$$

$$\frac{\partial^2 f}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial \theta}\right)$$

$$\frac{\partial^2 f}{\partial \theta^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \theta}\right)\left(\frac{\partial x}{\partial \theta}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \theta}\right)\left(\frac{\partial y}{\partial \theta}\right) \rightarrow \text{chain Rule}$$

$$\frac{\partial^2 f}{\partial \theta^2} = \frac{\partial}{\partial x}\left(-\frac{\partial f}{\partial x}r\sin(\theta) + \frac{\partial}{\partial y}r\cos(\theta)\right)(-r\sin(\theta))$$
$$+ \frac{\partial}{\partial y}\left(-\frac{\partial f}{\partial x}r\sin(\theta) + \frac{\partial f}{\partial y}r\cos(\theta)\right)(r\cos(\theta))$$

$$\frac{\partial^2 f}{\partial \theta^2} = \frac{\partial^2 f}{\partial x^2} r^2\sin^2(\theta) - 2\frac{\partial^2 f}{\partial x \partial y} r^2 \sin(\theta)\cos(\theta) + \frac{\partial^2 f}{\partial y^2} r^2 \cos(\theta)$$



6. Compute $\frac{df}{ds}$ for the function $f(x,y) = xy + x + y$ along the curve $x(s) = r\cos(\frac{s}{r})$, $y(s) = r\sin(\frac{s}{r})$. Verify that $\frac{dx}{ds}\underset{\sim}{i} + \frac{dy}{ds}\underset{\sim}{j}$ is a unit vector.

$\downarrow$ grad f

$\frac{df}{ds} = \nabla f \cdot \underset{\sim}{t}$ $\leftarrow$ must be unit vector

$\frac{\partial f}{\partial x} = y+1$

$\frac{\partial f}{\partial y} = x+1$

$\nabla f = \begin{pmatrix} y+1 \\ x+1 \end{pmatrix}$

$\frac{dx}{ds} = r\left[-\sin\left(\frac{s}{r}\right)\right]\left(\frac{1}{r}\right) = -\sin\left(\frac{s}{r}\right)$

$\frac{dy}{ds} = r\left[\cos\left(\frac{s}{r}\right)\right]\left(\frac{1}{r}\right) = \cos\left(\frac{s}{r}\right)$

$\frac{df}{ds} = \nabla f \cdot \underset{\sim}{t}$

$\nabla f = \frac{\partial f}{\partial x}\underset{\sim}{i} + \frac{\partial f}{\partial y}\underset{\sim}{j}$

$\underset{\sim}{t} = \frac{dx}{ds}\underset{\sim}{i} + \frac{dy}{ds}\underset{\sim}{j}$

$\underset{\sim}{t} = \begin{pmatrix} -\sin\left(\frac{s}{r}\right) \\ \cos\left(\frac{s}{r}\right) \end{pmatrix}$

$|\underset{\sim}{t}| = \sqrt{\sin^2\left(\frac{s}{r}\right) + \cos^2\left(\frac{s}{r}\right)}$

$= 1$ (unit vector)

$\frac{df}{ds} = \begin{pmatrix} y+1 \\ x+1 \end{pmatrix} \cdot \begin{pmatrix} -\sin(s/r) \\ \cos(s/r) \end{pmatrix}$

$\frac{df}{ds} = \left(r\sin\left(\frac{s}{r}\right)+1\right)\left(-\sin\left(\frac{s}{r}\right)\right) + \left(r\cos\left(\frac{s}{r}\right)+1\right)\left(\cos\left(\frac{s}{r}\right)\right)$

Directional Derivative

7. Compute the directional derivative for each of the following functions in the stated direction. Be sure that you use a unit vector.

(a) $f(x,y) = 2x + 3y$ at $(1,2)$ $\quad \underset{\sim}{t} = \frac{3}{5}\underset{\sim}{i} + \frac{4}{5}\underset{\sim}{j}$

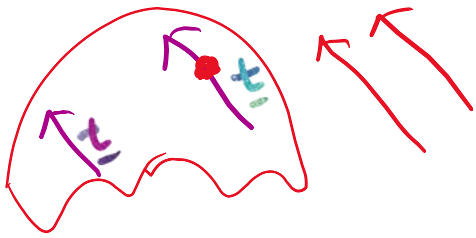
(b) $g(x,y) = 4 - x^2 - y^2$ at $(1,1)$ $\quad \underset{\sim}{t} = \frac{1}{\sqrt{2}}\underset{\sim}{i} + \frac{1}{\sqrt{2}}\underset{\sim}{j}$

(c) $h(x,y) = \sin(x)\cos(y)$ at $(\frac{\pi}{4}, \frac{\pi}{4})$ $\quad \underset{\sim}{t} = (\frac{1}{\sqrt{2}}\underset{\sim}{i} + \frac{1}{\sqrt{2}}\underset{\sim}{j})$

(d) $q(x,y,z) = \ln(x^2 + y^2 + z^2)$ at $(1,0,1)$ $\quad \underset{\sim}{t} = \underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$

(e) $r(x,y) = -xye^{-x^2-y^2}$ at $(1,1)$ $\quad \underset{\sim}{t} = 4\underset{\sim}{i} - 3\underset{\sim}{j}$

(f) $u(x,y,z) = \sqrt{1 - x^2 - y^2 - z^2}$ at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ $\quad \underset{\sim}{t} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$



$\underset{\sim}{i} + \underset{\sim}{j}$

(d) $\frac{df}{ds} = \nabla f \cdot \underset{\sim}{t}$ $\leftarrow$ unit vector

$\nabla f = \frac{\partial f}{\partial x}\underset{\sim}{i} + \frac{\partial f}{\partial y}\underset{\sim}{j} + \frac{\partial f}{\partial z}\underset{\sim}{k}$

$\frac{\partial q}{\partial x} = \frac{2x}{x^2+y^2+z^2}$, $\qquad$ At $(1,0,1)$ $\quad \frac{\partial q}{\partial x} = 1$

$\frac{\partial q}{\partial y} = \frac{2y}{x^2+y^2+z^2}$ $\qquad \frac{\partial q}{\partial y} = 0$

$\frac{\partial q}{\partial z} = \frac{2z}{x^2+y^2+z^2}$ $\qquad \frac{\partial q}{\partial z} = 1$

$\nabla f = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\underset{\sim}{t} = \underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k} \rightarrow |\underset{\sim}{t}| = \sqrt{1^2+1^2+1^2}$

$= \sqrt{3}$ NOT unit vector

$\boxed{\frac{df}{ds} = \nabla f \cdot \underset{\sim}{t}}$

$\hat{\underset{\sim}{t}} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix}$

$\boxed{\hat{\underset{\sim}{t}} = \frac{\underset{\sim}{t}}{|\underset{\sim}{t}|}}$

$\frac{df}{ds} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix} = 0$

(a) $f(x,y) = 2x + 3y$, $(1,2)$, $\underset{\sim}{t} = \frac{3}{5}\underset{\sim}{i} + \frac{4}{5}\underset{\sim}{j}$

$\frac{\partial f}{\partial x} = 2$ $\qquad \frac{\partial f}{\partial y} = 3$ $\qquad |\underset{\sim}{t}| = \sqrt{\left(\frac{9}{25}\right) + \frac{16}{25}} = 1$ unit

$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=1 \\ y=2}} = 2$ $\qquad \left.\frac{\partial f}{\partial x}\right|_{\substack{x=1 \\ y=2}} = 3$ $\qquad \nabla f = 2\underset{\sim}{i} + 3\underset{\sim}{j} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$$\frac{df}{ds} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} = \frac{6+12}{5} = \frac{18}{5}$$

(b) $g(x,y) = 4 - x^2 - y^2$ at $(1,1)$

$\frac{\partial g}{\partial x} = -2x$ $\qquad \frac{\partial g}{\partial y} = -2y$

$\left.\frac{\partial g}{\partial x}\right|_{\substack{x=1 \\ y=1}} = -2$ $\qquad \left.\frac{\partial g}{\partial y}\right|_{\substack{x=1 \\ y=1}} = -2$

$\nabla f = -2\underset{\sim}{i} - 2\underset{\sim}{j} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$

$\underset{\sim}{t} = \frac{1}{\sqrt{2}}\underset{\sim}{i} + \frac{1}{\sqrt{2}}\underset{\sim}{j}$

$|\underset{\sim}{t}| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$
$= 1$

$$\frac{df}{ds} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = -\frac{4}{\sqrt{2}} = -2\sqrt{2}$$

(c) $h(x,y) = \sin(x)\cos(y)$ $\qquad \left(\frac{\pi}{4}, \frac{\pi}{4}\right)$

$\frac{\partial h}{\partial x} = \cos(x)\cos(y)$ $\qquad \underset{\sim}{t} = \frac{1}{\sqrt{2}}\underset{\sim}{i} + \frac{1}{\sqrt{2}}\underset{\sim}{j}$

$\frac{\partial h}{\partial y} = -\sin(x)\sin(y)$ $\qquad |\underset{\sim}{t}| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$

$$\frac{df}{ds} = \nabla f \cdot \underset{\sim}{t} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = 0$$

7(e) $r(x,y) = -xy\, e^{-x^2-y^2}$ at $(1,1)$

$$\frac{\partial f}{\partial x} = -y e^{-x^2-y^2} + 2x^2 y\, e^{-x^2-y^2}$$

$$\frac{\partial f}{\partial y} = -x e^{-x^2-y} + 2xy^2 e^{-x-y^2}$$

$\underset{\sim}{t} = 4\underset{\sim}{i} - 3\underset{\sim}{j}$

$|t| = \sqrt{4^2 + 3^2}$

$|\underset{\sim}{t}| = 5$ units^2

$$\nabla f_{(1,1)} = \begin{pmatrix} e^{-2} \\ e^{-2} \end{pmatrix}$$

$\hat{\underset{\sim}{t}} = \frac{4}{5}\underset{\sim}{i} - \frac{3}{5}\underset{\sim}{j}$

$$\frac{df}{ds} = \nabla f \cdot \hat{\underset{\sim}{t}} = \begin{pmatrix} e^{-2} \\ e^{-2} \end{pmatrix} \cdot \begin{pmatrix} 4/5 \\ -3/5 \end{pmatrix} = \frac{1}{5} e^{-2}$$

7(f) $u(x,y,z) = \sqrt{1-x^2-y^2-z}$ at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

$$\frac{\partial u}{\partial x} = \frac{-x}{\sqrt{1-x^2-y^2-z^2}}, \quad \frac{\partial u}{\partial y} = \frac{-y}{\sqrt{1-x^2-y^2-z^2}}$$

$$\frac{\partial u}{\partial z} = \frac{-z}{\sqrt{1-x^2-y^2-z^2}} \qquad \nabla f = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$\underset{\sim}{t} = \sqrt{4+1+1} = \sqrt{6}, \quad \hat{t} = \begin{pmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$$

$$\frac{df}{ds} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \begin{pmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} = -\frac{2}{\sqrt{6}}$$

Extrema

8a. Compare the three functions

$$f(x,y)=x^2+y^2$$
$$g(x,y)=x^2-y^2 \rightarrow \text{saddle} \quad D<0$$
$$h(x,y)=-x^2-y^2 \rightarrow \text{local max} \quad D>0 \quad \frac{\partial^2 f}{\partial x^2}<0$$

in regards to their extrema. Sketch the graph of each function as part of your comparison.

b. What extrema can you find for the function $r(x,y)=x^2-y^3$? $\rightarrow$ No conclusion

(a) $f(x,y)=x^2+y^2$

$\frac{\partial f}{\partial x}=2x \quad \frac{\partial f}{\partial y}=2y \quad \frac{\partial^2 f}{\partial x^2}=2 \quad \frac{\partial^2 f}{\partial y^2}=2 \quad \frac{\partial^2 f}{\partial x \partial y}=0$

For critical point, $\frac{\partial f}{\partial x}=0$ & $\frac{\partial f}{\partial y}=0$ $(0,0)$

$x=0 \quad y=0$

$f(0,0)=0 \Rightarrow (0,0,0) \Rightarrow$ critical point

Determine nature * $D=\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right)-\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2=4-0=4>0$

2nd Derivative Test

$\frac{\partial^2 f}{\partial x^2}=2>0$

$\Rightarrow$ local minimum at $(0,0,0)$

$g(x,y)=x^2-y^2$

$g_x=2x \quad g_y=-2y$

$g_{xx}=2$
$g_{yy}=-2$
$g_{xy}=0$.

For extrema,

$2x=0$, $-2y=0$

$x=0$, $y=0$, $g(0,0)=0 \Rightarrow (0,0,0)$

$D=2\times -2-(0)^2=-2<0$

$\Rightarrow$ saddle point

$h(x,y) = -x^2 - y^2$

$h_x = -2x \qquad h_y = -2y$

$x = 0, \; y = 0 \Rightarrow (0,0,0)$

$h_{xx} = -2$
$h_{yy} = -2$
$h_{xy} = 0$

$D = (-2)(-2) - (0)^2 = 4 > 0, \; h_{xx} < 0$

$\Rightarrow$ local maximum point $(0,0,0)$

(b) $r(x,y) = x^2 - y^3$

$r_x = 2x \qquad r_y = -3y^2$

$x = 0, \; y = 0, \; z = 0 \Rightarrow (0,0,0)$

$r_{xx} = 2$
$r_{yy} = -6y$
$r_{xy} = 0$

$D = (2)(-6y) - (0)^2 = -12y$

at $(0,0,0) \rightarrow \boxed{D = 0}$ point $(0,0,0)$

$\Rightarrow$ No conclusion

9. Find the critical points (if any) for each of the following functions.

(a) $f(x,y) = x - x^3 + y^2$

(b) $g(x,y) = e^{x^2+y^2}$

(c) $h(x,y) = xye^{-x^2-y^2}$

(d) $p(x,y) = (2 - x^2)e^{-y}$

(e) $q(x,y) = \arctan(x^2 + y^2)$

(f) $r(x,y,z) = 4x^2 + 3y^2 + z^2$

(g) $s(x,y,z) = \arctan((x-1)^2 + y^2 + z^2)$

10. Identify the nature of the extrema found in each of the functions from Question $9(a-e)$. Can you find a way to classify the extrema in Question $9f$ and $9g$?

~~Q9~~ Q10

(a) $f(x,y) = x - x^3 + y^2$

$f_x = 1-3x^2$ $\quad f_y = 2y$ $\quad f_{xx} = -6x$ $\quad f_{yy} = 2$ $\quad f_{xy} = 0$

Critical point(s)

$f_x = 0$ & $f_y = 0$

$x = \pm\sqrt{\frac{1}{3}}$ ✓ $\quad y = 0$

$f(\sqrt{\frac{1}{3}}, 0) = \frac{1}{\sqrt{3}} - (\frac{1}{\sqrt{3}})^3$

$f(-\sqrt{\frac{1}{3}}, 0) = \frac{1}{\sqrt{3}} + (\frac{1}{\sqrt{3}})^3$

~~Ob~~ Check Nature

At $(\sqrt{\frac{1}{3}}, 0, \frac{1}{\sqrt{3}} - (\frac{1}{\sqrt{3}})^3)$

$$\boxed{D = -12x}$$

$D = (-6x)(2) - (0)^2 = -12x$ ✓

$x=\sqrt{\frac{1}{3}}$ $\quad D = -12(\sqrt{\frac{1}{3}}) < 0 \Rightarrow$ saddle point

$x=-\sqrt{\frac{1}{3}}$ $\quad$ At $(-\sqrt{\frac{1}{3}}, 0, \frac{1}{\sqrt{3}} + (\frac{1}{\sqrt{3}})^3)$, $\quad f_{xx} = -6x$

$D = -12(-\sqrt{\frac{1}{3}}) > 0$ $\quad f_{xx} = -6(-\sqrt{\frac{1}{3}}) = 6\sqrt{\frac{1}{3}} > 0$

$\Rightarrow$ local Minimum.

(b) $g(x,y) = e^{x^2+y^2}$

$g_x = 2xe^{x^2+y^2}$ $\quad g_y = 2ye^{x^2+y^2}$

$g_x = 0$ $[e^{x^2+y^2} \neq 0]$ $\quad g_y = 0$

$2x = 0$ $\quad 2y = 0$

$x = 0$ $\quad y = 0$

$g(0,0) = e^0 = 1 \Rightarrow (0,0,1)$

$g_{xx} = 2e^{x^2+y^2} + 4x^2 e^{x^2+y^2}$

$g_{yy} = 2e^{x^2+y^2} + 4y^2 e^{x^2+y^2}$

$g_{xy} = 4xy\, e^{x^2+y^2}$

$g_{xx}(0,0,1) = 2$, $g_{yy}(0,0,1) = 2$

$g_{xy}(0,0,1) = 0$

$D = (2)(2) - (0)^2 = 4 > 0$, $g_{xx} > 0$

local min at $(0,0,1)$

(c) $h(x,y) = xye^{-x^2-y^2}$

$h_x = ye^{-x^2-y^2} - 2x^2ye^{-x^2-y^2} = ye^{-x^2-y^2}(1-2x^2)$

$h_y = xe^{-x^2-y^2} - 2xy^2e^{-x^2-y^2} = xe^{-x^2-y^2}(1-2y^2)$

$h_x = 0$ $\quad$ $h_y = 0$ $\quad$ $(0,0,0)$ $\quad$ $(\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}, \tfrac{1}{2}e^{-1})$

$x = \pm\sqrt{\tfrac{1}{2}}$ $\quad$ $y = \pm\sqrt{\tfrac{1}{2}}$ $\quad$ $(0, \sqrt{\tfrac{1}{2}}, 0)$ $\quad$ $(\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}, -\tfrac{1}{2}e^{-1})$

$y = 0$ $\quad$ $x = 0$ $\quad$ $(0, -\sqrt{\tfrac{1}{2}}, 0)$ $\quad$ $(-\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}, -\tfrac{1}{2}e^{-1})$

$(\sqrt{\tfrac{1}{2}}, 0, 0)$ $\quad$ $(-\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}, \tfrac{1}{2}e^{-1})$

$(-\sqrt{\tfrac{1}{2}}, 0, 0)$

(d) $P(x,y) = (2-x^2)e^{-y}$

$P_x = -2xe^{-y}$ $\quad$ $P_y = -(2-x^2)e^{-y}$

$P_x = 0$ $\quad$ $P_y = 0$ $\Rightarrow$ In this case, we cannot find any value(s) of y that fullfil the condition $P_x = 0$ & $P_y = 0$

$x = 0$ $\quad$ $x = \pm\sqrt{2}$

$\Rightarrow$ No critical point

(e) $g(x,y) = \tan^{-1}(x^2+y^2)$

$g_x = \dfrac{2x}{1+(x^2+y^2)^2}$ $\quad$ $g_y = \dfrac{2y}{1+(x^2+y^2)^2}$

$g_{xx} = \dfrac{2[1+(x^2+y^2)^2] - 8x^2(x^2+y^2)}{(1+(x^2+y^2)^2)^2}$

$g_{yy} = \dfrac{2[1+(x^2+y^2)^2] - 8y^2(x^2+y^2)}{[1+(x^2+y^2)^2]^2}$

$g_{xy} = \dfrac{-(2x)[2(x^2+y^2)](2y)}{[1+(x^2+y^2)^2]^2}$

$g_x = 0$ $\quad$ $g_y = 0$

$2x = 0$ $\quad$ $2y = 0$

$x = 0$ $\quad$ $y = 0$

$g(0,0) = 0 \Rightarrow (0,0,0)$

$g_{xx}(0,0,0) = 2$ $\quad$ $g_{yy}(0,0,0) = 2$ $\quad$ $g_{xy}(0,0,0) = 0$

$D = (2)(2) - (0)^2 = 4 > 0$ $\quad$ $g_{xx}(0,0,0) > 0$

$\Rightarrow (0,0,0)$ is local minimum

(f) $r(x,y,z) = 4x^2+3y^2+z^2$

$r_x = 8x \quad r_y = 6y \quad r_z = 2z$

For critical point, $r_x = r_y = r_z = 0 \Rightarrow (0,0,0)$

$r(0,0,0) = 0 \Rightarrow (0,0,0,0)$

$D \Rightarrow$ Need to use eigenvalues of matrix of

$$\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} \Rightarrow \text{we will get } (0,0,0,0) \text{ local min.}$$

(g) $S(x,y,z) = \tan^{-1}((x-1)^2+y^2+z^2)$

$$S_x = \frac{2(x-1)}{1+[(x-1)^2+y^2+z^2]^2} \quad S_y = \frac{2y}{1+[(x-1)^2+y^2+z^2]^2} \quad S_z = \frac{2z}{1+[(x-1)^2+y^2+z^2]^2}$$

$S_x = 0$, $S_y = 0$, $S_z = 0$

$x = 1$, $y = 0$, $z = 0 \Rightarrow (1,0,0,0)$

$S(1,0,0) = 0$

Use eigenvalue to determine

$\Rightarrow (1,0,0,0)$ is local minimum