Chapter 1

Vectors, Lines and Planes

1.1 Introduction to Vectors

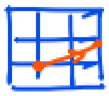
1.1.1 Notation and definition

Common forms of vector notation are bold symbols ($\mathbf{v}$), arrow notation ($\vec{v}$) and tilde notation ($\underset{\sim}{v}$). Throughout this Study Guide we will use the tilde notation. This notation compares suitably with the handwritten notation for vectors. Points in space are represented by a capital letter (for example the point P). Note that capital letters are also used for matrices, but considering each of these objects, this does not lead to ambiguity.

Vectors can be defined in (at least) two ways - **algebraically** as objects like

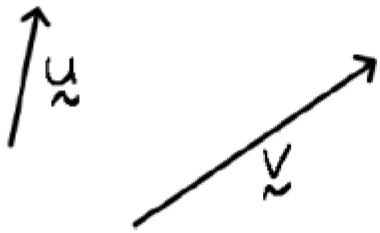
$$\underset{\sim}{v} = (1, 7, 3)$$
$$\underset{\sim}{u} = (2, -1, 4)$$

$\underset{\sim}{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$



$\underset{\sim}{u} = 2\hat{\underset{\sim}{i}} - 5\hat{j} + 4k$

or **geometrically** as arrows in space.



Note that vectors have both **magnitude** and **direction**. A quantity specified only by a number (but no direction) is known as a **scalar**.

How can we be sure that these two definitions actually describe the same object? Equally, how do we convert from one form to the other? That is, given $\underset{\sim}{v} = (1, 2, 7)$ how do we draw the arrow and likewise, given the arrow how do we extract the numbers $(1, 2, 7)$?

Suppose we are give two points P and Q. Suppose also that we find the change in coordinates from P to Q is (say) $(1, 2, 7)$. We could also draw an arrow from P to Q. Thus we have two ways of recording the path from P to Q, either as the numbers $(1, 2, 7)$ or the arrow.

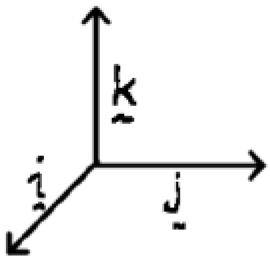
Suppose now that we have another pair of points R and S and further that we find the change in coordinates to be $(1, 2, 7)$. Again, we can join the points with an arrow. This arrow will have the same direction and length as that for P to Q.

In both cases, the **displacement**, from start to finish, is represented by either the numbers $(1, 2, 7)$ or the arrow – thus we can use either form to represent the vector. Note that this means that a vector does not live at any one place in space – it can be moved anywhere provided its length and direction are unchanged.

To extract the numbers $(1, 2, 7)$ given just the arrow, simply place the arrow somewhere in the x, y, z space, and the measure the change in coordinates from tail to tip of the vector. Equally, to draw the vector given the numbers $(1, 2, 7)$, choose $(0, 0, 0)$ as the tail then the point $(1, 2, 7)$ is the tip.

The **components** of a vector are just the numbers we use to describe the vector. In the above, the components of $\underset{\sim}{v}$ are 1,2 and 7.

Another very common way to write a vector, such as $\underset{\sim}{v} = (1, 7, 3)$ for example, is $\underset{\sim}{v} = 1\underset{\sim}{i} + 7\underset{\sim}{j} + 3\underset{\sim}{k}$. The three vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$ are a simple way to remind us that the three numbers in $\underset{\sim}{v} = (1, 7, 3)$ refer to directions parallel to the three coordinate axes (with $\underset{\sim}{i}$ parallel to the x-axis, $\underset{\sim}{j}$ parallel to the y-axis and $\underset{\sim}{k}$ parallel to the z-axis).



In this way we can always write down any 3-dimensional vector as a linear combination of the vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$ and thus these vectors are also known as **basis vectors**.

1.1.2 Linear independence

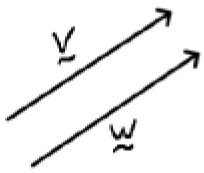
Two or more vectors are **linearly independent** if we cannot take any one of the vectors and write it as a **linear combination** of the others. We cannot write $\underset{\sim}{i}$ as a linear combination of $\underset{\sim}{j}$ and $\underset{\sim}{k}$. In other words there are no non-zero scalars α and β such that $\underset{\sim}{i} = \alpha \underset{\sim}{j} + \beta \underset{\sim}{k}$. Thus the basis vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$ are linearly independent.

If we can take a vector and write it as a linear combination of other vectors, then those vectors are known as **linearly dependent**. For example the vectors $\underset{\sim}{u} = (7, 17, -3)$, $\underset{\sim}{v} = (1, 2, 3)$ and $\underset{\sim}{w} = (3, 7, 1)$ are linearly dependent as $\underset{\sim}{u} = 3\underset{\sim}{w} - 2\underset{\sim}{v}$.

1.1.3 Algebraic properties

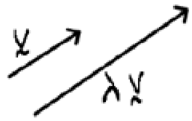
What rules must we observe when we are working with vectors?

- **Equality**
  $\underset{\sim}{v} = \underset{\sim}{w}$ only when the arrows for $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are identical.

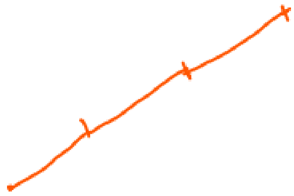
- **Stretching (scalar multiple)**
  The vector $\lambda \underset{\sim}{v}$ is parallel to $\underset{\sim}{v}$ but is stretched by a factor λ. The

magnitude of $\lambda \underset{\sim}{v}$ is $|\lambda|$ times the magnitude of $\underset{\sim}{v}$.

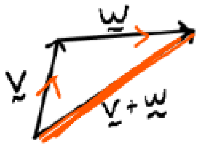


$\lambda \underset{\sim}{v}$

$3 \underset{\sim}{v}$

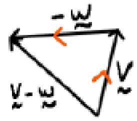


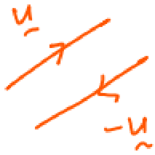
- **Addition**
  To add two vectors $\underset{\sim}{v}$ and $\underset{\sim}{w}$ arrange the two so that they are tip to tail. Then $\underset{\sim}{v} + \underset{\sim}{w}$ is the vector that starts at the first tail and ends at the second tip. Thus the sum of two vectors $\underset{\sim}{v}$ and $\underset{\sim}{w}$ is the *displacement* vector resulting from first applying $\underset{\sim}{v}$ then $\underset{\sim}{w}$.



- **Subtraction**
  The difference $\underset{\sim}{v} - \underset{\sim}{w}$ of two vectors $\underset{\sim}{v}$ and $\underset{\sim}{w}$ is the displacement vector resulting from first applying $\underset{\sim}{v}$ then $-\underset{\sim}{w}$. Note that $-\underset{\sim}{w}$ is simply the vector $\underset{\sim}{w}$ now pointing in the opposite direction to $\underset{\sim}{w}$.

$\underset{\sim}{v} + (-\underset{\sim}{w})$





Let $\underset{\sim}{u} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$ $\underset{\sim}{v} = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$

Example 1.1. Express each of the above rules in terms of the components of vectors (i.e. in terms of numbers like $(1, 2, 7)$ and (a, b, c)).

Scalar multiple: $2\underset{\sim}{u} = \begin{pmatrix} 2 \\ 4 \\ 14 \end{pmatrix}$

Addition: $\underset{\sim}{u} + \underset{\sim}{v} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 11 \end{pmatrix}$

Subtraction: $\underset{\sim}{u} - \underset{\sim}{v} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

Example 1.2. Given $\underset{\sim}{v} = (3, 4, 2)$ and $\underset{\sim}{w} = (1, 2, 3)$ compute $\underset{\sim}{v} + \underset{\sim}{w}$ and $2\underset{\sim}{v} + 7\underset{\sim}{w}$.

$\underset{\sim}{v} + \underset{\sim}{w} = \begin{pmatrix} 4 \\ 6 \\ 5 \end{pmatrix}$

$2\underset{\sim}{v} + 7\underset{\sim}{w} = \begin{pmatrix} 6 \\ 8 \\ 4 \end{pmatrix} + \begin{pmatrix} 7 \\ 14 \\ 21 \end{pmatrix} = \begin{pmatrix} 13 \\ 22 \\ 25 \end{pmatrix}$

Example 1.3. Given $\underset{\sim}{v} = (1, 2, 7)$ draw $\underset{\sim}{v}$, $2\underset{\sim}{v}$ and $-\underset{\sim}{v}$.

Example 1.4. Given $\underset{\sim}{v} = (1, 2, 7)$ and $\underset{\sim}{w} = (3, 4, 5)$ draw and compute $\underset{\sim}{v} - \underset{\sim}{w}$.

$\underset{\sim}{v} - \underset{\sim}{w} = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}$

1.2 Vector Dot Product

↳ multiplication

How do we multiply vectors? We have already seen one form, where we stretch $\underset{\sim}{v}$ by a scalar λ, i.e. $\underset{\sim}{v} \to \lambda \underset{\sim}{v}$. This is called **scalar multiplication**.

Another form is the **vector dot product**. Let $\underset{\sim}{v} = (v_x, v_y, v_z)$ and $\underset{\sim}{w} = (w_x, w_y, w_z)$ be a pair of vectors, then we *define* the dot product $\underset{\sim}{v} \cdot \underset{\sim}{w}$ by

$$\boxed{\underset{\sim}{v} \cdot \underset{\sim}{w} = v_x w_x + v_y w_y + v_z v_z\ .}$$

Example 1.5. Let $\underset{\sim}{v} = (1, 2, 7)$ and $\underset{\sim}{w} = (-1, 3, 4)$. Compute $\underset{\sim}{v} \cdot \underset{\sim}{v}$, $\underset{\sim}{w} \cdot \underset{\sim}{w}$ and $\underset{\sim}{v} \cdot \underset{\sim}{w}$

$\underset{\sim}{v} \cdot \underset{\sim}{v} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$

$\underset{\sim}{v} \cdot \underset{\sim}{v} = 1^2 + 2^2 + 7^2 = (1\times1) + (2\times2) + (7\times7) = 54$

$= 54$

$\underset{\sim}{w} \cdot \underset{\sim}{w} = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$

$= 1 + 9 + 16$

$= 26$

$\underset{\sim}{v} \cdot \underset{\sim}{w} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$

$= -1 + 6 + 28$

$= 33$

What do we observe?

$\underset{\sim}{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\underset{\sim}{v} \cdot \underset{\sim}{v} = a^2 + b^2 + c^2$

- $\underset{\sim}{v} \cdot \underset{\sim}{w}$ is a single number **not a vector** (i.e. it is a *scalar*)
- $\underset{\sim}{v} \cdot \underset{\sim}{w} = \underset{\sim}{w} \cdot \underset{\sim}{v}$
- $(\lambda \underset{\sim}{v}) \cdot \underset{\sim}{w} = \lambda(\underset{\sim}{v} \cdot \underset{\sim}{w})$
- $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot \underset{\sim}{v} = \underset{\sim}{a} \cdot \underset{\sim}{v} + \underset{\sim}{b} \cdot \underset{\sim}{v}$

$2(a+b) = 2a + 2b$

The last two cases display what we call **linearity**.

$\underset{\sim}{v} = (a, b, c)$

$\underset{\sim}{v} \cdot \underset{\sim}{v} = a^2 + b^2 + c^2$

1.2.1 Length of a vector

The **length of a vector** $\underset{\sim}{v}$ is defined by

$$|\underset{\sim}{v}| = \sqrt{\underset{\sim}{v} \cdot \underset{\sim}{v}}.$$

$|\underset{\sim}{v}| = \sqrt{a^2+b^2+c^2}$

$\underset{\sim}{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

The notation $|\underset{\sim}{v}|$ should be distinguished from the absolute value for a scalar (for example $|-5| = 5$). The length of a vector is one example of a **norm**, which is a quantity used in higher level mathematics.

$|\underset{\sim}{v}| = \sqrt{2^2+1^2}$

Example 1.6. Let $\underset{\sim}{v} = (1, 2, 7)$. Compute the distance from $(0, 0, 0)$ to $(1, 2, 7)$. Compare this with $\sqrt{\underset{\sim}{v} \cdot \underset{\sim}{v}}$.

$|\underset{\sim}{v}| = \sqrt{1+4+49}$
$= \sqrt{54}$

$\sqrt{\underset{\sim}{v} \cdot \underset{\sim}{v}} = \sqrt{(1\times1)+(2\times2)+(7\times7)}$
$= \sqrt{54}$

We can now show that

$$\underset{\sim}{v} \cdot \underset{\sim}{w} = |\underset{\sim}{v}||\underset{\sim}{w}| \cos\theta$$

where

$$|\underset{\sim}{v}| = \text{the length of } \underset{\sim}{v} = \left(v_x^2 + v_y^2 + v_z^2\right)^{1/2}$$

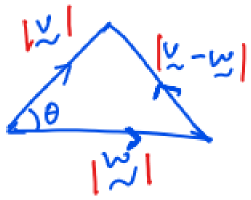
$$|\underset{\sim}{w}| = \text{the length of } \underset{\sim}{w} = \left(w_x^2 + w_y^2 + w_z^2\right)^{1/2}$$

and θ is the angle between the two vectors.

How do we prove this? Simply start with $\underset{\sim}{v} - \underset{\sim}{w}$ and compute its length,

$\underset{\sim}{v} \cdot \underset{\sim}{v}$

$$\begin{aligned}
\boxed{|\underset{\sim}{v} - \underset{\sim}{w}|^2} &= (\underset{\sim}{v} - \underset{\sim}{w}) \cdot (\underset{\sim}{v} - \underset{\sim}{w}) \\
&= \underset{\sim}{v} \cdot \underset{\sim}{v} - \underset{\sim}{v} \cdot \underset{\sim}{w} - \underset{\sim}{w} \cdot \underset{\sim}{v} + \underset{\sim}{w} \cdot \underset{\sim}{w} \\
&= |\underset{\sim}{v}|^2 + |\underset{\sim}{w}|^2 - 2\underset{\sim}{v} \cdot \underset{\sim}{w}
\end{aligned}$$



and from the Cosine Rule for triangles we know

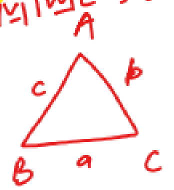
$$\boxed{|\underset{\sim}{v} - \underset{\sim}{w}|^2} = |\underset{\sim}{v}|^2 + |\underset{\sim}{w}|^2 - 2|\underset{\sim}{v}||\underset{\sim}{w}| \cos\theta$$

$\cancel{|\underset{\sim}{v}|^2} + \cancel{|\underset{\sim}{w}|^2} - \cancel{2}\,\underset{\sim}{v} \cdot \underset{\sim}{w} = \cancel{|\underset{\sim}{v}|^2} + \cancel{|\underset{\sim}{w}|^2} - \cancel{2}|\underset{\sim}{v}||\underset{\sim}{w}| \cos\theta$

Thus we have

$$\underset{\sim}{v} \cdot \underset{\sim}{w} = |\underset{\sim}{v}||\underset{\sim}{w}| \cos\theta$$

$\cos\theta = \dfrac{\underset{\sim}{v} \cdot \underset{\sim}{w}}{|\underset{\sim}{v}||\underset{\sim}{w}|}$



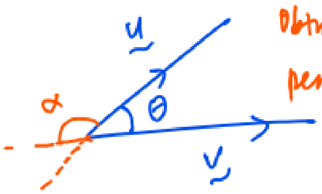
This gives us a convenient way to compute the angle between any pair of vectors. If we find $\cos\theta = 0$ then we can say that $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are **orthogonal** (perpendicular).

$a^2 = b^2 + c^2 - 2bc\cos A$

acute $(0<\theta<90^\circ)$ $\underset{\sim}{u} \cdot \underset{\sim}{v} \to +ve$

Obtuse $(90<\theta<180)$ $\underset{\sim}{u} \cdot \underset{\sim}{v} \to -ve$

perpendicular $(\theta=90^\circ)$ $\underset{\sim}{u} \cdot \underset{\sim}{v} = $ zero.



- Vectors $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are **orthogonal** when $\underset{\sim}{v} \cdot \underset{\sim}{w} = 0$ (provided neither $\underset{\sim}{v}$ nor $\underset{\sim}{w}$ are zero).

Example 1.7. Find the angle between the vectors $\underset{\sim}{v} = (2, 7, 1)$ and $\underset{\sim}{w} = (3, 4, -2)$

$$\cos\theta = \frac{\underset{\sim}{v} \cdot \underset{\sim}{w}}{|\underset{\sim}{v}||\underset{\sim}{w}|} = \frac{6+28-2}{\sqrt{54}\sqrt{29}}$$

$$\theta = \cos^{-1}\left(\frac{32}{\sqrt{54}\sqrt{29}}\right) = 36.04^\circ$$

1.2.2 Unit Vectors

A vector is said to be a **unit vector** if its length is one. That is, $\underset{\sim}{v}$ is a unit vector when $\underset{\sim}{v} \cdot \underset{\sim}{v} = 1$. The notation for a unit vector is $\hat{\underset{\sim}{v}}$ (called '$\underset{\sim}{v}$ hat').

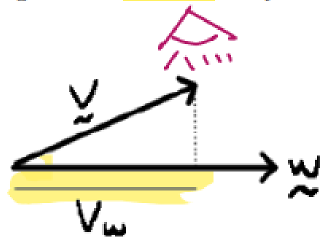
Unit vectors are calculated by: $\boxed{\hat{\underset{\sim}{v}} = \frac{\underset{\sim}{v}}{|\underset{\sim}{v}|}}$

$$\underset{\sim}{v} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} \qquad \hat{\underset{\sim}{v}} = \frac{1}{\sqrt{54}}\begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$$

1.2.3 Scalar projections

This is simply the length of the *shadow* cast by one vector onto another.

$$\hat{\underset{\sim}{v}} = \begin{pmatrix} 1/\sqrt{54} \\ 2/\sqrt{54} \\ 7/\sqrt{54} \end{pmatrix}$$



$\underset{\sim}{v}$ onto $\underset{\sim}{w}$

The **scalar projection**, v_w, of $\underset{\sim}{v}$ in the direction of $\underset{\sim}{w}$ is given by

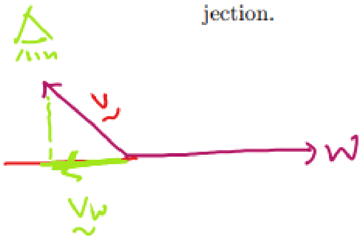
$$\boxed{v_w = \frac{\underset{\sim}{v} \cdot \underset{\sim}{w}}{|\underset{\sim}{w}|}}$$

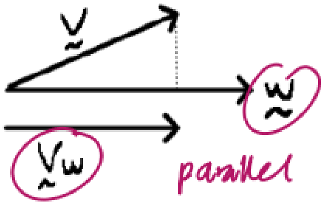
Example 1.8. What is the length (i.e. scalar projection) of $\underset{\sim}{v} = (1, 2, 7)$ in the direction of the vector $\underset{\sim}{w} = (2, 3, 4)$?

$$V_w = \frac{2+6+28}{\sqrt{29}} = \frac{36}{\sqrt{29}}$$

1.2.4 Vector projection

This time we produce a *vector shadow* with length equal to the scalar projection.





$\hat{w} = \frac{w}{|w|}$

$V_w = \frac{v \cdot w}{|w|}$

length of shadow

The **vector projection**, v_w, of v in the direction of w is given by

$V_w = \left(\frac{v \cdot w}{|w|}\right)\frac{w}{|w|}$

$$v_w = \left(\frac{v \cdot w}{|w|^2}\right) w$$

Example 1.9. Find the vector projection of $v = (1, 2, 7)$ in the direction of $w = (2, 3, 4)$

$V_w = \frac{36}{(\sqrt{29})^2}\begin{pmatrix}2\\3\\4\end{pmatrix}$

$V_w = \frac{36}{29}\begin{pmatrix}2\\3\\4\end{pmatrix}$

Example 1.10. This example shows how a vector may be **resolved** into its parts **parallel** and **perpendicular** to another vector.

Given $v = (1, 2, 7)$ and $w = (2, 3, 4)$ express v in terms of w and a vector perpendicular to w.

parallel to w

$\Rightarrow$ vector projection

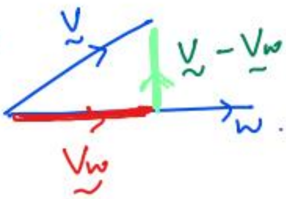
V_w

perpendicular to w

$V - V_w$

$= \begin{pmatrix}1\\2\\7\end{pmatrix} - \frac{36}{29}\begin{pmatrix}2\\3\\4\end{pmatrix}$

$= \begin{pmatrix}-\frac{43}{29}\\-\frac{50}{29}\\\frac{59}{29}\end{pmatrix}$

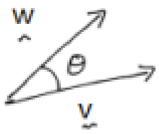


Vector Dot Product - Summary

Let $\underset{\sim}{v} = (v_x, v_y, v_z)$ and $\underset{\sim}{w} = (w_x, w_y, w_z)$. Then the **Dot Product** of $\underset{\sim}{v}$ and $\underset{\sim}{w}$ is the **scalar** defined by

$$\underset{\sim}{v} \cdot \underset{\sim}{w} = v_x w_x + v_y w_y + v_z v_z$$

Consider the angle θ between the two vectors such that $0 \leq \theta \leq \pi$:



$$\text{Then } \cos\theta = \frac{\underset{\sim}{v}.\underset{\sim}{w}}{|\underset{\sim}{v}||\underset{\sim}{w}|}$$

Two vectors are **orthogonal** if and only if

$$\underset{\sim}{v}.\underset{\sim}{w} = 0$$

The **scalar projection**, v_w, of $\underset{\sim}{v}$ in the direction of $\underset{\sim}{w}$ is given by

$$v_w = \frac{\underset{\sim}{v} \cdot \underset{\sim}{w}}{|\underset{\sim}{w}|}$$

The **vector projection**, $\underset{\sim}{v}_w$, of $\underset{\sim}{v}$ in the direction of $\underset{\sim}{w}$ is given by

$$\underset{\sim}{v}_w = \left(\frac{\underset{\sim}{v} \cdot \underset{\sim}{w}}{|\underset{\sim}{w}|^2}\right)\underset{\sim}{w}$$

1.3 Vector Cross Product

The vector cross product is another way to multiply vectors. We start with vectors $\underset{\sim}{v} = (v_x, v_y, v_z)$ and $\underset{\sim}{w} = (w_x, w_y, w_z)$. Then we define the **cross product** $\underset{\sim}{v} \times \underset{\sim}{w}$ by

$$\boxed{\underset{\sim}{v} \times \underset{\sim}{w} = (v_y w_z - v_z w_y, v_z w_x - v_x w_z, v_x w_y - v_y w_x)}$$

From this definition we can observe

$\underset{\sim}{v} \cdot \underset{\sim}{w} \Rightarrow$ scalar

$\underset{\sim}{v} \cdot \underset{\sim}{w} = \underset{\sim}{w} \cdot \underset{\sim}{v}$

- $\underset{\sim}{v} \times \underset{\sim}{w}$ is a **vector**
- $\underset{\sim}{v} \times \underset{\sim}{w} = -\underset{\sim}{w} \times \underset{\sim}{v}$
- $\underset{\sim}{v} \times \underset{\sim}{v} = \underset{\sim}{0} = (0, 0, 0)$ (the **zero** vector) parallel
- $(\lambda \underset{\sim}{v}) \times \underset{\sim}{w} = \lambda(\underset{\sim}{v} \times \underset{\sim}{w})$
- $(\underset{\sim}{a} + \underset{\sim}{b}) \times \underset{\sim}{v} = \underset{\sim}{a} \times \underset{\sim}{v} + \underset{\sim}{b} \times \underset{\sim}{v}$
- $(\underset{\sim}{v} \times \underset{\sim}{w}) \cdot \underset{\sim}{v} = (\underset{\sim}{v} \times \underset{\sim}{w}) \cdot \underset{\sim}{w} = 0$

$\underset{\sim}{v} \times \underset{\sim}{w} = \underset{\sim}{0}$

Example 1.11. Verify all of the above.

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Example 1.12. Given $\underset{\sim}{v} = (1, 2, 7)$ and $\underset{\sim}{w} = (-2, 3, 5)$ compute $\underset{\sim}{v} \times \underset{\sim}{w}$, and its dot product with each of $\underset{\sim}{v}$ and $\underset{\sim}{w}$.

	i	j	k
$\underset{\sim}{v}$	1	2	
$\underset{\sim}{w}$	-2	3	

direction $\searrow - \nearrow$

$$\underset{\sim}{v} \times \underset{\sim}{w} = \begin{pmatrix} (2\times5)-(3\times7) \\ -[(1\times5)-(-2\times7)] \\ (1\times3)-(-2\times2) \end{pmatrix} = \begin{pmatrix} -11 \\ -19 \\ 7 \end{pmatrix}$$

$\underset{\sim}{v} \cdot \underset{\sim}{v} = 54$

$\underset{\sim}{w} \cdot \underset{\sim}{w} = 38$

$\underset{\sim}{v} \cdot \underset{\sim}{w} = 39$

$\underset{\sim}{w} \cdot \underset{\sim}{v} = 39$

1.3.1 Interpreting the cross product

We know that $\underset{\sim}{v} \times \underset{\sim}{w}$ is a vector and we know how to compute it. But can we **describe** this vector? First we need a vector, so let's assume that $\underset{\sim}{v} \times \underset{\sim}{w} \neq \underset{\sim}{0}$. Then what can we say about the direction and length of $\underset{\sim}{v} \times \underset{\sim}{w}$?

The first thing we should note is that the cross product is a vector which is orthogonal to both of the original vectors. Thus $\underset{\sim}{v} \times \underset{\sim}{w}$ is a vector that is orthogonal to $\underset{\sim}{v}$ and to $\underset{\sim}{w}$. This fact follows from the definition of the cross product.

Thus we must have

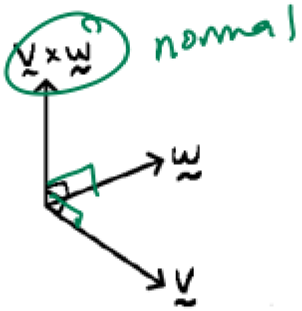
$$\underset{\sim}{v} \times \underset{\sim}{w} = \lambda \underset{\sim}{n}$$

where $\underset{\sim}{n}$ is a unit vector orthogonal to both $\underset{\sim}{v}$ and $\underset{\sim}{w}$ and λ is some unknown number (at this stage).

How do we construct $\underset{\sim}{n}$ and λ? Let's do it!

1.3.2 Right hand thumb rule

For any choice of $\underset{\sim}{v}$ and $\underset{\sim}{w}$ you can see that there are two choices for $\underset{\sim}{n}$ – one points in the opposite direction to the other. Which one do we choose? It's up to us to make a hard rule. This is it. Place your right hand palm so that your fingers curl over from $\underset{\sim}{v}$ to $\underset{\sim}{w}$. Your thumb then points in the direction of $\underset{\sim}{v} \times \underset{\sim}{w}$.



Now for λ, we will show that

$$|\underset{\sim}{v} \times \underset{\sim}{w}| = \lambda = |\underset{\sim}{v}||\underset{\sim}{w}| \sin\theta$$

How? First we build a triangle from $\underset{\sim}{v}$ and $\underset{\sim}{w}$ and then compute the cross product for each pair of vectors

$$\begin{aligned} \underset{\sim}{v} \times \underset{\sim}{w} &= \lambda_\theta \underset{\sim}{n} \\ (\underset{\sim}{v} - \underset{\sim}{w}) \times \underset{\sim}{v} &= \lambda_\phi \underset{\sim}{n} \\ (\underset{\sim}{v} - \underset{\sim}{w}) \times \underset{\sim}{w} &= \lambda_\rho \underset{\sim}{n} \end{aligned}$$

(one λ for each of the three vertices). We need to compute each λ.

Now since $(\beta \underset{\sim}{v}) \times \underset{\sim}{w} = \beta(\underset{\sim}{v} \times \underset{\sim}{w})$ for any number β we must have λ_θ in $\underset{\sim}{v} \times \underset{\sim}{w} = \lambda_\theta \underset{\sim}{n}$ proportional to $|\underset{\sim}{v}||\underset{\sim}{w}|$, likewise for the other λ's. Thus

$$\lambda_\theta = |\underset{\sim}{v}||\underset{\sim}{w}|\alpha_\theta$$

$$\lambda_\phi = |\underset{\sim}{v}||\underset{\sim}{v} - \underset{\sim}{w}|\alpha_\phi$$

$$\lambda_\rho = |\underset{\sim}{w}||\underset{\sim}{v} - \underset{\sim}{w}|\alpha_\rho$$

where each α depends only on the angle between the two vectors on which it was built (i.e. α_ϕ depends only on the angle ϕ between $\underset{\sim}{v}$ and $\underset{\sim}{v} - \underset{\sim}{w}$).

But we also have $\underset{\sim}{v} \times \underset{\sim}{w} = (\underset{\sim}{v} - \underset{\sim}{w}) \times \underset{\sim}{v} = (\underset{\sim}{v} - \underset{\sim}{w}) \times \underset{\sim}{w}$ which implies that $\lambda_\theta = \lambda_\phi = \lambda_\rho$ which in turn gives us

$$\frac{\alpha_\theta}{|\underset{\sim}{v} - \underset{\sim}{w}|} = \frac{\alpha_\phi}{|\underset{\sim}{w}|} = \frac{\alpha_\rho}{|\underset{\sim}{v}|}$$

But we also have the Sine Rule for triangles

$$\frac{\sin\theta}{|\underset{\sim}{v} - \underset{\sim}{w}|} = \frac{\sin\phi}{|\underset{\sim}{w}|} = \frac{\sin\rho}{|\underset{\sim}{v}|}$$

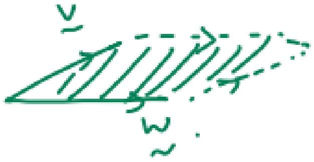
and so

$$\alpha_\theta = k\sin\theta, \qquad \alpha_\phi = k\sin\phi, \qquad \alpha_\rho = k\sin\rho$$

where k is a number that does not depend on any of the angles nor on any of lengths of the edges – the value of k is the same for every triangle. We can choose a trivial case to compute k, simply put $\underset{\sim}{v} = (1, 0, 0)$ and $\underset{\sim}{w} = (0, 1, 0)$. Then we find $k = 1$.

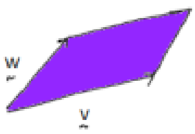
We have now found that

$$|\underset{\sim}{v} \times \underset{\sim}{w}| = |\underset{\sim}{v}||\underset{\sim}{w}|\sin\theta$$



Example 1.13. Show that $|\underset{\sim}{v} \times \underset{\sim}{w}|$ also equals the area of the parallelogram formed by $\underset{\sim}{v}$ and $\underset{\sim}{w}$.

tutorial

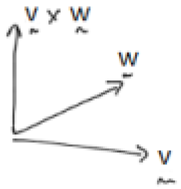


Vector Cross Product - Summary

Let $\underset{\sim}{v} = (v_x, v_y, v_z)$ and $\underset{\sim}{w} = (w_x, w_y, w_z)$. Then the **Cross Product** is defined by

$$\underset{\sim}{v} \times \underset{\sim}{w} = (v_y w_z - v_z w_y, v_z w_x - v_x w_z, v_x w_y - v_y w_x)$$

$\underset{\sim}{v} \times \underset{\sim}{w} = -\underset{\sim}{w} \times \underset{\sim}{v}$ gives a vector **orthogonal** to both $\underset{\sim}{v}$ and $\underset{\sim}{w}$, and defined by the right-hand rule:



If θ is the angle between $\underset{\sim}{v}$ and $\underset{\sim}{w}$, such that $0 \le \theta \le \pi$:

$$\sin\theta = \frac{|\underset{\sim}{v} \times \underset{\sim}{w}|}{|\underset{\sim}{v}||\underset{\sim}{w}|}.$$

Two vectors are parallel if and only if

$$\underset{\sim}{v} \times \underset{\sim}{w} = \underset{\sim}{0}.$$

The area of the parallelogram spanned by vectors $\underset{\sim}{v}$ and $\underset{\sim}{w}$ is:

$$A = |\underset{\sim}{v} \times \underset{\sim}{w}|.$$