MAT1830 - Discrete Mathematics for Computer Science
Tutorial Sheet #4 and Additional Practice Questions

Tutorial Questions

1. (a) Let $a_1, a_2, a_3, \ldots$ be a sequence of integers defined by $a_1 = 1$, $a_2 = 3$, and $a_t = a_{t-1} + a_{t-2}$ for each integer $t \geq 3$. (So the sequence goes $1, 3, 4, 7, 11, 18, \ldots$) How would you (informally) convince your friend Dwayne that every term in this sequence will be positive? How would you phrase this argument more formally?

   (b) Prove by induction that 3 divides $n^3 - 7n + 6$ for all integers $n \geq 0$.

2. (a) Are the following true or false?

   i. $\mathbb{N} \subseteq \mathbb{Q}$

   ii. $2 \subseteq \mathbb{N}$

   iii. $\{6, 7\} \subseteq \mathbb{N}$

   iv. $\{3\} \in \mathbb{Q}$

   v. $\mathbb{N} \subseteq \{x : x \in \mathbb{R}, x \geq 0\}$

   vi. $\{\} \subseteq \mathbb{N}$

   vii. $(a, d) \in \{a, b, c\} \times \{d, e\}$

   viii. $\mathbb{N} \times \mathbb{N} \subseteq \mathbb{Z} \times \mathbb{Z}$

   (b) Let $S = \{-1, 0, 1\}$. What is $\mathcal{P}(S)$?

   (c) Let $T = \{1, 2, \ldots, 10\}$. How many elements would $\mathcal{P}(T)$ have? Which of the following would be elements of $\mathcal{P}(T)$?

   i. $\emptyset$

   ii. $\{1, 4, 6\}$

   iii. 3

   iv. $\{2, 4, 12\}$

   v. $\{1, 2, \ldots, 10\}$

3. Let $A = \{1, 2\}$ and $B = \{-1, 0, 1\}$.

   (a) What is $A \cup B$?

   (b) What is $A \cap B$?

   (c) What is $A \times B$?

   (d) Is it true that, for all sets X, Y and Z, $(X \cup Y) \cap Z = X \cup (Y \cap Z)$? Why or why not?

   (e) Is it true that, for all sets Y and Z, $\mathcal{P}(Y) \cap \mathcal{P}(Z) = \mathcal{P}(Y \cap Z)$? Why or why not?

4. (a) Scrooge McDuck has created his own currency, duckbucks, which has only \$4 and \$7 notes. Using induction prove that, for any $n \geq 18$, n duckbucks can be made from \$4 and \$7 notes.

   (b) You're leaving Dwayne in charge of a store with a stack of \$4 and \$7 duckbuck notes. Assume all purchases will result in \$18 or more change. What (informal) instructions would you give to Dwayne so he can always make change for the customers?

(See over for practice questions.)

Practice Questions

1. Let U be some universal set and let P, Q and R be subsets of U. Let

$$S_1 = P \cup (\overline{Q \cap R})$$
$$S_2 = (P \cup (P \cap Q)) \cap \overline{R}.$$

   (a) Draw a Venn diagram to determine whether it is always the case that $S_1 = S_2$.

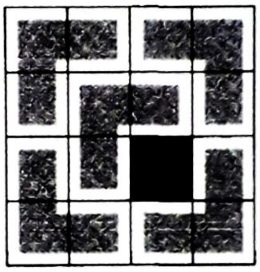
   (b) If this is not always the case, then does it ever happen? If so, when?

   (c) Let p, q and r be the propositions "$x \in P$", "$x \in Q$" and "$x \in R$". Find formulas in logic which mean "$x \in S_1$" and "$x \in S_2$". Find truth tables for these.

   (d) Do the truth tables contain the same information as you Venn diagrams? More? Less?

2. An L-tromino is like a domino, but made of three squares in the shape of an "L". Prove by induction that, for any n, if any one square is removed from a $2^n \times 2^n$ chess board, then the remaining squares can be completely covered by L-trominos.

   (The following picture shows an example on a 4×4 board.)



Hint: A $2^{k+1} \times 2^{k+1}$ board can be split into four $2^k \times 2^k$ boards.

3. (a) How would you formally define intersections and unions of infinitely many sets?

   (b) Can you find a collection of infinitely many sets of integers $A_1, A_2, A_3, \ldots$ such that each set contains infinitely many integers, $A_{i+1} \subseteq A_i$ for $i = 1, 2, 3, \ldots$, and $A_1 \cap A_2 \cap A_3 \cap \cdots = \emptyset$?

4. As in 1(a) on the previous page, define a sequence of integers $a_1, a_2, a_3, \ldots$ by $a_1 = 1$, $a_2 = 3$, and $a_{t+1} = a_t + a_{t-1}$ for each integer $t \geq 3$. Prove by strong induction that $a_n \leq (\frac{7}{4})^n$ for each integer $n \geq 1$.

Practice Questions (Tutorial sheet 4)

1) Universal set = U; subsets of U : P, Q, R

(a) not equal, venn diagrams are different

(b) idk

(c) $x \in S_1$ is TRUE except $x \notin P$, $x \in Q$, $x \in R$

while $x \in S_2$ is TRUE except $x \notin P$, $x \in R$

(d) same information; shaded = true, unshaded = false

2) when $n = 1$, $(2^1 \times 2^1) - 1 = 15$

$15/3 = 5$ so $P(n)$ = true where $n = 1$

For some integer $k \geq 1$, $P(k)$ can fit to $(2^k \times 2^k)$ board.

when $P(k+1) = (2^{k+1} \times 2^{k+1}) - 1 \rightsquigarrow ((2^k)(2^1) \times (2^k)(2^1)) - 1$

$= 4(2^k \times 2^k) - 1 \leftsquigarrow\rightsquigarrow = (2 \times 2) \times (2^k \times 2^k) - 1$

$P(2+1) = P(3) = 4(2^2 \times 2^2) - 1$

$= 4(16) - 1$

$= 64 - 1$ $\quad 63/3 = 21$

$= 63$ $\quad \therefore P(k+1)$ is TRUE

3)(a) define intersections ($\cap$) and unions ($\cup$) of infinitely many sets

intersection: $x \in \bigcap_{i \in I} A_i$ exactly when $\forall i (i \in I \rightarrow x \in A_i)$

union: $x \in \bigcup_{i \in I} A_i$ exactly when $\exists i (i \in I \wedge x \in A_i)$

(b) $A_i = \{i, i+1, i+2, i+3 \ldots i+n\}$ where n = last element in set

4) $a_{t+1} = a_t + a_{t-1}$ for each integer $t \geq 3$; $a_n \leq (\frac{7}{4})^n$ for $n \geq 1$ (integer)

$n = 1$, $a_1 = 1$; $(\frac{7}{4})^1 = \frac{7}{4} \longrightarrow P(1) \equiv T$

$n = 2$, $a_2 = 3$; $(\frac{7}{4})^2 = \frac{49}{16} \longrightarrow P(2) \equiv T$

assume $P(n) \equiv T$ for $n \leq k$

$n = 3$, $a_3 = a_2 + a_1 = \frac{7}{4} + \frac{49}{16} = \frac{77}{16}$

$\vdots$

$a_{k+1} \leq (\frac{7}{4})^k + (\frac{7}{4})^{k-1}$

$a_{k+1} \leq \frac{4}{7}((\frac{7}{4})^{k+1}) + \frac{16}{49}((\frac{7}{4})^{k+1})$

$a_{k+1} \leq (\frac{7}{4})^{k+1} (\frac{4}{7} + \frac{16}{49})$

$a_{k+1} \leq (\frac{7}{4})^{k+1} (\frac{44}{49}) \longrightarrow \frac{44}{49} \leq 1$

HENCE, $a_{k+1} \leq (\frac{7}{4})^{k+1}$

so $P(k+1) \equiv T$

$\cancel{a_{n+1} \leq (\frac{7}{4})^{n+1}}$

$a_{k+1} \leq (\frac{7}{4})^{k+1}$ for $k \geq 1$

$a_{k+2} = a_{k+1} + a_k$

$\hookrightarrow a_k = -a_{k+1} + a_{k+2}$

conclusion:

$P(1) \cap P(2) \rightarrow P(3)$

$P(1) \cap P(2) \cap P(3) \rightarrow P(4)$

therefore

$P(n)$ is true for $n \geq 1$