Date:

$\boxed{\text{5.2 Partial Derivatives}}$ ⇒ only do part of it

first derivative principle:

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \left(\frac{f(x+\Delta x) - f(x)}{\Delta x} \right)$$

— small change in x value, Δx

$x \to x + \Delta x$

for $f(x,y)$ ⇒ one variable at a time

*partial derivative:

$$\boxed{\frac{\partial f}{\partial x}} = \lim_{\Delta x \to 0} \left(\frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \right)$$

→ y is treated as a constant

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \left(\frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} \right)$$

→ x is treated as a constant

$$\boxed{f_x = \frac{\partial f}{\partial x} \;;\; f_y = \frac{\partial f}{\partial y}}$$

*product rule for partial derivative:

$$\frac{\partial (fg)}{\partial x} = g\frac{\partial f}{\partial x} + f\frac{\partial g}{\partial x}$$

$\frac{\partial f}{\partial x}$ ⇒ treat y as constant & $\frac{d}{dx} f(x,y)$

$$\frac{\partial (fg)}{\partial y} = g\frac{\partial f}{\partial y} + f\frac{\partial g}{\partial y}$$

$\frac{\partial f}{\partial y}$ ⇒ treat x as constant & $\frac{d}{dy} f(x,y)$

tangent line $\Rightarrow y - y_1 = m_t(x - x_1)$ for $y = f(x)$

$$\boxed{\text{tangent plane} \Rightarrow z - c = f_x(a,b)(x-a) + f_y(a,b)(y-b) \quad \text{for } z = f(x,y), \text{ at } (a,b,c)}$$

$z - c = f_x(a,b)(x-a) + f_y(a,b)(y-b)$

vector form :

can swap position

$$\underset{\sim}{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} p \\ q \\ r \end{pmatrix} t + \begin{pmatrix} l \\ m \\ n \end{pmatrix} s$$

gradient / tangent vector v/f x-axis, f_x

parameter

gradient vector v/f y-axis, f_y

cartesian form:

$$z = f(a,b) + f_x(a,b) \cdot (x-a) + f_y(a,b) \cdot (y-b)$$

Taylor polynomial Recall

$x=a$

$$T_n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f^2(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n-1)}(a)}{n!}(x-a)^n$$

$$T_1(x) = f(a) + \frac{f'(a)}{1!}(x-a) \Leftarrow \text{one variable}$$

1st derivative

$x=a,\ y=b$

partial diff

$$*\ T_n(x,y) = f(a,b) + \boxed{\frac{f_x(a,b)}{1!}(x-a) + \frac{f_y(a,b)}{1!}(y-b)} +$$

second order of partial diff

$$\boxed{\frac{f_{xx}(a,b)}{2!}(x-a)^2 + \frac{f_{xy}(a,b)}{2!}(x-a)(y-b)} +$$

$$\boxed{\frac{f_{yx}(a,b)}{2!}(x-a)(y-b) + \frac{f_{yy}(a,b)}{2!}(y-b)^2} + \dots$$

linear approx:

$$T_1(x,y) = f(a,b) + \frac{f_x(a,b)}{1!}(x-a) + \frac{f_y(a,b)}{1!}(y-b)$$

$$T_1(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$\Rightarrow$ same as tangent plane.

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

linear approximation, L $\Rightarrow$ $L(x,y) = T_1(x,y)$ = tangent plane eqn

eg 5-11 Derive linear approx $T_1(x,y)$ for $f(x,y) = \sqrt{3x-y}$ at point $(4,3)$

$$f(x,y) = (3x-y)^{1/2}$$

$$f_x = \tfrac{1}{2}(3x-y)^{-\frac{1}{2}}(3) = \frac{3}{2\sqrt{3x-y}}$$

$$f_y = \tfrac{1}{2}(3x-y)^{-\frac{1}{2}}(-1) = \frac{-1}{2\sqrt{3x-y}}$$

no need to add $\pm$

$$f(4,3) = \sqrt{12-3} = 3$$

point $= (4,3,3)$

$$f_x(4,3) = \frac{3}{2(3)} = \frac{1}{2}$$

$$f_y(4,3) = \frac{-1}{2(3)} = -\frac{1}{6}$$

$$L(x,y) = T_1(x,y) = 3 + \tfrac{1}{2}(x-4) - \tfrac{1}{6}(y-3)$$

tangent plane $\Rightarrow$ $z - 3 = \tfrac{1}{2}(x-4) - \tfrac{1}{6}(y-3)$

Eg 5-14 show that for curve $x(s) = s$, $y(s) = 2$, we get $\frac{df}{ds} = \frac{\partial f}{\partial s}$

$f(x, y)$ $\qquad$ $x(s) = s$ $\qquad$ $y(s) = 2$

$\swarrow \; \searrow$ $\qquad$ $\frac{dx}{ds} = 1$ $\qquad$ $\frac{dy}{ds} = 0$

$\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y}$

chain rule:

$$\frac{df}{ds} = \left(\frac{\partial f}{\partial x} \times \frac{dx}{ds}\right) + \left(\frac{\partial f}{\partial y} \times \frac{dy}{ds}\right)$$

$$= \frac{\partial f}{\partial x}(1) + 0$$

$$\frac{df}{ds} = \frac{\partial f}{\partial x}$$

#

eg 5-15 show that for curve $x(s) = -1$, $y(s) = s$, we get $\frac{df}{ds} = \frac{\partial f}{\partial y}$

$f(x, y)$ $\qquad$ $x(s) = -1$ $\qquad$ $y(s) = s$

$\swarrow \; \searrow$ $\qquad$ $\frac{dx}{ds} = 0$ $\qquad$ $\frac{dy}{ds} = 1$

$\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y}$

$$\frac{df}{ds} = \left(\frac{\partial f}{\partial x} \times \frac{dx}{ds}\right) + \left(\frac{\partial f}{\partial y} \times \frac{dy}{ds}\right)$$

$$= \frac{\partial f}{\partial x}(0) + \frac{\partial f}{\partial y}(1)$$

$$\frac{df}{ds} = \frac{\partial f}{\partial y}$$ #

*Formula:

$$\boxed{\frac{df}{ds} = \frac{\partial f}{\partial x} \times \frac{dx}{ds} + \frac{\partial f}{\partial y} \times \frac{dy}{ds}}$$

given that $f = f(x, y)$ whr the func is parametrized by $x = x(s)$ & $y = y(s)$

Date:

5.4 Chain Rule

$z = f(x,y)$

parametric eqn

$x(t)$ $\quad$ $y(t)$

$\underbrace{x(t) \quad y(t)}_{z(t)}$

$f = f(x(t), y(t)) = f(t)$

differentiation $\Rightarrow \frac{df}{dt}$

$f(x(t), y(t))$

eg 5-13 curve w/ $x(s) = 2s$

$y(s) = 4s^2$ whr $-1 < s < 1$

$f(x,y) = 5x - 7y + 2$

$\rightarrow$ compute $\frac{df}{ds}$ at $s=0$

① Direct method $f(x,y) \rightarrow f(s) \rightarrow \frac{df}{ds}$

$f(x,y) \rightarrow f(s) = 5(2s) - 7(4s^2) + 2$

$f(s) = 10s - 28s^2 + 2$

$\frac{df}{ds} = 10 - 56s$ #

★ The Chain Rule:

$f(x,y)$

$x(s)$ $\quad$ $y(s)$

$f(x,y) \rightarrow \frac{\boxed{\partial} f}{\partial x}, \frac{\partial f}{\partial y}$

$x(s) \rightarrow \frac{\boxed{d} x}{ds}$

$y(s) \rightarrow \frac{dy}{ds}$

$\frac{df}{ds} = \left(\frac{\partial f}{\partial x} \times \frac{dx}{ds}\right) + \left(\frac{\partial f}{\partial y} \times \frac{dy}{dx}\right)$

$\frac{d}{dx} x(s)?$ $\quad$ $\frac{d}{dy} y(s)?$

② Chain Rule method

$\frac{dx}{ds} = 2$ $\quad$ $\frac{dy}{ds} = 8s$

$\frac{\partial f}{\partial x} = 5$ $\quad$ $\frac{\partial f}{\partial y} = -7$

$\frac{df}{ds} = (5)(2) + (-7)(8s)$

$= 10 - 56s$ #

eg 5.15 show tht for curve $x(s) = -1$, $y(s) = s$, we get $\frac{df}{ds} = \frac{\partial f}{\partial y}$

$f(x,y)$ $\quad\quad$ $x(s) = -1$ $\quad\quad$ $y(s) = s$

$\swarrow \quad \searrow$ $\quad\quad$ $\frac{dx}{ds} = 0$ $\quad\quad$ $\frac{dy}{ds} = 1$

$\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y}$

$$\frac{df}{ds} = \left(\frac{\partial f}{\partial x} \times \frac{dx}{ds}\right) + \left(\frac{\partial f}{\partial y} \times \frac{dy}{ds}\right)$$

$$= \frac{\partial f}{\partial x}(0) + \frac{\partial f}{\partial y}(1)$$

$$\frac{df}{ds} = \frac{\partial f}{\partial y} \#$$

* Formula:

$$\frac{df}{ds} = \frac{\partial f}{\partial x} \times \frac{dx}{ds} + \frac{\partial f}{\partial y} \times \frac{dy}{ds}$$

given that $f = f(x,y)$ whr the func is parameterized by $x = x(s)$ & $y = y(s)$

$f(x,y)$

$\swarrow \quad \searrow$

$x(s,t) \quad y(s,t)$

$\underbrace{\qquad\qquad}_{f(s,t)}$

→ cannot get $\frac{df}{ds}$

→ will get $\frac{\partial f}{\partial s}$, $\frac{\partial f}{\partial t}$

* **Formula:**

Given that $f = f(x,y)$ whr

$x = x(s,t)$, $y = y(s,t)$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial t}$$

eg 5-16 $\quad f = f(x,y)$ and $x = 2s + 3t$, $y = s - 2t$, compute $\frac{\partial f}{\partial t}$

$f(x,y) \rightarrow f(s,t)$

$f(x,y)$

$\swarrow \quad \searrow$

$\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y}$

$\frac{\partial x}{\partial s} = 2 \qquad \frac{\partial x}{\partial t} = 3$

$\frac{\partial y}{\partial s} = 1 \qquad \frac{\partial y}{\partial t} = -2$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial t}$$

$$= 3\frac{\partial f}{\partial x} - 2\frac{\partial f}{\partial y} \ \#$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial s}$$

$$= 2\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \ \#$$

If s is distance, the ratio ≈ 1

$|\underset{\sim}{t}| = 1$

Directional Derivative:

define if $|\underset{\sim}{t}| = 1$ and $f(x,y)$

$$\underset{\sim}{t} \cdot \nabla f = \nabla_{\underset{\sim}{t}} f$$

is the derivative of f in direction $\underset{\sim}{t}$

Directional derivative ($\underset{\sim}{t}$) can be found if you tell me which vector direction you want to respect to.

→ diff vector direction produce diff directional derivative.

Note: if $\underset{\sim}{t}$ is the tangent vector to the curve $(x(s), y(s))$,

then $\boxed{\nabla_{\underset{\sim}{t}} f} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot \left(\frac{dx}{ds}, \frac{dy}{dx}\right)$

$$\underbrace{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dx}{ds}, \frac{dy}{dx}\right)}_{\underset{\sim}{t}}$$

directional derivative of the curve func in the vector $\underset{\sim}{t}$

$$\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \frac{dx}{ds} \\ \frac{dy}{ds} \end{pmatrix} = \left(\frac{\partial f}{\partial x} \cdot \frac{dx}{ds}\right) + \left(\frac{\partial f}{\partial y} \cdot \frac{dy}{ds}\right)$$

$\boxed{\text{chain rule}}$

Recap

1st partial differentiation

$f(x,y) \longrightarrow \frac{\partial f}{\partial x}$ / f_x

$\longrightarrow \frac{\partial f}{\partial y}$ / f_y

2nd partial derivative

$\frac{\partial f}{\partial x} \rightarrow \frac{\partial^2 f}{\partial x^2}$ / f_{xx}

$\rightarrow \frac{\partial^2 f}{\partial x \partial y}$ / f_{xy}

$\frac{\partial f}{\partial y} \rightarrow \frac{\partial^2 f}{\partial y^2}$ / f_{yy}

$\rightarrow \frac{\partial^2 f}{\partial y \partial x}$ / f_{yx}

same result

Second Derivative Test

$$D = \frac{\partial^2 f}{\partial x^2} \times \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

if $D > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0 \Rightarrow$ local minima

$D > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0 \Rightarrow$ local maxima

$D < 0 \Rightarrow$ saddle point

$D = 0$ inconclusive

eg 5.29 find stationary point & determine nature.

$f(x, y) = x^2 + y^2 - 2x - 6y + 14$

$f_x = 2x - 2$ $\quad$ $f_y = 2y - 6$

$f_{xx} = 2$ $\quad$ $f_{yy} = 2$ $\quad$ $f_{xy} = 0$

critical point

$f_x = 0$ $\quad$ $f_y = 0$

$2x - 2 = 0$ $\quad$ $2y - 6 = 0$

$x = 1$ $\quad$ $y = 3$

RMB to fill z-value!

$f(1,3) = 1 + 9 - 2 - 18 + 14 = 4$

critical point $\Rightarrow (1, 3, 4)$

nature

$D = f_{xx} f_{yy} - (f_{xy})^2$

$= (2)(2) - 0^2 = 4 > 0$

$f_{xx} = 2 > 0$

$\therefore$ since $D > 0$ and $f_{xx} > 0$,

local minimum at $(1, 3, 4)$ #