Problem Set Six: Trigonometric, Exponential and Logarithmic Function Derivatives

$\frac{d}{dx}\sin x = \cos x$ $\frac{d}{dx}\tan x = \sec^2 x$

$\frac{d}{dx}\cos x = -\sin x$

Trigonometric Functions

1. Differentiate the following trigonometric functions:

(a) $f(x) = \sin(3x-2)$

(b) $g(x) = \cos^2(3x)$

(c) $h(x) = x\sin(x)$ product Rule

(d) $f(z) = \tan^3(z)$

(a) $f'(x) = 3\boxed{\cos(3x-2)}$

WRONG sol'n — NOTE $\frac{d}{dx}\sin(3x-2)$ $= \cos(3x-2)(3)$ $= \cos(9x-6)$ ✗

(b) $g(x) = [\cos(3x)]^2$ chain Rule

$g'(x) = 2[\cos(3x)]'(-3\sin(3x)) = -6\sin(3x)\cos(3x)$ ✓

$= -3(2\sin(3x)\cos(3x))$

$= -3(\sin 6x)$

product rule (c) $h'(x) = (1)\sin(x) + x\cos(x) = \sin(x) + x\cos(x)$

(d) $f(z) = [\tan(z)]^3$

$f'(z) = 3[\tan(z)]^2 \sec^2(z) = 3\tan^2(z)\sec^2(z)$

Exponential Functions

$\frac{d}{dx}e^x = e^x$

2. Find the first derivative with respect to x of the following exponential functions:

(a) $f(x) = e^{2x}$

(b) $f(x) = e^{x^2+x}$

(c) $f(x) = (3x-2)e^{-x}$ product Rule.

(d) $f(x) = \frac{e^x}{1+e^x}$

$f(x) = e^x(1+e^x)^{-1}$ product Rule

(a) $f'(x) = 2e^{2x}$

(b) $f'(x) = (2x+1)e^{x^2+x}$

(c) $f'(x) = 3e^{-x} + (3x-2)(-e^{-x})$

$= 5e^{-x} - 3xe^{-x}$

$= e^{-x}(5-3x)$

(d) $f(x) = \frac{e^x}{1+e^x}$ "top-bottom"

$f'(x) = \frac{(e^x)(1+e^x) - (e^x)(e^x)}{(1+e^x)^2}$

$= \frac{e^x + \cancel{e^{2x}} - \cancel{e^{2x}}}{(1+e^x)^2}$

$= \frac{e^x}{(1+e^x)^2}$

Logarithmic Functions

Law Rule

$\frac{d}{dx}\ln x = \frac{1}{x}$

$\log_e x = \ln x$ $[\log \neq \log_{10}]$ $\log \rightarrow \log_e$

3. Differentiate the following logarithmic functions with respect to x:

$\ln(\frac{a}{b})$

$= \ln(a) - \ln(b)$

$\ln(ab)$

$= \ln a + \ln b$

(a) $f(x) = \ln(3x-2)$

(b) $g(x) = \ln\left(\frac{x-3}{x-2}\right)$

(c) $h(x) = \frac{1}{x}\ln x$

(a) $f'(x) = \frac{1}{3x-2}(3) = \frac{3}{3x-2}$

(b) $g(x) = \ln\left(\frac{x-3}{x-2}\right) = \ln(x-3) - \ln(x-2)$ ✓

$g'(x) = \frac{1}{x-3} - \frac{1}{x-2} = \frac{\cancel{x}-2-\cancel{x}+3}{(x-3)(x-2)} = \frac{1}{(x-3)(x-2)}$

Alternative Method

(b) $g'(x) = \frac{1}{\left(\frac{x-3}{x-2}\right)}\left[\frac{(1)(x-2)-(1)(x-3)}{(x-2)^2}\right]$ ✓

$= \left(\frac{\cancel{x-2}}{x-3}\right)\left(\frac{1}{(x-2)^{\cancel{2}}}\right) = \frac{1}{(x-3)(x-2)}$

(c) $h(x) = \frac{1}{x}\ln x = x^{-1}\ln x$ product Rule

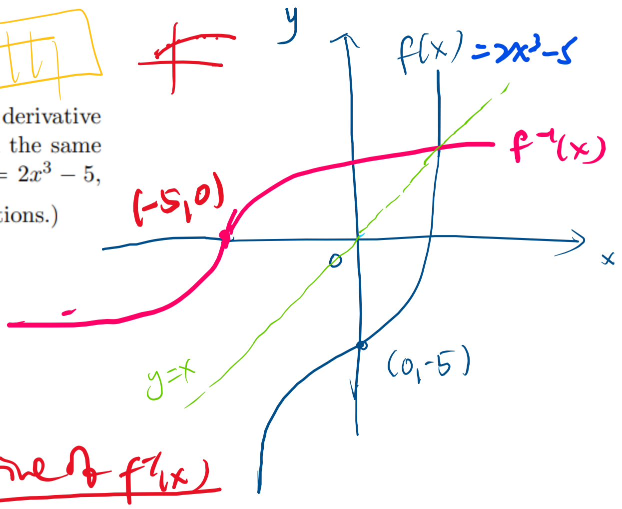
$= \frac{\ln x}{x}$

$h'(x) = -x^{-2}\ln x + x^{-1}\left(\frac{1}{x}\right)$ $\quad x^{-1} = \frac{1}{x}$

$h'(x) = \frac{-\ln x}{x^2} + \frac{1}{x^2} = \frac{1-\ln x}{x^2}$

Inverse Functions

4. Make a quick sketch of the function $f(x) = 2x^3 - 5$. Find the derivative of $f(x)$. Also find $f^{-1}(x)$ (the inverse of $f(x)$). Plot $f^{-1}(x)$ on the same axis as $f(x)$. Find the derivative of $f^{-1}(x)$. (Show that for $y = 2x^3 - 5$, $\frac{dy}{dx} = \frac{1}{dx/dy}$ which you can see from your sketch of the two functions.)



$y = f(x) = 2x^3 - 5$

✓ $f'(x) = 6x^2$ ✓

$\frac{dy}{dx} = 6x^2$

Inverse

$y = 2x^3 - 5$

$2x^3 = y + 5$

$x^3 = \frac{1}{2}(y+5)$

make x subject ⇒ $x = \sqrt[3]{\frac{1}{2}(y+5)}$

$f^{-1}(x) = \sqrt[3]{\frac{1}{2}}\,(x+5)^{\frac{1}{3}}$

derivative of $f^{-1}(x)$

$x = \sqrt[3]{\frac{1}{2}}\,(y+5)^{\frac{1}{3}}$

$\frac{dx}{dy} = \sqrt[3]{\frac{1}{2}}\left[\left(\frac{1}{3}\right)(y+5)^{-2/3}(1)\right]$

$\frac{dx}{dy} = \left(\frac{1}{3}\right)\left(\frac{1}{2^{1/3}}\right)\left[\frac{1}{(y+5)^{2/3}}\right]$ ← $\frac{d}{dx}f^{-1}(x)$

$\frac{dx}{dy} = \left(\frac{1}{3}\right)\left(\frac{1}{2^{1/3}}\right)\left[\frac{1}{(2x^3 \cancel{-5} \cancel{+5})^{2/3}}\right]$ $(ab)^m = a^m b^m$ → $(2x^3)^{2/3}$

$\frac{dx}{dy} = \left(\frac{1}{3}\right)\left(\frac{1}{2^{1/3}}\right)\left(\frac{1}{2^{2/3}\cdot x^2}\right)$

$\frac{dx}{dy} = \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{x^2}\right) = \frac{1}{6x^2}$ ⇔ $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$

$\frac{dy}{dx} = f'(x) = 6x^2$ → $1/\frac{dx}{dy} = 1/\frac{1}{6x^2} = 6x^2 = \frac{dy}{dx}$

$\boxed{\frac{dy}{dx} = 1/\frac{dx}{dy}}$

5. By using the definition of the inverse function, establish the identity

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}.$$

Refer lecture Note! Demonstrated in lecture!!

$\boxed{\sin^{-1}x \neq \frac{1}{\sin x}}$

6. Show that $\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}(x) = \frac{1}{1+x^2}$.

let $y = \tan^{-1}(x)$

$x = \tan(y)$

$\frac{dx}{dy} = \sec^2(y)$

$\frac{dx}{dy} = 1 + \tan^2 y$

$\frac{dx}{dy} = 1 + x^2$

$\sin^2 y + \cos^2 y = 1$

$(\div \cos^2 y)$ $\tan^2 y + 1 = \sec^2 y$

$\frac{dy}{dx} = 1 / \frac{dx}{dy}$ *

$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$

$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$

7. Find $\frac{\mathrm{d}f}{\mathrm{d}x}$ for these inverse circular functions:

(a) $f(x) = \sin^{-1}\left(\frac{x}{2}\right)$

(b) $f(x) = \tan^{-1}(3x)$

(c) $f(x) = \sqrt{(1+x^2)}\tan^{-1} x$

$\boxed{\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}}$

(a) $f'(x) = \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\left(\frac{1}{2}\right) = \frac{1}{2\sqrt{1-\frac{x^2}{4}}}$ ✓

$= \frac{1}{2\sqrt{\frac{4-x^2}{4}}}$ ✓ $= \frac{1}{2\frac{\sqrt{4-x^2}}{\sqrt{4}}}$ ✓

$= \frac{1}{\sqrt{4-x^2}}$ ✓

(b) $f'(x) = \frac{1}{1+(3x)^2}(3)$

$= \frac{3}{1+9x^2}$

(c) $f(x) = (1+x^2)^{\frac{1}{2}}\tan^{-1} x$

$f'(x) = \frac{1}{\cancel{2}}(1+x^2)^{-\frac{1}{2}}(\cancel{2}x)\tan^{-1} x + (1+x^2)^{\frac{1}{2}}\left(\frac{1}{1+x^2}\right)$

$f'(x) = \frac{x\tan^{-1}(x)}{\sqrt{1+x^2}} + \frac{\sqrt{1+x^2}}{1+x^2} = \frac{x\sqrt{1+x^2}\tan^{-1}(x) + \sqrt{1+x^2}}{1+x^2}$

Higher Order Derivatives

8. Find $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ when y is given by:

a. $y = x^3 + 3x^2 - 5x + 1$

b. $y = \sqrt{(1+x^2)}$

(a) $\frac{dy}{dx} = 3x^2 + 6x - 5$

$\frac{d^2y}{dx^2} = 6x + 6$

$f''(x) / f^{(2)}(x)$

$y = \sqrt{1+x^2}$

$y = (1+x^2)^{\frac{1}{2}}$

(b) $\frac{dy}{dx} = \frac{1}{\cancel{2}}(1+x^2)^{-\frac{1}{2}}(\cancel{2}x) = x(1+x^2)^{-\frac{1}{2}} = \frac{x}{\sqrt{1+x^2}} = \frac{x}{(1+x^2)^{\frac{1}{2}}}$

$\frac{d^2y}{dx^2} = \frac{(1)\sqrt{1+x^2} - (x)\left[\frac{1}{\cancel{2}}(1+x^2)^{-\frac{1}{2}}\right](\cancel{2}x)}{1+x^2}$

$= \frac{\left(\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}}\right)\sqrt{1+x^2} - \frac{x^2}{\sqrt{1+x^2}}}{1+x^2} = \frac{\frac{1+\cancel{x^2} - \cancel{x^2}}{\sqrt{1+x^2}}}{1+x^2}$

$= \frac{1}{\sqrt{1+x^2}} \times \left(\frac{1}{1+x^2}\right) = \frac{1}{(1+x^2)^{3/2}}$

9 Find $\frac{d^2y}{dx^2}$ for the function $y = \ln(x^2+x+1)$ (Note: $\ln(x) = \log_e(x)$.)

$\ln = \log$

$\frac{dy}{dx} = \frac{2x+1}{x^2+x+1}$

$\frac{d^2y}{dx^2} = \frac{(2)(x^2+x+1) - (2x+1)(2x+1)}{(x^2+x+1)^2}$

$= \frac{2x^2+2x+2-4x^2-4x-1}{(x^2+x+1)^2}$

$= \frac{1-2x-2x^2}{(x^2+x+1)^2}$

10. Find $f^{(1)}(x)$, $f^{(2)}(x)$ and $f^{(3)}(x)$ for the following functions:

a. $f(x) = e^{3x}$. What is a general formula for $f^{(n)}(x)$?

b. $f(x) = \ln(x+2)$. Can you find a general formula for $f^{(n)}(x)$? (This one is a bit more of a challenge.)

(a) $f(x) = e^{3x}$ $\quad 3^0$

$n=1,\ f^{(1)}(x) = 3e^{3x}$ $\quad 3^1$

$n=2,\ f^{(2)}(x) = 9e^{3x}$ $\quad 3^2$

$n=3,\ f^{(3)}(x) = 27e^{3x}$ $\quad 3^3$

$\vdots$

$f^{(n)}(x) = 3^n e^{3x}$

$n \geq 1$

(b) $f(x) = \ln(x+2)$ $\quad (-1)^2 \quad (-1)^0$

$f^{(1)}(x) = \frac{1}{x+2} = (x+2)^{-1} = (-1)^?(1)(x+2)^{-1}$

$f^{(2)}(x) = -(x+2)^{-2}$ (1.1) $= (-1)^?(1)(x+2)^{-2}$

$f^{(3)}(x) = 2(x+2)^{-3}$ (1.2) $= (-1)^?(2)(x+2)^{-3}$

$f^{(4)}(x) = -6(x+2)^{-4}$ (1.2.3) $= (-1)^?(6)(x+2)^{-4}$

$f^{(5)}(x) = 24(x+2)^{-5}$ (1.2.3.4) $\quad 24$

$f^{(n)}(x) = (-1)^{n-1}(n-1)!\,(x+2)^{-n}$

$n \geq 1$ $\quad 0! = 1$