MAT1830 - Discrete Mathematics for Computer Science
Tutorial Sheet #10 and Additional Practice Questions

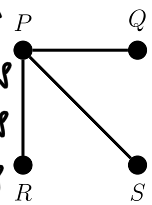
Tutorial Questions

1. Find recursive definitions for the following.
   (a) The sequence $a_0, a_1, a_2, \ldots$ where $a_n = 2^n$ for $n \geq 0$.
   (b) The sequence $b_0, b_1, b_2, \ldots$ where $b_n = n^2$ for $n \geq 0$. (Your recurrence may involve n, but not n^2.)

2. Give the vertex sets and edge sets for the following graphs.

vertex set $\{P, Q, R, S\}$ (a)

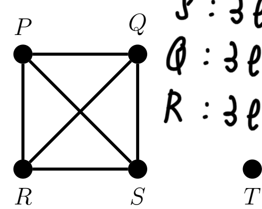
P: 3 edges
S: 1 edges
Q: 1 edges
R: 1 edges



edge set $= \{PQ, PS, PR\}$

vertex set $\{P, Q, R, S, T\}$ (b)

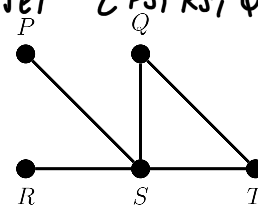
P: 3 edges
S: 3 edges
Q: 3 edges
R: 3 edges



edge set $= \{PQ, PS, PR, QR, QS, RS\}$

vertex set $\{P, Q, R, S, T\}$ (c)

edge set $= \{PS, RS, QS, ST, QT\}$



P: 1 edge
R: 1 edge
S: 4 edge
T: 2 edge
Q: 2 edge

3. Let s_n be the number of ways (order being important) of writing n as a sum of 1s and 2s. For example $s_4 = 5$ because 4 can be written in five ways:
$$1+1+1+1, \quad 1+1+2, \quad 1+2+1, \quad 2+1+1, \quad 2+2.$$
Find a recurrence for s_n. mirror image = undirected "—"

disconnected graph

$V_1 — V_2$
$V_3 — V_4$

4. (a) Draw the simple graph with adjacency matrix

$V_1 \rightleftarrows V_2$
$V_3 \rightleftarrows V_4$

"to"

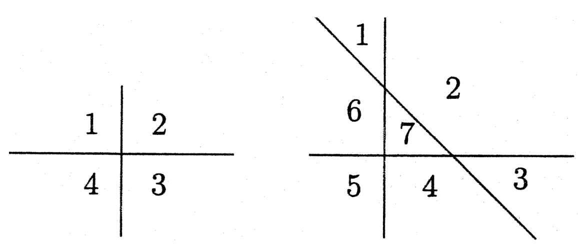
$$\text{"from"} \quad M = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \begin{array}{c} \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

4x4 = 4 vertice graph

⇒ all 0, no loops

using V_1, V_2, V_3, V_4 as the names for the vertices corresponding to columns 1, 2, 3, 4 respectively.
   (b) Find the number of walks of length 3 from V_1 to V_2 in the graph.
   (c) Without any calculation show that the top row of M^n for any even $n \geq 2$ is "1 0 0 0".

5. Let r_n be the number of regions created when the plane is divided by n straight (infinite) lines, with no two lines parallel and no three meeting in a single point. For example,



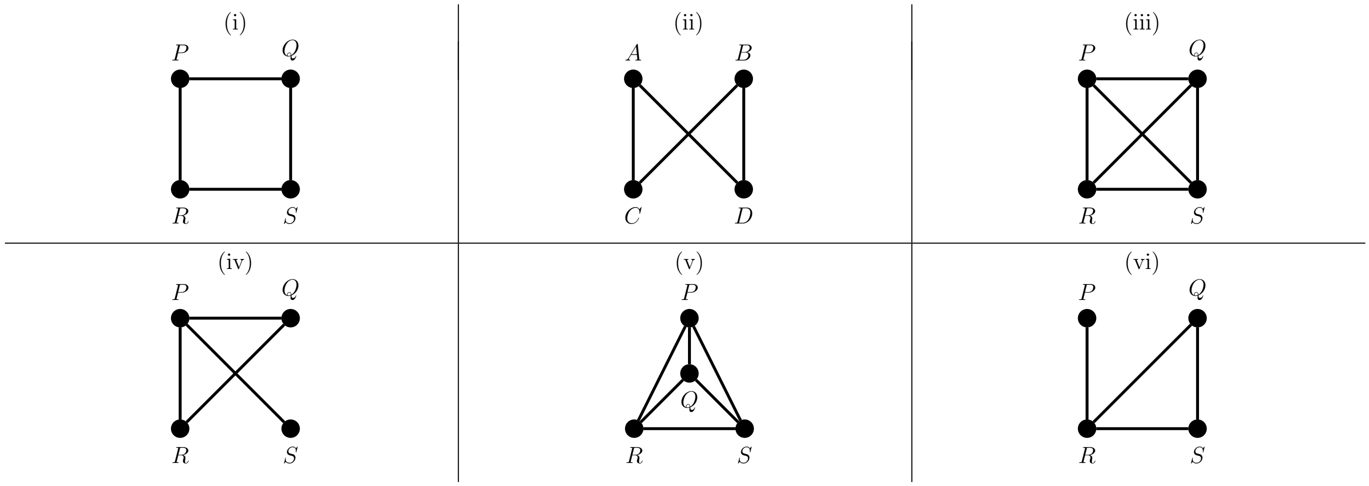
$r_2 = 4$ $\quad$ $r_3 = 7$

Find a recurrence for r_n.

(See over for practice questions.)

Practice Questions

1. Two graphs are equal if they have exactly the same vertex and edge sets. They are isomorphic if we can "rename" the vertices of one graph to make it equal to the other. Which of the following graphs are equal? Which are isomorphic? How would you prove this?



2. How would you change the definition of isomorphic graphs given above to make it more formal?

3. (a) What would you do if you saw someone drive past in a car that was isomorphic to yours?
   (b) What would you do if you saw someone drive past in a car that was equal to yours?

4. Suppose you want to network some computers together in such a way that

   - each computer is directly connected to at most three others; and
   - any two computers are either directly connected or are both directly connected to some third computer.

   Can you find a way to network 7 computers like this? 8? 9? 10?

   (For a way to do this for 10 computers, google "Petersen graph")

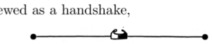
5. Let $S(n,k)$ be the number of equivalence relations on the set $\{1, 2, \ldots, n\}$ with exactly k (non-empty) equivalence classes. Prove that $S(n,k) = kS(n-1,k) + S(n-1,k-1)$ for all integers n and k such that $n > k > 1$.

   ($S(n,k)$ are sometimes called *Stirling numbers of the second kind.*)

31.1 The handshaking lemma

In any graph,
sum of degrees = 2× number of edges.

The reason for the name is that if each edge is viewed as a handshake,



then at each vertex V

degree(V) = number of hands.

Hence

sum of degrees
= total number of hands
= 2 × number of handshakes

An important consequence
The handshaking lemma implies that *in any graph the sum of degrees is even* (being 2×something). Thus it is impossible, e.g. for a graph to have degrees 1,2,3,4,5.

May

1) Recursive definition

(a) sequence $a_0, a_1, a_2, \ldots$; $a_n = 2^n$ for $n \geq 0$

$a_n = 2^n$
$a_{n+1} = 2^{n+1}$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{2^n}$$

$$= 2$$

HENCE, $\underbrace{2a_{n+1} = 2a_n}$

↳ recursive relationship (homogenuous)

"first order"

(b) $b_n = n^2$; $n \geq 0$

$b_{n+1} = (n+1)^2 = n^2 + 2n + 1$

$b_{n+1} = b_n + 2n + 1$

$b_{n+2} = b_{n+1} + 2(n+1) + 1$

* All version are correct

$b_{n+2} - b_{n+1} = b_{n+1} - b_n + 2$

$b_{n+2} = 2b_{n+1} - b_n + 2$

3rd order

$b_{n+1} = 2b_n - b_{n-1} + 2$

$b_{n+2} - b_{n+1} = 2b_{n+1} - 2b_n - b_n + b_{n-1}$

* $b_{n+2} = 3b_{n+1} - 3b_n + b_{n+1}$

$b_0 = 0$; $b_1 = 1$; $b_2 = 4$

S_n = number of ways to write n as sum of 1s and 2s

* order is important: Q: Find recurrance for S_n

$n = 4$

S_4 = 1+1+1+1 kk 1+2+1 kk 1+1+2 kk 2+1+1 kk 2+2

① ② ③ ④ ⑤

HENCE, $S_4 = 5$

eg. $n=1$; $S_1 = 1$

$n=2$; S_2 = 1+1 kk 2

$n=3$; S_3 = 1+1+1 kk 2+1 kk 1+2

$n=4$; S_4 = 1+1+1+1 kk 2+1+1 kk 1+2+1 kk 1+1+2 kk 2+2

HENCE, $n=1$; $S_1 = 1$

$n=2$; $S_2 = 2$

$n=3$; $S_3 = 3$

$n=4$; $S_4 = 5$

Type 1 $\rightarrow S_{n-1}$

Type 2 $\rightarrow S_{n-2}$

$S_n = (S_{n+1}) + (S_{n+2})$ $\boxed{S_1 = 1 \quad S_2 = 2}$

In the Tower of Hanoi game, you are given 3 identical vertical stands where circular disks of different sizes can be slotted onto the vertical pole of the stand.

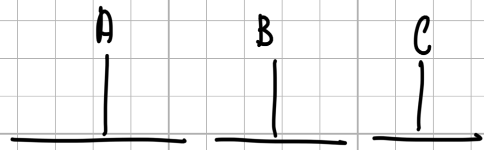
At the start of the game, you have n disks (all of them of different sizes) stacked on one of the pole, smallest disk on top and largest disk at the bottom of the stack.

You are asked to ultimately "transfer" the entire stack of disks onto any of the other 2 poles, but every one move you make can only move one disk at a time, AND a larger disk must not be placed on top a smaller disk at any time.

(i) Using recursive reasoning, show that we can do the transfer for any number of disks on the original stack given us.
Work out the recursive relationship applicable to move n disks.

(ii) Propose a closed form formula to move n disks

rules :
- only small on top of big, not big on top of small
- 1 move 1 disk



move disks from A to C

t_n = num of moves to shift a stook (?) of n disks

propose close form formula

$\boxed{t_{n+1} = 2t_n + 1}$ $\Rightarrow t_1 = 1,\ t_2 = 2$

t_n = # of moves to shift a stook of n disks

$\boxed{t_{n+1} = 2t_n + 1,\ t_1 = 1}$

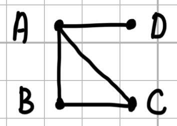
$t_n = ?$

n	t_n
1	1
2	3
3	7
4	15

Lecture 29: Graphs

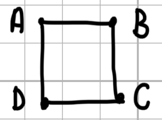
Question 29.1 :

(a)



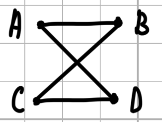
vertex sets $\Rightarrow \{A, B, C, D\}$
edge sets $\Rightarrow \{AB, BC, AC, AD\}$

(b)



vertex sets $\Rightarrow \{A, B, C, D\}$
edge sets $\Rightarrow \{AB, BC, CD, AD\}$

(c)



vertex sets $\Rightarrow \{A, B, C, D\}$
edge sets $\Rightarrow \{AB, BC, AD, CD\}$

HENCE, (b) and (c) are the same graphs

Question 29.2 :

(a) vertex $\Rightarrow \{A, B, C, D\}$; edge $\Rightarrow \{AB, BC, BD\}$

(b) vertex $\Rightarrow \{A, B, C, D, E\}$; edge $\Rightarrow \{AB, BC, CA, DE\}$