<u>**MAT1841 Revision Discussion Questions**</u>

1. Given $\mathbf{u} = (3, 4, -1)$ and $\mathbf{v} = (1, -2, 1)$

(a) Calculate $\mathbf{u} \cdot \mathbf{v}$.

$$\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 3 - 8 - 1 = -6$$

1 mark

$$\begin{matrix} 3 & 4 & -1 \\ 1 & -2 & 1 \end{matrix}$$

(b) Calculate $\mathbf{u} \times \mathbf{v}$ and hence find $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$

(a) $\begin{pmatrix} 2 \\ 4 \\ 10 \end{pmatrix}, 0$

(b) $\begin{pmatrix} 2 \\ -4 \\ 10 \end{pmatrix}, 1$

(c) $\begin{pmatrix} 2 \\ -4 \\ 10 \end{pmatrix}, -1$

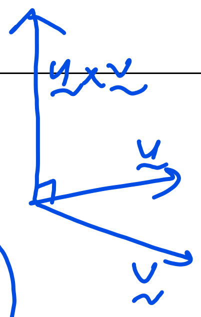
(d) $\begin{pmatrix} 2 \\ -4 \\ -10 \end{pmatrix}, 0$

(e) $\begin{pmatrix} 2 \\ 4 \\ -10 \end{pmatrix}, 0$

$$\underset{\sim}{u} \times \underset{\sim}{v} = \begin{pmatrix} 4 - (2) \\ -[3 - (-1)] \\ -6 - 4 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ -10 \end{pmatrix}$$

$$\underset{\sim}{u} \times \underset{\sim}{v} = \begin{pmatrix} \boxed{2} \\ \boxed{-4} \\ \boxed{-10} \end{pmatrix}$$

$$\underset{\sim}{u} \cdot (\underset{\sim}{u} \times \underset{\sim}{v}) = 0$$



2 mark

(c) Calculate the vector projection of $\mathbf{u}$ onto $\mathbf{v}$, i.e. $\mathbf{u}_v$.

$$\underset{\sim}{u}_v = \left(\frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{v}|^2} \right) \cdot \underset{\sim}{v}$$

$$|\underset{\sim}{v}| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$$

$$= \left(\frac{-6}{6} \right) \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \text{ or } (-1, 2, -1) \text{ or } -\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$$

2 marks

(d) Calculate the vector form of the line through the points $(4, 0, 2)$ and $(-2, 1, 1)$.

$$\underset{\sim}{d} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix}$$

$$\underset{\sim}{r} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix} \quad \text{or} \quad \underset{\sim}{r} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix}$$

or $$\underset{\sim}{r} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} -6 \\ 1 \\ -1 \end{pmatrix} \quad \text{or} \quad \underset{\sim}{r} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} -6 \\ 1 \\ -1 \end{pmatrix}$$

1 marks

(e) Consider the lines

$$r(\overset{s}{\cancel{t}}) = \begin{pmatrix}2\\-1\\4\end{pmatrix} + s\begin{pmatrix}1\\1\\-1\end{pmatrix}$$

and

$$r(t) = \begin{pmatrix}1\\1\\-2\end{pmatrix} + t\begin{pmatrix}0\\1\\-1\end{pmatrix}$$

Determine if they intersect. If yes, find the point of intersection.

$$\begin{pmatrix}2+s\\-1+s\\4-s\end{pmatrix} = \begin{pmatrix}1\\1+t\\-2-t\end{pmatrix} \Rightarrow \begin{array}{l}\text{line } 1 \Rightarrow s=-1\\ \text{line } 2 \Rightarrow -1+(-1)=1+t\\ \qquad\Rightarrow t=-3\end{array}$$

Subs $s=-1$ & $t=-3$ into line 3.

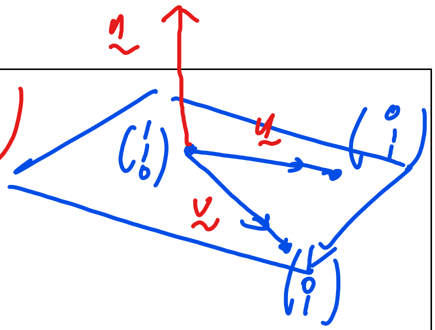
L.H.S $4-s \Rightarrow 4-(-1)=5$ Not equal

R.H.S $-2-(-3)=1$

$\Rightarrow$ These two lines have no intersection

2 marks

(f) Calculate the equation of the plane passing through the three points (0, 1, 1), (1, 0, 1) and (1, 1, 0), and state a normal vector to the plane.



$\underset{\sim}{u} = \begin{pmatrix}1\\1\\0\end{pmatrix} - \begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}$ $\quad \underset{\sim}{v} = \begin{pmatrix}1\\1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix}$

$\underset{\sim}{n} = \underset{\sim}{u} \times \underset{\sim}{v} = \begin{pmatrix}1\\0\\-1\end{pmatrix} \times \begin{pmatrix}0\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$

eqn of plane $\Rightarrow (\underset{\sim}{r} - \underset{\sim}{a}) \cdot \underset{\sim}{n} = 0$

$$\left[\begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix}\right] \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} = 0$$

$$\begin{pmatrix}x-1\\y-1\\z\end{pmatrix} \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} = 0 \qquad x+y+z=2$$

$x-1+y-1+z=0$

3 marks

$ax+by+cz=d$

$a=\boxed{1}$ $\quad b=\boxed{1}$ $\quad c=\boxed{1}$ $\quad d=\boxed{2}$

2. Given the following matrices

$$A = \begin{bmatrix} 2 & 3 \\ -1 & -1 \end{bmatrix},\ B = \begin{bmatrix} 3 & -1 \\ 0 & 2 \\ 0 & -1 \end{bmatrix} \text{ and } C = \begin{bmatrix} -2 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix};$$

(a) Calculate $B^T + C$.

$$\begin{pmatrix} 3 & 0 & 0 \\ -1 & 2 & -1 \end{pmatrix} + \begin{pmatrix} -2 & -1 & 1 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 4 & 0 \end{pmatrix}$$

2 marks

(b) Calculate BB^T.

$$\begin{pmatrix} 3 & -1 \\ 0 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ -1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 9+1 & 0+(-2) & 0+1 \\ 0+(-2) & 0+4 & 0+(-2) \\ 0+1 & 0+(-2) & 0+1 \end{pmatrix}$$

$3 \times 2 \qquad 2 \times 3$ ✓

$$= \begin{pmatrix} 10 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

2 marks

(c) Calculate $(CB)^{-1}$

$$CB = \begin{pmatrix} -2 & -1 & 1 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 0 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -6 & -1 \\ 0 & 3 \end{pmatrix}$$

Show Ans

$$(CB)^{-1} = \frac{1}{-18-0}\begin{pmatrix} 3 & 1 \\ 0 & -6 \end{pmatrix} = -\frac{1}{18}\begin{pmatrix} 3 & 1 \\ 0 & -6 \end{pmatrix} \qquad (CB)^{-1} = \frac{1}{\square}\begin{pmatrix} \square & 1 \\ 0 & \square \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{6} & -\frac{1}{18} \\ 0 & \frac{1}{3} \end{pmatrix}$$

3 marks

(d) Given

$$A = \begin{bmatrix} -2 & -1 & -1 & 2 \\ 0 & 3 & 2 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 0 & -1 & 0 \end{bmatrix}$$

$(-1)^{i+j} a_{ij} S_{ij}$

Calculate the determinant of A.

$$\det(A) = (-1)^{1+4}(-1)\begin{vmatrix} -1 & -1 & 2 \\ 3 & 2 & -1 \\ -1 & -1 & 1 \end{vmatrix} + (-1)^{3+4}(-1)\begin{vmatrix} -2 & -1 & 2 \\ 0 & 3 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= (-2-6-1) - (-4-1-3) + (-6+0+1) - (6-2+0)$$

$$= -10$$

3 marks

3. Given the linear system

$$3x - y + 2z = 3$$
$$x + y - z = 2$$
$$2x - 2y + 3z = b$$

(a) Write the Coefficient matrix and the augmented matrix. .

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 1 & 1 & -1 \\ 2 & -2 & 3 \end{pmatrix}$$

Augmented matrix $(A|b)$

$$\left(\begin{array}{ccc|c} 3 & -1 & 2 & 3 \\ 1 & 1 & -1 & 2 \\ 2 & -2 & 3 & b \end{array}\right)$$

2 marks

(b) Reduce the augmented matrix to echelon form using Gaussian elimination. State the rank of the coefficient matrix.

$$\left(\begin{array}{ccc|c} 3 & -1 & 2 & 3 \\ 1 & 1 & -1 & 2 \\ 2 & -2 & 3 & b \end{array}\right) \xrightarrow[R_3 \Leftarrow 2R_2 - R_3]{R_2 \leftarrow R_1 - 3R_2} \left(\begin{array}{ccc|c} 3 & -1 & 2 & 3 \\ 0 & -4 & 5 & -3 \\ 0 & 4 & -5 & 4-b \end{array}\right)$$

$$\xrightarrow{R_3 \Leftarrow R_2 + R_3} \left(\begin{array}{ccc|c} 3 & -1 & 2 & 3 \\ 0 & -4 & 5 & -3 \\ 0 & 0 & 0 & 1-b \end{array}\right) \text{echelon}$$

$\text{Rank}(A) = 2 = \#(\text{pivot})$

(iii) No sol^n.

Last row if it has illogical situation

$1 - b \neq 0$

$b \neq 1$

2 marks

(c) For what values if b will the system have
   (i) Unique solution
   (ii) Infinite solution
   (iii) No solution

(i) Impossible to have unique soln. rank(A) < # of variables

(ii) Last row must have all entries "zero"

$\Rightarrow 1-b=0$

$b=1$

(iii) Refer to part (b)

with c under part (b)

3 marks

(d) Hence, solve the linear system for **x** by using result in part (c), or otherwise.

$b=1$ $\left(\begin{array}{ccc|c} 3 & -1 & 2 & 3 \\ 0 & -4 & 5 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$ Back substitution method.

let $z=\alpha$ (free variable)

2nd row $\Rightarrow$ $-4y+5z=-3$

$y=\frac{5\alpha+3}{4}$

1st row $\Rightarrow$ $3x-\left(\frac{5\alpha+3}{4}\right)+2\alpha=3$

$x=\frac{5-\alpha}{4}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} \frac{5-\alpha}{4} \\ \frac{5\alpha+3}{4} \\ \alpha \end{pmatrix} \Rightarrow \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}+\alpha\begin{pmatrix} -1 \\ 5 \\ 1 \end{pmatrix}$

3 marks

4. Calculate the following.

(a) Using first principles, calculate the derivatives of $f(x) = \sqrt{x}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \left[\frac{\sqrt{x+h} - \sqrt{x}}{h} \right] \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} = \frac{1}{\sqrt{x} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

2 marks

(b) Calculate the derivative of $g(x) = \frac{e^{2x-1} - \ln(x^2)}{\cos(x)}$.

Quotient Rule

$$g'(x) = \frac{\cos(x)\left[2e^{2x-1} - \frac{2x}{x^2}\right] - (-\sin x)\left[e^{2x-1} - \ln(x)^2\right]}{\cos^2(x)}$$

$$g'(x) = \frac{\cos(x)\left[2e^{2x-1} - \frac{2}{x}\right] + \sin(x)\left[e^{2x-1} - \ln(x^2)\right]}{\cos^2(x)}$$

2 marks

$\frac{dx}{dt} = 2t-2 \qquad \frac{dy}{dt} = 3t^2+3$

(c) Consider the parametric curve defined as $x = t^2 - 2t$ and $y = t^3 + 3t + 2$. Derive an equation for the tangent line to this curve when $t = 3$.

Chain Rule $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt}$

$\frac{dy}{dx} = \frac{3t^2+3}{2t-2}$

$\left.\frac{dy}{dx}\right|_{t=3} = \frac{15}{2} \, (m_t)$

$t=3, \quad x=3, \quad y=38$

$(3, 38) \quad m = \frac{15}{2}$

tangent line

$y - y_1 = m_t(x - x_1)$

$y - 38 = \frac{15}{2}(x-3)$

$y = \frac{15}{2}x + \frac{31}{2}$

$2y - 15x = 31$

2 marks

(d) Calculate the absolute minimum and maximum for the function $f(x) = \frac{x^2-1}{e^x}$ over the interval $[-2 \leq x \leq 2]$.

$f'(x) = \frac{2x e^x - e^x(x^2-1)}{e^{2x}} = \frac{e^x(2x - x^2 + 1)}{e^{2x}}$

$f'(x) = \frac{-x^2+2x+1}{e^x}$

For min/max point $f'(x) = 0$

$\frac{-x^2+2x+1}{e^x} = 0 \qquad (e^x \neq 0)$

$-x^2+2x+1 = 0$

$x = \frac{2 \pm \sqrt{4+4}}{2} \approx 1+\sqrt{2} \approx 2.4\ldots$

$f(-2) = \frac{3}{e^{-2}} = 3e^2$

$f(2) = \frac{3}{e^2} = e^{-2}$

Absolute max

$(-2, 3e^2)$

Absolute min

$\left(1-\sqrt{2}, \frac{2-2\sqrt{2}}{e^{1-\sqrt{2}}}\right)$

4 marks

Power series table $e^{-x} = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \dots$

$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots$

5. (a) Calculate the first **four** non-zero terms in the Taylor series expansion of $f(x) = e^{-x}sin\,(x)$ about $x = \frac{\pi}{2}$.

$f(x) = e^{-x}\sin(x)$ $\quad f(\frac{\pi}{2}) = e^{-\frac{\pi}{2}}$

$f'(x) = -e^{-x}\sin(x) + e^{x}\cos(x)$

$= e^{-x}(-\sin(x) + \cos(x))$ $\quad f'(\frac{\pi}{2}) = -e^{-\frac{\pi}{2}}$

$f''(x) = -e^{-x}(-\sin(x) + \cos(x)) + e^{-x}(-\cos(x) - \sin(x))$ $\quad f''(\frac{\pi}{2}) = 0$

$= -2e^{-x}\cos(x)$

$f^{(3)}(x) = 2e^{-x}\cos(x) + 2e^{-x}\sin(x)$ $\quad f^{3}(\frac{\pi}{2}) = 2e^{-\frac{\pi}{2}}$

$= 2e^{-x}(\cos(x) + \sin(x))$

$f^{(4)}(x) = -2e^{-x}(\cos(x) + \sin(x)) + 2e^{-x}(-\sin(x) + \cos(x))$ $\quad f^{(4)}(\frac{\pi}{2})$

$= -4e^{-x}\sin(x)$ $\quad = -4e^{-\frac{\pi}{2}}$

$e^{-x}\sin(x) \approx e^{-\frac{\pi}{2}} + \left(\frac{-e^{-\frac{\pi}{2}}}{1!}\right)(x - \frac{\pi}{2}) + \left(\frac{2e^{-\frac{\pi}{2}}}{3!}\right)(x - \frac{\pi}{2})^3 + \left(\frac{-4e^{-\frac{\pi}{2}}}{4!}\right)(x - \frac{\pi}{2})^4$

6 marks

(b) The following questions relate to Taylor series. **A table of useful power series is provided in the formulae section of this paper.**

Compute the Taylor series expansions, around $x = 0$, for ln $(1 + x)$ and ln $(1 - x)$. Hence obtain a Taylor series for $f(x) = \ln\left(\frac{1+x}{1-x}\right)$, express in the summation form such as $\sum_{n=0}^{\infty} 2x^{n-1}$.

$f(x) = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$

power series table,

$\ln(1+x) = x - \cancel{\frac{1}{2}x^2} + \frac{1}{3}x^3 - \cancel{\frac{1}{4}x^4} + \frac{1}{5}x^5 - \cancel{\frac{1}{6}x^6} + \dots$

$(-)\ \ln(1-x) = -x - \cancel{\frac{1}{2}x^2} - \frac{1}{3}x^3 - \cancel{\frac{1}{4}x^4} - \frac{1}{5}x^5 - \cancel{\frac{1}{6}x^6} + \dots$

$\ln(1+x) - \ln(1-x) = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \dots$

$\ln\left(\frac{1+x}{1-x}\right) \approx 2(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots)$

$\approx \sum_{n=0}^{\infty} 2\left(\frac{1}{2n+1}\right)x^{2n+1}$

4 marks

6. Calculate the following indefinite integrals.

(a) $I = \int \cos(x) e^{3\sin(x)}\ dx$

Substitution

Let $u = \sin(x)$

$du = \cos(x)\ dx$

$I = \int e^{3u}\ du$

$I = \frac{1}{3} e^{3u} + c$

$I = \int \cos(x) e^{3\sin(x)}\ dx = \frac{1}{3} e^{3\sin(x)} + c$

3 marks

(b) $I = \int x\sqrt{2x-1}\ \ dx$ by parts.

Let $f = x$ $\quad g' = (2x-1)^{\frac{1}{2}}$

$f' = 1$ $\quad g = \frac{(2x-1)^{3/2}}{(2)\left(\frac{3}{2}\right)} = \frac{1}{3}(2x-1)^{3/2}$

$I = \frac{x}{3}(2x-1)^{3/2} - \int \frac{1}{3}(2x-1)^{3/2}\ dx$

$I = \frac{x}{3}(2x-1)^{3/2} - \frac{1}{3}\left[\frac{(2x-1)^{5/2}}{(2)\left(\frac{5}{2}\right)}\right] + c$

$I = \frac{x}{3}(2x-1)^{3/2} - \frac{1}{15}(2x-1)^{5/2} + c$

3 marks

(c) Use Integration by Parts twice to calculate $I = cos\,(x)\, e^x dx$.

4ATG .

let $u = e^x$ $\quad dv = \cos(x)$

$\frac{du}{dx} = e^x$ $\quad v = \sin(x)$

$I = e^x \sin(x) - \int e^x \sin(x)\, dx \rightarrow$

$f = e^x$

$\frac{df}{dx} = e^x$

$g' = \sin(x)$

$g = -\cos(x)$

$\int e^x \cos(x)\, dx = e^x \sin(x) + e^x \cos x - \int e^x \cos(x)\, dx$

$2\int e^x \cos(x)\, dx = e^x \sin(x) + e^x \cos(x)$

$\int e^x \cos(x)\, dx = \frac{1}{2} e^x [\sin(x) + \cos(x)] + c$

4 marks

(d) Use a substitution and then integration by parts to solve

$$I = \int \frac{\ln(\sqrt{x})}{\sqrt{x}} dx$$

subs. let $u = \sqrt{x} = x^{\frac{1}{2}}$

$du = \frac{1}{2} x^{-\frac{1}{2}} dx$

$du = \frac{1}{2\sqrt{x}} dx$

$2du = \frac{1}{\sqrt{x}} dx$

$\rightarrow I = \int 2\ln(u)\, du$

$I = 2\int \ln(u)\, du$

by parts. Let $f = \ln(u)$ $\quad g' = 1$

$\frac{df}{du} = \frac{1}{u}$ $\quad g = u$

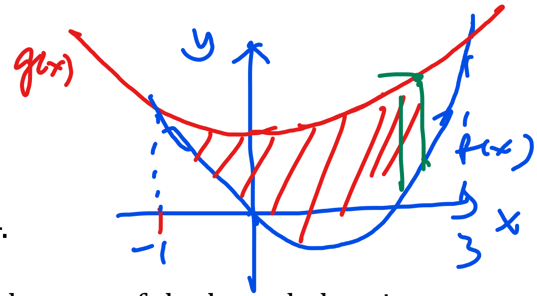
$I = 2\left[u\ln(u) - \int 1\, du\right]$

$I = 2u\ln(u) - u + c$

$I = 2\sqrt{x}\ln(\sqrt{x}) - \sqrt{x} + c$

4 marks

7. Given the two functions $f(x) = x^2 - x$ and $g(x) = \frac{1}{2}x^2 + \frac{3}{2}$.



(a) Use the Fundamental Theorem of Calculus to calculate the area of the bounded region between the curves. Note that the curves intersect at the points (-1, 2) and (3, 6).

$$A = \int_{-1}^{3} [g(x) - f(x)]\, dx$$

$$A = \int_{-1}^{3} \left(-\frac{1}{2}x^2 + x + \frac{3}{2}\right) dx$$

$$A = \left[-\frac{1}{6}x^3 + \frac{x^2}{2} + \frac{3}{2}x\right]_{-1}^{3}$$

$$A = \frac{16}{3} \text{ units}^2 \; / \; 5\frac{1}{3} \text{ units}^2$$

If swap of f(x)

$$A = \int_{-1}^{3} [f(x) - g(x)]\, dx$$

$\vdots$

$$A = -\frac{16}{3}$$

5 marks

(b) Approximate the area between the curves using the Trapezoidal rule with $n = 4$. Leave as a sum of terms on a common denominator.

parallel lengths for the trapeziums

$h(x) = -\frac{1}{2}x^2 + x + \frac{3}{2}$

$x = -1$ to $x = 3$,

$h = \frac{3-(-1)}{4} = 1$

x	$h(x) = -\frac{1}{2}x^2 + x + \frac{3}{2}$
-1	0
0	$3/2$
1	2
2	$3/2$
3	0

$A \approx \frac{1}{2}(1)\left[0 + 0 + 2\left[\frac{3}{2} + 2 + \frac{3}{2}\right]\right]$

≈ 5 units^2

5 marks

8. Find all first and second partial derivatives of the function $f\ (x, y) = x^2 \sin(x + y)$.

$f_x = 2x\sin(x+y) + x^2\cos(x+y)$

$f_{xx} = 2\sin(x+y) + 2x\cos(x+y) + 2x\cos(x+y) - x^2\sin(x+y)$

$= (2-x^2)\sin(x+y) + 4x\cos(x+y)$

$f_y = x^2\cos(x+y)$

$f_{yy} = -x^2\sin(x+y)$

$f_{yx} = f_{xy} = 2x\cos(x+y) - x^2\sin(x+y)$

10 marks

9. Find the equations of the approximations to the surface $f(x, y) = x^2y + xy^2 + y^3$ at the point $(x, y) = (-2, 1)$.

(a) Find the tangent plane, $T_1(x, y)$.

$f_x = 2xy + y^2$ $\qquad$ $f_y = x^2 + 2xy + 3y^2$

$f_x(-2,1) = 3$ $\qquad$ $f_y(-2,1) = -3$

$f(-2,1) = 3 \Rightarrow (-2, 1, 3) = (a, b, c)$

Tangent plane eqn.

$T_1(x,y) = z = \underline{3 + 3(x+2) - 3(y-1)}$

4 marks

(b) Find the second order Taylor expansion, $T_2(x, y)$. You do not need to simplify.

$f_{xx} = 2y$ $\qquad$ $f_{xy} = 2x + 2y$ $\qquad$ $f_{yy} = 2x + 6y$

$f_{xx}(-2,1) = 2$ $\qquad$ $f_{xy}(-2,1) = -2$ $\qquad$ $f_{yy}(-2,1) = 2$

$$T_2(x,y) = T_1(x,y) + \frac{f_{xx}}{2!}(x+2)^2 + \frac{f_{xy}}{2!}(x+2)(y-1) + \frac{f_{yy}}{2!}(y-1)^2$$

$$T_2(x,y) = 3 + 3(x+2) - 3(y-1) + (x+2)^2 - 2(x+2)(y-1) + (y-1)^2$$

6 marks

$\frac{\partial f}{\partial x} = 2xy^2 + 2y$ $\qquad$ $\frac{dx}{ds} = e^s$

10. Calculate the following:

$\frac{\partial f}{\partial y} = 2x^2y + 2x + 1$ $\qquad$ $\frac{dy}{ds} = \frac{1}{2}e^{\frac{s}{2}}$

(a) Compute $\frac{df}{ds}$ for $f(x, y) = x^2y^2 + 2xy + y$, where $x = e^s$ and $y = e^{\frac{s}{2}}$.

$f(x,y) \Rightarrow f(s)$

$f(s) = e^{2s}e^s + 2e^{\frac{3}{2}s} + e^{\frac{s}{2}} = e^{3s} + 2e^{\frac{3}{2}s} + e^{\frac{s}{2}}$

$\frac{df}{ds} = 3e^{3s} + 3e^{\frac{3}{2}s} + \frac{1}{2}e^{\frac{s}{2}}$

Chain Rule $\frac{df}{ds} = \frac{\partial f}{\partial x} \times \frac{dx}{ds} + \frac{\partial f}{\partial y} \times \frac{dy}{ds}$

$= 3e^{3s} + 3e^{3/2 s} + \frac{1}{2}e^{\frac{s}{2}}$

4 marks

(b) Find the directional derivative of the function $f(x, y) = x^2y^2 + 2xy + y$ in the direction of the vector $\boldsymbol{t} = 3\boldsymbol{i} + 4\boldsymbol{j}$ at the point $(x, y) = (-2, 1)$.

$\frac{df}{ds} = \nabla f \cdot \hat{\underset{\sim}{t}}$

$\frac{\partial f}{\partial x} = 2xy^2 + 2y$ $\qquad$ $\frac{\partial f}{\partial y} = 2x^2y + 2x + 1$

$\frac{\partial f}{\partial x}\Big|_{(-2,1)} = -2$ $\qquad$ $\frac{\partial f}{\partial y}\Big|_{(-2,1)} = 5$

$\underset{\sim}{t} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$

$|\underset{\sim}{t}| = 5$

$\hat{\underset{\sim}{t}} = \frac{1}{5}(3\underset{\sim}{i} + 4\underset{\sim}{j})$

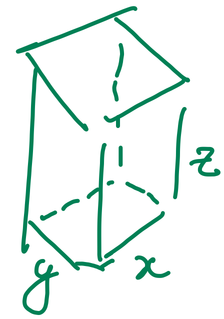
$\hat{\underset{\sim}{t}} = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$

$\frac{df}{ds} = \begin{pmatrix} -2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} = \frac{14}{15}$

6 marks

volume

(c) A container with an open top is to have $10\ m^3$ capacity and be made of thin sheet metal. Calculate the dimensions of the box if it is to use the minimum possible amount of metal so that the cost is least, where the cost function is given as $M(x, y, z) = 2xy + 3xz + 3yz$.



$V = xyz = 10$

$z = \frac{10}{xy}$

$M(x, y, z) = 2xy + 3xz + 3yz$

$M(x, y) = 2xy + 3x\left(\frac{10}{xy}\right) + 3y\left(\frac{10}{xy}\right)$

$M(x, y) = 2xy + \frac{30}{y} + \frac{30}{x}$

optimization, $M_x = 2y - \frac{30}{x^2}$, $M_y = 2x - \frac{30}{y^2}$

set $M_x = 0$, $M_y = 0$ solve for x & y

$2y - \frac{30}{x^2} = 0$ $\qquad$ $2x - \frac{30}{y^2} = 0$

$y = \frac{15}{x^2}$ $\xrightarrow{\text{subs}}$

$2x - \frac{30}{\left(\frac{225}{x^4}\right)} = 0$

$2x - \frac{2x^4}{15} = 0$

$2x\left(1 - \frac{x^3}{15}\right) = 0$

$x = 0$ (rejected) $\qquad x = \sqrt[3]{15}$

$x = \sqrt[3]{15}$, $y = \frac{15}{(\sqrt[3]{15})^2} = \sqrt[3]{15}$, $z = \frac{10}{\sqrt[3]{15}\sqrt[3]{15}} = \frac{10}{(15)^{2/3}}$

Verification $\quad M_x = 2y - \frac{30}{x^2} \qquad M_y = 2x - \frac{30}{y^2} \qquad M_{xy} = 2$

$M_{xx} = \frac{60}{x^3} \qquad M_{yy} = \frac{60}{y^3}$

Use 2nd derivative, $D = M_{xx}M_{yy} - (M_{xy})^2$

$D = 12 > 0$

10 marks

when $x = \sqrt[3]{15}$, $M_{xx} = 4 > 0$ $\Rightarrow$ minimum cost.