Partial Derivatives

1) First Partial Derivatives

- Given $f(x,y)$, we can write partial derivatives of f:

(i) $$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = f_x$$

y is treated as constant

x is treated as constant

(ii) $$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} = f_y$$

* the ∂ symbol is used instead of d to remind that in computing this derivative all other variables are held constant

* if for a function with one variable, its derivative is known as ordinary derivative

- If z is a function of two independent variables x and y (ie $z = f(x,y)$) then there are two independent rates of change
  (i) rate of change of f with respect to the variable x
  (ii) rate of change of f with respect to the variable y

- Product rule for partial derivatives

$$\boxed{\frac{\partial (fg)}{\partial x} = g\frac{\partial f}{\partial x} + f\frac{\partial g}{\partial x} \qquad \frac{\partial (fg)}{\partial y} = g\frac{\partial f}{\partial y} + f\frac{\partial g}{\partial y}}$$

- Rules for finding partial derivatives

(i) to find $\frac{\partial f}{\partial x}$, treat y as constant and differentiate $f(x,y)$ with respect to x only
(ii) to find $\frac{\partial f}{\partial y}$, treat x as constant and differentiate $f(x,y)$ with respect to y only

- example:

1. If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2,1)$ and $f_y(2,1)$

$f_x = 3x^2 + 2xy^3$

$f_y = 3x^2y^2 - 4y$

$f_x(2,1) = 3(2)^2 + 2(2)(1)^3 = 16$

$f_y(2,1) = 3(2)^2(1)^2 - 4(1) = 8$

② If $f(x,y) = \sin x \cos y$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

$f_x = \cos x \cos y$ $\qquad$ $f_y = \sin x(-\sin y)$

$= \frac{\partial f}{\partial x}$ $\qquad$ $= -\sin x \sin y$

$= \frac{\partial f}{\partial y}$

③ If $g(x,y,z) = e^{-x^2-y^2-z^2}$, find $\frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial y}$ and $\frac{\partial g}{\partial z}$

$\frac{\partial g}{\partial x} = -2xe^{-x^2-y^2-z^2}$ $\qquad$ $\frac{\partial g}{\partial z} = -2ze^{-x^2-y^2-z^2}$

$\frac{\partial g}{\partial y} = -2ye^{-x^2-y^2-z^2}$

② Second Partial Derivatives

- Notation:

$$\boxed{\begin{array}{ll} \frac{\partial^2 f}{\partial x^2} = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) & \frac{\partial^2 f}{\partial x \partial y} = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \\ \frac{\partial^2 f}{\partial y^2} = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) & \frac{\partial^2 f}{\partial y \partial x} = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \end{array}}$$

- Example:

① Given $f(x,y) = 3x^2 + 2xy$, compute $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$

$\frac{\partial f}{\partial x} = 6x + 2y$ $\qquad$ $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 2$

$\frac{\partial f}{\partial y} = 2x$ $\qquad$ $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2$

SAME (follows Clairaut's Theorem)

Clairaut's Theorem
→ order of partial derivatives does not matter
→ if $f(x,y)$ is a twice differentiable function whose second order mixed partial derivatives are continuous, then the order in which its mixed partial derivatives are calculated does not matter.
→ each ordering will yield the same function
→ therefore $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

Tangent Plane

recall equation of tangent line :

$\boxed{y - y_1 = m_T(x - x_1)}$ where m_T is the gradient

equation of tangent plane :

(i) with two variables

$c = f(a,b)$ $f(x,y)$ at (a,b)

$$\boxed{z - c = f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

(the final equation will be in terms of x, y, z)

(ii) with three variables

$f(x,y,z)$ at (a,b,c)

$$d = f(a,b,c) \quad \boxed{w - d = f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)}$$

(the final equation will be in terms of x, y, z, w)

$f_x = \frac{\partial f}{\partial x}$ = slope of tangent line to surface $z = f(x,y)$ in x-direction

$f_y = \frac{\partial f}{\partial y}$ = slope of tangent line to surface $z = f(x,y)$ in y-direction

The tangent plane to surface at a point P is the plane that contains both of the tangent lines T_1 and T_2

recall equation of plane (vector form)

$$\underset{\sim}{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} p \\ q \\ r \end{pmatrix} t + \begin{pmatrix} l \\ m \\ n \end{pmatrix} s$$

gradient vector with y-axis

gradient vector with x-axis

order can exchange

• example :

① Find the equation of the tangent plane to the surface $z = 2x^2 + y^2$ at the point $(a,b) = (1,1)$

$f(x,y) = 2x^2 + y^2$

$f_x = 4x$

$f_y = 2y$

$f(1,1) = 2(1)^2 + 1^2$
$= 3$

$f_x(1,1) = 4(1) = 4$

$f_y(1,1) = 2(1) = 2$

equation of tangent plane :

$z - 3 = 4(x-1) + 2(y-1)$

$z = 4x + 2y - 4 - 2 + 3$

$z = 4x + 2y - 3$

Linear Approximation

- The equation of the tangent plane to the surface $z=f(x,y)$ at the point (a,b) is also the equation for the linear approximation to $z=f(x,y)$ for points (x,y) near (a,b)
- We can regard the tangent plane equation
  $$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
  as the natural extension to functions of two variables (x,y) of the Taylor polynomial of degree one

$$\boxed{\begin{aligned} \therefore\ z &= T_1(x,y) \\ &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &= L(x,y) \end{aligned}}$$

- example:

① Derive the linear approximation function $T_1(x,y)$ for the function $f(x,y) = \sqrt{3x-y}$ at the point $(4,3)$

$f(x,y) = (3x-y)^{\frac{1}{2}}$

$f_x = \frac{1}{2}(3x-y)^{-\frac{1}{2}}(3) = \frac{3}{2}(3x-y)^{-\frac{1}{2}}$

$f_y = \frac{1}{2}(3x-y)^{-\frac{1}{2}}(-1) = -\frac{1}{2}(3x-y)^{-\frac{1}{2}}$

$f(4,3) = (12-3)^{\frac{1}{2}} = \sqrt{9} = 3$

$f_x(4,3) = \frac{3}{2\sqrt{12-3}} = \frac{1}{2}$

$f_y(4,3) = -\frac{1}{2\sqrt{12-3}} = -\frac{1}{6}$

$$\begin{aligned} T_1(x,y) &= 3 + \tfrac{1}{2}(x-4) - \tfrac{1}{6}(y-3) \\ &= \tfrac{1}{2}x - \tfrac{1}{6}y + 3 - 2 + \tfrac{1}{2} \\ &= \tfrac{1}{2}x - \tfrac{1}{6}y + \tfrac{3}{2} \end{aligned}$$

② Find the tangent line to the function $f(x) = \sin x$ at $x = \frac{\pi}{4}$ and use the result to estimate $\sin x \sin y$ at $(\frac{5\pi}{16}, \frac{5\pi}{16})$

$f'(x) = \cos x$

$f(\frac{\pi}{4}) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$f'(\frac{\pi}{4}) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$y - y_1 = m(x - x_1)$

$y - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(x - \frac{\pi}{4})$

$y = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}(1 - \frac{\pi}{4})$

$f(x,y) = \sin x \sin y$ $\qquad f(\frac{\pi}{4}, \frac{\pi}{4}) = \frac{\sqrt{2}}{2}(\frac{\sqrt{2}}{2}) = \frac{1}{2}$

$f_x = \cos x \sin y$ $\qquad f_x(\frac{\pi}{4}, \frac{\pi}{4}) = \frac{1}{2}$

$f_y = \sin x \cos y$ $\qquad f_y(\frac{\pi}{4}, \frac{\pi}{4}) = \frac{1}{2}$

$L(x,y) = \frac{1}{2} + \frac{1}{2}(x - \frac{\pi}{4}) + \frac{1}{2}(y - \frac{\pi}{4})$

at $(\frac{5\pi}{16}, \frac{5\pi}{16})$

$L(\frac{5\pi}{16}, \frac{5\pi}{16}) = \frac{1}{2} + \frac{1}{2}(\frac{5\pi}{16} - \frac{\pi}{4}) + \frac{1}{2}(\frac{5\pi}{16} - \frac{\pi}{4})$

≈ 0.6963

original value $= \sin\frac{5\pi}{16}\cos\frac{5\pi}{16}$

≈ 0.6913

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③ Derive the linear approximation $T_1(x,y,z)$ for the function $f(x,y,z) = x^2+y^2+z^2$ at the point $(1,1,1)$

$f(x,y,z) = x^2+y^2+z^2$ $\qquad$ $f(1,1,1) = 1^2+1^2+1^2 = 3$

$f_x = 2x$ $\qquad$ $f_x(1,1,1) = 2(1) = 2$

$f_y = 2y$ $\qquad$ $f_y(1,1,1) = 2(1) = 2$

$f_z = 2z$ $\qquad$ $f_z(1,1,1) = 2(1) = 2$

$$T_1(x,y,z) = 3 + 2(x-1) + 2(y-1) + 2(z-1)$$
$$= 2x + 2y + 2z - 3$$

Chain Rule

- Recall chain rule :

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

- Given our curve is described by parametric equations

$x = x(s)$ $\qquad$ $y = y(s)$

$$\Downarrow$$

$$f = f(x(s), y(s))$$

- we can compute $\frac{df}{ds}$ in terms of partial derivatives by chain rule,

$$\boxed{\frac{df}{ds} = \left(\frac{\partial f}{\partial x} \cdot \frac{dx}{ds}\right) + \left(\frac{\partial f}{\partial y} \cdot \frac{dy}{ds}\right)}$$

From

$f(x,y)$
$x(s)$ $\qquad$ we can get $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$, $\frac{dx}{ds}$, $\frac{dy}{ds}$
$y(s)$

- example :

① Given the curve $x(s) = 2s$, $y(s) = 4s^2$ $\quad -1 < s < 1$ and the function $f(x,y) = 5x - 7y + 2$, compute $\frac{df}{ds}$ at $s=0$

$\frac{dx}{ds} = 2$ $\qquad$ $\frac{\partial f}{\partial x} = 5$

$\frac{dy}{ds} = 8s$ $\qquad$ $\frac{\partial f}{\partial y} = -7$

$$\frac{df}{ds} = 5(2) + (-7)(8s)$$
$$= 10 - 56s$$

Alternative method :

$f(x,y) = 5x - 7y + 2$

$f(s) = 5(2s) - 7(4s^2) + 2 = 10s - 28s^2 + 2$

$\frac{df}{ds} = f'(s) = 10 - 56s$

• If there are two parameters?

Let $f = f(x,y)$ be a differentiable function. If $x = x(u,v)$ and $y = y(u,v)$ then

$$\boxed{\begin{aligned} \frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial u}\right) + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial u}\right) \\ \frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial v}\right) + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial u}\right) \end{aligned}}$$

- example:

① Given $f = f(x,y)$ and $x = 2s + 3t$, $y = s - 2t$ compute $\frac{\partial f}{\partial t}$ directly and by the way of chain rule

$\frac{\partial x}{\partial s} = 2 \quad \frac{\partial x}{\partial t} = 3 \quad \frac{\partial y}{\partial s} = 1 \quad \frac{\partial y}{\partial t} = -2$

by chain rule,

$$\begin{aligned} \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial t}\right) \\ &= \frac{\partial f}{\partial x}(3) + \frac{\partial f}{\partial y}(-2) \\ &= 3\frac{\partial f}{\partial x} - 2\frac{\partial f}{\partial y} \end{aligned}$$

② Given $f(x,y) = 2xy$ and $x(r,\theta) = r\cos\theta$, $y(r,\theta) = r\sin\theta$, compute $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$

$\frac{\partial f}{\partial x} = 2y \quad \frac{\partial x}{\partial r} = \cos\theta \quad \frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial f}{\partial y} = 2x \quad \frac{\partial x}{\partial \theta} = -r\sin\theta \quad \frac{\partial y}{\partial \theta} = r\cos\theta$

$$\begin{aligned} \frac{\partial f}{\partial r} &= 2y(\cos\theta) + 2x(\sin\theta) \\ &= 2r\sin\theta\cos\theta + 2r\cos\theta\sin\theta \\ &= 4r\sin\theta\cos\theta \end{aligned}$$

$$\begin{aligned} \frac{\partial r}{\partial \theta} &= 2y(-r\sin\theta) + 2x(r\cos\theta) \\ &= 2r\sin\theta(-r\sin\theta) + 2r\cos\theta(r\cos\theta) \\ &= -2r^2\sin^2\theta + 2r^2\cos^2\theta \\ &= 2r^2(\cos^2\theta - \sin^2\theta) \end{aligned}$$

Gradient and Directional Derivative

Gradient of f / grad f for a function $f(x,y)$ of two variables :

$$\boxed{\nabla f = \frac{\partial f}{\partial x}\underset{\sim}{i} + \frac{\partial f}{\partial y}\underset{\sim}{j}}$$ ← gradient vector for $f(x,y)$

claim : $\left(\frac{dx}{ds}, \frac{dy}{ds}\right)$ is tangent to the curve $\underset{\sim}{t}$

* if $\underset{\sim}{t}$ is the tangent vector to $(x(s), y(s))$, then

$$\nabla_{\underset{\sim}{t}} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot \left(\frac{dx}{ds}, \frac{dy}{ds}\right)$$

$$= \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \frac{dx}{ds} \\ \frac{dy}{ds} \end{pmatrix}$$

$$= \frac{\partial f}{\partial x}\left(\frac{dx}{ds}\right) + \frac{\partial f}{\partial y}\left(\frac{dy}{ds}\right)$$ (chain rule)

∴ The equation for directional derivative can be written as

$$\boxed{\frac{df}{ds} = \underset{\sim}{t} \cdot \nabla f \quad \text{or} \quad \frac{df}{ds} = \nabla_{\underset{\sim}{t}} f}$$

* if given t is not having magnitude 1 then we need to find the unit vector $\underset{\sim}{t}$

$$\hat{\underset{\sim}{t}} = \frac{\underset{\sim}{t}}{|\underset{\sim}{t}|}$$

- example:

① Given $f(x,y) = \sin x \cos y$, compute the directional derivative of f in the direction $\underset{\sim}{t} = \frac{\underset{\sim}{i} + \underset{\sim}{j}}{\sqrt{2}}$

$$f_x = \frac{\partial f}{\partial x} = \cos x \cos y$$

$$f_y = \frac{\partial f}{\partial y} = -\sin x \sin y$$

$$\underset{\sim}{t} = \frac{1}{\sqrt{2}}\underset{\sim}{i} + \frac{1}{\sqrt{2}}\underset{\sim}{j}$$

$$|\underset{\sim}{t}| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$

$$\frac{df}{ds} = \begin{pmatrix} \cos x \cos y \\ -\sin x \sin y \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}(\cos x \cos y - \sin x \sin y)$$

SUMMARY (directional derivative)

The directional derivative $\frac{df}{ds}$ of a function f in the direction $\underset{\sim}{t}$ is given by

$$\frac{df}{ds} = \underset{\sim}{t} \cdot \nabla f = \nabla_{\underset{\sim}{t}} f$$

where the gradient ∇f is defined by

$$\nabla f = \frac{\partial f}{\partial x}\underset{\sim}{i} + \frac{\partial f}{\partial y}\underset{\sim}{j}$$

and $\underset{\sim}{t}$ is a unit vector, $\underset{\sim}{t} \cdot \underset{\sim}{t} = 1$

a vector that has magnitude of 1 is a unit vector

Taylor polynomials of higher degree

Given that taylor polynomial of degree one,

$$T_1(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

The taylor polynomial of degree two,

$$T_2(x,y) = T_1(x,y) + \frac{1}{2}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right]$$

$T_2(x)$ is known as quadratic approximation for $f(x)$

Stationary Points

method : to find the points, let $f_x = 0$, $f_y = 0$ $(f_z = 0)$
to determine the nature, use 2nd derivative and D

results :

① A local maximum
→ occurs when $f(x,y) \leq f(a,b)$ for all (x,y) close to (a,b)
→ when $D > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$ $(f_{xx} < 0)$

② A local minimum
→ occurs when $f(x,y) \geq f(a,b)$ for all (x,y) close to (a,b)
→ when $D > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$ $(f_{xx} > 0)$

③ A saddle point
→ if the point is neither a maximum or minimum
→ $D < 0$

④ Inconclusive
→ when $D = 0$

· 2nd derivative test,
If $0 = \nabla f$ at a point P then, at P compute

$$D = \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

· example :
① Classify the stationary points of $f(x,y) = x^2 + y^2 - 2x - 6y + 14$

$f_x = \frac{\partial f}{\partial x} = 2x - 2$

$f_y = \frac{\partial f}{\partial y} = 2y - 6$

$f_{xx} = 2$ $\quad f_{yy} = 2$

$f_{xy} = 0$

$z = f(1,3)$
$= 1 + 9 - 2 - 18 + 14$
$= 4$

let $f_x = 0$ and $f_y = 0$
$2x - 2 = 0$ $\quad 2y - 6 = 0$
$x = 1$ $\quad y = 3$
critical point at $(1,3)$

$D = (2)(2) - 0^2 = 4 > 0$

∴ Since $D > 0$ and $f_{xx} > 0$ hence the point $(1,3,4)$ is a local minimum