Problem Set Three: Matrices and Gaussian Elimination

Matrices

1. Evaluate each of the following matrix operations.

(a)

$$2\begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & -8 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ -1 & -9 \end{pmatrix}$$

(b)

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} = \begin{pmatrix} 2+3 & -1+1 \\ 2-12 & -1-4 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ -10 & -5 \end{pmatrix}$$

(c)

$$BA = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix} = \begin{pmatrix} 2-1 & 2+4 \\ 3+1 & 3-4 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 4 & -1 \end{pmatrix}$$

$AB \neq BA$

(d)

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & -4 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{pmatrix} 2+3+3 & -1+1+6 \\ 2-12+2 & -1-4+4 \end{pmatrix} = \begin{pmatrix} 8 & 6 \\ -8 & -1 \end{pmatrix}$$

$2\times 3 \quad 3\times 2$

2. Evaluate each of the following matrix operations, if possible. If not possible, state why.

(a)

$$\begin{bmatrix} 2 & -1 \\ 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}$$

$3\times 2 \quad 2\times 2$

$$= \begin{pmatrix} 2-1 & 2+4 \\ 3+1 & 3-4 \\ 1+2 & 1-8 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 4 & -1 \\ 3 & -7 \end{pmatrix}$$

(b)

$$\begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 1 \\ 1 & 2 \end{bmatrix}$$

$2\times 2 \quad 3\times 2$ ✗

# of columns in 1st matrix is not equal to the # of rows in 2nd matrix

(c)

$$\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}\begin{bmatrix} 2 & 3 & 7 \end{bmatrix}$$

$3\times 1 \quad 1\times 3$

$$= \begin{pmatrix} 6 & 9 & 21 \\ 2 & 3 & 7 \\ 10 & 15 & 35 \end{pmatrix}$$

(d)

$$\begin{bmatrix} 2 & 3 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$

$1\times 3 \quad 3\times 1$

$$= (6+3+35) = (44)$$

What is another name for this type of matrix multiplication (in Question 2d)?

scalar matrix eg. $A = (2)$, 1×1; $B = (2\ 4\ 6)$, 1×3

$2B = 2(2\ 4\ 6) = (4\ 8\ 12)$ | $AB = (2)(2\ 4\ 6) = (4\ 8\ 12)$

3. Rewrite the systems of equations in Problem Set Two from Questions 7a and b, in matrix form. Hence write down the coefficient and augmented matrices for these systems.

(a)

$$x + y = 5$$
$$2x + 3y = 1$$

matrix form

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

Coefficient matrix $A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$

Augmented matrix : $\left(\begin{array}{cc|c} 1 & 1 & 5 \\ 2 & 3 & 1 \end{array}\right)$

(b)

$$x + 2y - z = 6$$
$$2x + 5y - z = 13$$
$$x + 3y - 3z = 4$$

matrix form

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ 1 & 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 13 \\ 4 \end{pmatrix}$$

Coefficient Matrix $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ 1 & 3 & -3 \end{pmatrix}$

Augmented Matrix : $\left(\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 2 & 5 & -1 & 13 \\ 1 & 3 & -3 & 4 \end{array}\right)$

Gaussian Elimination

4. Repeat the row-operations part of Question 7b from Problem Set Two using matrix notation (should be easy).

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 2 & 5 & -1 & 13 \\ 1 & 3 & -3 & 4 \end{array}\right) \xrightarrow[R_3 \leftarrow R_1 - R_3]{R_2 \leftarrow R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 2 & 2 \end{array}\right)$$

$$\xrightarrow{R_3 \leftarrow R_1 + R_3} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 3 \end{array}\right)$$

echelon form + back sub.

$R_3 \to z = 1$

$R_2 \to y = 0$

$R_1 \to x = 7$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 1 \end{pmatrix}$$

Reduced echelon form $\to$ direct sol'n

$$\xrightarrow{R_3 \leftarrow \frac{1}{3}R_3} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow[R_1 \leftarrow R_1 + R_3]{R_2 \leftarrow R_2 - R_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 \leftarrow R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right) \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 1 \end{pmatrix}$$

5. Consider the system of linear equations

$$x + y = 5$$
$$2x + 3y = 1$$

a. Write down the coefficient matrix and the augmented matrix of the system.

b. Use Gaussian elimination to bring the augmented matrix to row echelon form.

c. Write the augmented matrix in reduced row echelon form.

d. What is the rank of the coefficient matrix?

e. Identify the leading variables and the free variables.

f. Solve the system.

g. Give a geometrical interpretation of your answer; include a sketch to help you.

Tut#2 ⇒ Q7(a)

(a) coeff matrix $A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$

Augmented matrix $\left(\begin{array}{cc|c} 1 & 1 & 5 \\ 2 & 3 & 1 \end{array}\right)$

(b) $\xrightarrow{R_2 \leftarrow 2R_1 - R_2} \left(\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & -1 & 9 \end{array}\right)$

(c) $\xrightarrow[R_2 \leftarrow -R_2]{R_1 \leftarrow R_1 + R_2} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left(\begin{array}{cc|c} 1 & 0 & 14 \\ 0 & 1 & -9 \end{array}\right) \end{array}$

(d) rank(A) = #(pivots) = 2

rank(A) = total # variables ⇒ unique sol'n

(e) Leading variables : x, y

Free variable : None

(f) Echelon form $\left(\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & -1 & 9 \end{array}\right)$

Back subs

$R_2 \to y = -9$

$R_1 \to x - 9 = 5$

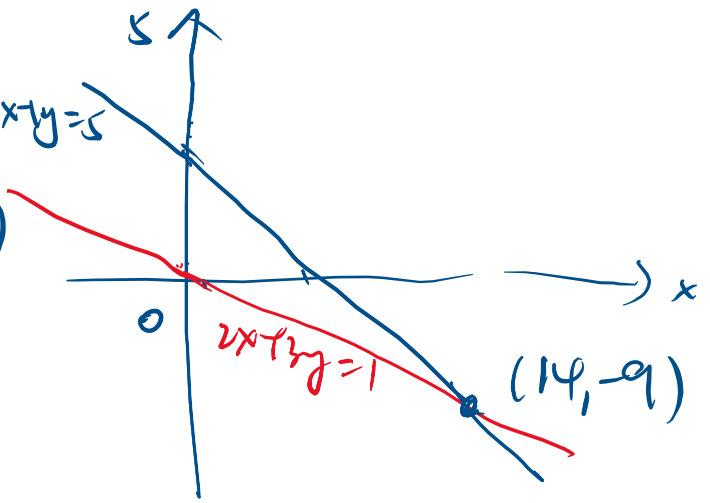
$x = 14$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ -9 \end{pmatrix}$

Reduced echelon form $\left(\begin{array}{cc|c} 1 & 0 & 14 \\ 0 & 1 & -9 \end{array}\right)$

Direct sol'n ⇒ $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ -9 \end{pmatrix}$

(g) Two lines have intersect at (14, −9)



when reach echelon form

(a) unique sol'n

△ cannot be zero

$\left(\begin{array}{ccc|c} \circ & & & \\ 0 & 0 & \triangle & \end{array}\right)$

6. Consider the following system of linear equations

$$3x - y + 2z = 3$$
$$x + 2y - z = 2$$
$$2x - 3y + az = b$$

Find conditions on a and b ($a, b \in R$) such that the system has

(b) No sol'n

△ must be zero, ☆ cannot zero

$\left(\begin{array}{ccc|c} \circ & & & \\ 0 & 0 & \triangle & ☆ \end{array}\right)$

(c) Infinite sol'n

△ & ☆ must be zero.

$\left(\begin{array}{ccc|c} \circ & & & \\ 0 & 0 & \triangle & ☆ \end{array}\right)$

$$\left(\begin{array}{ccc|c} 3 & -1 & 2 & 3 \\ 1 & 2 & -1 & 2 \\ 2 & -3 & a & b \end{array}\right) \xrightarrow[R_3 \leftarrow R_3 - 2R_2]{R_2 \leftarrow 3R_2 - R_1}$$

$$\left(\begin{array}{ccc|c} 3 & -1 & 2 & 3 \\ 0 & 7 & -5 & 3 \\ 0 & -7 & a+2 & b-4 \end{array}\right) \xrightarrow{R_3 \leftarrow R_2 + R_3} \left(\begin{array}{ccc|c} 3 & -1 & 2 & 3 \\ 0 & 7 & -5 & 3 \\ 0 & 0 & a-3 & b-1 \end{array}\right) \text{ echelon form}$$

a. No solution.

b. One solution.

c. Infinitely many solutions.

(a) No sol'n / Last row ⇒ 0 0 0 ≠0

$a-3=0$ and $b-1 \neq 0$

$a=3$ and $b \neq 1$

(b) one sol'n Last row ⇒ 0 0 ≠0 Anything

$a-3 \neq 0$ $\quad b-1 \in \mathbb{R}$

$a \neq 3$ $\quad b \in \mathbb{R}$

(c) Infinite Sol'n Last row ⇒ 0 0 0 0

$a-3=0$ and $b-1=0$

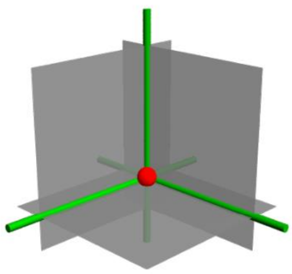
$a=3$ and $b=1$

7. In how many different ways (or combinations) can three planes intersect?
Make a quick sketch of each of these.

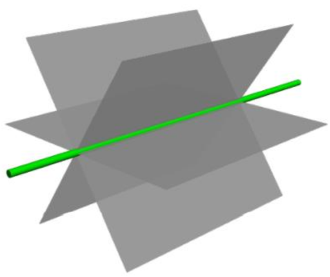
There are total of 8 possibilities, somehow will be still ended up ONLY three situations. (See attachments in the Teams – channel week 4)

No point of intersection (no solution)

One point of intersection (unique solution)



Intersection in a common line (infinite solutions)



8. If a system of equations has maximal rank, what does this tell us about the number of solutions to the system?

rank(A) has the value equal to the total # of variable(s) in the system eqs

rank(A) = #(pivots)

(maximal) rank(A) = total # of variable(s) ⇒ unique sol'n

rank(A) < Total # of variable(s) ⇒ ① No sol'n or ② infinite sol'n

9. Solve the following system of equations

$$x + 2y + 2z + 3w = 3$$
$$2x + 4y + 4z + 7w = 5$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 3 \\ 2 & 4 & 4 & 7 & 5 \end{array}\right)$$

$R_2 \leftarrow 2R_1 - R_2$

$$\begin{array}{cccc} x & y & z & w \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 3 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right)$$ echelon form

Leading variables: x, w
Free variables: y, z

Rank(A) = #(pivots) = 2
Rank(A) = 2 < 4

Let $y = \alpha$, $z = \beta$ (Free parameters)

* Back subs ⇒ $w = -1$

$R_1 \Rightarrow x + 2\alpha + 2\beta - 3 = 3$

$x = 6 - 2\alpha - 2\beta$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 6 - 2\alpha - 2\beta \\ \alpha \\ \beta \\ -1 \end{pmatrix}$$

10. Solve the following system

$$\begin{aligned} x+y+z &= 0 \\ -2x+5y+2z &= 0 \\ -7x+7y+z &= 0 \end{aligned}$$

homogeneous
passes through origin

What is the geometric interpretation of this solution?

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 0 \\ -7 & 7 & 1 & 0 \end{array}\right) \xrightarrow[R_3 \leftarrow 7R_1 + R_3]{R_2 \leftarrow 2R_1 + R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 0 \\ 0 & 14 & 8 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 \leftarrow 2R_2 - R_3} \begin{array}{cccc} x & y & z & \\ \end{array} \left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 0 \\ 0 & \boxed{7} & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \text{Infinite soln.}$$

z: free variable

Let $z = \alpha$

$y = -\frac{4}{7}\alpha$

$x = -\frac{3}{7}\alpha$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{3}{7}\alpha \\ -\frac{4}{7}\alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -3/7 \\ -4/7 \\ 1 \end{pmatrix}$$

passes through Origin