MAT1830 - Discrete Mathematics for Computer Science
Tutorial Sheet #8 and Additional Practice Questions

Tutorial Questions

1. A fair coin is flipped three times. Let X be the number of times heads is flipped. Find the probability distribution of X.

2. An ultrasound shows that a woman is pregnant with twins. About three in every ten pairs of twins worldwide is a pair of identical twins.

   30% => 3/10 chance

   (a) The ultrasound does not give any further information. The woman asks her doctor what the chance is that her twins are identical. What should the doctor answer? 3 in every 10 pairs

   (b) The ultrasound also reveals that both of the twins are male. The woman asks her doctor what the chance is that her twins are identical. What should the doctor answer? 15/100 = 15% Chance

   (Assume that 50% of pairs of identical twins are two females and 50% are two males. Assume that 25% of pairs of non-identical twins are two females, 50% are one female and one male, and 25% are two males.)

   (This question approximates real life, but it's a simplification. Real life is more complicated and interesting: using IVF changes things, twinning rates vary in different regions, a significant number of people are intersex, etc.)

3. (a) Find the expected value and variance of a random variable X with probability distribution given by the table below.

x	0	1	2
$\Pr(X = x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

   (b) Let p and q be real numbers. A random variable Y has expected value 2 and its probability distribution is given by the table below.

y	0	1	2	3
$\Pr(Y = y)$	p	$\frac{1}{12}$	$\frac{1}{3}$	q

   Find p and q.

4. A binary string of length 8 is chosen uniformly at random. Let X be the number of 1s in the string and Y be the number of 0s in the string.

   (a) Are X and Y independent random variables?

   (b) Let $Z = X + Y$. Find the probability distribution for Z.

5. (a) Invent an example of a random variable X such that $\Pr(X = E[X]) = 0$.

   (b) Invent an example of a random variable Y such that E$[Y]$ is negative but Y is extremely likely to be positive.

   (c) Invent an example of a random variable Z that is never negative, is not always 0, and has a high probability of being at least 3E$[Z]$ (where "high" means as high as you can make it).

(See over for practice questions.)

Practice Questions

1. (a) A number X is chosen uniformly at random from the set $\{100, 101, \ldots, 199\}$.
   What is $\mathrm{E}[X]$? What is $\mathrm{Var}[X]$?

   (b) A number Y is chosen uniformly at random from the set $\{101, 102, \ldots, 200\}$.
   What is $\mathrm{E}[Y]$? What is $\mathrm{Var}[Y]$?

2. On a game show there are three identical looking boxes. One contains a prize and the other two are empty. The contestant doesn't know which box contains the prize, but the host does. The game goes like this.

   - The contestant is asked to choose one of the boxes – this box is not opened yet, though.
   - The host then opens one of the two boxes that the contestant did not choose. She never opens the box with the prize.
   - The contestant is then given the option to "stay" with the box they originally chose or to "switch" to the other unopened box. They win if their final choice contains the prize.

   What is the contestant's probability of winning if they stay? What if they switch?

   (To find more takes on this question than you could ever want to read, google "Monty Hall problem".)

3. A Bayes spam filter is being trained on a library of spam emails and a library of legitimate emails. It finds that 10% of the spam emails contain the word "winner" but only 1% of the legitimate emails do. The filter assumes that 53% of emails are spam. Use Bayes' theorem to calculate probability that an email is spam given that it contains "winner".

   (Bayes spam filters are a thing – google them. That 53% of emails are spam is in line with recent statistics, but the "winner" percentages are just made up.)

4. Let $S = \{1, 2, \ldots, 10\}$. Out of all the equivalence relations on S that have exactly 2 equivalence classes, an equivalence relation is chosen uniformly at random. What is the probability that it has two equivalence classes of size 5?

1) Fair coin flipped 3 times **BINOMIAL THEOREM**

50% = heads ($1/2$)
50% = tails ($1/2$)

X	0	1	2	3
Pr(X)	1/8	3/8	3/8	1/8

$X=0 \Rightarrow {}^3C_0\,(1/2)^3(1/2)^0 = 1/8$
$X=1 \Rightarrow {}^3C_1\,(1/2)^2(1/2)^1 = 3/8$
$X=2 \Rightarrow {}^3C_2\,(1/2)^1(1/2)^2 = 3/8$
$X=3 \Rightarrow {}^3C_3\,(1/2)^0(1/2)^3 = 1/8$

$1/8 + 3/8 + 3/8 + 1/8 = 1$

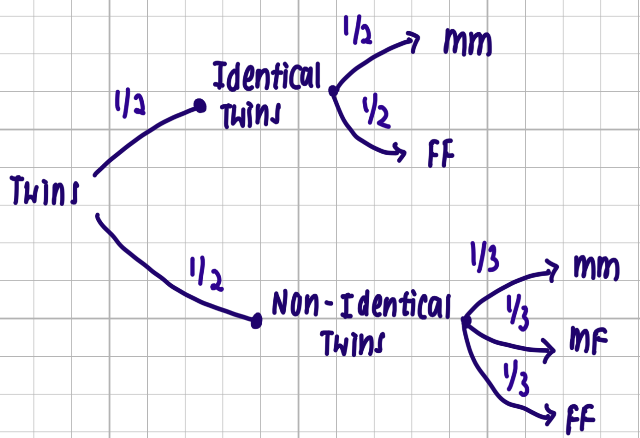
$\therefore X \sim Bin(3, 1/2)$

SOLVE IT BY BASE THEOREM

2) 3 in every 10 pairs is identical twins **CONDITIONAL PROBABILITY**

(a) Pregnant women's probability of having identical twins

* Identical twins can be both males (mm) or both females (FF)
* Non-identical twins can be mm, MF, FF



BIG NO NO !

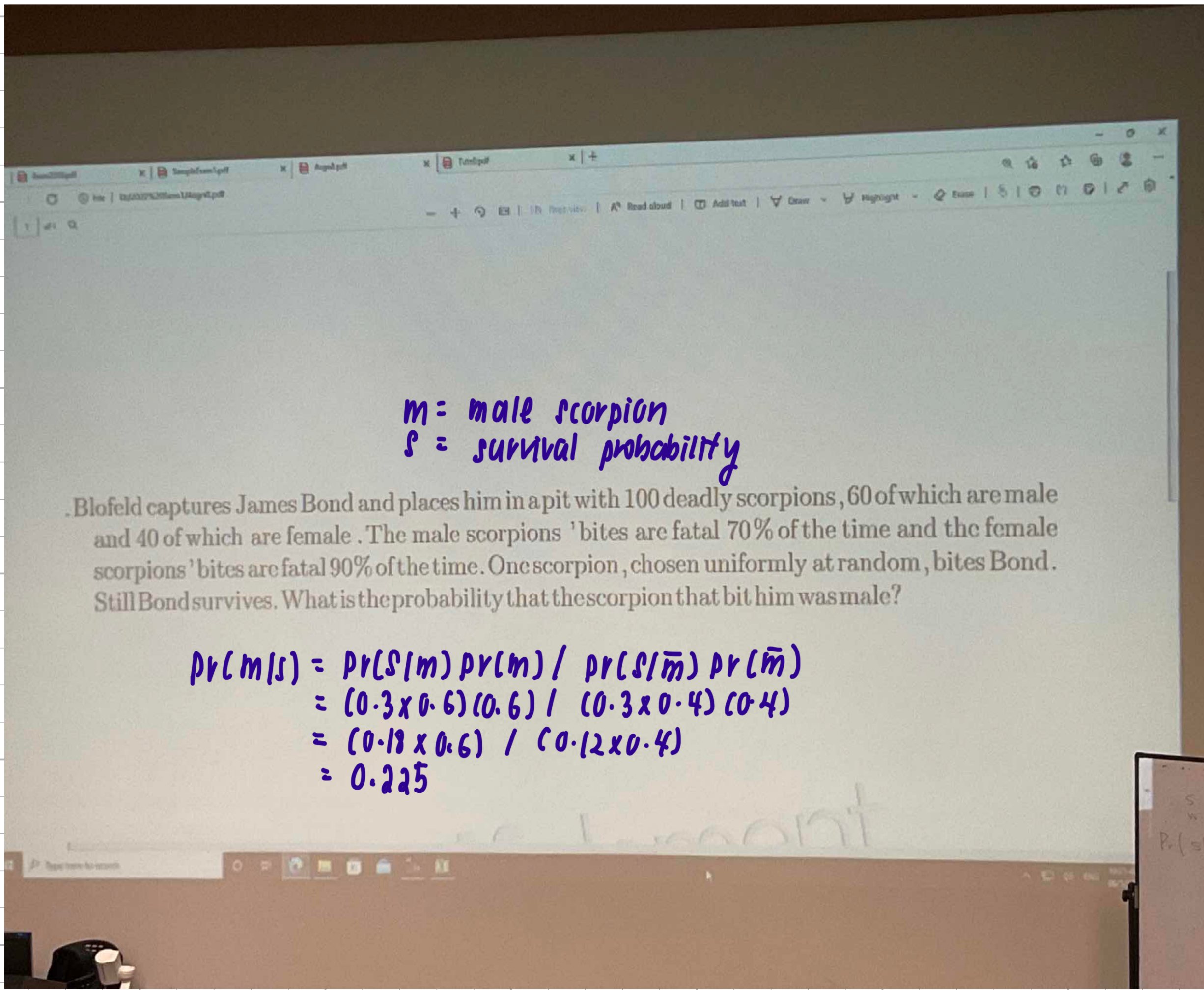
$Pr(\text{Identical twins} / mm)$

$$= \frac{Pr(\text{Identical twins} \cap mm)}{Pr(\text{Identical twins} \cap mm) + Pr(\text{Non-Identical Twins} \cap mm)}$$

$$= \frac{Pr(mm/ID) + Pr(ID)}{Pr(mm|ID) + Pr(ID) + Pr(mm + \overline{ID}) + Pr(\overline{ID})}$$

$$= \frac{0.5 \times 0.3}{(0.5 \times 0.3) + (0.25 \times 0.7)}$$

(b) Pregnant women's probability of having identical twins that's male gender



3) (a) Find $E(X)$ & $Var(X)$

x	0	1	2
Pr	1/2	1/3	1/6

$u: E(X) = x_i p_i$

$Var(X) = p_i(x_i - u)^2$

$E(X) = 0 + 1/3 + 2/6$

$= 0 + 1/3 + 1/3$

$= 2/3$

$Var(X) = 1/2(0 - 2/3)^2 + 1/3(1 - 2/3)^2 + 1/6(2 - 2/3)^2$

(b)

y	0	1	2	3
Pr(Y)	2/12	1/12	1/3	q

$p = 2/12$

$q = 1 - (2/12 + 1/12 + 4/12)$

$q = 1 - 7/12$

$q = 5/12$

$\therefore p = 2/12;\ q = 5/12$

$Var(x) = E(y^2) = [E(y)]^2$

(6) (a) Let p and q be real numbers. A random variable X has expected value -1 and its probability distribution is given by the table below.

x	-4	0	1	2
$\Pr(X = x)$	p	q	$\frac{1}{3}$	$\frac{1}{6}$

(i) Find p and q. [2]

(ii) Find the variance of X. [2]

(b) A shipwrecked sailor has a 10% chance of being rescued by a passing ship on any given day. Let

Let X and Y be independent random variables with distributions given in the tables below.

x	0	1	2
$\Pr(X = x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

y	0	1	2
$\Pr(Y = y)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

Let Z be the random variable defined by $Z = XY$.

(a) Give the probability distribution of Z.

(b) Find $\Pr(X = 1 \wedge Z = 2)$.

(c) Using your answer to (b), find $\Pr(X = 1 \mid Z = 2)$.

(d) Using your answer to (b), show that X and Z are not independent random variables.