MAT1841
Continuous Mathematics for Computer Science

Applied Class Questions, Semester 1 2022

Problem Set One: Linear Algebra - Vectors, Dot Product, Cross Product

Vectors

1. Consider the vectors $\underset{\sim}{u} = (2, -2, 3)$ and $\underset{\sim}{v} = (3, 2, -2)$.

a. Compute the following:

$$\underset{\sim}{u} + \underset{\sim}{v}$$
$$\underset{\sim}{u} + 2\underset{\sim}{v}$$
$$\underset{\sim}{u} - \underset{\sim}{v}$$
$$\underset{\sim}{v} - \underset{\sim}{u}$$

b. Show that $2\underset{\sim}{u} + 2\underset{\sim}{v} = 2(\underset{\sim}{u} + \underset{\sim}{v})$.

c.i. Find the length of $\underset{\sim}{u}$.
ii. Find the length of $\underset{\sim}{v}$.
iii. Find a unit vector in the direction of $\underset{\sim}{u}$.

Vector Dot Product

2. For each of the pairs of vectors given below, calculate the vector dot product and the angle θ between the vectors.

a. $\underset{\sim}{v} = (3, 2, -2)$ and $\underset{\sim}{w} = (1, -2, -1)$

b. $\underset{\sim}{v} = (0, -1, 4)$ and $\underset{\sim}{w} = (4, 2, -2)$

c. $\underset{\sim}{v} = (2, 0, 2)$ and $\underset{\sim}{w} = (-3, -2, 0)$

What does it mean (geometrically) when the dot product is negative?

3. Given the two vectors $\underset{\sim}{v} = (\cos(\theta), \sin(\theta), 0)$ and $\underset{\sim}{w} = (\cos(\phi), \sin(\phi), 0)$, use the dot product to derive the trigonometric identity

$$\cos(\theta - \phi) = \cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi).$$

4. Use the dot product to determine which of the following vectors are perpendicular to one another: $\underset{\sim}{u} = (3, 2, -2), \underset{\sim}{v} = (1, 2, -2), \underset{\sim}{w} = (2, -1, 2)$.

5. Find the scalar projection of the vector $\underset{\sim}{v}$ in the direction of $\underset{\sim}{w}$ (i.e. find v_w) for the following pairs of vectors:

a. $\underset{\sim}{v} = (3, 2, -2)$, $\underset{\sim}{w} = (1, -2, -1)$
b. $\underset{\sim}{v} = (0, -1, 4)$, $\underset{\sim}{w} = (4, 2, -2)$
c. $\underset{\sim}{v} = (2, 0, 2)$, $\underset{\sim}{w} = (-3, -2, 0)$

What does it mean (geometrically) when the scalar projection is negative?

6. Find the vector projection of the vector $\underset{\sim}{v}$ in the direction of $\underset{\sim}{w}$ (i.e. find $\underset{\sim}{v}_w$) for the pairs of vectors in Question $5(a - c)$.

Vector Cross Product

7. For each pair of vectors in Question $5(a - c)$:

Calculate the vector cross product, $\underset{\sim}{v} \times \underset{\sim}{w}$, and find the area of the parallelogram spanned by the two vectors $\underset{\sim}{v}$ and $\underset{\sim}{w}$.

What can you say about vectors $\underset{\sim}{v}$ and $\underset{\sim}{w}$ if $|\underset{\sim}{v} \times \underset{\sim}{w}| = 0$?

8. For the vectors $\underset{\sim}{v} = (3, 2, -1)$ and $\underset{\sim}{w} = (2, 1, 3)$, verify that

a. $\underset{\sim}{v} \times \underset{\sim}{w} = -\underset{\sim}{w} \times \underset{\sim}{v}$.

b. $\underset{\sim}{v} \times \underset{\sim}{v} = 0$.

9. For the vectors $\underset{\sim}{u} = (1, -2, -1)$, $\underset{\sim}{v} = (3, 2, -1)$ and $\underset{\sim}{w} = (2, 1, 3)$, find

a. $(\underset{\sim}{u} \times \underset{\sim}{v}).\underset{\sim}{w}$ and $(\underset{\sim}{u} \times \underset{\sim}{w}).\underset{\sim}{v}$.

b. $\underset{\sim}{u} \times (\underset{\sim}{v} + \underset{\sim}{w})$ and $(\underset{\sim}{u} \times \underset{\sim}{v}) + (\underset{\sim}{u} \times \underset{\sim}{w})$.

What do you notice? Do you think this is true for all vectors $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$?

10. For the vectors $\underset{\sim}{u} = (1, 0, 0)$, $\underset{\sim}{v} = (0, 1, 0)$ and $\underset{\sim}{w} = (0, 0, 1)$, find $\underset{\sim}{u} \times \underset{\sim}{v}$.

Without performing any calculations, what would you expect $\underset{\sim}{v} \times \underset{\sim}{w}$ to be? What about $\underset{\sim}{u} \times \underset{\sim}{w}$?

Problem Set Two: Linear Algebra - Lines and Planes, Systems of Equations

Lines and Planes

1. Consider the points $(1, 2, -1)$ and $(2, 0, 3)$.

a. Find a vector equation of the line through these points in parametric form.

b. Find the distance between this line and the point (1, 0, 1).

2. Find an equation of the plane that passes through the points $(1, 2, -1)$, $(2, 0, 3)$ and $(-1, -1, 0)$.

3. Consider a plane defined by the equation $3x + 4y - z = 2$ and a line defined by the following vector equation (in parametric form)

$$x(t) = 2 - 2t, \qquad y(t) = -1 + 3t, \qquad z(t) = -t.$$

a. Find the point where the line intersects the plane. (Hint: Substitute the parametric form into the equation of the plane.)

b. Find a normal vector to the plane.

c. Find the angle at which the line intersects the plane. (Hint: Use the dot product.)

4. Find the minimum distance between the two lines defined by

$$x(t) = 1 + t, \qquad y(t) = 1 - 3t, \qquad z(t) = -2 + 2t$$

and

$$x(s) = 3s, \qquad y(s) = 1 - 2s, \qquad z(s) = 2 - s$$

(Hint: Use scalar projection as demonstrated in the lecture notes. Alternatively, first define the lines within parallel planes.)

5. Find the distance between the parallel planes defined by the equations $2x - y + 3z = -4$ and $2x - y + 3z = 24$.

6. Consider two planes defined by the equations $3x + 4y - z = 2$ and $-2x + y + 2z = 6$.

(a) Find where the planes intersect the x, y and z axes.

(b) Find normal (i.e. perpendicular) vectors for the planes.

(c) Find an equation of the line defined by the intersection of these planes. (Hint: Use the normal vectors to define the direction of the line.)

(d) Find the angle between these two planes.

Systems of Equations

7. Solve each of the following linear systems using elementary row operations and back substitution.
(a)

$$\begin{aligned} x + y &= 5 \\ 2x + 3y &= 1 \end{aligned}$$

(b)

$$\begin{aligned} x + 2y - z &= 6 \\ 2x + 5y - z &= 13 \\ x + 3y - 3z &= 4 \end{aligned}$$

8. Find all possible solutions for the following (under-determined) system of equations

$$\begin{aligned} x + 2y - z &= 6 \\ 2x + 5y - z &= 13 \\ 2x + 4y - 2z &= 12 \end{aligned}$$

(Hint: You have two equations but three unknowns. You will need to introduce one free parameter.)

9. Find all possible solutions for the system (yes, this system has only one equation) of equations

$$x + 2y - z = 6$$

(Hint: You have one equation but three unknowns. You will need to introduce two free parameters.)

10. Consider two planes defined by $3x+4y-z=2$ and $-2x+y+2z=6$. Find the intersection of the planes using linear systems (i.e. solve the equations for x, y and z, without taking the cross product of the vectors normal to each plane).

Problem Set Three: Matrices and Gaussian Elimination

Matrices

1. Evaluate each of the following matrix operations.

(a)

$$2\begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & -4 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 1 \\ 1 & 2 \end{bmatrix}$$

2. Evaluate each of the following matrix operations, if possible. If not possible, state why.

(a)

$$\begin{bmatrix} 2 & -1 \\ 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 1 \\ 1 & 2 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}\begin{bmatrix} 2 & 3 & 7 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 2 & 3 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$

What is another name for this type of matrix multiplication (in Question $2d$)?

3. Rewrite the systems of equations in Problem Set Two from Questions $7a$ and b, in matrix form. Hence write down the coefficient and augmented matrices for these systems.

Gaussian Elimination

4. Repeat the row-operations part of Question $7b$ from Problem Set Two using matrix notation (should be easy).

5. Consider the system of linear equations

$$\begin{aligned} x + y &= 5 \\ 2x + 3y &= 1 \end{aligned}$$

a. Write down the coefficient matrix and the augmented matrix of the system.

b. Use Gaussian elimination to bring the augmented matrix to row echelon form.

c. Write the augmented matrix in reduced row echelon form.

d. What is the rank of the coefficient matrix?

e. Identify the leading variables and the free variables.

f. Solve the system.

g. Give a geometrical interpretation of your answer; include a sketch to help you.

6. Consider the following system of linear equations

$$\begin{aligned} 3x - y + 2z &= 3 \\ x + 2y - z &= 2 \\ 2x - 3y + az &= b \end{aligned}$$

Find conditions on a and b $(a, b \in R)$ such that the system has

a. No solution.

b. One solution.

c. Infinitely many solutions.

7. In how many different ways (or combinations) can three planes intersect? Make a quick sketch of each of these.

8. If a system of equations has maximal rank, what does this tell us about the number of solutions to the system?

9. Solve the following system of equations

$$\begin{aligned} x + 2y + 2z + 3w &= 3 \\ 2x + 4y + 4z + 7w &= 5 \end{aligned}$$

10. Solve the following system

$$\begin{aligned} x + y + z &= 0 \\ -2x + 5y + 2z &= 0 \\ -7x + 7y + z &= 0 \end{aligned}$$

What is the geometric interpretation of this solution?

Problem Set Four: Matrices - Inverse, Transpose and Determinant

Matrix Inverse

1. Compute the inverse of the following matrices, or state why the inverse does not exist.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 3 \\ 1 & 3 \\ 1 & 2 \end{bmatrix}$$

Verify that $A^{-1}A = I$ and $AA^{-1} = I$. What is $(A^{-1})^{-1}$?

2. Show that the matrix $B = \begin{bmatrix} 2 & 5 \\ 6 & 15 \end{bmatrix}$ is not invertible. What are the implications of this for the following linear system?

$$\begin{aligned} 2x + 5y &= 7 \\ 6x + 15y &= 8 \end{aligned}$$

3. Calculate B^{-1}, B^2 and $(B^2)^{-1}$ for the matrix $B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$.

Verify that $(B^2)^{-1} = B^{-1}B^{-1}$.

Matrix Transpose

4a. For the matrix $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 3 \\ 1 & 2 & -1 \end{bmatrix}$ compute A^T. What is $(A^T)^T$?

4b. For the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & 3 \\ 1 & 2 \end{bmatrix}$ compute B^T. Also compute BB^T and B^TB.

5. Consider the following matrices

$$A = \begin{bmatrix} 4 & -2 \\ 1 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$$

Compute A^T, B^T and $A^T B^T$. Verify that $(AB)^T = B^T A^T$.

6. For the matrices A and B from Question 5, compute the following:

(a) $(A+B)^T$

(b) $A^T + B^T$

(c) $A + A^T$ and $B + B^T$

(d) $(A+A^T)^T$ and $(B+B^T)^T$

(e) $(A^T)^{-1}$

(f) $(A^{-1})^T$

(g) $(AB)^{-1}$

(h) $B^{-1}A^{-1}$

Matrix Determinant

7. Consider the following matrices

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 3 \\ 1 & 2 & -1 \end{bmatrix}$$

a. Compute the determinant for each matrix.

b. Compute AB and find $\det(AB)$.

c. Compute BA and find $\det(BA)$.

d. What is $\det(A)\det(B)$?

e. Compute A^T. What is $\det(A^T)$?

f. Compute A^{-1}. What is $\det(A^{-1})$?

g. What is $\det(A)\det(A^{-1})$?

8. Write down the coefficient matrix, C, for each of the under determined systems. What do you notice about the determinant of each?

a.

$$\begin{aligned} x + 2y - z &= 6 \\ x + 3y + 0z &= 7 \\ 2x + 5y - z &= 13 \end{aligned}$$

b.

$$x + 2y - z = 6$$

9. Consider the matrix C from Question 7.

a. Use the Gauss-Jordan method to find C^{-1}. Check your answer by showing $CC^{-1} = I$.

b. You have calculated $\det(C)$ in Question 7. Use this, and the formula

$$a_{ji} = (-1)^{i+j} \frac{\det S_{ij}}{\det C}$$

to compute C^{-1}, where S_{ij} is the 2×2 minor matrix of C with column i and row j removed.

10. Use Cramer's rule to solve the following system of equations:

$$\begin{aligned} 2x + 3y - z &= 4 \\ x + y + 3z &= 1 \\ x + 2y - z &= 3 \end{aligned}$$

Problem Set Five: Calculus - Differentiation

1. The definition of the derivative is given below.

$$\frac{\mathrm{d}f}{\mathrm{d}x} := \lim_{\Delta x \to 0} \left(\frac{f(x+\Delta x) - f(x)}{\Delta x} \right)$$

Use this definition (i.e. use **first principles**) to find $\frac{\mathrm{d}f}{\mathrm{d}x}$ when:

a. $f(x) = x^3 + 2x$

b. $f(x) = \frac{1}{x^2}$

2. Differentiate the following polynomial functions:

(a) $f(x) = 5x^2 - 2x + 1$

(b) $f(x) = (4x - 2)^7$

(c) $f(x) = (3x^2 - x + 1)^3$

(d) $f(x) = (2x + 4)^7 + (3x - 2)^5$

3. Find the slope of the curve $y = x^2 + 6$ at the point $(x, y) = (3, 15)$. Use this to find the equation of the tangent line at this point. Make a quick sketch of $y = x^2 + 6$ and the tangent, to support your answer.

4. State and use the appropriate rules of differentiation (i.e. Product rule, Quotient rule, Chain rule) to obtain $\frac{\mathrm{d}f}{\mathrm{d}x}$ for the following functions:

(a) $f(x) = (2x + 4)^7(3x - 2)^5$

(b) $f(x) = (5x^2 - 2x + 1)(4x - 2)^7$

(c) $f(x) = \sqrt{1 + 2x^2}$

(d) $f(x) = \frac{1}{1 + x^2}$

(e) $f(x) = \frac{x}{1 + x^2}$

5. Find the gradient of the slope of the function $f(x) = (4x - 2)^7$ at the point $x = 0$. Use this to find the equation of the tangent line at this point.

6. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when:

(a) $y = \dfrac{x-3}{x-2}$

(b) $y = \dfrac{1}{\sqrt{x}}$

(c) $y = \left(\dfrac{1}{\sqrt{x}} + \sqrt{x}\right)^2$

(d) $y = \dfrac{x}{\sqrt{x^2 - 1}}$

7. Find the slope of the tangent line for the function $y = \dfrac{x-3}{x-2}$ at the point $x = 0$. Use this to find the equation of the tangent line at this point. Make a sketch to help support your answer.

8. For the function $f(x) = 5x^2 - 2x + 1$, find the value of c such that $f'(c) = 0$. Find the value of $f(x)$ at c, i.e. find $f(c)$. What is the significance of the point $(c, f(c))$? Use this point to sketch the graph of $f(x)$.

9. Find the interval/s where the function $f(x) = x^3 - 2x$ is (a) increasing, (b) decreasing, and (c) constant.

10. Identify the local maxima and minima for the function $y = x^3 + 4x^2 + 5x + 2$ over the interval R.

Problem Set Six: Trigonometric, Exponential and Logarithmic Function Derivatives

Trigonometric Functions

1. Differentiate the following trigonometric functions:

(a) $f(x) = \sin(3x - 2)$

(b) $g(x) = \cos^2(3x)$

(c) $h(x) = x\sin(x)$

(d) $f(z) = \tan^3(z)$

Exponential Functions

2. Find the first derivative with respect to x of the following exponential functions:

(a) $f(x) = e^{2x}$

(b) $f(x) = e^{x^2+x}$

(c) $f(x) = (3x - 2)e^{-x}$

(d) $f(x) = \dfrac{e^x}{1 + e^x}$

Logarithmic Functions

3. Differentiate the following logarithmic functions with respect to x:

(a) $f(x) = \ln(3x - 2)$

(b) $g(x) = \ln\left(\dfrac{x-3}{x-2}\right)$

(c) $h(x) = \dfrac{1}{x}\ln x$

Inverse Functions

4. Make a quick sketch of the function $f(x) = 2x^3 - 5$. Find the derivative of $f(x)$. Also find $f^{-1}(x)$ (the inverse of $f(x)$). Plot $f^{-1}(x)$ on the same axis as $f(x)$. Find the derivative of $f^{-1}(x)$. (Show that for $y = 2x^3 - 5$, $\dfrac{dy}{dx} = \dfrac{1}{dx/dy}$ which you can see from your sketch of the two functions.)

5. By using the definition of the inverse function, establish the identity

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}.$$

6. Show that $\dfrac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}(x) = \dfrac{1}{1+x^2}$.

7. Find $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ for these inverse circular functions:

(a) $f(x) = \sin^{-1}\left(\dfrac{x}{2}\right)$

(b) $f(x) = \tan^{-1}(3x)$

(c) $f(x) = \sqrt{(1+x^2)}\,\tan^{-1} x$

Higher Order Derivatives

8. Find $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ when y is given by:

a. $y = x^3 + 3x^2 - 5x + 1$

b. $y = \sqrt{(1+x^2)}$

9 Find $\dfrac{d^2y}{dx^2}$ for the function $y = \ln(x^2 + x + 1)$ (Note: $\ln(x) = \log_e(x)$.)

10. Find $f^{(1)}(x)$, $f^{(2)}(x)$ and $f^{(3)}(x)$ for the following functions:

a. $f(x) = e^{3x}$. What is a general formula for $f^{(n)}(x)$?

b. $f(x) = \ln(x+2)$. Can you find a general formula for $f^{(n)}(x)$? (This one is a bit more of a challenge.)

Problem Set Seven: Parametric Curves and Parametric Differentiation

Parametric Curves

1. Find a parametric form of the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

That is, find functions $x(t)$, $y(t)$ for $0 \leq t < c$ such that every point on the ellipse is described by the given functions for at least one value of t (and ideally, to each point there is exactly one value of t).

2. What do the following parametric curves represent? Sketch each one separately.

(a) $x(v) = 3$, $y(v) = 4$, $z(v) = v$ for $-\infty < v < \infty$

(b) $x(u) = 5$, $y(u) = 3u - 2$, $z(u) = 5u + 1$ for $-1 < u < 1$

(c) $x(t) = 3t^2$, $y(t) = 4t$, $z(t) = 5$ for $1 \leq t \leq 7$

(d) $x(t) = 3\cos(t)$, $y(t) = 2\sin(t)$, $z(t) = 3t - 1$ for $0 \leq t < 2\pi$

(e) $x(w) = w\sin(w)$, $y(w) = w\cos(w)$, $z(w) = w$ for $0 < w < \infty$

(f) $x(t) = \sin(t)$, $y(t) = 0$, $z(t) = t$ for $0 < t < 2\pi$

3. By finding the value of the parameter t, show that each of the following curves passes through the given point.

(a) $x(t) = 4t - 1$, $y(t) = 7t + 3$, $z(t) = t^2 - 1$ for the point $(-1, 3, -1)$

(b) $x(t) = 2\cos(t)$, $y(t) = 3\sin(t)$, $z(t) = 2 + t$ for the point $(2, 0, 2)$

Show that the point $(-1, 0, 1)$ is on neither of the above curves.

Parametric Differentiation

4. Compute the derivative $\dfrac{dy}{dx}$ for each of the following curves at $t = 0$. You can assume that each curve is defined over the interval $-\pi < t < \pi$.

(a) $x(t) = \sin(t)$, $y(t) = \cos(t)\sin(t)$

(b) $x(t) = 2(t - \cos(t))$, $y(t) = \sin(t)$

(c) $x(t) = 3t^2 - t$, $y(t) = 5t - 3$

(d) $x(t) = 5t - 3$, $y(t) = 3t^2 - t$

5. The equations $x = t\cos(t)$ and $y = t\sin(t)$ are the parametric equations for a spiral curve known as the Spiral of Archimedes.[1] Find $\frac{dy}{dx}$ in terms of t. Can you find the Cartesian form of this spiral? (You need to find a relation between x and y that does not involve t.)

6. Show that the following pair of curves intersect.

Curve A: $x(v) = 3$, $y(v) = 4$, $z(v) = v$, for $0 < v < 2$.
Curve B: $x(u) = 3 + \sin(u)$, $y(u) = 4 - u$, $z(u) = 1 - u$, for $-1 < u < 1$.

Hence construct a plane that is tangent to both curves at the point of intersection.

7. Consider the following parametric equations for the half circle:

$$x(t) = \cos(t) \qquad y(t) = \sin(t) \qquad \text{for } 0 \leq t \leq \pi$$

(a) Find $\frac{dy}{dt}$ and $\frac{dx}{dt}$.

(b) Rearrange $x^2 + y^2 = 1$ (for non negative values of y only) and find $\frac{dy}{dx}$.

(c) Show that $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt}$.

8. Find the equation of the tangent line to

$$x(t) = 4\sin(t) \qquad y(t) = 2\cos(t) \qquad \text{for } 0 \leq t \leq 2\pi \qquad \text{at } t = \frac{\pi}{3}.$$

Sketch both the curve and the tangent line.

[1] The Spiral of Archimedes was studied by Archimedes himself, around 225 BC. Notably this spiral can be used to trisect an angle, and square the circle (both of which are classical mathematics problems from early Greek times, along with the problem of doubling the cube). In polar form, the spiral has the equation $r = a\theta$.

9. Consider the parametric equations:

$$x(t) = \frac{1-t^2}{1+t^2}$$
$$y(t) = \frac{2t}{1+t^2}$$

(a) Show that this is equivalent to the equation of a circle $x^2 + y^2 = 1$.

(b) What is unusual about the point $(-1, 0)$ using the above parametrisation?

10a. Find the tangent vector to each of the following curves given by the position vector $\underset{\sim}{r}(t) = (x(t), y(t), z(t))$ where:

i. $x(t) = 3t^2 \qquad y(t) = 4t \qquad z(t) = 5$

ii. $x(t) = 3\cos(t) \qquad y(t) = 2\sin(t) \qquad z(t) = t$

10b. Let $\underset{\sim}{r}(t) = (x(t), y(t), z(t))$ be the position vector along some curve for $-\infty < t < \infty$. Suppose t has been chosen so that $1 = \underset{\sim}{r}' \cdot \underset{\sim}{r}'$ for all t. Show that $0 = \underset{\sim}{r}' \cdot \underset{\sim}{r}''$.

What does this tell you about the relationship of the vector $\underset{\sim}{r}''$ to the curve? (Hint: Use the Product rule.)

Problem Set Eight: Function Approximation using Taylor Series and Cubic Splines

Taylor Series

1. Calculate $f^{(1)}$, $f^{(2)}$, $f^{(3)}$ and $f^{(4)}$ for the function $f(x) = e^{-x}$. Now calculate the values of each of these derivatives at $x = 0$ and calculate $a_n = \dfrac{f^{(n)}(0)}{n!}$ to construct the first five partial sums of the Taylor series, $T_0(x)$, $T_1(x)$, $T_2(x)$, $T_3(x)$ and $T_4(x)$.

2. Construct a Taylor series for each of the following functions, centred at $x = 0$.

(a) $f(x) = \ln(1 - x)$

(b) $f(x) = \sin(x)$

(c) $f(x) = e^{-x}\sin(x)$

3. Construct the Taylor series (up to $T_3(x)$ is sufficient) for the function $f(x) = \sin^{-1}(x)$, centred at $x = 0$.

4. Find the first four non-zero terms of the Taylor series (about $x = 0$) for each of the following functions

(a) $f(x) = \cos(x)$

(b) $f(x) = \sin(2x)$

(c) $f(x) = e^{x}$

(d) $f(x) = \arctan(x)$

5. Use the results of the previous question to obtain the first two non-zero terms of the Taylor series (about $x = 0$) for the following functions

(a) $f(x) = \cos(x)\sin(2x)$

(b) $f(x) = e^{-x^2}$

(c) $f(x) = \arctan(\arctan(x))$

6. Compute the Taylor polynomial T_n, about the given point, for each of the following functions:

(a) $f(x) = e^x$, about $a = 1$.

(b) $f(x) = e^x$, about $a = -1$.

7. Compute the Taylor series, around $x = 0$, for $\log(1 + x)$ and $\log(1 - x)$. Hence obtain a Taylor series for $f(x) = \log\left(\dfrac{1+x}{1-x}\right)$.

Linear Approximation

8. Write down the linear approximation to $f(x) = \sqrt{1+x}$ at $x = 0$. Use this to find an approximation for $f(x)$ when $x = 1$. Is this a reasonable approximation for $\sqrt{2}$? Explain.

9. Write down the linear approximation to $f(x) = \sin(x)$ at $x = 0$. Sketch the graphs of $f(x)$ and $L(x)$ (the linear approximation) on the same set of axes. Is $L(x)$ a reasonable approximation for $f(x) = \sin(x)$? Explain.

Cubic Splines

10a. Find the cubic spline approximation for the function $f(x) = x + \frac{1}{x}$, using the points on the graph of $f(x)$ corresponding to $x_1 = \frac{1}{2}$, $x_2 = 1$, $x_3 = \frac{3}{2}$ and $x_4 = 2$.

10b. Check that the following conditions are met for the three cubic equations found above:

i. Interpolation condition: $y_i = \tilde{y}_i(x_i)$.

ii. Continuity of the function: $\tilde{y}_{i-1}(x_i) = y_i$.

iii. Continuity of the first derivative: $\tilde{y}'_{i-1}(x_i) = \tilde{y}'_i(x_i)$.

iv. Continuity of the second derivative: $\tilde{y}''_{i-1}(x_i) = \tilde{y}''_i(x_i)$.

Problem Set Nine: Integration, Area Under Curve, Trapezoidal Rule

Integration

1. Use a suitable substitution to integrate the following functions:

(a) $f(x) = 2x\cos(x^2 + 2)$

(b) $f(x) = 3x\sin(x^2 + 2)$

(c) $f(x) = 4x^2\sqrt{x^3 - 5}$

(d) $f(x) = \cos(x)e^{2\sin(x)}$

Check the above integrals are correct by differentiating each one.

2. Evaluate each of the following using integration by parts. Recall that

$$\int f\frac{dg}{dx}\,dx = fg - \int g\frac{df}{dx}\,dx$$

(a) $\int x\cos(x)\,dx$

(b) $\int xe^{-x}\,dx$

(c) $\int y\sqrt{y+1}\,dy$

(d) $\int x^2\ln(x)\,dx$

Check the above integrals are correct by differentiating each one.

3. Use integration by parts to find $\int \sin^2(x)\,dx$ and $\int \cos^2(x)\,dx$.

4. Use integration by parts twice to find $\int e^x\sin(x)\,dx$ and $\int e^x\cos(x)\,dx$.

5. Use a substitution and integration by parts to evaluate each of the following:

(a) $\int \cos(x)\sin(x)e^{\cos(x)}\,dx$

(b) $\int e^{2x}\cos(e^x)\,dx$

(c) $\int \pi\, e^{\sqrt{x}}\,dx$

6. Spot the error in the following calculation.

We wish to compute $\int \frac{1}{x}dx$. For this we will use integration by parts with $f = \frac{1}{x}$ and $g' = 1$. This gives us $f' = -\frac{1}{x^2}$ and $g = x$. Thus using $\int f\,g' = fg - \int g\,f'$ we find

$$\int \frac{1}{x}\,dx = 1 + \int \frac{1}{x}\,dx$$

and rearranging this gives $0 = 1$.

Area under the Curve

7. Find the area bounded by the graphs of $y = \sin^2(x)$ and $y = \cos^2(x)$ for $0 \leq x \leq \frac{\pi}{2}$ by using methods of integration.

Trapezoidal Rule

8a. Use the Trapezoidal rule to approximate $\int_0^4 x^2\,dx$ for the following:

i. $n = 1$.

ii. $n = 2$.

iii. $n = 4$.

iv. $n = 8$.

b. Compare each of these with the definite integral $\int_0^4 x^2\,dx$ and calculate the error in each case.

9. Use the Trapezoidal rule with $n = 4$ to find an approximate value of $\int_0^\pi x\cos(x)\,dx$. Use Question 2a to evaluate $\int x\cos(x)\,dx$ between $x = 0$ and $x = \pi$. How does this compare with your Trapezoidal approximation?

10a. Choose an appropriate n and use the Trapezoidal rule to estimate the value of $\int_0^1 \frac{4}{1+x^2}\,dx$.

b. How is this a fair approximation for π?

Problem Set Ten: Functions of Several Variables, Partial Derivatives, Tangent Planes

Functions of Several Variables

1. Sketch the graphs of the following functions of several variables:

(a) $f(x,y) = x^2 + y^2 - 2$

(b) $f(x,y) = 2 - x^2 - y^2$

(c) $f(x,y) = (x+2)^2 + (y-1)^2$

(d) $f(x,y) = \dfrac{(x+2)^2}{4} + \dfrac{(y-1)^2}{9}$

2. Identify the surface given by the following equations. Remember you are trying to combine the three equations into one equation involving x, y and z but not u and v.

$$x = 4u + 3v + 5 \qquad y = 2u + v + 1 \qquad z = u + v + 1$$

Partial Derivatives

3. For the function $f(u,v) = uv(1 - u^2 - v^2)$, evaluate $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$.

4. Evaluate $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ for each of the following functions:

(a) $f(x,y) = \cos(x)\cos(y)$

(b) $f(x,y) = \sin(xy)$

(c) $f(x,y) = \frac{\log(1+x)}{\log(1+y)}$

(d) $f(x,y) = \frac{x+y}{x-y}$

(e) $f(x,y) = e^x y$

5. If $f(x,y) = 2x^4 - 3x^3y^2 + 2x^2y$, find $f_x(1,2)$ and $f_y(1,2)$. What do these two values represent?

6. If $f(x,y,z) = \sin(xy)\cos(yz^2)$, find f_x, f_y and f_z.

Tangent Planes

7. If $f(x,y) = \dfrac{x^2}{4} + \dfrac{y^2}{9}$, find f_x and f_y. Use this to find the equation of the tangent plane at the point $(-2,3)$. Sketch the function and the tangent plane.

8. Compute the tangent plane approximation for each of the following functions at the stated point.

(a) $f(x,y) = 2x + 3y$ at the point $(1,2)$

(b) $g(x,y) = 4 - x^2 - y^2$ at the point $(1,1)$

(c) $h(x,y) = \sin(x)\cos(y)$ at the point $(\frac{\pi}{4}, \frac{\pi}{4})$

(d) $q(x,y,z) = \ln(x^2 + y^2 + z^2)$ at the point $(1,0,1)$

(e) $r(x,y) = -xye^{-x^2-y^2}$ at the point $(1,1)$

(f) $u(x,y,z) = \sqrt{1 - x^2 - y^2 - z^2}$ at the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

9. Use the tangent plane approximations found in the previous question to estimate the function at the stated points. Compare your estimate with that given by a calculator using the function itself.

(a) $f(x,y)$ at the point $(1.1, 1.9)$

(b) $g(x,y)$ at the point $(0.1, 1.1)$

(c) $h(x,y)$ at the point $(\frac{3\pi}{16}, \frac{5\pi}{16})$

(d) $q(x,y,z)$ at the point $(0.8, 0.1, 0.9)$

(e) $r(x,y)$ at the point $(0.9, 1.1)$

(f) $u(x,y,z)$ at the point $(0.6, 0.4, 0.6)$

10. Derive the linear approximation $T_1(x,y,z)$ for the function $f(x,y,z) = x^2 + y^2 + z^2$ at the point $(1,1,1)$. What does this function, and its linear approximation, represent?

Problem Set Eleven: Chain Rule, Directional Derivative, Stationary Points

Chain Rule

1. Given $f(x, y) = 2x^2 + 4y - 2$ and $x(s) = 3s$, $y(s) = 2s^2$, compute $\frac{df}{ds}$ by direct substitution (i.e. by first constructing $f(s)$) and also by the Chain rule.

2. Given $f(x, y, z) = 2x^2 + 4y + 3z^3 - 5$ and $x(s) = 3s$, $y(s) = 2s^2$, $z(s) = s^3$ compute $\frac{df}{ds}$ using the Chain rule.

3. For the function $f(x, y) = y^2 \sin(x)$ verify that

$$\frac{\partial}{dx}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

4. Given $f(x, y) = 2xy$ and $x(r, \theta) = r\cos(\theta)$, $y(r, \theta) = r\sin(\theta)$, compute $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$.

5. Let $f = f(x, y)$ be an arbitrary function of (x, y), and $x(r, \theta) = r\cos(\theta)$, $y(r, \theta) = r\sin(\theta)$.

a. Use the Chain rule to express $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$ in terms of the first partial derivatives of $f(x, y)$.

b. Use the Chain rule again to express $\frac{\partial^2 f}{\partial r^2}$ and $\frac{\partial^2 f}{\partial \theta^2}$ in terms of the second partial derivatives of $f(x, y)$.

6. Compute $\frac{df}{ds}$ for the function $f(x, y) = xy + x + y$ along the curve $x(s) = r\cos(\frac{s}{r})$, $y(s) = r\sin(\frac{s}{r})$. Verify that $\frac{dx}{ds}\underset{\sim}{i} + \frac{dy}{ds}\underset{\sim}{j}$ is a unit vector.

Directional Derivative

7. Compute the directional derivative for each of the following functions in the stated direction. Be sure that you use a unit vector.

(a) $f(x,y) = 2x + 3y$ at $(1,2)$ $\quad \utilde{t} = \frac{3}{5}\utilde{i} + \frac{4}{5}\utilde{j}$

(b) $g(x,y) = 4 - x^2 - y^2$ at $(1,1)$ $\quad \utilde{t} = \frac{1}{\sqrt{2}}\utilde{i} + \frac{1}{\sqrt{2}}\utilde{j}$

(c) $h(x,y) = \sin(x)\cos(y)$ at $(\frac{\pi}{4}, \frac{\pi}{4})$ $\quad \utilde{t} = (\frac{1}{\sqrt{2}}\utilde{i} + \frac{1}{\sqrt{2}}\utilde{j})$

(d) $q(x,y,z) = \ln(x^2 + y^2 + z^2)$ at $(1,0,1)$ $\quad \utilde{t} = \utilde{i} + \utilde{j} - \utilde{k}$

(e) $r(x,y) = -xye^{-x^2-y^2}$ at $(1,1)$ $\quad \utilde{t} = 4\utilde{i} - 3\utilde{j}$

(f) $u(x,y,z) = \sqrt{1 - x^2 - y^2 - z^2}$ at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ $\quad \utilde{t} = 2\utilde{i} - \utilde{j} + \utilde{k}$

Extrema

8a. Compare the three functions

$$\begin{aligned} f(x,y) &= x^2 + y^2 \\ g(x,y) &= x^2 - y^2 \\ h(x,y) &= -x^2 - y^2 \end{aligned}$$

in regards to their extrema. Sketch the graph of each function as part of your comparison.
b. What extrema can you find for the function $r(x,y) = x^2 - y^3$?

9. Find the critical points (if any) for each of the following functions.

(a) $f(x,y) = x - x^3 + y^2$

(b) $g(x,y) = e^{x^2+y^2}$

(c) $h(x,y) = xye^{-x^2-y^2}$

(d) $p(x,y) = (2 - x^2)e^{-y}$

(e) $q(x,y) = \arctan(x^2 + y^2)$

(f) $r(x,y,z) = 4x^2 + 3y^2 + z^2$

(g) $s(x,y,z) = \arctan((x-1)^2 + y^2 + z^2)$

10. Identify the nature of the extrema found in each of the functions from Question $9(a - e)$. Can you find a way to classify the extrema in Question $9f$ and $9g$?