MACM 101

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Septemper 5, 2024

1 Counting

1.1 The Rules of Sums and Products

Be careful of initial conditions (duplicates and assumptions)
Rules of Sums

If task A can be performed in m ways, while task B can be performed in n ways and A and B cannot be done simultaneously, then performing either task can be done in any one of $m + n$ ways
Rules of Products

A procedure P can be broken down into A and B stage. If A has m outcomes and B has n outcomes, P can be carried out in $m * n$ ways.

1.2 Permutations

- Distinct Objects

- Linear arrangement objects, i.e. the *order* of objects is important

Definition 1.1. *Factorials*

For integer $n \geq 0$,

$$n! = \begin{cases} 1 & n = 0 \\ n * (n-1)! & n \geq 1 \end{cases}$$

Definition 1.2.

If there are n distinct objects and $1 \leq r \leq n$, then, by rule of product, the number of permutations of size r for the n objects is

$$P(n, r) = \frac{n!}{(n - r)!}$$

1.3 Combinations

Definition 1.3.

If there are n distinct objects and $1 \leq r \leq n$, then the number of combinations of size r for the n objects is

$$\binom{n}{k} = C(n, r) = \frac{n!}{(n - r)!r!}$$

You can use a combinatorial argument in proofs.

Proposition 1.3.1. *For positive integers n and k with $n = 2k$, $\frac{n!}{2!^k}$ is an integer.*

Proof. Consider the n symbols: $x_1, x_1, x_2, x_2, \cdots, x_k, x_k$. The number of arrangements of all these $n = 2k$ symbols is an integer that equals

$$\underbrace{\frac{n!}{2!2! \cdots 2!}}_{\text{k factors of 2!}} = \frac{n!}{2!^k}$$

Definition 1.4. *Sigma notation*

$$a_m + a_{m+1} + a_{m+2} + \cdots + a_{m+n} = \sum_{i=m}^{m+n} a_i$$

Definition 1.5. *Weight*

Weight of a string $X = x_1 x_2 \ldots x_n$ is defiined as $\text{wt}(X) = \sum_{i=1}^{n} x_i$

Theorem 1.1. *Binomial Theorem*

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Corollary 1.1.1.

Set $x = y = 1$, then it follows that

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

Corollary 1.1.2.

Similary, set $x = -1$ and $y = 1$, then it follows that

$$\sum_{i=0}^{n} -1^i \binom{n}{i} = 0$$

Theorem 1.2. *Multinomial Theorem*

With integers $n, t > 0$, the coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$ in the expansion of $(x_1 + x_2 + \cdots + x_t)^n$ is

$$\frac{n!}{n_1! n_2! \cdots n_t!} = \binom{n}{n_1, n_2, \cdots n_t}$$

where each n_i is an integer with $0 \leq n_i \leq n$, for all $1 \leq i \leq t$, and $n_1 + n_2 + \cdots + n_t = n$.

Proof. Choose x_1 from n_1 out of n factors, then choose x_2 from n_2 out of $n - n_1$ factors, and so on. This gives

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{t-1}}{n_t}$$
$$= \frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\cdots\frac{(n-n_1-n_2-\cdots-n_{t-1})!}{n_t!(n-n_1-n_2-\cdots-n_{t-1}-n_t)!}$$
$$= \frac{n!}{n_1!n_2!\cdots n_t!}$$

1.4 Combinations with Repetition

The number of ways to select r of n distinct objects with repetitions is

$$\binom{n+r-1}{r}$$

It is equivalent to the number of ways to separate r identical stones with $n-1$ identical sticks where there are n slots to represent how many times the nth object was chosen with the number of stones.

Same logic can be used for counting how many ways r objects can be distributed to n containers, or how many ways n nonnegative integers can add up to r (order matters).

You can also count the number of execution of such codes:

```
counter := 0;
for i = 1 to n do
    for j = 1 to i do
        for k = 1 to j do
            counter := counter + 1;
```

It is equivalent to counting how many triples of (i, j, k) satisfy $1 \leq k \leq j \leq i \leq n$, which is choosing 3 numbers from n numbers with repetitions. $counter$ would be $\binom{n+3-1}{3}$.

2 Fundamentals of Logic

2.1 Basic Connectives and Truth Tables

Definition 2.1. Declarative sentences that are either true or false are called *statements* (or *propositions*), and we use lowercase letters of the alphabet to represent such statements.

Primitive statements cannot be broken down into anything simpler, and new statements can be obtained from existing ones in two ways.

1. Transform a given statement p to $\neg p$ (Not p).

2. Combine two or more statements into a *compound* statement, using one of the *logical connectives*.

   (a) Conjunction: $p \wedge q$ (p and q)

   (b) Disjunction:

       i. $p \vee q$ (p or q)
       ii. $p \veebar q$

   (c) Implication: $p \rightarrow q$ (p implies q)

   (d) Biconditional: $p \leftrightarrow q$ (p if and only if q)

Here is the truth table.[1]

p	q	$p \wedge q$	$p \vee q$	$p \veebar q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	T	T	F	T	T
T	F	F	T	T	F	F
F	T	F	T	T	T	F
F	F	F	F	F	T	T

Definition 2.2. A compound statement is called a *tautology* if it is always true. If it is always false, it is called a *contradiction.*

We use the symbol T_0 to denote any tautology and the symbol F_0 to denote any contradiction.

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[1]Sometimes, 0 and 1 are used for F and T instead, similar to bit-logic.

2.2 Logical Equivalence: The Laws of Logic

Definition 2.3. Two statements s_1, s_2 are said to be *logically equivalent* when $s_1 \leftrightarrow s_2$, and we write $s_1 \Leftrightarrow s_2$.

If 2 statements are not logically equivalent, we write $s_1 \not\Leftrightarrow s_2 \quad (\neg(s_1 \Leftrightarrow s_2))$.

The Laws of Logic

1)	$\neg\neg p \Leftrightarrow p$	Law of Double Negation
2)	$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$	DeMorgan's Laws
	$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	
3)	$p \wedge q \Leftrightarrow q \wedge p$	Commutative Laws
	$p \vee q \Leftrightarrow q \vee p$	
4)	$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$	Associative Laws
	$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	
5)	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	Distributive Laws
	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	
6)	$p \vee p \Leftrightarrow p$	Idempotent Laws
	$p \wedge p \Leftrightarrow p$	
7)	$p \vee F_0 \Leftrightarrow p$	Identity Laws
	$p \wedge T_0 \Leftrightarrow p$	
8)	$p \vee \neg p \Leftrightarrow T_0$	Inverse Laws
	$p \wedge \neg p \Leftrightarrow F_0$	
9)	$p \wedge F_0 \Leftrightarrow F_0$	Domination Laws
	$p \vee T_0 \Leftrightarrow T_0$	
10)	$p \vee (p \wedge q) \Leftrightarrow p$	Absorption Laws
	$p \wedge (p \vee q) \Leftrightarrow p$	

Following statements are also equivalent.

1. $p \rightarrow q \Leftrightarrow \neg p \vee q$

2. $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p) \Leftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$

3. $p \veebar q \Leftrightarrow (p \vee q) \wedge \neg(p \wedge q)$

Using the above logival equivalences, we can eliminate those three connectives($\rightarrow$, $\leftrightarrow$, $\veebar$) from any logical compound statements.

Definition 2.4. Let s be a statement containing only $\land$ and $\lor$ as logical connectives. The dual of s, denoted s^d, is derived from s by replacing each $\land$ with $\lor$, $\lor$ with $\land$, T_0 with F_0, and F_0 with T_0.

If p is a primitive statement, then p_d is the same as p.

$$p = p^d$$

Theorem 2.1. *The Principle of Duality.* Let s and t be statements containing no logical connectives other than $\land$ and $\lor$. If $s \Leftrightarrow t$, then $s^d \Leftrightarrow t^d$.

2.2.1 The Subtitution Rules

1. Suppose, compound statement P is a tautology. If p is a primitive statement that appears in P and we replace all occurrences of p by an arbitrary statement q, then the resulting compound statement P_1 is also a tautology.

2. P is a compound statement, p is an arbitrary statement that appears in P, let q be a statement such that $p \Leftrightarrow q$. Now replace 1 or more occurrences of p by q, this yields P_1. Now, $P_1 \Leftrightarrow P$.

2.2.2 Relatives of the Implication $p \to q$

Inverse: $\neg p \to \neg q$
Converse: $q \to p$
Contrapositive: $\neg q \to \neg p$

1)	$p \to q \Leftrightarrow \neg q \to \neg p$	Implication $\Leftrightarrow$ Contrapositive
2)	$q \to p \Leftrightarrow \neg p \to \neg q$	Converse $\Leftrightarrow$ Inverse
3)	$p \to q \not\Leftrightarrow q \to p$	Implication $\not\Leftrightarrow$ Converse
4)	$\neg p \to \neg q \not\Leftrightarrow \neg q \to \neg p$	Contrapositive $\not\Leftrightarrow$ Inverse

An implication is logically equivalent to its contrapositive, but not to its inverse or converse. At the same time, its inverse and converse are logically equivalent.

(*Note:* The negation of an it-then statement (in words) does not begin with the word if. i.e. it is not another implication.)

2.2.3 Applications: Simplifying Switching Networks

A switching network is made up of wires and switches connecting two Terminals T_1 and T_2. In such a network, each switch is either open (0) so current doesn't flow, or closed (1) so current does flow through it. Switches in *parallel* are represented by $\vee$, and switches in *series* are represented by $\wedge$.
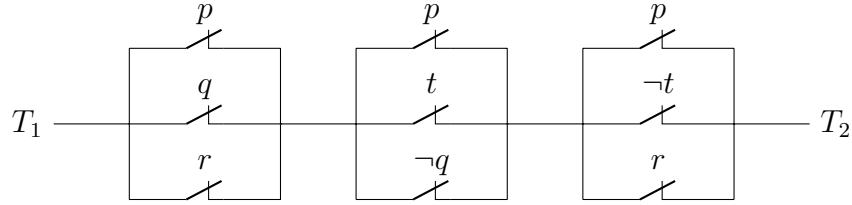
Such network can be represented by the statement $(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r)$. This can be simplified using the laws of logic.

$$
\begin{array}{ll}
(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r) & \textbf{Reasons} \\
\Leftrightarrow \quad (p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r) & \text{d}
\end{array}
$$

2.3 Logical Implication: Rules of Inference

We show an argument with premises $(p_1, p_2, \cdots, p_n)$ and conclusion q is valid by showing the following implication is a tautology:

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$$

To show the following implication is a tautology, we need to show that q is true if all the premises are true.

You can incorporate this into an automatic "inference engine", the basic component of an AI expert system. These systems combine basic facts to develop more facts.

Definition 2.5. If p, q are arbitrary statements such that $p \rightarrow q$ is a tautology, then we say that p *logically implies* q and we write $p \Rightarrow q$ to denote this situation.

The notation $p \not\Rightarrow q$ is used to indicate that $p \rightarrow q$ is *not* a tautology — so the given implication $(p \rightarrow q)$ is not a logical implication. If $p \Leftrightarrow q$, then $p \leftrightarrow q$ is a tautology. This means $p \rightarrow q$ and $q \rightarrow p$ are tautologies, too, thus $p \Rightarrow q$ and $q \Rightarrow p$. The converse is also true.

When establishing the validity of an argument, the rules of inferences will enable us to consider only the cases wherein all the premises are true. They are fundamental in the development of a step-by-step validation of how the conclusion q logically follows from the premises $p_1, p_2, \cdots, p_n$ in an implication of the form

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to q.$$

This development will establish the validity of the given arguement, for it will show how the truth of the conclusion can be deduced from the truth of the premises.

	Rule of Inference	**Related Logical Implication**	**Name of Rule**
1)	p $p \rightarrow q$ $\therefore \quad q$	$[p \wedge (p \rightarrow q)] \rightarrow q$	Rule of Detachment
2)	$p \rightarrow q$ $q \rightarrow r$ $\therefore \quad p \rightarrow r$	$[(p \rightarrow q) \vee (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Law of the Syllogism
3)	$p \rightarrow q$ $\neg q$ $\therefore \quad \neg p$	$[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$	Modus Tollens
4)	p q $\therefore \quad p \wedge q$		Rule of Conjunction
5)	$p \vee q$ $\neg p$ $\therefore \quad q$	$[(p \vee q) \wedge \neg p] \Rightarrow q$	Rule of Disjunctive Syllogism
6)	$\dfrac{\neg p \rightarrow F_0}{\therefore \quad p}$	$(\neg p \rightarrow F_0) \rightarrow p$	Rule of Contradiction
7)	$\dfrac{p \wedge q}{\therefore \quad p}$	$(p \wedge q) \rightarrow p$	Rule of Conjunctive Simplification
8)	$\dfrac{p}{\therefore \quad p \vee q}$	$p \rightarrow p \vee q$	Rule of Disjunctive Amplification
9)	$p \wedge q$ $p \rightarrow (q \rightarrow r)$ $\therefore \quad r$	$[(p \wedge q) \wedge [p \rightarrow (q \rightarrow r)]] \rightarrow r$	Rule of Conditional Proof
10)	$p \rightarrow r$ $q \rightarrow r$ $\therefore \quad (p \vee q) \rightarrow r$	$[(p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow [(p \vee q) \rightarrow r]$	Rule for Proof Cases
11)	$p \rightarrow q$ $r \rightarrow s$ $p \vee r$ $\therefore \quad q \vee s$	$[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee r)] \rightarrow (q \vee s)$	Rule of the Constructive Dilemma
12)	$\neg p \rightarrow p$ $\neg r \rightarrow s$ $\neg q \vee \neg s$ $\therefore \quad \neg p \vee \neg r$	$[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (\neg q \vee \neg s)] \rightarrow (\neg p \vee \neg r)$	Rule of Contradiction

The Rule of Contradiction is the basis of a method of *Proof by Contradiction*, or *Reductio ad Absurdum*. In a Proof of Contradiction, we establish the validity of the argument

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$$

by establishing the validity of the logically equivalent argument

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n \wedge q) \rightarrow F_0.$$

We first assume that what we are trying to prove is actually false, then produce a contradiction of the form $s \wedge \neg s$, for some statement s. This contradiction concludes that the statement that was assumed to be false is in fact true, and this validates the argument (or completes the proof).

2.4   Predicate Logic and Quantifiers

Definition 2.6. A declarative sentence is an *open statement* if

1. It contains one or more variables, and

2. It is not a statement, but

3. It becomes a statement when the variables in it are replaced by certain allowable choices.

All of these are open statements:

$$\text{The number } x - 5 \text{ is an even integer.}$$
$$x - 5 = 7$$
$$3x + y > 7$$

The allowable choices of variable constitute what is called the *universe*[2] or *universe of discourse* for the open statement. Open statements are denoted by $p(x), q(x, y)$, etc. (also called predicates).

We can *quantify* an open statement with two types of quantifiers:

1. existential quantifier: $\exists$
   $\exists x$:   For some x
           For at least one x
           There exists an x such that ...

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[2]This is an example of a *set*

2. universal quantifier: $\forall$

$\forall x$: For all x
       For any x
       For every x

Variable x is called a *free* variable in an open statement and a *bound* variable in a quantified open statement.

   Let $p(x)$ denote any open statement with a prescribed *nonempty* universe. Then,

$$\forall x\ p(x) \Rightarrow \exists x\ p(x).$$