MACM 101

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1 Counting

1.1 The Rules of Sums and Products

Be careful of initial conditions (duplicates and assumptions)
Rules of Sums

If task A can be performed in m ways, while task B can be performed in n ways and A and B cannot be done simultaneously, then performing either task can be done in any one of $m + n$ ways
Rules of Products

A procedure P can be broken down into A and B stage. If A has m outcomes and B has n outcomes, P can be carried out in $m * n$ ways.

1.2 Permutations

- Distinct Objects
- Linear arrangement objects, i.e. the *order* of objects is important

Definition 1.1. *Factorials*

For integer $n \geq 0$,

$$n! = \begin{cases} 1 & n = 0 \\ n * (n-1)! & n \geq 1 \end{cases}$$

Definition 1.2.

If there are n distinct objects and $1 \leq r \leq n$, then, by rule of product, the number of permutations of size r for the n objects is

$$P(n, r) = \frac{n!}{(n-r)!}$$

1.3 Combinations

Definition 1.3.

If there are n distinct objects and $1 \leq r \leq n$, then the number of combinations of size r for the n objects is

$$\binom{n}{k} = C(n, r) = \frac{n!}{(n-r)!r!}$$

You can use a combinatorial argument in proofs.

Proposition 1.3.1. *For positive integers n and k with $n = 2k$, $\frac{n!}{2!^k}$ is an integer.*

Proof. Consider the n symbols: $x_1, x_1, x_2, x_2, \cdots, x_k, x_k$. The number of arrangements of all these $n = 2k$ symbols is an integer that equals

$$\frac{n!}{\underbrace{2!2!\cdots 2!}_{\text{k factors of 2!}}} = \frac{n!}{2!^k}$$

Definition 1.4. *Sigma notation*

$$a_m + a_{m+1} + a_{m+2} + \cdots + a_{m+n} = \sum_{i=m}^{m+n} a_i$$

Definition 1.5. *Weight*

Weight of a string $X = x_1 x_2 \ldots x_n$ is defiined as $\mathrm{wt}(X) = \sum_{i=1}^{n} x_i$

Theorem 1.1. *Binomial Theorem*

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Corollary 1.1.1.

Set $x = y = 1$, then it follows that

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

Corollary 1.1.2.

Similary, set $x = -1$ and $y = 1$, then it follows that

$$\sum_{i=0}^{n} -1^i \binom{n}{i} = 0$$

Theorem 1.2. *Multinomial Theorem*

With integers $n, t > 0$, the coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$ in the expansion of $(x_1 + x_2 + \cdots + x_t)^n$ is

$$\frac{n!}{n_1! n_2! \cdots n_t!} = \binom{n}{n_1, n_2, \cdots n_t}$$

where each n_i is an integer with $0 \leq n_i \leq n$, for all $1 \leq i \leq t$, and $n_1 + n_2 + \cdots + n_t = n$.

Proof. Choose x_1 from n_1 out of n factors, then choose x_2 from n_2 out of $n - n_1$ factors, and so on. This gives

$$\begin{aligned} & \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{t-1}}{n_t} \\ = \quad & \frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\cdots\frac{(n-n_1-n_2-\cdots-n_{t-1})!}{n_t!(n-n_1-n_2-\cdots-n_{t-1}-n_t)!} \\ = \quad & \frac{n!}{n_1!n_2!\cdots n_t!} \end{aligned}$$

1.4 Combinations with Repetition

The number of ways to select r of n distinct objects with repetitions is

$$\binom{n+r-1}{r}$$

It is equivalent to the number of ways to separate r identical stones with $n-1$ identical sticks where there are n slots to represent how many times the nth object was chosen with the number of stones.

Same logic can be used for counting how many ways r objects can be distributed to n containers, or how many ways n nonnegative integers can add up to r (order matters).

You can also count the number of execution of such codes:

$counter := 0$;
for $i = 1$ **to** n **do**
  for $j = 1$ **to** i **do**
    for $k = 1$ **to** j **do**
      $counter := counter + 1$;

It is equivalent to counting how many triples of (i, j, k) satisfy $1 \leq k \leq j \leq i \leq n$, which is choosing 3 numbers from n numbers with repetitions. *counter* would be $\binom{n+3-1}{3}$.

2 Fundamentals of Logic

2.1 Basic Connectives and Truth Tables

Definition 2.1. Declarative sentences that are either true or false are called *statements*(or *propositions*), and we use lowercase letters of the alphabet to represent such statements.

Primitive statements cannot be broken down into anything simpler, and new statements can be obtained from existing ones in two ways.

1. Transform a given statement p to $\neg p$ (Not p).

2. Combine two or more statements into a *compound* statement, using one of the *logical connectives.*
   (a) Conjunction: $p \wedge q$ (p and q)
   (b) Disjunction:
      i. $p \vee q$ (p or q)
      ii. $p \veebar q$
   (c) Implication: $p \rightarrow q$ (p implies q)
   (d) Biconditional: $p \leftrightarrow q$ (p if and only if q)

Here is the truth table.[1]

p	q	$p \wedge q$	$p \vee q$	$p \veebar q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	T	T	F	T	T
T	F	F	T	T	F	F
F	T	F	T	T	T	F
F	F	F	F	F	T	T

Definition 2.2. A compound statement is called a *tautology* if it is always true. If it is always false, it is called a *contradiction.*

We use the symbol T_0 to denote any tautology and the symbol F_0 to denote any contradiction.

[1] Sometimes, 0 and 1 are used for F and T instead, similar to bit-logic.

2.2 Logical Equivalence: The Laws of Logic

Definition 2.3. Two statements s_1, s_2 are said to be *logically equivalent* when $s_1 \leftrightarrow s_2$, and we write $s_1 \Leftrightarrow s_2$.

The Laws of Logic

1)	$\neg\neg p \Leftrightarrow p$	Law of Double Negation
2)	$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$	DeMorgan's Laws
	$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	
3)	$p \wedge q \Leftrightarrow q \wedge p$	Commutative Laws
	$p \vee q \Leftrightarrow q \vee p$	
4)	$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$	Associative Laws
	$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	
5)	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	Distributive Laws
	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	
6)	$p \vee p \Leftrightarrow p$	Idempotent Laws
	$p \wedge p \Leftrightarrow p$	
7)	$p \vee F_0 \Leftrightarrow p$	Identity Laws
	$p \wedge T_0 \Leftrightarrow p$	
8)	$p \vee \neg p \Leftrightarrow T_0$	Inverse Laws
	$p \wedge \neg p \Leftrightarrow F_0$	
9)	$p \wedge F_0 \Leftrightarrow F_0$	Domination Laws
	$p \vee T_0 \Leftrightarrow T_0$	
10)	$p \vee (p \wedge q) \Leftrightarrow p$	Absorption Laws
	$p \wedge (p \vee q) \Leftrightarrow p$	

Following statements are also equivalent.

1. $p \to q \Leftrightarrow \neg p \vee q$
2. $p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p) \Leftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$
3. $p \veebar q \Leftrightarrow (p \vee q) \wedge \neg(p \wedge q)$

Using the above logival equivalences, we can eliminate those three connectives($\to$, $\leftrightarrow, \veebar$) from any logical compound statements.