Math Journal

Andrew Ha

Combinatorial Proof

09/10/2024

Proposition 1. *For positive integers n and k with $n = 2k$, $\frac{n!}{2!^k}$ is an integer.*

Proof. Consider the n symbols: $x_1, x_1, x_2, x_2, \cdots, x_k, x_k$. The number of arrangements of all these $n = 2k$ symbols is an integer that equals

$$\frac{n!}{\underbrace{2!2!\cdots 2!}_{\text{k factors of 2!}}} = \frac{n!}{2!^k}$$

I learned this from the first day of class in MACM 101. This is an example of proving that a value is an integer by obtaining that value from counting something. Researching further, I also found out about double counting. It is based on the idea that counting the same objects in two different ways results in two different expressions, which must be equal to each other.

Identity 1.

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

For example, this identity can be proved by counting the number of subsets of a set A with n elements. One way to count this is noticing how you can either include or exclude each element, giving 2 choices for each of the n elements. This gives 2^n subsets. Second way to count the number of subsets is summing up the number of subsets with k elements where k can be any integer from 0 to n. For each case, you would be choosing k elements out of n elements, so there are $\binom{n}{k}$ subsets. Summing up, the total number of subsets is $\sum_{k=0}^{n} \binom{n}{k}$. Two expressions, 2^n and $\sum_{k=0}^{n} \binom{n}{k}$, must be equal since they represent the same thing.

Putnam 2002 B1

09/11/2024

Here is a fun problem I wanted to share from the 2002 William Lowell Putnam Mathematics Competition.

> B1. Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?

This problem doesn't require much of advanced mathematics, yet it is simple and tricky. Here's a solution I like:

Solution. The probability of nth success is the proportion of shots she has hit so far, which is $\frac{n-1}{\#\text{ of shots so far}}$. On the other hand, the probability of nth miss is $1 - \frac{\#\text{ of successes so far}}{\#\text{ of shots so far}} = \frac{\#\text{ of shots so far}}{\#\text{ of shots so far}} - \frac{\#\text{ of shots so far}-(n-1)}{\#\text{ of shots so far}} = \frac{n-1}{\#\text{ of shots so far}}$. For her to hit exactly 50 of her first 100 shots, she has to hit exactly 49 shots and miss exactly 49 shots. Consider the probability of one case where she hits 49 shots consecutively then misses 49 shots consecutively:

$$\underbrace{\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} \times \cdots \times \frac{49}{50}}_{\text{49 successes}} \times \underbrace{\frac{1}{51} \times \frac{2}{52} \times \frac{3}{53} \times \frac{4}{54} \cdots \times \frac{49}{99}}_{\text{49 misses}} = \frac{49!^2}{99!}$$

Here, the arrangement of shots does not affect the overall probability since the numerator will always be $49!^2$ as long as she hits 49 shots and misses 49 shots, and the denominator will always be $99!$. Therefore, multiplying the probability of single case by the number of arrangements will give us the answer. The number of arrangements of 49 shots and 49 misses is $\frac{98!}{49!^2}$ since there are 49 duplicates of 2 cases. Multiplying two numbers yields

$$\frac{49!^2}{99!} \cdot \frac{98!}{49!^2} = \frac{1}{99}$$

The probability she hits exactly 50 of her first 100 shots is $\frac{1}{99}$.

Two Random Distributions

09/16/2024
This is from a YouTube video I watched recently. It talks about a surprising fact that choosing a random number between 0 and 1 and calculating its square root is actually the same as choosing two random numbers between 0 and 1 and taking the larger one. To see why, we look at the distribution of two functions, $\max(X, Y)$ and $\sqrt{X}$. X and Y are independent and random numbers between 0 and 1. Since two distributions are continuous, the probability of getting a specific number is 0. Instead, we look at the probability of getting a number less than or equal to n. For $\max(X, Y)$, both X and Y must be less than or equal to n. Notice that the probability of getting a number less than or equal to n is n since it is between 0 and 1.

$$P(\max(X, Y) \leq n) = P(X \leq n) \cdot P(Y \leq n) = n^2$$

Now, for $\sqrt{X}$, X has to be less than or equal to n^2 for $\sqrt{X}$ to be less than or equal to n.

$$P(\sqrt{X} \leq n) = P(X \leq n^2) = n^2$$

This can actually be generalized that choosing a max value of m random numbers between 0 and 1 and taking the mth root of a random number between 0 and 1 are identical.

$$\begin{aligned} P(\max(X_1, X_2, \cdots, X_m) \leq n) = && P(X_1 \leq n)P(X_2 \leq n) \cdots P(X_m \leq n) && = n^m \\ P(\sqrt[m]{X} \leq n) = && P(X \leq n^m) && = n^m \end{aligned}$$

This reminds me of what I learned in statistics about probabilities.

$\wedge$

09/17/2024

I had a funny thought while walking to school today. I imagined two bicycles coming from opposite sides of a road that narrows suddenly in the middle of them. There would be two types of people, people who slow down for the other person to pass first (call them D) and people who speed up to pass faster than the other person (call them U). If D and D meet, or if U and D meet, there will be no accident. However, if U and U meet, there will be an accident. I thought of the truth table for AND ($\wedge$).

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Asking "accident?", D say no (F), and U say yes (T). Two meeting in the road is the AND operation, so the result comes out as the truth table suggests.