Geometric Optics

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1. Ray theory

- Optical path length: $L_0 = \int_P^{P'} n(\vec{r})\,ds = cst$
- Fermat's principle $\delta \int_P^{P'} n(\vec{r})\,ds = 0$

  The path taken by a ray of light has an extremal optical path length with respect to neighboring paths.

- Reflection & Transmission

  Perpendicular incidence ($\theta = 0$): $R = \left(\frac{n-n'}{n+n'}\right)^2$ $\quad T = \frac{4nn'}{(n+n')^2}$

  Application: beamsplitter, beam combiners

- The ray equation: $\frac{d}{ds}\left(n\frac{d\vec{r}}{ds}\right) = \nabla n$

  Paraxial approximation: $s \simeq z$ $\quad \frac{d}{dz}\left(n\frac{d\vec{r}}{dz}\right) = \nabla n$

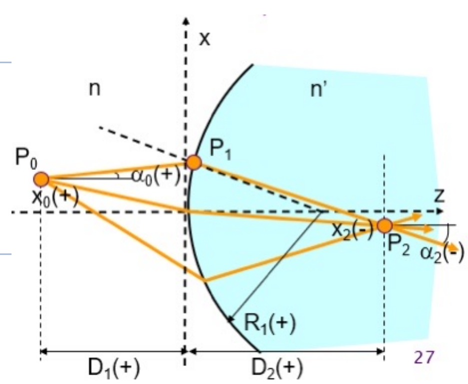
  Parabolic index profile

  - Period is determined only by the index profile
  - Valid only for meridional rays
  - Parabolic profile does not extend infinitely
  - Applications: Graded Index fibers and lenses

2. Imaging systems

① Paraxial approximation: $\sin\theta \simeq \theta$ $\qquad$ Snell's law: $n\theta = n'\theta'$

Imaging: $\frac{n}{D_1} + \frac{n'}{D_2} = \frac{n'-n}{R_1}$



Magnification:

- Lateral: $m_x = \frac{x_2}{x_0} = -\frac{n}{n'}\frac{D_2}{D_1}$
- Angular: $m_\alpha = \frac{\Delta\alpha_2}{\Delta\alpha_0} = -\frac{D_1}{D_2}$

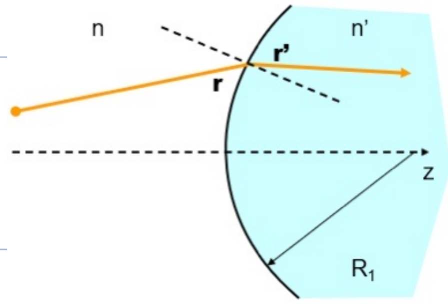
$\Big\}\ m_x \cdot m_\alpha = \frac{n}{n'}$

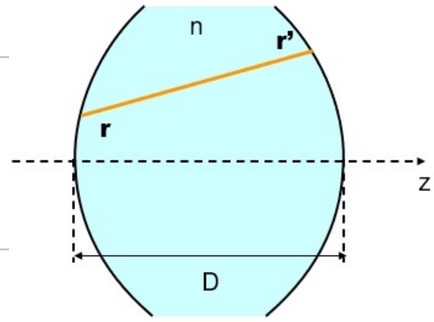
- Lagrange-equation: $n'x_2\Delta\alpha_2 = nx_0\Delta\alpha_0$

- For non-paraxial angles: $n'x_2\sin\theta_2 = nx_0\sin\theta_0$ (Abbe sine-relation)

[θ: between chief and margin rays]

② Matrix formalism





- $r = \begin{bmatrix} x \\ n\alpha \end{bmatrix}$

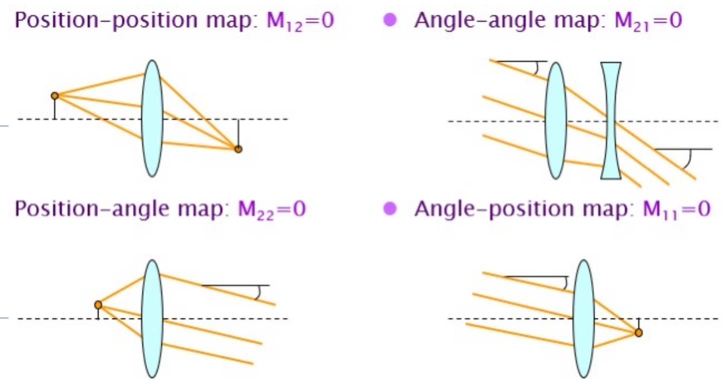
  $r' = \begin{bmatrix} x' \\ n'\alpha' \end{bmatrix}$

$r' = Rr \quad R = \begin{bmatrix} 1 & 0 \\ -P & 1 \end{bmatrix}$ $\qquad r' = Tr \quad T = \begin{bmatrix} 1 & D/n \\ 0 & 1 \end{bmatrix}$

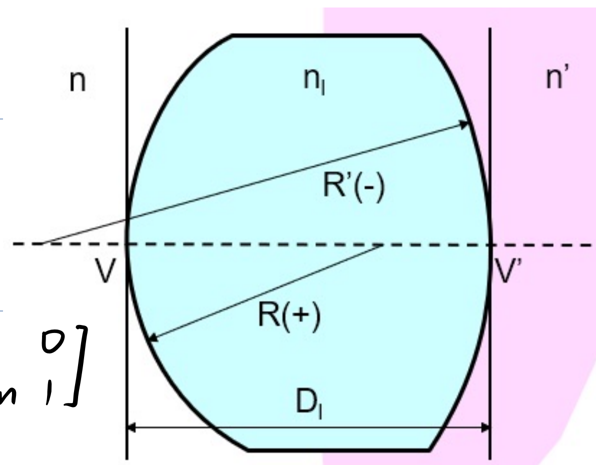
Refractive power: $P = \dfrac{n'-n}{R}$

Imaging: $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$

$\det(M) = 1$



- Lens: $P = \dfrac{n_l - n}{R} \qquad P' = \dfrac{n' - n_l}{R}$



$$M = R'TR = \begin{bmatrix} 1 - \dfrac{PD_l}{n_l} & \dfrac{D_l}{n_l} \\ \dfrac{PP'D_l}{n_l} - P - P' & 1 - \dfrac{P'D_l}{n_l} \end{bmatrix}$$

thin lens: $D_l = 0 \qquad M = \begin{bmatrix} 1 & 0 \\ -(P'+P) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -P_{thin} & 1 \end{bmatrix}$

$P_{thin} = \dfrac{n_l - n}{R} - \dfrac{n_l - n'}{R'}$

focal length: $f' = \dfrac{n'}{P_{thin}} \qquad f = \dfrac{n}{P_{thin}} \Rightarrow \dfrac{f'}{n'} = \dfrac{f}{n}$

translation over a distance S:

$$M = \begin{bmatrix} 1 & S'/n' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -P & 1 \end{bmatrix} \begin{bmatrix} 1 & S/n \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 - PS/n & \frac{S}{n} + \frac{S'}{n'} - PSS'/nn' \\ -P & 1 - PS/n \end{bmatrix}$$

$\left(\dfrac{n}{S} + \dfrac{n'}{S'} = P_{thin} = \dfrac{n'}{f'} = \dfrac{n}{f} \right)$

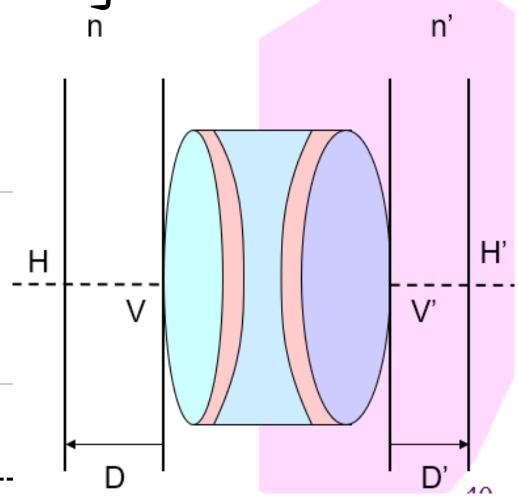
③ Complex lens system = thin lens

Transformation into a "thin lens"

$$M' = T'MT = \begin{bmatrix} M_{11} + \dfrac{M_{21}D'}{n'} & \dfrac{M_{22}D'}{n'} + \dfrac{M_{21}D}{n} \cdot \dfrac{D'}{n'} + M_{12} + \dfrac{M_{11}D}{n} \\ M_{21} & M_{22} + \dfrac{M_{21}D}{n} \end{bmatrix}$$

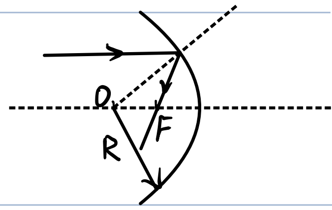
Thin lens matrix: $M' = \begin{bmatrix} 1 & 0 \\ M_{21} & 1 \end{bmatrix}$

- $\frac{D}{n} = \frac{1-M_{22}}{M_{21}}$ $\quad \frac{D'}{n'} = \frac{1-M_{11}}{M_{21}}$ (Principal planes)



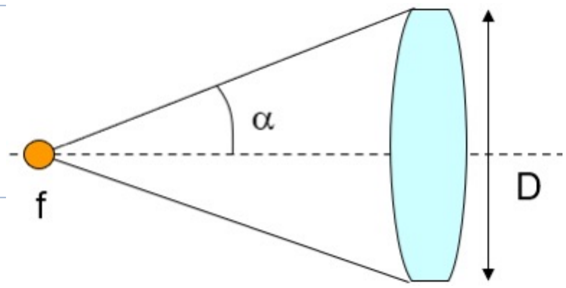
④ Spherical mirrors

- paraxial approximation:
  $P = \frac{2n}{R}$
- focal length: $f = \frac{R}{2}$



3. Optical systems

- relative aperture: $f\text{-number} = \frac{f}{D}$
- numerical aperture: $NA = n\sin\alpha = \frac{1}{2(f\text{-number})}$
- Field stop / Field of view
  Aperture stop / Entrance pupil / Exit pupil
  Depth of field



Larger aperture $\begin{cases} \text{gathering more light (higher intensity)} \\ \text{lower depth of focus} \\ \text{more aberration} \end{cases}$

- Aberrations
  ① aberrations which result in non-stigmatic image:
  spherical aberration, astigmatism, coma
  ② aberrations with distorted stigmatic image: (3rd order)
  field curvature, distortion
  ③ chromatic aberrations:
  dispersion of the material

depend on $\begin{cases} \text{the lens system} \\ \text{magnification} \end{cases}$

spherical aberration solution:
{ lens with the best shape $\quad$ shape factor: $q = \frac{R_2+R_1}{R_2-R_1}$

combination of lenses K_2-K_1
aspherical lens

achromatic doublet / symmetric doublets / aspheric lenses

Chromatic aberration:

Abbe number: $V=\frac{n_Y-1}{n_B-n_R}$ (Glass: $\lambda\downarrow \Rightarrow n\uparrow$ $V>0$)

Dispersion:

Abbe number: $V=\frac{n_Y-1}{n_B-n_R}=\frac{P_Y}{P_B-P_R}$ ($V\downarrow \rightarrow$ strong dispersion)

- Anti-reflection coating:

$\begin{cases} \text{thickness}: d=\frac{\lambda}{4} \\ \text{refraction index}: n_{AR}=\sqrt{n_1 n_2} \end{cases}$

$$\boxed{M=-\frac{s'}{s}=\frac{|s'|}{|s|}=1+\frac{|s'|}{f}}$$

4. Objectives

① Microscope

$M_{ob}=-\frac{s'}{s}=1-\frac{s'}{f_1}$ $s'\gg f_1$

$M_{tot}=M_{ob}\cdot M_{oc}$

② Keplerian telescope

$M=\frac{\alpha'}{\alpha}=\frac{f_1}{f_2}$

③ Galilean telescope:

negative eyepiece, positive magnification

④ Camera

real image

⑤ Pentaprism

rays reflected by 3 planes

$\begin{cases} \text{left-right inversion} \\ \text{upside-down inversion} \end{cases}$

⑥ Binoculars

⑦ Slide projector

⑧ GRIN lenses

⑨ Fresnel lenses

{ discontinuous surface
 angle changes continuously

⑩ "Cat's eye"

Scalar Wave Photonics

1. Monochromatic waves

- $u(\vec{r},t) = a(\vec{r})\cos[2\pi\nu t + \varphi(\vec{r})]$

  $U(\vec{r},t) = a(\vec{r})\,e^{j\varphi(\vec{r})}e^{j2\pi\nu t}$ $\quad u(\vec{r},t) = \mathrm{Re}\{U(\vec{r},t)\}$ $\quad U(\vec{r}) = a(\vec{r})e^{j\varphi(\vec{r})}$

- Helmholtz equation

  $(\nabla^2 + k^2)U(\vec{r}) = 0 \quad [k = \frac{2\pi n\nu}{c} = \frac{n\omega}{c}]$

  wave fronts: $\varphi(\vec{r}) = \text{const}$ (often const $= 2q\pi$) $\quad$ intensity: $I(\vec{r}) = |U(\vec{r})|^2$

① Plane waves

$U(\vec{r}) = Ae^{-j\vec{k}\cdot\vec{r}} = Ae^{-j(k_x x + k_y y + k_z z)}$ $[k_x^2 + k_y^2 + k_z^2 = k^2 = (\frac{n\omega}{c})^2]$

- k & n real: $|\vec{k}| = \sqrt{k_x^2 + k_y^2 + k_z^2} = k = \frac{n\omega}{c}$
- $\vec{k} = \vec{k}_R + j\vec{k}_I$ { propagation in the direction $\vec{k}_R$
  exponential decrease in the direction $\vec{k}_I$

Evanescent plane waves

① if $\vec{k}_R \parallel \vec{k}_I \parallel z$-axis: $U(\vec{r}) = Ae^{k_I z}e^{-jk_R\cdot z}$

decreasing amplitude as the wave propagates further

② if $\vec{k}_I \perp \vec{k}_R \parallel z$-axis: $U(\vec{r}) = Ae^{-k_I y}e^{-jk_R\cdot z}$

propagation in the z-axis and decrease in the y-axis

example: total internal reflection

② Spherical waves

$U(\vec{r}) = \frac{A}{r}e^{-jkr}$

- Radial propagation { $-jkr$: away from the origin

[$+jkr$: to the origin

③ Parabolic waves

Taylor expansion for r: $r \simeq z + \frac{x^2+y^2}{2z}$

$$U(\vec{r}) = \frac{A}{z} e^{-jkz} e^{-jk\frac{x^2+y^2}{2z}}$$

∘ Paraxial waves

$$\frac{\partial A}{\partial z} << kA \qquad \frac{\partial^2 A}{\partial z^2} << k^2 A$$

$$\nabla_T^2 A(\vec{r}) - 2jk\frac{\partial A(\vec{r})}{\partial z} = 0 \quad \text{with } \nabla_T^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

2. Interference

$$\begin{cases} I_1 \neq I_2 : I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi \\ I_1 = I_2 : I = 4I_0\cos^2\left(\frac{\phi}{2}\right) \end{cases} \qquad [\phi = \phi_2 - \phi_1]$$

① Interferometer

$$\begin{cases} U_1 = \sqrt{I_0} e^{-jkz} \\ U_2 = \sqrt{I_0} e^{-jk(z-d)} \end{cases} \longrightarrow I = 2I_0\left[1 + \cos\left(2\pi\frac{d}{\lambda}\right)\right]$$

examples: Mach-Zehnder, Michelson, Sagnac

② Interference of two plane waves

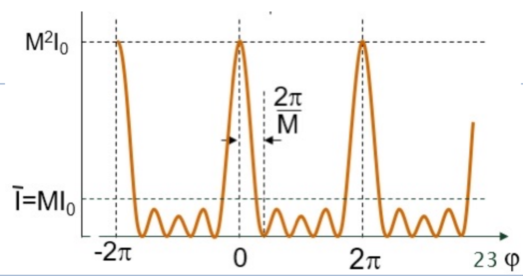
$$\begin{cases} U_1 = \sqrt{I_0} e^{-jkz} \\ U_2 = \sqrt{I_0} e^{-j(k\cos(\theta)z + k\sin(\theta)x)} \end{cases} \longrightarrow I = 2I_0[1 + \cos(k\sin(\theta)x)]$$

③ Interference of multiple waves

∘ $U_m = \sqrt{I_0} e^{[j(m-1)\phi]}$, $m = 1, 2, \cdots, M$

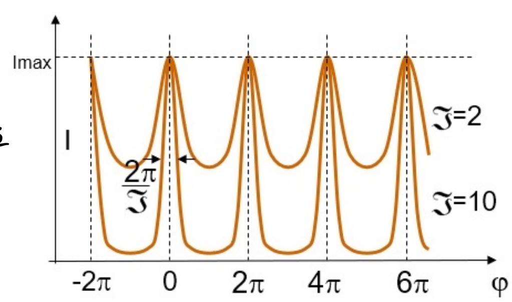
$$\downarrow$$

$$I = I_0 \frac{\sin^2(M\phi/2)}{\sin^2(\phi/2)}$$



∘ $\begin{cases} U_1 = \sqrt{I_0},\ U_2 = hU_1,\ U_3 = h^2 U_1, \cdots \\ h = re^{j\phi},\ r < 1 \end{cases} \longrightarrow U = \frac{\sqrt{I_0}}{1 - re^{j\phi}}$

$$\longrightarrow I = \frac{I_{max}}{1 + (2\mathfrak{F}/\pi)^2\sin^2(\phi/2)} \qquad \begin{cases} I_{max} = \frac{I_0}{(1-r)^2} \\ \mathfrak{F} = \frac{\pi\sqrt{r}}{1-r} \end{cases}$$



④ Michelson Interferometer

∘ phase difference: $\Delta\varphi = \frac{4\pi d}{\lambda}\cos\theta$

∘ Intensity: $I(\Delta\varphi) = 4I_1\cos^2\frac{\Delta\varphi}{2}$ → concentric rings

⑤ Sagnac Interferometer

③ Sagnac interferometer

- phase difference: $\Delta\varphi = 2\pi\nu\Delta t$
- time difference: $\Delta t = \frac{2\pi R}{c+wR} - \frac{2\pi R}{c-wR} = 4\frac{Aw}{c^2}$
- application: measurement of rotations (optical gyroscope)

- Eikonal equation

$$|\nabla S|^2 \approx n^2 \qquad U(\vec{r}) = a(\vec{r})e^{-jk_0 S(\vec{r})}$$

- The ray equation

$$\frac{d}{ds}\left(n\frac{d\vec{r}}{ds}\right) = \nabla n$$

Gaussian Beam Optics

1. Monochromatic light beam

$$\nabla^2 U + k^2 U = 0 \qquad U(x,z) = A(x,z)e^{-jkz}$$

paraxial approximation: $\left|\frac{\partial^2 A}{dz^2}\right| \ll k\left|\frac{\partial A}{dz}\right| \longrightarrow \frac{\partial^2 A}{\partial x^2} - 2jk\frac{\partial A}{\partial z} = 0 \longrightarrow A(x, z=0) = e^{-\frac{x^2}{W_0^2}}$

- beam width: $2W_0$

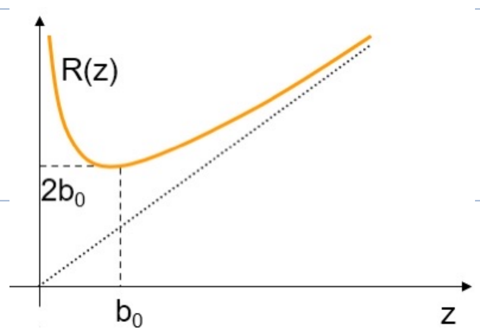
$$z>0:\ A(x,z) = e^{-j\left(p(z) + k\frac{x^2}{2q(z)}\right)}$$

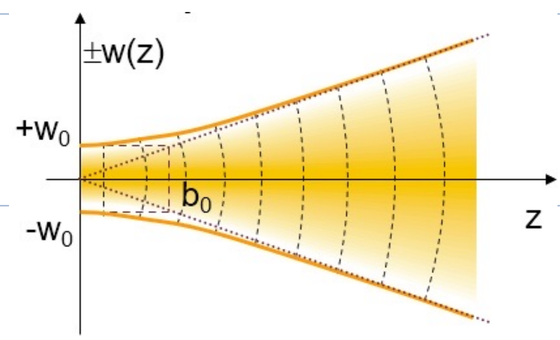
$$q(z) = z + j\frac{kW_0^2}{2} \qquad \frac{1}{q(z)} \triangleq \frac{1}{R(z)} - j\frac{2}{kW^2(z)}$$

$\begin{cases} R(z)\text{: radius of curvature of the phase front} \\ W(z)\text{: half } 1/e^2 \text{ of a Gaussian beam at } z \end{cases}$

$\begin{cases} R(z) = z\left(1+\frac{b_0^2}{z^2}\right) \\ W(z) = W_0\sqrt{1+\frac{z^2}{b_0^2}} \end{cases}$ [the Rayleigh range: $b_0 = \frac{kW_0^2}{2} = \frac{\pi W_0^2}{\lambda}$]

$$A(x,z) = \sqrt{\frac{W_0}{W(z)}}\, e^{-\frac{x^2}{W(z)^2}}\, e^{-j\frac{kx^2}{2R(z)}}\, e^{\frac{j}{2}\arctan\frac{z}{b_0}}$$





$\begin{cases} R(0) = \infty \\ R(b_0) = 2b_0 \end{cases}$

$\begin{cases} z<0:\ W(z) = W_0 \\ z>0:\ W(z) \approx z\frac{W_0}{\ldots} \end{cases}$

$R(\infty)=z$ $\qquad$ $z \gg b_0$: $W(z) \approx z\frac{w_0}{b_0}$

Solution in 3D: $U(x,y,z)=A(x,y,z)e^{-jkz}$

$$A(x,y,z)=\frac{W_0}{W(z)}e^{-\frac{x^2+y^2}{W(z)^2}}e^{-j\frac{k(x^2+y^2)}{2R(z)}}e^{j\arctan\frac{z}{b_0}}$$

2. Lens systems

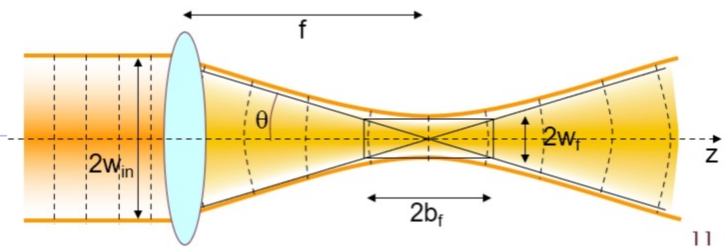
① focusing a Gaussian beam

- incident beam:

$$\begin{cases}\text{plane phase front}\\ \text{width: } W_{in}\end{cases}$$

- outgoing beam:

$$\begin{cases}\text{converge } \theta=\frac{W_{in}}{f}\\ \theta=\frac{\lambda}{\pi W_f}\end{cases}$$



width in the focal plane: $2W_f=\frac{2\lambda f}{\pi W_{in}}$

- if the beam as wide as lens: $\theta=NA_{lens}$ $\quad 2W_f=\frac{2\lambda}{\pi NA_{lens}}$

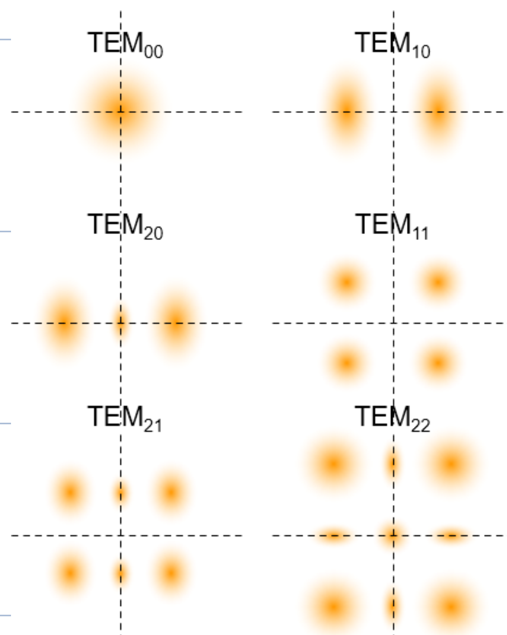
depth of field: $2b_f=kW_f^2$

$$\boxed{2W_f=0.64\lambda/NA\approx\lambda/NA}$$

② telescopes

$$\boxed{\Delta\alpha\approx 0.64\frac{\lambda/NA}{f_1}=1.28\frac{\lambda}{D}}$$

3. Hermite-Gaussian beams



4. M^2 factor

Gaussian beams: $\pi\theta\frac{W_0}{\lambda}=1$

Non-Gaussian beams: $\pi\theta\frac{W_0}{\lambda}\geqslant 1$

$$M^2=\pi\theta\frac{W_0}{\lambda}\geqslant 1$$

Electromagnetism

1. Maxwell equations

no free charges $\begin{cases} \nabla\times\vec{H}=\frac{\partial\vec{D}}{\partial t} \\ \nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \\ \nabla\cdot\vec{D}=0 \\ \nabla\cdot\vec{B}=0 \end{cases}$ $\quad\begin{cases} \vec{D}=\varepsilon_0\vec{E}+\vec{P} \\ \vec{B}=\mu_0(\vec{H}+\vec{M}) \end{cases}$

Poynting vector: $\vec{P}=\vec{E}\times\vec{H}$ (W/m^2)

- homogenous, linear, non-dispersive, non-magnetic and isotropic media:

$$\begin{cases} \nabla\times\vec{H}=\varepsilon\frac{\partial\vec{E}}{\partial t} \\ \nabla\times\vec{E}=-\mu_0\frac{\partial\vec{H}}{\partial t} \\ \nabla\times\vec{E}=0 \\ \nabla\times\vec{H}=0 \end{cases} \qquad \begin{cases} \vec{P}=\varepsilon_0\chi\vec{E} \\ \vec{D}=\varepsilon\vec{E} \end{cases} \text{ with } \varepsilon=\varepsilon_0(1+\chi)$$

$$\nabla^2 u-\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}=0 \text{ with } v^2=\frac{1}{\varepsilon\mu_0} \ \& \ n=\frac{c}{v}$$

2. Polarization

$$\vec{E}(z,t)=Re\left[\vec{A}e^{2j\pi\nu(t-\frac{z}{c})}\right] \qquad \vec{A}=A_x\vec{e}_x+A_y\vec{e}_y$$

- elliptically polarization

$$A_x=a_xe^{j\phi_x} \qquad A_y=a_ye^{j\phi_y}$$

$$\vec{E}(z,t)=E_x\vec{e}_x+E_y\vec{e}_y\begin{cases} E_x=a_x\cos[2\pi\nu(t-\frac{z}{c})+\phi_x] \\ E_y=a_y\cos[2\pi\nu(t-\frac{z}{c})+\phi_y] \end{cases} \Rightarrow \frac{E_x^2}{a_x^2}+\frac{E_y^2}{a_y^2}-2\cos\phi\frac{E_xE_y}{a_xa_y}=\sin^2\phi$$

- linear polarization

$a_x=0$

$\phi=0/\pi \quad E_y=\pm\frac{a_y}{a_x}x$

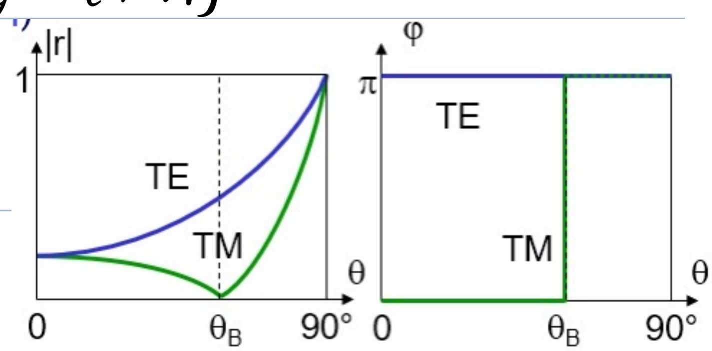
3. Reflection & Transmission

① $r_{TE}=\frac{n\cos\theta-n'\cos\theta'}{n\cos\theta+n'\cos\theta'}$ $\quad r_{TM}=\frac{n'\cos\theta-n\cos\theta'}{n'\cos\theta+n\cos\theta'}$ $\quad (n'>n)$

perpendicular case: $R=\left(\frac{n-n'}{n+n'}\right)^2$

$T=1-R=\frac{4nn'}{(n+n')^2}$

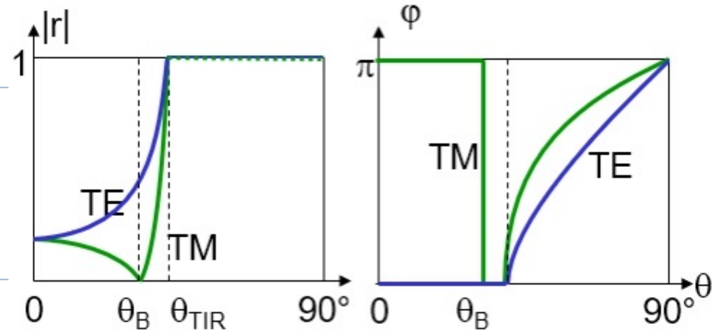
Brewster's angle: $tg\,\theta_B=\frac{n'}{n}$



② $r_{TE} = \frac{n\cos\theta - n'\cos\theta'}{n\cos\theta + n'\cos\theta'}$ $\quad r_{TM} = \frac{n'\cos\theta - n\cos\theta'}{n'\cos\theta + n\cos\theta'}$ $(n > n')$

Brewster's angle: $tg\theta_B = \frac{n'}{n}$

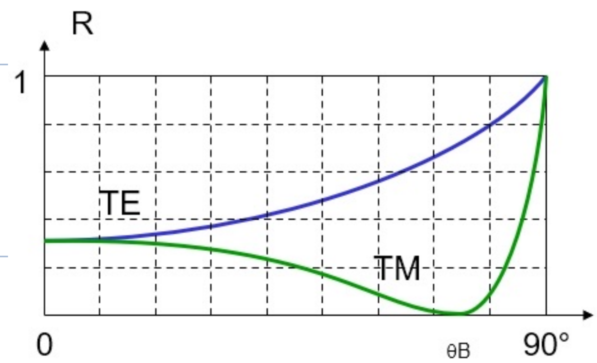
total internal reflection (TIR): $\sin\theta_{crit} = \frac{n'}{n}$



③ power reflection and transmission

$R_{TE} = r_{TE}^2 \qquad T_{TE} = \frac{n'\cos\theta'}{n\cos\theta} t_{TE}^2$

$R_{TM} = r_{TM}^2 \qquad T_{TM} = \frac{n'\cos\theta'}{n\cos\theta} t_{TM}^2$



4. Absorption & Dispersion

① absorption:

complex susceptibility: $\chi = \chi_R + j\chi_I$

$k = k_0\sqrt{(1+\chi_R + j\chi_I)} = \beta - \frac{j}{2}\alpha \qquad e^{-jkz} = e^{-\frac{1}{2}\alpha z} e^{-j\beta z}$

$\begin{cases} \alpha: \text{attenuation or absorption coefficient} \\ \beta: \text{propagation constant} \end{cases}$

② dispersion:

$\chi(\omega)$ & $n(\omega)$ & $\beta(\omega) = k_0 n(\omega)$

phase velocity: $v_p = \frac{c}{n(\omega)} = \frac{k_0 c}{\beta(\omega)} = \left(\frac{\beta(\omega)}{\omega}\right)^{-1}$

group velocity: $v_g = \left(\frac{d\beta(\omega)}{d\omega}\right)^{-1}$

group refractive index: $N = \frac{c}{v_g} = n + \omega\frac{dn}{d\omega} = n - \lambda\frac{dn}{d\lambda}$

5. Layered structures

① symmerical three-layer structure

reflection and transmission of 1 interface:

$R_1 = |r_{12}|^2,\ T_1 = |t_{12}t_{21}|\ (T_1 = 1 - R_1)$

transmission of a plate:

$I = \frac{\quad}{1+R_1^2-2R_1\cos 2\phi} = \frac{\quad}{1+F\sin^2\phi}$ $\left[F=\frac{4R_1}{(1-R_1)^2}\right]$ Airy equation

perpendicular incidence:

$\phi = m\pi \rightarrow d = m\frac{\lambda}{2n_2} \rightarrow$ minimal transmission: $T_{min} = \frac{1}{1+F}$

② coatings

- anti-reflection coating

  interfere destructively

  $n_1<n_2<n_3$ (phase shift π): $\boxed{d=\frac{1}{4}\frac{\lambda_0}{n_2}=\frac{\lambda_2}{4}}$

  $\boxed{n_2=\sqrt{n_1 n_3}}$

- highly reflective coating

  interfere constructively

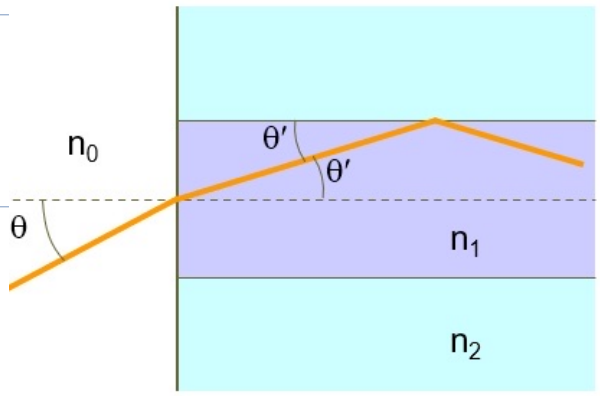
  $n_H d_H = n_L d_L = \frac{\lambda_0}{4}$

  $R=\left[\frac{1-\left(\frac{n_H}{n_L}\right)^{2N}}{1+\left(\frac{n_H}{n_L}\right)^{2N}}\right]^2$ $\begin{cases}\text{converges to 1 for } N\uparrow \\ \text{better converges for } \frac{n_H}{n_L}\uparrow\end{cases}$

Waveguides

1. Waveguide theory

① step-index waveguide



- core: n_1 cladding: n_2 $n_1>n_2$
- TIR: $\theta'_{max} = a\cos\frac{n_1}{n_2}$
- maximal incidence angle:

  $n_0\sin\theta_{max} = \sqrt{n_1^2-n_2^2} \cong \sqrt{2n\Delta n}$

- numerical aperture: $NA = n_0\sin\theta_{max}^{(\prime)}$

② graded-index waveguide

- parabolic index profile

  $\rightarrow$ sinusoidal rays & all have the same period

③ bends in waveguides

- radius R should be large

  $(n_1-n_2)\uparrow \rightarrow R\downarrow$ can be

- $(n_1-n_2)\uparrow \rightarrow$ losses $\downarrow$ $R\uparrow \rightarrow$ losses $\downarrow$

2. Layered waveguides

① longitudinally invariant waveguides

$$n(\vec{r}) = n(x,y) \longrightarrow \begin{cases} \vec{E}(x,y,z) = \vec{e}(x,y)\, e^{-j\beta z} \\ \vec{H}(x,y,z) = \vec{h}(x,y)\, e^{-j\beta z} \end{cases}$$

- effective refractive index: $n_{eff} = \beta / k_0$
- effective dielectric constant: $\varepsilon_{eff} = n_{eff}^2$

② lossless waveguides

- $n_{eff} > n_{max}$: no eigenmodes
- $n_{max} > n_{eff} > \max(n_{cladding})$: guided modes
- $n_{eff} < \max(n_{cladding})$: radiation modes

  propagating radiating modes: $\varepsilon_{eff} > 0 \rightarrow$ real n_{eff}

  evanescent radiating modes: $\varepsilon_{eff} < 0 \rightarrow$ imaginary n_{eff}

③ three-layer slab waveguides

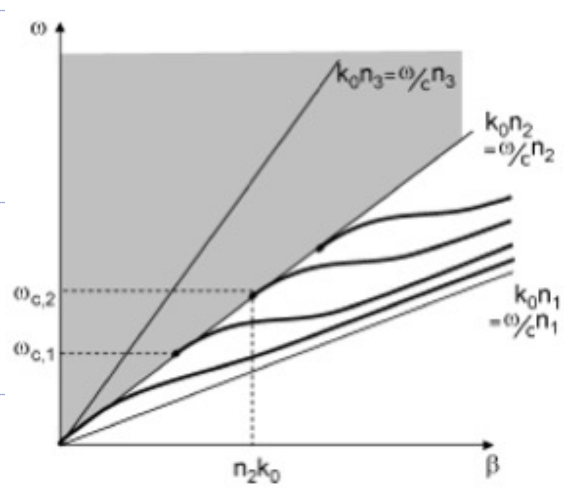
$$\begin{cases} e_{y,i} = A e^{-\delta x} & x \geq 0 \\ e_{y,i} = A\cos\kappa x + B\sin\kappa x & -d \leq x \leq 0 \\ e_{y,i} = (A\cos\kappa d - B\sin\kappa d)\, e^{\gamma(x+d)} & x \leq -d \end{cases}$$

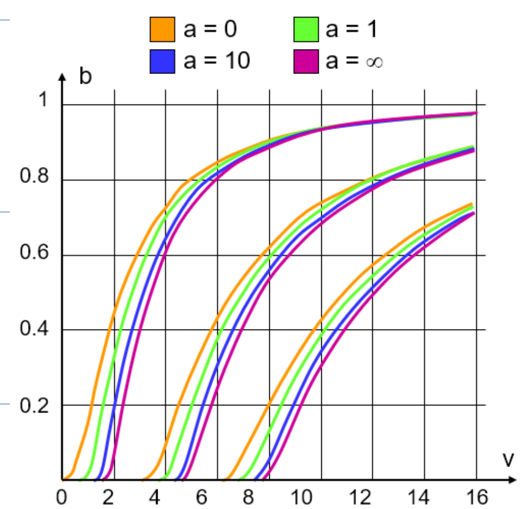
n_3, n_1, n_2; x: 0, $-d$

$$\begin{cases} \delta = \sqrt{\beta^2 - n_3^2 k_0^2} \\ \kappa = \sqrt{n_1^2 k_0^2 - \beta^2} \\ \gamma = \sqrt{\beta^2 - n_2^2 k_0^2} \end{cases}$$

boundary conditions: $\tan\kappa d = \dfrac{\kappa(\gamma+\delta)}{\kappa^2 - \gamma\delta}$

Normalized quantities
$$\begin{cases} \text{normalized frequency: } V = k_0 d\sqrt{n_1^2 - n_2^2} \\ \text{relative effective index: } b = \dfrac{n_{eff}^2 - n_2^2}{n_1^2 - n_2^2} \\ \text{asymmetry parameter: } a^{TE} = \dfrac{n_2^2 - n_3^2}{n_1^2 - n_2^2} \end{cases}$$





Photon Optics

① photon energy

- $E = h\nu = \hbar\omega \ (h = 6.626\times10^{-34}\,\mathrm{Js})$
- $E_n = (n+\frac{1}{2})h\nu, \ n = 0, 1, 2 \cdots$

② photon momentum

- $p = \frac{E}{c} = \hbar k = \frac{h}{\lambda}$

③ photon spin

spin $S = \pm\frac{h}{2\pi}$

④ photon fluxes

power density $[W/m^2]$ $\longrightarrow$ average photon flux density $[\text{photons}/s\cdot m^2]$

$\boxed{\text{For } \lambda = 200\,nm\,(UV) \qquad 1\,nW = 1\ \text{photon}/ns}$

coherent light probability of n photons arriving in a given time shot:

$$p(n) = \frac{\bar{n}^n e^{\bar{n}}}{n!}$$

Lasers

1. Gain medium

① Einstein relations

- $\begin{cases} \text{Absorption}: \frac{dN_2}{dt} = -\frac{dN_1}{dt} = P_{abs}N_1 = B_{21}\rho(\nu_0)N_1 \\ \text{Spontaneous emission}: \frac{dN_2}{dt} = -\frac{dN_1}{dt} = -P_{sp}N_2 = -A_{21}N_2 \\ \text{Stimulated emission}: \frac{dN_2}{dt} = -\frac{dN_1}{dt} = -P_{st}N_2 = -B_{21}\rho(\nu_0)N_2 \end{cases}$
- Einstein relation: $g_1B_{12} = g_2B_{21} \qquad A_{21} = B_{12}\frac{8\pi h\nu^3}{c^3}$
- Boltzmann distribution: $\frac{N_1}{N_2} = \frac{g_1}{g_2}\exp\left(\frac{E_2-E_1}{kT}\right) = \frac{g_1}{g_2}\exp\left(\frac{h\nu}{kT}\right)$

② absorption or amplication of a field

- Intensity $I(x)$ $[W/m^2]$: $I = \rho_\nu\frac{c}{n} = N_f h\nu\frac{c}{n}$
- $\frac{dI}{dx}(x) = \frac{dI}{dx}\Big|_{st} + \frac{dI}{dx}\Big|_{abs} = \rho_\nu(N_2-N_1)h\nu B_{21} = I(x)\frac{n}{c}(N_2-N_1)h\nu B_{21}$
- $I(x) = I_0e^{gx} \qquad \boxed{g = (N_2-N_1)\frac{h\nu n}{c}B_{21}}$

③ pump in a two-level system

- absorption = emission
- $(N_1-N_2)\rho_\nu B - N_2A_{21} = 0 \longrightarrow N_2 = \frac{N_1}{\quad}$

$(N_1-N_2)f\nu B_{21} \quad N_2A_{21} \quad \ldots \quad N_2 \quad 1+\frac{A_{21}}{B_{21}N_f h\nu}$

$N_2<N_1$ never population inversion

④ pump in a three-level system

- population inversion between E_2 and E_1
- drawback: need larger pump power

  pump requires high N_1

  population inversion requires low N_1

⑤ pump in a four-level system

- population inversion between E_2 and E_1
- drawback: high pump energy (E_3-E_0)

⑥ inhomogeneous broadening

- doppler broadening:

$$\Delta\nu_D=2\nu_0\sqrt{\frac{2kT}{Mc^2}\ln 2} \qquad g(\nu)=\frac{2\sqrt{\ln 2}}{\sqrt{\pi}\Delta\nu_D}\exp\left[-4\ln 2\left(\frac{\nu-\nu_0}{\Delta\nu_D}\right)^2\right]$$

⑦ homogeneous broadening

- Heisenberg: $\delta(E_2-E_1)\tau=\frac{h}{2\pi}$

  uncertainty on the frequency: $\delta\nu=\frac{1}{2\pi\tau}$

- Lorentz-distribution:

$$g(\nu)=\frac{\Delta\nu}{2\pi\left[(\nu-\nu_0)^2+\left(\frac{\Delta\nu}{2}\right)^2\right]}$$

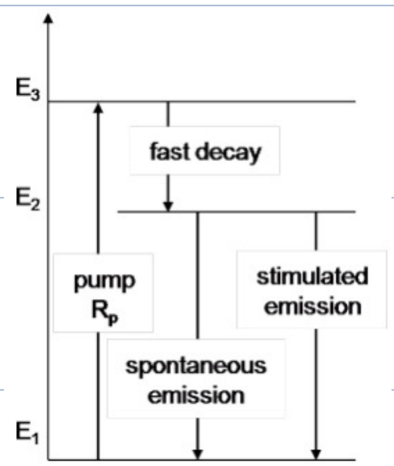
⑧ homogeneous & inhomogeneous

- time between collisions $\begin{bmatrix} << \\ >> \end{bmatrix}$ lifetime excited atoms $\begin{cases}\text{homogeneous broadening} \\ \text{inhomogeneous broadening}\end{cases}$

2. Laser cavity

① rate equations

$$\begin{cases} \frac{dN_2}{dt}=R-A_{21}N_2-N_f h\nu B_{21}(N_2-N_1) \\ \frac{dN_1}{dt}=-\frac{dN_2}{dt} \\ \frac{dN_f}{dt}=N_f h\nu B_{21}(N_2-N_1)+\beta A_{21}N_2-\frac{N_f}{\tau_p} \end{cases}$$



- neglect spontaneous emission:

$$\frac{dN_f}{dt}=N_f\left(cg-\frac{1}{\tau_p}\right) \longrightarrow \begin{cases}\text{gain (per s)}: cg-\frac{1}{\tau_p}\end{cases}$$

gain (per m): $g - \frac{1}{c\tau_p}$

- passing through the cavity (length L)

  loop gain $= e^{(g - \frac{1}{c\tau_p})2L}$

$$\begin{cases} \frac{dN_2}{dt} = R - A_{21}N_2 - cgN_f \\ \frac{dN_f}{dt} = cgN_f + \beta A_{21}N_2 - \frac{N_f}{\tau_p} \\ g = \frac{B_{21}h\nu}{c}(2N_2 - N) \end{cases}$$

gain threshold: $g = \frac{1}{c\tau_p}$

② power transmission

$T = |t|^2 = \frac{T_{max}}{1 + F\sin^2\phi}$ $\quad$ $T_{max} = 1$ if $r_1 = r_2$: symmetrical structure

$F = \frac{4r^2}{(1-r^2)^2}$

maxima for $2\phi = 2m\pi$ / $L = \frac{m\lambda}{2n}$

distance between maxima $\Delta\lambda = \frac{\lambda^2}{2nL}$ / $\Delta\nu = \frac{c}{2nL}$

③ quality factor

finesse $= \frac{\text{distance between maxima}}{\text{3dB width of peak}} = \frac{\Delta\nu}{\Delta\nu_{3dB}}$

④ stability condition

$0 \le (1 - \frac{P_1 L}{2n})(1 - \frac{P_2 L}{2n}) \le 1 \qquad P = \frac{2}{R}$

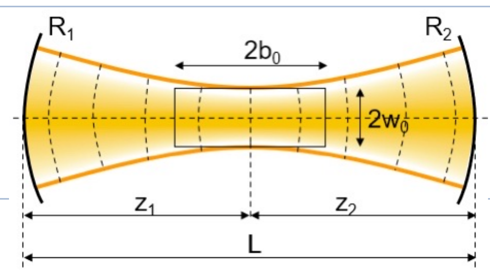
⑤ Gaussian beam

$R_1 = z_1 + \frac{b_0^2}{z_1}$ & $R_2 = z_2 + \frac{b_0^2}{z_2}$

if $R_1 = R_2 = R$:

$z_1 = z_2 = \frac{L}{2} \qquad b_0^2 = \frac{L}{2}(R - \frac{L}{2})$

$w_0 = \sqrt{\frac{2}{kn}\sqrt{\frac{L}{2}(R - \frac{L}{2})}}$



3. Properties of laser beams

① coherence

degree of coherence $\gamma_{12}(\tau) = \frac{\langle E_1(t+\tau)E_2^*(t)\rangle}{\sqrt{\langle|E_1(t)|^2\rangle\langle|E_2(t)|^2\rangle}}$ $\quad$ between 0 and 1

$\gamma_{11}(\tau)$, τ = coherence time $\qquad L = \tau c$ = coherence length

② monochromatic

③ directional

④ intense

④ intense

4. Pulsed lasers

① mode locking

- intensity distribution: $I(t) = E_0^2 \left(\frac{\sin(\pi N \Delta\nu t)}{\sin(\pi \Delta\nu t)} \right)^2$
- maximum number of modes: $N_{max} = \frac{\Delta\nu_{gain}}{\Delta\nu}$

minimal pulse duration: $\tau_{min} = \frac{1}{\Delta\nu_{gain}}$

② Q-switching

Semiconductor light sources

1. Optical properties

① absorption

$\begin{cases} \text{photon energy} \ll \text{bandgap: little absorption} \\ \text{photon energy} \sim \text{bandgap: large increase in absorption} \end{cases}$

$\begin{cases} \text{direct band structure: sudden increase of absorption} \\ \text{indirect band structure: slow increase of absorption} \end{cases}$

② stimulated emission

stimulated emission > absorption

- absorption coefficient $\alpha < 0$
- occurs when $\hbar\omega > E_g$

③ refractive index

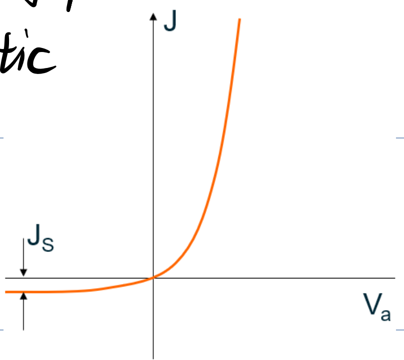
- large bandgap → low n
- increases with photon energy

2. The PN-junction

① recombination

- recombination rate $U = B(n_0 \Delta p + p_0 \Delta n)$
- $\begin{cases} \text{indirect bandgap transition: radiative recombination improbable} \\ \text{direct bandgap transition: radiative recombination dominates} \end{cases}$

② I-V characteristic



3. Semiconductor light sources

① LEDs

- internal quantum efficiency

$$\eta_i = \frac{U_r}{U} = \frac{1/\tau_r}{1/\tau} \qquad \phi_i = \eta_i G V = \eta_i \frac{V \Delta n}{\tau} = \frac{V \Delta n}{\tau_r}$$

- extraction efficiency

improve extraction $\begin{cases} \text{less TIR} \\ \text{works as collimating lens} \end{cases}$

② modulation bandwidth

- transfer of current variation to light variation

$$R(f) = \frac{\Delta P}{\Delta I} = \frac{R(0)}{\sqrt{1+4\pi^2 f^2 \tau^2}}$$

- 3dB bandwidth $\quad f_{3dB} = \frac{1}{2\pi\tau}$

③ amplication

$$g_p = \alpha\left(\frac{J}{J_T} - 1\right)$$