Lasers 2022-2023 Exam

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Problem

In the seventieth there was an active search for stable lasers operating in the far-infrared (FIR) range to bridge the gap between the infrared (IR) range and the millimeter-wave range. CO_2 lasers emitting in the IR, namely around $10\,\mu m$, were already quite well developed and used to optically pump FIR lasers made of molecular gas. Here, we will study the output power of a FIR laser using the $570.5\,\mu m$ transition line of methanol vapor CH_3OH. The laser is described by a four-level system as represented in Fig.1. The level 0 corresponds to some rotational level in the vibrational ground state, 1 and 2 are adjacent rotational levels in a low-lying excited vibrational state and 3 represents the lumped effect of those rotational levels in the upper vibrational manifold, which are not directly involved in the laser transition $2 \rightarrow 1$.

The molecules are excited from 0 to 2 at a pumping rate w_p by a CO_2 laser tuned to the $0 \rightarrow 2$ absorption line. Collisions between the molecules, occurring at a rate w_r, are very efficient in establishing thermal equilibrium among the rotational levels of a given vibrational state. The rate w_r can therefore be assumed to be much larger than the rate w_ν of relaxation back to the vibrational ground state.

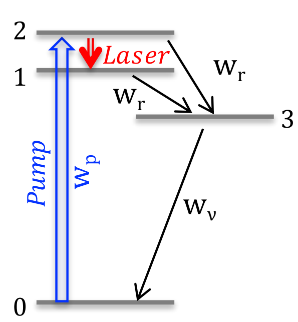

Figure 1: Four-level model for the optically pumped FIR laser

1) Briefly explain what are vibrational and rotational states. What are their corresponding wavelength ranges?

2) For which energy distance from the ground state, the level 1 and 2 can be considered as unpopulated without the CO_2 pumping. Give an order of magnitude at room temperature. What is the physical effect at play? We will consider in the following that the levels 1 and 2 are sufficiently above the ground state to neglect the impact of this effect.

3) Compare the spontaneous emission rate in free space at visible wavelengths, namely for a wavelength around $\lambda = 0.5\,\mu m$ and that in the FIR range $\lambda = 500\,\mu m$. Considering that the typical spontaneous radiative lifetime is $1\,ns$ at $\lambda = 0.5\,\mu m$, what is the spontaneous radiative lifetime at $\lambda = 500\,\mu m$. In view of this result, we will neglect the spontaneous emission in the rate equations governing the laser dynamics.

4) Write the rate equations governing the total occupation numbers per unit of volume N of the various molecular levels, and the FIR photon density n. Use the Einstein coefficient B_{21}, the degeneracy g_1 and g_2 of the level 1 and 2, and τ_c the photon cavity lifetime. The total number of molecules is N.

5) Show that in the steady state the FIR photon density is given by:

$$n = \frac{g_1}{g_1 + g_2} \frac{w_r}{B_{21}} \Big(\frac{N B_{21} \tau_c w_p}{w_r(1 + w_p/w_\nu)} - 1 \Big) \tag{1}$$

6) Determine the small signal gain per unit length.

7) Demonstrate that the saturated intensity is given by:

$$I^{sat} = c \times h\nu \frac{g_1}{g_1 + g_2} \frac{w_r}{B_{21}} \tag{2}$$

We now want to study the impact of the gas pressure p and of the pump power U on the laser characteristic, in particular on the FIR photon density n given in Eq.(1). We assume that the $1 \rightarrow 2$ transition has a homogeneous linewidth $\Delta\nu_{12}$ that varies linearly with the pressure p as $\Delta\nu_{12} = Cp$ with C a constant. The spontaneous radiative lifetime of this transition is τ_{sp}. The active volume of the laser is V.

1) What is the lineshape of the $1 \rightarrow 2$ transition in view of the physics at play? The normalized lineshape is noted $f(\nu)$.

2) Why does the Einstein coefficient $B_{21}(\nu)$ depend on the pressure?

3) Express $B_{21}(\nu)$ as a function of $f(\nu)$, τ_{sp} and $\Delta\nu_{12}$ and any other relevant parameters. How does $B_{21}(\nu)$ varies with the pressure p?

4) Give the equation between the rotational relaxation rate w_r and the pressure p.

5) The pump power absorbed in the gas is denoted by U_{abs} and we assume that a fraction ξ of this power goes into the $0 \to 2$ transition. Express ξU_{abs} as a function of w_p and the pump frequency ν_p.

6) Write the rate equation for the pump intensity I in the cavity considering that Γ measure the loss rate of the pump radiation.

7) The pump rate w_p is proportional to the intensity I of the pump radiation in the cavity with ζ the coefficient of proportionality. Considering 1) the steady state of the rate equation for the pump intensity found in the previous question, 2) the expression of U_{abs} versus w_p, and 3) the ideal gas law, determine w_p as a function of U and p. Introduce the parameters $p_s = 2\pi kT\xi\Gamma/h\nu_p f_0 \zeta V$ and $U_s = (1 + p/p_s)2\pi\Gamma w_\nu/\zeta$, where k is the Boltzmann constant and f_0 the fraction of the total number of molecules being in the 0 state in the absence of pumping.

8) Show that

$$\frac{N w_p}{1 + w_p/w_\nu} = \frac{\xi}{V h\nu_p} \frac{p/p_s}{1 + p/p_s} \frac{U}{1 + U/U_s} \tag{3}$$

if $p \ll p_s$ or $U \ll U_s$. Show that this expression is also valid if $U \gg U_s$.

9) Determine the small signal gain and the FIR output power as a function of the pressure p and the pump power U. Comment.

I. 2) $\Delta E_{01} > k_BT$ $\Delta E_{02} > k_BT$ cf. Boltzman statistics

$$\frac{N_i}{N_0} = \frac{g_i}{g_0} e^{-h\nu/k_BT}$$ at $300K \Rightarrow k_BT \simeq 25\,meV$

$h\nu = \Delta E_{0i}$

3) $A_{21} \propto \nu\rho(\nu)$ & $\rho(\nu) = \frac{8\pi\nu^2}{c^3}$ $\Rightarrow A_{21} \propto \frac{1}{\lambda^3}$

$$A_{21}^{FIR} = \frac{\lambda_{VIS}^3}{\lambda_{FIR}^3} A_{21}^{VIS} \qquad A_{21}^{VIS} = \frac{1}{\tau_{21}^{VIS}} = 10^9 s^{-1} \qquad A_{21}^{FIR} = 1 s^{-1}$$

4) $$\frac{dN_2}{dt} = W_p N_0 - n B_{21}\left(N_2 - \frac{g_2}{g_1} N_1\right) - W_r N_2$$

$$\frac{dN_1}{dt} = n B_{21}\left(N_2 - \frac{g_2}{g_1} N_1\right) - W_r N_1$$

$$\frac{dN_3}{dt} = W_r N_2 + W_r N_1 - W_v N_3$$

$$\frac{dN_0}{dt} = W_v N_3 - W_p N_0 \qquad N = N_0 + N_1 + N_2 + N_3$$

$$\frac{dn}{dt} = n B_{21}\left(N_2 - \frac{g_2}{g_1} N_1\right) - \frac{n}{\tau_c}$$

5) $\frac{dN_2}{dt} = \frac{dN_1}{dt} = \frac{dN_3}{dt} = \frac{dN_0}{dt} = 0$ (steady state)

$\frac{dn}{dt} = 0$ (gain = loss)

from $\frac{dN_0}{dt} = 0 \Rightarrow N_3 = \frac{W_p}{W_v} N_0$

from $\frac{dN_3}{dt} = 0 \Rightarrow N_0 = \frac{W_r}{W_p}(N_1 + N_2)$

$$N = N_0 + \frac{W_p}{W_r} N_0 + \frac{W_p}{W_v} N_0 = \left(1 + \frac{W_p}{W_r} + \frac{W_p}{W_v}\right) N_0$$

$$W_r \gg W_v \Rightarrow N \simeq \left(1+\frac{W_p}{W_v}\right) N_0$$

from $g_2 \times \frac{dN_1}{dt} - g_1 \times \frac{dN_2}{dt} = 0$

$$\Rightarrow -g_1 W_p N_0 + (g_1+g_2) n B_{21} \left(N_2 - \frac{g_2}{g_1} N_1\right) + W_r g_1 N_2 - W_r g_2 N_1 = 0$$

$$\left[(g_1+g_2) n B_{21} + g_1 W_r\right] \underbrace{\left(N_2 - \frac{g_2}{g_1} N_1\right)}_{\Downarrow\, \text{gain} = \text{loss} \; \frac{1}{B_{21}\tau_c}} - g_1 W_p N_0 = 0$$

$$\Rightarrow n = \frac{g_1}{g_1+g_2} \times \frac{W_r}{B_{21}} \left[\frac{N B_{21} \tau_c W_p}{W_r (1 + W_p/W_v)} - 1\right]$$

6) $\gamma = \sigma\left(N_2 - \frac{g_2}{g_1} N_1\right) \qquad B_{21} = \frac{\sigma}{h\nu} c$

$$\left(N_2 - \frac{g_2}{g_1} N_1\right) = \frac{g_1 W_p N_0}{(g_1+g_2) n B_{21} + g_1 W_r}$$

$$\Rightarrow \gamma = \frac{\frac{h\nu}{c} B_{21} g_1 W_p N_0}{g_1 W_r + (g_1+g_2) n B_{21}}$$

$$\gamma = \underbrace{\boxed{\frac{h\nu}{c} B_{21} \frac{W_p}{W_r} \frac{N}{\left(1+\frac{W_p}{W_v}\right)}}}_{\gamma_0} \times \frac{1}{1 + \underbrace{\boxed{\frac{(g_1+g_2) B_{21}}{c h\nu g_1 W_r}}}_{\frac{1}{n_{sat}}} n}$$

II. 1) $B_{21}(\nu) = \frac{c^3}{8\pi h\nu^3} A_{21} g(\nu)$ $\qquad$ $g(\nu) \to$ Lorentzian

$$g(\nu) = \frac{\gamma}{(\frac{\gamma}{2})^2 + (2\pi(\nu-\nu_0))^2} \qquad g(\nu) = \frac{4}{\gamma} f(\nu) \qquad \gamma = 2\pi\Delta\nu_{12}$$

2)

3) $B_{21}(\nu) = \frac{1}{h\nu} \times \frac{c^3}{4\pi^2\nu^2 T_{sp}\Delta\nu_{12}} f(\nu)$ $\qquad$ $\Delta\nu_{12} = Cp$ $\qquad$ $B_{21}(\nu) \propto \frac{1}{p}$

4) $w_\nu = \frac{1}{2}\gamma = \pi\Delta\nu_{12} = \pi C p$

5) $\xi U_{abs} = w_p N_0 V h\nu_p = h\nu_p N \frac{w_p}{1+w_p/w_\nu} V$

6) $\frac{dI}{dt} = U - 2\pi\Gamma I - U_{abs}$

7) $w_p = f(p, U)$ $\qquad$ $\frac{dI}{dt} = U - 2\pi\Gamma I - U_{abs} = 0$

$$U - 2\pi\Gamma\frac{w_p}{\zeta} - \frac{h\nu_p}{\xi} V \frac{N w_p}{1+w_p/w_\nu} = 0$$

ideal gas law : $N = \frac{p}{kT}$ $\qquad$ $p_s = \frac{2\pi k T \xi \Gamma}{h\nu_p \zeta V}$

$$\frac{w_p/w_\nu}{1+w_p/w_\nu} \times \frac{p}{p_s} + \frac{w_p}{w_\nu} = \frac{V}{V_s}(1+\frac{p}{p_s})$$

$$V_s = (1+\frac{p}{p_s})\frac{2\pi\Gamma w_\nu}{\zeta} \qquad \gamma = \frac{w_p}{w_\nu}$$

$$\gamma^2 + \gamma(1+\frac{p}{p_s})(1-\frac{U}{U_s}) - \frac{U}{U_s}(1+\frac{p}{p_s}) = 0$$

$$\Delta = (1+\frac{p}{p_s})^2(1-\frac{U}{U_s})^2 + 4\frac{U}{U_s}(1+\frac{p}{p_s})$$

$$\gamma = \frac{1}{2}\left[(1+\frac{p}{p_s})(\frac{U}{U_s}-1) + \sqrt{\Delta}\right]$$

① $P \ll P_s \Rightarrow \gamma \simeq \frac{(\frac{U}{U_s}-1)+(\frac{U}{U_s}+1)}{2} = \frac{U}{U_s}$

② $U \ll U_s \Rightarrow \gamma \simeq \frac{U}{U_s}$

③ $U \gg U_s \Rightarrow \gamma \simeq \frac{1}{2}(1+\frac{P}{P_s})\frac{U}{U_s} + (1+\frac{P}{P_s})\frac{U}{U_s}\sqrt{1+\frac{U_s}{U}\frac{4}{(1+1/P_s)}}$

$$\gamma \simeq (1+\frac{P}{P_s})\frac{U}{U_s}$$

8) $\frac{NW_P}{1+W_P/W_V} = W_V\frac{P}{KT} \times \frac{\gamma}{1+\gamma} = W_V\frac{P}{KTU_s}\frac{U}{1+U/U_s}$

$$= \frac{\xi}{Vh\nu_P}\frac{P/P_s}{1+P/P_s} \times \frac{U}{1+U/U_s}$$

$$\frac{NW_P}{1+W_P/W_V} = W_V\frac{P}{\quad}$$

9) $\gamma_0 = \frac{h\nu}{c}B_{21} \times \frac{1}{W_r}\frac{NW_P}{(1+\frac{W_P}{W_V})}$

$$\gamma_0 = \frac{\xi B_{21}}{cV} \times \frac{1}{W_r} \times \frac{P/P_s}{1+P/P_s} \times \frac{U/U_s}{1+U/U_s}$$

$$B_{21}(\nu) = \frac{1}{h\nu_P}\frac{c^3}{4\pi^2\nu^2 T_{sp}}$$