Lasers 2021-2022 Homework

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Problem

The purpose of this problem is to model Tm:YVO_4 lasers that are made of thulium ions Tm^{3+} in a yttrium orthovanadate crystal YVO_4. Such solid-state lasers emit in the near infrared. They are of great interest for various applications ranging from ground or space based remote sensing applications (lidar) that request eye-safe spectral emission to medical applications making use of the strong water absorption in the near infrared.

1) Give the labeling of the fundamental state of the thulium atom.

2) Determine the fundamental state of the thulium ion Tm^{3+} as well as its four first excited states in free space.

3) What is the impact of the crystallin environment when the thulium ions are in the YVO_4 matrix? Considering that the first excited state has an angular momentum (state orbital) two times smaller than that of the ground state whereas the second and third excited states have the same angular momentum as the ground state, label the first four states, ground state included.

4) What is the degeneracy of the different states?

5) Among the four first states, ground state included, what are the possible radiative transitions according to the selection rules?

We consider only the first four energy levels of the thulium ions in the YVO_4 crystal. The crystal is optically pumped at $798\,nm$, which corresponds to the transition between the ground state (level 1) and the fourth level (level 4). The laser transition occurs between the energy level 2 and the ground state with a wavelength $\lambda_L = 1920\,nm$. The absorption coefficient at $798\,nm$ is $\alpha = 32\,cm^{-1}$. The laser emission cross section is $\sigma_e = 6.7 \times 10^{-22}\,cm^{-2}$. The lifetime τ_2 of level 2 is assumed to be the same as in the case of Tm:YAG, namely $\tau_2 = 12\,ms$. The lifetime of level 4 is noted τ_4. The photon **densities** at the pump and laser wavelength are Φ_P and Φ_L, respectively.

1) Determine the absorption scattering cross σ_a for the transition between level 1 and level 4 knowing that the molar mass and density of the YVO_4 crystal are $204\,g.mol^{-1}$ and $4.24\,g.cm^{-3}$.

2) Considering only transitions involving level 1, level 2 and level 4, write the rate equations for the populations densities of thulium ions N_2 and N_4 in level 2 and level 4.

3) Assuming $N_2 \gg N_3$, N_4, determine the steady state equation for N_2 where only the total density of thulium ions N_T is involved.

4) Introducing normalized intensity densities $I_L^r = \frac{I_L}{I_L^{sat}}$ and $I_P^r = \frac{I_P}{I_P^{sat}}$, where I_L and I_P are the intensity densities of the laser emission and of the pump beam, respectively, and I_L^{sat} and I_P^{sat} are constants to be defined, show that:

$$I_L^r = \frac{1}{2}(\frac{N_T}{\Delta N^{th}} + 1)\left[\frac{\frac{N_T}{\Delta N^{th}} - 1}{\frac{N_T}{\Delta N^{th}} + 1} I_P^r - 1\right] \tag{1}$$

with ΔN^{th} a constant involving the cavity photon lifetime τ_p.

5) Determine the value of the pump intensity at threshold I_P^{th} by considering an optical cavity made of two mirrors separated by a distance $L = 1\,mm$ and reflection coefficient $R_1 = 0.995$, $R_2 = 0.98$ at the laser wavelength. The intrinsic loss is $\alpha_i = 0.024\,cm^{-1}$ at the laser wavelength.

6) On which parameters and how can we operate to decrease the pump intensity as much as possible?

7) What is the value of the laser intensity density for a pump intensity densities $I_P = 2 \times I_P^{th}$ and $I_P = 10 \times I_P^{th}$. How to increase the slope of the laser characteristic?

We will now determine the waist of the laser mode inside the cavity. Due to thermal effects resulting from the pump and laser beam profile we assume that the refractive index inside the crystal is given by $n(z) = n_0(z) - \frac{1}{2}n_2(z)r_\perp^2$ with z the coordinate along the light propagation axis (optical axis) and $\vec{r}_\perp$ the transverse vector defining the position perpendicular to the optical axis. We assume that the mirrors are flat and made by coatings deposited on the end facets of the YVO_4 crystal.

1) Show that the ray equation $\frac{d}{ds}(n\frac{d\vec{r}}{ds}) = \overrightarrow{grad}\, n$ can be written in the paraxial approximation as $\frac{d}{dz}\left[n_0(z)\frac{dr_\perp(z)}{dz}\right] + n_2(z)r_\perp(z)$. The parameter s parametrizes the curve of the ray and $\vec{r}$ defines the position of a point of the ray in the space: $\vec{r} = r_x(s)\vec{x} + r_y(s)\vec{y} + r_z(s)\vec{z}$ with $(\vec{x}, \vec{y}, \vec{z})$ the standard basis vectors of the three dimensional spatial space.

2) We neglect the z variation of the refractive index. Defining $r'_\perp(z) = n_0 \frac{dr_\perp(z)}{dz}$ where $\frac{dr_\perp(z)}{dz}$ is nothing else than the ray angle with the optical axis, show that the transfer matrix between $\begin{pmatrix} r_\perp(z) \\ r'_\perp(z) \end{pmatrix}$ and $\begin{pmatrix} r_\perp(0) \\ r'_\perp(0) \end{pmatrix}$ is given by:

$$\begin{pmatrix} \cos\gamma z & (n_0\gamma)^{-1}\sin\gamma z \\ -n_0\gamma\sin\gamma z & \cos\gamma z \end{pmatrix} \tag{2}$$

where γ is a parameter to define.

3) Write the transfer matrix of the laser optical cavity.

4) For which condition is the cavity stable?

5) Determine the radius of curvature and the waist of the fundamental Gaussian beam that can be excited inside the cavity.

6) Using the dispersion relationship and the eikonal equation, retrieve the ray equation. (Hint: use $\overrightarrow{\nabla}(\vec{A}\cdot\vec{B}) = \vec{A}\times(\overrightarrow{\nabla}\times\vec{B}) + \vec{B}\times(\overrightarrow{\nabla}\times\vec{A}) + (\vec{B}\cdot\overrightarrow{\nabla})\vec{A} + (\vec{A}\cdot\overrightarrow{\nabla})\vec{B}$ where $\times$ stands here for vectorial product and $\cdot$ for the scalar product).

Lanthanide ions are subject to energy transfers between them. For instance up-conversion processes can happen: when one ion relaxes from level 2 to the ground state level 1, the corresponding energy can be used to excite a close-by ion from level 2 to level 4 with a rate constant k_{2124}. The rate constant k_{2124} depends on the distance between the ions, *i.e.* on their concentration in the crystal. Note that cross-relaxation processes can also happen: when one ion relaxes from level 4 to level 2, the corresponding lost energy can be used to excite a close-by ion from the ground state level 1 to the level 2 with a rate constant k_{4212}. Level 3 can also be involved in other cross-relaxation processes. In the following we consider only the k_{2124} process.

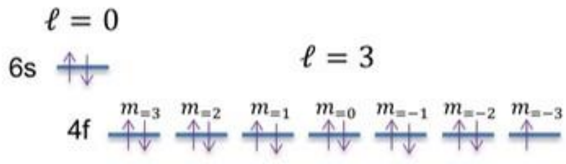
1) Write the new rate equations for the populations densities of thulium ions N_2 and N_4.

2) Find the new laser equation I_L versus I_P. What is the impact of the up-conversion process? As typical values, $k_{2124} \sim 4\times10^{-18}\, cm^3 s^{-1}$ and $\tau_4 \sim 1\, ms$ for a 5% atomic concentration of thulium ions.

LASERS 2022-2023 HOMEWORK: 1ST PART

1) Give the labeling of the fundamental state of the thulium atom

Tm (4f[13] 6s[2])



$$L = |2\times3 + 2\times2 + 2\times1 - 2\times1 - 2\times2 - 1\times3| = 3$$

$$S = 1/2$$

$$|L - S| = 5/2 \qquad L + S = 7/2$$

$$5/2 \leq J \leq 7/2$$

$\Rightarrow$ $^2F_{7/2}$

$J = |L + S|$

Orbital more than half filled

LASERS 2022-2023 HOMEWORK: 1ST PART

2) Determine the fundamental state of the thulium ion Tm^{3+} as well as its four first excited states in free space.

States of lower energy:

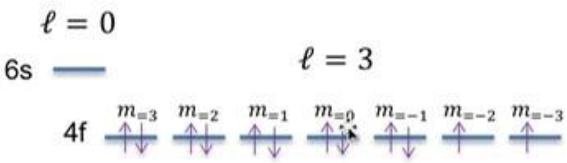
First

$S\ max$

And then

$$L = \left|\sum m_z\right|\ max$$

Tm^{3+} (4f[12])



FUNDAMENTAL STATE

$$L = |2\times3 + 2\times2 + 2\times1 - 2\times1 - 1\times2 - 1\times3| = 5$$

$$S = 1$$

$$|L - S| = 4 \qquad L + S = 6$$

$$4 \leq J \leq 6$$

3 times degenerated

$\Rightarrow$ $^3H_6$

$J = |L + S|$

Orbital more than half filled

LASERS 2022-2023 HOMEWORK: 1ST PART

2) Determine the fundamental state of the thulium ion Tm^{3+} as well as its four first excited states in free space.

States of lower energy:

First

$S\ max$

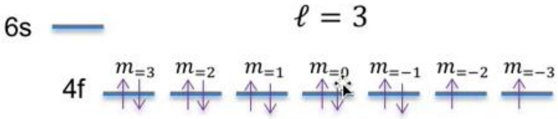
And then

$L = \left|\sum m_z\right|\ max$

Tm^{3+} ($4f^{12}$)

$\ell = 0$

$\ell = 3$



FUNDAMENTAL STATE

$L = |2\times3 + 2\times2 + 2\times1 - 2\times1 - 1\times2 - 1\times3| = 5$

$S = 1$

$|L - S| = 4 \qquad L + S = 6$

$4 \leq J \leq 6$

3 times degenerated

$\Rightarrow$ $^3H_6$

$\uparrow$

$J = |L + S|$

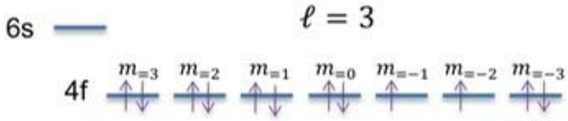
Orbital more than half filled

LASERS 2022-2023 HOMEWORK

2) The first excited states in free space.

$\ell = 0$

$\ell = 3$



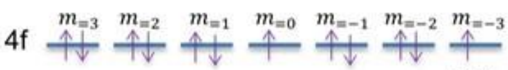
$^3F_4$

$S = 1$

$2 \leq J \leq 4$

3 times degenerated

$\ell = 3$

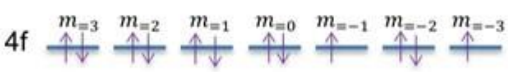


$^3F_4$

$2 \leq J \leq 4$

3 times degenerated

$\ell = 3$

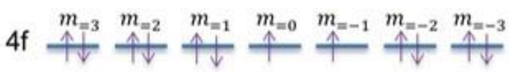


$^3G_5$

$3 \leq J \leq 5$

3 times degenerated

$\ell = 3$

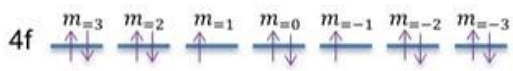


$^3P_2$

$0 \leq J \leq 2$

3 times degenerated

$\ell = 3$

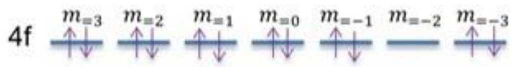


$^3S_1$

$1 \leq J \leq 1$

3x1 times degenerated

$\ell = 3$

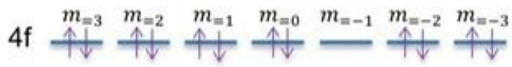


$^1G_4$

$S = 0$

1 times degenerated

$\ell = 3$

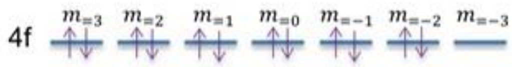


$^1D_2$

$S = 0$

1 times degenerated

$\ell = 3$

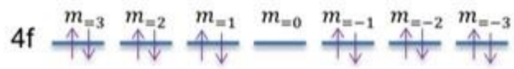


$^1I_6$

$S = 0$

1 times degenerated

$\ell = 3$



$^1S_1$

$S = 0$

1 times degenerated

LASERS 2022-2023 HOMEWORK: 1ST PART

3) What is the impact of the crystallin environment when the thulium ions are in the YVO4 matrix?

Lift the degeneracy:

$^3H_6$

$4 \leq J \leq 6$

3 times degenerated

$\Rightarrow$

$^3H_4$

$^3H_5$

$^3H_6$

LASERS 2022-2023 HOMEWORK: 1ST PART

3) Considering that the first excited state has an angular momentum (state orbital) two times smaller than that of the ground state whereas the second and third excited states have the same angular momentum as the ground state, label the first four states, ground state included.

$^3H_4$

$^3H_5$

$^3F_4$

$^3H_6$

Depends on the crystalline matrix

Silica germanate glass

$^1I_6$

$^1D_2$

$^1G_4$

$^3F_2$

$^3F_3$

$^3H_4$

$^3H_5$

$^3F_4$

$^3H_6$

ZBLAN fiber (fluoride glass)

$^3P_2$

$^1I_6$

$^1D_2$

$^1G_4$

$^3F_2$

$^3F_3$

$^3F_4$

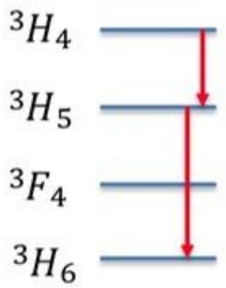
$^3H_4$

$^3H_5$

$^3H_6$

LASERS 2022-2023 HOMEWORK: 1ST PART

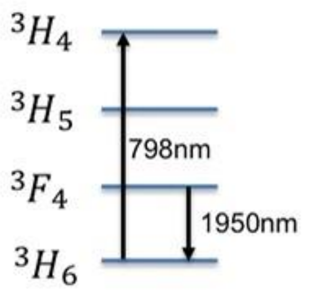
5) Among the four first states, ground state included, what are the possible radiative transitions according to the selection rules?



transition allowed only with
Δn arbitray
$\Delta L = 0 \; or \pm 1$
$\Delta J = \pm 1$

No transition from singlet to triplet states for ideal $\vec{L}\vec{S}$ coupling

"Forbidden transitions"



The crystal field partially relaxes the selection rules

LASERS 2022-2023 HOMEWORK: 2ND PART

1)

$\sigma_a = \frac{\alpha}{N}$ with N the number of emitters per cm^3

$\alpha = 32 \; cm^{-1}$

N_{YVO_4} : number of atoms per cm^3 in the crystal

$$N_{YVO_4} = \frac{\overset{[g \cdot cm^{-3}]}{4{,}24}}{\underset{[g \cdot mol^{-1}]}{204}} \times \underbrace{6{,}022 \cdot 10^{23}}_{\text{Avogadro Number } [mol^{-1}]} \simeq 0{,}125 \cdot 10^{23} \; atoms/cm^3$$

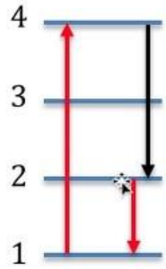
$N = \frac{5}{100} N_{YVO_4} \Rightarrow N \simeq 6{,}3 \cdot 10^{20} \; cm^{-3}$

$$\boxed{\sigma_a \simeq 5{,}1 \cdot 10^{-20} \; cm^2}$$

2)

$$\frac{dN_2}{dt} = -\frac{N_2}{\tau_2} - \sigma_e c \phi_L (N_2 - N_1) + \frac{N_4}{\tau_4}$$

$$\frac{dN_4}{dt} = \sigma_p c \phi_p (N_1 - N_4) - \frac{N_4}{\tau_4}$$



3)

$$N_1 \approx N_T - N_2 \quad (N_2 \gg N_3, N_4)$$

$$N_1 - N_4 = N_T - N_2 - N_4 \simeq N_T - N_2$$

$$0 = -\frac{N_2}{\tau_2} - \sigma_e c \phi_L (2N_2 - N_T) + \sigma_p c \phi_p (N_T - N_2)$$

LASERS 2022-2023 HOMEWORK: 2ND PART

4)

$$I_L = h\nu_L c \phi_L \qquad I_L^{sat} = \frac{h\nu_L}{\sigma_e \tau_2}$$

$$I_p = h\nu_p \phi_p \qquad I_p^{sat} = \frac{h\nu_p}{\sigma_p \tau_2}$$

$$N_2 + \frac{I_L}{I_L^{sat}}(2N_2 - N_T) - \frac{I_p}{I_p^{sat}}(N_T - N_2) = 0$$

LASERS 2022-2023 HOMEWORK: 2ND PART

4)

$$I_L^R = \frac{I_L}{I_L^{sat}} \qquad I_p^R = \frac{I_p}{I_p^{sat}}$$

$$N_2(1 + 2I_L^R + I_p^R) = N_T(I_L^R + I_p^R)$$

$$\frac{N_2}{N_T} = \frac{I_L^R + I_p^R}{1 + 2I_L^R + I_p^R} = x$$

$$\gamma = \sigma_e \Delta N = \sigma_e(N_2 - N_1) = \sigma_e(2N_2 - N_T)$$
$$= \sigma_e N_T(2x - 1)$$

4)

Threshold condition

$$2L\gamma = \alpha_i 2L - \ln R_1 R_2$$

$$\gamma = \alpha_i - \frac{1}{2L}\ln R_1 R_2$$

$$\gamma = \frac{1}{c\tau_p}$$

LASERS 2022-2023 HOMEWORK: 2ND PART

4)

$$\sigma_e \Delta N = \sigma_e N_T (2X - 1) = \frac{1}{c\tau_p}$$

$$2X - 1 = \frac{1}{N_T \sigma_e c \tau_p} \qquad \Delta N^{th} = (\sigma_e c \tau_p)^{-1}$$

photon lifetime inside the cavity

$$2X - 1 = \frac{\Delta N^{th}}{N_T}$$

$$2X - 1 = \frac{2I_L^R + 2I_P^R - 1 - 2I_L^R - I_P^R}{1 + 2I_L^R + I_P^R} = \frac{I_P^R - 1}{1 + 2I_L^R + I_P^R}$$

LASERS 2022-2023 HOMEWORK: 2ND PART

4)

$$\frac{I_p^R - 1}{1+2I_L^R+I_p^R} = \frac{\Delta N^{Th}}{N_T}$$

$$1+2I_L^R+I_p^R = \frac{N_T}{\Delta N^{Th}}\left(I_p - 1\right)$$

$$2I_L^R = \frac{N_T}{\Delta N^{th}}\left(I_p^R - 1\right) - 1 - I_p^R$$

$$2I_L^R = \left(\frac{N_T}{\Delta N^{th}} - 1\right) I_p^R - \frac{N_T}{\Delta N^{th}} - 1$$

$$\boxed{I_L^R = \frac{1}{2}\left(\frac{N_T}{\Delta N^{th}}+1\right)\left[\left(\frac{\frac{N_T}{\Delta N^{th}} - 1}{\frac{N_T}{\Delta N^{th}}+1}\right) I_p^R - 1\right]}$$

LASERS 2022-2023 HOMEWORK: 2ND PART

5)

at threshold $I_L^R \neq 0$

$$I_L^R \geq 0 \iff \left(\frac{N_T/\Delta N^{th} - 1}{N_T/\Delta N^{th} + 1}\right) I_p^R \geq 1$$

$$N_T > \Delta N^{th} \implies I_p^R \geq \frac{N_T/\Delta N^{th} + 1}{N_T/\Delta N^{th} - 1}$$

$$I_p \geq \left(\frac{N_T/\Delta N^{th} + 1}{N_T/\Delta N^{th} - 1}\right) I_p^{sat}$$

$$I_p^{th} = \left(\frac{N_T \sigma_e c \tau_p + 1}{N_T \sigma_e c \tau_p - 1}\right) \times \frac{h\nu_p}{\sigma_p \tau_2} \qquad \sigma_p = \sigma_a$$

5)

$$c\tau_p = \frac{1}{\alpha_i - \frac{1}{2L}\ln R_1R_2} \Rightarrow c\tau_p = \frac{1}{0{,}024 - \frac{1}{2\cdot 10^{-3}}\ln(0{,}995\times 0{,}98)}\ [cm]$$

$$c\tau_p \simeq 0{,}079\ [cm]$$

$$\alpha_i - \frac{1}{2L}\ln R_1R_2 \simeq 12{,}6\,[cm^{-1}]$$

$$\sigma_e = \frac{\gamma_0}{N_T} \quad \text{with } N_T = 6{,}3\cdot 10^{20}\ cm^{-3} \quad (\text{see question 1}))$$

per definition

for $\sigma_e = 6{,}7\cdot 10^{-22}\,[cm^2]$, $\gamma \simeq 0{,}42\,[cm^{-1}]$

$< \alpha_i - \frac{1}{2L}\ln R_1R_2$

No laser emission possible

for $\sigma_e = 6{,}7\cdot 10^{-20}\,[cm^2]$, $\gamma \simeq 42{,}2\,[cm^{-1}]$

$> \alpha_i - \frac{1}{2L}\ln R_1R_1$

laser emission possible

LASERS 2022-2023 HOMEWORK: 2ND PART

5) $N_T\sigma_e c\tau_p = \gamma \times c\tau_p \simeq \frac{42{,}2\,[cm^{-1}]}{12{,}6\,[cm^{-1}]} \simeq 3{,}35$

$$I_p^{th} \simeq \left(\frac{3{,}35+1}{3{,}35-1}\right)\times\frac{h\nu_p}{\sigma_p\tau_2} \simeq 1{,}85\times\frac{h\nu_p}{\sigma_p\tau_2}$$

$$\frac{h\nu_p}{\sigma_p\tau_2} = \frac{hc}{\sigma_a\tau_2\lambda_p} \qquad \sigma_p = \sigma_a$$

$\sigma_a = 5{,}1\cdot 10^{-20}\ [cm^2]$

$\tau_2 = 12\cdot 10^{-3}\ [s]$

$\lambda_p = 0{,}798\cdot 10^{-6}\ [m]$

$c \simeq 3\cdot 10^{8}\ [m.s^{-1}]$

$h \simeq 6{,}63\cdot 10^{-34}\ [J.s]$

$$\frac{hc}{\lambda_p} \simeq 24{,}9\cdot 10^{-20}\,[J]$$

$$\frac{hc}{\tau_2\lambda_p} \simeq 2{,}1\cdot 10^{-17}\,[W]$$

$$I_p^{sat} = \frac{hc/\tau_2\lambda_p}{\sigma_a} \simeq \frac{2{,}1\cdot 10^{-17}}{5{,}1\cdot 10^{-20}} \simeq 4{,}1\cdot 10^{2}\,[W.cm^{-2}]$$

$$\boxed{I_p^{th} \simeq 7{,}6\cdot 10^{2}\,[W.cm^{-2}]}$$

For a pump beam with a radius of 200 µm, $I^{th}_{P_{total}} \simeq 191\,[mW]$

LASERS 2022-2023 HOMEWORK: 2ND PART

6)

$$I_p^{th} = \underbrace{\frac{X+1}{X-1}}_{\substack{\downarrow \\ \text{decreasing function} \\ \text{of } X}} \cdot \frac{h\nu_p}{\sigma_a \tau_2} \quad \text{with } X = N_t \sigma_e c \tau_p = \frac{N_T}{\Delta N^{th}}$$

I_p^{th} is minimum if X is maximum

As a result we want $\frac{N_T}{\Delta N^{th}}$ as large as possible, i.e. ΔN^{th} as small as possible or equivalently $\sigma_e c \tau_p$ as large as possible, i.e. a strong stimulated scattering cross section σ_e and a large photon lifetime τ_p.

Besides $\sigma_a \tau_2$ should be also as large as possible

7)

$$I_p^R = \frac{I_p}{I_p^{sat}} = \frac{a\, I_p^{th}}{I_p^{sat}} = \left(\frac{X+1}{X-1}\right) a \quad \text{with } a = 2 \text{ or } a = 10$$

$$\Rightarrow I_L^R = \frac{1}{2}(X+1)\left[a-1\right] \quad \text{remember } X \simeq 3{,}55$$

$$\Rightarrow \begin{cases} I_L^R \simeq 2{,}2 \text{ for } a = 2 \Rightarrow I_L \simeq 2{,}2 \times I_L^{sat} \\ I_L^R \simeq 19{,}6 \text{ for } a = 10 \Rightarrow I_L \simeq 19{,}6 \times I_L^{sat} \end{cases}$$

$$I_L^{sat} = \frac{h\nu_L}{\sigma_e \tau_2} = \frac{hc}{\sigma_e \tau_2 \lambda_L} \qquad \lambda_L = 1{,}920 \cdot 10^{-6} \,[m] \qquad \frac{hc}{\lambda_p} \simeq 10{,}4 \cdot 10^{-20} \,[J]$$

$$I_L^{sat} \simeq 1{,}7 \cdot 10^{2} \,[W \cdot cm^{-2}]$$

LASERS 2022-2023 HOMEWORK: 2ND PART

7)

for $I_p = 2 \times I_p^{th}$ the laser intensity density is $I_L \simeq 3{,}7 \cdot 10^{2} [W \cdot cm^{-2}]$

for $I = 10 \times I_p^{th}$ " " " $I_L \simeq 18{,}7 \cdot 10^{2} [W \cdot cm^{-2}]$

The slopes of the laser characteristic is given by:

$$S = \frac{1}{2}\left(\frac{N_T}{AN^{th}} - 1\right) \times \frac{I_L^{sat}}{I_p^{sat}}$$

$$\boxed{S = \frac{1}{2}\left(\sigma_e c \tau_p N_T - 1\right) \times \frac{\sigma_p}{\sigma_e} \frac{\nu_L}{\nu_p}}$$

to maximize $S \Rightarrow \sigma_e \uparrow \quad N_T \uparrow$

$\tau_p \uparrow \quad \frac{\nu_L}{\nu_p} \uparrow$

$\frac{\sigma_p}{\sigma_e} \uparrow$

LASERS 2022-2023 HOMEWORK: 3RD PART

1)

Ray equation $\frac{d}{ds}\left(n \frac{d\vec{r}}{ds}\right) = \vec{grad}\, n$

$\vec{r}_\perp$ and $\vec{z}$ unit vectors

$$\frac{d\vec{r}}{ds} = \frac{dr_\perp}{ds}\vec{r}_\perp + \frac{dz}{ds}\vec{z} \simeq \frac{dr_\perp}{ds}\vec{r}_\perp + \vec{z}$$

$\vec{r}_\perp$ and $\vec{z}$ unit vectors. $\quad \frac{dz}{ds} \sim 1$ see paraxial approximation

$$\frac{d}{ds}\left(n \frac{d\vec{r}}{ds}\right) = \frac{dn}{ds}\frac{d\vec{r}}{ds} + n\left(\frac{d^2 r_\perp}{ds^2}\vec{r}_\perp + \frac{d^2 z}{ds^2}\vec{z}\right)$$

$$\simeq \frac{dn}{ds}\frac{d\vec{r}}{ds} + n\frac{d^2 r_\perp}{ds^2}\vec{r}_\perp$$ see $\frac{d^2 z}{ds^2} \sim 0$ due to paraxial approximation

$$= \frac{dn}{ds}\frac{dr_\perp}{ds}\vec{r}_\perp + \frac{dn}{ds}\vec{z} + n\frac{d^2 r_\perp}{ds^2}\vec{r}_\perp$$

LASERS 2022-2023 HOMEWORK: 3RD PART

1)

reminder on differential calculus:

$s = s(r_\perp, z)$

$df(s) = \frac{\partial f}{\partial s} ds$

$df(s(r_\perp, z)) = df(r_\perp, z) = \frac{\partial f}{\partial r_\perp} dr_\perp + \frac{\partial f}{\partial z} dz$

$$df(s(r_\perp, z)) = dg(r_\perp, z) = \frac{\partial g}{\partial r_\perp} dr_\perp + \frac{\partial g}{\partial z} dz$$

$$= \frac{df}{ds}\frac{\partial s}{\partial r_\perp} dr_\perp + \frac{df}{ds}\frac{\partial s}{\partial z} dz$$ chain rule

$$\Rightarrow \begin{cases} \frac{\partial f}{\partial r_\perp} = \frac{df}{ds}\frac{\partial s}{\partial r_\perp} \\ \frac{\partial f}{\partial z} = \frac{df}{ds}\frac{\partial s}{\partial z} \end{cases}$$

Paraxial approximation $\Rightarrow \frac{\partial s}{\partial z} \simeq 1$

$$\Rightarrow \frac{\partial f}{\partial z} = \frac{df}{ds}$$

LASERS 2022-2023 HOMEWORK: 3RD PART

1)

$$\frac{d}{ds}\left(n\frac{d\vec{r}}{ds}\right) = \frac{dn}{ds}\frac{dr_\perp}{ds}\vec{r_\perp} + n\frac{d^2 r_\perp}{ds^2}\vec{r_\perp} + \frac{dn}{ds}\vec{z}$$

$\frac{dr_\perp}{ds} \simeq \frac{\partial r_\perp}{\partial z}$ and $\frac{dn}{ds} = \frac{\partial n}{\partial z}$ see $\frac{\partial f}{\partial z} = \frac{df}{ds}$ in the paraxial approximation

$$\frac{d}{ds}\left(n\frac{d\vec{r}}{ds}\right) = \left(\frac{\partial n}{\partial z}\frac{\partial r_\perp}{\partial z} + n\frac{\partial^2 r_\perp}{\partial z^2}\right)\vec{r_\perp} + \frac{\partial n}{\partial z}\vec{z}$$

$$= \frac{\partial}{\partial z}\left(n\frac{\partial r_\perp}{\partial z}\right)\vec{r_\perp} + \frac{\partial n}{\partial z}\vec{z}$$

Besides, $\vec{\nabla} n = \text{grad}\, n = \frac{\partial n}{\partial r_\perp}\vec{r_\perp} + \frac{\partial n}{\partial z}\vec{z}$

1)

As a result, the ray equation leads to:

$$\frac{\partial}{\partial z}\left(n \frac{\partial r_\perp}{\partial z}\right)\vec{r}_\perp = \frac{\partial n}{\partial r_\perp}\vec{r}_\perp$$

$$\Leftrightarrow \quad \frac{\partial}{\partial z}\left(\underbrace{\left(n_0(z) - \tfrac{1}{2} n_2 r_\perp^2\right)}_{\text{second order in } r_\perp \text{ (see paraxial approximation)}} \frac{\partial r_\perp}{\partial z}\right) + n_2(z) r_\perp(z) = 0$$

$$\Leftrightarrow \quad \boxed{\frac{\partial}{\partial z}\left(n_0(z)\frac{\partial r_\perp}{\partial z}\right) + n_2(z)\, r_\perp(z) = 0}$$

$$\Leftrightarrow \quad \frac{d}{dz}\left(n_0(z)\frac{d r_\perp}{dz}\right) + n_2(z)\, r_\perp(z) = 0 \quad \text{as } r_\perp = r_\perp(z)$$

LASERS 2022-2023 HOMEWORK: 3RD PART

2)

$$r_\perp'(z) = n_0 \frac{d r_\perp}{dz}$$

using the previous form of the ray equation:

$$\Rightarrow \quad \frac{d r_\perp'}{dz} = -n_2 r_\perp$$

$$\begin{cases} \dfrac{d r_\perp}{dz} = \dfrac{r_\perp'}{n_0} \\ \dfrac{d r_\perp'}{dz} = -n_2 r_\perp \end{cases} \Rightarrow \quad \frac{d^2 r_\perp}{dz^2} + \frac{n_2}{n_0} r_\perp = 0$$

$$\gamma = \sqrt{\frac{n_2}{n_0}}$$

2)

solution: $r_\perp(z) = Ae^{i\gamma z} + Be^{-i\gamma z}$

with $r_\perp(0) = A + B$

$\frac{dr_\perp}{dz}(0) = i\gamma A - i\gamma B$

$\Rightarrow r_\perp(z) = r_\perp(0)\cos(\gamma z) + \frac{1}{\gamma}\frac{dr_\perp(0)}{dz}\sin(\gamma z)$

$$\boxed{r_\perp(z) = r_\perp(0)\cos(\gamma z) + \frac{1}{n_0\gamma}r'_\perp(0)\sin(\gamma z)}$$

taking the derivative:

$r'_\perp(z) = n_0\frac{dr_\perp}{dz} = -n_0\gamma\, r_\perp(0)\sin(\gamma z) + r'_\perp(0)\cos(\gamma z)$

as a result:

$$\begin{bmatrix} r_\perp(z) \\ r'_\perp(z) \end{bmatrix} = \begin{bmatrix} \cos(\gamma z) & (n_0\gamma)^{-1}\sin(\gamma z) \\ -n_0\gamma\sin(\gamma z) & \cos(\gamma z) \end{bmatrix} \begin{bmatrix} r_\perp(0) \\ r'_\perp(0) \end{bmatrix}$$

LASERS 2022-2023 HOMEWORK: 3RD PART

3)

We set:

$A_L = \cos\gamma L = D_L$

$B_L = (n_0\gamma)^{-1}\sin\gamma L$ where L is the cavity length

$C_L = -n_0\gamma\sin\gamma L$

Considering that the mirrors are flat, the transfer matrix over one round trip is:

$$\begin{bmatrix} A_L & B_L \\ C_L & D_L \end{bmatrix}\begin{bmatrix} A_L & B_L \\ C_L & D_L \end{bmatrix} = \begin{bmatrix} A_{2L} & B_{2L} \\ C_{2L} & D_{2L} \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\gamma z) & (n_0\gamma)^{-1}\sin(2\gamma z) \\ -n_0\gamma\sin(2\gamma z) & \cos(2\gamma z) \end{bmatrix}$$

LASERS 2022-2023 HOMEWORK: 3RD PART

4)

The cavity is stable if $\left|\frac{A_{2L}+D_{2L}}{2}\right| \leq 1$

i.e. $\cos(2\gamma z) \leq 1$

This last condition is always fulfilled as γ is real.

5)

$R(M) = \frac{2B_{2L}}{D_{2L}-A_{2L}}$ as $D_{2L} = A_{2L}$ we have $R(M) = \infty$
$\forall M$

point on the optical axis at z

$$w(M) = \left(\frac{\lambda}{\pi}\right)^{1/2} \frac{(n_0\gamma)^{-1}\sqrt{\sin(2\gamma z)}}{(1-\cos^2(2\gamma z))^{1/4}} = \left(\frac{\lambda}{\pi}\right)^{1/2} \times \frac{1}{\sqrt{n_0 n_2}}$$

The scalar field of the gaussian mode is :

$$U(r_\perp, z) = A_0\, e^{-ik\frac{r_\perp^2}{2q(z)}}\, e^{-i\phi(z)}$$ with $q(z)$ pure complex number

LASERS 2022-2023 HOMEWORK: 3RD PART

6)

Dispersion relationship :

$$\omega^2 = k^2\left(\frac{c}{n}\right)^2 \iff k^2 = \left(\frac{2\pi}{\lambda_0}\right)^2 n^2$$

$$\iff k^2 = k_0^2 n^2$$

$$k^2 = k_0^2 n^2 \iff \vec{\nabla}_{\vec{r}}(k^2) = k_0^2\, \vec{\nabla}_{\vec{r}} n^2$$

$$\Rightarrow \vec{\nabla}_{\vec{r}}(\vec{k}\cdot\vec{k}) = 2 k_0^2\, n\, \vec{\nabla}_{\vec{r}} n$$

$$\vec{\nabla}_{\vec{r}}(\vec{k}\cdot\vec{k}) = 2\vec{k}\times(\vec{\nabla}_{\vec{r}}\times\vec{k}) + 2(\vec{k}\cdot\vec{\nabla}_{\vec{r}})\vec{k}$$

LASERS 2022-2023 HOMEWORK: 3RD PART

6)

Per definition $\vec{k} = \vec{\text{grad}}\, S(\vec{r}) = \vec{\nabla}_{\vec{r}} S(\vec{r})$

with $S(\vec{r})$ the eikonal

ae $\vec{E}(\vec{r},t) = \vec{E}(\vec{r})e^{ik_0 S(\vec{r})}e^{-i\omega t}$

$\vec{\nabla}_{\vec{r}}\times(\vec{\nabla}_{\vec{r}} S(\vec{r})) = 0 \quad (\vec{\text{rot}}(\vec{\text{grad}}) = 0)$

As a result :

$$\vec{\nabla}_{\vec{r}}(\vec{k}\cdot\vec{k}) = 2(\vec{k}\cdot\vec{\nabla}_{\vec{r}})\vec{k}$$

Light rays are defined as $n\vec{s} = n\frac{d\vec{r}}{ds} = \vec{\text{grad}}\, S(\vec{r})$

with s the length of arc of the ray

$\vec{r}(s)$ the position of a point on the ray

$\vec{s}$ the unit vector tangent to the ray at the position $\vec{r}(s)$.

It follows : $\vec{k} = k_0 n \vec{s}$

and $(\vec{k}\cdot\vec{\nabla}_{\vec{r}}) = k_0 n\, \vec{s}\cdot\vec{\nabla}_{\vec{r}} = k_0 n \frac{d}{ds}$

Then $(\vec{k}\cdot\vec{\nabla}_{r})\vec{k} = k_0^2 n \vec{\nabla}_r n \iff n\frac{d}{ds}(n\vec{s}) = n\vec{\nabla}_r n$

$$\iff \boxed{\frac{d}{ds}\left(n\frac{d\vec{r}}{ds}\right) = \vec{\nabla}_r n}$$

LASERS 2022-2023 HOMEWORK: 3RD PART

Remarks on the geometrical optics approximation

general representation of the fields several λ away from the source:

$$\vec{E}(\vec{r},t) = \vec{e}(\vec{r})\, e^{ik_0 S(\vec{r})}\, e^{-i\omega t}$$

$$\vec{H}(\vec{r},t) = \vec{h}(\vec{r})\, e^{ik_0 S(\vec{r})}\, e^{-i\omega t}$$

writing the Maxwell's equations to the first order in $\frac{1}{k_0}$ i.e. for very short λ_0, the eikonal equation is obtained:

$$\left(\vec{\mathrm{grad}}\, S(\vec{r})\right)^2 = n^2$$

from the eikonal equation, a light ray can be defined as:

$$n\vec{s} = n\frac{d\vec{r}}{ds} = \vec{\mathrm{grad}}\, S(\vec{r})$$

Taylor expansion of $S(\vec{r})$:

$$S(\vec{r}) = S(\vec{0}) + \left(\vec{\nabla} S(\vec{r})\right)_{\vec{r}=\vec{0}} \vec{r} + \cdots$$

The wave vector is defined as $\vec{k} = k_0 \vec{\nabla} S(\vec{r})$

With the geometrical optics approximation: $\vec{k} = k_0 n \vec{s}$

$$FT\{\vec{E}(\vec{r},t)\}_{\vec{k}} = \int \vec{e}(\vec{r})\, e^{ik_0 S(\vec{r})}\, e^{-i\vec{k}\vec{r}}\, d\vec{r}\; e^{-i\omega t}$$

$$\simeq e^{ik_0 S(\vec{0})} \times e^{-i\omega t} \times \int e^{i(k_0 \vec{\nabla} S(\vec{r}) - \vec{k})}\, d\vec{r}$$

Note that the eikonal equation is still valid for a second order approximation in $\frac{1}{k_0}$ of the Maxwell's equations

LASERS 2022-2023 HOMEWORK: 3RD PART

Remarks on differential calculus:

$$g(\lambda) = f(x^1(\lambda), \cdots, x^n(\lambda))$$

$$\left.\frac{dg}{d\lambda}\right|_P = \sum_i \frac{dx^i}{d\lambda}\frac{\partial f}{\partial x^i}$$

$$\left.\frac{d}{d\lambda}\right|_P = \sum_i \frac{dx^i}{d\lambda}\frac{\partial}{\partial x^i}$$

The directional derivatives along curves, like $d/d\lambda$, form a vector space at P

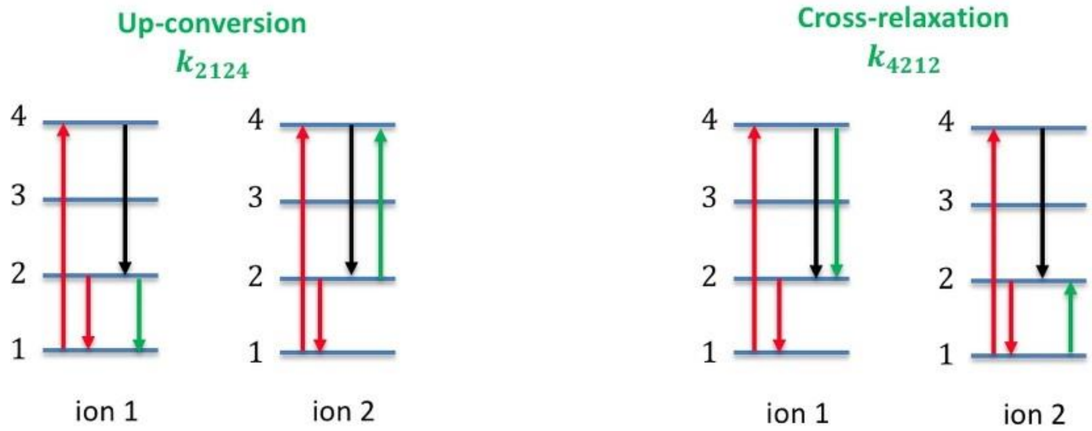
$\{\partial/\partial x^i\}$ are a basis for this vector space

$\left\{\frac{dx^i}{d\lambda}\right\}$ are the components of $\frac{d}{d\lambda}$ on this basis

The space of all tangent vectors at P and the space of all derivatives along curves at P are in 1-1 correspondence. For this reason $\frac{d}{d\lambda}$ is said to be the tangent vector to the curve $x^i(\lambda)$.

LASERS 2022-2023 HOMEWORK: 4TH PART

1)



LASERS 2022-2023 HOMEWORK: 4TH PART

1)

$$\frac{dN_2}{dt} = -2\, k_{2124} N_2^2 - \frac{N_2}{\tau_2} - \sigma_e c \phi_L (N_2 - N_1) + \frac{N_4}{\tau_4}$$

loss of 2 ions excited in level 2

$$\frac{dN_4}{dt} = \sigma_p c \phi_p (N_1 - N_4) + k_{2124} N_2^2 - \frac{N_4}{\tau_4}$$

gain of 1 ion excited in level 4

Lasers 2022-2023 Homework: 4TH part

2)

Steady state:

$$0 = -2k_{2124} N_2^2 - \sigma_e c \phi_L (2N_2 - N_T) - \frac{N_2}{\tau_2} + \sigma_p c \phi_p (N_T - N_2) + k_{2124} N_2^2$$

$$\tau_2 k_{2124} N_2^2 + N_2 + I_L^R (2N_2 - N_T) - I_P^R (N_T - N_2) = 0$$

$$\tau_2 k_{2124} N_2^2 + (1 + 2I_L^R + I_P^R) N_2 - (I_L^R + I_P^R) N_T = 0$$

$$x = \frac{N_2}{N_T}$$

$$\tau_2 k_{2124} N_T x^2 + (1 + 2I_L^R + I_P^R) x - (I_L^R + I_P^R) = 0$$

$$u = \tau_2 k_{2124} N_T$$

$$x > 0 \Rightarrow \boxed{x = \frac{-(1+2I_L^R+I_P^R) + \sqrt{(1+2I_L^R+I_P^R)^2 + 4u(I_L^R+I_P^R)}}{2u}}$$

Lasers 2022-2023 Homework: 4TH part

2)

$$\gamma = \sigma_e N_T (2x - 1) = \frac{1}{c\tau_p} \Rightarrow 2x - 1 = \frac{\Delta N^{th}}{N_T}$$

$$\boxed{x = \frac{-(1+2I_L^R+I_P^R) + \sqrt{(1+2I_L^R+I_P^R)^2 + 4u(I_L^R+I_P^R)}}{2u}}$$

$$-(1 + 2I_L^R + I_P^R) + \sqrt{[-]} - u = \frac{\Delta N^{th}}{N_T} u$$

$$\sqrt{[-]} = \left(\frac{\Delta N^{th}}{N_T} + 1\right) u + (1 + 2I_L^R + I_P^R)$$

$$[-] = \left(\left(\frac{\Delta N^{th}}{N_T} + 1\right) u + (1 + 2I_L^R + I_P^R)\right)^2$$

$$4u(I_L^R + I_P^R) = \left(\frac{\Delta N^{th}}{N_T} + 1\right)^2 u^2 + 2\left(\frac{\Delta N^{th}}{N_T} + 1\right) u (1 + 2I_L^R + I_P^R)$$

LASERS 2022-2023 HOMEWORK: 4TH PART

2)

$$Y = \frac{\Delta N^{th}}{N_T}$$

$$-4uY\, I_L^R = (Y+1)^2 u^2 + 2(Y+1)u + 2(Y-1)u\, I_P^R$$

$$I_L^R = -\frac{(Y+1)^2 u + 2(Y+1)}{4Y} + \frac{-Y+1}{2Y} I_P^R$$

$$I_L^R = \frac{1}{2}(Y+1) \times \frac{1}{Y} \cdot \left[\frac{-Y+1}{Y+1} I_P^R - 1 - (Y+1)\frac{u}{2} \right]$$

$$\boxed{I_L^R = \frac{1}{2}\left(\frac{N_T}{\Delta N^{th}} + 1\right)\left[\left(\frac{\frac{N_T}{\Delta N^{th}} - 1}{\frac{N_T}{\Delta N^{th}} + 1}\right) I_P^R - 1 - \left(\frac{\Delta N^{th}}{N_T} + 1\right)\tau_2 k_{2124} N_T\right]}$$

LASERS 2022-2023 HOMEWORK: 4TH PART

2)

$$\boxed{I_L^R = \frac{1}{2}\left(\frac{N_T}{\Delta N^{th}} + 1\right)\left[\left(\frac{\frac{N_T}{\Delta N^{th}} - 1}{\frac{N_T}{\Delta N^{th}} + 1}\right) I_P^R - 1 - \left(\frac{\Delta N^{th}}{N_T} + 1\right)\tau_2 k_{2124} N_T\right]}$$

Instead of $I_P^{th} = \left(\frac{N_T/\Delta N^{th} + 1}{N_T/\Delta N^{th} - 1}\right) I_P^{sat}$ we have:

$$I_P^{th} = \left(\frac{N_T/\Delta N^{th} + 1}{N_T/\Delta N^{th} - 1}\right)\left(1 + \left(\frac{\Delta N^{th}}{N_T} + 1\right)\tau_2 k_{2124} N_T\right) I_P^{sat}$$

The intensity pump value at threshold is increased by a factor $\left(1 + \left(\frac{\Delta N^{th}}{N_T} + 1\right)\tau_2 k_{2124} N_T\right) = f$

$$f \simeq \left(1 + \left(\frac{1}{3{,}35} + 1\right) 12 \cdot 10^{-3} \times 4 \cdot 10^{-19} \times 6{,}3 \cdot 10^{20}\right)$$

$$f \simeq 1 + 39{,}3$$

$$f \simeq 40$$