On the complexity of $\mathbb H$-coloring for special oriented trees

Article *in* European Journal of Combinatorics · July 2014

DOI: 10.1016/j.ejc.2017.10.001 · Source: arXiv

1 author:

Jakub Bulín
University of Colorado Boulder

arXiv:1407.1779v2 [math.CO] 26 Jul 2014

ON THE COMPLEXITY OF $\mathbb{H}$-COLORING FOR SPECIAL ORIENTED TREES

JAKUB BULÍN

Abstract. For a fixed digraph $\mathbb{H}$, the $\mathbb{H}$-coloring problem is the problem of deciding whether a given input digraph $\mathbb{G}$ admits a homomorphism to $\mathbb{H}$. The CSP dichotomy conjecture of Feder and Vardi is equivalent to proving that, for any $\mathbb{H}$, the $\mathbb{H}$-coloring problem is in in **P** or **NP**-complete. We confirm this dichotomy for a certain class of oriented trees, which we call special trees (generalizing earlier results on special triads and polyads). Moreover, we prove that every tractable special oriented tree has bounded width, i.e., the corresponding $\mathbb{H}$-coloring problem is solvable by local consistency checking. Our proof relies on recent algebraic tools, namely characterization of congruence meet-semidistributivity via pointing operations and absorption theory.

1. Introduction

The Constraint Satisfaction Problem (CSP) provides a common framework for various problems from theoretical computer science as well as for many real-life applications (e.g. in graph theory, database theory, artificial intelligence, scheduling). Its history dates back to 1970s and it has been central to the development of theoretical computer science in the past few decades.

For a fixed (finite) relational structure $\mathbb{A}$, the Constraint Satisfaction Problem with template $\mathbb{A}$, or CSP($\mathbb{A}$) for short, is the following decision problem:

INPUT: A relational structure $\mathbb{X}$ (of the same type as $\mathbb{A}$).
QUESTION: Is there a homomorphism from $\mathbb{X}$ to $\mathbb{A}$?

For a (directed) graph $\mathbb{H}$, CSP($\mathbb{H}$) is also commonly referred to as the $\mathbb{H}$-coloring problem.

A lot of interest in this class of problems was sparked by a seminal work of Feder and Vardi [20], in which the authors established a connection to computational complexity theory: they conjectured a large natural class of **NP** decision problems avoiding the complexity classes strictly between **P** and **NP**-complete (assuming that **P**$\neq$**NP**). Many natural decision problems, such as k-SAT, graph k-colorability or solving systems of linear equations over finite fields belong to this class. They also proved that each problem from this class can be reduced in polynomial time to CSP($\mathbb{A}$), for some relational structure $\mathbb{A}$. Hence their conjecture can be formulated as follows.

Conjecture 1 (The CSP dichotomy conjecture)**.** *For every (finite) relational structure $\mathbb{A}$,* CSP($\mathbb{A}$) *is in **P** or **NP**-complete.*

At that time this conjecture was supported by two major cases: Schaefer's dichotomy result for two-element domains [33] and the dichotomy theorem for undirected graphs by Hell and Nešetřil [26]. A major breakthrough followed the work of Jeavons, Cohen and Gyssens [29], later refined by Bulatov, Jeavons and Krokhin [15], which uncovered an intimate connection between the constraint satisfaction

This research was supported by the grant projects GAČR 201/09/H012 & 13-01832S, GA UK 67410 & 558313, MŠMT ČR 7AMB 13P10 13 and SVV-2014-260107.

problem and universal algebra. This connection brought a better understanding of the known results as well as a number of new results which seemed out of reach for pre-algebraic methods. The most important results include dichotomy for three-element domains [14] and for conservative structures (i.e., containing all subsets as unary relations) [13] by Bulatov (see also [1]), a characterization of solvability by the *few subpowers* algorithm (a generalization of Gaussian elimination) by Berman et al [11, 28] and solvability by local consistency checking (so-called *bounded width*) by Barto and Kozik [6] (conjectured in [31]). Larose and Tesson [30] successfully applied the theory to study finer complexity classes of CSPs.

The connection between CSPs and algebras turned out to be fruitful in both directions; it has lead to a discovery of important structural properties of finite algebras. Of particular importance to us is the theory of absorption by Barto and Kozik [4, 9] and a characterization of congruence meet-semiditributivity via pointing operations by Barto, Kozik and Stanovský [5, 9].

In the paper [20], Feder and Vardi also constructed, for every structure $\mathbb{A}$, a directed graph $\mathcal{D}(\mathbb{A})$ such that $\mathrm{CSP}(\mathbb{A})$ and $\mathrm{CSP}(\mathcal{D}(\mathbb{A}))$ are polynomial-time equivalent. Hence the CSP dichotomy conjecture is equivalent to its restriction to digraphs. A variant of this reduction (which is, in fact, logspace) is studied by the author, Delić, Jackson and Niven in [17, 18], where we prove that most properties relevant to the CSP carry over from $\mathbb{A}$ to $\mathcal{D}(\mathbb{A})$. As a consequence, the algebraic conjectures characterizing CSPs solvable in **P** [15], **NL** and **L** [30] are equivalent to their restrictions to digraphs. The digraphs $\mathcal{D}(\mathbb{A})$ are, in fact, *special balanced digraphs* in the terminology of this paper, a generalization of special triads, special polyads and special trees discussed below.

Using the algebraic approach, Barto, Kozik and Niven confirmed the conjecture of Bang-Jensen and Hell and proved dichotomy for *smooth digraphs* (i.e., digraphs with no sources and no sinks) [8]. The dichotomy was also established for a number of other classes of digraphs, e.g. oriented paths (which are all tractable) [22] or oriented cycles [19].

This paper is concerned with $\mathbb{H}$-coloring for oriented trees. In the class of all digraphs, oriented trees are in some sense very far from smooth digraphs, and the algebraic tools seem to be not yet developed enough to deal with them. Hence oriented trees serve as a good field-test for new methods.

Except the oriented paths, the simplest class of oriented trees are *triads* (i.e., oriented trees with one vertex of degree 3 and all other vertices of degree 2 or 1); the CSP dichotomy remains open even for triads. Among the triads, Hell, Nešetřil and Zhu [24, 25] identified a (fairly restricted) subclass, for which they coined the term *special triads* and which allowed them to handle at least some examples. For instance, they constructed a special triad with **NP**-complete $\mathbb{H}$-coloring.

In [7], Barto et al used algebraic methods to prove that every special triad has **NP**-complete $\mathbb{H}$-coloring, or a compatible majority operation (so-called *strict width 2*) or compatible totally symmetric idempotent operations of all arities (so-called *width 1*). In [2], the author and Barto established the CSP dichotomy conjecture for *special polyads*, a generalization of special triads where the one vertex of degree > 2 is allowed to have an arbitrary degree. In particular, every tractable core special polyad has bounded width. However, there are special polyads which have bounded width, but neither bounded strict width nor width 1.

In this paper we study *special trees*, a broad generalization of special triads and special polyads. Special trees have an underlying structure of a height 1 oriented tree (see the definition in Section 2) and while for special triads it has only 7 vertices and for special polyads it has radius 2, for general special trees it can be arbitrary.

We confirm the CSP dichotomy conjecture for special trees and, moreover, prove that every tractable core special tree has bounded width. The proof uses modern tools from the algebraic approach to the CSP (in particular, absorption and pointing operations [9]) and is somewhat simpler and more natural than the proofs in [7] and [2]. Therefore we believe that there is hope for further generalization. In particular, we conjecture that tractability implies bounded width for all oriented trees.

2. Special trees & the main result

In this section we define special trees and state the main result of this paper. The notions used here will be defined later, in Sections 3 and 4.

Definition 2.1. An oriented path $\mathbb{P}$ with initial vertex a and terminal vertex b is *minimal* if

- $\mathrm{lvl}(a) = 0$,
- $\mathrm{lvl}(b) = \mathrm{hgt}(\mathbb{P})$, and
- $0 < \mathrm{lvl}(v) < \mathrm{hgt}(\mathbb{P})$ for every $v \in P \setminus \{a, b\}$.

Minimal paths have the property that their *net length* (the number of forward edges minus the number of backward edges) is strictly greater than the net length of any of their subpaths; hence the name. An example of a minimal path is depicted in Figure 1 below.

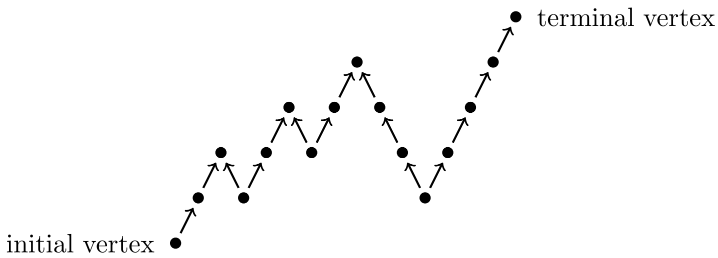


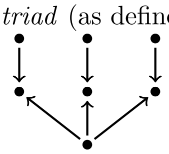
FIGURE 1. A minimal path

We will need the following well known fact. The proof can be found in [27].

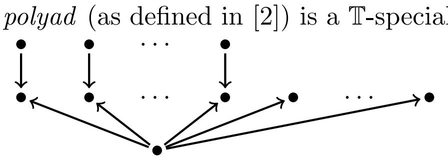
Lemma 2.2. *Let $\mathbb{P}_1, \mathbb{P}_2, \dots \mathbb{P}_k$ be minimal paths of the same height h. There exists a minimal path $\mathbb{Q}$ of height h such that for every $i \in [k]$ there exists an onto homomorphism $\mathbb{Q} \to \mathbb{P}_i$.*

Definition 2.3. Let $\mathbb{T} = (T; E)$ be an oriented tree of height 1. A $\mathbb{T}$-*special tree* of height h is an oriented tree obtained from $\mathbb{T}$ by replacing every edge $(a, b) \in E$ with some minimal path $\mathbb{P}_{(a,b)}$ of height h, preserving orientation. (That is, identifying the initial vertex of $\mathbb{P}_{(a,b)}$ with a and the terminal vertex with b. We require the vertex sets of the minimal paths to be pairwise disjoint and also disjoint with T.)

- A *special triad* (as defined in [7]) is a $\mathbb{T}$-special tree with

$\mathbb{T} =$ 

- A *special polyad* (as defined in [2]) is a $\mathbb{T}$-special tree with

$\mathbb{T} =$ 

- A *special tree* is simply a $\mathbb{T}$-special tree for some height 1 oriented tree $\mathbb{T}$.

As an example, in Figure 2 below we present a special triad constructed in [7], which has **NP**-complete $\mathbb{H}$-coloring (and is conjectured to be the smallest oriented tree with this property). The vertices from the bottom and top level are marked by ■ and □, respectively.

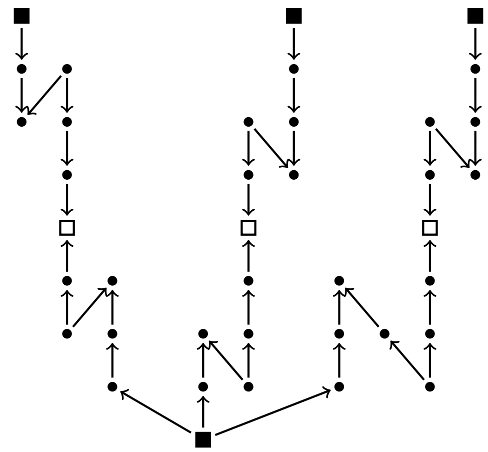


FIGURE 2. A special triad; the smallest known oriented tree with **NP**-complete $\mathbb{H}$-coloring problem (39 vertices).

The following theorem is the main algebraic result of our paper.

Theorem 2.4. *Let $\mathbb{H}$ be a special tree. If the algebra of idempotent polymorphisms of $\mathbb{H}$ is Taylor, then it is congruence meet-semidistributive.*

As a consequence, we confirm the dichotomy of $\mathbb{H}$-coloring for special trees.

Corollary 2.5. *The CSP dichotomy conjecture holds for special trees. For any core special tree $\mathbb{H}$,* $\mathrm{CSP}(\mathbb{H})$ *is* ***NP****-complete or $\mathbb{H}$ has bounded width.*

We will prove Theorem 2.4 and Corollary 2.5 in Section 5.

3. Preliminaries

In this section we introduce basic notions and fix notation used throughout the paper. We assume the reader possesses some knowledge of graph theory and basic universal algebra.

We recommend [23] for a detailed exposition of digraphs, relational structures (under the name "general relational systems") and their homomorphisms as well as an introduction to graph coloring and constraint satisfaction. For an introduction to the notions from universal algebra that are not explained in detail in this paper we invite the reader to consult [10]. Primary source for the algebraic approach to the CSP is the paper [15].

Our aim is to make the paper accessible to a wider audience outside of universal algebra. Thus we refrain from using specialist terminology wherever possible, or move it to explanatory remarks which the reader may skip.

3.1. **Notation.** For a positive integer n we denote the set $\{1, 2, \ldots, n\}$ by $[n]$; we set $[0] = \emptyset$. We write tuples using boldface notation, e.g., $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in A^k$. When ranging over tuples we use superscripts, e.g. $(\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^n) \in (A^k)^n$, where $\mathbf{a}^i = (a^i_1, a^i_2, \ldots, a^i_k)$, for $i \in [n]$. We sometimes write $\langle a_1 a_2 \ldots \rangle$ to denote a sequence of elements.

3.2. **Relational structures.** An *n-ary relation* on a set A is a subset $R \subseteq A^n$. A (finite) *relational structure* $\mathbb{A}$ is a finite, nonempty set A equipped with finitely many relations $R_1 \dots R_m$ on A; we write $\mathbb{A} = (A; R_1, \dots, R_m)$.

Let $\mathbb{B} = (B; S_1, \dots, S_m)$ be a relational structure of the same *type* as $\mathbb{A}$ (i.e., same number of relations and corresponding relations have the same arity). A mapping $\varphi : A \to B$ is a *homomorphism* from $\mathbb{A}$ to $\mathbb{B}$, if for each $i \in [m]$ and $\mathbf{a} \in R_i$ (say k-ary) we have $(\varphi(a_1), \dots, \varphi(a_k)) \in S_i$. We write $\varphi : \mathbb{A} \to \mathbb{B}$ to mean that φ is a homomorphism from $\mathbb{A}$ to $\mathbb{B}$, and $\mathbb{A} \to \mathbb{B}$ to mean that there exists a homomorphism from $\mathbb{A}$ to $\mathbb{B}$.

For every $\mathbb{A}$ there exists a relational structure $\mathbb{A}'$ such that $\mathbb{A} \to \mathbb{A}'$ and $\mathbb{A}' \to \mathbb{A}$ and $\mathbb{A}'$ is of minimal size with respect to these properties; that structure $\mathbb{A}'$ is called the *core of* $\mathbb{A}$ (it is unique up to isomorphism); $\mathbb{A}$ is a *core* if it is the core of itself.

We will be almost exclusively interested in a special type of relational structures: *directed graphs*.

3.3. **Digraphs.** A *digraph* (short for "*directed graph*") is a relational structure $\mathbb{G} = (G; \to)$ with a single binary relation $\to \subseteq G^2$. We call $u \in G$ and $(u, v) \in \to$ (usually written as $u \to v$) *vertices* and *edges* of $\mathbb{G}$, respectively. A digraph $\mathbb{G}' = (G'; \to')$ is a *subgraph* of $\mathbb{G}$, if $G' \subseteq G$ and $\to' \subseteq \to$. It is an *induced subgraph* if $\to' = \to \cap (G')^2$.

An *oriented path* is a digraph $\mathbb{P}$ which consists of a non-repeating sequence of vertices $\langle v_0 v_1 \dots v_k \rangle$ (allowing for the degenerate case $k = 0$) such that precisely one of $(v_{i-1}, v_i), (v_i, v_{i-1})$ is an edge, for each $i \in [k]$. We require oriented paths to have a fixed direction, and thus an *initial* and a *terminal* vertex.

For $a, b \in G$ we say that a is *connected* to b in $\mathbb{G}$ via an oriented path $\mathbb{P}$, if $\mathbb{P}$ is a subgraph of $\mathbb{G}$ and a and b are the initial and terminal vertex of $\mathbb{P}$, respectively. The *distance* of a and b in $\mathbb{G}$ is then the number of edges in the shortest oriented path $\mathbb{P}'$ connecting a to b in $\mathbb{G}$. Connectivity is an equivalence relation, its classes are *components of connectivity* of $\mathbb{G}$ and $\mathbb{G}$ is *connected* if it consists of a single component of connectivity.

For $n > 0$, the *nth direct power of* $\mathbb{G}$ is the digraph $\mathbb{G}^n = (G^n, \to^n)$, i.e., its vertices are n-tuples of vertices of $\mathbb{G}$ and the edge relation is

$$\{(\mathbf{u}, \mathbf{v}) \in (G^n)^2 \mid u_i \to v_i \text{ for all } i \in [n]\}.$$

Connectivity in direct powers of digraphs will play an important role.

An *oriented tree* is a connected digraph containing no oriented cycles. Equivalently, it is a digraph in which every two vertices are connected via a unique oriented path. Oriented paths and trees are natural examples of balanced digraphs: a connected digraph is *balanced* if it admits a *level function* $\mathrm{lvl} : G \to \mathbb{N} \cup \{0\}$, where $\mathrm{lvl}(b) = \mathrm{lvl}(a) + 1$ whenever (a, b) is an edge, and the minimum level is 0. The maximum level is called *height* and denoted by $\mathrm{hgt}(\mathbb{G})$.

3.4. **Algebras.** A *k-ary operation* on a set A is a mapping $f \colon A^k \to A$. By an *algebra* we mean a pair $\mathbf{A} = (A; \mathcal{F})$, where A is a nonempty set and $\mathcal{F}$ is a set of operations on A (so-called *basic operations* of $\mathbf{A}$). We denote by $\mathrm{Clo}(\mathbf{A})$ the set of all *term operations* of $\mathbf{A}$ (i.e., operations obtained from $\mathcal{F}$ together with the *projection* operations by composition).

A subset $B \subseteq A$ is a *subuniverse* of $\mathbf{A}$ (denoted by $B \leq \mathbf{A}$) if it is closed under all (basic, or equivalently term) operations of $\mathbf{A}$. A nonempty subuniverse B is an algebra in its own right, equipped with operations of $\mathbf{A}$ restricted to B, i.e., $(B; \{f|_B \mid f \in \mathcal{F}\})$. We will frequently use the fact that an intersection of subuniverses is again a subuniverse.

An operation is *idempotent* if $f(x, x, \dots, x) = x$ for all $x \in A$. An algebra is idempotent if all of its (basic, or equivalently term) operations are idempotent. Note that an algebra $\mathbf{A}$ is idempotent, if and only if $\{a\} \leq \mathbf{A}$ for every $a \in A$.

For $n > 0$, the *nth power* of $\mathbf{A}$ is the algebra $\mathbf{A}^n = (A^n; \{f \times \cdots \times f \mid f \in \mathcal{F}\})$ where $f \times \cdots \times f$ means that f is applied to n-tuples of elements coordinatewise.

We write $C \leq B \leq \mathbf{A}$ to mean that both B and C are subuniverses of $\mathbf{A}$ and $C \subseteq B$. In particular, if B and C are subuniverses of $\mathbf{A}$, then $E \leq B \times C$ means that E is a subuniverse of $\mathbf{A}^2$ contained in $B \times C$ (which is a subuniverse of $\mathbf{A}^2$ as well).

All algebras we will work with will be subuniverses of a certain finite idempotent algebra (or rarely of its 2nd power): the *algebra of idempotent polymorphisms* of some fixed relational structure.

3.5. **Algebra of idempotent polymorphisms.** Note that a digraph homomorphism is simply an edge-preserving mapping. The notion of digraph *polymorphism* is a natural generalization to higher arity operations:

Let $\mathbb{G} = (G; \rightarrow)$ be a digraph. A k-ary ($k > 0$) operation φ on G is a *polymorphism* of $\mathbb{G}$, if it is a homomorphism from $\mathbb{G}^k$ to $\mathbb{G}$. This means that φ preserves edges in the following sense: if $a_i \rightarrow b_i$ for $i \in [k]$, then $\varphi(\mathbf{a}) \rightarrow \varphi(\mathbf{b})$. The notions of kth direct power, preserving a relation, and polymorphism generalize naturally to relational structures.

Let $\mathbb{A}$ be a relational structure. The *algebra of idempotent polymorphisms* of $\mathbb{A}$ is the algebra $\mathbf{alg}\,\mathbb{A} = (A; \mathrm{IdPol}(\mathbb{A}))$, where $\mathrm{IdPol}(\mathbb{A})$ denotes the set of all idempotent polymorphisms of $\mathbb{A}$; we write $\mathrm{IdPol}_k(\mathbb{A})$ to denote its k-ary part.

A relation $S \subseteq A^n$ is *primitive positive definable* from $\mathbb{A}$ *with constants*, if it is definable by an existentially quantified conjunction of atomic formulæ of the form $x_i = a$ or $R(x_{i_1}, \ldots, x_{i_j})$, where $a \in A$ and R is one of the relations of $\mathbb{A}$. The following fact, based on the Galois correspondence between clones and relational clones [12, 21] is central to the algebraic approach to the CSP.

Lemma 3.1 (see [15, Proposition 2.21])**.** *A relation $S \subseteq A^n$ is primitive positive definable from $\mathbb{A}$ with constants, if and only if S is a subuniverse of $(\mathbf{alg}\,\mathbb{A})^n$.*

The connection between universal algebra and constraint satisfaction is discussed in detail in [15, 16].

4. Algebraic tools

In this section we introduce the universal algebraic tools we will use in our proof. Recall that for a fixed relational structure $\mathbb{A}$, the *Constraint satisfaction problem* for $\mathbb{A}$ is membership problem for the set $\mathrm{CSP}(\mathbb{A}) = \{\mathbb{X} \mid \mathbb{X} \rightarrow \mathbb{A}\}$. Note that if $\mathbb{A}'$ is the core of $\mathbb{A}$, then $\mathrm{CSP}(\mathbb{A}) = \mathrm{CSP}(\mathbb{A}')$.

Of particular importance to the CSP are the following two well known classes of finite algebras: *Taylor* algebras (called "active" in [10]) and *congruence meet-semidistributive* (*SD*($\wedge$)) algebras[1]. Instead of providing direct definitions, we present the following characterization from [32].

Definition 4.1. A *weak near-unanimity* (*WNU*) on a set A is an n-ary ($n \geq 2$) idempotent operation ω such that for all $x, y \in A$,

$$\omega(x, \ldots, x, y) = \omega(x, \ldots, x, y, x) = \cdots = \omega(y, x, \ldots, x).$$

Theorem 4.2 ([32])**.** *Let $\mathbf{A}$ be a finite algebra.*

- $\mathbf{A}$ *is Taylor, if and only if there exists a WNU operation* $\omega \in \mathrm{Clo}(\mathbf{A})$.
- $\mathbf{A}$ *is* $\mathrm{SD}(\wedge)$*, if and only if there exists n_0 such that for all $n \geq n_0$ there exists an n-ary WNU operation* $\omega_n \in \mathrm{Clo}(\mathbf{A})$.

[1] Taylor and SD($\wedge$) algebras are also commonly referred to as "omitting type 1" and "omitting types 1, 2"; this terminology comes from Tame Congruence Theory (see [10, Chapter 8]).

The Algebraic CSP dichotomy conjecture ([15], see also [16, Conjecture 1]) asserts that being Taylor is what distinguishes (algebras of idempotent polymorphisms of) tractable core relational structures from the **NP**-complete ones; the hardness part is known.

Theorem 4.3 ([15])**.** *Let $\mathbb{A}$ be a core relational structure. If* $\mathbf{alg}\,\mathbb{A}$ *is not Taylor, then* $\mathrm{CSP}(\mathbb{A})$ *is* ***NP****-complete.*

A relational structure $\mathbb{A}$ is said to have *bounded width* [20], if $\mathrm{CSP}(\mathbb{A})$ is solvable by "local consistency checking" algorithm (or rather algorithmic principle). We refer the reader to [6] for a detailed exposition. This property is characterized (for cores) by congruence meet-semidistributivity; the characterization was conjectured, and the "only if" part proved, in [31].

Theorem 4.4 ([6],"Bounded Width Theorem")**.** *A core relational structure $\mathbb{A}$ has bounded width (implying that* $\mathrm{CSP}(\mathbb{A})$ *is in* ***P****), if and only if* $\mathbf{alg}\,\mathbb{A}$ *is* $\mathrm{SD}(\wedge)$*.*

The proof of the Bounded Width Theorem uncovered a new characterization of $\mathrm{SD}(\wedge)$ algebras via so-called *pointing operations* as well as the concept of *absorbing subuniverse*, which turned out to be quite useful even outside of the realm of congruence meet-semidistributivity (see [4, 9]).

4.1. **Pointing operations.** Pointing operations were first used in [5]. More details as well as a proof of the characterization theorem we need are in the manuscript [9].

Definition 4.5. Let f be an n-ary idempotent operation on a set A and X, Y nonempty subsets of A. We say that f *weakly points* X to Y, if there exist $\mathbf{a^1}, \dots, \mathbf{a^n} \in A^n$ such that for every $i \in [n]$ and $x \in X$ we have

$$f(a^i_1, \dots, a^i_{i-1}, x, a^i_{i+1}, \dots, a^i_n) \in Y$$

(where x is in the ith place). We refer to $\mathbf{a^1}, \dots, \mathbf{a^n}$ as *witnessing tuples.*

The word "weakly" means that we can have different witnessing tuples for different coordinates, as opposed to (strongly) pointing operations from [9]. For $f : A^k \to A$ and $g : A^n \to A$, we denote by $g \lessdot f$ the kn-ary operation on A defined by

$$(g \lessdot f)(x_1, \dots, x_{kn}) = g(f(x_1, \dots, x_k), f(x_{k+1}, \dots, x_{2k}), \dots, f(x_{(n-1)k+1}, \dots, x_{nk})).$$

We will need the following easy observation.

Observation 4.6. *If $f : A^k \to A$ weakly points X to Y and $g : A^n \to A$ weakly points Y to Z, then $g \lessdot f$ weakly points X to Z.*

Proof. Let the witnessing tuples for f weakly pointing X to Y and g weakly pointing Y to Z be $\mathbf{a^1}, \dots, \mathbf{a^k}$ and $\mathbf{b^1}, \dots, \mathbf{b^n}$, respectively. For $i \in [n]$ and $j \in [k]$ define $\mathbf{c^{i,j}} \in A^{nk}$ to be the following tuple:

$$\begin{aligned}\mathbf{c^{i,j}} = (&b^i_1, b^i_1, \dots, b^i_1, b^i_2, b^i_2, \dots, b^i_2, \dots, b^i_{i-1}, b^i_{i-1}, \dots, b^i_{i-1},\\ &a^j_1, a^j_2, \dots, a^j_k, b^i_{i+1}, b^i_{i+1}, \dots, b^i_{i+1}, \dots, b^i_n, b^i_n, \dots, b^i_n),\end{aligned}$$

where b^i_l appears k-times for every $l \in [n] \setminus \{i\}$. It is straightforward to verify (using idempotency of f) that $g \lessdot f$ weakly points X to Z with witnessing tuples $\mathbf{c^{1,1}}, \mathbf{c^{1,2}}, \dots, \mathbf{c^{1,k}}, \mathbf{c^{2,1}}, \dots, \mathbf{c^{n,k}}$. $\square$

Of particular interest are term operations weakly pointing the whole algebra (or a subuniverse) to a singleton, due to the following characterization of congruence meet-semidistributivity.

Definition 4.7. Let $\mathbf{A}$ be a finite idempotent algebra. We say that $\mathbf{A}$ *has a weakly pointing operation*, if there exists $\tau \in \mathrm{Clo}\,\mathbf{A}$ and $a \in A$ such that τ weakly points A to $\{a\}$.

Theorem 4.8 ([9, Theorem 1.3])**.** *A finite idempotent algebra* $\mathbf{A}$ *is* $\mathrm{SD}(\wedge)$*, if and only if every nonempty subuniverse* $B \leq \mathbf{A}$ *has a weakly pointing operation.*

Remark. Using this characterization it is easy to prove that given a finite idempotent algebra $\mathbf{A}$, the class of all $\mathrm{SD}(\wedge)$ members of the pseudovariety generated by $\mathbf{A}$ (that is, quotients of subuniverses of finite powers of $\mathbf{A}$) is closed under taking products, subalgebras and quotients. In particular, we will need the following fact.

Lemma 4.9 ([9, Proposition 2.1(6)])**.** *Let* $\mathbf{A}$ *be a finite idempotent algebra and* B, C *its nonempty subuniverses. If* B *and* C *are* $\mathrm{SD}(\wedge)$*, then* $B \times C$ *(considered as a subuniverse of* $\mathbf{A}^2$*) is* $\mathrm{SD}(\wedge)$ *as well.*

4.2. **Absorbing subuniverses.** We briefly introduce basic notions and facts from the theory of absorption of Barto and Kozik. For more details see [4, 9].

Definition 4.10. Let $\mathbf{A}$ be an algebra and $B \leq \mathbf{A}$ a nonempty subuniverse. We say that B is an *absorbing subuniverse* of $\mathbf{A}$, and write $B \trianglelefteq \mathbf{A}$, if there exists an idempotent $\tau \in \operatorname{Clo} \mathbf{A}$ such that

$$\begin{gathered}\tau(A, B, B, \dots, B, B) \subseteq B,\\ \tau(B, A, B, \dots, B, B) \subseteq B,\\ \vdots\\ \tau(B, B, B, \dots, B, A) \subseteq B.\end{gathered}$$

We also say that B absorbs $\mathbf{A}$ *via* τ and call τ an *absorbing operation.*

Note that B absorbs $\mathbf{A}$ *via* τ (say n-ary), if and only if τ (strongly) points A to B and any tuple $\mathbf{b} \in B^n$ can serve as a witnessing tuple for that. Hence absorption is somewhat stronger than pointing operations.

In applications of absorption theory an important role is played by algebras with no proper absorbing subuniverses, the *absorption-free* algebras.

Definition 4.11. An algebra $\mathbf{A}$ is *absorption-free*, if $|A| > 1$, and $B \trianglelefteq \mathbf{A}$ implies that $B = A$.

The following corollary, which is an easy consequence of Theorem 4.8, will be applied several times in our proof.

Corollary 4.12 (see [9, Corollary 2.13])**.** *A finite idempotent algebra* $\mathbf{A}$ *is* $\mathrm{SD}(\wedge)$*, if and only if every absorption-free subuniverse* $B \leq \mathbf{A}$ *has a weakly pointing operation.*

We will use without further notice the following easy facts about absorption:

Lemma 4.13 ([4, Proposition 2.4])**.** *Let* $\mathbf{A}$ *be a finite idempotent algebra.*

- *If* $B \trianglelefteq \mathbf{A}$ *and* $C \trianglelefteq B$*, then* $C \trianglelefteq \mathbf{A}$*.*
- *If* $B \trianglelefteq \mathbf{A}$ *(via* τ*) and* $C \leq A$ *and* $B \cap C \neq \emptyset$*, then* $B \cap C \trianglelefteq C$ *(via* $\tau|_C$*).*

5. The proof

Let us start by introducing notation used throughout the proof. Let $\mathbb{T} = (T; E)$ be an oriented tree of height 1, with $T = A \dot\cup B$ and $E \subseteq A \times B$. We will sometimes write $a \dashrightarrow b$ to mean $(a, b) \in E$. Let $\mathbb{H} = (H; \rightarrow)$ be a $\mathbb{T}$-special tree of height h such that $\mathbf{alg}\, \mathbb{H}$ is Taylor. Our aim is to prove that $\mathbf{alg}\, \mathbb{H}$ is $\mathrm{SD}(\wedge)$. We divide the proof into several steps organized into subsections.

5.1. **Reduction to the top and bottom levels.** Our first step is to show that we can focus only on the top and bottom level of $\mathbb{H}$, i.e., the sets (indeed, subuniverses) A and B. This is the property that justifies the definition of special trees. The reduction was already described in detail in [2] (although the construction there is different).

Lemma 5.1. *Both A and B are subuniverses of $\mathbf{alg}\,\mathbb{H}$. Moreover, $E \leq A \times B$ ($\leq (\mathbf{alg}\,\mathbb{H})^2$).*

Proof. By Lemma 3.1, it is enough to show that A, B and E are primitive positive definable from $\mathbb{H}$ with constants (although in fact, we will not need the constants). Let $\mathbb{Q}$ be a minimal oriented path of height h which maps homomorphically onto $\mathbb{P}_e$ for all $e \in E$, given by Lemma 2.2. Let us denote by u and v the initial and terminal vertex of $\mathbb{Q}$, respectively. The binary relation E is equal to the set

$$\{(\varphi(u), \varphi(v)) \mid \varphi : \mathbb{Q} \to \mathbb{H} \text{ is a homomorphism}\},$$

which can be expressed by a primitive positive formula. Consequently, $(\exists y)(x \dashrightarrow y)$ and $(\exists y)(y \dashrightarrow x)$ provides us with primitive positive definitions of A and B, respectively. $\square$

It is useful to observe that an n-ary polymorphism can be defined on different components of connectivity of $\mathbb{H}^n$ independently; to verify that it preserves the edges one has to be concerned with inputs from one component at a time only. Among the components a prominent one is the component containing the diagonal: For $n > 0$ we denote by Δ_n the component of connectivity of the digraph $\mathbb{H}^n$ containing the diagonal (i.e., the set $\{(v, \ldots, v) : v \in H\}$).

Lemma 5.2. *For any $n > 0$, $(A^n \cup B^n) \subseteq \Delta_n$.*

Proof. It is easily seen that the set $(A^n \cup B^n)$ is connected in the digraph $\mathbb{T}^n$. Let $(\mathbf{a}, \mathbf{b})$ be an edge in $\mathbb{T}^n$ (i.e., $a_i \dashrightarrow b_i$ for $i \in [n]$). Let $\mathbb{Q}$ be a minimal oriented path of height h which maps homomorphically onto all the paths $\{\mathbb{P}_{(a_i,b_i)} \mid i \in [n]\}$, whose existence is provided by Lemma 2.2. For every $i \in [n]$ let $\varphi_i : \mathbb{Q} \to \mathbb{P}_{(a_i,b_i)}$ be a homomorphism. Then the mapping $\varphi : \mathbb{Q} \to \mathbb{H}^n$ given by $\varphi(\mathbf{x}) = (\varphi_1(x_1), \ldots, \varphi_n(x_n))$ is also a homomorphism and it maps the initial and terminal vertex of $\mathbb{Q}$ to $\mathbf{a}$ and $\mathbf{b}$, respectively. This shows that $\mathbf{a}$ and $\mathbf{b}$ are connected in $\mathbb{H}^n$ (via $\varphi(\mathbb{Q})$). Consequently, the whole set $(A^n \cup B^n)$ is connected in $\mathbb{H}^n$. As it intersects the diagonal, it follows that $(A^n \cup B^n) \subseteq \Delta_n$. $\square$

In the next lemma we prove that every polymorphism which is a WNU on the top and bottom levels can be modified to obtain a polymorphism satisfying the WNU property everywhere. In Corollary 5.4 below we combine this fact with Theorem 4.2 to obtain the desired result. The assumption that $n > 2$ is there only to avoid a technical nuisance; in fact, the claim is true for $n = 2$ as well (see [2]).

Lemma 5.3. *Let $n \geq 3$ and let $\tau \in \mathrm{IdPol}_n(\mathbb{H})$ be such that $\tau|_A$ and $\tau|_B$ are WNU operations on A and B, respectively. Then there exists $\tau' \in \mathrm{IdPol}_n(\mathbb{H})$ which is a WNU on H.*

Proof. Let us fix an arbitrary linear order $\leq_E$ of the set E. We define the following linear order $\sqsubseteq$ on the set $H \setminus (A \cup B)$: for $x \in \mathbb{P}_{(a,b)}$ and $y \in \mathbb{P}_{(a',b')}$ we put $x \sqsubset y$ if

- $(a, b) <_E (a', b')$, or
- $(a, b) = (a', b')$ and x is closer to a than y (in $\mathbb{H}$).

We split the definition of τ' into several cases. Fix $\mathbf{x} \in H^n$.

(1) If $\mathbf{x} \in A^n \cup B^n$, then we set $\tau'(\mathbf{x}) = \tau(\mathbf{x})$.
(2) If $\mathbf{x} \in \Delta_n \setminus (A^n \cup B^n)$, then

(a) if $\{x_1, \dots, x_n\} \subseteq \mathbb{P}_{(a,b)}$ for some $(a, b) \in E$, then we define $\tau'(\mathbf{x})$ to be the $\sqsubseteq$-minimal element from $\{x_1, \dots, x_n\}$,
(b) if there exists $i \in [n]$ and $e \neq e' \in E$ such that $x_i \in \mathbb{P}_e$ and $x_j \in \mathbb{P}_{e'}$ for all $j \neq i$, then we define

$$\tau'(\mathbf{x}) = \tau(x_i, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n),$$

(c) in all other cases we set $\tau'(\mathbf{x}) = \tau(\mathbf{x})$.

(3) If $\mathbf{x} \notin \Delta_n$, then
(a) if $\operatorname{lvl}(x_1) = \operatorname{lvl}(x_2) = \cdots = \operatorname{lvl}(x_n)$, then we define $\tau'(\mathbf{x})$ to be the $\sqsubseteq$-minimal element from $\{x_1, \dots, x_n\}$,
(b) if there exists $i \in [n]$ and $k \neq l$ such that $\operatorname{lvl}(x_i) = k$ and $\operatorname{lvl}(x_j) = l$ for all $j \neq i$, then we define $\tau'(\mathbf{x}) = x_i$,
(c) in all other cases we define $\tau'(\mathbf{x}) = x_1$.

Let us first comment on subcase (2b) of the construction. Since τ is a polymorphism, for any $(a_i, b_i) \in E$, $i \in [n]$, it induces a homomorphism from $\Delta_n \cap \prod_{i=1}^n \mathbb{P}_{(a_i,b_i)}$ (as an induced subgraph of $\mathbb{H}^n$) to $\mathbb{P}_{(\tau(\mathbf{a}),\tau(\mathbf{b}))}$. However, typically there are many such homomorphisms. Even if $\tau(\mathbf{a}) = \tau(\mathbf{a}')$, $\tau(\mathbf{b}) = \tau(\mathbf{b}')$ and $\mathbf{a}', \mathbf{b}'$ are just permutations of $\mathbf{a}, \mathbf{b}$, the two corresponding homomorphisms induced by τ can be different. That is why we cannot simply define $\tau'(\mathbf{x}) = \tau(\mathbf{x})$ in subcase (2b); the WNU property might not hold.

We divide the proof into two separate claims.

Claim. τ' is a polymorphism of $\mathbb{H}$.

Let $(\mathbf{x}, \mathbf{y})$ be an edge in $\mathbb{H}^n$. For every $i \in [n]$ let $e_i = (a_i, b_i) \in E$ be such that $x_i, y_i \in \mathbb{P}_{e_i}$. If $\mathbf{x}$ falls under case (1) of the construction, then $\tau'(\mathbf{x}) = \tau(\mathbf{a})$, and $\mathbf{y}$ falls under one of the subcases of (2). If it is (2a), then $e_1 = \cdots = e_n = e$ for some $e = (a, b) \in E$ and $y_1 = \cdots = y_n = \tau'(\mathbf{y}) = y$, where y is the unique vertex from $\mathbb{P}_e$ such that $a \to y$. Hence $\tau'(\mathbf{x}) \to \tau'(\mathbf{y})$ holds. If it is subcase (2b), then $\mathbf{x} = \mathbf{a} = (a, \dots, a, a', a, \dots, a)$ for some $a, a' \in A$ (where a' is in the ith coordinate) and $\mathbf{y} = (y, \dots, y, y', y, \dots, y)$. Using both that τ is a WNU and a polymorphism we get that $\tau'(\mathbf{x}) = \tau(\mathbf{a}) = \tau(a', a, \dots, a) \to \tau(y', y, \dots, y) = \tau'(\mathbf{y})$. If $\mathbf{y}$ falls under subcase (2c), then for every $i \in [n]$, y_i is the unique vertex from $\mathbb{P}_{e_i}$ such that $a_i \to y_i$ and since τ is a polymorphism we get that $\tau'(\mathbf{x}) = \tau(\mathbf{a}) \to \tau(\mathbf{y}) = \tau'(\mathbf{y})$.

The argument is similar when $\mathbf{y}$ falls under case (1) (and so $\mathbf{x}$ under (2)). In all other situations both $\mathbf{x}$ and $\mathbf{y}$ fall under the same subcase of the construction. Note that since $x_i \to y_i$ and $x_i \notin A$, $y_i \notin B$ (for all $i \in [n]$), it follows that there is an edge between the $\sqsubseteq$-minimal element of $\{x_1, \dots, x_n\}$ and of $\{y_1, \dots, y_n\}$. This implies $\tau'(\mathbf{x}) \to \tau'(\mathbf{y})$ for cases (2a) and (3a).

In cases (2c) and (3c) the polymorphism condition follows immediately from the fact that τ is a polymorphism (in (2c)) and that $x_1 \to y_1$ (in (3c)). For the remaining cases, (2b) and (3b), we have to add the observation that the distinguished coordinate $i \in [n]$ is the same for both $\mathbf{x}$ and $\mathbf{y}$.

Claim. τ' is a WNU on H.

Let $x, y \in H$ be arbitrary. Note that all of the tuples $(y, x, \dots, x)$, $(x, y, x, \dots, x)$, $\dots$, $(x, \dots, x, y)$ fall under the same case (and subcase) of the construction, and that it can be neither (2c) nor (3c). In case (1) the WNU property follows from the fact that τ is a WNU on A and B while in cases (2a) and (3a) from the fact that the construction in these cases is independent of order and repetition of elements. In case (2b) the result is $\tau(y, x, \dots, x)$ for all the tuples in question while in case (3b) the result is always y. $\square$

Corollary 5.4. *If both A and B are* $\mathrm{SD}(\wedge)$*, then* $\mathbf{alg}\,\mathbb{H}$ *is* $\mathrm{SD}(\wedge)$.

Proof. By Lemma 4.9, $A \times B$ ($\leq (\mathbf{alg}\,\mathbb{H})^2$) is SD($\wedge$) as well. Hence, by Theorem 4.2, there exists n_0 such that for every $n \geq n_0$ there exists $\tau_n \in \mathrm{IdPol}_n(\mathbb{H})$ such that $(\tau_n \times \tau_n)|_{A\times B}$ is a WNU on $A \times B$. This implies that the restrictions of τ_n to A and B are WNUs. Using Lemma 5.3 we obtain, for every $n \geq \max(n_0, 3)$, a WNU $\tau_n' \in \mathrm{IdPol}_n(\mathbb{H})$. The proof concludes by another application of Theorem 4.2. □

5.2. **Singleton absorbing subuniverse.** Our next step is to prove that either A or B has a singleton absorbing subuniverse. This is the one and only place where we use the assumption that $\mathbf{alg}\,\mathbb{H}$ is Taylor.

Since $\mathbf{alg}\,\mathbb{H}$ is Taylor, by Theorem 4.2 there exists a WNU operation $\omega \in \mathrm{IdPol}(\mathbb{H})$. Let $\circ : H^2 \to H$ be the *binary polymer* of the WNU ω, that is,

$$x \circ y = \omega(x, x, \dots, y) = \dots = \omega(y, x, \dots, x)$$

for $x, y \in H$. Note that $\circ \in \mathrm{IdPol}_2(\mathbb{H})$.

We can and will assume that ω is *special* in the sense of [3, Definition 6.2], that is, satisfies $x \circ (x \circ y) = x \circ y$. (Here the word *special* is unrelated to our definition of *special* trees.) This property can be enforced by an iterated composition of ω with itself (i.e., $\omega \lessdot \omega \lessdot \dots \lessdot \omega$, $|H|!$-times, see [3, Lemma 6.4]).

For $x, y \in A \cup B$ we denote by $\mathrm{dist}_E(a, b)$ the *distance* of x and y in $\mathbb{T}$. For a subset $C \subseteq A$ we define the *E-neighbourhood* of C, denoted by $E_+(C)$, to be the set $\{b \in B \mid c \dashrightarrow b \text{ for some } c \in C\}$. Similarly, the E-neighbourhood of $D \subseteq B$ is the set $E_-(D) = \{a \in A \mid a \dashrightarrow d \text{ for some } d \in D\}$. For brevity we write $E_+(c)$, $E_-(d)$ instead of $E_+(\{c\})$, $E_-(\{d\})$. Moreover, for every $k \geq 0$, $C \subseteq A$ and $D \subseteq B$ we inductively define the sets $E_k(C)$ and $E_k(D)$ as follows:

- $E_0(C) = C$ and $E_0(D) = D$,
- $E_1(C) = E_+(C)$ and $E_1(D) = E_-(D)$, and
- $E_k(C) = E_1(E_{k-1}(C))$ and $E_k(D) = E_1(E_{k-1}(D))$ for $k > 1$.

Note that the above definition can be reformulated as follows:

$$E_k(C) = \{x \in A \cup B \mid (\exists\, c \in C)\ \mathrm{dist}_E(x, c) \leq k \ \&\ \mathrm{dist}_E(x, c) \equiv k \pmod 2\},$$

and similarly for $E_k(D)$. We will frequently use the following easy facts (as well as the obvious "dual" versions for $D \leq D' \leq B$), which are all consequences of the fact that $E \leq (\mathbf{alg}\,\mathbb{H})^2$. We leave the proof to the reader.

Observation 5.5. *If $C \leq C' \leq A$, then the following holds:*

- $E_+(C) \leq E_+(C') \leq B$,
- *$E_k(C) \leq A$ for k even and $E_k(C) \leq B$ for k odd,*
- *if $k \leq l$ and $l - k$ is even, then $E_k(C) \leq E_l(C)$,*
- *if $C \neq 0$, then there exists k such that $E_k(C) = A$ and $E_{k+1}(C) = B$, and*
- *if $C \trianglelefteq C'$, then for every $k \geq 0$, $E_k(C) \trianglelefteq E_k(C')$ as well and, moreover, the absorption is via the same $\tau \in \mathrm{IdPol}(\mathbb{H})$.* [2]

We are now ready to prove that either A or B has a singleton absorbing subuniverse and, moreover, that this absorption is realized via the WNU operation ω.

Lemma 5.6. *There exists $o \in A \cup B$ such that $\{o\} \trianglelefteq E_2(o)$ via ω.*

Proof. Suppose for contradiction that no such element exists. It follows that for every $u \in A \cup B$ there exists $w \in E_2(u)$ such that $u \circ w = v \neq u$. Since the WNU ω is special, we have that $u \circ v = u \circ (u \circ w) = u \circ w = v$. Consider the binary relation $\gg$ on $A \cup B$ defined by setting $u \gg v$ if and only if $v \in E_2(u) \setminus \{u\}$ and $u \circ v = v$. We have proved that for every $u \in A \cup B$ there exists v such that $u \gg v$.

[2] Technically, the absorbing operation is $\tau|_{C'}$ in the first case while it is $\tau|_{E_k(C')}$ in the second case, but we will neglect this formality.

Let k be maximal such that there exists a sequence $\langle u_0 u_1 \dots u_k \rangle$ of elements of $A \cup B$ with the following properties:

(1) $\mathrm{dist}_E(u_0, u_i) = i$ for all $i \in [k]$, and
(2) $u_i \gg u_{i+2}$ for all $0 \leq i \leq k-2$.

Note that (1) ensures that the sequence is non-repeating and thus, by finiteness of $A \cup B$, such a maximal k exists. The previous paragraph shows that $k \geq 2$: just take $\langle a, b, a' \rangle$ for any $a, a' \in E_-(b)$ such that $a \gg a'$.

Let us assume that $u_k \in A$; the proof for $u_k \in B$ is analogous. Let $u'_k \in A$ and $u'_{k+1} \in B$ be such that $u_{k-1} \gg u'_{k+1}$ and $u_{k-1}, u'_{k+1} \in E_+(u'_k)$ (see the figure below). We will prove that the sequence $\langle u_0 u_1 \dots u_{k-1} u'_k u'_{k+1} \rangle$ also satisfies properties (1) and (2); a contradiction with maximality of k.

$$\begin{tikzcd}
 & & & u_k \arrow[dl, dashed] & \\
\cdots & u_{k-2} \arrow[l, dashed] \arrow[r, dashed] & u_{k-1} & & \\
 & & & u'_k \arrow[ul, dashed] \arrow[r, dashed] & u'_{k+1}
\end{tikzcd}$$

First we prove (1). From $u_{k-1} \gg u'_{k+1}$ we get that $\mathrm{dist}_E(u_{k-1}, u'_k) = 1$ and $\mathrm{dist}_E(u_{k-1}, u'_{k+1}) = 2$. Since $\mathbb{T}$ is a tree, it suffices to rule out the possibility that $u'_k = u_{k-2}$. In that case $u_{k-2} \dashrightarrow u'_{k+1}$, $u_{k-2} \dashrightarrow u_{k-1}$ and $u_k \dashrightarrow u_{k-1}$ would give

$$\omega(u_{k-2}, u_{k-2}, \dots, u_{k-2}, u_k) \dashrightarrow \omega(u'_{k+1}, u_{k-1}, \dots, u_{k-1}, u_{k-1}).$$

The left hand side is $u_{k-2} \circ u_k = u_k$ while the right hand side is $u_{k-1} \circ u'_{k+1} = u'_{k+1}$; and so we get $u_k \dashrightarrow u'_{k+1}$. But $u_k \in E_-(u_{k-1}) \cap E_-(u'_{k+1})$ would imply that $u_k = u'_k = u_{k-2}$ which contradicts $u_{k-2} \gg u_k$.

To prove (2) we only need to establish $u_{k-2} \gg u'_k$. From $u_{k-2} \dashrightarrow u_{k-1}$, $u'_k \dashrightarrow u'_{k+1}$ and the fact that $\circ$ preserves E we get

$$u_{k-2} \circ u'_k \dashrightarrow u_{k-1} \circ u'_{k+1} = u'_{k+1}.$$

On the other hand, $\{u_{k-2}, u'_k\} \subseteq E_-(u_{k-1})$, which is a subuniverse, and thus $u_{k-2} \circ u'_k \dashrightarrow u_{k-1}$. It follows that $u_{k-2} \circ u'_k = u'_k$; and $u_{k-2} \neq u'_k$ is proved above. $\square$

Fix $o \in A \cup B$ given by the previous lemma. To simplify the exposition we choose that $o \in A$. The proofs are essentially the same in the other case (moreover, note that reversing edges of $\mathbb{H}$ does not change $\mathbf{alg}\,\mathbb{H}$).

Since $\mathbb{H}$ is an oriented tree, it follows that for every $v \in H$ there exists a unique oriented path $\mathbb{Q}_{o,v}$ connecting o to v in $\mathbb{H}$. We define a partial order $\preceq$ on H by setting $u \preceq v$ if and only if $u \in \mathbb{Q}_{o,v}$. Note that o is the minimum element in this order. Furthermore, for $u, v \in A \cup B$, $u \preceq v$ implies $\mathrm{dist}_E(o, u) \leq \mathrm{dist}_E(o, v)$.

Lemma 5.7. *If $a, a' \in A$ and $a \preceq a'$, then $a \circ a' = a$ (and similarly for $b, b' \in B$). In particular, $\{o\} \trianglelefteq A$ via ω.*

Proof. If $a = a'$, then $a \circ a' = a$ follows trivially from idempotency of ω. Else, there exists $k \geq 0$ such that $a \in E_k(o)$ and $a' \in E_{k+2}(o) \setminus E_k(o)$. From Lemma 5.6 and the last item of Observation 5.5 it follows that $E_k(o) \trianglelefteq E_{k+2}(o)$ via ω and so $a \circ a' \in E_k(o)$. In particular, $a \circ a' \neq a'$.

Note that $l = \mathrm{dist}_E(a, a')$ is even and that there exists a unique vertex $u \in \mathbb{Q}_{o,a'} \cap (A \cup B)$ such that $\mathrm{dist}_E(a, u) = \mathrm{dist}_E(u, a') = l/2$. Since $a, a' \in E_{l/2}(u)$, which is a subuniverse, we have $a \circ a' \in E_{l/2}(u)$ while a is the $\preceq$-minimal element of $E_{l/2}(u)$. It follows that $a \preceq a \circ a'$.

Suppose for contradiction that $a \neq a \circ a'$. Then repeating the arguments from the first paragraph with $a \circ a'$ in the role of a' yields $a \circ (a \circ a') \neq a \circ a'$ which contradicts the fact that ω is a special WNU.

Hence we have proved that $a \circ a' = a$. The proof for $b \preceq b'$ is essentially the same. The fact that $\{o\} \trianglelefteq A$ via ω now follows immediately from the definition of absorption and the fact that o is the $\preceq$-minimum element of A. □

Remark. Incidentally, the Absorption Theorem of Barto and Kozik [4, Theorem 2.3] applied to A, B and E immediately yields that either A or B has a singleton absorbing subuniverse. We need a slightly stronger fact for our proof (namely that the absorbing operation is a WNU); it is however likely that the claim of the Absorption theorem can be strengthened to replace the above ad hoc argument. Our argument can be viewed as a proof of a special case of the Absorption Theorem, where the relation E is acyclic.

Existence of the singleton absorbing subuniverse $\{o\}$ already significantly restricts living space for possible absorption-free subuniverses in A and B, as we can see in the next lemma. (Of course, the dual version for $D \leq B$ is also true.)

Lemma 5.8. *If $C \leq A$ is absorption-free, then there exists $k > 0$ such that $\operatorname{dist}_E(o, c) = k$ for all $c \in C$.*

Proof. Let k be the minimum from the set $\{\operatorname{dist}_E(o, c) \mid c \in C\}$. Since $\{o\} \trianglelefteq A$, by Observation 5.5 we have $E_k(o) \trianglelefteq E_k(A) = A$, and thus also $C \cap E_k(o) \trianglelefteq C \cap A = C$. Since C is absorption-free, it follows that $C \cap E_k(o) = C$.

We have proved that $k \leq \operatorname{dist}_E(o, c) \leq k$ for all $c \in C$. Note that $k > 0$, since otherwise $C = \{o\}$ which is not absorption-free by definition. □

5.3. **E-neighbourhoods of singletons are** SD($\wedge$)**.** In this subsection we prove that E-neighbourhoods of elements from $A \cup B$ are SD($\wedge$). Our strategy is to show that whenever they have an absorption-free subuniverse, it must have a weakly pointing operation (and then apply Corollary 4.12). For the rest of this subsection we fix $b \in B$ and an absorption-free subuniverse $C \leq E_-(b)$. (The proof for $D \leq E_+(a)$ is analogous.)

From Lemma 5.8 and the fact that $|C| > 1$ (and that $\mathbb{T}$ is a tree) we see that $b \prec c$ for all $c \in C$. In the first step we prove that elements from B which are $\preceq$-above C are "absorbed by b" via a certain binary operation $\star$. (Note that such elements do not need to form a subuniverse, and so it is not absorption in the sense we defined.) Later we will use this operation to construct various binary polymorphisms and then build up a weakly pointing operation for C from them.

Let us denote by $\star$ the binary idempotent polymorphism of $\mathbb{H}$ given by

$$x \star y = (\ldots(((x \underbrace{\circ y) \circ y) \circ \ldots \circ}_{|H|\times} y),$$

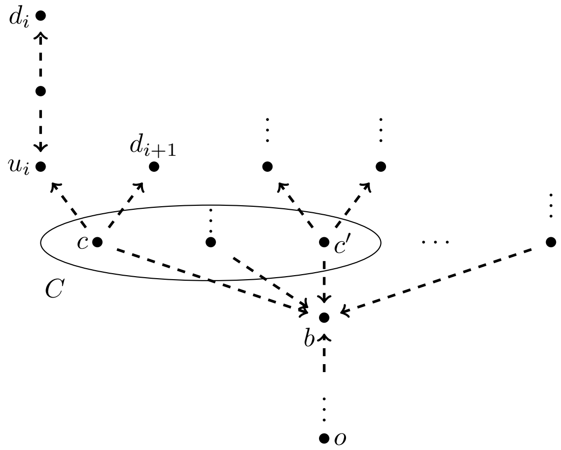
where the operation $\circ$ appears $|H|$-times (just for good measure).

Lemma 5.9. *If $d \in B$ is such that $c \prec d$ for some $c \in C$, then $b \star d = d \star b = b$.*

Proof. From Lemma 5.7 we get that $b \circ d = b$ and thus also $b \star d = b$. To prove the other equality, fix $d \in B$ and $c \in C$ with $b \prec c \prec d$ and consider the sequence $\langle d_0 d_1 \ldots d_{|H|} \rangle$ of elements of B defined inductively by setting $d_0 = d$ and $d_i = d_{i-1} \circ b$ for $i \in [|H|]$. Observe that $d_{|H|} = d \star b$ which we want to equate to b.

Let k_i denote the distance $\operatorname{dist}_E(d_i, b)$. We will prove that for every $0 \leq i \leq |H|$, $b \preceq d_i$ and $k_i \leq k_{i-1}$ (we set $k_{-1} = k_0$). The proof uses induction on i; the case $i = 0$ is trivial. Assume that the claim holds for some $i < |H|$. Following the same argument as in the proof of Lemma 5.7, there exists $u_i \in A \cup B$ such that

$\operatorname{dist}_E(u_i, b) = \operatorname{dist}_E(u_i, d_i) = k_i/2$. Since $b \preceq d_i$, it follows that b is the $\preceq$-minimal (and d_i a $\preceq$-maximal) element of $E_{k_i/2}(u_i)$. Consequently, $d_{i+1} = d_i \circ b \in E_{k_i/2}(u_i)$ implies that $b \preceq d_{i+1}$ and $k_{i+1} \leq k_i$ (see the figure below).



Note that $k_0 < |H|$, and so there must exist $i < |H|$ such that $k_i = k_{i+1}$. Denote this distance by k and suppose for contradiction that $k \neq 0$ (and so $k \geq 2$, since k is even). Pick any $c' \in C$. Since $c \in E_{k-1}(d_i)$, $c' \in E_-(b) \leq E_{k-1}(b)$, $d_i \circ b = d_{i+1}$ and $\circ$ preserves E, it follows that $c \circ c' \in E_{k-1}(d_{i+1})$. But we also have $c \circ c' \in C$ and $E_{k-1}(d_{i+1}) \cap C = \{c\}$. Thus we have proved that $c \circ c' = c$ for all $c' \in C$, which means that $\{c\} \trianglelefteq C$ via ω, a contradiction with C being absorption-free. Therefore it must be the case that $k = 0$, which means $d_i = b$ and thus by idempotency of $\circ$ also $d_{|B|} = d \star b = b$. ☐

Let us denote by $\mathcal{F}$ the smallest set of binary operations on H satisfying

- $x \star y \in \mathcal{F}$, $y \star x \in \mathcal{F}$, and
- if $\varphi(x, y) \in \mathcal{F}$, then $\{x \star \varphi(x, y), y \star \varphi(x, y), \varphi(x, y) \star x, \varphi(x, y) \star y\} \subseteq \mathcal{F}$,
- if $\varphi(x, y), \varphi'(x, y) \in \mathcal{F}$, then $(\varphi(x, y) \star \varphi'(x, y)) \in \mathcal{F}$.

From Lemma 5.9 and the construction of $\mathcal{F}$ we immediately obtain the following:

Corollary 5.10. *If $d \in B$ is such that $c \prec d$ for some $c \in C$, then $\varphi(b, d) = \varphi(d, b) = b$ for every $\varphi \in \mathcal{F}$.*

For every $c, c' \in C$ let $S_{c,c'}$ be the set $\{\varphi(c, c') \mid \varphi(x, y) \in \mathcal{F}\} \subseteq C$. We will use the following easy facts:

- $S_{c,c} = \{c\}$,
- $S_{c,c'} = S_{c',c}$,
- both $S_{c,c'}$ and $S_{c,c'} \cup \{c, c'\}$ are closed under the operation $\star$,
- in particular, if $x, y \in S_{c,c'}$, then $S_{x,y} \subseteq S_{c,c'}$.

Remark. Alternatively, using terminology from universal algebra, we could have defined $\mathcal{F}$ to be the set of all binary terms in the binary operation symbol $\star$ which contain both the variables x and y. Then $S_{c,c'}$ would be the image of $\mathcal{F}$ under the homomorphism from the absolutely free two-generated algebra to $(C; \{\star\})$ given by $x \mapsto c$ and $y \mapsto c'$.

Note that $\mathcal{F} \subseteq \operatorname{IdPol}_2(\mathbb{H})$. In the next lemma we prove that, in fact, $\mathbb{H}$ has many more binary idempotent polymorphisms.

Lemma 5.11. *Let $\gamma : C^2 \to C$ be any binary operation such that $\gamma(c, c') \in S_{c,c'}$ for all $c, c' \in C$. Then there exists $\tau \in \operatorname{IdPol}_2(\mathbb{H})$ extending γ (i.e., $\tau|_C = \gamma$).*

Proof. For every $c, c' \in C$ we fix some $\varphi_{c,c'}(x,y) \in \mathcal{F}$ witnessing that $\gamma(c,c') \in S_{c,c'}$. For $x, y \in H$ we define $\tau(x,y)$ in the following way:

(1) If there exist $c, c' \in C$ such that
- $b \prec x \prec c$ or $c \preceq x$,
- $b \prec y \prec c'$ or $c' \preceq y$, and
- $\mathrm{lvl}(x) = \mathrm{lvl}(y)$,

then we set $\tau(x,y) = \varphi_{c,c'}(x,y)$.

(2) Else, we define $\tau(x,y) = x \star y$.

It follows immediately from the construction that τ is idempotent and $\tau|_C = \gamma$. To prove that $\tau \in \mathrm{IdPol}(\mathbb{H})$, let $x \to u$, $y \to v$ be arbitrary edges of $\mathbb{H}$. Note that since $\star$ and $\varphi_{c,c'}$ (for any $c, c' \in C$) are polymorphisms of $\mathbb{H}$, $\tau(x,y) \to \tau(u,v)$ follows immediately if both $\{x,y\}$ and $\{u,v\}$ fall under the same case of the construction. If they do not, then it must be the case that $\{x,y\}$ falls under case (1) while $\{u,v\}$ under case (2) (it cannot be the opposite, since if $b \prec u$ and $x \to u$, then $b \prec x$ as well, and similarly for the other conditions). Thus $\tau(x,y) = \varphi_{c,c'}(x,y)$ for some $c, c' \in C$ and $\tau(u,v) = u \star v$. Moreover it must be the case that $b \preceq u$, $b \preceq v$ and $b \in \{u,v\}$. As $\mathrm{lvl}(x) = \mathrm{lvl}(y)$ implies that $\mathrm{lvl}(u) = \mathrm{lvl}(v)$, we get $u, v \in B$. It follows that $\tau(u,v) = u \star v = b = \varphi_{c,c'}(u,v)$, either by Corollary 5.10 or by idempotency (in case that $u = v = b$). We conclude that $\tau(x,y) \to \tau(u,v)$ in this case as well. ☐

As an easy consequence of this lemma, we can prove that C has a binary idempotent commutative operation (i.e., a binary WNU).

Corollary 5.12. *There exists $\varphi \in \mathrm{IdPol}_2(\mathbb{H})$ such that $\varphi|_C$ is commutative.*

Proof. For every $c, c' \in C$ define $\gamma(c,c') = \gamma(c',c)$ to be an arbitrary element from $S_{c,c'}$ thus making γ commutative, and then apply Lemma 5.11. ☐

The above corollary implies that $|C| > 2$, since a binary WNU on a 2-element set is a semilattice operation which would violate absorption-freeness. Unfortunately, a binary WNU is not enough to construct a weakly pointing operation for C; we need a slightly more involved argument.

Lemma 5.13. *C has a weakly pointing operation.*

Proof. We start by showing that every two-element set is weakly pointed to a singleton by some operation, with an additional "symmetry" property.

Claim. For every $x, y \in C$ there exist $\varphi \in \mathrm{IdPol}(\mathbb{H})$ (say it is n-ary), $z \in C$, $\mathbf{c^1}, \ldots, \mathbf{c^n} \in C^n$ and $\alpha : C \to C$ such that the following hold:

(1) $\varphi|_C$ weakly points $\{x,y\}$ to $\{z\}$ with witnessing tuples $\mathbf{c^1}, \ldots, \mathbf{c^n}$.
(2) For every $i \in [n]$ and $u \in C$, $\varphi(c^i_1, c^i_2, \ldots, c^i_{i-1}, u, c^i_{i+1}, \ldots, c^i_n) = \alpha(u)$.

We will prove the claim by induction on $|S_{x,y} \cup \{x,y\}|$. Assume first that $S_{x,y} \cap \{x,y\} \neq \emptyset$, say $x \in S_{x,y}$ (the argument for $y \in S_{x,y}$ is analogous). In that case we can apply Lemma 5.11 to construct $\varphi \in \mathrm{IdPol}_2(\mathbb{H})$ such that $\varphi(x,y) = \varphi(y,x) = \varphi(x,x) = x$ and $\varphi|_C$ is commutative (see the proof of Corollary 5.12). The claim follows since $\varphi|_C$ weakly points $\{x,y\}$ to $\{x\}$, the witnessing tuple is (x,x) for both coordinates and $\alpha(u) = \varphi(u,x)$ for all $u \in C$. This also covers the base step of our induction (i.e., $S_{x,y} \subseteq \{x,y\}$).

We can now assume that $S_{x,y} \cap \{x,y\} = \emptyset$. Let us define $c = x \star y$, $x' = x \star c$ and $y' = y \star c$. Using Lemma 5.11 we can construct $\varphi \in \mathrm{IdPol}_2(\mathbb{H})$ such that $\varphi(x,c) = \varphi(c,x) = x'$ and $\varphi(y,c) = \varphi(c,y) = y'$ and $\varphi|_C$ is commutative. In particular, $\varphi|_C$ points $\{x,y\}$ to $\{x',y'\}$, the witnessing tuple is (c,c) for both coordinates.

Since $x', y' \in S_{x,y}$, it follows that $S_{x',y'} \cup \{x', y'\} \subseteq S_{x,y} \subsetneq S_{x,y} \cup \{x, y\}$. Hence, by induction assumption, the claim holds for x', y'. Let it be witnessed by $\psi \in \mathrm{IdPol}(\mathbb{H})$ weakly pointing $\{x', y'\}$ to $\{z\}$ and let $\alpha' : C \to C$ be the corresponding mapping from (2).

Using Observation 4.6 we get that $(\psi \lessdot \varphi)|_C$ weakly points $\{x, y\}$ to $\{z\}$ and it is not hard to see from its proof that (2) holds as well, with $\alpha : C \to C$ given by $\alpha(u) = \alpha'(\varphi(u, c))$, for $u \in C$. We leave the verification to the reader.

We will now compose the operations from this claim to construct a weakly pointing operation for C; we use another induction argument.

Claim. For every nonempty $X \subseteq C$ there exists $c \in C$ and $\varphi \in \mathrm{IdPol}(\mathbb{H})$ such that $\varphi|_C$ weakly points X to $\{c\}$.

We prove the claim by induction on $|X|$. If $X = \{x\}$, then the claim is trivial: take any $\varphi \in \mathrm{IdPol}(\mathbb{H})$, $z = x$ and witnessing tuple $(x, x, \ldots, x)$ for all coordinates. Let $|X| = k > 1$ and assume that the claim holds for all at most $(k-1)$-element subsets of C. Pick any $x, y \in X$, $x \neq y$ and let $\varphi \in \mathrm{IdPol}(\mathbb{H})$ (say n-ary), $z \in C$ and $\alpha : C \to C$ be the objects given by the previous claim applied to x and y. It is easy to see that $\varphi|_C$ weakly points X to $Y = \{\alpha(x) \mid x \in X\}$ (this is why we need the "symmetry" property from the previous claim). Since $\alpha(x) = z = \alpha(y)$, it follows that $|Y| < |X|$. By induction assumption, there exists $\psi \in \mathrm{IdPol}(\mathbb{H})$ and $c \in C$ such that $\psi|_C$ weakly points Y to $\{c\}$. Using Observation 4.6 we get that $(\psi \lessdot \varphi)|_C$ weakly points X to $\{c\}$ which concludes the proof.

□

We have achieved the goal of this subsection, i.e., the following corollary.

Corollary 5.14. *For every $b \in B$, $E_-(b)$ is $\mathrm{SD}(\wedge)$. Similarly for $a \in A$ and $E_+(a)$.*

Proof. By Lemma 5.13, every absorption-free subuniverse $C \leq E_-(b)$ has a weakly pointing operation and so we can apply Corollary 4.12. The proof for $a \in A$ is analogous. □

5.4. **Absorption-free subuniverses are** $\mathrm{SD}(\wedge)$**.** The last step of our proof is to show that every absorption-free subuniverse C of A or B has a weakly pointing operation. Theorem 2.4 will then follow from Corollary 4.12 and Corollary 5.4.

Lemma 5.15. *Every absorption-free subuniverse C of A or B has a weakly pointing operation.*

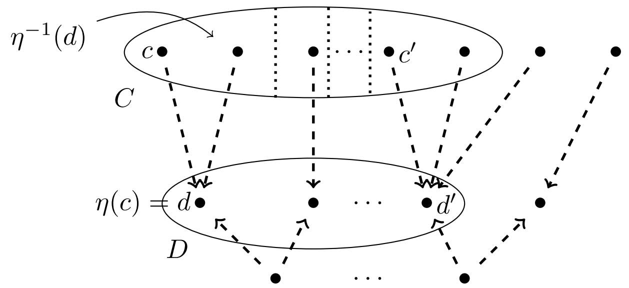
Proof. Recall that by Lemma 5.8, for every absorption-free subuniverse C of A or B there exists $k > 0$ such that $\mathrm{dist}_E(c, o) = k$ for all $c \in C$. We will proceed by induction on this distance k. The base step, $k = 1$, follows from Lemma 5.13 from the previous subsection, since in that case $C \leq E_+(o)$.

Let $k > 1$ and assume that $C \leq A$ (the proof for $C \leq B$ is analogous). Let us denote by D the subuniverse $D = E_+(C) \cap E_{k-1}(o) \leq B$. If $D = \{d\}$ for some $d \in B$, then $C \leq E_-(d)$ and C has a weakly pointing operation by Lemma 5.13. Thus we can assume that $|D| > 1$.

The binary relation $E \cap (C \times D)$ induces an onto mapping $\eta : C \to D$ defined by $\eta(c) = d$, where $d \in D$ is unique such that $(c, d) \in E$ (this is because $\mathbb{T}$ is a tree; see the figure below).

The relation $E \cap (C \times D)$ is preserved by every $\varphi \in \mathrm{IdPol}(\mathbb{H})$ (see Lemma 5.1). The following are easy consequences of this fact:

- for every $D' \leq D$ the set $\eta^{-1}(D')$ is a subuniverse of C,
- if $D' \trianglelefteq D$, then $\eta^{-1}(D') \trianglelefteq C$ (the absorbing polymorphism is the same),
- for $D' \leq D$, $D' = D$ if and only if $\eta^{-1}(D') = C$ (since η is onto).



Combining these facts together with the fact that C is absorption-free yields that D is absorption-free. Hence by induction assumption D has a weakly pointing operation.

Let $\varphi \in \mathrm{IdPol}(\mathbb{H})$ (say n-ary) be such that $\varphi|_D$ weakly points D to $\{d\}$ with witnessing tuples $\mathbf{d^1}, \dots, \mathbf{d^n}$. It is easy to verify that $\varphi|_C$ weakly points C to $\eta^{-1}(d)$; any $\mathbf{c^1}, \dots, \mathbf{c^n} \in C^n$ such that $\eta(c^i_j) = d^i_j$ (for $i, j \in [n]$) can serve as witnessing tuples.

Since $\eta^{-1}(d) \le E_-(d)$, it follows from Corollary 5.14 and Theorem 4.8 that $\eta^{-1}(d)$ has a weakly pointing operation. Let $\psi \in \mathrm{IdPol}(\mathbb{H})$ and $c \in \eta^{-1}(d)$ be such that $\psi|_{\eta^{-1}(d)}$ weakly points $\eta^{-1}(d)$ to $\{c\}$. In particular, $\psi|_C$ weakly points $\eta^{-1}(d)$ to $\{c\}$ and thus by Observation 4.6, $(\psi \lessdot \varphi)|_C$ weakly points C to c. □

Remark. In the language of universal algebra, the relation $E \cap (C \times D)$ is the graph of an onto homomorphism $\eta : C \to D$ and thus, by the First Isomorphism Theorem, D is isomorphic to the quotient of C over the kernel of η. The induction step in the previous lemma follows easily from this observation.

Proof of Theorem 2.4 and Corollary 2.5. Let $\mathbb{H}$ be a special tree such that $\mathbf{alg}\,\mathbb{H}$ is Taylor. In Lemma 5.15 we proved that every absorption-free subuniverse of A or B has a weakly pointing operation. By Corollary 4.12, both A and B are $\mathrm{SD}(\wedge)$ and thus it follows from Corollary 5.4 that $\mathbf{alg}\,\mathbb{H}$ is $\mathrm{SD}(\wedge)$.

It is easy to see that the core of a special tree is again a special tree. If $\mathbb{H}$ is a core, then either $\mathbf{alg}\,\mathbb{H}$ is not Taylor, in which case $\mathrm{CSP}(\mathbb{H})$ is **NP**-complete by Theorem 4.3, or $\mathbf{alg}\,\mathbb{H}$ is $\mathrm{SD}(\wedge)$ and $\mathbb{H}$ has bounded width by Theorem 4.4. □

6. Discussion

We believe that given the evidence, it is reasonable to conjecture that our result generalizes to all oriented trees. Moreover, we hope that the techniques developed in this paper will be useful in pursuit of the proof.

Conjecture 2. *For every oriented tree $\mathbb{H}$, either $\mathbf{alg}\,\mathbb{H}$ is not Taylor or it is $\mathrm{SD}(\wedge)$. In particular, if $\mathbb{H}$ is a core, then $\mathbb{H}$ has bounded width or $\mathrm{CSP}(\mathbb{H})$ is **NP**-complete.*

The reader may wonder why we need two different characterizations of $\mathrm{SD}(\wedge)$ algebras, i.e., why we use WNU operations for the proof of Corollary 5.4. The reason is that our techniques used later in the proof are not well suited to deal with non-diagonal components of connectivity of powers of $\mathbb{H}$. This is one obstacle to generalizing the result to all oriented trees.

Another shortcoming is that we cannot get a good handle of polymorphisms of higher arities than binary. For example, it follows from Corollary 5.12 that neither A nor B can have a two-element absorption-free subuniverse and in fact, we can prove that $\mathbf{alg}\,\mathbb{H}$ (if it is Taylor) cannot have a two-element absorption-free subuniverse at all (we will not present the argument here, but it is similar in spirit to the proof of Lemma 5.3). We do not know if this result can be extended to more than two elements. Hence the following open problem.

Problem. Let $\mathbb{H}$ be a (special, or any oriented) tree such that $\mathbf{alg}\,\mathbb{H}$ is Taylor. Is $\mathbf{alg}\,\mathbb{H}$ *always absorbing*?

A finite idempotent algebra $\mathbf{A}$ is *always absorbing*, if for every nonempty $B \leq \mathbf{A}$ there exists $b \in B$ such that $\{b\} \trianglelefteq B$. (Equivalently, there are no absorption-free algebras in the pseudovariety generated by $\mathbf{A}$, see [9, Proposition 2.1].) By Corollary 4.12, always absorbing algebras are SD($\wedge$). A positive answer to this problem would significantly simplify our proof.

Special balanced digraphs, a natural relaxation of the definition of special trees to balanced digraphs, appear naturally in the reduction of constraint satisfaction problems to digraph $\mathbb{H}$-coloring [17, 18, 20]. The reader may notice similarities with some of the proofs in [18]. Can our techniques be adapted to obtain interesting results about special balanced digraphs?

Acknowledgements

The author would like to thank Libor Barto for thoughtful comments and discussions and valuable inputs which improved this paper in several places.

References

[1] Barto, L.: The dichotomy for conservative constraint satisfaction problems revisited. In: In Proceedings of the 26th Annual IEEE Symposium on Logic in Computer Science, LICS 2011, pp. 21–24. IEEE Computer Society (2011)

[2] Barto, L., Bulín, J.: CSP dichotomy for special polyads. International Journal of Algebra and Computation **23**(5), 1151–1174 (2013)

[3] Barto, L., Kozik, M.: Constraint satisfaction problems of bounded width. In: 50th Annual IEEE Symposium on Foundations of Computer Science, 2009. FOCS '09, pp. 595–603 (2009)

[4] Barto, L., Kozik, M.: Absorbing subalgebras, cyclic terms, and the constraint satisfaction problem. Logical Methods in Computer Science **8**(1), 1:07, 27 (2012)

[5] Barto, L., Kozik, M.: Robust satisfiability of constraint satisfaction problems. In: Proceedings of the Forty-fourth Annual ACM Symposium on Theory of Computing, STOC '12, pp. 931–940. ACM, New York, NY, USA (2012)

[6] Barto, L., Kozik, M.: Constraint satisfaction problems solvable by local consistency methods. Journal of the ACM **61**(1), Art. 3, 19 (2014)

[7] Barto, L., Kozik, M., Maróti, M., Niven, T.: CSP dichotomy for special triads. Proceedings of the American Mathematical Society **137**(9), 2921–2934 (2009)

[8] Barto, L., Kozik, M., Niven, T.: The CSP dichotomy holds for digraphs with no sources and no sinks (a positive answer to a conjecture of Bang-Jensen and Hell). SIAM Journal on Computing **38**(5), 1782–1802 (2008)

[9] Barto, L., Kozik, M., Stanovský, D.: Mal'tsev conditions, lack of absorption, and solvability. Manuscript (2013). URL `http://www.karlin.mff.cuni.cz/~stanovsk/math/maltsev_absorption_solvability.pdf`

[10] Bergman, C.: Universal Algebra: Fundamentals and Selected Topics, 1st edn. Chapman and Hall/CRC, Boca Raton (2011)

[11] Berman, J., Idziak, P., Marković, P., McKenzie, R., Valeriote, M., Willard, R.: Varieties with few subalgebras of powers. Transactions of the American Mathematical Society **362**(3), 1445–1473 (2010)

[12] Bodnarčuk, V.G., Kalužnin, L.A., Kotov, V.N., Romov, B.A.: Galois theory for post algebras. i, II. Otdelenie Matematiki, Mekhaniki i Kibernetiki Akademii Nauk Ukrainskoi SSR. Kibernetika (3), no. 5, 1–9 (1969)

[13] Bulatov, A.: Tractable conservative constraint satisfaction problems. In: Proceedings of the 18th Annual IEEE Symposium on Logic in Computer Science, LICS '03, p. 321. IEEE Computer Society, Washington, DC, USA (2003)

[14] Bulatov, A.: A dichotomy theorem for constraint satisfaction problems on a 3-element set. J. ACM **53**(1), 66–120 (2006)

[15] Bulatov, A., Jeavons, P., Krokhin, A.: Classifying the complexity of constraints using finite algebras. SIAM Journal on Computing **34**(3), 720–742 (2005)

[16] Bulatov, A., Valeriote, M.: Recent results on the algebraic approach to the CSP. In: Complexity of Constraints, pp. 68–92 (2008)
[17] Bulín, J., Delić, D., Jackson, M., Niven, T.: On the reduction of the CSP dichotomy conjecture to digraphs. In: C. Schulte (ed.) Principles and Practice of Constraint Programming, no. 8124 in Lecture Notes in Computer Science, pp. 184–199. Springer Berlin Heidelberg (2013)
[18] Bulín, J., Delić, D., Jackson, M., Niven, T.: A finer reduction of constraint problems to digraphs. arXiv:1406.6413 [cs, math] (2014). URL `http://arxiv.org/abs/1406.6413`
[19] Feder, T.: Classification of homomorphisms to oriented cycles and of k-partite satisfiability. SIAM J. Discret. Math. **14**(4), 471–480 (2001)
[20] Feder, T., Vardi, M.Y.: The computational structure of monotone monadic SNP and constraint satisfaction: a study through datalog and group theory. SIAM Journal on Computing **28**(1), 57–104 (electronic) (1999)
[21] Geiger, D.: Closed systems of functions and predicates. Pacific Journal of Mathematics **27**, 95–100 (1968)
[22] Gutjahr, W., Welzl, E., Woeginger, G.: Polynomial graph-colorings. Discrete Applied Mathematics. The Journal of Combinatorial Algorithms, Informatics and Computational Sciences **35**(1), 29–45 (1992)
[23] Hell, P., Nešetřil, J.: Graphs and Homomorphisms. Oxford University Press, Oxford; New York (2004)
[24] Hell, P., Nešetřil, J., Zhu, X.: Complexity of tree homomorphisms. Discrete Applied Mathematics **70**(1), 23–36 (1996)
[25] Hell, P., Nešetřil, J., Zhu, X.: Duality and polynomial testing of tree homomorphisms. Transactions of the American Mathematical Society **348**(4), 1281–1297 (1996)
[26] Hell, P., Nešetřil, J.: On the complexity of H-coloring. J. Comb. Theory Ser. B **48**(1), 92–110 (1990)
[27] Häggkvist, R., Hell, P., Miller, D.J., Neumann Lara, V.: On multiplicative graphs and the product conjecture. Combinatorica. An International Journal of the János Bolyai Mathematical Society **8**(1), 63–74 (1988)
[28] Idziak, P., Mckenzie, R., Willard, R.: Tractability and learnability arising from algebras with few subpowers. In: In LICS'07, pp. 213–224. IEEE Computer Society (2007)
[29] Jeavons, P., Cohen, D., Gyssens, M.: Closure properties of constraints. J. ACM **44**(4), 527–548 (1997)
[30] Larose, B., Tesson, P.: Universal algebra and hardness results for constraint satisfaction problems. Theoretical Computer Science **410**(18), 1629–1647 (2009). Automata, Languages and Programming (ICALP 2007)
[31] Larose, B., Zádori, L.: Bounded width problems and algebras. Algebra universalis **56**(3-4), 439–466 (2007)
[32] Maróti, M., McKenzie, R.: Existence theorems for weakly symmetric operations. Algebra Universalis **59**(3-4), 463–489 (2008)
[33] Schaefer, T.J.: The complexity of satisfiability problems. In: Proceedings of the Tenth Annual ACM Symposium on Theory of Computing, STOC '78, pp. 216–226. ACM, New York, NY, USA (1978)

Department of Algebra, Faculty of Mathematics and Physics, Charles University in Prague, Czech Republic
E-mail address: `jakub.bulin@gmail.com`