What is... a Modular Curve

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Overview

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- Modular Curves
- Higher Dimensions

Modular Forms

The *modular group* $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ as follows:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathcal{H} \to \mathcal{H}$$
$$\tau \mapsto \frac{a\tau + b}{c\tau + d}$$

Definition

A *modular form* of weight k is a holomorphic map $f : \mathcal{H} \to \mathbb{C}$ such that:

1. $f(\gamma(\tau)) = (c\tau + d)^k f(\tau)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ y $\tau \in \mathcal{H}$,
2. f es "holomorphic at ∞".

Fourier Expansion

Condition 1 applied to the matrix

$$\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z})$$

tells us $f(\tau+1) = f(\tau)$ for all $\tau \in \mathcal{H}$. Then $f(\tau) = g(e^{2\pi i \tau})$. We say f is holomorphic at ∞ iff g admits a holomorphic extension to 0, i.e., f has a Fourier expansion

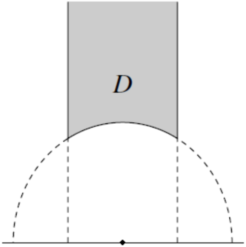
$$f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n \tau}$$

It is in these Fourier coefficients that we often find information relevant to some number theoretic problems.

Moduli of elliptic curves

Each $\tau \in \mathcal{H}$ determines an elliptic curve $E_\tau = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$, and one has $E_\tau \cong E_{\tau'}$ if and only if $\tau' = \gamma(\tau)$ for some $\gamma \in \mathsf{SL}_2(\mathbb{Z})$.

Hence the quotient $Y := \mathsf{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ parametrizes elliptic curves up to isomorphism. This is an example of a *modular curve*.



Dominio fundamental

The quotient $X := \mathsf{SL}_2(\mathbb{Z})\backslash(\mathcal{H} \cup \mathbb{Q} \cup \infty)$ is a compactification of Y.

Congruence subgroups

Let $N \in \mathbb{Z}_{\geq 1}$. The *principal congruence subgroup of level N* is

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (mod \;\; N) \right\}$$

A subgroup $\Gamma \leq \mathrm{SL}_(\mathbb{Z})$ is called a *congruence subgroup* of level N iff $\Gamma(N) \subseteq \Gamma$.

Examples

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (mod \;\; N) \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} (mod \;\; N) \right\}$$

One has $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$

Modular Curves

Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup, acting on the upper half plane $\mathcal{H}$ from the left. The *modular curve*$Y(\Gamma)$ is defined as the quotient space of orbits under Γ,

$$Y(\Gamma) = \Gamma \backslash \mathcal{H} = \{\Gamma\tau : \tau \in \mathcal{H}\}$$

These are compactified by adding the "cusps", which are the Γ-equivalence classes of points in $\mathbb{Q} \cup \{\infty\}$.

Of particular importance are the modular curves

$$Y_0(N) = \Gamma_0(N) \backslash \mathcal{H}, \quad Y_1(N) = \Gamma_1(N) \backslash \mathcal{H}, \quad Y(N) = \Gamma(N) \backslash \mathcal{H}$$

The action on the Siegel upper half plane

Let $g \geq 2$. We consider the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$, consisting of the matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A, B, C, D \in \mathrm{Mat}(g \times g, \mathbb{Z})$ satisfy $AB^t = BA^t, CD^t = DC^t$, $AD^t - BC^t = 1_g$. We also consider

$$\mathcal{H}_g := \{\tau \in \mathrm{Mat}(g \times g, \mathbb{C}) \mid \tau^t = \tau, \Im(\tau) > 0\},$$

and the action $\mathrm{Sp}(2g, \mathbb{Z}) \times \mathcal{H}_g \to \mathcal{H}_g$ given by

$$\begin{aligned} \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathcal{H}_g &\to \mathcal{H}_g \\ \tau &\mapsto (A\tau + B)(C\tau + D)^{-1} \end{aligned}$$

Siegel Modular Forms

Definition

A *Siegel modular form* of genus g and weight k is a holomorphic function $f : \mathcal{H}_g \to \mathbb{C}$ such that

$$f(\gamma(\tau)) = \det(C\tau + D)^k f(\tau)$$

for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z})$ and all $\tau \in \mathcal{H}_g$.They constitute a vector space M_k^g.

Moduli of Abelian Varieties

Each $\tau \in \mathcal{H}_g$ determines a principally polarized abelian variety $A_\tau := \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$, and one has $A_\tau \cong A_{\tau'}$ if and only if $\tau' = \gamma(\tau)$ for some $\gamma \in \mathsf{Sp}(2g, \mathbb{Z})$.

The quotient $\mathcal{A}_g := \mathsf{Sp}(2g, \mathbb{Z})\backslash\mathcal{H}_g$ is the moduli space of principally polarized abelian varieties.