Problem 1.1. Let k be a field.

(a) Let R be a k-algebra which is a domain, not necessarily of finite type over k. Show that its Krull dimension is at most the transcendence degree of the quotient field of R, i.e. we have
$$\dim R \;\leq\; \operatorname{trdeg}_k(\operatorname{Quot}(R)).$$

(b) Illustrate by an example that in general the above inequality can be strict.

Problem 1.2. Let k be a field. Let C be a normal curve over k that is birational to $\mathbb{P}^1_k$.

(a) Show that C is isomorphic to an open subset of $\mathbb{P}^1_k$.

(b) Deduce that if C is not proper, then it is affine and $\Gamma(C, \mathcal{O}_C)$ is a factorial ring.

Problem 1.3. Let k be an algebraically closed field with $\operatorname{char}(k) \neq 2$.

(a) Show that $A = k[x,y]/(y^2 - x^3 - x)$ is a domain which is integral over $R = k[x]$.

(b) Define an automorphism $\sigma\colon A \to A$ by $\sigma_{|R} = \mathrm{id}$ and $\sigma(y) = -y$. Show that we have a multiplicative map
$$N\colon \quad A \;\to\; R, \quad N(a) \;:=\; a \cdot \sigma(a).$$

(c) Show that $A^\times = k^\times$ and that x, y are irreducible elements in A. Deduce that A is not factorial, and that $\operatorname{Spec}(A)$ is not isomorphic to an open subset of the affine line over k.

Problem 1.4. Let $C \subset \mathbb{P}^2_k$ be the projective closure of $C_0 = \operatorname{Spec}(A) \subset \mathbb{A}^2_k$ from problem 1.3.

(a) Show that C is normal and the complement $C \setminus C_0$ consists of a single point o.

(b) Let $p, q, r \in C$ be distinct closed points. Show that the following are equivalent:

- There is a rational function $f \in k(C)^\times$ with $\operatorname{div}(f) = [p] + [q] + [r] - 3[o]$.
- The three points p, q, r are collinear, i.e. they lie on a common line $\ell \subset \mathbb{P}^2_k$.

(c) What does this say about the Picard group $\operatorname{Pic}(C)$?