Wavelets Homework 1

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Contents

- Mathematical foundation: Haar transform theory
- Implementation: Fast Haar Transform algorithms
- Analysis: Understanding coefficient patterns
- Applications: Compression and approximation

Mathematical Foundation

- Haar basis functions and their properties
- Matrix construction for discrete Haar transform
- Complexity analysis: $\mathcal{O}(n \log n)$ vs $\mathcal{O}(n)$
- Theoretical bounds and properties

Exercise 1: Take T_2 as a start

We compute T_2:

$$\begin{bmatrix} s & s & 0 & 0 \\ s & -s & 0 & 0 \\ 0 & 0 & s & s \\ 0 & 0 & s & -s \end{bmatrix} \begin{pmatrix} c_{20} \\ c_{21} \\ c_{22} \\ c_{23} \end{pmatrix} = \begin{pmatrix} c_{10} \\ h_{10} \\ c_{11} \\ h_{11} \end{pmatrix}$$

$$\begin{bmatrix} s & 0 & s & 0 \\ s & 0 & -s & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} c_{10} \\ h_{10} \\ c_{11} \\ h_{11} \end{pmatrix} = \begin{pmatrix} c_{00} \\ h_{00} \\ h_{10} \\ h_{11} \end{pmatrix}$$

$$\begin{bmatrix} s & 0 & s & 0 \\ s & 0 & -s & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s & s & 0 & 0 \\ s & -s & 0 & 0 \\ 0 & 0 & s & s \\ 0 & 0 & s & -s \end{bmatrix} = \begin{bmatrix} s^2 & s^2 & s^2 & s^2 \\ s^2 & s^2 & -s^2 & -s^2 \\ s & -s & 0 & 0 \\ 0 & 0 & s & -s \end{bmatrix}$$

Exercise 1: Construction of T_j

When computing T_3, we get the following matrix construction:

$$\left[\begin{array}{cccccccc}
* & * & * & * & * & * & * & * \\
\hline
* & * & * & * & * & * & * & * \\
\hline
* & * & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & * & * \\
\hline
* & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & *
\end{array}\right]$$

With this idea in mind, we can compute the amount of nonzero entries for all T_j:

- For every box in the matrix T_j, there are 2^j nonzero entries;
- There are $j+1$ boxes;
- The total amount of nonzero entries is $(j+1)\cdot 2^j$.

Exercise 2: Complexity of T_j

- Take the amount of nonzeros of $T_j := (j+1) \cdot 2^j$;
- Assume $2^j = n$;
- Note that we apply also $(j+1) \cdot 2^j$ multiplications;
- Then the complexity of T_j is $\mathcal{O}(n \log n)$.

Exercise 3: How about T_j^{-1}

As we've seen in the lecture, $T_j^{-1} = T_j^T, \forall j \in \mathbb{N}$, thus:

$$T_2^{-1} = \begin{bmatrix} s^2 & s^2 & s & 0 \\ s^2 & s^2 & -s & 0 \\ s^2 & -s^2 & 0 & s \\ s^2 & -s^2 & 0 & -s \end{bmatrix} \qquad T_3^{-1} : \left[\begin{array}{c|c|cc|cccc} * & * & * & 0 & * & 0 & 0 & 0 \\ * & * & * & 0 & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & * & 0 & 0 \\ * & * & * & 0 & 0 & * & 0 & 0 \\ * & * & 0 & * & 0 & 0 & * & 0 \\ * & * & 0 & * & 0 & 0 & * & 0 \\ * & * & 0 & * & 0 & 0 & 0 & * \\ * & * & 0 & * & 0 & 0 & 0 & * \end{array}\right]$$

Furthermore, The amount of nonzeros entries hasn't changed, thus this is still equal to $(j+1) \cdot 2^j$.

Code Inspection: DHT Implementation

- `DHT` algorithm: sparse matrix multiplication
- Inverse transform: `iDHT`

Code Inspection: DHT Implementation

- DHT algorithm: sparse matrix multiplication
- Inverse transform: `iDHT`

$$
\begin{array}{cc}
\left[\begin{array}{cccccccc}
s^3 & s^3 & s^3 & s^3 & s^3 & s^3 & s^3 & s^3 \\ \hline
s^3 & s^3 & s^3 & s^3 & -s^3 & -s^3 & -s^3 & -s^3 \\ \hline
s^2 & s^2 & -s^2 & -s^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & s^2 & s^2 & -s^2 & -s^2 \\ \hline
s & -s & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & s & -s & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & s & -s & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & s & -s
\end{array}\right]
&
\begin{array}{c}
\\ (0) \\ \\ (0) \\ \\ \\ \\ (0)
\end{array}
\end{array}
$$

We want a loop that accesses exactly these entries from a virtual matrix.

Code Inspection: DHT Implementation

- DHT algorithm: sparse matrix multiplication
- Inverse transform: `iDHT`

Within and outside of the intervals [`start`, `mid`) and [`mid`, `end`), the values of the matrix are constant.

$$
\begin{array}{cc}
\left[\begin{array}{cccccccc}
s^3 & s^3 & s^3 & s^3 & s^3 & s^3 & s^3 & s^3 \\
\hline
s^3 & s^3 & s^3 & s^3 & -s^3 & -s^3 & -s^3 & -s^3 \\
\hline
s^2 & s^2 & -s^2 & -s^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & s^2 & s^2 & -s^2 & -s^2 \\
\hline
s & -s & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & s & -s & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & s & -s & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & s & -s
\end{array}\right]
&
\begin{array}{c}
\\
(0) \\
\\
(0) \\
\\
\\
\\
(0)
\end{array}
\end{array}
$$

We want a loop that accesses exactly these entries from a virtual matrix.

Exercise 4: Multiplications of the FHT

- Take $j \in \mathbb{N}$. The first matrix is of size $2^j \times 2^j$;
- The amount of multiplications in the first step of the FHT is $2 \cdot 2^j$ (the amount of nonzero elements);
- Going one layer lower leaves a matrix of size $2^{j-1} \times 2^{j-1}$;
- This gives us $2 \cdot 2^{j-1}$ multiplications;
- the scaler difference of mulitplications of going from j to $j-1$ is a factor of 2.

$$\text{As for } T_3\text{:} \quad \left[\begin{array}{c|c|c|c} S & 0 & 0 & 0 \\ \hline 0 & S & 0 & 0 \\ \hline 0 & 0 & S & 0 \\ \hline 0 & 0 & 0 & S \end{array}\right]$$

Exercise 4: Multiplications and complexity

What is the total amount of multiplications needed?

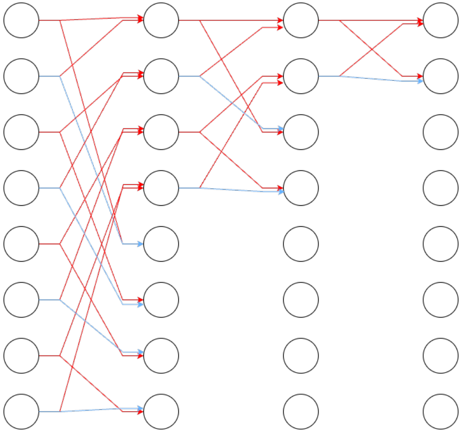
- We want to add up all the multiplications $2 \cdot 2^k$, where $1 \leq k \leq j$;
- This is equal to $\sum_{k=1}^{j} 2^{k+1} = 2^{j+2} - 2$.

What is the corresponding complexity?

- Assume again $2^j = n$;
- This gives $4n - 2$, which has complexity $\mathcal{O}(n)$.

Code Inspection: FHT Implementation

- FHT function implementation
- Inverse transform: `iFHT`



Live Demo: Function Reconstruction

- Sample a test function
- Apply `FHT` / `DHT` to get Haar coefficients
- Reconstruct using inverse `FHT` / `DHT`

Exercise 7: Hölder continuity

We eventually want to show that $|\langle f, H_{jk}\rangle| \leq C2^{-j(\alpha+1/2)}$, $C \in \mathbb{R}$.

- We are given that f is Hölder continuous with coefficient $\alpha \in [0, 1)$;
- $\exists c \in \mathbb{R}$ such that $|f(x) - f(y)| \leq c|x - y|^{\alpha}$, $\forall x, y \in \mathbb{R}$;
- We also know that H_{jk} is non-zero on the interval $[2^{-j}k, 2^{-j}(k+1)]$ and $|h_{jk}| = |\langle f, H_{jk}\rangle|$

Throughout the computation, we apply the substution $u = t - 2^{-j-1}$, which we denote as 'sub.'

Exercise 7: Proof of the inequality

$$
\begin{aligned}
|\langle f, H_{jk}\rangle| &= \left|\int_{2^{-j}k}^{2^{-j}(k+1)} f(t)H_{jk}(t)\,dt\right| \\
&= 2^{j/2}\left|\int_{2^{-j}k}^{2^{-j}(k+1/2)} f(t)\,dt - \int_{2^{-j}(k+1/2)}^{2^{-j}(k+1)} f(t)\,dt\right| \\
&\overset{\text{sub.}}{=} 2^{j/2}\left|\int_{2^{-j}(k+1/2)}^{2^{-j}(k+1)} f(u+2^{-j-1}) - f(u)\,du\right| \\
&\leq 2^{j/2}\int_{2^{-j}(k+1/2)}^{2^{-j}(k+1)} \left|f(u+2^{-j-1}) - f(u)\right|\,du \\
&\overset{\text{Höl.}}{\leq} c\cdot 2^{j/2}2^{(-j-1)\alpha} \\
&= C\cdot 2^{-j(\alpha+1/2)}.
\end{aligned}
$$

Analysis: Understanding Coefficient Patterns

- Multi-resolution structure of Haar coefficients
- Coefficient decay across scales
- Theoretical bounds for Hölder continuous functions
- Empirical verification and sampling methods

Live Demo: Coefficient Analysis

$$|\langle f, H_{jk} \rangle| \leq C 2^{-j(\alpha+1/2)}$$

- Visualize coefficients at different layers
- Compute ratios of consecutive layers
- Compare theoretical bounds with empirical results
- Effect of sampling methods on bounds

Exercise 10: Invariance under translation and dialation

Define l such that $\forall 0 \leq q \leq l$, $\int_{\mathbb{R}} x^q W(x)\, dx = 0$. Does a translation or dialation on $W(x)$ gives back zero?

- We start by writing $2^{j/2} W(2^{-j}x - k)$ for $W(x)$ as the translation and dialation;
- This results in $2^{j/2} \int_{\mathbb{R}} x^q W(2^{-j}x - k)\, dx$, which we want to solve.

We start by applying the substitution: $x \longmapsto x + \frac{k}{2^{-j}}$. This gives us:

$$2^{j/2} \int_{\mathbb{R}} \left(x + \frac{k}{2^{-j}}\right)^q W(x)\, dx.$$

Exercise 10: Invariance under translation and dialation

To go further, we see:

$$\left(x + \frac{k}{2^{-j}}\right)^q = x^q + \frac{k}{2^{-j}} x^{q-1} + \cdots + \frac{k^q}{2^{-qj}}$$

As integrals are linear, we can rewrite the previous integral as:

$$2^{j/2} \int_{\mathbb{R}} x^q W(x)\, dx + \frac{2^{j/2} k}{2^{-j}} \int_{\mathbb{R}} x^{q-1} W(x)\, dx + \cdots + \frac{2^{j/2} k^q}{2^{-qj}} \int_{\mathbb{R}} W(x)\, dx.$$

We already knew that $\int_{\mathbb{R}} x^q W(x)\, dx = 0$, $\forall 0 \leq q \leq l$. Which implies:

$$2^{j/2} \int_{\mathbb{R}} \left(x + \frac{k}{2^{-j}}\right)^q W(x)\, dx = 0.$$

Applications: Compression and Approximation

- N-term approximation in Haar basis
- Coefficient sorting and thresholding
- Error analysis and compression efficiency
- Practical applications to discontinuous functions

Live Demo: Compression

- Apply FHT to discontinuous function
- Sort coefficients by magnitude
- Compute best N-term approximation error
- Visualize compressed reconstruction

Thank You

Presentation Script: Part 1 - Foundation

- **Mathematical Foundation**: Introduce Haar basis functions, matrix construction, and complexity analysis as the theoretical foundation for our implementation work.
- **Exercise 1: Matrix construction T_j**: Show how T_2 matrix construction reveals the recursive pattern that leads to the general T_j structure with (j+1)·2^j nonzero entries.
- **Exercise 2: Complexity analysis O(n log n)**: Derive O(n log n) complexity for the direct matrix approach by counting multiplications in the T_j matrix operations.
- **Exercise 3: Inverse transform properties**: Demonstrate that T_j^{-1} = T_j^T due to orthogonality, maintaining the same complexity but enabling efficient inverse transforms.
- **Code Inspection: DHT Implementation**: Show the DHT algorithm implementation with visual matrix pattern, demonstrating how the T_j structure maps to sparse matrix multiplication with colored intervals.

Presentation Script: Part 2 - Implementation

- **Exercise 4: Multiplications and complexity**: Complete the FHT complexity analysis showing total multiplications and O(n) complexity, bridging from DHT theory to FHT implementation.
- **Code Inspection: FHT Implementation**: Show the FHT algorithm implementation with visual diagram, demonstrating the iterative approach and comparing with DHT matrix-based method.
- **Live Demo: Function Reconstruction**: Sample a test function and apply FHT/DHT to get Haar coefficients, then reconstruct using inverse transforms. Display visual comparison showing how the mathematical theory translates into accurate function approximation.

Presentation Script: Part 3 - Analysis

- **Exercise 7: Hölder continuity**: Establish theoretical bounds $|\langle f, H_{jk} \rangle| \leq C2^{-j(\alpha+1/2)}$ for Hölder continuous functions, providing the foundation for understanding coefficient decay across scales.
- **Exercise 7: Proof of the inequality**: Complete the mathematical proof of the Hölder continuity bounds, showing the step-by-step derivation and substitution techniques.
- **Analysis: Understanding Coefficient Patterns**: Bridge from theoretical bounds to empirical analysis by introducing multi-resolution coefficient patterns and explaining how the Hölder continuity bounds provide the theoretical foundation.
- **Live Demo: Coefficient Analysis**: Visualize coefficients at different layers using draw_coefficients_at_layer() and compute ratios using compute_ratios(). Compare theoretical bounds with empirical results, showing how sampling methods affect the bounds.
- **Exercise 10: Invariance properties**: Prove invariance properties of

Presentation Script: Part 4 - Applications

- **Applications: Compression and Approximation**: Introduce N-term approximation and compression as practical applications of the theoretical foundation, explaining how the coefficient decay properties enable effective compression strategies.
- **Live Demo: Compression**: Apply FHT to a discontinuous function, sort coefficients by magnitude, and compute best N-term approximation error. Visualize compressed reconstruction showing how the mathematical theory enables practical compression with quantifiable error bounds.
- **Thank You**: Conclude the presentation and open for questions.