Inputs: $A^{n\times m}$, $v^{m\times 1}$. Outputs: $L^{n\times k}$, $R^{m\times k}$, $B^{k\times k}$, $c^{1\times 1}$, $r^{m\times 1}$.

Theorem 1. *Assume $AR = LB$, $L^\top A = BR^\top + e_k\phi^\top$ with invertible B, let all columns in L and R have unit length, and assume $I_\gg \circ B = 0$. Then, the following two three conditions are equivalent:*

1. *$I_< \circ (L^\top L - I) = 0$ (which is equivalent to $L^\top L = I$)*
2. *$I_< \circ B = 0$*
3. *$I_< \circ (R^\top R - I) = 0$ (which is equivalent to $R^\top R = I$).*

Proof. As a preliminary, consider:

$$L^\top LB = L^\top AR = BR^\top R + e_k\phi^\top R. \tag{1}$$

We prove the theorem by induction over k.

Let $k = 1$. We assume that the columns in L have unit length, so the first condition is always true. B is a single element, so the second condition is always true. We assume that the columns in R have unit length, so the third condition is always true. Since all three conditions are always true, they are equivalent (though the term "equivalent" is a bit of a farce here).

For the induction step, assume that the three conditions are equivalent for some k. Let L, R, and B be as before, and let ℓ^n, r^m, b^k, c^k, and a^1 be as follows:

$$\hat{L} = \begin{pmatrix} L & \ell \end{pmatrix}, \quad \hat{R} = \begin{pmatrix} R & r \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} B & b \\ c^\top & a \end{pmatrix}. \tag{2}$$

Then,

$$\hat{L}^\top \hat{L} = \begin{pmatrix} L^\top L & L^\top \ell \\ \ell^\top L & 1 \end{pmatrix} \quad \hat{R}^\top \hat{R} = \begin{pmatrix} R^\top R & R^\top r \\ r^\top R & 1 \end{pmatrix} \tag{3}$$

and

$$\hat{L}^\top \hat{L}\hat{B} = \begin{pmatrix} \star & L^\top Lb + L^\top \ell a \\ \star & \star \end{pmatrix}, \quad \hat{B}\hat{R}^\top \hat{R} - e_{k+1}\hat{\phi}^\top \hat{R} = \begin{pmatrix} \star & BR^\top r + b \\ \star & \star \end{pmatrix} \tag{4}$$

where "$\star$" represents unimportant values. Summarise the preliminary equation as

$$\begin{pmatrix} \star & L^\top Lb + L^\top \ell a \\ \star & \star \end{pmatrix} = \begin{pmatrix} \star & BR^\top r + b \\ \star & \star \end{pmatrix}. \tag{5}$$

Now, prove that $\hat{L}^\top \hat{L} = I$ implies $\hat{R}^\top \hat{R} = I$. To this end, assume $L^\top L = I$ and $L^\top \ell = 0$. Then, $R^\top R = I$ by the induction hypothesis, and it remains to show $R^\top r = 0$. The northeast block becomes $L^\top \ell a = BR^\top r$, and since B is invertible, $R^\top r = 0$. With the same technique, we can show that $R^\top R = I$ and $R^\top r = 0$ implies $L^\top L = I$ and $L^\top \ell = 0$. In summary, the conditions $\hat{R}^\top \hat{R} = I$ and $\hat{L}^\top \hat{L} = I$ are equivalent.

To show equivalence between $\hat{L}^\top \hat{L} = I$ and $\hat{B}$ being bidiagonal, consider $\hat{B} = \hat{L}^\top A \hat{R}$,

$$\begin{pmatrix} B & b \\ c^\top & a \end{pmatrix} = \begin{pmatrix} \star & \star \\ \ell^\top AR & \star \end{pmatrix}. \tag{6}$$

Zoom in on the southwest block and use $AR = LB$,

$$c^\top = \ell^\top AR = \ell^\top LB. \tag{7}$$

Now, if $\hat{L}^\top \hat{L} = I$ holds, $\ell^\top L = 0$ holds, thus $c^\top$ must be zero since B is invertible. The reverse is also true. Therefore, $I_< \circ B$ is equivalent to $L^\top L = I$, and the proof is complete. $\square$

Here is an application of the above result. Usually, we require $I_< \circ B = 0$. But what follows is a derivation of the adjoints where $I_< \circ (R^\top R - I) = 0$ holds.

Constraints:

$$\begin{aligned}
AR - LB &= 0^{n\times k} & (8)\\
A^\top L - RB^\top - re_k^\top &= 0^{m\times k} & (9)\\
Re_1 - vc &= 0^{m\times 1} & (10)\\
I_= \circ (L^\top L - I) &= 0^{k\times k} & (11)\\
I_\leq \circ (R^\top R - I) &= 0^{k\times k} & (12)\\
I_\gg \circ B &= 0^{k\times k} & (13)
\end{aligned}$$

Differentiate:

$$\begin{aligned}
(\mathrm{d}A)R + A\mathrm{d}R - (\mathrm{d}L)B - L\mathrm{d}B &= 0^{n\times k} & (14)\\
(\mathrm{d}A)^\top L + A^\top \mathrm{d}L - (\mathrm{d}R)B^\top - R(\mathrm{d}B)^\top - (\mathrm{d}r)e_k^\top &= 0^{m\times k} & (15)\\
(\mathrm{d}R)e_1 - (\mathrm{d}v)c - v\mathrm{d}c &= 0^{m\times 1} & (16)\\
I_= \circ ((\mathrm{d}L)^\top L + L^\top \mathrm{d}L) &= 0^{k\times k} & (17)\\
I_\leq \circ ((\mathrm{d}R)^\top R + R^\top \mathrm{d}R) &= 0^{k\times k} & (18)\\
I_\gg \circ (\mathrm{d}B) &= 0^{k\times k} & (19)
\end{aligned}$$

Lagrange multipliers: $\Lambda_1^{n\times k}$, $\Lambda_2^{m\times k}$, $\lambda_3^{m\times 1}$, $\Lambda_4^{k\times k}$, $\Lambda_5^{k\times k}$, $\Lambda_6^{k\times k}$:

$$\begin{aligned}
\mathrm{d}\mu &= \langle \nabla_L, \mathrm{d}L\rangle + \langle \nabla_R, \mathrm{d}R\rangle + \langle \nabla_B, \mathrm{d}B\rangle + \langle \nabla_c, \mathrm{d}c\rangle + \langle \nabla_r, \mathrm{d}r\rangle && (20)\\
&= \langle \nabla_L, \mathrm{d}L\rangle + \langle \nabla_R, \mathrm{d}R\rangle + \langle \nabla_B, \mathrm{d}B\rangle + \langle \nabla_c, \mathrm{d}c\rangle + \langle \nabla_r, \mathrm{d}r\rangle && (21)\\
&\quad + \langle \Lambda_1, (\mathrm{d}A)R + A\mathrm{d}R - (\mathrm{d}L)B - L\mathrm{d}B\rangle && (22)\\
&\quad + \langle \Lambda_2, (\mathrm{d}A)^\top L + A^\top \mathrm{d}L - (\mathrm{d}R)B^\top - R(\mathrm{d}B)^\top - (\mathrm{d}r)e_k^\top\rangle && (23)\\
&\quad + \langle \lambda_3, (\mathrm{d}R)e_1 - (\mathrm{d}v)c - v\mathrm{d}c\rangle && (24)\\
&\quad + \langle \Lambda_4, I_= \circ ((\mathrm{d}L)^\top L + L^\top \mathrm{d}L)\rangle && (25)\\
&\quad + \langle \Lambda_5, I_\leq \circ ((\mathrm{d}R)^\top R + R^\top \mathrm{d}R)\rangle && (26)\\
&\quad + \langle \Lambda_6, I_\gg \circ (\mathrm{d}B)\rangle && (27)\\
&= \langle Z_L, \mathrm{d}L\rangle + \langle Z_R, \mathrm{d}R\rangle + \langle Z_B, \mathrm{d}B\rangle + \langle z_c, \mathrm{d}c\rangle + \langle z_r, \mathrm{d}r\rangle + \langle Z_A, \mathrm{d}A\rangle + \langle z_v, \mathrm{d}v\rangle && (28)
\end{aligned}$$

with the adjoint system determined by

$$\begin{aligned}
Z_L &= \nabla_L - \Lambda_1 B^\top + A\Lambda_2 + 2L(I_= \circ \Lambda_4) && (29)\\
Z_R &= \nabla_R + A^\top \Lambda_1 + \Lambda_2 B + \lambda_3 e_1^\top + R[(I_\leq \circ \Lambda_5) + (I_\leq \circ \Lambda_5)^\top] && (30)\\
Z_B &= \nabla_B - L^\top \Lambda_1 - \Lambda_2^\top R + I_\gg \circ \Lambda_6 && (31)\\
z_c &= \nabla_c - v^\top \lambda_3 && (32)\\
Z_r &= \nabla_r - \Lambda_2 e_k && (33)
\end{aligned}$$

and the gradients

$$\begin{aligned}
Z_A &= \Lambda_1 R^\top + L\Lambda_2^\top && (34)\\
Z_v &= \lambda_3 c^\top (= \lambda_r c) && (35)
\end{aligned}$$

Solution? $Z_r = 0$ yields $\Lambda_2 e_k$. Also,

$$\begin{aligned}
L^\top Z_L &= L^\top \nabla_L - L^\top \Lambda_1 B^\top + L^\top A\Lambda_2 + 2(I_= \circ \Lambda_4) && (36)\\
&= L^\top \nabla_L - (\nabla_B - \Lambda_2^\top R - I_\gg \circ \Lambda_6)B^\top + L^\top A\Lambda_2 + 2(I_= \circ \Lambda_4) && (37)\\
&= L^\top \nabla_L - \nabla_B B^\top + \Lambda_2^\top R B^\top + (I_\gg \circ \Lambda_6)B^\top + L^\top A\Lambda_2 + 2(I_= \circ \Lambda_4) && (38)
\end{aligned}$$

and since the diagonal of $(I_\gg \circ \Lambda_g)B^\top$ must be zero, we get

$$I_= \circ \Lambda_4 = I_= \circ \left(L^\top \nabla_L - \nabla_B B^\top + \Lambda_2^\top R B^\top + L^\top A\Lambda_2\right). \tag{39}$$

Since we have the k-th column of Λ_2, we get the k-th diagonal entry of Λ_4, and this pattern proceeds for all other indices. In other words, $I_= \circ \Lambda_4$ is sorted. Next, get the next column of Λ_1 via

$$Z_L B^{-\top} = \nabla_L - \Lambda_1 + A\Lambda_2 B^{-\top} + 2L(I_= \circ \Lambda_4)B^{-\top} \tag{40}$$

(which should work out, I'm pretty certain...).

Next:

$$R^\top Z_R = R^\top \nabla_R + R^\top A^\top \Lambda_1 + R^\top \Lambda_2 B + R^\top \lambda_3 e_1^\top + (I_\leq \circ \Lambda_5) + (I_\leq \circ \Lambda_5)^\top \tag{41}$$

(todo: use $Z_B = 0$ constraint to eliminate $R\Lambda_2^\top$, select the lower triangular part to eliminate all influence of Λ_6, and get a subset of Λ_5. With this Λ_5, get the next column of Λ_2 via $Z_R = 0$ and repeat the process ad absurdum. Does this give us what we want?)